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Modelling Dependent Lives

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Title: Modelling dependent lives

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Abstract: In this thesis, we model the dependence between the remaining lifetimes of a husband and wife using a specific Markov model. We examined the impact of the dependence on the net single premium using the specific Markov model that captures the long-term dependence between lifetimes of the two considered lives. Using this model we have calculated 10-year joint-life annuity due and 10-year last-survivor annuity due considering the age rage $(37, 80)$ in case of dependence and also independence of the two considered lives. The calculations were based on the dataset related to the Czech population in 2015. The impact of the dependence between the remaining lifetimes of the husband and wife was found to be not significant.

Keywords: positive quadrant depedence, multiple life insurance premiums, dependent lifetimes, joint-life annuity, last-survivor annuity, joint-life and last-survivor models

Název práce: Modelování závislých životů

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Abstrakt: V této práci modelujeme závislost mezi zbývajícími délkami života manželů s použitím specifického Markovského modelu. Kvantifikovali jsme dopad této závislosti na jednorázové netto pojistné s použitím specifického Markovského modelu, který zachycuje závislost dlouhodobého spolužití manželského páru. V tomto specifickém Markovském modelu jsme spočítali jednorázové netto pojistné pro důchod sdružených životů s durací 10 let a důchod posledního přežívajícího, taktéž s durací 10 let a to ve věkovém rozpětí $(37, 80)$ za předpokladů závislosti i nezávislosti zbývajících délek životů manželů. Pri výpočtech byla použita data pro Českou populaci v roce 2015. Zhodnotili jsme, že dopad závislosti mezi zbývajícími délkami života manželů na jednorázové pojistné u již zmíněných důchodů není signifikantní.

Klíčová slova: pozitivní závislost, pojistné více-stavových pojištění, závislé životy, důchod sdružených životů, důchod posledního přežívajícího, modely sdružených životů a posledního přežívajícího

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Contents

Introduction

Some insurance products provide benefits that are contingent on the combined status of a couple or group of people and these kind of products are known as multiple life insurance policies. A very good example of a such kind of product is a contract issued to a married couple.

The computation of multiple life premiums in a standard actuarial theory is based on the unrealistic assumption of independence of the remaining lifetimes of a husband and wife. However, this assumption is mainly used because of its computational convenience. Several empirical studies suggest that there is a considerable dependence between the lifetimes of a married couple. Denuit et al. (2001) argue that a husband and wife are more or less exposed to the same risks since they share a common way of life. Parkes et al. (1969) showed that there is a significant increase in mortality rate among widowers during the first six months after the deaths of their wives in comparison with the rate for married men of the same age. They also call this kind of death a “broken heart” and interpret it as a figure of speech which reflects a bygone belief that grief could kill, and kills through the heart. Another type of dependence is known as a “common shock” effect, which reflects simultaneous deaths of a couple from a common disaster, such as road accidents or airplane crash. Overall, the dependence between the future lifetimes of a husband and wife can be categorized as the long-term association between lifetimes, the short-term impact of spousal death and the instantaneous dependence due to a catastrophic event that affects both lives.

A number of articles about insurance contracts on two lives, which allow for dependence between two lifetimes, have been published over the past years. One way how to model the dependence between lifetimes is using Markov models with a finite number of states. Depending on the properties of the transition intensities, we can use Markov or semi-Markov models. Transition intensities in Markov models depend only on the current state, while in semi-Markov models, transition intensities depend on the current state and also on the time elapsed since the last transition. From a multiple state model, we can clearly see how the change of a particular state impacts mortality. Thus, the advantage of using Markov and semi-Markov models is its high transparency. Markov multiple state models have been applied to different areas in actuarial science. Dickson et al. (2009) explain how Markov models can be used for modelling various insurance benefits such as joint-life, last-survivor, critical illness, permanent disability or accidental death. The first application of the joint-life mortality is probably done by Norberg (1989). His work extends Spreeuw and Wang (2008) by considering mortality to vary with the time elapsed since the death of a spouse.

Another approach to modelling the dependence between remaining lifetimes is through copulas. Using copulas has the advantage that the correlation structure of the remaining lifetime variables can be estimated separately from their distributions. On the other hand, it is not easy to quantify whether the dependence structure of a particular copula fits appropriately. Frees and Valdez (1988) explore in their paper some practical applications including estimation of joint life mortality. Copula models were also applied by Denuit et al. (2001)

to classical insurance contracts issued to married couples. We would like to mention that lifetime dependence can be modelled not only through Markovian approaches or copulas but also using multivariate distributions. A multivariate gamma distribution for incorporating the dependence is applied by Alai et al. (2012) and a multivariate Pareto distribution is applied later again by Alai et al. (2016).

In this thesis, we will model the dependence between remaining lifetimes of a husband and wife using a specific Markov model. Our main aim is to examine the effect of a possible dependence of remaining lifetimes on the amount of a net single premium related to insurance products sold to married couples. We shall deal with a joint-life and last-survivor life insurance. We would like to mention that, to our best knowledge, up to now these products are not sold in the Czech Republic. One can arrange this kind of insurances e.g. in the United Kingdom, the United States of America or India.

The thesis consist of three chapters. The first two chapters are devoted to theory. A practical application can be found in Chapter 3.

Chapter 1 introduces the joint-life and last-survivor status, and it states formulae of the net single premiums for specific insurances. The concern of Chapter 1 lies also in fundamental graduation approaches in life insurance and in the concept of positive quadrant dependence.

In Chapter 2, we deal with a specific Markov model that can be used for calculation premiums and benefits for the joint-life or last-survivor insurance. This chapter is a base for the practical part in Chapter 3. Considering the specific Markov model, we derive probabilities related to joint-life and last-survivor insurances. We will be employed with relationships among transition intensities and remaining lifetimes of the husband and wife. Moreover, we will pay our attention to a special case of the considered specific Markov model which assumes independence of remaining lifetimes. Finally, we state two extensions of the specific Markov model.

Chapter 3 aims to examine the impact of dependence between the remaining lifetimes of the husband and wife on the amount of premium for joint-life and last-survivor insurances by calculating net single premiums. Calculations are done in Excel and Mathematica.

1. Multiple Life Insurance and Other Related Aspects

The theory for the analysis of financial benefits contingent on the time of the death of a single life can be extended to benefits involving several lives. A very good example of the application of this extension is a joint-life or last-survivor insurance. In this chapter, we shall introduce the joint-life and last-survivor status. We shall also state the formulas of net single premiums for particular insurances considering general case, as well as the case of independence. Furthermore, we devote our attention to fundamental graduation approaches by which a survival model can be constructed. Moreover, the concept of positive quadrant dependence will be brought forward and its importance for actuarial computations in case of multiple life insurances will be explicated.

Let us start with some notation. We denote a life aged x by (x), where $x \geq 0$. In single life insurance theory, we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x). In multiple life theory we consider m lives with initial ages $x_1, x_2, \ldots, x_m$. Further, we denote the future lifetime of the kth life by T_{x_k} for $k = 1, \ldots, m$. On the basis of the remaining lifetimes of m considered lives we shall define a status u with a continuous future lifetime $T(u)$. Accordingly, a standard actuarial notation is used here with the subscript listing several ages rather than a single age. For example, $A_{x_1 x_2 \ldots x_m}$ and ${}_tp_{x_1 x_2 \ldots x_m}$ have the same meaning for the joint-life insurance status $(x_1 x_2 \ldots x_m)$ as A_x and ${}_tp_x$ for the single life x. Elementary life insurance theory can be found in Gerber (1997).

In this thesis, we will restrict ourselves to considering only two lives, since our main interest lies in analysing premiums relating to products sold to married couples. In such a case, let us refer to x as the age of the husband and y as the age of the wife. Obviously, all actuarial quantities denoted with subscript x (resp. y) shall refer to the husband (resp. wife) if not stated differently.

1.1 Joint-Life Status

The status

$$u = xy \tag{1.1}$$

is defined to exist as long as both considered lives survive. The failure time of this joint-life status is

$$T(u) = min(T_x, T_y). \tag{1.2}$$

The probability ${}_tp_{xy}$ of the status (1.1) surviving beyond time t is given by

$$\begin{aligned}
{}_tp_{xy} &= P[T(xy) > t] = \\
&= P[min(T_x, T_y) > t] = \\
&= P[T_x > t \cap T_y > t] = && \text{assuming independence} \quad (1.3) \\
&= P[T_x > t]P[T_y > t] = \\
&= {}_tp_x \, {}_tp_y, && t \geq 0. \quad (1.4)
\end{aligned}$$

The probability that the joint-life status (1.1) fails before time t, denoted by ${}_tq_{xy}$, is then given by

$$\begin{aligned}
{}_tq_{xy} &= P[T(xy) \le t] = P[min(T_x, T_y) \le t] = \\
&= 1 - P[min(T_x, T_y) > t] = \\
&= 1 - {}_tp_{xy} = && \text{assuming independence} \quad (1.5) \\
&= 1 - {}_tp_x\, {}_tp_y = && (1.6) \\
&= 1 - (1 - {}_tq_x)(1 - {}_tq_y) = \\
&= {}_tq_x + {}_tq_y - {}_tq_x\, {}_tq_y, && t \ge 0. \quad (1.7)
\end{aligned}$$

Let us define $K(u) = \lfloor T(u) \rfloor$, a random variable representing a number of completed future years survived by a status u - in other words, the curtate future "lifetime" of a status u. Whenever it is clear by context, we denote the remaining lifetime of a status u simply by T. Now the probability distribution of this integer-valued random variable K is given by

$$\begin{aligned}
P[K(u) = k] &= P[k \le T < k+1] = P[k < T \le k+1] = \\
&= P[T \le k+1] - P[T < k] = \\
&= 1 - P[T > k+1] - (1 - P[T \ge k]) = \\
&= P[T \ge k] - P[T > k+1] = \\
&= {}_kp_u - {}_{k+1}p_u = {}_kp_u - {}_kp_u\, p_{u+k} = \\
&= {}_kp_u(1 - p_{u+k}) = {}_kp_u\, q_{u+k}, && k = 0, 1, \ldots . \quad (1.8)
\end{aligned}$$

Under our assumption that $T(u)$ is a continuous random variable it holds that $P[T(u) = k] = P[T(u) = k+1] = 0$. Thus, the interchanging of inequalities in above stated relations is possible. The identity

$$ {}_{k+1}p_u = {}_kp_u\, p_{u+k} \tag{1.9}$$

has an intuitive interpretation. The probability that both the husband aged x and the wife aged y survive more than $k+1$ years is the same as the probability that the status xy survives beyond its "age" $u+k+1$, having survived to "age" $u+k$. Further, the quantity q_{u+k} can be interpreted as the failure of a status u between "ages" $u+k$ and $u+k+1$.

Now we can apply the principles of life insurance theory for a single life in order to calculate the net single premium for specific insurances in a multiple life theory. Let us consider a whole life insurance and a life annuity-due. Obviously, a similar study can be obtained for other insurance products as well. In case we extend a whole life insurance for the joint-life status, it provides a payment of 1 at the end of the year of the first death and its net single premium is defined by

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k:y+k} \tag{1.10}$$

using (1.8). By extending a life annuity-due for a joint-life status, that is a guaranteed contract promising to provide a regular income over the lifetime(s) of individuals, we obtain a joint-life annuity-due. This product pays 1 at the end of the years $1, 2, \ldots$ as long as both lives survive and its net single premium has the form

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \,{}_kp_{xy}. \tag{1.11}$$

Obviously, we can also consider net single premiums of the joint-life status for temporary contracts. Let us denote by $\overline{n}|$ the status which fails exactly at time n, i.e.

$$T(\overline{n}|) = n. \tag{1.12}$$

Then we can write

$$T(u : \overline{n}|) = min(T(u), n). \tag{1.13}$$

Considering a special case for a single life (x), we obtain $T(x : \overline{n}|) = min(T(x), n)$ and so it is apparent that the net single premium symbols e.g. $A_{x:\overline{n}|}$ for term insurance or $\ddot{a}_{x:\overline{n}|}$ for n-year temporary life annuity-due are in accordance with the notation in the multiple life theory. Further, in order to indicate that the corresponding amount of premium is computed under the independence of the remaining lifetimes of the husband aged x and the wife aged y, we set the superscript "$\perp$". More precisely, ${}^{\perp}\mathrm{A}^{1}_{xy:\overline{n}|}$ and ${}^{\perp}\ddot{a}_{xy:\overline{n}|}$ are given by

$${}^{\perp}A^{1}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x \,{}_kp_y (q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k}) \tag{1.14}$$

using (1.7) and (1.4),

$${}^{\perp}\ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x \,{}_kp_y \tag{1.15}$$

using (1.4).

1.2 Last-Survivor Status

The status

$$u = \overline{xy} \tag{1.16}$$

is defined to exist while at least one of the two lives survives, so the failure time of this last-survivor status happens with the last death and can be written in the form

$$T(u) = max(T_x, T_y). \tag{1.17}$$

In the following we shall use the inclusion-exclusion principle of probability, that is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1.18}$$

Defining A as $\{T(x) \leq t\}$ and B as $\{T(y) \leq t\}$, we have $A \cap B = \{T(\overline{xy}) \leq t\}$ and $A \cup B = \{T(xy) \leq t\}$, which can be useful in deriving the probability of the status (1.16) surviving beyond "age" $u+t$. This probability is denoted by ${}_tp_{\overline{xy}}$ and given by

$$\begin{aligned}
{}_tp_{\overline{xy}} &= P[T(\overline{xy}) > t] = P[max(T_x, T_y) > t] = \\
&= P[T_x > t \cup T_y > t] = && \text{using} \quad (1.18)\\
&= (P[T_x > t] + P[T_y > t] - P[T_x > t \cap T_y > t]) = \\
&= {}_tp_x + {}_tp_y - {}_tp_{xy} = && \text{assuming independence} \quad (1.19)\\
&= {}_tp_x + {}_tp_y - {}_tp_x\, {}_tp_y, && t \geq 0. \quad (1.20)
\end{aligned}$$

The probability ${}_tq_{\overline{xy}}$ that the last-survivor status (1.16) fails before time t is then given by

$$\begin{aligned}
{}_tq_{\overline{xy}} &= P[T(\overline{xy}) \leq t] = P[max(T_x, T_y) \leq t] = \\
&= 1 - P[max(T_x, T_y) > t] = 1 - {}_tp_{\overline{xy}} = && \text{using} \quad (1.19) \quad (1.21)\\
&= 1 - ({}_tp_x + {}_tp_y - {}_tp_{xy}) = \\
&= 1 - (1 - {}_tq_x) - (1 - {}_tq_y) + (1 - {}_tq_{xy}) = \\
&= {}_tq_x + {}_tq_y - {}_tq_{xy} = && (1.22)\\
&= {}_tq_x + {}_tq_y - (1 - {}_tp_{xy}) = && \text{assuming independence}\\
&= {}_tq_x + {}_tq_y - (1 - (1 - {}_tq_x)(1 - {}_tq_y)) = \\
&= {}_tq_x\, {}_tq_y, && t \geq 0. \quad (1.23)
\end{aligned}$$

We know that using (1.18), it holds

$$P[T_x > t \cup T_y > t] + P[T_x > t \cap T_y > t] = P[T_x > t] + P[T_y > t].$$

Since there exists a discrete version of the above stated equality for the curtate future lifetimes, we can write

$$P[K(\overline{xy}) > k] + P[K(xy) > k] = P[K(x) > k] + P[K(y) > k]. \quad (1.24)$$

From (1.24) it follows that

$$P[K(\overline{xy}) = k] + P[K(xy) = k] = P[K(x) = k] + P[K(y) = k]. \quad (1.25)$$

Then from (1.25) and applying (1.8), the probability distribution of the integer-valued random variable $K(\overline{xy})$, where $K(\overline{xy})$ represents the number of completed future years survived by the last-survivor status, is determined by

$$P[K(\overline{xy}) = k] = {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}. \quad (1.26)$$

In case of independent lives, (1.4) and (1.7) allow us to write (1.26) as

$$\begin{aligned}
P[K(\overline{xy}) = k] &= {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_x\, {}_kp_y (q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k}) = \\
&= (1 - {}_kp_y){}_kp_x\, q_{x+k} + (1 - {}_kp_x){}_kp_y\, q_{y+k} + {}_kp_x\, {}_kp_y\, q_{x+k}\, q_{y+k}.
\end{aligned} \quad (1.27)$$

In the last equality stated above, the first two terms are the probabilities for which the first death occurs before reaching the age $y+k$ (resp. $x+k$) and the second one occurs between ages $x+k$ and $x+k+1$ (resp. $y+k$ and $y+k+1$).

The third term represents the probability that both lives end up between times k and $k+1$.

Now we are able to specify net single premiums for particular insurances. By allowing for the extension of whole life insurance for the last-survivor status, we obtain an insurance which provides a payment of 1 at the end of the year of the last death and can be expressed as

$$\begin{aligned} A_{\overline{xy}} &= \sum_{k=0}^{\infty} v^{k+1} P[K(\overline{xy}) = k] = \\ &= \sum_{k=0}^{\infty} v^{k+1} ({}_kp_x \, q_{x+k} + {}_kp_y \, q_{y+k} + {}_kp_{xy} \, q_{x+k:y+k}) \end{aligned} \tag{1.28}$$

using (1.26). Further, we consider the last-survivor annuity for a married couple. This product pays 1 at the end of the years $1, 2, \ldots$ as long as at least one of the spouses survives and it is defined by

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k ({}_kp_x + {}_kp_y - {}_kp_{xy}) \tag{1.29}$$

using (1.19). Moreover, taking into account n-year contracts under the assumption of indepedence, the corresponding formulas for ${}^{\perp}\mathrm{A}^{1}_{\overline{xy}:\overline{n}|}$ and ${}^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|}$ using (1.27) and (1.20) are given by

$${}^{\perp}A^{1}_{\overline{xy}:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}((1 - {}_kp_y){}_kp_x \, q_{x+k} + (1 - {}_kp_x){}_kp_y \, q_{y+k} + {}_kp_x \, {}_kp_y \, q_{x+k} \, q_{y+k}) \tag{1.30}$$

$${}^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{k=0}^{n-1} v^k ({}_kp_x + {}_kp_y - {}_kp_x \, {}_kp_y). \tag{1.31}$$

1.3 Graduation via Mortality Laws

The term graduation is defined by Haberman and Renshaw (1996) as the set of principles and methods by which a set of observed (or crude) probabilities is adjusted to provide a suitable basis for making practical inferences and calculations of premiums. One of its principal applications is to construct a survival model which is usually represented in the form of a life table. In order to show how to mathematically represent the force of mortality[1], as a quantity of life tables, by graduation, we shall introduce Gompertz and Gompertz-Makeham mortality laws.

In the literature, a lot of various graduation methods are suggested that are also used in practice. In particular, two broad fundamental categories can be classified, i.e. parametric approaches adjusting data to a function, and non-parametric approaches that avoid adjusting data to a functional form. Non-parametric methods involve, for example, Kaplan-Meier estimates. Gompertz and Makeham mortality laws are recognized as parametric graduation methods.

[1]The force of mortality is also known as the hazard function.

The use of mathematical functions was historically the first approach to the task of constructing a survival model. Its purpose is to find a mathematical representation of the force of mortality that achieves a proper fit to empirical data that can be relied upon as exact. This approach seeks to find functions which capture the age-effects of mortality properly, since the force of mortality is a function of age. The first useful approach was proposed by Benjamin Gompertz (see Gompertz (1825)) and is known as Gompertz law. Gompertz formula is a function of age x given by

$$\mu_x = Bc^x, \qquad x \geq 0, B > 0, c > 1. \tag{1.32}$$

Thus, this model assumes that mortality grows exponentially with age. Moreover, Gompertz noted in his work that there might exist another component of mortality that is independent of age, e.g. because of accidents. However, this age-independent component had been added to the Gompertz model as a constant term by William Makeham (see Makeham (1866)). Then the model known as Gompertz-Makeham law is given by

$$\mu_x = A + Bc^x, \qquad x \geq 0, B > 0, c > 1, A \geq -B. \tag{1.33}$$

Both approaches are parsimonious, in a sense of simplicity, and nowadays are still used in actuarial work.

Let us further derive ${}_tp_x$ of (x) under Gompertz-Makeham law. Using the relationship of a survival probability ${}_tp_x$ and a force of mortality μ_x from a single life theory, we can write

$$\begin{aligned}
{}_tp_x &= \exp\Big\{-\int_x^{x+t} \mu_s ds\Big\} = \\
&= \exp\Big\{-\int_x^{x+t} (A + Bc^s) ds\Big\} = \\
&= \exp\Big\{-\Big[As + \frac{Bc^s}{\ln c}\Big]_x^{x+t}\Big\} = \\
&= \exp\Big\{-A(x+t-x) - \Big(\frac{B}{\ln c}(c^{x+t} - c^x)\Big)\Big\} = \\
&= \exp\{-At\} \exp\Big\{-\frac{B}{\ln c}c^x(c^t - 1)\Big\}.
\end{aligned} \tag{1.34}$$

Since Gompertz law is a special case of Gompertz-Makeham law with $A = 0$, under Gompertz formula we have

$${}_tp_x = \exp\Big\{-\frac{B}{\ln c}c^x(c^t - 1)\Big\}.$$

We shall devote ourselves to the derivation of survival probabilities for the specific Markov model in the next chapter.

1.4 Positive Quadrant Dependence

A positive quadrant dependence is a kind of dependence between two random variables, which describes the characteristic that large values of the one variable are associated with large values of the second variable. In this section, we give an intuitive interpretation of positive quadrant dependence for remaining lifetimes of the husband and wife. Furthermore, we explain how significant this dependence is for actuarial computations.

The concept of positive quadrant dependence is introduced in Lehmann (1966). Let X and Y be real random variables defined on some probability space. We say that X and Y are *positive quadrant dependent* if

$$P[X > s, Y > t] \geq P[X > s]P[Y > t], \qquad \text{for all } s, t \in \mathbb{R}. \tag{1.35}$$

Similarly, *negative quadrant dependence* of two random variables can be defined if (1.35) holds with the reversed sign of the inequality.

Remark 1. *Note that* (1.35) *is equivalent to*

$$P[X \leq s, Y \leq t] \geq P[X \leq s]P[Y \leq t], \qquad \textit{for all } s, t \in \mathbb{R},$$

since

$$P[X > s, Y > t] = 1 - P[X \leq s] - P[Y \leq t] + P[X \leq s, Y \leq t]$$

and

$$P[X > s]P[Y > t] = 1 - P[X \leq s] - P[Y \leq t] + P[X \leq s]P[Y \leq t].$$

In this section, we shall assume that the remaining lifetimes T_x and T_y are positive quadrant dependent. The following property, which was stated and proved by Denuit et al. (2001, page 25, Property 4.7), provides an intuitive interpretation of positive quadrant dependence.

Property 1. *If T_x and T_y are positive quadrant dependent then the inequalities*

$$E[T_y \mid T_x > s] \geq E[T_y] \qquad \textit{for all } s \geq 0$$

and

$$E[T_x \mid T_y > t] \geq E[T_x] \qquad \textit{for all } t \geq 0$$

both hold true.

Proof. We will prove only the first inequality, since the proof of the second one is analogous. We have

$$\begin{aligned}
E[T_y \mid T_x > s] &= \int_0^\infty P[T_y > t \mid T_x > s]dt = \\
&= \frac{1}{P[T_x > s]} \int_0^\infty P[T_y > t, T_x > s]dt \geq \qquad \text{using} \quad (1.35) \\
&\geq \frac{1}{P[T_x > s]} \int_0^\infty P[T_y > t]P[T_x > s]dt = \\
&= \int_0^\infty P[T_y > t]dt = E[T_y].
\end{aligned} \tag{1.36}$$

□

Property (1) tells us that the expected remaining lifetime of one of the spouses grows with the information that the second one of the two considered lives is still alive at some time. Intuitively, we realise from the introduction that the assumption of positive quadrant dependence for the remaining lifetimes of married couples seems to be a natural assumption.

A relevancy of positive quadrant dependence assumption for actuarial calculations lies in the height of a premium. Assuming positive quadrant dependence, the joint-life annuity values are higher than annuities under independence, e.g. for n-year joint-life annuity-due we have

$$\ddot{a}_{xy:\overline{n}|} \geq {}^{\perp}\ddot{a}_{xy:\overline{n}|}. \tag{1.37}$$

The above relation is obvious, since it holds that ${}_tp_{xy} = P[T_x > t, T_y > t]$ and ${}_tp_x\, {}_tp_y = P[T_x > t]P[T_y > t]$ (see (1.4) and (1.3)). Considering last-survivor annuities, there is a reverse situation. Thus, these annuities are lower than or equal that annuity values under independence, e.g. in case of n-year last-survivor annuity due it holds

$$\ddot{a}_{\overline{xy}:\overline{n}|} \leq {}^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|}. \tag{1.38}$$

To sum up, considering positive quadrant dependence in the case of joint-life insurance there is an underestimated premium, and in the case of last-survivor insurance there is an overestimated premium in comparison with the dependent "situation".

Obviously, a similar study can be achieved for whole life or term insurance. In case of whole life insurance, we have inequalities

$$A_{xy} \leq A^{\perp}_{xy} \quad \text{and} \quad A_{\overline{xy}} \geq A^{\perp}_{\overline{xy}}. \tag{1.39}$$

The resulting inequality for n-year joint-life term insurance has the form

$$A^{1}_{xy:\overline{n}|} \leq {}^{\perp}A^{1}_{xy:\overline{n}|}. \tag{1.40}$$

Comparing (1.26) and (1.27), we obtain for n-year last-survivor term insurance the relation

$$A^{1}_{\overline{xy}:\overline{n}|} \geq {}^{\perp}A^{1}_{\overline{xy}:\overline{n}|}. \tag{1.41}$$

2. Joint-Life and Last-Survivor Model

Multiple state models based on Markov and also semi-Markov stochastic processes constitute a powerful mathematical instrument that can be used to form a general attentive approach for describing and analysing insurance premiums and benefits. It can be applied not only to joint-life or last-survivor types of insurances but also to several others, such as disability, long-term care or critical illness insurance. A fine overview of multiple state models for life and other contingencies using Markov chains, in both the time-continuous and the time-discrete case, is provided by Haberman and Pitacco (1998).

Markov model which considers forces of mortality depending on marital status was probably proposed by Norberg (1989, page 247). We devote this chapter to this Norberg's original model. Let us further refer to this Markov model as to the joint-life and last-survivor model. Further, in order to show how to calculate premiums for joint-life and last-survivor insurances, we derive corresponding probabilities based on the joint-life and last-survivor model. We will be occupied with relationships among transition intensities and remaining lifetimes. Moreover, we examine the probabilities of a special case of the joint-life and last-survivor model which assumes independence of remaining lifetimes of the husband and wife. Finally, we state two extensions of the joint-life and last-survivor model.

We would like to mention that all the derivations and proofs that are contained in this chapter are done just for the joint-life and last-survivor model, even though it could be derived in a general case. The reason why we restrict ourselves only to the derivations for the joint-life and last-survivor model is that the results obtained in this chapter will be directly used in Chapter 3.

2.1 Description of Model

Since the husband and wife share a common way of life, they are mostly exposed to the same risk. The joint-life and last-survivor model captures this interdependence between spouses just by taking into account forces of mortality which depend on whether the other partner is still alive. To be specific, if say the wife is alive, the force of mortality depends on her exact age as well as on the age of the husband. Reflecting this information into a notation, we have that $\mu^{01}_{x+t:y+t}$ denotes the force of mortality for the wife aged $y+t$, given that her husband aged $x+t$ is still alive. However, if the wife died, then the force of mortality for the husband, denoted by μ^{13}_{x+t} , depends only on the present age and the fact that his wife died, but definitely not on how long she has been dead. Analogously, we denote by $\mu^{02}_{x+t:y+t}$ the force of mortality of the $(x+t)$-year-old husband whose spouse is still alive and by μ^{23}_{y+t} the force of mortality of the $(y+t)$-year-old wife in case her husband is dead. Thus, the future development of the marital status for the x-year-old husband and the y-year-old wife can be seen as *time-continuous Markov process* $\{X_t, t \geq 0\}$ with the state space consisting of four states and forces of mortality as shown in Figure (2.1).

Further, we state four important assumptions of this model, introduce some notation and also specify transition probabilities related to the Markov process.

Assumption 1. *Obviously, we assume that the above mentioned time-continuous Markov process fulfills Markov property, i.e. we assume that the conditional probability* $P[X_{t+s} = j \mid X_t = i]$, *for any states* $i, j \in \{0, 1, 2, 3\}$ *and any times* $s, t \geq 0$, *is well defined in a sense that its value does not depend on any knowledge about the process that happened before time* t.

This basically means that the probabilities of future events do not depend on the position of the past states of the process, but are entirely determined by knowing the present state.

Assumption 2. *We assume that for any interval of time* $h > 0$,

$$P[\textit{two transitions within a time period of length } h] = o(h), h \to 0.$$

Let us recall that $o(.)$ is a function that satisfies $\lim_{h \to 0} \frac{o(h)}{h} = 0$, i.e. $o(h)$ converges to 0 faster than h. Assumption 2 tells us that for a small interval of time $h > 0$, the probability of two transitions in that interval is so small that it can be neglected. Note that in case of more complicated models which consist of more states and transitions than the considered model shown in Figure (2.1), we would have to consider not only the probability of two transitions but also more than two transitions.

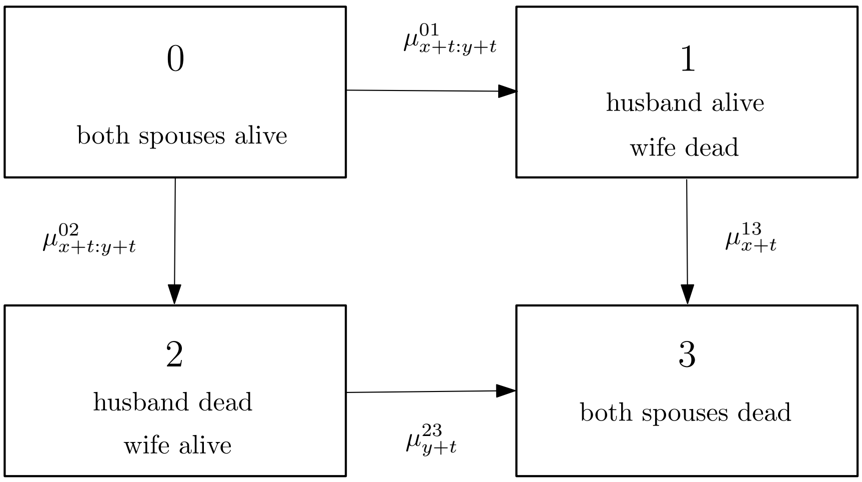

Figure 2.1: The joint-life and last-survivor model with the forces of mortality depending on the marital status

Let us denote the transition probabilities of Markov process by

$${}_tp^{ij}_s = P[X_{s+t} = j \mid X_s = i] \text{ for } i, j \in \{0, 1, 2, 3\} \text{ and } s, t \geq 0.$$

The probability ${}_tp^{ij}_s$ can be interpreted as the conditional probability that the married couple is in state j at time $s + t$, given that it was in state i at time s. Clearly, for any $s, t \geq 0, 0 \leq {}_tp^{ij}_s \leq 1$ for all $i, j \in \{0, 1, 2, 3\}$, ${}_0p^{ij}_s = 1$ if $i = j$ and it is equal to 0 otherwise. Also it holds that $\sum_j {}_tp^{ij}_s = 1$ for all $i \in \{0, 1, 2, 3\}$.

From Figure (2.1) it can be also observed that for example the probability ${}_tp_s^{10}$ is equal to 0, since the backward transition is not allowed as indicated by the direction of the arrow between states 0 and 1. Furthermore, note that we adapt the notation of transition probabilities to the notation of forces of mortality. Concretely, say ${}_tp_{x+s}^{13}$ denotes the probability that the husband aged $x+s$, whose wife has died, does not survive beyond age $x+s+t$. For this probability, the age of the wife's death is not a part of the notation, since it is assumed to be irrelevant.

Now, we can state the third assumption.

Assumption 3. *We assume that the process* $\{X_t, t \geq 0\}$ *is time-inhomogeneous process.*

The following definition gives a relation of forces of mortality to the transition probabilities.

Definition 1. *For* $s \geq 0$*, we define* μ_s^{ij} *as the transition intensity or force of transition between states* i *and* j *by*

$$\mu_s^{ij} = \lim_{h \to 0+} \frac{{}_hp_s^{ij}}{h} \quad \text{for} \quad i \neq j. \tag{2.1}$$

Assumption 4. *For all states* $i, j \in \{0, 1, 2, 3\}$ *and all* $s \geq 0$*, we assume that* ${}_tp_s^{ij}$ *is a continuously differentiable function of* t*.*

Assumption 4 is a technical assumption that is needed to guarantee that mathematics proceeds smoothly.

Note that (2.1) for $h > 0$ can be also written in the form

$${}_hp_s^{ij} = h\mu_s^{ij} + o(h), h \to 0. \tag{2.2}$$

Thus, we can say that for small $h > 0$

$${}_hp_s^{ij} \approx h\mu_s^{ij}. \tag{2.3}$$

2.2 Derivation of Probabilities

In this section, we assume that the forces of transition are known and we derive the probabilities associated with the joint-life and last-survivor model. It is important to note the fact that all the probabilities can be expressed in terms of transition intensities. This fact tells us that by knowing transition intensities, we can determine all that is necessary to know about the joint-life and last-survivor model.

Firstly, let us prove the following result.

Problem 1. *Show that for the joint-life and last-survivor model and for* $h > 0$ *it holds*

$${}_hp_s^{ii} = 1 - h \sum_{j \in \{0,1,2,3\}, j \neq i} \mu_s^{ij} + o(h), \quad h \to 0, \quad i \in \{0, 1, 2, 3\}. \tag{2.4}$$

Solution. Note that $1 - {}_hp_s^{ii}$ is the probability that Markov process leaves state i between times s and $s+h$. Obviously, any state i in the model cannot be reentered, and so we can write

$$\begin{aligned} 1 - {}_hp_s^{ii} &= \sum_{j\in\{0,1,2,3\}, j\neq i} {}_hp_s^{ij} = && \text{using} \quad (2.2) \\ &= h \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_s^{ij} + o(h), && h \to 0, \end{aligned}$$

which proves (2.4).

□

Now we can proceed to the derivation of probabilities in Markov process in terms of transition intensities.

Problem 2. *For the probability that the process does not leave the state i between times s and $s+t$ in the joint-life and last-survivor model it holds*

$$ {}_tp_s^{ii} = \exp\left\{ -\int_0^t \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_{s+\tau}^{ij} d\tau \right\}, \qquad i \in \{0,1,2,3\}. \tag{2.5}$$

Solution. For $h > 0$, consider the probability ${}_{t+h}p_s^{ii}$, i.e. the probability that the process stays in state i between times s and $s+t+h$, given that the process was in state i at time s. Clearly, the event can be splitted into two events, the process stays in state i between times s and $s+t$, given that it was in state i at time s, and the process stays in state i between times $s+t$ and $s+t+h$, given that it was in state i at time $s+t$. Thus, using the chain rule for probabilities, we can write

$$ {}_{t+h}p_s^{ii} = {}_tp_s^{ii}\, {}_hp_{s+t}^{ii}. $$

Using the solution of Problem 1, the above relation can be rewritten as

$$ {}_{t+h}p_s^{ii} = {}_tp_s^{ii}\Big(1 - h \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_{s+t}^{ij} + o(h)\Big), \quad h \to 0. $$

Rearranging this equation, we obtain

$$ \frac{{}_{t+h}p_s^{ii} - {}_tp_s^{ii}}{h} = -{}_tp_s^{ii} \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_{s+t}^{ij} + \frac{o(h)}{h}, \quad h \to 0, $$

and since $h \to 0$, we get the differential equation

$$ \frac{d}{dt}{}_tp_s^{ii} = -{}_tp_s^{ii} \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_{s+t}^{ij}. $$

Integrating over $(0,t)$, we have

$$ \log {}_tp_s^{ii} - \log {}_0p_s^{ii} = -\int_0^t \sum_{j\in\{0,1,2,3\}, j\neq i} \mu_{s+\tau}^{ij} d\tau. $$

And finally, by exponentiating both sides, the solution has the form

$$ {}_tp_s^{ii} = {}_0p_s^{ii} \exp\left\{ -\int_0^t \sum_{j\in\{0,1,2,3\},j\neq i} \mu_{s+\tau}^{ij} d\tau \right\}. $$

This proves (2.5), since ${}_0p_s^{ii} = 1$.

□

Writing (2.5) in the notation that we have adapted for transition probabilities, we have for the husband aged x and the wife aged y

$$ {}_tp_{xy}^{00} = \exp\left\{ -\int_0^t \left(\mu_{x+\tau:y+\tau}^{01} + \mu_{x+\tau:y+\tau}^{02}\right) d\tau \right\}, \tag{2.6} $$

$$ {}_tp_x^{11} = \exp\left\{ -\int_0^t \mu_{x+\tau}^{13} d\tau \right\} \qquad \text{and} \tag{2.7} $$

$$ {}_tp_y^{22} = \exp\left\{ -\int_0^t \mu_{y+\tau}^{23} d\tau \right\}. \tag{2.8} $$

In order to derive the rest of the probabilities in the joint-life and last-survivor model, we solve Problem 3 and state the following remark.

Remark 2. *Note that in the joint-life and last-survivor model by stating, for example, that the x-year-old husband and the y-year-old wife are in state 1, we mean that the x-year-old husband is alive and the wife died before reaching age y.*

Problem 3. *The probability given that the husband aged $x+u$ and the wife aged $y+u$ are in state 1, given the husband aged x and the wife aged y were in state 0, considering $t, h, u > 0$, in the joint-life and last-survivor model has the form*

$$ {}_up_{xy}^{01} = \int_0^u {}_tp_{xy}^{00}\, \mu_{x+t:y+t}^{01}\, {}_{u-t}p_{x+t}^{11} dt. \tag{2.9} $$

Obviously, the analogous formula for the married couple of movement from state 0 to state 2,

$$ {}_up_{xy}^{02} = \int_0^u {}_tp_{xy}^{00}\, \mu_{x+t:y+t}^{02}\, {}_{u-t}p_{y+t}^{22} dt, \tag{2.10} $$

also holds. Furthermore, the above stated formulae for the probabilities can also be derived intuitively.

Solution. It is sufficient to derive (2.9), since (2.10) can be obtained analogously. For $x, y \geq 0$ and $t, h, u > 0$, we start by proving the formula

$$ {}_{t+h}p_{xy}^{01} = {}_tp_{xy}^{01}\, {}_hp_{x+t}^{11} + {}_tp_{xy}^{00}\, h\, \mu_{x+t:y+t}^{01} + o(h), \quad h \to 0. \tag{2.11} $$

The probability on the left-hand side of (2.11) is the probability that the husband aged $x+t+h$ and the wife aged $y+t+h$ are in state 1, given that the husband aged x and the wife aged y were in state 0. This probability can be splitted into two paths. Either the couple with corresponding ages $x+t$ and $y+t$ is in state 1, given that it was in state 0 at ages x and y (the probability ${}_tp_{xy}^{01}$) and then stays at state 1 until the husband reaches the age $x+t+h$ (the probability ${}_hp_{x+t}^{11}$) or the considered couple with ages $x+t$ and $y+t$ stays in state 0, given that it was in state 0 at ages x and y (the probability ${}_tp^{00}_{xy}$) and then occupies state 1 at the husband's age $x+t+h$ (the wife is already dead), given that it was in state 0 at the husband's age $x+t$ (the probability $h\,\mu^{01}_{x+t:y+t}+o(h)$, $h\to 0$, using (2.2)).

Notice that the probability of both lives being alive at ages $x+t$ and $y+t$, then considering the death of the wife before her age $y+t+h$ and the death of the husband before his age $x+t+h$ (i.e. Markov process would reach state 3) is equal to $o(h), h\to 0$, since it involves two transitions in a time interval of length h.

Using the solution of Problem 1, we can rewrite the probability (2.11) as

$$ {}_{t+h}p^{01}_{xy} = {}_tp^{01}_{xy}\,(1-h\,\mu^{13}_{x+t}) + {}_tp^{00}_{xy}\,h\,\mu^{01}_{x+t:y+t} + o(h), \quad h\to 0. \tag{2.12} $$

After rearranging the equation (2.12) and dividing it by h we obtain

$$ \frac{{}_{t+h}p^{01}_{xy} - {}_tp^{01}_{xy}}{h} = {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t} - {}_tp^{01}_{xy}\,\mu^{13}_{x+t} + \frac{o(h)}{h}, \quad h\to 0, $$

and since $h\to 0$, we get

$$ \frac{d}{dt}\,{}_tp^{01}_{xy} = {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t} - {}_tp^{01}_{xy}\,\mu^{13}_{x+t}. \tag{2.13} $$

Further, we calculate the derivation

$$ \begin{aligned} &\frac{d}{dt}\left({}_tp^{01}_{xy}\,\exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}\right) = \\ &\quad = \exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}\frac{d}{dt}{}_tp^{01}_{xy} + {}_tp^{01}_{xy}\exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}\mu^{13}_{x+t} = \qquad \text{using} \quad (2.13) \\ &= \exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}\left({}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t} - {}_tp^{01}_{xy}\,\mu^{13}_{x+t} + {}_t\,p^{01}_{xy}\mu^{13}_{x+t}\right) = \\ &= {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t}\,\exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}. \end{aligned} $$

Integrating both sides of the equation over (0, u) and realising that ${}_0p^{01}_{xy}=0$, we have

$$ {}_up^{01}_{xy}\,\exp\left\{\int_0^u \mu^{13}_{x+s}ds\right\} = \int_0^u {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t}\,\exp\left\{\int_0^t \mu^{13}_{x+s}ds\right\}dt. $$

Multiplying both sides of the equation by $\exp\left\{-\int_0^u \mu^{13}_{x+s}ds\right\}$ gives

$$ {}_up^{01}_{xy} = \int_0^u {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t}\,\exp\left\{-\int_t^u \mu^{13}_{x+s}ds\right\}dt. $$

For proving the formula (2.9), it suffices to realise that using (2.5), we obtain

$$ {}_{u-t}p^{11}_{x+t} = \exp\left\{-\int_t^u \mu^{13}_{x+s}ds\right\}. \tag{2.14} $$

Lastly, we derive the formula (2.9) intuitively. For the married couple with corresponding ages x and y to transfer from state 0 to state 1 before reaching corresponding ages $x+u$ and $y+u$, the couple must occupy state 0 until some ages $x+t$ and $y+t$, move to state 1 between the husband's ages $x+t$ and $x+t+dt$, whereas dt is small, and then stay in state 1 (which means that the wife is already dead) from the husband's age $x+t+dt$ to age $x+u$. These events can be illustrated as shown in Figure (2.2).

We can write the infinitesimal probability of this sequence of events as

$$ {}_tp^{00}_{xy}\,\mu^{01}_{x+t:y+t}\,dt\;{}_{u-t}p^{11}_{x+t}. $$

Note that it could also be possible to write ${}_{u-t-dt}p^{11}_{x+t}$ instead of ${}_{u-t}p^{11}_{x+t}$, but it is not necessary, since both terms are for small dt approximately the same. Considering that the transfer from state 0 to state 1 can happen anytime between the husband's ages x and $x+u$, we have that the "sum" (i.e. integral) of these probabilities over the interval $(0,u)$ represents the total probability ${}_up^{01}_{xy}$.

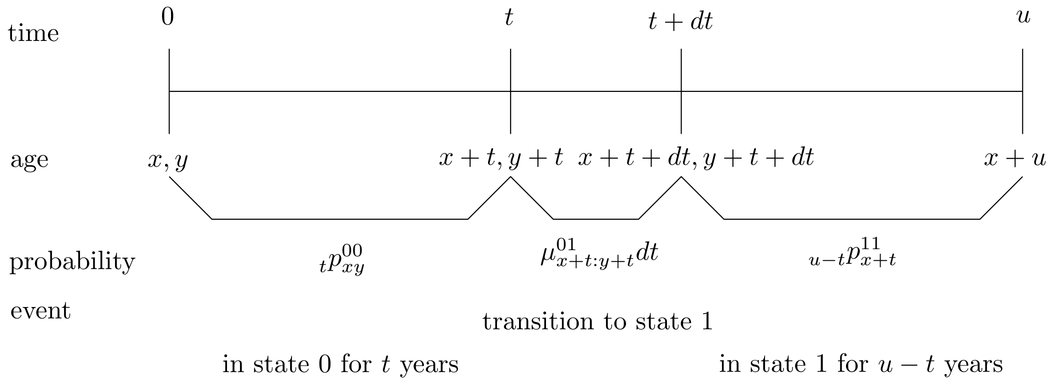

Figure 2.2: Illustration for an intuitive derivation of the probability ${}_up^{01}_{xy}$ in the joint-life and last-survivor model

□

We have derived formulae for probabilities in the joint-life and last-survivor model that are expressed in terms of transition intensities. Similar derivation for probabilities can be found in Dickson et al. (2009), but in terms of a disability model.

Moreover, we consider important to introduce the set of equations for Markov process known as Kolmogorov's forward equations. These equations are used to calculate probabilities in multiple state models as we have already encountered them in the solutions of Problem 2 and Problem 3. Problem 4 will be devoted to the derivation of Kolmogorov's forward equations.

Problem 4. *For $s,t,h \geq 0$, in the joint-life and last-survivor model, we derive the formula*

$$ \frac{d}{dt}\,{}_tp^{ij}_s = \sum_{k\in\{0,1,2,3\},k\neq j} \Big({}_tp^{ik}_s\,\mu^{kj}_{s+t} - {}_tp^{ij}_s\,\mu^{jk}_{s+t}\Big), \qquad i,j\in\{0,1,2,3\}. \tag{2.15} $$

This formula consists of a set of equations for Markov process and is known as Kolmogorov's forward equations.

Solution. In order to derive this set of equations, we proceed as we did in Problem 2 and Problem 3. Let us think of the probability being in state j at time $s+t+h$, given that the process is in some state at time $s+t$. So the process is either already in state j or in some state, say k, and the movement to j is needed before time $s+t+h$. Therefore, we have

$$ {}_{t+h}p_s^{ij} = {}_tp_s^{ij} \; {}_hp_{s+t}^{jj} + \sum_{k\in\{0,1,2,3\},k\neq j} {}_tp_s^{ik} \; {}_hp_{s+t}^{kj} $$

Using formulae (2.2) and (2.4), we can rewrite the above stated equality as

$$ {}_{t+h}p_s^{ij} = {}_tp_s^{ij}\Big(1 - h \sum_{k\in\{0,1,2,3\},k\neq j} \mu_{s+t}^{jk} - o(h)\Big) + h \sum_{k\in\{0,1,2,3\},k\neq j} {}_tp_s^{ik} \; {}_h\mu_{s+t}^{kj} + o(h), h \to 0. $$

Rearranging the equation gives

$$ {}_{t+h}p_s^{ij} = {}_tp_s^{ij} + h \sum_{k\in\{0,1,2,3\},k\neq j} \Big({}_tp_s^{ik} \; {}_h\mu_{s+t}^{kj} - {}_tp_s^{ij} \; \mu_{s+t}^{jk}\Big) + o(h), \quad h \to 0, \quad (2.16) $$

again rearranging, then dividing it by h and letting $h \to 0$ proves (2.15).

□

2.3 Formulae for Transition Intensities

In Section 2.2, we have derived the formulae for probabilities in the joint-life and last-survivor model in terms of transition intensities assuming that transition intensities are known. Now we can write the joint survival function of remaining lifetimes of the husband aged x and the wife aged y and also related marginal distribution functions. The joint survival function of (T_x, T_y) is given by

$$ P[T_x > s, T_y > t] = \begin{cases} {}_sp_{xy}^{00} + {}_tp_{xy}^{00} \; {}_{s-t}p_{x+t:y+t}^{01}, & s \geq t \geq 0, \\ {}_tp_{xy}^{00} + {}_sp_{xy}^{00} \; {}_{t-s}p_{x+s:y+s}^{02}, & t > s \geq 0. \end{cases} \quad (2.17) $$

In the following, we omit "dv" terms in particular integrals in order to simplify formulae, i.e. we will use

$$ e^{-\int_0^t \mu_{x+v:y+v}^{ij} dv} \equiv e^{-\int_0^t \mu_{x:y}^{ij}}, \quad i,j \in \{0,1,2,3\}. $$

Then using (2.6),(2.9) and (2.10), we can write

$$ P[T_x > s, T_y > t] = $$
$$ \begin{cases} {}_sp_{xy}^{00} + {}_tp_{xy}^{00} \int_t^s {}_{\tau-t}p_{x+t:y+t}^{00} \; \mu_{x+\tau:y+\tau}^{01} \; {}_{s-\tau}p_{x+\tau:y+\tau}^{11} d\tau, & s \geq t \geq 0, \\ {}_tp_{xy}^{00} + {}_sp_{xy}^{00} \int_s^t {}_{\tau-t}p_{x+s:y+s}^{00} \; \mu_{x+\tau:y+\tau}^{02} \; {}_{t-\tau}p_{x+\tau:y+\tau}^{22} d\tau, & t > s \geq 0. \end{cases} $$

Using (2.6), (2.8) and (2.7), we obtain

$$ P[T_x > s, T_y > t] = $$
$$ \begin{cases} e^{-\int_0^s \mu_{xy}^{01}+\mu_{xy}^{02}} + e^{-\int_0^t \mu_{xy}^{01}+\mu_{xy}^{02}} \int_t^s e^{-\int_t^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{01} e^{-\int_\tau^s \mu_x^{13}} d\tau, & s \geq t \geq 0, \\ e^{-\int_0^t \mu_{xy}^{01}+\mu_{xy}^{02}} + e^{-\int_0^s \mu_{xy}^{01}+\mu_{xy}^{02}} \int_s^t e^{-\int_s^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{02} e^{-\int_\tau^t \mu_y^{23}} d\tau, & t > s \geq 0, \end{cases} $$

and simplifying, we have

$$P[T_x > s, T_y > t] =$$
$$\begin{cases} e^{-\int_0^s \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_t^s e^{-\int_0^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{01} e^{-\int_\tau^s \mu_x^{13}} d\tau, & s \geq t \geq 0, \\ e^{-\int_0^t \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_s^t e^{-\int_0^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{02} e^{-\int_\tau^t \mu_y^{23}} d\tau, & t > s \geq 0. \end{cases} \tag{2.18}$$

The marginal survival functions of T_x and T_y are given by

$$P[T_x > s] = {}_sp_{xy}^{00} + {}_sp_{xy}^{01}, \qquad s \geq 0, \tag{2.19}$$
$$P[T_y > t] = {}_tp_{xy}^{00} + {}_tp_{xy}^{02}, \qquad t \geq 0. \tag{2.20}$$

Again applying (2.6),(2.9) and (2.10) and omitting particular "dv" terms, we can write

$$P[T_x > s] = e^{-\int_0^s \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_0^s {}_\tau p_{xy}^{00}\, \mu_{x+\tau:y+\tau}^{01}\, {}_{s-\tau}p_{x+\tau:y+\tau}^{11} d\tau, \qquad s \geq 0,$$
$$P[T_y > t] = e^{-\int_0^t \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_0^t {}_\tau p_{xy}^{00}\, \mu_{x+\tau:y+\tau}^{02}\, {}_{t-\tau}p_{x+\tau:y+\tau}^{22} d\tau, \qquad t \geq 0,$$

and using (2.6), (2.8) and (2.7), we obtain

$$P[T_x > s] = e^{-\int_0^s \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_0^s e^{-\int_0^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{01} e^{-\int_\tau^s \mu_x^{13}} d\tau, \qquad s \geq 0, \tag{2.21}$$
$$P[T_y > t] = e^{-\int_0^t \mu_{xy}^{01}+\mu_{xy}^{02}} + \int_0^t e^{-\int_0^\tau \mu_{xy}^{01}+\mu_{xy}^{02}} \mu_{x+\tau:y+\tau}^{02} e^{-\int_\tau^t \mu_y^{23}} d\tau, \qquad t \geq 0. \tag{2.22}$$

In Section 1.3, we have introduced Gompertz and Gompertz-Makeham laws of mortality in case of single life theory representing force of mortality of a single life as a function of age. In order to show how to calculate premiums for joint-life and last-survivor insurances, it remains to express forces of transition in the joint-life and last-survivor model in some functional form, so that they could be estimated from data. And that is the aim of this section. Let us start with the concept of a right tail increase for two random variables.

Let X and Y be continuous non-negative real random variables defined on some probability space and $s, t \geq 0$. We say that X is *right tail increasing* (resp. *decreasing*) in Y if $P[X > s \mid Y > t]$ is increasing (resp. decreasing) in t for all $s \geq 0$. The following lemma clarifies the relationship between right tail increasing variables and variables that are positively quadrant dependent.

Lemma 1. *If say X is right tail increasing in Y then X and Y are positive quadrant dependent.*

Proof. Note that X and Y are positive quadrant dependent if

$$P[Y > t \mid X > s] \geq P[Y > t], \qquad s, t \geq 0. \tag{2.23}$$

Obviously, the assumption that X is right tail increasing in Y is equivalent to the statement that

$$\frac{P[Y > t, X > s]}{P[Y > t]} \frac{1}{P[X > s]} \quad \text{is increasing in t for all } s \geq 0. \tag{2.24}$$

We choose $t = 0$ for fixed $s \geq 0$ and we rewrite the term in (2.24) as follows

$$\frac{P[Y > 0, X > s]}{P[Y > 0]} \frac{1}{P[X > s]}. \tag{2.25}$$

Since we assume that Y is a continuous non-negative random variable, we have that $P(Y > 0) = 1$, and then the term (2.25) is equal to 1. Since the term in (2.24) is increasing in t for all $s \geq 0$, we have for $t > 0$ that

$$\frac{P[Y > t, X > s]}{P[X > s]} \frac{1}{P[Y > t]} > 1,$$

which proves (2.23).

$\square$

Now we present the result proved by Norberg (1989, page 249, Theorem 4.1) which will lead to a natural suggestion for a functional form of transition intensities in the joint-life and last-survivor model. This result characterizes relationships between transition intensities and remaining lifetimes.

Theorem 2. *In the joint-life and last-survivor model it holds:*

$$\begin{aligned}
&i)\ \mu^{02}_{x+\tau:y+\tau} \leq \mu^{13}_{x+\tau} \text{ and } \mu^{01}_{x+\tau:y+\tau} \leq \mu^{23}_{y+\tau} \quad \text{for all} \quad \tau \geq 0 \Longrightarrow \\
&\qquad T_x \text{ and } T_y \text{ are positively quadrant dependent,} \\
&ii)\ \mu^{02}_{x+\tau:y+\tau} \geq \mu^{13}_{x+\tau} \text{ and } \mu^{01}_{x+\tau:y+\tau} \geq \mu^{23}_{y+\tau} \quad \text{for all} \quad \tau \geq 0 \Longrightarrow \\
&\qquad -T_x \text{ and } T_y \text{ are positively quadrant dependent,} \\
&iii)\ \mu^{02}_{x+\tau:y+\tau} = \mu^{13}_{x+\tau} \text{ and } \mu^{01}_{x+\tau:y+\tau} = \mu^{23}_{y+\tau} \quad \text{for all} \quad \tau \geq 0 \Longleftrightarrow \\
&\qquad T_x \text{ and } T_y \text{ are independent.}
\end{aligned}$$

Proof. Since the proof is quite massive and contains straightforward but tedious calculations, we shall prove it only partially. We start by proving item *i*). Using Lemma 1, it is sufficient to establish that

$$\mu^{02}_{x+\tau:y+\tau} \leq \mu^{13}_{x+\tau} \text{ and } \mu^{01}_{x+\tau:y+\tau} \leq \mu^{23}_{y+\tau} \quad \text{for all} \quad \tau \geq 0$$

implies that say T_x is right tail increasing in T_y.

Let us proceed further for $s < t$. From (2.18) and (2.22), we can write

$$\begin{aligned}
&P[T_x > s \mid T_y > t] = \\
&\qquad = \frac{P[T_x > s, T_y > t]}{P[T_y > t]} = \\
&\qquad = \frac{e^{-\int_0^t \mu^{01}_{xy} + \mu^{02}_{xy}} + \int_s^t e^{-\int_0^\tau \mu^{01}_{xy} + \mu^{02}_{xy}} \mu^{02}_{x+\tau:y+\tau} e^{-\int_\tau^t \mu^{23}_y} d\tau}{e^{-\int_0^t \mu^{01}_{xy} + \mu^{02}_{xy}} + \int_0^t e^{-\int_0^\tau \mu^{01}_{xy} + \mu^{02}_{xy}} \mu^{02}_{x+\tau:y+\tau} e^{-\int_\tau^t \mu^{23}_y} d\tau} = \\
&\qquad = \frac{1 + \int_s^t e^{-\int_\tau^t \mu^{01}_{xy} + \mu^{02}_{xy} - \mu^{23}_y} \mu^{02}_{x+\tau:y+\tau} d\tau}{1 + \int_0^t e^{-\int_\tau^t \mu^{01}_{xy} + \mu^{02}_{xy} - \mu^{23}_y} \mu^{02}_{x+\tau:y+\tau} d\tau.}
\end{aligned}$$

Using the rule for differentiating the fraction, the numerator of the derivation $\frac{d}{dt}P[T_x > s \mid T_y > t]$ has the form

$$\begin{aligned}&\Big(1+\int_0^t e^{-\int_\tau^t \mu_{xy}^{01}+\mu_{xy}^{02}-\mu_y^{23}}\mu_{x+\tau:y+\tau}^{02}d\tau\Big)\\&\Big[\mu_{x+t:y+t}^{02}+\int_s^t e^{-\int_\tau^t \mu_{xy}^{01}+\mu_{xy}^{02}-\mu_y^{23}}\mu_{x+\tau:y+\tau}^{02}d\tau\Big(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23}\Big)\Big]\\&-\Big(1+\int_s^t e^{-\int_\tau^t \mu_{xy}^{01}+\mu_{xy}^{02}-\mu_y^{23}}\mu_{x+\tau:y+\tau}^{02}d\tau\Big)\\&\Big[\mu_{x+t:y+t}^{02}+\int_0^t e^{-\int_\tau^t \mu_{xy}^{01}+\mu_{xy}^{02}-\mu_y^{23}}\mu_{x+\tau:y+\tau}^{02}d\tau\Big(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23}\Big)\Big],\end{aligned} \tag{2.26}$$

We realise that (2.26) has the same sign as $\frac{d}{dt}P[T_x > s \mid T_y > t]$, since the denominator of the derivation is positive due to the second power. Now, putting

$$A_t = \int_0^t e^{-\int_\tau^t \mu_{xy}^{01}+\mu_{xy}^{02}-\mu_y^{23}}\mu_{x+\tau:y+\tau}^{02}d\tau, \tag{2.27}$$

we can rewrite (2.26) as

$$\begin{aligned}&(1+A_t)\Big[\mu_{x+t:y+t}^{02}+(A_t-A_s)(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\Big]\\&\qquad-(1+A_t-A_s)\Big[\mu_{x+t:y+t}^{02}+A_t(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\Big]=\\&=(1+A_t)\Big[\mu_{x+t:y+t}^{02}+A_t(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\Big]\\&\qquad-(1+A_t)A_s(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\\&\qquad-(1+A_t)\Big[\mu_{x+t:y+t}^{02}+A_t(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\Big]\\&\qquad+A_s\Big[\,\mu_{x+t:y+t}^{02}+A_t(\mu_{x+t:y+t}^{01}+\mu_{x+t:y+t}^{02}-\mu_{y+t}^{23})\Big]=\\&=-A_s\,\mu_{x+t:y+t}^{01}-A_s\,\mu_{x+t:y+t}^{02}+A_s\,\mu_{y+t}^{23}\\&\qquad-A_tA_s\,\mu_{x+t:y+t}^{01}-A_tA_s\,\mu_{x+t:y+t}^{02}+A_tA_s\,\mu_{y+t}^{23}\\&\qquad+A_s\,\mu_{x+t:y+t}^{02}+A_sA_t\,\mu_{x+t:y+t}^{01}+A_sA_t\,\mu_{x+t:y+t}^{02}-A_sA_t\,\mu_{y+t}^{23}=\\&=A_s(\,\mu_{y+t}^{23}-\;\mu_{x+t:y+t}^{01}).\end{aligned} \tag{2.28}$$

Since the term A_s is positive for all s, it can be seen from (2.28) that $\mu_{y+t}^{23} \geq \mu_{x+t:y+t}^{01}$ implies $\frac{d}{dt}P[T_x > s \mid T_y > t] \geq 0$ for $s < t$. It can also be showed that $\mu_{x+t}^{13} \geq \mu_{x+t:y+t}^{02}$ implies $\frac{d}{dt}P[T_x > s \mid T_y > t] \geq 0$ for $s \geq t$. For the proof, we refer to Norberg (1989, page 249, Theorem 4.1). Overall, for all s and t, we have that T_x is right tail increasing in T_y, which proves item i). Item ii) follows immediately from item i). Item iii) follows by noting that T_x is right tail increasing in T_y and also T_x is right tail decreasing in T_y is together equivalent to independence.

□

In the introduction, we have mentioned that the husband and wife are more or less exposed to the same risks, since they share a common way of life. Therefore, we consider forces of mortality that depend on the fact whether the other spouse is still alive or not. Taking this into account and also in the view of items $i) - iii)$

in Theorem 2, it seems natural to suggest that for $t \geq 0$,

$$\mu^{01}_{x+t:y+t} = (1-\alpha_{01})\mu_{y+t}, \quad (2.29)$$
$$\mu^{23}_{y+t} = (1+\alpha_{23})\mu_{y+t}, \quad (2.30)$$
$$\mu^{02}_{x+t:y+t} = (1-\alpha_{02})\mu_{x+t} \quad \text{and} \quad (2.31)$$
$$\mu^{13}_{x+t} = (1+\alpha_{13})\mu_{x+t}, \quad (2.32)$$

whereas α_{0j}'s $\in [0,1)$ and α_{ij}'s are non-negative as has been proposed by Denuit et al. (2001, page 27). Obviously, μ_{x+t} (resp. μ_{y+t}) denotes force of mortality of a single life for the husband aged x (resp. the wife aged y). So, we have derived functional formulae for transition intensities in the joint-life and last-survivor model and we are able to calculate premiums for joint-life and last-survivor insurances which will be performed in Chapter 3 using a specific dataset.

2.4 Independent Joint-Life and Last-Survivor Model

Item *iii*) in Theorem 2 has suggested a special case of the joint-life and last-survivor model, where the lives of the husband and wife are independent. It says that remaining lifetimes of the husband and wife are independent if and only if

$$\mu^{02}_{x+t:y+t} = \mu^{13}_{x+t} \quad \text{and} \quad \mu^{01}_{x+t:y+t} = \mu^{23}_{y+t} \quad \text{for all} \quad t \geq 0.$$

Thus, forces of mortality in this model do not depend on whether the other partner is still alive or not. It is an important model, since it is often used in practise. Let us further refer to this special case of the joint-life and last-survivor model with the assumption of independent lives as to the independent joint-life and last-survivor model.

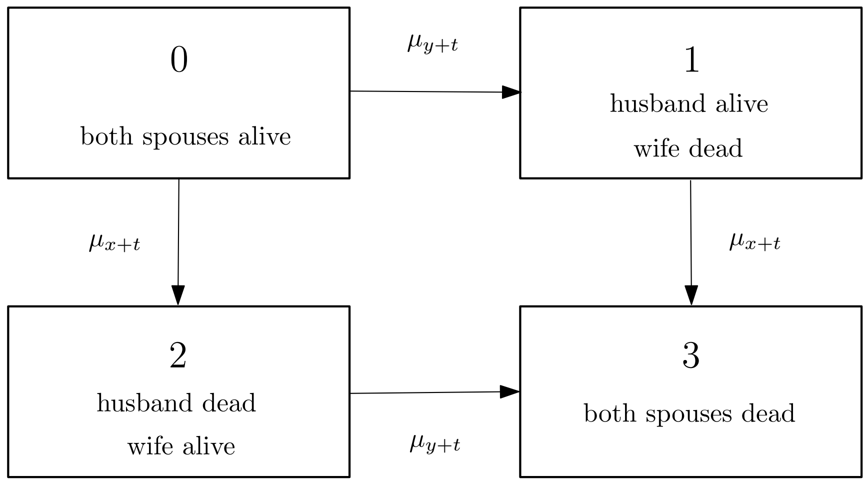

Figure 2.3: The independent joint-life and last-survivor model

By solving Problem 5, which assumes independence in the joint-life and last-survivor model, we show that probabilities of joint events are the product of the probabilities of events for each life separately.

Problem 5. *Suppose that in the joint-life and last-survivor model it holds that*

$$\mu^{02}_{x+t:y+t} = \mu^{13}_{x+t} \quad \textit{and} \quad \mu^{01}_{x+t:y+t} = \mu^{23}_{y+t} \quad \textit{for all} \quad t \geq 0.$$

We show that
i) the probability that both partners are alive at time t is

$$\exp\left\{-\int_0^t \mu^{13}_{x+s} ds\right\} \exp\left\{-\int_0^t \mu^{23}_{y+s} ds\right\}, \tag{2.33}$$

ii) the probability that the husband is alive and the wife is dead at time t is

$$\exp\left\{-\int_0^t \mu^{13}_{x+s} ds\right\}\left(1-\exp\left\{-\int_0^t \mu^{23}_{y+s} ds\right\}\right), \tag{2.34}$$

iii) the probability that the husband is alive at time t has the form

$$\exp\left\{-\int_0^t \mu^{13}_{x+s} ds\right\}, \tag{2.35}$$

iv) and finally the probability that both partners are dead at time t is

$$\left(1-\exp\left\{-\int_0^t \mu^{13}_{x+s} ds\right\}\right)\left(1-\exp\left\{-\int_0^t \mu^{23}_{y+s} ds\right\}\right). \tag{2.36}$$

Solution. i) Obviously, the probability that both partners are alive at time t in the joint-life and last-survivor model is ${}_tp^{00}_{xy}$ and using the assumption of independence, it can be rewritten as

$$\begin{aligned}
{}_tp^{00}_{xy} &= \exp\left\{-\int_0^t \left(\mu^{01}_{x+\tau:y+\tau} + \mu^{02}_{x+\tau:y+\tau}\right) d\tau\right\} = \\
&= \exp\left\{-\int_0^t \mu^{01}_{x+\tau:y+\tau} d\tau\right\} \exp\left\{-\int_0^t \mu^{02}_{x+\tau:y+\tau} d\tau\right\} = \\
&= \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} \exp\left\{-\int_0^t \mu^{23}_{y+\tau} d\tau\right\}.
\end{aligned}$$

ii) The probability that the husband is alive and the wife is dead at time t is given by (2.9) and can be rewritten as

$$\begin{aligned}
{}_tp^{01}_{xy} &= \int_0^t \exp\left\{-\int_0^s \left(\mu^{01}_{x+\tau:y+\tau} + \mu^{02}_{x+\tau:y+\tau}\right) d\tau\right\} \mu^{01}_{x+s:y+s} \\
&\quad \exp\left\{-\int_s^t \mu^{13}_{x+\tau} d\tau\right\} ds = \\
&= \int_0^t \exp\left\{-\int_0^s \left(\mu^{23}_{y+\tau} + \mu^{13}_{x+\tau}\right) du\right\} \mu^{23}_{y+s} \exp\left\{-\int_s^t \mu^{13}_{x+\tau} d\tau\right\} ds = \\
&= \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} \int_0^t \exp\left\{-\int_0^s \mu^{23}_{y+\tau} d\tau\right\} \mu^{23}_{y+s}\, ds = \\
&= \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} \int_0^t \frac{d}{ds}\left(-\exp\left\{-\int_0^s \mu^{23}_{y+\tau} d\tau\right\}\right) ds = \\
&= \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} \frac{d}{ds}\left(-\int_0^t \exp\left\{-\int_0^s \mu^{23}_{y+\tau} d\tau\right\} ds\right) = \\
&= \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\}\left(1-\exp\left\{-\int_0^t \mu^{23}_{y+\tau} d\tau\right\}\right),
\end{aligned}$$

which proves (2.34).
iii) The probability that the husband is alive at time t is ${}_tp^{00}_{xy} + {}_tp^{01}_{xy}$. Applying the results from items i) and ii), we obtain

$$
\begin{aligned}
&{}_tp^{00}_{xy} + {}_tp^{01}_{xy} = \\
&\qquad = \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} \exp\left\{-\int_0^t \mu^{23}_{y+\tau} d\tau\right\} + \\
&\qquad + \exp\left\{-\int_0^t \mu^{13}_{x+u} du\right\}\left(1 - \exp\left\{-\int_0^t \mu^{23}_{y+\tau} d\tau\right\}\right) = \\
&\qquad = \exp\left\{-\int_0^t \mu^{13}_{x+u} du\right\} \qquad (2.37)
\end{aligned}
$$

as required.
iv) The probability that both the husband and the wife are dead at time t, can be rewritten as

$$1 - {}_tp^{00}_{xy} - {}_tp^{01}_{xy} - {}_tp^{02}_{xy},$$

using (2.37) and the corresponding formula for ${}_tp^{02}_{xy}$, we get the probability

$$1 - \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\} - \exp\left\{-\int_0^t \mu^{23}_{y+\tau} d\tau\right\}\left(1 - \exp\left\{-\int_0^t \mu^{13}_{x+\tau} d\tau\right\}\right),$$

which gives (2.36) after adjustment.

□

Moreover, note that

$$\mu^{02}_{x+t:y+t} = \mu^{13}_{x+t} = \mu_{x+t}, \qquad t \geq 0$$

and

$$\mu^{01}_{x+t:y+t} = \mu^{23}_{y+t} = \mu_{y+t}, \qquad t \geq 0.$$

The independent joint-life and last-survivor model is illustrated by Figure (2.3).

2.5 Extended Models

In this section we would like to state two extensions of the joint-life and last-survivor model. In the introduction, we mentioned that there are three types of possible dependencies between the two considered lives, i.e, the long-term association between lifetimes, the short-term impact of a spousal death and the instantaneous dependence due to a catastrophic event.

The long-term association is captured by the joint-life and last-survivor model by taking into account forces of mortality which depend on whether the other partner is alive. Now, we modify the joint-life and last-survivor model by considering the instantaneous dependence due to a catastrophic event that effects both lives and by considering the short-term impact of a spousal death.

In order to extend the joint-life and last-survivor model shown in (2.1) by considering a “common shock” type of dependence, which stands for a

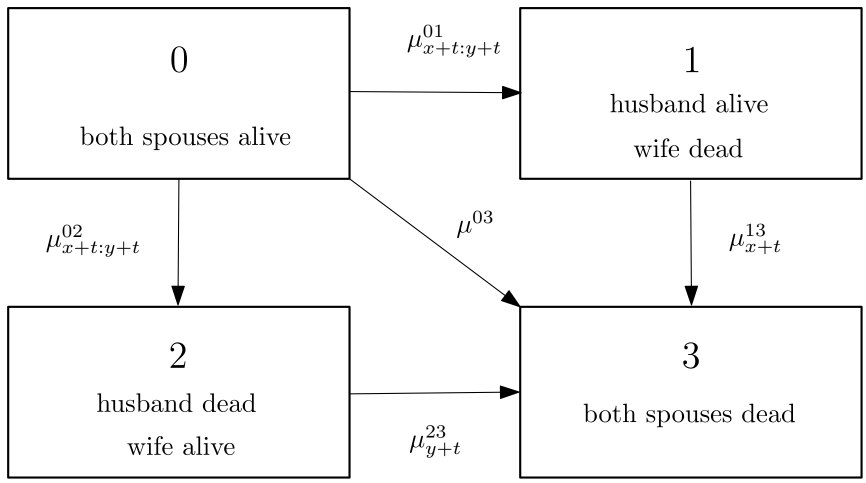

Figure 2.4: The joint-live and last-survivor model with the "common shock" effect

simultaneous deaths of the couple due to a common catastrophic event, we simply add the transition from state 0 to state 3, i.e. μ^{03}. We assume that μ^{03} is independent of age (time), which is already reflected by the notation. The model is shown in Figure (2.4). The force of mortality of the $(y+t)$-year-old wife, in case her husband is dead, is denoted by μ^{23}_{y+t}. The force of mortality from all causes other than a "common shock" effect, for the wife aged $y+t$ with the husband aged $x+t$ still living, is denoted by $\mu^{01}_{x+t:y+t}$. Likewise, the husband's force of mortality at age $x+t$ is μ^{13}_{x+t} if he is widowed, while $\mu^{02}_{x+t:y+t}$ denotes the mortality for the still-married man aged $x+t$ from all causes other than a "common shock" effect. The use of "common shock" transitions means that the total force of mortality for the married woman aged $y+t$ is $\mu^{01}_{x+t:y+t}+\mu^{03}$, and similarly for the married men.

The joint-life and last-survivor model with the "common shock" effect differs from the joint-life and last-survivor model only in one equation from the set of all differential equations, to be concrete,

$$\frac{d}{dt}{}_tp^{00}_{xy} = -{}_tp^{00}_{xy}\Big(\mu^{01}_{x+t:y+t}+\mu^{02}_{x+t:y+t}+\mu^{03}\Big). \tag{2.38}$$

According to Problem 2, the solution of (2.38) has the form

$${}_tp^{00}_{xy} = \exp\bigg\{-\int_0^t \Big(\mu^{01}_{x+\tau:y+\tau}+\mu^{02}_{x+\tau:y+\tau}+\mu^{03}\Big)d\tau\bigg\}. \tag{2.39}$$

Further, we extend the joint-life and last-survivor model shown in (2.1) by considering the third kind of dependence, i.e. the short-term impact of the spousal death. This dependence represents the existence of appreciable increase in mortality rate among the widowers during the first six months after the deaths of their wives in comparison with the rate of married men of the same age. We can take this dependence into account by considering the transition intensities from state 1 to 3 and from state 2 to 3 to be dependent not only on the current state, but also on the time elapsed since the latest transition. So, we take $\mu^{13}_{x+t,r}$

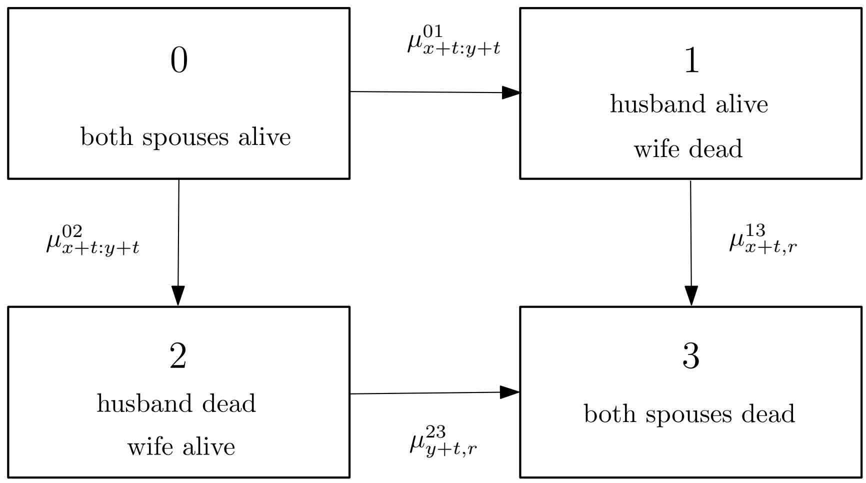

Figure 2.5: The joint-live and last-survivor model including the short-term impact of the spousal death

(resp. $\mu^{23}_{y+t,r}$) instead of μ^{13}_{x+t} (resp. μ^{23}_{y+t}), whereas r is the time elapsed since the latest transition into state 1 (resp. 2). Note that the forces of mortality $\mu^{01}_{x+t:y+t}$ and $\mu^{02}_{x+t:y+t}$ stay unchanged.

In order to write differential equations (also integro-differential equations) for the joint-life and last-survivor model including the short-term impact of the spousal death, we would have to consider semi-Markov theory, but it is not our purpose. We just want to emphasize that transition intensities (also probabilities) in Markov models depend only on the current state and the strength of semi-Markov theory is that it allows to built a model in which the intensities (also probabilities) depend not only on the current state but also on the time elapsed since the last transition to that state. The joint-life and last-survivor model including the short-term impact of the spousal death is shown in Figure (2.5).

3. Application of Joint-Life and Last-Survivor Model

This chapter aims to quantify the impact of dependence between the remaining lifetimes of the husband and wife on the amount of premiums considering the joint-life and last-survivor model. To do so, we calculate premiums for n-year joint-life and n-year last-survivor annuities due in both cases, assuming independence and also dependence of lifetimes of the husband and wife, and we compare them.

In this thesis, the application of the joint-life and last-survivor model will be based on a dataset collected by the Czech Statistical Office considering the population in the Czech Republic during the year 2015. This dataset can be partially found on the official webpage[1] of the Czech Statistical Office and was partially provided on request. The dataset description will be specified later in this chapter.

Let us proceed to the calculation of premiums in case of the dependency of the two insured lifes. We assume that the husband and wife buy the insurance contract in their ages x and y and we want to calculate n-year joint-life and n-year last-survivor annuities due, given by

$$\ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_k p_{xy} \qquad \text{and} \tag{3.1}$$

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{k=0}^{n-1} v^k (\, {}_k p_x + \, {}_k p_y - \, {}_k p_{xy}). \tag{3.2}$$

To do so, we need to calculate corresponding probabilities in the joint-life and last-survivor model. We already know that all the probabilities in the joint-life and last-survivor model can be written in terms of transition intensities. It tells us that all the information that is necessary to know about the joint-life and last-survivor model is fully determined by transition intensities. Thus, we will be concerned with the calculation of transition intensities that are given by (2.29) - (2.32). Before we deal with the estimation of non-negative parameters α_{ij}'s, that are contained in the above mentioned formulae, we will pay attention to modelling forces of mortality μ_{x+t} and μ_{y+t} using Gompertz-Makeham mortality law.

3.1 Gompertz-Makeham Law

Let us assume that forces of mortality for the woman aged $y + t$ and the man aged $x + t$ are given by Gompertz-Makeham law (1.33), i.e.

$$\mu_{y+t} = A_1 + B_1 c_1^{y+t}, \qquad y, t \geq 0, B_1 > 0, c_1 > 1, A_1 \geq 0, \tag{3.3}$$

$$\mu_{x+t} = A_2 + B_2 c_2^{x+t}, \qquad x, t \geq 0, B_2 > 0, c_2 > 1, A_2 \geq 0. \tag{3.4}$$

[1] https://www.czso.cz

Further, by denoting

$$A_i = -\ln s_i \quad \text{and} \quad B_i = -\ln c_i \ln g_i, \qquad s_i, g_i \in [0,1],\ i = 1,2, \tag{3.5}$$

we obtain

$$_tp_y = s_1^t g_1^{c_1^{y+t} - c_1^y}, \qquad t \geq 0, \tag{3.6}$$

$$_tp_x = s_2^t g_2^{c_2^{x+t} - c_2^x}, \qquad t \geq 0. \tag{3.7}$$

To realise that say (3.6) holds true, it is sufficient to plug A_1, B_1 and c_1 into (1.34) instead of A, B and c, obviously considering the woman aged y, and so we get the result immediately.

We have at hand life tables experienced in the Czech Republic during 2015 and based on this dataset, we estimate parameters $A_i, B_i, c_i, i = 1, 2$. The method of least squares will be used and accompanied by the following algorithm that was used by Denuit et al. (2001, page 21).

Let us proceed to the estimation of parameters say related to women. The formula (1.34), in case of one-year survival probability for the woman aged y using the above stated notation, can be written as

$$p_y = \exp\{-A_1\} \exp\Big\{ -\frac{B_1}{\ln c_1} c_1^y (c_1 - 1) \Big\}. \tag{3.8}$$

By applying logarithm on both sides of (3.8) and then multiplying the equation by minus one, we obtain

$$-\ln(p_y) = A_1 + \frac{B_1}{\ln c_1} c_1^y (c_1 - 1). \tag{3.9}$$

Let us further denote the empirical estimator of one-year survival probability at age y by $\hat{p}_y$ and define

$$\alpha_y = -\ln \hat{p}_y. \tag{3.10}$$

The empirical one-year survival probability $\hat{p}_y$ will be taken from life tables that we have at hand. And now, we plug (3.10) in (3.9) instead of $-\ln(p_y)$ and we obtain

$$\alpha_y = A_1 + \beta_1 c_1^y, \tag{3.11}$$

where

$$\beta_1 = -(c_1 - 1) \ln g_1. \tag{3.12}$$

We estimate parameters A_1, β_1, c_1 in the equation (3.11) for a chosen age range (ϑ_1, η_2) in such a way, that we decompose the age range into two parts, i.e. $(\vartheta_1, \vartheta_2)$ and (η_1, η_2), whereas $\eta_1 = \vartheta_2 + 1$. In (3.9), the term A_1 can be neglected for the range (η_1, η_2). Thus, by omitting A_1 in (3.9) and further by applying logarithm, we obtain a linear approximation

$$\ln(\alpha_y) \approx \ln \beta_1 + y \ln c_1. \tag{3.13}$$

The mentioned approximation will be useful for choosing the age η_1. On the other hand, the term A_1 may no longer be omitted in $(\vartheta_1, \vartheta_2)$. The method determines the estimates of parameters β_1, c_1, i.e. $\hat{\beta}_1, \hat{c}_1$, as the solution of

$$(\hat{\beta}_1, \hat{c_1}) = \underset{(\beta_1, c_1)}{\operatorname{argmin}} \sum_{y=\eta_1}^{\eta_2} \Big(\ln \alpha_y - \ln \beta_1 - y \ln c_1 \Big)^2, \tag{3.14}$$

and the estimate of A_1, i.e. $\hat{A}_1$, is determined by

$$\hat{A}_1 = \underset{A_1}{\operatorname{argmin}} \sum_{y=\vartheta_1}^{\vartheta_2} \Big(\alpha_y - A_1 - \hat{\beta}_1 \hat{c}_1^y \Big)^2. \tag{3.15}$$

In order to obtain estimates $\hat{\beta}_1, \hat{c}_1$, we just differentiate the sum in (3.14) with respect to β_1 and separately with respect to c_1, and then we set these derivatives to be equal to zero. We obtain the equations

$$\sum_{y=\eta_1}^{\eta_2} \Big(\ln \alpha_y - \ln \hat{\beta}_1 - y \ln \hat{c}_1 \Big) = 0, \tag{3.16}$$

$$\sum_{y=\eta_1}^{\eta_2} \Big(\ln \alpha_y - \ln \hat{\beta}_1 - y \ln \hat{c}_1 \Big) y = 0. \tag{3.17}$$

From (3.16), we have

$$(\eta_2 - \vartheta_2) \ln \hat{\beta}_1 + \ln \hat{c}_1 \sum_{y=\eta_1}^{\eta_2} y = \sum_{y=\eta_1}^{\eta_2} \ln \alpha_y,$$

and expressing $\ln \hat{\beta}_1$, we obtain

$$\ln \hat{\beta}_1 = \frac{\sum_{y=\eta_1}^{\eta_2} \ln \alpha_y - \ln \hat{c}_1 \sum_{y=\eta_1}^{\eta_2} y}{(\eta_2 - \vartheta_2)}. \tag{3.18}$$

After plugging (3.18) into (3.17), we have

$$\sum_{y=\eta_1}^{\eta_2} y \ln \alpha_y - \frac{\sum_{y=\eta_1}^{\eta_2} y \sum_{y=\eta_1}^{\eta_2} \ln \alpha_y - \ln \hat{c}_1 (\sum_{y=\eta_1}^{\eta_2} y)^2}{(\eta_2 - \vartheta_2)} - \ln \hat{c}_1 \sum_{y=\eta_1}^{\eta_2} y^2 = 0,$$

rearranging gives us

$$\ln \hat{c}_1 \Big(\sum_{y=\eta_1}^{\eta_2} y \Big)^2 - (\eta_2 - \vartheta_2) \ln \hat{c}_1 \sum_{y=\eta_1}^{\eta_2} y^2 = \sum_{y=\eta_1}^{\eta_2} y \sum_{y=\eta_1}^{\eta_2} \ln \alpha_y - (\eta_2 - \vartheta_2) \sum_{y=\eta_1}^{\eta_2} y \ln \alpha_y.$$

Finally, we have

$$\ln \hat{c}_1 = \frac{\sum_{y=\eta_1}^{\eta_2} y \sum_{y=\eta_1}^{\eta_2} \ln \alpha_y - (\eta_2 - \vartheta_2) \sum_{y=\eta_1}^{\eta_2} y \ln \alpha_y}{(\sum_{y=\eta_1}^{\eta_2} y)^2 - (\eta_2 - \vartheta_2) \sum_{y=\eta_1}^{\eta_2} y^2}. \tag{3.19}$$

Differentiating the sum in (3.15) with respect to A_1, applying estimates $\hat{\beta}_1, \hat{c}_1$, and setting it to zero gives

$$\sum_{y=\vartheta_1}^{\vartheta_2} \Big(\alpha_y - \hat{A}_1 - \hat{\beta}_1 \hat{c}_1^y \Big) = 0,$$

by rearranging, we obtain

$$\sum_{y=\vartheta_1}^{\vartheta_2} \alpha_y - (\eta_1 - \vartheta_1)\hat{A}_1 - \sum_{y=\vartheta_1}^{\vartheta_2} \hat{\beta}_1 \hat{c}_1^y = 0,$$

and then the estimate $\hat{A}_1$ takes the form

$$\hat{A}_1 = \frac{\sum_{y=\vartheta_1}^{\vartheta_2} (\alpha_y - \hat{\beta}_1 \hat{c}_1^y)}{(\eta_1 - \vartheta_1)}. \tag{3.20}$$

An analogous derivation can be proceeded for survival probability p_x related to a man aged x.

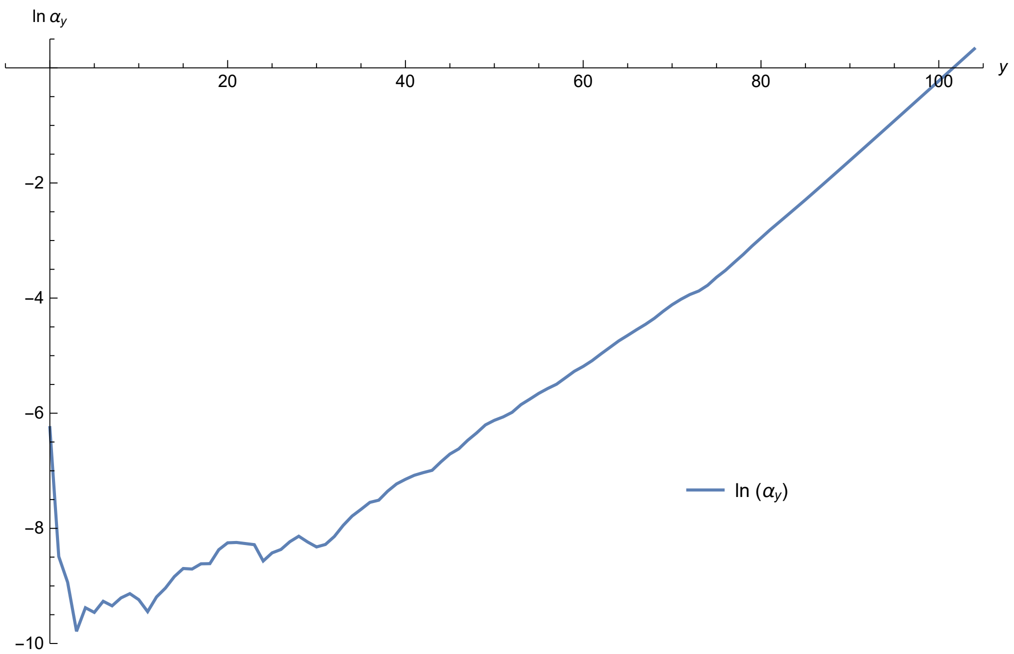

Figure 3.1: A graph of $\ln \alpha_y$ for the age range from 0 to 104

Clearly, the above stated method is sensitive to the choices of age ranges $(\vartheta_1, \vartheta_2)$ and (η_1, η_2). There is no rule how to determine them, therefore we do an analysis based on our data. And we start the analysis related to the female population 2015.

In order to determine η_1, we have plotted $\ln \hat{p_y}$ (see Figure (3.1)). By omitting term A_1 and applying logarithm in (3.11) for the range (η_1, η_2), we obtained the linear approximation (3.13). Thus, if the values of $\ln \alpha_y$ in our female population remind us a line from some age, we can take that age as η_1. When we look at Figure (3.1), we can see two "lines", i.e. the graph reminds us the line already from age 45 that ends at age 72, since there is a small blip and this "line" seems to have a different slope than the "line" starting from age 72. So, based on this observation, we can determine η_1 bigger or equal than 45 years.

Now, we would like to determine ages ϑ_1 and η_2, and also find good estimates of A_1, β_1(also B_1) and c_1. From the literature, we know (see for example Fiala (2005)) that in general the values of a Gompertz-Makeham fit may significantly differ from empirical values under age 60 and above age 75, when estimating forces

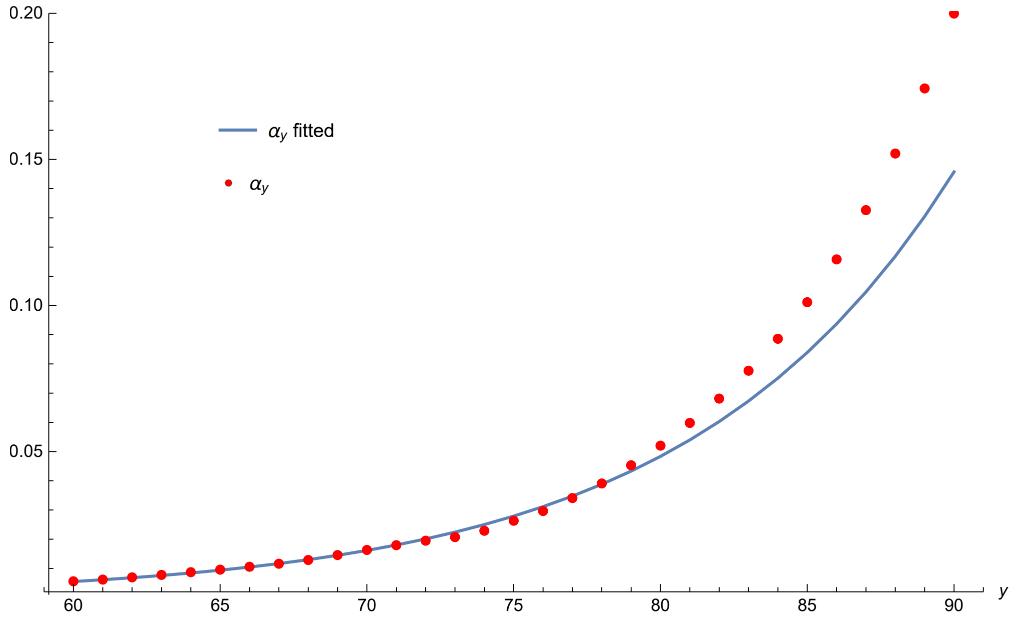

Figure 3.2: The empirical values of α_y and its fitted values for choice τ_y^2

of mortality, but it is not a rule. It also indicates that a force of mortality cannot be expressed by one simple function for all ages say from 0 to 104. Furthermore, when estimating forces of mortality, we know that age 85 is considered as reliable from the view of empirical data, because for older ages there is less observations (see for example Dotlačilová and Langhamrová (2015)), and therefore the upper age η_2 should be taken around 85.

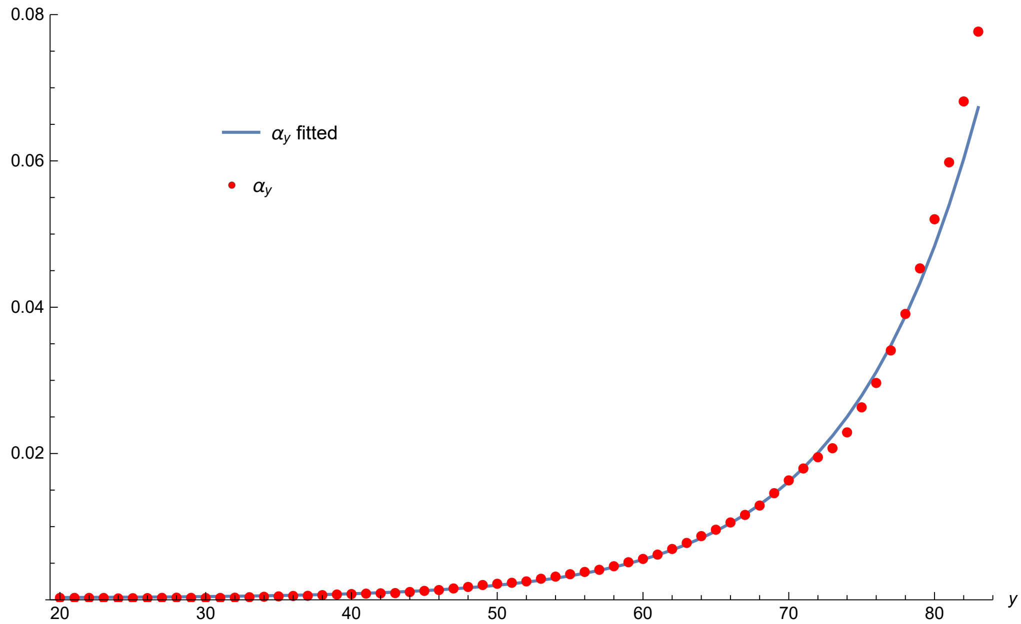

Figure 3.3: The empirical values of α_y and its fitted values for choice τ_y^1

Based on the above made analysis and the above stated knowledge from literature, we did several reasonable choices of $(\vartheta_1, \vartheta_2)$ and (η_1, η_2), we made

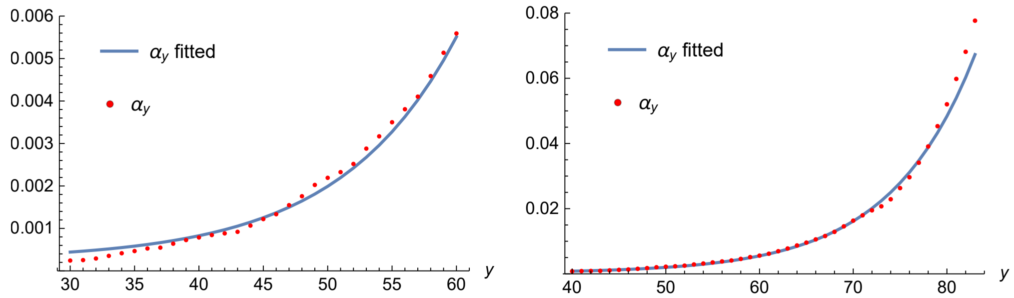

Figure 3.4: The empirical values of α_y and its fitted values for choice τ_y^1 using different scaling

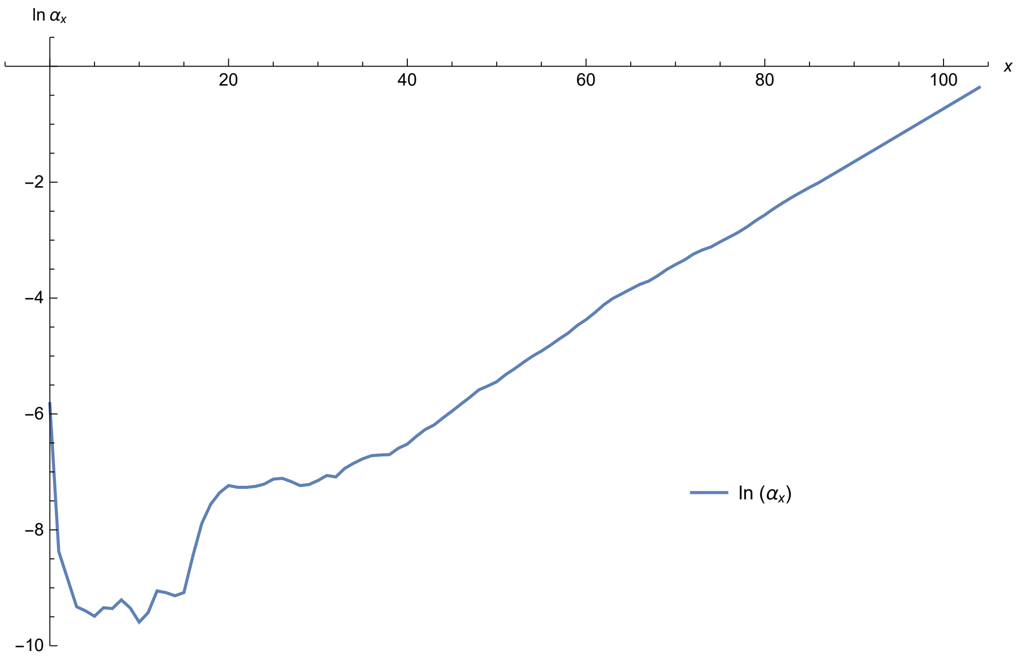

Figure 3.5: A graph of $\ln \alpha_x$ for the age range from 0 to 104

a fit for the choices and we compared empirical observations with fitted values. We decided to mention two choices that we will discuss in detail. Let us denote the i-th choice of age ranges by $\tau_y^i, i = 1, 2$, whereas the subscript y indicates that choices are made for population related to women. The age ranges for the two choices are in table (3.1).

age parameters\choices	τ_y^1	τ_y^2
ϑ_1	20	60
ϑ_2	65	65
η_1	66	66
η_2	80	80

Table 3.1: The choices of age ranges related to the female population 2015

Figure (3.2) shows the empirical values of α_y and its fitted values for choice

τ_y^2. The choices $\vartheta_1 = 60$ and $\eta_2 = 80$ were based on the recommendations from the above mentioned literature, and the choice $\eta_1 = 66$ was made on the basis of the comparison of graphs of empirical and fitted values for different choices of η_1. From Figure (3.2), we can clearly see that empirical values match with those fitted very well from 60 to 72, then they are very little overestimated for ages 73 to 75, and they are greatly underestimated from age 82 and higher. Moreover, we can compare Figure (3.1) and Figure (3.2), and notice that they correspond to each other. Since, in Figure (3.1), the "line" from age 60 to 72 seems to have a different slope than the "line" from age 72 and higher, in Figure (3.2), the fit is very good only until age 72.

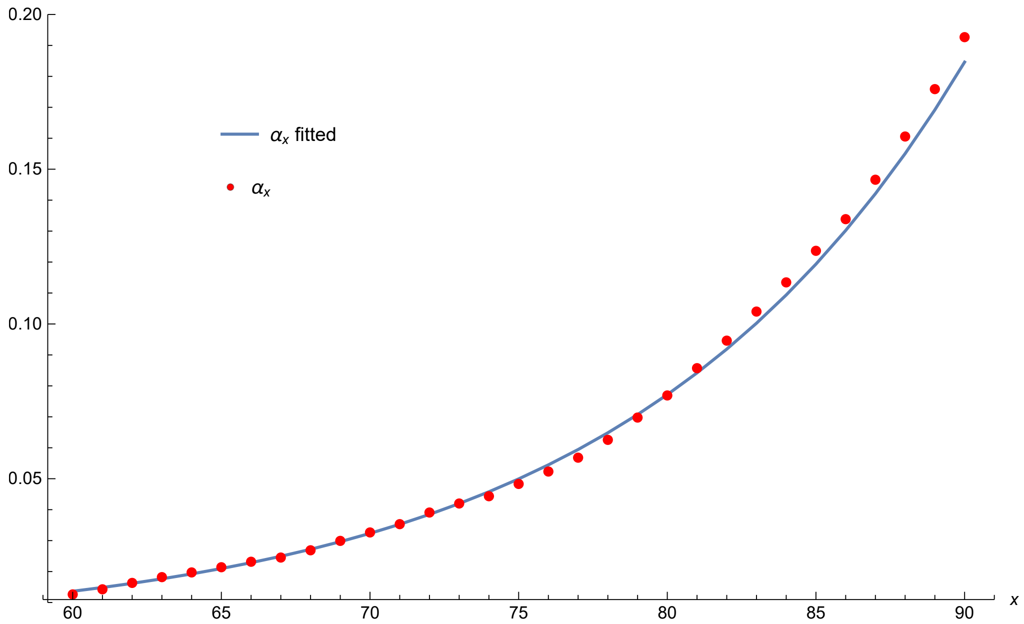

Figure 3.6: The empirical values of α_x and its fitted values for choice τ_y^2

We realise that for our purposes we would like to estimate A_1, β_1(also B_1) and c_1 for a wide range (ϑ_1, η_2), since later we need to plot joint-life and last-survivor annuities considering the range (ϑ_1, η_2), and therefore the age range $(60, 72)$ is quite short to be used later for plotting annuities. Moreover, the method that will be used for estimation of α's is very sensitive on the choice of (ϑ_1, η_2). We could have also considered the age range $(60, 80)$, since from the graph in Figure (3.2) we can still see good results, but we would like to make the age range (ϑ_1, η_2) as big as possible, even with the acception of some small misalignments, because we want to achieve the aim of this thesis, e.g. to examine the impact of dependence between the remaining lifetimes of the husband and wife on the amount of premiums considering the joint-life and last-survivor model by plotting and comparing annuities. It would be sufficient for us to choose ϑ around 30 and η_2 around 80.

From Figure (3.1), we see the "line" from age 45, therefore we can try to set ϑ_1 to lower ages around $20 - 30$ and have a look at the graph of empirical and fitted values. We did several choices of η_1 (from 45 to 72) with ϑ_1 at lower ages and η_2 around age 80, and we decided to choose the choice τ_y^1, since empirical values of α_y seems to match with fitted values best (see Figure (3.3)). From

Figure (3.3), it seems that there is a very good match of empirical values with those fitted already from age 20 to 72. Again, we can compare Figure (3.3) and Figure (3.1) and we realise that the fit in Figure (3.3) is really good up to age 72, since there is a blip in Figure (3.1) at age 72. We realise that an important role is played by scaling and therefore we decided to plot empirical and fitted values for the choice τ_y^1 using different scaling. Looking at the graphs in Figure (3.4), we can see that the match of empirical and fitted values is not so good as it seems to be from Figure (3.2). We can clearly see that empirical values of α_y

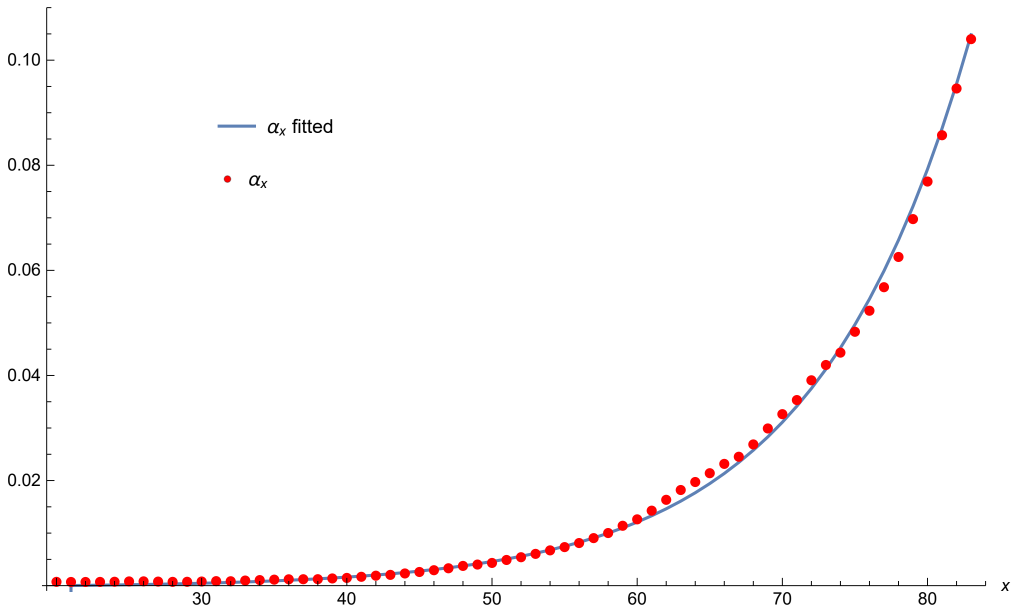

Figure 3.7: The empirical values of α_x and its fitted values for choice τ_x^1

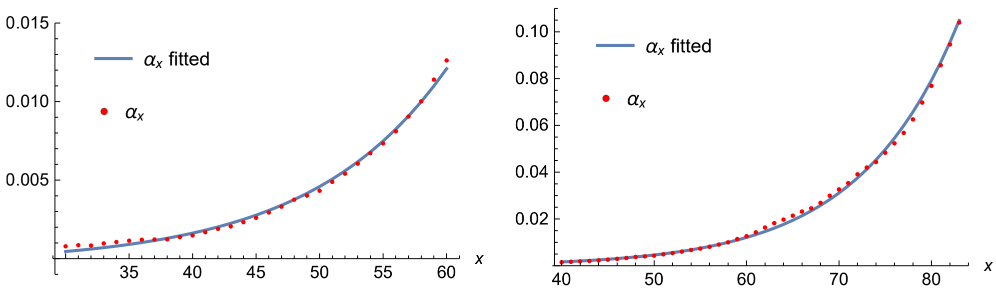

Figure 3.8: The empirical values of α_x and its fitted values for choice τ_x^1 using different scaling

are quite overestimated for ages lower than 35 and from ages 35 to 80 we can see some small over- or underestimations that we consider as acceptable (definitely not bad), since we require a wide age range. So, we decided to choose the age range $(35, 80)$ and using the choice τ_y^1, we obtained the estimates $\hat{A}_1, \hat{\beta}_1$ and $\hat{c}_1$, then the estimate $\hat{B}_i$ can be calculated from (3.5), whereas $\hat{g}_1$ can be obtained from (3.12) as

$$\hat{g}_1 = \exp\left\{\frac{-\hat{\beta}_1}{(\hat{c}_1 - 1)}\right\}.$$

All the estimates for the choice τ_y^1 are shown in Table (3.3).

Let us make a similar analysis related to the male population 2015. We want to determine age ranges $(\vartheta_1, \vartheta_2)$ and (η_1, η_2) and acceptable estimates of A_2, β_2 (also B_2) and c_2. We proceed analogously as we did for the female population. Firstly, we look at the graph in Figure (3.5). We see that the graph of $\ln \alpha_x$ reminds us the line from age 60, and this "line" seems to have a different slope than the "line" in the age range $(37, 60)$. So based on this graph, we can determine η_1 bigger than or equal to 37. Again, we did several reasonable choices of $(\vartheta_1, \vartheta_2)$ and (η_1, η_2), we made a fit for those choices and compared empirical values with fitted values. We will discuss in detail two specific choices τ_x^1, τ_x^2 whose age ranges are in Table (3.2). Figure (3.6) shows the empirical values of α_x and its fitted values for

age parameters\choices	τ_x^1	τ_x^2
ϑ_1	30	60
ϑ_2	51	65
η_1	52	66
η_2	83	83

Table 3.2: The choices of age ranges related to the male population 2015

choice τ_x^2. The choice τ_x^2 was determined in the same way as in case of the female population. From Figure (3.2), we can see that the empirical values match with those fitted quite well from 60 to 73, then they are very little overestimated for ages 74 to 78, and they are underestimated from age 82 and higher. Moreover, we can compare Figure (3.5) and Figure (3.6), and notice that they correspond to each other. The over- and underestimation in Figure (3.6) corresponds with very small blips that can be seen in Figure (3.5) over the range $(60, 80)$.

Since we would like to obtain a wide age range (ϑ_1, η_2), we have considered the choices with ϑ_1 at lower ages around $20 - 30$ similarly as in the case of female population. Finally, we decided to choose the choice τ_x^1 by comparing the empirical and fitted values with other possible choices. The empirical values of α_x and its fitted values for choice τ_x^1 are shown in Figure (3.7). From Figure (3.7), it seems that we have quite a good match of the empirical values with those fitted already from age 30 to 60. There is an underestimation in range $(61, 67)$ and an overestimation in range $(76, 80)$. Further, we look at the empirical and fitted values for the choice τ_x^1 using different scaling (see Figure (3.8)). We can see that there is an underestimation for ages lower than or equal to 36 and then around ages 45 and 50, there can be seen a small overestimation. A little disturbing seems to be an underestimation from 61 to 67. An overestimation in range $(76, 80)$ seems to be very small. Looking again at the graph in Figure (3.5), we realise that all over- and underestimations in Figure (3.7) correspond with blips in Figure (3.5). It is impossible to obtain an excellent fit of $\ln \alpha_x$ in the age range starting around 30 and ending about 85 by a simple function. We must say that the estimation results are a little worse for the male population than for the female population

2015, but are not bad and we consider them acceptable for our further work in this thesis. So, we decided to choose the age range $(37, 83)$ and using the choice τ_y^1, we obtained the estimates that are in Table (3.3).

parameter	women (i = 1)	men (i = 2)
$\hat{A}_i$	0.000252598	-0.000307324
$\hat{B}_i$	0.000006866	0.000046943
$\hat{c}_i$	1.117035884	1.097397160
$\hat{s}_i$	0.999747434	1.000307371
$\hat{g}_i$	0.999937965	0.999495040
$\hat{\beta}_i$	0.000006866	0.000046943

Table 3.3: Least squares estimates of all considered parameters related to the female (resp. male) population in 2015 with the choice τ_y^1 (resp. τ_x^1)

In order to consider the same age ranges for the female and male population, we make an intersection of final female and male age ranges and we obtain the range $(37, 80)$. We would like to emphasize that our aim in this section was not to obtain the best estimates of considered parameters, but to obtain the estimates that are good enough for further calculations and can be used for a wide age range.

To sum up, we have estimated parameters A_1, B_1 and c_1 in (3.3) (resp. A_2, B_2 and c_2 in (3.4)) using data related to the female (resp. male) population in the Czech Republic in 2015 and in this thesis we will assume for further calculation that the forces of mortality in forms

$$\mu_y = 0.000253 + 6.87 \cdot 10^{-6} \cdot 1.117036^y, \tag{3.21}$$

$$\mu_x = -0.000307 + 46.94 \cdot 10^{-6} \cdot 1.097397^x \tag{3.22}$$

hold for the age range $(37, 80)$. The calculations were done in Excel.

3.2 Estimation of Parameters

Now we can proceed to the estimation of four parameters $\alpha_{01}, \alpha_{13}, \alpha_{02}$ and α_{23} involved in (2.29) - (2.32). For estimation purposes, we have at hand a dataset related to the Czech population experienced during the year 2015, more precisely we have at hand the total population sorted according to sex, age and marital status on 31 December 2014 and also on 31 December 2015, as well as the number of divorces according to age and sex, marriages and deaths according to age, sex and marital status during the year 2015. The method of least squares, which will be used, can be found in Denuit et al. (2001, page 27). We start by a definition of transaction functions. We define the transition functions[2] as

$$\Omega_{s+t}^{ij} = \int_0^t \mu_{s+\tau}^{ij} d\tau, \quad t \geq 0, \tag{3.23}$$

where s represents the age of an insured person and t is the time elapsed since the person's age s. In our notation, for example, if $i = 0$ and $j = 1$, we have that

[2]The transition functions are known as cumulative hazard functions.

$s+t \equiv x+t : y+t$. Further, let $l \geq 0$, then we write the theoretical increments $\Delta\Omega^{ij}_{s+l}$ of Ω^{ij}_{s+l} as

$$\begin{aligned}
\Delta\Omega^{ij}_{s+l} &= \Omega^{ij}_{s+l+1} - \Omega^{ij}_{s+l} = \qquad (3.24)\\
&= \int_0^{l+1} \mu^{ij}_{s+\tau} d\tau - \int_0^{l} \mu^{ij}_{s+\tau} d\tau = \\
&= \int_l^{l+1} \mu^{ij}_{s+\tau} d\tau = \\
&= \int_0^{1} \mu^{ij}_{s+l+\tau} d\tau. \qquad (3.25)
\end{aligned}$$

We want to obtain estimates $\hat{\alpha}_{01}, \hat{\alpha}_{13}, \hat{\alpha}_{02}$ and $\hat{\alpha}_{23}$ by minimizing the sum of squared differences between the increments $\Delta\Omega^{ij}$ of the transition functions Ω^{ij} and their estimates $\Delta\hat{\Omega}^{ij}$. We make the substitution $k = s + l$, which gives

$$\Delta\Omega^{ij}_{k} = \int_0^{1} \mu^{ij}_{k+\tau} d\tau. \qquad (3.26)$$

To explain this substitution, let us consider a couple with ages $x = 60$ and $y = 60$. If we wanted to estimate say α_{01} using the increments $\Delta\Omega^{01}_{60+l:60+l}$, then estimates of the increments $\Delta\hat{\Omega}^{01}_{60+l:60+l}$ must have been based only on the data related to the groups consisting only of the couples being in state 0 at ages 60 and 60. To be concrete, for example, the number of 62-years old widowers must have contained only the widowers that had been married at age 60 and became widowers at age 62. This is not our case, since in our dataset the number of 62-years-old widowers may contain also the widowers who became widowers at the age of 60 or before. Moreover, using the increments $\Delta\Omega^{01}_{60+l:60+l}$ for estimation of α_{01} would mean that α_{01} would be dependent on the spouses's ages 60 and 60. But we are not interested in α_{01} just for specific ages. We want α_{01} to be usable for the age range 37 to 80 that we have determined in Section 3.1. Moreover, the integral from 0 to 1 from say $\mu^{01}_{60+\tau}$ with respect to τ gives the same value as the integral from 0 to 1 from $\mu^{01}_{40+20+\tau}$ with respect to τ.

Thus, because of the reasons mentioned above, it makes sense to make the substitution (3.26).

And now we determine estimates $\hat{\alpha}_{01}, \hat{\alpha}_{13}, \hat{\alpha}_{02}$ and $\hat{\alpha}_{23}$ as the solution of

$$\hat{\alpha}_{ij} = \underset{\alpha_{ij}}{\operatorname{argmin}} \sum_k \left(\Delta\hat{\Omega}^{ij}_{k} - \int_0^1 \mu^{ij}_{k+t} dt \right)^2. \qquad (3.27)$$

Before we deal with the estimates of the increments of the transition functions, we minimize the sum of the squared differences between the increments $\Delta\Omega^{ij}$ and their estimates $\Delta\hat{\Omega}^{ij}$ and state its explicit solution.

Let us calculate the explicit solution say of the estimate $\hat{\alpha}_{01}$. We start by adjusting the sum in (3.27), i.e.

$$\sum_k \left(\Delta\hat{\Omega}^{01}_{k} - \int_0^1 \mu^{01}_{k+t} dt \right)^2. \qquad (3.28)$$

Applying (2.29), (3.28) can be rewritten as

$$\sum_k \left(\Delta\hat{\Omega}^{01}_{k} - \int_0^1 (1-\alpha_{01}) \mu_{k+t} dt \right)^2,$$

and using (3.3), we have

$$\sum_k \left(\Delta\hat{\Omega}_k^{01} - (1-\alpha_{01}) \int_0^1 (A_1 + B_1 c_1^{k+t}) dt \right)^2.$$

After calculating the integral in the above stated expression, we obtain

$$\sum_k \left(\Delta\hat{\Omega}_k^{01} - (1-\alpha_{01}) \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right) \right)^2. \tag{3.29}$$

Differentiating the sum (3.29) with respect to α_{01} and setting it to zero gives

$$\sum_k \left(\Delta\hat{\Omega}_k^{01} - (1-\hat{\alpha}_{01}) \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right) \right) \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right) = 0,$$

by rearranging, we obtain

$$(1-\hat{\alpha}_{01}) \sum_k \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right)^2 = \sum_k \Delta\hat{\Omega}_k^{01} \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right).$$

And then the estimate $\hat{\alpha}_{01}$ takes the form

$$\hat{\alpha}_{01} = 1 - \frac{\sum_k \Delta\hat{\Omega}_k^{01} \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right)}{\sum_k \left(A_1 + B_1 c_1^k \frac{c_1 - 1}{\ln c_1} \right)^2}. \tag{3.30}$$

Since we have already calculated estimates $\hat{A}_1, \hat{B}_1$ and $\hat{c}_1$ in Section 3.1, we can just plug them into (3.30). Clearly, analogous calculation can be done to compute $\hat{\alpha}_{02}, \hat{\alpha}_{13}$ and $\hat{\alpha}_{23}$. Thus, in the joint-life and last-survivor model assuming (2.29) - (2.32), we derived the explicit forms of estimators $\hat{\alpha}_{ij}$ that are given by

$$\hat{\alpha}_{01} = 1 - \frac{\sum_k \Delta\hat{\Omega}_k^{01} \left(\hat{A}_1 + \hat{\beta}_1 \hat{c}_1^k \frac{\hat{c}_1 - 1}{\ln \hat{c}_1} \right)}{\sum_k \left(\hat{A}_1 + \hat{\beta}_1 \hat{c}_1^k \frac{\hat{c}_1 - 1}{\ln \hat{c}_1} \right)^2}, \tag{3.31}$$

$$\hat{\alpha}_{02} = 1 - \frac{\sum_k \Delta\hat{\Omega}_k^{02} \left(\hat{A}_2 + \hat{\beta}_2 \hat{c}_2^k \frac{\hat{c}_2 - 1}{\ln \hat{c}_2} \right)}{\sum_k \left(\hat{A}_2 + \hat{\beta}_2 \hat{c}_2^k \frac{\hat{c}_2 - 1}{\ln \hat{c}_2} \right)^2}, \tag{3.32}$$

$$\hat{\alpha}_{13} = \frac{\sum_k \Delta\hat{\Omega}_k^{13} \left(\hat{A}_2 + \hat{\beta}_2 \hat{c}_2^k \frac{\hat{c}_2 - 1}{\ln \hat{c}_2} \right)}{\sum_k \left(\hat{A}_2 + \hat{\beta}_2 \hat{c}_2^k \frac{\hat{c}_2 - 1}{\ln \hat{c}_2} \right)^2} - 1, \tag{3.33}$$

$$\hat{\alpha}_{23} = \frac{\sum_k \Delta\hat{\Omega}_k^{23} \left(\hat{A}_1 + \hat{\beta}_1 \hat{c}_1^k \frac{\hat{c}_1 - 1}{\ln \hat{c}_1} \right)}{\sum_k \left(\hat{A}_1 + \hat{\beta}_1 \hat{c}_1^k \frac{\hat{c}_1 - 1}{\ln \hat{c}_1} \right)^2} - 1. \tag{3.34}$$

Looking at expressions (3.31) - (3.34), we realise that it remains to determine the estimates of the increments of the transition functions, i.e. $\hat{\Omega}_k^{01}, \hat{\Omega}_k^{02}, \hat{\Omega}_k^{13}$ and $\hat{\Omega}_k^{23}$.

Let us denote the number of couples in state i at age t by L_t^i and the number of transitions from state i to state j over $[0, t]$ by L_t^{ij}. In order to do so, we will consider the Nelson-Aalen estimator. Notice that from (3.23) it holds

$$\begin{aligned} \Omega_{s+\tau+h}^{ij} - \Omega_{s+\tau}^{ij} &\approx h\mu_{s+\tau}^{ij} \\ &= P[X_r = j, r \in (s+\tau, s+\tau+h] \mid X_{s+\tau} = i]. \end{aligned}$$

It is natural to estimate $\Omega^{ij}_{s+\tau+h} - \Omega^{ij}_{s+\tau}$ by

$$\frac{L^{ij}_{s+\tau+h} - L^{ij}_{s+\tau}}{L^{i}_{s+\tau}}.$$

Summing these quantities over subintervals of $(0, t]$ and letting the subintervals get small enough so that they contain at most one event time, gives the Nelson-Aalen estimator

$$\hat{\Omega}^{ij}_{s+t} = \int_0^t \frac{\mathbb{1}(L^i_{s+\tau} > 0)}{L^i_{s+\tau}} dL^{ij}_{s+\tau}, \tag{3.35}$$

whereas $0 \leq \tau \leq t$, s is a fixed age and we consider the convention that the integral is given to be zero when $L^i_{s+\tau} = 0$. More details about Nelson-Aalen estimator can be found for example in Therneau and Grambsch (2000, page 7). Further, for $l \geq 0$, we can write the empirical increments $\Delta\hat{\Omega}^{ij}_{s+l}$ of $\hat{\Omega}^{ij}_{s+l}$ as

$$\begin{aligned}
\Delta\hat{\Omega}^{ij}_{s+l} &= \hat{\Omega}^{ij}_{s+l+1} - \hat{\Omega}^{ij}_{s+l} = &\quad(3.36)\\
&= \int_0^{l+1} \frac{\mathbb{1}(L^i_{s+\tau} > 0)}{L^i_{s+\tau}} dL^{ij}_{s+\tau} - \int_0^{l} \frac{\mathbb{1}(L^i_{s+\tau} > 0)}{L^i_{s+\tau}} dL^{ij}_{s+\tau} = \\
&= \int_l^{l+1} \frac{\mathbb{1}(L^i_{s+\tau} > 0)}{L^i_{s+\tau}} dL^{ij}_{s+\tau} = \\
&= \int_0^{t} \frac{\mathbb{1}(L^i_{s+l+\tau} > 0)}{L^i_{s+l+\tau}} dL^{ij}_{s+l+\tau}. &\quad(3.37)
\end{aligned}$$

We want the empirical estimated increments to correspond with the theoretical increments (3.24), therefore we make the substitution $k = s + l$, which gives

$$\Delta\hat{\Omega}^{ij}_{k+t} = \int_0^t \frac{\mathbb{1}(L^i_{k+\tau} > 0)}{L^i_{k+\tau}} dL^{ij}_{k+\tau}. \tag{3.38}$$

The above mentioned substitution is made also because of the structure of the dataset that we have at hand, and because we want to estimate α_{ij} for a range of ages 37 to 80 as we have already explained in the case of the substitution made in theoretical increments $\Delta\Omega^{ij}_{s+l}$. Now, we can proceed to the calculation of the integral (3.38) and its determination from the dataset.

In our dataset, the number of transitions is available only for a year, therefore we will use the assumption of linearity, i.e. we assume that for any integer k and $0 \leq t \leq 1$ it holds

$$\begin{aligned}
L^{ij}_{k+t} &= L^{ij}_k + t(L^{ij}_{k+1} - L^{ij}_k) \quad \text{and}\\
L^{i}_{k+t} &= L^{i}_k + t(L^{i}_{k+1} - L^{i}_k).
\end{aligned}$$

Then, we can write

$$\begin{aligned}
\Delta\hat{\Omega}^{ij}_{k+t} &= \int_0^1 \frac{\mathbb{1}(L^i_{k+\tau} > 0)}{L^i_k + \tau(L^i_{k+1} - L^i_k)} (L^{ij}_{k+1} - L^{ij}_k) d\tau = \\
&= (L^{ij}_{k+1} - L^{ij}_k) \frac{1}{L^i_{k+1} - L^i_k} \left[\ln(L^i_k + \tau(L^i_{k+1} - L^i_k))\right]_0^1 \\
&= \frac{L^{ij}_{k+1} - L^{ij}_k}{L^i_{k+1} - L^i_k} (\ln L^i_{k+1} - \ln L^i_k) = \\
&= \frac{L^{i:j}_k}{L^i_{k+1} - L^i_k} (\ln L^i_{k+1} - \ln L^i_k), &\quad(3.39)
\end{aligned}$$

where

$$L_k^{i:j} = L_{k+1}^{ij} - L_k^{ij}$$

represents the number of transitions from state i to state j observed for lives aged k. Further, we describe the precise estimation say of $\Delta\Omega_k^{01}$ and $\Delta\Omega_k^{13}$ considering the dataset related to the Czech population experienced during the year 2015. Let us start with $\Delta\Omega_k^{01}$ and analyse its components:

1. The numerator $L_k^{0:1}$ represents the number (#, in short) of married women aged k died during the year 2015 (this number is directly available from our dataset).

2. The denominator $L_{k+1}^0 - L_k^0$ is equal to:
- # of married women aged k died during 2015
    - # of married women aged k whose husband died during 2015
    + # of women aged k got married during 2015
    - # of married women aged k got divorced during 2015.

Since the number of married women aged k whose husband died during 2015 cannot be obtained directly from our dataset, we estimate it in the following way:
  + # of widows aged $k+1$ on 31 December 2015
    - (# of widows aged k on 31 December 2014
    - # of widows aged k died during 2015
    - # of widows aged k got married during 2015).

3. Lastly, considering the difference between the logarithms in (3.39), L_k^0 represents the number of married women aged k on 31 December 2014 and L_{k+1}^0 is easily deduced from above since $(L_{k+1}^0 - L_k^0) + L_k^0 = L_{k+1}^0$.

Let us continue with the examination of $\Delta\Omega_k^{13}$:

1. The numerator $L_k^{1:3}$ represents the number of widowers aged k died during 2015.

2. The denominator $L_{k+1}^1 - L_k^1$ is equal to:
- # of widowers aged k died during 2015
+ # of men aged k whose wife died during 2015
- # of widowers k got married during 2015.

Since the number of men aged k whose wife died during 2015 cannot be obtained directly from our dataset, we estimate it in the following way:
# of widowers aged $k+1$ on 31 December 2015
    - (# of widowers aged k on 31 December 2014
    - # of widowers aged k died during 2015
    - # of widowers aged k got married during 2015).

3. Lastly, considering the difference between the logarithms in (3.39), L_k^1 represents the number of widowers aged k on 31 December 2014 and L_{k+1}^1 is easily deduced from above since $(L_{k+1}^1 - L_k^1) + L_k^1 = L_{k+1}^1$.

Apparently, the estimation of $\Delta\Omega_k^{02}$ and $\Delta\Omega_k^{23}$ can be deduced from the above description simply by exchanging the roles of the two spouses. According to expressions (3.31) - (3.34), using the dataset provided by the Czech Statistical Office concerning the Czech population experienced during 2015 and considering age spread from 37 to 80, we calculated

$$\hat{\alpha}_{01} = 0.209245955, \quad \hat{\alpha}_{02} = 0.158489993 \tag{3.40}$$

$$\hat{\alpha}_{13} = 0.240952327, \quad \hat{\alpha}_{23} = 0.042490475. \tag{3.41}$$

The calculation was done in Excel and these numbers tell us that in comparison to the mortality experienced by the entire Czech population during 2015, there is an under-mortality about 21% for married women, 16% for married men, and an over-mortality about 24% for widowers and 4% for widows.

3.3 Calculation of Premium

In this section, we will deal with a calculation of the net single premium for n-year joint-life annuity due and the net single premium of n-year last-survivor annuity due in case of dependence and also independence. We start with the calculation of the premium in case of dependence.

We rewrite the probabilities in (3.1) and (3.2) in the notation used for the joint-life and last-survivor model. We realise that the survival probability ${}_kp_{xy}$ is equal to the probability ${}_kp_{xy}^{00}$, and so we can write

$$\ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_{xy}^{00} \tag{3.42}$$

Furher, say the probability ${}_kp_x$ represents the probability that the husband aged x will be alive at his age $x+k$. We realise that the husband is alive at his age $x+k$ in the joint-life and last-survivor model when the considered couple is in state 0 or in state 1, therefore the probability ${}_kp_x$ is equal to ${}_kp_{xy}^{00} + {}_kp_{xy}^{01}$. Analogously, we have that the ${}_kp_y$ is equal to the probability ${}_kp_{xy}^{00} + {}_kp_{xy}^{02}$, therefore we can write

$$\begin{aligned}\ddot{a}_{\overline{xy}:\overline{n}|} &= \sum_{k=0}^{n-1} v^k({}_kp_x + {}_kp_y - {}_kp_{xy}) = \\ &= \sum_{k=0}^{n-1} v^k({}_kp_{xy}^{00} + {}_kp_{xy}^{01} + {}_kp_{xy}^{00} + {}_kp_{xy}^{02} - {}_kp_{xy}^{00}) = \\ &= \sum_{k=0}^{n-1} v^k({}_kp_{xy}^{01} + {}_kp_{xy}^{02} + {}_kp_{xy}^{00}). \end{aligned} \tag{3.43}$$

So, in order to price joint-life and last-survivor annuities given by (3.42) and (3.43), we just need to determine the probabilities ${}_kp_{xy}^{00}$, ${}_kp_{xy}^{01}$ and ${}_kp_{xy}^{02}$.

We start with the calculation of the probability ${}_kp^{00}_{xy}$. We can rewrite (2.6) using (2.29) and (2.31) in the following way:

$$
\begin{aligned}
{}_kp^{00}_{xy} &= \exp\left\{ -\int_0^k \left(\mu^{01}_{x+\tau:y+\tau} + \mu^{02}_{x+\tau:y+\tau}\right)d\tau \right\} = \\
&= \exp\left\{ -(1-\alpha_{01})\int_0^k \mu_{y+\tau}d\tau - (1-\alpha_{02})\int_0^k \mu_{x+\tau}d\tau \right\} =
\end{aligned}
$$

and using (3.3) and (3.4), we have

$$
\begin{aligned}
&= \exp\left\{ -(1-\alpha_{01})\int_0^k (A_1 + B_1c_1^{y+\tau})d\tau - (1-\alpha_{02})\int_0^k (A_2 + B_2c_2^{x+\tau})d\tau \right\} = \\
&= \exp\left\{ -(1-\alpha_{01})\left[A_1\tau + B_1\frac{c_1^{y+\tau}}{\ln c_1}\right]_0^k - (1-\alpha_{02})\left[A_2\tau + B_2\frac{c_2^{x+\tau}}{\ln c_2}\right]_0^k \right\} = \\
&= \exp\left\{ -(1-\alpha_{01})\left[A_1k + \frac{B_1}{\ln c_1}c_1^y(c_1^k - 1)\right] \right\} \\
&\qquad \exp\left\{ -(1-\alpha_{02})\left[A_2k + \frac{B_2}{\ln c_2}c_2^x(c_2^k - 1)\right] \right\}.
\end{aligned}
\tag{3.44}
$$

The calculation of ${}_kp^{01}_{xy}$ is quite tedious, therefore we will not write every step in detail. Firstly, we need to calculate

$$
\begin{aligned}
{}_{k-t}p^{11}_{x+t} &= \exp\left\{ -\int_t^k \mu^{13}_{x+s}ds \right\} = \qquad \text{using} \quad (2.32) \\
&= \exp\left\{ -(1+\alpha_{13})\int_t^k \mu_{x+s}ds \right\} = \qquad \text{using} \quad (3.4) \\
&= \exp\left\{ -(1+\alpha_{13})\int_t^k (A_2 + B_2c_2^{x+s})ds \right\} = \\
&= \exp\left\{ -(1+\alpha_{13})\left[A_2(k-t) + \frac{B_2c_2^x}{\ln c_2}(c_2^k - c_2^t)\right] \right\}.
\end{aligned}
\tag{3.45}
$$

And now, we rewrite (2.9) using (3.44),(2.29), (3.45) and we obtain

$$
\begin{aligned}
{}_kp^{01}_{xy} = \int_0^k &\exp\left\{ -(1-\alpha_{01})\left[A_1t + \frac{B_1}{\ln c_1}c_1^y(c_1^t - 1)\right] \right\} \\
&\exp\left\{ -(1-\alpha_{02})\left[A_2t + \frac{B_2}{\ln c_2}c_2^x(c_2^t - 1)\right] \right\} \\
&\exp\left\{ -(1+\alpha_{13})\left[A_2(k-t) + \frac{B_2c_2^x}{\ln c_2}(c_2^k - c_2^t)\right] \right\}(1-\alpha_{01})\left[A_1 + B_1c_1^{y+t}\right]dt =
\end{aligned}
$$

Let us denote the exponent in the above written formula by $f(x,y,k,t)$, i.e.

$$
\begin{aligned}
f(x,y,k,t) \coloneqq &-(1-\alpha_{01})\left[A_1t + \frac{B_1}{\ln c_1}c_1^y(c_1^t - 1)\right] \\
&-(1-\alpha_{02})\left[A_2t + \frac{B_2}{\ln c_2}c_2^x(c_2^t - 1)\right] \\
&-(1+\alpha_{13})\left[A_2(k-t) + \frac{B_2c_2^x}{\ln c_2}(c_2^k - c_2^t)\right].
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
{}_kp^{01}_{xy} &= (1-\alpha_{01})A_1\int_0^k \exp\{f(x,y,k,t)\}dt \\
&\quad + (1-\alpha_{01})B_1c_1^y\int_0^k c_1^t\exp\{f(x,y,k,t)\}dt. \qquad (3.46)
\end{aligned}
$$

Let us denote

$$
(*) := (1-\alpha_{01})A_1\int_0^k \exp\{f(x,y,k,t)\}dt \qquad \text{and} \qquad (3.47)
$$

$$
(**) := (1-\alpha_{01})B_1c_1^y\int_0^k c_1^t\exp\{f(x,y,k,t)\}dt. \qquad (3.48)
$$

Firstly, we simplify $\exp\{f(x,y,k,t)\}$ and then we compute $(*)$ and $(**)$. So, for $\exp\{f(x,y,k,t)\}$, we can write

$$
\begin{aligned}
\exp\{f(x,y,k,t)\} &= \exp\left\{-A_2k + A_2t - \frac{B_2c_2^x}{\ln c_2}c_2^k + \frac{B_2c_2^x}{\ln c_2}c_2^t\right\} \\
&\exp\left\{-\alpha_{13}A_2k + \alpha_{13}A_2t - \alpha_{13}\frac{B_2c_2^x}{\ln c_2}c_2^k + \alpha_{13}\frac{B_2c_2^x}{\ln c_2}c_2^t\right\} \\
&\exp\left\{-A_1t - \frac{B_1c_1^y}{\ln c_1}c_1^t + \frac{B_1c_1^y}{\ln c_1} + \alpha_{01}A_1t + \alpha_{01}\frac{B_1c_1^y}{\ln c_1}c_1^t - \alpha_{01}\frac{B_1c_1^y}{\ln c_1}\right\} \\
&\exp\left\{-A_2t - \frac{B_2c_2^x}{\ln c_2}c_2^t + \frac{B_2c_2^x}{\ln c_2} + \alpha_{02}A_2t + \alpha_{02}\frac{B_2c_2^x}{\ln c_2}c_2^t - \alpha_{02}\frac{B_2c_2^x}{\ln c_2}\right\} = \\
&= \exp\left\{-(1+\alpha_{13})A_2k + \frac{B_2c_2^x}{\ln c_2}\Big(-c_2^k(1+\alpha_{13}) + 1 - \alpha_{02}\Big)\right\} \\
&\exp\left\{\frac{B_1c_1^y}{\ln c_1}(1-\alpha_{01})\right\} \\
&\exp\left\{t[A_1(\alpha_{01}-1) + A_2(\alpha_{02}+\alpha_{13})]\right\} \\
&\exp\left\{c_1^t\left[\frac{B_1c_1^y}{\ln c_1}(\alpha_{01}-1)\right] + c_2^t\left[\frac{B_2c_2^x}{\ln c_2}(\alpha_{02}+\alpha_{13})\right]\right\} = \\
&= h_{11}\exp\Big\{t\,h_{12} + c_1^th_{13} + c_2^th_{14}\Big\},
\end{aligned}
$$

whereas

$$
\begin{aligned}
h_{11} &:= \exp\left\{-(1+\alpha_{13})A_2k + \frac{B_2c_2^x}{\ln c_2}\Big(-c_2^k(1+\alpha_{13}) + 1 - \alpha_{02}\Big)\right\} \\
&\quad \exp\left\{\frac{B_1c_1^y}{\ln c_1}(1-\alpha_{01})\right\}, \\
h_{12} &:= [A_1(\alpha_{01}-1) + A_2(\alpha_{02}+\alpha_{13})], \\
h_{13} &:= \left[\frac{B_1c_1^y}{\ln c_1}(\alpha_{01}-1)\right] \quad \text{and} \\
h_{14} &:= \left[\frac{B_2c_2^x}{\ln c_2}(\alpha_{02}+\alpha_{13})\right].
\end{aligned}
$$

And we have

$$(*) = (1-\alpha_{01})A_1 h_{11}\int_0^k \exp\Big\{t\,h_{12} + c_1^t h_{13} + c_2^t h_{14}\Big\}dt \qquad \text{and} \tag{3.49}$$

$$(**) = (1-\alpha_{01})B_1 c_1^y h_{11}\int_0^k c_1^t \exp\Big\{t\,h_{12} + c_1^t h_{13} + c_2^t h_{14}\Big\}dt. \tag{3.50}$$

The integrals in (3.49) and (3.50) cannot be done in terms of any standard mathematical functions. For their calculation must numerical integration be used. So, we have obtained that

$$\begin{aligned} {}_kp_{xy}^{01} &= (1-\alpha_{01})A_1 h_{11}\int_0^k \exp\Big\{t\,h_{12} + c_1^t h_{13} + c_2^t h_{14}\Big\}dt \\ &\quad + (1-\alpha_{01})B_1 c_1^y h_{11}\int_0^k c_1^t \exp\Big\{t\,h_{12} + c_1^t h_{13} + c_2^t h_{14}\Big\}dt. \end{aligned} \tag{3.51}$$

The calculation of ${}_kp_{xy}^{02}$ can be done analogously. We start by the computation of ${}_{k-t}p_{y+t}^{22}$, i.e.

$$\begin{aligned} {}_{k-t}p_{y+t}^{22} &= \exp\left\{-\int_t^k \mu_{y+s}^{23}ds\right\} = \qquad \text{using} \quad (2.30) \\ &= \exp\left\{-(1+\alpha_{23})\int_t^k \mu_{y+s}ds\right\} = \qquad \text{using} \quad (3.3) \\ &= \exp\left\{-(1+\alpha_{23})\int_t^k (A_1 + B_1 c_1^{y+s})ds\right\} = \\ &= \exp\left\{-(1+\alpha_{23})\left[A_1(k-t) + \frac{B_1 c_1^y}{\ln c_1}(c_1^k - c_1^t)\right]\right\}. \end{aligned} \tag{3.52}$$

Now, we rewrite (2.10) using (3.44), (2.31), (3.52) and we obtain

$$\begin{aligned} {}_kp_{xy}^{02} = &\int_0^k \exp\left\{-(1-\alpha_{01})\left[A_1 t + \frac{B_1}{\ln c_1}c_1^y(c_1^t - 1)\right]\right\} \\ &\exp\left\{-(1-\alpha_{02})\left[A_2 t + \frac{B_2}{\ln c_2}c_2^x(c_2^t - 1)\right]\right\} \\ &\exp\left\{-(1+\alpha_{23})\left[A_1(k-t) + \frac{B_1 c_1^y}{\ln c_1}(c_1^k - c_1^t)\right]\right\}(1-\alpha_{02})\Big[A_2 + B_2 c_2^{x+t}\Big]dt = \\ = &\ (1-\alpha_{02})A_2 h_{21}\int_0^k \exp\Big\{t\,h_{22} + c_1^t h_{23} + c_2^t h_{24}\Big\}dt \\ &+ (1-\alpha_{02})B_2 c_2^x h_{21}\int_0^k c_2^t \exp\Big\{t\,h_{22} + c_1^t h_{23} + c_2^t h_{24}\Big\}dt, \end{aligned} \tag{3.53}$$

whereas

$$\begin{aligned} h_{21} &:= \exp\left\{-(1+\alpha_{23})A_1 k + \frac{B_1 c_1^y}{\ln c_1}\Big(-c_1^k(1+\alpha_{23}) + 1 - \alpha_{01}\Big)\right\} \\ &\quad \exp\left\{\frac{B_2 c_2^x}{\ln c_2}(1-\alpha_{02})\right\}, \\ h_{22} &:= [A_2(\alpha_{02}-1) + A_1(\alpha_{01}+\alpha_{23})], \\ h_{23} &:= \left[\frac{B_1 c_1^y}{\ln c_1}(\alpha_{01}+\alpha_{23})\right] \quad \text{and} \\ h_{24} &:= \left[\frac{B_2 c_2^x}{\ln c_2}(\alpha_{02}-1)\right]. \end{aligned}$$

The integrals in (3.53) must be again solved numerically.

Since we have already computed estimates of A_i, B_i, c_i and α_{0i} for $i = 1, 2$ in Section 3.1 and Section 3.2, we can plug those into (3.44) , (3.51), (3.53), and we obtain the values of ${}_kp^{00}_{xy}$, ${}_kp^{01}_{xy}$ and ${}_kp^{02}_{xy}$ for $k = 0, 1, \ldots, n-1$.

Now, we are able to calculate the net single premium of n-year joint-life annuity due given by (3.42) and the net single premium of n-year last-survivor annuity due given by (3.43). In order to obtain consistent results, we calculate n-year joint-life and n-year last-survivor annuities due assuming independence given by (1.15) and (1.31) using relations

$$ {}_kp_y = \exp\left\{ -\int_0^k \mu_{y+\tau} d\tau \right\} \tag{3.54} $$

and

$$ {}_kp_x = \exp\left\{ -\int_0^k \mu_{x+\tau} d\tau \right\}. \tag{3.55} $$

Using (3.3), we can rewrite (3.54) as

$$ \begin{aligned} {}_kp_y &= \exp\left\{ -\int_0^k (A_1 + B_1 c_1^{y+\tau}) d\tau \right\} = \\ &= \exp\left\{ -\left[A_1\tau + B_1 \frac{c_1^{y+\tau}}{\ln c_1} \right]_0^k \right\} = \\ &= \exp\left\{ -\left[A_1 k + \frac{B_1}{\ln c_1} c_1^y (c_1^k - 1) \right] \right\}. \end{aligned} $$

Analogously, using (3.4), we rewrite (3.55) and we obtain

$$ {}_kp_x = \exp\left\{ -\left[A_2 k + \frac{B_2}{\ln c_2} c_2^x (c_2^k - 1) \right] \right\}. $$

3.4 Results and Discussion

In order to examine the effect of a possible dependence of remaining lifetimes on the amount of the net single premium, we have plotted the net single premiums of n-year joint-life annuity due and n-year last-survivor annuity due in case of the dependence and also independence. We assume that the duration of contracts is 10 years and the annual interest rate is 4% for annuities in all the graphs in this section. For the sake of simplicity, we assume that $x = y$. The calculation of annuities was done in Mathematica and the main part of the code can be found in the Appendix.

Firstly, we look at joint-life annuities (see Figure (3.9)). We can see that 10-year joint-life annuity due assuming independence is lower than 10-year "dependent" joint-life annuity due for all considered ages and it can be explained as follows. Since all $\hat{\alpha}_{ij}$'s are non-negative, we have from Theorem 2 that the remaining lifetimes of the husband and wife T_x and T_y are positive quadrant dependent. With positive quadrant dependent remaining lifetimes, the couple

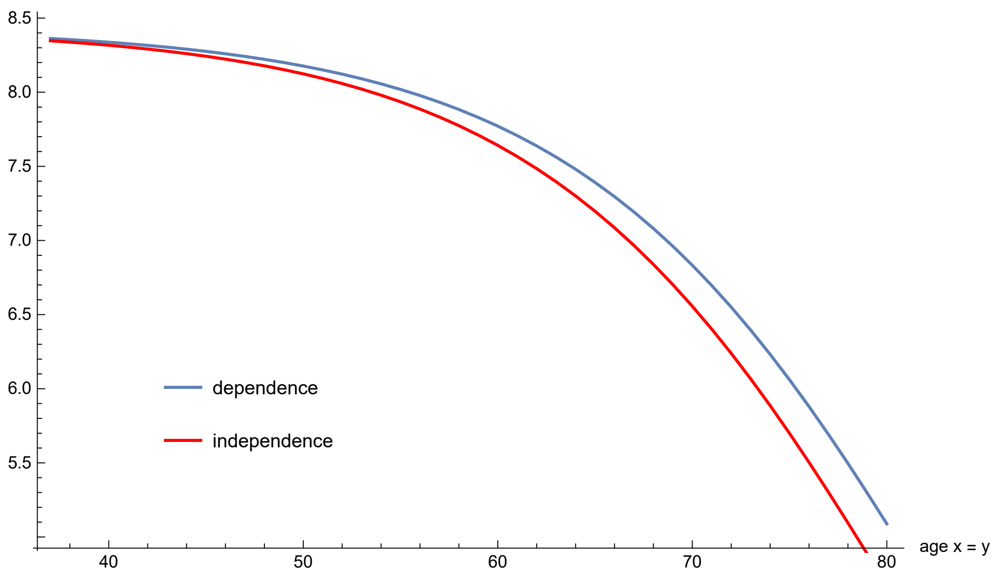

Figure 3.9: 10-year joint-life annuities due

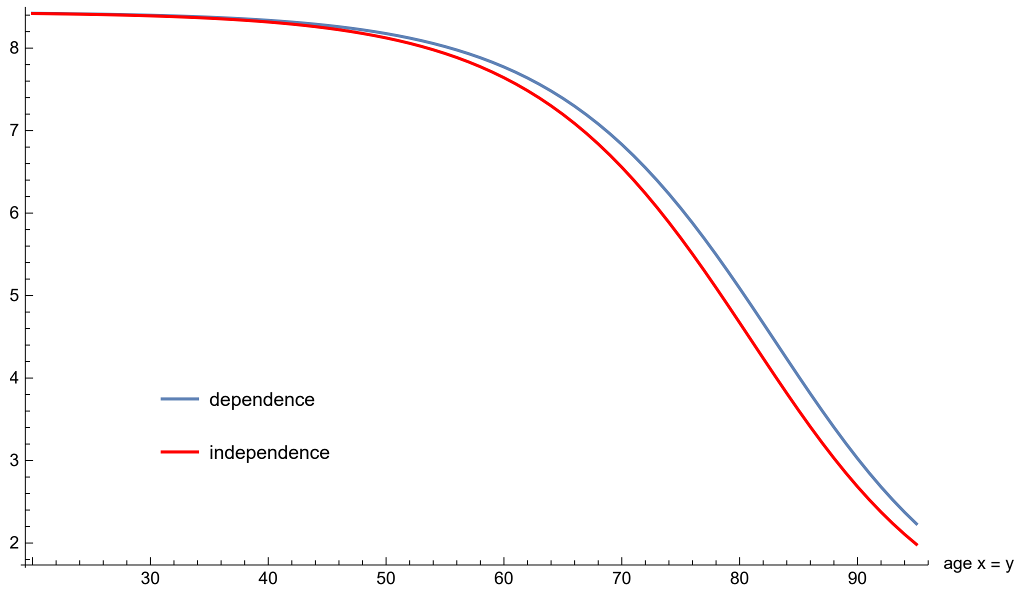

Figure 3.10: An illustration of 10-year joint-life annuities due

stays longer in state 0 (thus there is a longer time for annuity payments) and shorter in states 1 or 2 (no payments). Further, we calculated that the net single premium ${}^{\perp}\ddot{a}_{xy:\overline{n}|}$ is roughly of about 99% of the net single premium $\ddot{a}_{xy:\overline{n}|}$. The result is clearly not significant. The "exact" value of the premium $\ddot{a}_{xy:\overline{n}|}$ itself is important for actuaries, since it can help them for example to decide whether or not to grant a discount to an assured couple. It is also meaningful when determining a safety loading. In our case, it tells us that underestimation of ${}^{\perp}\ddot{a}_{xy:\overline{n}|}$ is not so significant, therefore actuaries should avoid using excessive safety margins.

Note that we assume that all the annuities plotted in this section hold for the age range $(37, 80)$ as we have determined in Section 3.1. Just for illustration, we decided to plot joint-life annuities on the age scale $(20, 95)$ (see Figure (3.10)).

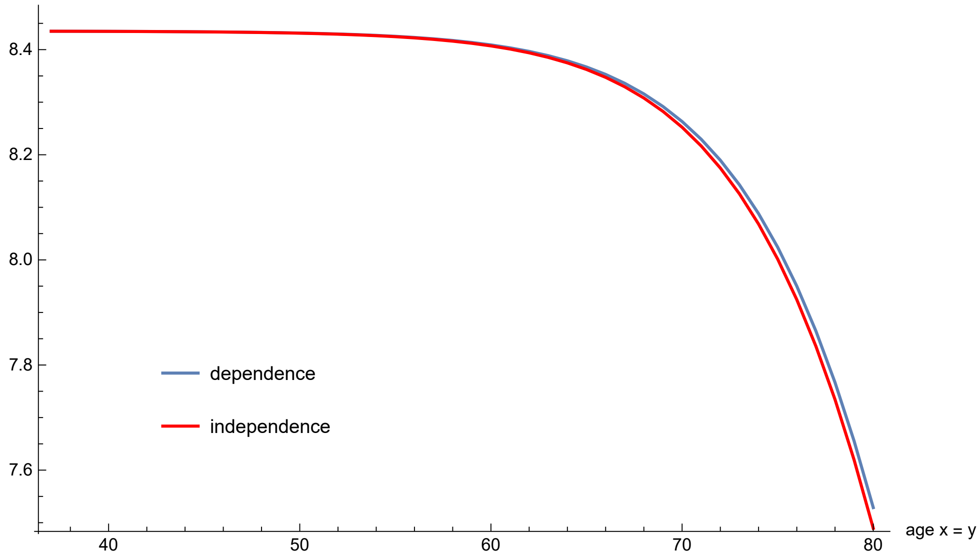

Figure 3.11: 10-year last-survivor annuities due

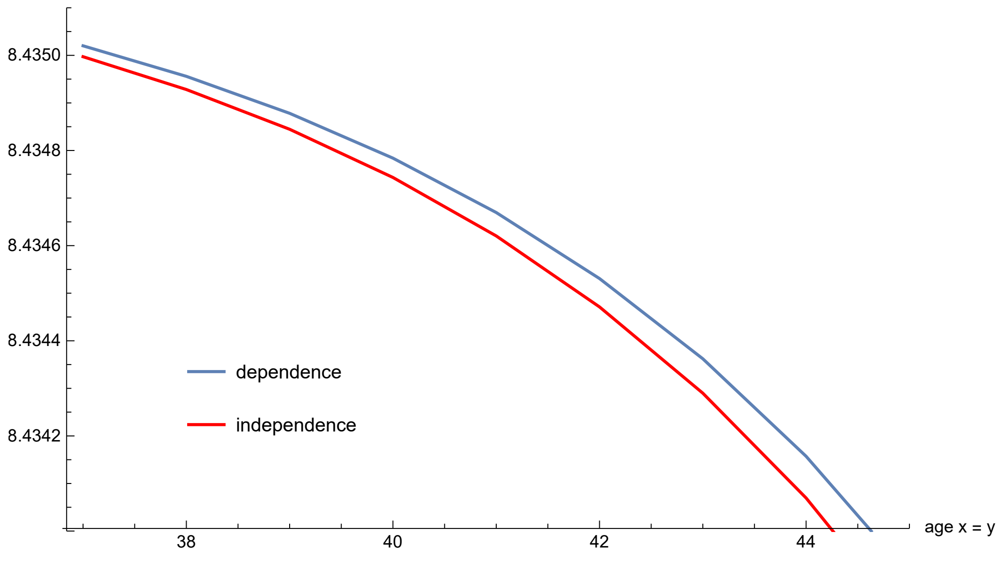

Figure 3.12: 10-year last-survivor annuities due with the age scale from 37 to 45

Let us look at last-survivor annuities that are shown in Figure (3.11). From Figure (3.11), it is not clear whether the 10-year last-survivor annuity due assuming independence is lower than 10-year "dependent" last-survivor annuity due for all considered ages, therefore we plotted last-survivor annuities using different scaling of x axis (see Figure (3.12) and Figure (3.13)). We can clearly see that the premium $\ddot{a}_{\overline{xy}:\overline{n}|}$ is bigger than the premium ${}^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|}$ for all considered ages. Again, since all $\hat{\alpha}_{ij}$'s are non-negative, we have from Theorem 2 that the remaining lifetimes of the husband and wife T_x and T_y are positive quadrant

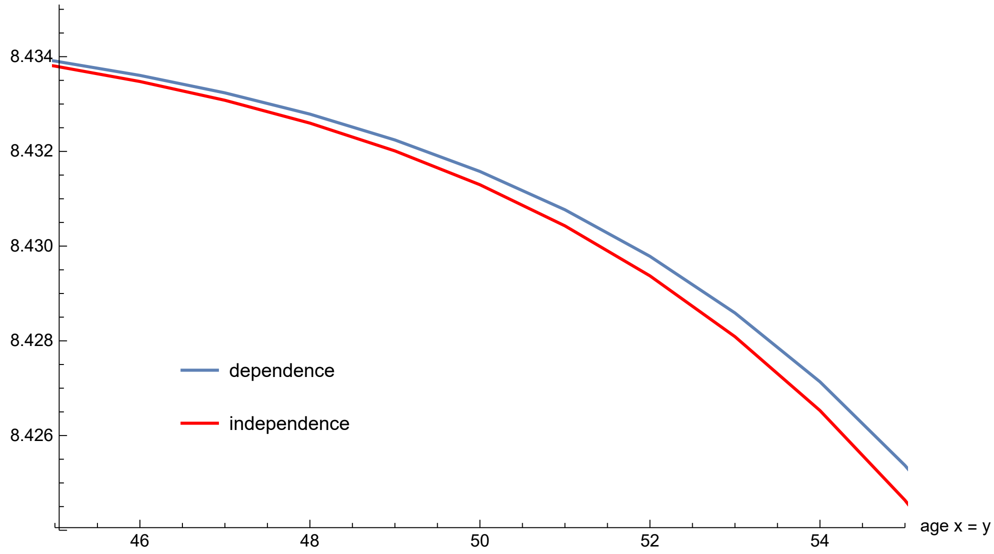

Figure 3.13: 10-year last-survivor annuities due with the age scale from 45 to 55

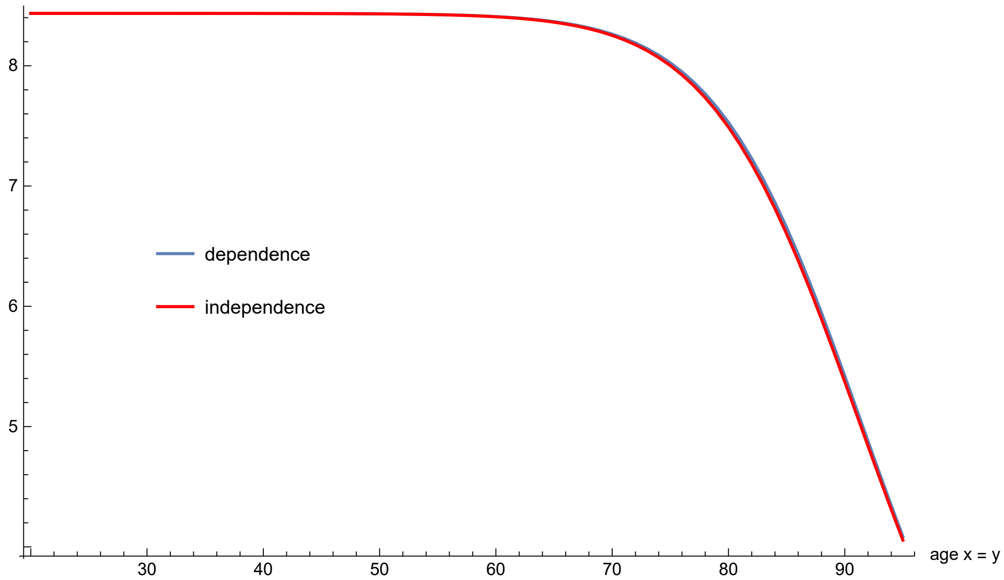

Figure 3.14: An illustration of 10-year last-survivor annuities due

dependent. For last-survivor annuity, the premium is paid when the couple is in states $0, 1$ or 2. Therefore, using positive quadrant dependence the relationship between $\ddot{a}_{\overline{xy}:\overline{n}|}$ and $^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|}$ cannot be seen straightforward. By comparing Figure (3.9) and Figure (3.11), we realise that the net single premium $^{\perp}\ddot{a}_{\overline{xy}:\overline{n}|}$ must be roughly of about 99.9% of the net single premium $\ddot{a}_{\overline{xy}:\overline{n}|}$. Again the result is not significant.

To sum up, we calculated and plotted the net single premiums of 10-year joint-life annuity due and 10-year last-survivor annuity due in case of the dependence and also independence in order to quantify the effect of a possible dependence of the remaining lifetimes on the amount of the net single premium. The calculations were based on the dataset related to the Czech population 2015. We have concluded that the differences between net single premiums of 10-year joint-life annuity due and 10-year last-survivor annuity due are not significant for the Czech population in 2015. For actuaries, it would mean for example that they should avoid excessive safety margins.

Conclusion

In this thesis, our main interest was the dependence between remaining lifelengths of the husband and wife. We mentioned that there are three types of possible dependencies between the two considered lives, i.e, the long-term association between lifetimes, the short-term impact of a spousal death and the instantenaous dependence due to a catastrophic event.

The long-term association between lifetimes was captured by the joint-life and last-survivor model. Considering this model, we calculated and plotted 10-year joint-life annuities due and 10-year last survivor annuities due in case of dependence and also independence. The calculations were based on the dataset related to the Czech population in 2015. By comparing the annuities we have concluded that the effect of the possible dependence of remaining lifetimes on the amount of the net single premium is not significant for both 10-year joint-life annuity due and 10-year last-survivor annuity due. Hypothetically, as a result, actuaries should for example avoid excessive safety margins.

The short-term impact of a spousal death and the instantenaous dependence due to a catastrophic event were captured by the extensions of the joint-life and last-survivor model.

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Appendix

Main Part of the Source Code of Joint-Life and Last-Survivor Annuities

```
A1 = 0.000252597703303867;
B1 = 0.0000068662152719738l;
c1 = 1.11703588412242;
A2 = -0.000307324024515891;
B2 = 0.0000469433916408876;
c2 = 1.09739715992391;
alpha01 = 0.209245955040946;
alpha13 = 0.240952327076487;
alpha02 = 0.158489993441526;
alpha23 = 0.0424904747821149;

i = 0.04;
v = 1/(1 + i);

p00 [x_, y_, k_] :=
 Exp[-(1 - alpha01) (A1*k + B1/Log[c1]*c1^y (c1^k - 1)) - (1 -
      alpha02) (A2*k + B2/Log[c2]*c2^x (c2^k - 1))]

h11[x_, y_, k_] :=
 Exp[-(1 + alpha13) A2*k + (B2*c2^x)/
     Log[c2] (-c2^k (1 + alpha13) + 1 - alpha02)]*
  Exp[(B1*c1^y)/Log[c1] (1 - alpha01)]
h12[x_, y_, k_] := A1*(alpha01 - 1) + A2*(alpha02 + alpha13)
h13[x_, y_, k_] := (B1*c1^y)/Log[c1] (alpha01 - 1)
h14[x_, y_, k_] := (B2*c2^x)/Log[c2] (alpha02 + alpha13)
p01 [x_, y_,
  k_] := (1 - alpha01)*A1*h11[x, y, k]*
   N[Integrate[
     Exp[t*h12[x, y, k] + c1^t*h13[x, y, k] + c2^t*h14[x, y, k]], {t,
      0, k}] ] + (1 - alpha01)*B1*c1^y*h11[x, y, k]*
   N[Integrate[
     c1^t*Exp[
       t*h12[x, y, k] + c1^t*h13[x, y, k] + c2^t*h14[x, y, k]], {t,
      0, k}] ]

h21[x_, y_, k_] :=
 Exp[-(1 + alpha23) A1*k + (B1*c1^y)/
     Log[c1] (-c1^k (1 + alpha23) + 1 - alpha01)]*
  Exp[(B2*c2^x)/Log[c2] (1 - alpha02)]
h22[x_, y_, k_] := A2*(alpha02 - 1) + A1*(alpha01 + alpha23)
h23[x_, y_, k_] := (B1*c1^y)/Log[c1] (alpha01 + alpha23)
h24[x_, y_, k_] := (B2*c2^x)/Log[c2] (alpha02 - 1)
p02 [x_, y_,
  k_] := (1 - alpha02)*A2*h21[x, y, k]*
   N[Integrate[
     Exp[t*h22[x, y, k] + c1^t*h23[x, y, k] + c2^t*h24[x, y, k]], {t,
      0, k}] ] + (1 - alpha02)*B2*c2^x*h21[x, y, k]*
   N[Integrate[
     c2^t*Exp[
       t*h22[x, y, k] + c1^t*h23[x, y, k] + c2^t*h24[x, y, k]], {t,
```

```
      0, k}] ]

JointLife[x_, y_, n_] := Sum[v^k*p00[x, y, k], {k , 0, n - 1}]

LastSurvivor[x_, y_, n_] :=
 Sum[v^k*(p00[x, y, k] + p01[x, y, k] + p02[x, y, k]), {k , 0, n - 1}]

py[x_, y_, k_] := Exp[-(A1*k + B1/Log[c1]*c1^y (c1^k - 1))]
px[x_, y_, k_] := Exp[-(A2*k + B2/Log[c2]*c2^x (c2^k - 1))]

JointLifeIndep[x_, y_, n_] :=
 Sum[v^k*px[x, y, k]*py[x, y, k], {k , 0, n - 1}]

LastSurvivorIndep[x_, y_, n_] :=
 Sum[v^k*(px[x, y, k] + py[x, y, k] - px[x, y, k]*py[x, y, k]), {k ,
   0, n - 1}]
```