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Boundedness of the average operator on Orlicz sequence spaces

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague on July 19 2017 Jan Krejčí

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Title: Boundedness of the average operator on Orlicz sequence spaces

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Abstract: The goal of this thesis is to characterize the average operator on Orlicz sequence spaces and to give a condition equivalent to Δ_2^0.

Keywords: Orlicz space, Δ_2^0, average operator, bounded operator, Young function

Contents

Introduction

The main goal of this thesis is to study boundedness of the average operator on Orlicz sequence spaces. For a sequence $a = (a_i)_{i=1}^{\infty}$ of real numbers we define the average operator as follows

$$A(a) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)_{n=1}^{\infty} .$$

Primarily based on the article [1] written by D. Gallardo, whose result for a similar Hardy-Littlewood operator on Orlicz spaces of functions is that the complementary function of the one which the space is associated with, must be Δ_2. In the last chapter, similar result – complementary function of the one, which the *sequence* space is associated with, must be Δ_2^0 for the average operator to be bounded.

In the first chapter, basic theory behind convex functions, which give rise to the Orlicz spaces, is given (cited mainly from [2] and [3]) and then the Orlicz space is defined. In the chapter, both Orlicz and Luxemburg norms are introduced, because they are needed in proofs of sufficiency and necessity of the Δ_2^0-condition respectively.

In the second chapter, the Δ_2^0-condition is defined and equivalent characterization is given. Basically, instead of requirement for the function to satisfy the condition on a certain interval, it suffices to find a sequence with some properties and prove that the condition is met on every term of it. Theorem 2.1 is original and to the knowledge of both the supervisor and the author has not yet been published.

1. Introduction to Orlicz spaces

In the beginning, to understand the behaviour of the average operator, we shall study it on ℓ^p spaces. Then the concept of an Orlicz space will be introduced as well as some basic tools, which will be used to give the necessary and sufficient conditions for the operator A to be bounded on a given Orlicz space.

Theorem 1.1. *The operator $A : \ell^p \to \ell^p$ is bounded if and only if $p > 1$.*

Proof. For $p = 1$, consider the sequence $e_1 = (1, 0, 0, ...)$ from ℓ^1, then the sequence $A(e_1) = (\frac{1}{n})_{n=1}^{\infty}$, being a harmonic series, is not in ℓ^1.

For $p = \infty$, take a bounded sequence y. From the estimate of the nth term,

$$|(A(y))_n| = |\frac{1}{n} \cdot \sum_{i=1}^{n} y_i| \leq \frac{1}{n} \sum_{i=1}^{n} \sup_{k \in \mathbb{N}} |y_k| = \|y\|_\infty ,$$

it follows that $\|A(y)\|_\infty \leq \|y\|_\infty < \infty$.

When $p \in (1, \infty)$ and $z \in \ell^p$, then

$$\|A(z)\|_p = \sum_{i=1}^{\infty} |\frac{1}{i} \sum_{n=1}^{i} z_n|^p \leq \sum_{i=1}^{\infty} (\frac{1}{i} \sum_{n=1}^{i} |z_n|)^p.$$

From Hardy's inequality[1] the last expression has an upper estimate and so we get

$$\|A(z)\|_p \leq (\frac{p}{p-1})^p \sum_{n=1}^{\infty} |z_n|^p = (\frac{p}{p-1})^p \|z\|_p .$$

□

Definition 1.1 (Young function)**.** *A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if it meets the following conditions:*

i) *Φ is a continuous increasing and convex function,*

ii) $\lim_{x \to 0^+} \frac{\Phi(x)}{x} = 0$,

iii) $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$.

Remark 1. It is quite straightforward that any given function defined on any interval $[0, c]$ satisfying the properties i) and ii) from Definition 1.1 can be continuously extended to the interval $[0, \infty)$ in such way that the third property is met as well. One of the ways is to continuously extend the function by an appropriate quadratic function with positive leading coefficient on the interval (c, ∞). From now on, we will consider this extended function.

Definition 1.2 (Orlicz space)**.** *For a Young function Φ, we shall call $(\ell^\Phi, \|\cdot\|_{\ell^\Phi}^{Lux})$ the Orlicz space associated with Φ, where*

$$\ell^\Phi = \{x \in \mathbb{R}^{\mathbb{N}}, \exists \lambda > 0, \sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{\lambda}) \leq 1\}$$

and

$$\|x\|_{\ell^\Phi}^{Lux} = \inf\{\lambda > 0, \sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{\lambda}) \leq 1\}.$$

[1] As mentioned in [4, page 1], for a non-negative sequence (a_n), which is not identically zero, and $p > 1$ it is true that $\sum_{n=1}^{\infty} (\frac{1}{n} \sum_{m=1}^{n} a_m)^p \leq (\frac{p}{p-1})^p \sum_{n=1}^{\infty} a_n^p$.

Remark 2. The function $\|\cdot\|_{\ell^\Phi}^{Lux}$ is called the *Luxemburg norm.* A proof that it is a norm is pretty straightforward[2] and so it is omitted here.

As proven in [2, Prop. 4.a.5], any two Young functions which are equivalent[3] at 0 produce the same Orlicz space, thus regarless of the extension in Remark 1 the associated Orlicz space will be always the same.

Orlicz spaces constitute a generalization of ℓ^p spaces in the sense that ℓ^p is equivalent to ℓ^Φ for $\Phi(x) = \frac{x^p}{p}$. However the Luxemburg norm and the ℓ^p norms are only equivalent (and not always equal).

Theorem 1.2. *[3, chapter 1]*
For any Young function Φ there is a non-decreasing right-continuous function $p : [0, \infty) \to [0, \infty)$ satisfying $p(0) = 0$ and $\lim_{x\to\infty} p(x) = \infty$ such that

$$\Phi(x) = \int_0^x p(s)ds.$$

The function p is called the *density function* of Φ.

Remark 3. Let's denote by q the right generalized inverse of p,

$$q(x) = \sup\{t, p(t) \leq x\}.$$

One can verify that q is a density function. Put $\widetilde{\Phi}(x) = \int_0^x q(s)ds$, then $\widetilde{\Phi}(x)$ is also a Young function and it is called the *complementary function to* Φ. It is true that $\widetilde{\widetilde{\Phi}} = \Phi$ for every Φ.

Theorem 1.3 (Young's inequality)**.** *[2, page 147]*
Let Φ be a Young function, $\widetilde{\Phi}$ the complementary function to Φ and x, $y \geq 0$. Then

$$xy \leq \Phi(x) + \widetilde{\Phi}(y).$$

The equality holds if and only if $y = p(x)$ or $x = q(y)$, where p and q are the right derivatives of Φ and $\widetilde{\Phi}$ respectively.

Definition 1.3 (Orlicz norm)**.** *Let Φ be a Young function and $\widetilde{\Phi}$ the complementary function to Φ. Then for $x \in \ell^\Phi$ we define the Orlicz norm*

$$\|x\|_{\ell^\Phi} = \sup\{|\sum_{n=1}^{\infty} x_n y_n|, \sum_{n=1}^{\infty} \widetilde{\Phi}(|y_n|) \leq 1\}.$$

The sequences y satisfying $\sum_{n=1}^{\infty} \widetilde{\Phi}(|y_n|) \leq 1$ will be referred to as *test sequences.*

Proposition 1.4. *Let Φ be a Young function and $x \in \ell^\Phi$, then*

$$\|x\|_{\ell^\Phi}^{Lux} \leq \|x\|_{\ell^\Phi} \leq 2\,\|x\|_{\ell^\Phi}^{Lux}\,.$$

[2]The proof can be found in Proposition 4.7 (for $\mathbb{F} = \mathbb{R}$) here: `https://www.math.leidenuniv.nl/scripties/BachClaassens.pdf`.

[3]Young functions Φ_1 and Φ_2 are equivalent at 0 if there are positive constants K_1, K_2 and t_0 satisfying for all $t \in [0, t_0]$ inequality $K_1^{-1}\Phi_2(K_2^{-1}t) \leq \Phi_1(t) \leq K_1\Phi_2(K_2 t)$.

Proof. For the second inequality, let's suppose there is a $\lambda > 0$ from the definition of the Luxemburg norm (otherwise the inequality holds trivially). Let y be a sequence such that $\sum_{n=1}^{\infty} \widetilde{\Phi}(|y_n|) \leq 1$. Applying Young's inequality for non-negative pairs of numbers, $\frac{|x_n|}{\lambda}$ and $|y_n|$, we obtain that

$$\frac{|x_n y_n|}{\lambda} \leq \Phi(\frac{|x_n|}{\lambda}) + \widetilde{\Phi}(|y_n|), \ n \in \mathbb{N}.$$

Summing over $n \in \mathbb{N}$, we get that

$$|\sum_{n=1}^{\infty} x_n z_n| \leq \sum_{n=1}^{\infty} |x_n y_n| \leq 2\lambda.$$

Passing to the infimum of λ and then to the supremum of the values of $|\sum_{n=1}^{\infty} x_n y_n|$ the second inequality is verified.

The first inequality is a bit more difficult. First of all, note that for a zero sequence the inequality holds. For the rest, following [2, p. 147], let a sequence x be such that $\|x\|_{\ell^\Phi} = 1$ and set $y_n = p(|x_n|)$, where p is the right derivative of Φ. Then, using Young's inequality, $|x_n| y_n = \Phi(|x_n|) + \widetilde{\Phi}(y_n)$.

If there is $k \in \mathbb{N}$ such that $\sum_{n=1}^{k} \widetilde{\Phi}(y_n) > 1$ then from the convexity,

$$\widetilde{\Phi}(\frac{y_n}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)}) \leq \frac{\widetilde{\Phi}(y_n)}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)}$$

and consequently

$$\sum_{n=1}^{k} \widetilde{\Phi}(\frac{y_n}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)}) \leq \frac{\sum_{n=1}^{k} \widetilde{\Phi}(y_n)}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)} = 1.$$

From the choice of x,

$$\sum_{n=1}^{k} |x_n| \frac{y_n}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)} \overset{x_i, y_j \geq 0}{\leq} \sum_{n=1}^{\infty} |x_n| \frac{y_n}{\sum_{m=1}^{k} \widetilde{\Phi}(y_m)} \leq 1.$$

Thus we obtain:

$$\sum_{n=1}^{k} \widetilde{\Phi}(y_n) \geq \sum_{n=1}^{k} |x_n| y_n = \sum_{n=1}^{k} \Phi(|x_n|) + \sum_{n=1}^{k} \widetilde{\Phi}(y_n)$$

and that is a contradiction. Hence $\sum_{n=1}^{\infty} \widetilde{\Phi}(y_n) \leq 1$ and

$$1 \geq \sum_{n=1}^{\infty} |x_n| y_n = \sum_{n=1}^{\infty} \Phi(|x_n|) + \sum_{n=1}^{\infty} \widetilde{\Phi}(y_n) \geq \sum_{n=1}^{\infty} \Phi(|x_n|),$$

which means that $\|x\|_{\ell^\Phi}^{Lux} \leq 1 = \|x\|_{\ell^\Phi}$. Given a non-zero sequence $y \in \ell^\Phi$ we can conclude that

$$\|y\|_{\ell^\Phi} = \|y\|_{\ell^\Phi} \left\| \frac{y}{\|y\|_{\ell^\Phi}} \right\|_{\ell^\Phi} \geq \|y\|_{\ell^\Phi} \left\| \frac{y}{\|y\|_{\ell^\Phi}} \right\|_{\ell^\Phi}^{Lux} = \|y\|_{\ell^\Phi}^{Lux}.$$

□

Norms of characteristic sequences

For the purpose of proving a necessary condition for an operator A to be bounded on a space ℓ^Φ, we shall have a quick glance at characteristic sequences. On some occasions throughout the text, we will refer to the measures of sets. Especially in this chapter counting measure is used with the notation of $|N|$ for the measure of N.

Definition 1.4 (Characteristic sequence)**.** *For a set $M \subset \{x \in \mathbb{R}, x \geq 0\}$ we define its characteristic sequence $\chi_M = \sum_{i \in M \cap \mathbb{N}} e_i$. For $n \in \mathbb{N}$, denote $\chi_n = \chi_{[1,n]}$.*

Theorem 1.5. *For a given $M \subset \{x \in \mathbb{R}, x \geq 0\}$ and $\chi_M \neq 0$,*

$$\|\chi_M\|_{\ell^\Phi} = |\{i \in \mathbb{N} \cap M\}| \cdot \widetilde{\Phi}^{-1}\left(\frac{1}{|\{i \in \mathbb{N} \cap M\}|}\right),$$

$$\|\chi_M\|_{\ell^\Phi}^{Lux} = \frac{1}{\Phi^{-1}\left(\frac{1}{|\{i \in \mathbb{N} \cap M\}|}\right)}.$$

Proof. Denote $\emptyset \neq N = \{i \in \mathbb{N} \cap M\}$ and assume that $|N| < \infty$. Certainly a sequence $x = \sum_{i \in N} \widetilde{\Phi}^{-1}(\frac{1}{|N|}) e_i$ is a test sequence from the definition of Orlicz norm, so

$$\|\chi_M\|_{\ell^\Phi} \geq \sum_{i=1}^{\infty} (\chi_M)_i x_i = |N| \widetilde{\Phi}^{-1}(\frac{1}{|N|}).$$

For the converse inequality, take any test sequence y from the definition of the norm, then

$$\widetilde{\Phi}^{-1}(\frac{1}{|N|} \sum_{i \in N} \widetilde{\Phi}(y_i)) \leq \widetilde{\Phi}^{-1}\left(\frac{1}{|N|}\right). \tag{1.1}$$

It follows from the fact that y is a test sequence and $\widetilde{\Phi}^{-1}$ is non-decreasing. By concavity of $\widetilde{\Phi}^{-1}$ and the fact that $|N| \geq 1$, we have

$$\frac{1}{|N|} \sum_{i \in N} y_i \leq \widetilde{\Phi}^{-1}\left(\frac{1}{|N|} \sum_{i \in N} \widetilde{\Phi}(y_i)\right). \tag{1.2}$$

Putting it together,

$$\frac{1}{|N|} \sum_{i=1}^{\infty} (\chi_M)_i y_i = \frac{1}{|N|} \sum_{i \in N} y_i \overset{(1.2)}{\leq} \widetilde{\Phi}^{-1}\left(\frac{1}{|N|} \sum_{i \in N} \widetilde{\Phi}(y_i)\right) \overset{(1.1)}{\leq} \widetilde{\Phi}^{-1}\left(\frac{1}{|N|}\right).$$

Finally passing to the supremum of the expression on the left, we obtain the second inequality. For $|N| = \infty$ denote $N_n = N \cap \{1, \dots n\}$. Then

$$|N| \widetilde{\Phi}^{-1}(\frac{1}{|N|}) = \lim_{n \to \infty} |N_n| \widetilde{\Phi}^{-1}(\frac{1}{|N_n|}) = \lim_{n \to \infty} \frac{\widetilde{\Phi}^{-1}(\frac{1}{|N_n|})}{\widetilde{\Phi}(\widetilde{\Phi}^{-1}(\frac{1}{|N_n|}))} \overset{\text{Heine}}{=} \lim_{t \to 0+} \frac{t}{\widetilde{\Phi}(t)} \overset{\text{D.1.1}}{=} \infty.$$

The equality for the Luxemburg norm is easier. The assumptions remain the same, $\emptyset \neq N = \{i \in \mathbb{N} \cap M\}$ and $|N| < \infty$. Due to N being finite there exists λ so that

$$|N| \Phi(\frac{1}{\lambda}) = \sum_{i \in N} \Phi(\frac{1}{\lambda}) \leq 1.$$

Now for every such λ we can divide the inequality by $|N|$ and apply Φ^{-1}, because it is increasing. After rearranging the terms, we get

$$\frac{1}{\Phi^{-1}\left(\frac{1}{|N|}\right)} \leq \lambda.$$

For $\lambda = \frac{1}{\Phi^{-1}\left(\frac{1}{|N|}\right)}$ it is easy to verify that $\sum_{i \in N} \Phi(\frac{1}{\lambda}) = 1$. In the case when $|N| = \infty$ we denote $N_n = N \cap \{1, \dots, n\}$ and obtain the result by passing to the limits again. □

2. Δ_2^0-condition

The Δ_2^0-condition of the complementary function $\widetilde{\Phi}$ is both necessary and sufficient for the boundedness of average operator on ℓ^Φ as shall be proven further on. Now, we will define the condition itself and subsequently prove an interesting and somehow more manageable equivalent condition.

Definition 2.1 (Δ_2^0-condition)**.** *A Young function Φ satisfies the Δ_2^0-condition, i.e. $\Phi \in \Delta_2^0$, if there is a constant $C > 0$ such that*

$$\Phi(2t) \leq C\Phi(t), \ t \in (0, 1].$$

Remark 4. In order for a continuous function f to satisfy the Δ_2^0-condition, it is sufficient that the condition $f(2t) \leq Cf(t)$ holds on a fixed interval $(0, t] \subset (0, 1]$. The rest follows from the compactness of $[t, 1]$ and the continuity of the function f.

Definition 2.2 (Δ_2^0-fundamental sequence)**.** *A non-negative sequence x is called Δ_2^0-fundamental if*

$$(\exists C > 0 \ \forall n \in \mathbb{N}, \Phi(2x_n) \leq C\Phi(x_n)) \Rightarrow \Phi \in \Delta_2^0.$$

As shown in Theorem 2.1, for the function f to satisfy the Δ_2^0-condition it is sufficient to find a sequence (with certain properties) and prove that the condition $f(2t) \leq Cf(t)$ holds true just for terms of the sequence.

Theorem 2.1. *Let Φ be a Young function. Then a non-negative decreasing sequence x convergent to 0 is Δ_2^0-fundamental if and only if*

$$\limsup \frac{x_n}{x_{n+1}} \leq 2.$$

Proof. Let x be a non-negative decreasing sequence with the limit 0. Denote $K = \limsup \frac{x_n}{x_{n+1}}$. Without loss on generality assume that $x_1 = 1$. We shall prove the equivalence by discussing three possibilities – $K > 2$, $K = 2$ and $K < 2$.

Let $K > 2$. We shall find a function $\Phi \notin \Delta_2^0$which for some $C > 0$ satisfies

$$\Phi(2x_n) \leq C\Phi(x_n), \ n \in \mathbb{N}.$$

First assume that

$$\frac{x_n}{x_{n+1}} \geq 2c, \ n \in \mathbb{N}, \ c > 1. \tag{2.1}$$

Put $\varphi_0(x) = 2x$, $\varphi_1(x) = 1$, $\varphi_n(x) = \frac{\varphi_{n-1}(x)}{n}$ for $n \in \mathbb{N}$, $n \geq 2$, and define

$$\varphi = \varphi_0 \chi_{[2,\infty)} + \sum_{n=1}^{\infty} \varphi_n \chi_{(2x_{n+1}, 2x_n]}.$$

It is easy to verify that φ is a non-decreasing right-continuous function and $\varphi(0) = 0$, so $\Phi(x) = \int_0^x \varphi(s)ds$ is a Young function by the merit of Theorem 1.2. Regarding the desired properties of Φ, denote

$$F(x) = \frac{\Phi(2x)}{\Phi(x)} = 1 + \frac{\int_x^{2x} \varphi}{\int_0^x \varphi}.$$

For any non-decreasing function f, the following estimates hold:

$$xf(x) \leq \int_x^{2x} f \leq xf(2x), \tag{2.2}$$

$$\left(1 - \frac{1}{c}\right) xf\left(\frac{x}{c}\right) \leq \int_{\frac{x}{c}}^{x} f \leq \int_0^x f,\ c > 1. \tag{2.3}$$

Now, we want to use the estimates for the function φ in the expressions $F(x_n)$ and $F(2x_n)$. The role of c from estimate (2.3) is taken by c from assumption (2.1). Also note that $\varphi(4x_n) = \varphi_{n-1}$ and $\varphi(x_n) = \varphi(2x_n) = \varphi_n$ (coming from the choice of φ and the assumption (2.1)). Consequently,

$$F(x_n) \leq 1 + \frac{x_n\varphi_n}{(1 - \frac{1}{c})x_n\varphi_n} = \frac{2c-1}{c-1} < \infty$$

and furthermore,

$$F(2x_n) \geq 1 + \frac{2x_n\varphi_{n-1}}{2x_n\varphi_n} \geq 1 + \frac{\varphi_{n-1}}{\varphi_n} = 1 + n.$$

The last equality is from the recurrent definition of φ_n. Hence we found a function, for which $F(x_n) < C$ and yet $\Phi \notin \Delta_2^0$, due to $F(2x_n) \to \infty,\ n \to \infty$.

If assumption (2.1) does not hold, then put $c = \frac{1+\frac{K}{2}}{2}$ and arrange all indeces k, for which $x_k \geq (2+c)x_{k+1}$, into an increasing sequence (n_l). Put

$$M_0 = (2x_{n_1}, \infty],\ M_1 = (2x_{n_1+1}, 2x_{n_1}] \text{ and } M_i = (2x_{n_{i+1}+1}, 2x_{n_i+1}],\ i > 1,$$

$$\varphi_0(x) = x,\ \varphi_1(x) = 2x_{n_1} \text{ and } \varphi_{i+1}(x) = \tfrac{\varphi_i(x)}{i+1},\ i > 1.$$

Finally set $\varphi = \sum_{i=0}^{\infty} \varphi_i \chi_{M_i}$ and $\Phi_1(x) = \int_0^x \varphi$. The function Φ_1 is a Young function and there is a constant $C > 0$ such that

$$F(x_{n_k}) < C,\ k \in \mathbb{N} \tag{2.4}$$

and

$$F(2x_{n_k}) \to \infty,\ k \to \infty.$$

The proof is very similar to the one for Φ. It remains to be shown that there is a constant $C_1 > 0$ such that

$$F(x_n) < C_1,\ n \in \mathbb{N}.$$

Let $k \in \mathbb{N}$, $k \neq n_l$, $l \in \mathbb{N}$. Then there is m such that $n_{m+1} < l < n_m$. In addition, from the construction of (x_{n_k}),

$$x_l \geq x_{n_{m+1}} \geq 2cx_{n_{m+1}+1}.$$

Hence,

$$0 \leq \frac{x_l}{x_l - x_{n_{m+1}+1}} = 1 + \frac{x_{n_{m+1}+1}}{x_l - x_{n_{m+1}+1}} \leq 1 + \frac{1}{2c-1} < \infty.$$

Putting it together, we arrive at the following inequality

$$\int_0^{x_l} \varphi \geq \int_{x_{m+1}+1}^{x_l} \varphi = \varphi_m(x_l - x_{n_{m+1}+1}).$$

Consequently we obtain upper estimate of $F(x_l)$,

$$F(x_l) \leq 1 + \frac{x_l \varphi_m}{\varphi_m(x_l - x_{n_{m+1}+1})} \leq 2 + \frac{1}{2c-1} < \infty.$$

Choosing $C_1 = \max\{C, 2 + \frac{1}{2c-1}\}$, it is true that $F(x_n) < C_1$, $n \in \mathbb{N}$.

For $K < 2$ the proof is straightforward. Suppose that

$$F(x_k) < C, \ k > 0. \tag{2.5}$$

One can find $n_0 \in \mathbb{N}$ such that $x_n \leq 2x_{n+1}$, $n \geq n_0$. For $y \in (0, x_{n_0}]$ there is an index k such that $y \in (x_{k+1}, x_k]$ and following inequality holds,

$$\Phi(2y) \overset{\Phi \text{ mon.}}{\leq} \Phi(2x_k) \leq C\Phi(x_k) \leq C\Phi(2x_{k+1}) \leq C^2\Phi(x_{k+1}) \overset{\Phi \text{ mon.}}{\leq} C^2\Phi(y). \tag{2.6}$$

The rest comes from Remark 4.

Finally suppose that $K = 2$. We can safely assume that for all n, $\frac{x_n}{x_{n+1}} \geq 2$. If the converse inequality holds, then for every $y \in [x_{n+1}, x_n]$ we can use estimate (2.6). So, due to the assumption and fact that $2 = \limsup \frac{x_n}{x_{n+1}}$,

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 2. \tag{2.7}$$

In addition, integral $\int_0^{x_n} \varphi$ can be estimated as follows,

$$\int_0^{x_n} \varphi \leq \int_0^{2x_{n+1}} \varphi + \int_{2x_{n+1}}^{x_n} \varphi \overset{(2.2)}{\leq} 2x_{n+1}\varphi(2x_{n+1}) + (x_n - 2x_{n+1})\varphi(x_n),$$

furthermore,

$$\frac{\varphi(x_n)x_n}{(2x_{n+1}\varphi(2x_{n+1}) + (x_n - 2x_{n+1})\varphi(x_n)} = 1 + \frac{\varphi(x_n) - \varphi(x_{n+1})}{\varphi(x_{n+1}) + (\frac{x_n}{2x_{n+1}} - 1)\varphi(x_n)}.$$

As a result,

$$C \overset{(2.5)}{>} F(2x_n) - 1 = \frac{\int_{x_n}^{2x_n} \varphi}{\int_0^{x_n} \varphi} \geq 1 + \frac{\varphi(x_n) - \varphi(x_{n+1})}{\varphi(x_{n+1}) + (\frac{x_n}{2x_{n+1}} - 1)\varphi(x_n)}. \tag{2.8}$$

Note that if $1 + \frac{\int_x^{2x} \varphi}{\int_0^x \varphi} < C$, then $C > 2$, because φ is non-decreasing and so the fraction on the left side is at least 1. Since $\frac{x_n}{x_{n+1}} \to 2_+$, we can find n_0 so large that for every $n \geq n_0$,

$$1 \geq 1 + (2 - C)(\frac{x_n}{2x_{n+1}} - 1) > \frac{1}{2},$$

then by rearranging the inequality (2.8),

$$\frac{\varphi(x_n)}{\varphi(x_{n+1})} \leq \frac{C-1}{1 + (2-C)(\frac{x_n}{2x_{n+1}} - 1)} < \frac{2(C-1)}{3} < \infty, \ n \geq n_0. \tag{2.9}$$

Now, we shall prove that ratio of $\Phi(x_n)$ and $\Phi(2x_{n+1})$ is bounded.

$$\frac{\Phi(x_n)}{\Phi(2x_{n+1})} - 1 = \frac{\int_{2x_{n+1}}^{x_n} \varphi}{\int_0^{2x_{n+1}} \varphi} \leq \frac{\int_{2x_{n+1}}^{x_n} \varphi}{\int_{x_{n+1}}^{2x_{n+1}} \varphi} \overset{\varphi \text{ non-dec.}}{\leq} \frac{x_n - 2x_{n+1}}{x_{n+1}} \frac{\varphi(x_n)}{\varphi(x_{n+1})}.$$

From the inequality (2.9) we have that

$$\frac{\varphi(x_n)}{\varphi(x_{n+1})} < \frac{2(C-1)}{3}, \; n \geq n_0$$

and from the assumption (2.7), there is n_1 such that

$$\frac{x_n - 2x_{n+1}}{x_{n+1}} < \frac{1}{2}, \; n \geq n_1.$$

Put $n_2 = \max\{n_0, \; n_1\}$, then for $n \geq n_2$ the ratio $\frac{\Phi(x_n)}{2x_{n+1}}$ is bounded by $L = 1 + \frac{C-1}{3}$. Ultimately, for $y \in (0, x_{n_2}]$, there is index k such that $y \in (x_{k+1}, x_k]$ and

$$\Phi(2y) \overset{\Phi \text{ mon.}}{\leq} \Phi(2x_k) \leq C\Phi(x_k) \leq CL\Phi(2x_{k+1}) \leq C^2 L\Phi(x_{k+1}) \overset{\Phi \text{ mon.}}{\leq} C^2 L\Phi(y).$$

The rest follows from Remark 4. □

Definition 2.3 (Δ_2-condition)**.** *Let Φ be a Young function, then Φ satisfies Δ_2-condition in $[0, \infty)$, if there is a constant $C > 0$ such that*

$$\frac{\Phi(2x)}{\Phi(x)} < C, \; x > 0$$

Theorem 2.2. *[1, Proposition 1.4.]*

Let Φ be a Young function, then $\widetilde{\Phi}$ satisfies the Δ_2-condition in $[0, \infty)$ if and only if

$$\inf_{s>0} \frac{s\varphi(s)}{\Phi(s)} > 1,$$

where φ is the density function of Φ.

3. Characterization of the average operator on Orlicz spaces

We finally have all the necessities to give the desired characterization. To prove it, we will start with some technical lemmas and then prove the characterization in the end of the chapter.

Lemma 3.1. *For a convex function f and a real number $a > 1$,*

$$f(at) \geq af(t),\ t > 0.$$

Proof. As we know, $f(t) = \int_0^t p(s)ds$, where p is a non-decreasing right-continuous function, $t > 0$. So to prove the lemma, it suffices to prove that

$$\int_0^{at} p(s)ds - a\int_0^t p(s)ds = \int_t^{at} p(s)ds - (a-1)\int_0^t p(s)ds \geq 0.$$

Using the change of variables $y = \frac{s-t}{a-1}$ the right side of the inequality has the form $(a-1)\int_0^t p((a-1)s+t) - p(s)ds$. We have

$$(a-1)s + t \geq s,\ s \in [0,1],$$

and $p((a-1)s+t) \geq p(s)$, because the function p is non-decreasing. □

Definition 3.1. *For two functions f and g, we shall denote $f \approx g$ on a given set M, if there are constants K, $k > 0$ such that*

$$kg(x) \leq f(x) \leq Kg(x),\ x \in M.$$

Proposition 3.2. *For $a \geq 0$ and a positive integer N the following estimate holds*

$$\sum_{i=n}^{N} \frac{\log^a(i)}{i} \approx \log^{a+1}(N).$$

Proof. The function $g(x) = \frac{\log^a(x)}{x}$ is decreasing for $x > a$. So for positive integer $n_1 \geq a$

$$\int_{n_1}^{N} g(x)dx \leq \sum_{j=n_1}^{N} g(j) \leq g(n_1) + \int_{n_1}^{N} f(x)dx.$$

Using the change of variables $t = \log(x)$,

$$\int_{n_1}^{N} g(x)dx = \frac{1}{a+1}\left(\log\left(\frac{N}{n_1}\right)\right).$$

Consequently,

$$\frac{\log^{a+1}(\frac{N}{n_1})}{a+1} \leq \sum_{j=n_1}^{N} g(j) \leq \frac{\log^{a+1}(\frac{N}{n_1})}{a+1} + \frac{\log^a(n_1)}{n_1}.$$

Now, the fraction $\frac{log^a(n_1)}{n_1}$ is finite as well as the sum $\sum_{i=1}^{n_1} g(j)$. Furthermore, $\log^{a+1}(\frac{N}{n})$ is nonzero for every $N > 1$ and for fixed n, $\lim_{N\to\infty} \log^{a+1}(\frac{N}{n}) = \infty$. So there are positive constants k and K, such that

$$k\log^{a+1}(N) \leq \sum_{i=1}^{N} g(j) \leq K\log^{a+1}(N).$$

□

In the proofs of the following theorems, we will use arguments in the spirit of [1, Th.2.1], adopted to sequence spaces.

Theorem 3.3. *If the operator* $A : \ell^\Phi \to \ell^\Phi$ *is bounded, then* $\widetilde{\Phi} \in \Delta_2^0$.

Proof. For $n, m \in \mathbb{N}$ consider the sequence $x = \chi_n$, then

$$\|Ax\|_{\ell^\Phi} \geq \left\|\chi_{[n,mn]} Ax\right\|_{\ell^\Phi} = \sup\{\sum_{k=n}^{mn} (Ax)_k y_k, \sum_{k=1}^{\infty} \widetilde{\Phi}(|y_k|) \leq 1\}. \tag{3.1}$$

Take $y = \widetilde{\Phi}^{-1}\left(\frac{1}{mn}\right) \chi_{mn}$. Consequently $\sum_{k=1}^{\infty} \widetilde{\Phi}(|y_k|) = 1$ and

$$\|Ax\|_{\ell^\Phi} \overset{(3.1)}{\geq} \sum_{k=n}^{mn} \frac{n}{k} \widetilde{\Phi}^{-1}\left(\frac{1}{mn}\right) = \widetilde{\Phi}^{-1}\left(\frac{1}{mn}\right) n \sum_{k=n}^{mn} \frac{1}{k} \overset{\text{Prop.3.2}}{\approx} \widetilde{\Phi}^{-1}\left(\frac{1}{mn}\right) n \log(m).$$

From the assumption that the operator A is bounded and the last string of inequalities we obtain that for some $C > 0$:

$$n \log m \cdot \widetilde{\Phi}^{-1}\left(\frac{1}{mn}\right) \leq C \|x\|_{\ell^\Phi} \overset{\text{Th.1.5}}{=} Cn \cdot \widetilde{\Phi}^{-1}\left(\frac{1}{n}\right).$$

Now take $m \geq e^{2C}$ and denote $t_n = \frac{1}{mn}$, then

$$2Cn\widetilde{\Phi}^{-1}(t_n) \leq n \log m \widetilde{\Phi}^{-1}(t_n) \leq Cn\widetilde{\Phi}^{-1}(mt_n),$$

consequently

$$2\widetilde{\Phi}^{-1}(t_n) \leq \widetilde{\Phi}^{-1}(mt_n)$$

and because the function $\widetilde{\Phi}$ is increasing,

$$\widetilde{\Phi}(2\widetilde{\Phi}^{-1}(t_n)) \leq mt_n,$$

that is

$$\widetilde{\Phi}(2s_n) \leq m\widetilde{\Phi}(s_n), \;\; s_n = \widetilde{\Phi}^{-1}(t_n).$$

The last step to prove the theorem is to show that the sequence

$$(s_n) = (\widetilde{\Phi}^{-1}(\frac{1}{mn}))_{n=1}^{\infty}$$

is Δ_2^0-fundamental. Obviously, the sequence is non-negative decreasing and convergent with the limit at 0. Moreover,

$$\widetilde{\Phi}(2\widetilde{\Phi}^{-1}\left(\frac{1}{m(n+1)}\right)) \overset{\text{L.3.1}}{\geq} \frac{2}{m(n+1)} \geq \frac{1}{mn}, \;\; n \in \mathbb{N}.$$

After applying an increasing function $\widetilde{\Phi}^{-1}$ and rearranging the inequality, we get that

$$\frac{\widetilde{\Phi}^{-1}(\frac{1}{mn})}{\widetilde{\Phi}^{-1}(\frac{1}{m(n+1)})} \leq 2.$$

Hence by Theorem 2.1 the sequence (s_n) is Δ_2^0-fundamental and thus $\widetilde{\Phi} \in \Delta_2^0$. $\square$

Theorem 3.4. *Let T be an operator defined on*[1] $\ell^1 + \ell^\infty$ *and there is a constant C such that $\forall y \in \ell^1$, $\forall z \in \ell^\infty$, $\lambda > 0$:*

1) $|\{n \in \mathbb{N}, |(Ty)_n| > \lambda\}| \leq C\lambda^{-1} \sum_{i=1}^{\infty} |y_i|$

2) $\|Ty\|_\infty \leq C \|y\|_\infty$

3) $|T(y+z)| \leq C(|Ty| + |Tz|)$.

Let Φ be a Young function, whose complementary function satisfies Δ_2-condition, then there are positive constants A and m such that

$$\sum_{i=1}^{\infty}(\Phi(|(Tx)_i|)) \leq A \sum_{i=1}^{\infty} \Phi(m|x_i|),\ x \in \ell^\Phi.$$

Proof. For $x \in \ell^\Phi$ and $\lambda > 0$ denote $x_\lambda = (x_i(\chi_{A(\frac{\lambda}{2C^2})})_i)_{i=1}^\infty$ and $x^\lambda = x - x_\lambda$, where $A(\alpha) = \{x_i, |x_i| > \alpha\}$.

Obviously, $x^\lambda \in \ell^\infty$ and $x_\lambda \in \ell^1$, because

$$\left\|x^\lambda\right\|_\infty \leq \frac{\lambda}{2C^2}$$

and

$$\sum_{i=1}^{\infty} |(x_\lambda)_i| \leq \|x\|_\infty |\{i \in \mathbb{N}, x_i > \frac{\lambda}{2C^2}\}| \overset{1)}{<} \infty.$$

It is quite straightforward that

$$\sum_{i=1}^{\infty} \frac{2C^2}{\lambda}(x_\lambda)_i \geq \sum_{i=1}^{\infty}(\chi_{A(\frac{\lambda}{2C^2})})_i). \tag{3.2}$$

Now to the inequality,

$$\begin{aligned}\sum_{i=1}^{\infty} \Phi((Tx)_i) &= \int_0^\infty \varphi(\lambda)|\{i \in \mathbb{N}, (Tx)_i > \lambda\}|d\lambda \\ &\overset{(3.2)}{\leq} 2C^2 \int_0^\infty \frac{\varphi(\lambda)}{\lambda} \sum_{i=1}^{\infty}(x_\lambda)_i = 2C^2 \sum_{i=1}^{\infty}(x_\lambda)_i \int_0^{2C^2(x_\lambda)_i} \frac{\varphi(\lambda)}{\lambda} d\lambda.\end{aligned}$$

Let's focus on the last expression – integrating by parts,

$$\int_0^s \frac{\varphi(\lambda)}{\lambda} d\lambda = \frac{\Phi(s)}{s} + \int_0^s \frac{\Phi(\lambda)}{\lambda^2} d\lambda. \tag{3.3}$$

Due to Theorem 2.2 and fact that $\widetilde{\Phi} \in \Delta_2$, there is $\beta > 1$ such that

$$\beta\Phi(s) < s\varphi(s),\ s > 0. \tag{3.4}$$

If $\lambda \in (0,1)$, then

$$\frac{\Phi(\lambda)}{\lambda^2} \leq \Phi(1)\lambda^{\beta-2}, \tag{3.5}$$

[1] $x \in \ell^1 + \ell^\infty$ iff there is a decomposition such that $x = y + z$, $y \in \ell^1$ and $z \in \ell^\infty$.

because by integrating (3.4) over the interval $[\lambda, 1]$ we have

$$\int_\lambda^1 \frac{\beta}{s} ds < \int_\lambda^1 \frac{\varphi(s)}{\Phi(s)} ds$$

that is

$$\log \frac{1}{\lambda^\beta} < \log \frac{\Phi(1)}{\Phi(\lambda)}.$$

From the fact, that log is an increasing function on $\mathbb{R}^+$ we arrive at

$$\frac{1}{\lambda^\beta} < \frac{\Phi(1)}{\Phi(\lambda)}$$

and finally by rearranging the inequality we obtain (3.5). Hence $\int_0^s \frac{\Phi(\lambda)}{\lambda^2}$ is finite for $s > 0$ – it follows directly from (3.5) for $s \in (0, 1)$ and then from the continuity of the function $\frac{\Phi(\lambda)}{\lambda}$ on $[1, s]$ for $s > 1$. Additionally, from (3.5) for $s,\ t > 0$

$$\frac{1}{\beta - 1} \frac{t\varphi(t) - \beta\Phi(t)}{t^2} > 0,$$

then by subtracting $\frac{\Phi(t)}{t^2}$ and integrating both sides of the inequality over the interval $(0, s)$ we obtain the following,

$$\frac{1}{\beta - 1} \int_0^s \frac{t\varphi(t) - \Phi(t)}{t^2} dt > \int_0^s \frac{\Phi(t)}{t^2} dt$$

and consequently,

$$\frac{\Phi(s)}{(\beta - 1)s} > \int_0^s \frac{\Phi(t)}{t^2} dt.$$

Thus, from (3.3) and the last inequality we get that

$$\int_0^s \frac{\varphi(\lambda)}{\lambda} d\lambda < (1 + \frac{1}{\beta - 1}) \frac{\Phi(s)}{s} = \frac{\beta}{\beta - 1} \frac{\Phi(s)}{s},\ \ s > 0.$$

So we obtain the theorem with $A = \frac{\beta}{\beta-1}$ and $m = 2C^2$. ☐

Theorem 3.5. *If Φ is a Young function, for which $\widetilde{\Phi} \in \Delta_2^0$, then $A : \ell^\Phi \to \ell^\Phi$ is bounded.*

Proof. First of all, assume that $\widetilde{\Phi} \in \Delta_2$. We will show that the operator A satisfies the assumptions of Theorem 3.4. Conditions 2) and 3) obviously hold. For $x \in \ell^1$,

$$|(Ax)_i| \leq \frac{1}{i} \sum_{j=1}^{i} |x_j| \leq \frac{\|x\|_1}{i} \overset{i\to\infty}{\to} 0.$$

Consider the smallest n_0 such that $\forall n > n_0, |(Ax)_i| \leq \lambda$. If n_0=0, then the last condition holds trivially. Otherwise from the choice of n_0,

$$|\{i \in \mathbb{N}, |(Tx)_i| > \lambda\}| \leq n_0$$

and

$$\sum_{i=1}^{\infty} |x_i| \geq |\sum_{i=1}^{n_0} x_i| > n_0\lambda.$$

Putting it all together, the operator A satisfies the assumptions of Theorem 3.4 for $C = 1$.

If $x \in \ell^\Phi$, then $\frac{x}{m\max\{B,1\}\|x\|_{\ell^\Phi}^{Lux}} \in \ell^\Phi$, where B and m are the constants from Theorem 3.4. Moreover, we have

$$\sum_{i=1}^{\infty}\left(\Phi\left(\frac{|(Ax)_i|}{m\max\{B,1\}\,\|x\|_{\ell^\Phi}^{Lux}}\right)\right) \leq B\sum_{i=1}^{\infty}\Phi\left(\frac{m|x_i|}{m\max\{B,1\}\,\|x\|_{\ell^\Phi}^{Lux}}\right)$$
$$\overset{\Phi\text{ conv.}}{\leq} \frac{B}{\max\{B,1\}}\sum_{i=1}^{\infty}\Phi\left(\frac{|x_i|}{\|x\|_{\ell^\Phi}^{Lux}}\right) \leq 1.$$

So,

$$\sum_{i=1}^{\infty}\left(\Phi\left(\frac{|(Ax)_i|}{m\max\{B,1\}\,\|x\|_{\ell^\Phi}^{Lux}}\right)\right) \leq 1$$

and that means

$$\|Ax\|_{\ell^\Phi}^{Lux} \leq m\max\{B,1\}\,\|x\|_{\ell^{\bar\Phi}}^{Lux},\ x \in \ell^\Phi. \tag{3.6}$$

Now, let $\widetilde{\Phi} \in \Delta_2^0$ and consider the function $\widetilde{\Phi}_1(x)$, which is a Young function, satisfies Δ_2-condition and equals $\widetilde{\Phi}$ on the interval $[0, \Phi^{-1}(1)]$ (the construction might be done in the spirit of Remark 1 – continuously extending the function $\widetilde{\Phi}|_{[0,\Phi^{-1}(1)]}$ by a suitable quadratic polynomial). Denote $\bar\Phi$ the complementary function to $\widetilde{\Phi}_1$. From the choice of $\widetilde{\Phi}_1$, it is also true that

$$\Phi(x) = \bar\Phi(x),\ x \in [0, \Phi^{-1}(1)]. \tag{3.7}$$

Now, consider a sequence $x \in \ell^\Phi$ and $\lambda > 0$, for which

$$\sum_{n=1}^{\infty}\Phi(\frac{|x_n|}{\lambda}) \leq 1.$$

From the fact that the function Φ is non-negative follows that

$$\Phi(\frac{|x_n|}{\lambda}) \leq 1,\ n \in \mathbb{N}$$

and the monotonicity of Φ^{-1} implies that

$$\frac{|x_n|}{\lambda} \leq \Phi^{-1}(1),\ n \in \mathbb{N}.$$

Finally from (3.7)

$$\sum_{n=1}^{\infty}\Phi(\frac{|x_n|}{\lambda}) = \sum_{n=1}^{\infty}\bar\Phi(\frac{|x_n|}{\lambda}).$$

Passing to infimum of λ, we obtain that

$$\|x\|_{\ell^\Phi}^{Lux} \geq \|x\|_{\ell^{\bar\Phi}}^{Lux},\ x \in \ell^\Phi$$

Due to $\Phi^{-1}(1) = \bar\Phi^{-1}(1)$, we can prove the converse inequality by repeating the same procedure, but with the functions Φ and $\bar\Phi$ switched around. So

$$\|x\|_{\ell^\Phi}^{Lux} = \|x\|_{\ell^{\bar\Phi}}^{Lux},\ x \in \ell^\Phi$$

and for $\bar\Phi$ we can use the result (3.6).

$\square$

To illustrate that the result is restrictive for the function associated with the Orlicz space, we shall find a family of functions, for which the complementary functions are not Δ_2^0.

Proposition 3.6. *The function*

$$\Phi_\alpha(x) = \begin{cases} 0, & x = 0, \\ \frac{x}{\log^\alpha \frac{1}{x}}, & 0 < x < \frac{1}{2}, \end{cases}$$

satisfies properties i) and ii) from Definition 1.1 for every $\alpha > 0$. Moreover, $\widetilde{\Phi} \notin \Delta_2^0$ and

$$\lim_{x \to 0^+} \frac{\Phi_\alpha(x)}{x^p} = \infty, \ p > 1.$$

Proof. The function Φ_α obviously has the second property. For the first one, by differentiating f_α twice we obtain:

$$\frac{d\Phi_\alpha}{dx} = \frac{\log(\frac{1}{x}) + \alpha}{\log^{\alpha+1}(\frac{1}{x})}$$

and

$$\frac{d^2\Phi_\alpha}{dx^2} = \frac{\alpha \cdot \log(\frac{1}{x}) + \alpha^2 + \alpha}{x \cdot \log^{\alpha+2}(\frac{1}{x})}.$$

Fractions on the right sides of both derivatives are positive for any $\alpha > 0$ and $x < 1$. So, from the correspondence of the sign of derivatives and monotonicity and convexity, we get that Φ_α is a Young function. In addition,

$$\lim_{x \to 0^+} \frac{\Phi_\alpha(x)}{x^p} = \frac{1}{x^{p-1} \cdot \log^\alpha(\frac{1}{x})} = \infty,$$

due to the fact that $p - 1 > 0$. From now on, we will assume that all variables are sufficiently small so that following estimates hold. For the last property,

$$\varphi_\alpha(s) = \frac{1}{\log^\alpha(\frac{1}{x})} + \frac{\alpha}{\log^{\alpha+1}(\frac{1}{x})}.$$

From the fact that

$$\lim_{x \to 0_+} \frac{\log^\alpha(\frac{1}{x})}{\log^{\alpha-1}(\frac{1}{x})} = \infty$$

follows $\varphi_\alpha(x) \approx \frac{1}{\log^\alpha(\frac{1}{x})}$. Consequently $\psi_\alpha(y) = e^{-y^{-\frac{1}{\alpha}}}$ and $\widetilde{\Phi}(t) = \int_0^t \psi_\alpha$. Ultimately,

$$\frac{\widetilde{\Phi}_\alpha(2x)}{\widetilde{\Phi}_\alpha(x)} = 1 + \frac{\int_x^{2x} \psi_\alpha}{\int_0^x \psi_\alpha} \geq 1 + \frac{\frac{x}{2}\psi_\alpha(\frac{3}{2}x)}{x\psi_\alpha(x)} = 1 + \frac{1}{2} e^{\frac{3^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}}}{(3x)^{\frac{1}{\alpha}}}} \overset{x \to 0_+}{\longrightarrow} \infty.$$

$\square$

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