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Structural properties of hereditary permutation classes

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Title: Structural properties of hereditary permutation classes

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Abstract: A permutation class C is splittable if it is contained in a merge of its two proper subclasses, and it is 1-amalgamable if given two permutations $\sigma, \tau \in C$, each with a marked element, we can find a permutation $\pi \in C$ containing both σ and τ such that the two marked elements coincide. In this thesis, we study both 1-amalgamability and splittability of permutation classes. It was previously shown that unsplittability implies 1-amalgamability. We prove that unsplittability and 1-amalgamability are not equivalent properties of permutation classes by showing that there is a permutation class that is both splittable and 1-amalgamable. Moreover, we show that there are infinitely many such classes. Our construction is based on the concept of LR-inflations or more generally on hereditary 2-colorings, which we both introduce here and which may be of independent interest.

Keywords: permutation classes, splittability, amalgamability, 1-amalgamability

Contents

Introduction

Let S_n be the set of permutations of the letters $\{1, 2, \ldots, n\} = [n]$. We write a permutation $\pi \in S_n$ as a sequence $\pi_1, \ldots, \pi_n$. Note that we omit all punctuation in the case of short permutations, e.g., we write simply 123. A permutation $\pi \in S_n$ *contains* a pattern $\sigma \in S_k$ if there is a subsequence of $\pi_1, \ldots, \pi_n$ in the same relative order as $\sigma_1, \ldots, \sigma_k$. Otherwise we say that π *avoids* the pattern σ. We let $\mathrm{Av}(\sigma)$ denote the set of all σ-avoiding permutations and $\mathrm{Av}(\sigma_1, \ldots, \sigma_k)$ the set of permutations that avoid σ_i for every $i \in [k]$. Pattern avoidance is the central topic in the study of permutations, see, e.g., a book by Bóna [4] for an exhaustive overview.

In the study of permutation classes, a notable interest has recently been directed towards the operation of merging. We say that a permutation π is a *merge* of σ and τ if the elements of π can be colored red and blue so that the red elements form a copy of σ and the blue elements form a copy of τ. For instance, Claesson, Jelínek and Steingrímsson [9] showed that every 1324-avoiding permutation can be merged from a 132-avoiding permutation and a 213-avoiding permutation, and used this fact to prove that there are at most 16^n 1324-avoiding permutations of length n. There have been other instances of using merges to establish bounds on the growth rates of certain permutation classes, e.g. [3, 5, 6].

A general problem that follows naturally is how to identify when a permutation class C has proper subclasses A and B, such that every element of C can be obtained as a merge of an element of A and an element of B. We say that such permutation class C is *splittable*. In the first paper focused mainly on splittability, Jelínek and Valtr [10] showed that every inflation-closed class is unsplittable and the class of σ-avoiding permutations, where σ is a direct sum of two nonempty permutations and has length at least four, is splittable (we refer reader to Chapter 1 for proper formal definitions of inflations and sums). Furthermore, they mentioned the connection of splittability to more general structural properties of classes of relational structures studied in the area of Ramsey theory, most notably the notion of 1-amalgamability. We say that a permutation class C is π-*amalgamable* if given two permutations $\sigma, \tau \in C$, each with a marked occurrence of π, we can find a permutation $\rho \in C$ containing both σ and τ such that the two marked occurrences of π coincide. Moreover, a class is k-*amalgamable* if it is π-amalgamable for every π of length at most k.

Not much is known about 1-amalgamability of permutation classes. Jelínek and Valtr [10, Lemma 1.5], using a more general result from Ramsey theory, showed that unsplittability implies 1-amalgamability, and they raised the question whether there is a permutation class that is both splittable and 1-amalgamable. In this thesis, we answer their question by showing that the class $\mathrm{Av}(1423, 1342)$ has both properties and furthermore we actually generate infinite number of its superclasses that are also both 1-amalgamable and splittable.

Outline

We properly define and introduce the concepts of splittability and amalgamability in Chapter 2. First, we provide a short survey of previously known-results about splittability and amalgamability. We continue by stating some basic observations and lemmas about 1-amalgamability, which are often analogous to the results that were already known about unsplittability. And finally, we conclude this chapter by characterizing all 2-amalgamable classes of separable permutations. We delay a formal structural definition of separable permutations to Chapter 1, but alternatively a permutation π is *separable* if it avoids both 2413 and 3142.

In Chapter 3, we will introduce a slightly weaker property than being inflation-closed, that is closed under inflating just the elements that are left-to-right minima. We say that an element of permutation π is a *left-to-right minimum*, or just LR-minimum, if it is smaller than all the elements preceding it. We shall prove that certain properties of a permutation class C imply that its closure under inflating LR-minima is splittable and 1-amalgamable.

We utilize these results in Chapter 4. We show that the class $\mathrm{Av}(1423, 1342)$ is in fact a closure of $\mathrm{Av}(123)$ under inflating LR-minima and that $\mathrm{Av}(123)$ has all the desired properties that imply 1-amalgamability and splittability of its closure. Moreover, we use this result to generate infinitely many classes that are 1-amalgamable and splittable.

Finally, we present a generalization of inflating LR-minima in Chapter 5. We prove claims similar to the ones from Chapter 3 which might hint that these can be used to prove 1-amalgamability and splittability of different permutation classes. And even though we were not able to achieve this goal we include them in this thesis since we think that they might be of independent interest.

1. Preliminaries

A *permutation* π of length $n \geq 1$ is a sequence of all the n distinct numbers from the set $[n] = \{1, 2, \ldots, n\}$. We denote the i-th element of π as π_i. Note that we omit all punctuation when writing out short permutations, e.g., we write 123 instead of 1, 2, 3. The set of all permutations of length n is denoted S_n.

We say that two sequences of distinct numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are *order-isomorphic* if for every two indices $i < j$ we have $a_i < a_j$ if and only if $b_i < b_j$. Given two permutations $\pi \in S_n$ and $\sigma \in S_k$, we say that π *contains* σ if there is a k-tuple $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the sequence $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ is order-isomorphic to σ and we say that such sequence is an *occurrence* of σ in π. Furthermore, we say that the corresponding function $f : [k] \to [n]$ defined as $f(j) = i_j$ is an *embedding* of σ into π. In the context of permutation containment, we often refer to the permutation σ as a *pattern*.

A permutation that does not contain σ is σ*-avoiding* and we let $\mathrm{Av}(\sigma)$ denote the set of all σ-avoiding permutations. Similarly, for a set of permutations F , we let $\mathrm{Av}(F)$ denote the set of permutations that avoid all elements of F. Note that for small sets F we omit the curly braces, e.g., we simply write $\mathrm{Av}(\sigma, \rho)$ instead of $\mathrm{Av}(\{\sigma, \rho\})$.

We say that a set of permutations C is a *permutation class* if for every $\pi \in C$ and σ contained in π, σ belongs to C as well. Observe that a set of permutations C is a permutation class if and only if there is a set F such that $C = \mathrm{Av}(F)$. Moreover, for every permutation class C, there is a unique inclusionwise minimal set F such that $C = \mathrm{Av}(F)$; this set F is known as the *basis* of C. A class is said to be *principal* if its basis has a single element, i.e., if the class has the form $\mathrm{Av}(\sigma)$ for a permutation σ.

The *direct sum* $\pi \oplus \sigma$ of two permutations $\pi \in S_n$ and $\sigma \in S_m$ is the permutation $\pi_1, \pi_2, \ldots, \pi_n, \sigma_1 + n, \sigma_2 + n, \ldots, \sigma_m + n$ of length $n + m$. Similarly, the *skew sum* $\pi \ominus \sigma$ is the permutation $\pi_1 + m, \pi_2 + m, \ldots, \pi_n + m, \sigma_1, \sigma_2, \ldots, \sigma_m$. A permutation is *decomposable* if it is a direct or skew sum of two nonempty permutations.

We say that permutation is *separable* if it can be obtained by direct and skew sums of singleton permutations. We use Sep to denote the class of separable permutations. Equivalently, Sep can be defined as the smallest non-empty class closed under taking both direct and skew sums or alternatively through a simple basis as $\mathrm{Av}(2413, 3142)$ (see, e.g., Bose et al. [7]).

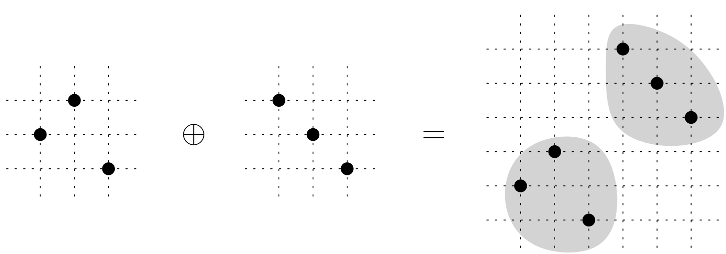


Figure 1.1: An example of a direct sum: $231 \oplus 321 = 231654$.

Suppose that $\pi \in S_n$ is a permutation, let $\sigma_1, \ldots, \sigma_n$ be an n-tuple of non-empty permutations, and let m_i be the order of σ_i for $i \in [n]$. The *inflation* of π by the sequence $\sigma_1, \ldots, \sigma_n$, denoted by $\pi[\sigma_1, \ldots, \sigma_n]$, is the permutation of order $m_1 + \cdots + m_n$ obtained by concatenating n sequences $\overline{\sigma}_1 \overline{\sigma}_2 \cdots \overline{\sigma}_n$ with these properties:

- for each $i \in [n]$, $\overline{\sigma}_i$ is order-isomorphic to σ_i, and
- for each $i, j \in [n]$, if $\pi_i < \pi_j$, then all the elements of $\overline{\sigma}_i$ are smaller than all the elements of $\overline{\sigma}_j$.

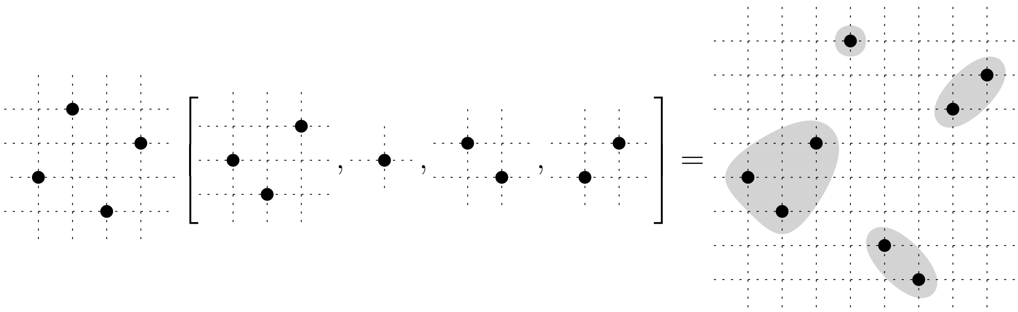


Figure 1.2: An example of an inflation: $2413[213, 1, 21, 12] = 43582167$.

Note that we can use inflations to define direct and skew sums as $\pi \oplus \sigma = 12[\pi, \sigma]$ and similarly $\pi \ominus \sigma = 21[\pi, \sigma]$.

For two sets of permutations A and B, we let $A[B]$ denote the set of all the permutations that can be obtained as inflation of a permutation from A by a sequence of permutations from B. We say that a set of permutations A is $\cdot[B]$-*closed* if $A[B] \subseteq A$, and similarly a set of permutations B is $A[\cdot]$-*closed* if $A[B] \subseteq B$. Finally, we say that a set of permutations C is *inflation-closed* if $C[C] \subseteq C$.

There is a nice way to characterize an inflation-closed class through its basis. We say that a permutation π is *simple* if it cannot be obtained by inflation from smaller permutations, except for the trivial inflations $\pi[1, \ldots, 1]$ and $1[\pi]$. Inflation-closed permutation classes are precisely the classes whose basis only contains simple permutations [1, Proposition 1].

We say that *interval* in a permutation is a contiguous set of positions on which the values also form a contiguous set.

2. Splittability and amalgamability

In this chapter, we introduce the properties of splittability and amalgamability of permutation classes, which are two examples of what we often call Ramsey-type properties. In the case of splittability, we mostly state or rephrase results that were already known. However, we provide some new result about amalgamability, as not much was previously known about amalgamability of permutation classes.

2.1 Splittability

We say that a permutation π is a *merge* of permutations τ and σ, if it can be partitioned into two disjoint subsequences, one of which is an occurrence of σ and the other is an occurrence of τ. For two permutation classes A and B, we write $A \odot B$ for the class of all merges of a (possibly empty) permutation from A with a (possibly empty) permutation from B. Trivially, $A \odot B$ is again a permutation class.

Conversely, we say that a multiset of permutation classes $\{P_1, \ldots, P_m\}$ forms a *splitting* of a permutation class C if $C \subseteq P_1 \odot \cdots \odot P_m$. We call P_i the *parts* of the splitting. The splitting is *nontrivial* if none of its parts is a superset of C, and the splitting is *irredundant* if no proper submultiset of $\{P_1, \ldots, P_m\}$ forms a splitting of C. A permutation class C is then *splittable* if C admits a nontrivial splitting.

The following simple lemma is due to Jelínek and Valtr [10, Lemma 1.3].

Lemma 2.1. *For a class C of permutations, the following properties are equivalent:*

(a) C is splittable.

(b) C has a nontrivial splitting into two parts.

(c) C has a splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial splitting into two parts, in which each part is a principal class.

Following the previous Lemma 2.1, we can characterize a splittable class C by the splittings of the form $\{\mathrm{Av}(\pi), \mathrm{Av}(\sigma)\}$, where both π and σ are permutations from C. Therefore, we want to identify permutations inside C that cannot define any such splitting.

Definition 2.2. Let C be a permutation class. We say that a permutation $\pi \in C$ is *unavoidable in C*, if for any permutation $\tau \in C$, there is a permutation $\sigma \in C$ such that any red-blue coloring of σ has a red copy of τ or a blue copy of π. We let U_C denote the set of all unavoidable permutations in C.

It is easy to see that a permutation π is unavoidable in C if and only if C has no nontrivial splitting into two parts with one part being $\mathrm{Av}(\pi)$. A more detailed overview of the properties of unavoidable permutations was provided by Jelínek and Valtr [10, Observation 2.2-3], we will mention only the observations needed for our results.

Note that for a nonempty permutation class C, the set of unavoidable permutations U_C is in fact a nonempty permutation class contained in the class C. We can use the class of unavoidable permutations to characterize the unsplittable permutation classes.

Observation 2.3. *A permutation class C is unsplittable if and only if $U_C = C$.*

Furthermore, we can show that if C is closed under certain inflations then also U_C is closed under the same inflations. Again, the following result is due to Jelínek and Valtr [10, Lemma 2.4].

Lemma 2.4 (Jelínek and Valtr [10])**.** *Let C be a permutation class. If, for a set of permutations X, the class C is closed under $\cdot[X]$, then U_C is also closed under $\cdot[X]$, and if C is closed under $X[\cdot]$, then so is U_C. Consequently, if C is inflation-closed, then $U_C = C$ and C is unsplittable.*

Furthermore, Albert and Jelínek [2] proved that if permutation classes C and D are both unsplittable then their inflation $C[D]$ must be unsplittable as well. We can slightly strengthen this into a claim about unavoidable permutations.

We say that a permutation class C is *atomic* if it does not have proper subclasses A and B such that $C = A \cup B$. Observe that a permutation class that is not atomic is clearly splittable. For us, the important property of an atomic class C is that for any $\pi, \sigma \in C$ there is a permutation $\tau \in C$ that contains both π and σ (this is known as the *joint-embedding property*).

Lemma 2.5. *If C is a permutation class and D is an atomic permutation class, then $U_C[U_D] \subseteq U_{C[D]}$. Consequently, if C and D are any unsplittable permutation classes, then so is $C[D]$.*

Proof. First, it is easy to see how the second claim follows from the first due to Observation 2.3 as any unsplittable class is trivially atomic.

Let π be an unavoidable permutation in C of length k and $\sigma_1, \dots, \sigma_k$ unavoidable permutations in D. We aim to show that $\pi[\sigma_1, \dots, \sigma_k]$ is unavoidable in $C[D]$. Let τ be any permutation in $C[D]$. We can express τ as $\alpha[\beta_1, \dots, \beta_l]$ for some $\alpha \in C$ and $\beta_1, \dots, \beta_l \in D$. Since D is atomic, there is a permutation $\beta \in D$ that contains all the $\beta_1, \dots, \beta_l$. It suffices to show that there is a permutation ψ such that any red-blue coloring of ψ has a red copy of $\pi[\sigma_1, \dots, \sigma_k]$ or a blue copy of $\alpha[\beta, \dots, \beta]$.

Since π is unavoidable in C, there is a permutation $\rho \in C$ such that any red-blue coloring of ρ has either a red copy of π or a blue copy of α. Analogously, for every $i \in [k]$ there is a permutation $\gamma_i \in D$ such that any red-blue coloring of γ_i has either a red copy of σ_i or a blue copy of β. Moreover, there is a permutation $\gamma \in D$ containing all the $\gamma_1, \dots, \gamma_k$ due to the atomicity of D.

We claim that $\psi = \rho[\gamma, \dots, \gamma]$ is the desired permutation. Fix an arbitrary red-blue coloring of ψ. We can view ψ as a concatenation of blocks, each being a copy of γ. It is easy to see that every block contains a blue copy of β or a red copy of σ_i for every $i \in [k]$. We now define a coloring of ρ in the following way: an element ρ_i is blue if and only if the i-th block of ψ contains a blue copy of β. First, suppose that this coloring of ρ contains a blue copy of α. In this case, the original coloring of ψ must contain a blue copy of $\alpha[\beta, \ldots, \beta]$. Similarly, if the coloring of ρ contains a red copy of π then we get a red copy of $\pi[\sigma_1, \ldots, \sigma_k]$ since each of the corresponding blocks contain red copies of σ_i for every $i \in [k]$. □

Moreover, it was observed by Albert and Jelínek [2] that under some further assumptions, an inflation that involves splittable class is itself splittable. We include a proof for completeness since it was stated in [2] only as an observation.

Lemma 2.6. *If C, D, C_1 and C_2 are permutation classes such that $C \subseteq C_1 \odot C_2$ then $C[D] \subseteq C_1[D] \odot C_2[D]$ and $D[C] \subseteq D[C_1] \odot D[C_2]$. Consequently,*

- *if $C_1[D]$ and $C_2[D]$ do not contain the whole class $C[D]$ then $C[D]$ is splittable into parts $C_1[D]$ and $C_2[D]$,*
- *if $D[C_1]$ and $D[C_2]$ do not contain the whole class $D[C]$ then $D[C]$ is splittable into parts $D[C_1]$ and $D[C_2]$.*

Proof. The second part of the lemma clearly follows from the first as it just straight-forwardly guarantees that we obtained a nontrivial splitting.

Let π be a permutation in $C[D]$ which means there is a permutation $\tau \in C$ and permutations $\sigma_1, \ldots, \sigma_k \in D$ such that $\pi = \tau[\sigma_1, \ldots, \sigma_k]$. We have 2-coloring of τ such that the red elements create a permutation from C_1 and the blue elements a permutation from C_2. Looking at π as k blocks with i-th block being a copy of σ_i, we simply color every element in i-th block using the color of τ_i. It is easy to see that the red elements form a copy of permutation from $C_1[D]$ and the blue elements form a copy of permutation from $C_2[D]$.

Similarly, let π be a permutation in $D[C]$ which means there is a permutation $\tau \in D$ and permutations $\sigma_1, \ldots, \sigma_k \in C$ such that $\pi = \tau[\sigma_1, \ldots, \sigma_k]$. We have 2-colorings of each σ_i such that the red elements form a permutation from C_1 and the blue elements a permutation from C_2. In this case, we simply color i-th block of π according to the 2-coloring of σ_i. Again, it is easy to see that the red elements form a copy of permutation from $D[C_1]$ and the blue elements form a copy of permutation from $D[C_2]$. □

Let us provide a short summary of the currently known results about splittability. As we will see later in Subsection 2.2.1, these are also relevant when studying 1-amalgamability. Jelínek and Valtr [10] studied hereditary classes in particular and obtained the following results.

Theorem 2.7 (Jelínek and Valtr [10])**.** *For a single pattern σ the following holds:*

- *if σ is simple then $Av(\sigma)$ is unsplittable,*
- *if σ is* 12, 213 *or one of their symmetries then $Av(\sigma)$ is unsplittable,*
- *if σ is any other decomposable permutation then $Av(\sigma)$ is splittable.*

Later, Albert and Jelínek [2] characterized all the unsplittable subclasses of the class of separable permutations.

Theorem 2.8 (Albert and Jelínek [2])**.** *A proper subclass of* Sep *is unsplittable if and only if it can be obtained by iterated inflations of the classes $Av(12)$, $Av(132)$ and their symmetries.*

2.2 Amalgamability

Now let us introduce the concept of amalgamation, which comes from the general study of relational structures.

We say that a permutation class C is π*-amalgamable* if for any two permutations $\tau_1, \tau_2 \in C$ and any two mappings f_1 and f_2, where f_i is an embedding of π into τ_i, there is a permutation $\sigma \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of τ_i into σ and $g_1 \circ f_1 = g_2 \circ f_2$. We also say, for $k \in \mathbb{N}$ that a permutation class C is k-amalgamable if it is π-amalgamable for every π of order at most k. Furthermore, a permutation class C is amalgamable if it is k-amalgamable for every k.

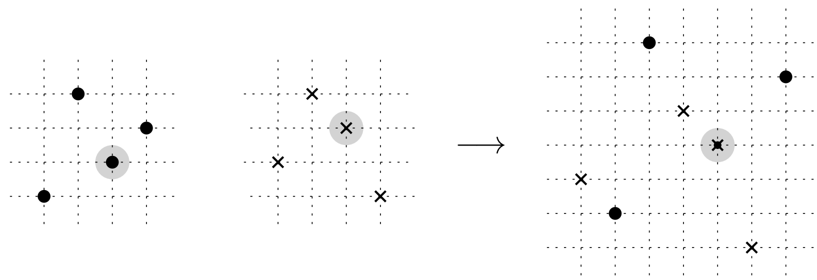


Figure 2.1: One possible 1-amalgamation of 1423 and 2431 with highlighted embeddings of the singleton permutations is the permutation 3275416.

Note that k-amalgamability implies $(k-1)$-amalgamability, so we have an infinite number of increasingly stronger properties. However, the situation is quite simple in case of the permutation classes. As shown by Cameron [8], there are only five infinite amalgamable classes.

Theorem 2.9 (Cameron [8])**.** *A non-empty permutation class is amalgamable if and only if it is one of the following:* $Av(12)$*,* $Av(21)$*, the class of all layered permutations* $Av(231, 312)$*, the class of its complements* $Av(213, 132)$ *or the class of all permutations.*

These are also the only permutation classes that are 3-amalgamable, implying that for any $k \geq 3$, a permutation class is k-amalgamable if and only if it is amalgamable. In contrast, very little is known about 1-amalgamable and 2-amalgamable permutation classes. In the rest of this section, we will try to shed some light on this subject.

2.2.1 1-Amalgamability

Definition 2.10. Let C be a permutation class. We say that a permutation $\pi \in C$ is *1-amalgamable in* C, if for every $\sigma \in C$ and every prescribed pair of embeddings f_1 and f_2 of the singleton permutation 1 into π and τ there is a permutation $\tau \in C$ and embeddings g_1 and g_2 of π and τ to σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We use A_C to denote the set of all 1-amalgamable permutations in C.

Trivially, A_C is a permutation class contained in C. Moreover, the properties of A_C are largely analogous to those of U_C, as shown by the next several results.

Observation 2.11. *A permutation class C is 1-amalgamable if and only if $A_C = C$.*

Similarly to U_C, the set A_C is closed under the same inflations as the original class C.

Lemma 2.12. *Let C be a permutation class. If, for a set of permutations X, the class C is closed under $\cdot[X]$, then A_C is also closed under $\cdot[X]$, and if C is closed under $X[\cdot]$, then so is A_C. Consequently, if C is inflation-closed, then $A_C = C$ and C is 1-amalgamable.*

Proof. Suppose that C is closed under $\cdot[X]$. We can assume that X itself is inflation-closed since if C is closed under $\cdot[X]$, it is also closed under $\cdot[X[X]]$.

Let $\pi \in A_C$ be a 1-amalgamable permutation of order k and let $\rho_1, \dots, \rho_k$ be permutations from X. Our goal is to prove that also $\pi[\rho_1, \dots, \rho_k]$ belongs to A_C. We can assume, without loss of generality, that all ρ_i are actually equal to a single permutation ρ. Otherwise, we could just take $\rho \in X$ that contains every ρ_i (this is possible since X is inflation-closed) and prove the stronger claim that $\pi[\rho, \dots, \rho]$ belongs to A_C. Let us use $\pi[\rho]$ as a shorthand notation for $\pi[\rho, \dots, \rho]$.

It is now sufficient to show that $\pi[\rho]$ belongs to A_C for every $\pi \in A_C$ and $\rho \in X$. Fix a permutation $\tau \in C$ and two embeddings f_1 and f_2 of the singleton permutation into $\pi[\rho]$ and τ. We aim to find a permutation $\sigma \in C$ and two embeddings g_1 and g_2 of $\pi[\rho]$ and τ into σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We can straightforwardly decompose f_1 into an embedding h_1 of the singleton permutation into π, by simply looking to which inflated block order-isomorphic to ρ belongs the image of f_1, and an embedding h_2 of the singleton permutation into ρ, determined by restricting f_1 only to that copy of ρ. Since π belongs to A_C, there is a permutation σ' with embeddings g'_1 and g'_2 of π and τ such that $g'_1 \circ h_1 = g'_2 \circ f_2$.

Define $\sigma = \sigma'[\rho]$, and view σ as a concatenation of l blocks, each a copy of ρ. Let us define mapping g_1 by simply using g'_1 to map blocks of $\pi[\rho]$ to the blocks of σ, each element in $\pi[\rho]$ gets mapped to the same element of the corresponding copy of ρ in σ. Then define mapping g_2 by using g'_2 to map its elements to the blocks of σ and then within the copy of ρ to the single element in the image of h_2. It is easy to see that g_1 and g_2 are in fact embeddings of $\pi[\rho]$ and τ into σ. Also the images of $g_1 \circ f_1$ and $g_2 \circ f_2$ must lie in the same block of σ. And finally these images must be equal since we used h_2 to place the single element from the image of g_2 inside each block of σ.

We now show that if C is closed under $X[\cdot]$ then so is A_C. Fix a permutation $\rho \in X$ of order k, and a k-tuple $\pi_1, \dots, \pi_k$ of permutations from A_C. We will show that $\rho[\pi_1, \dots, \pi_k]$ belongs to A_C.

Fix a permutation $\tau \in C$ and two embeddings f_1 and f_2 of the singleton permutation into $\rho[\pi_1, \dots, \pi_k]$ and τ. We aim to find a permutation $\sigma \in C$ and two embeddings g_1 and g_2 of $\rho[\pi_1, \dots, \pi_k]$ and τ into σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We again view $\rho[\pi_1, \dots, \pi_k]$ as a concatenation of k blocks, the i-th block being order-isomorphic to π_i. Suppose that the image of f_1 is in the j-th block. Let us decompose f_1 into an embedding h_1 of the singleton permutation into ρ whose image is the j-th element of ρ, and an embedding h_2 of the singleton permutation into π_j. Since π_j belongs to A_C, there is a permutation σ' with embeddings g'_1 and g'_2 of π_j and τ such that $g'_1 \circ h_2 = g'_2 \circ f_2$.

Define $\sigma = \rho[\pi_1, \ldots, \pi_{j-1}, \sigma', \pi_{j+1}, \ldots, \pi_k]$ and let us define mapping g_1 in the following way. Every block of $\rho[\pi_1, \ldots, \pi_k]$ except for the j-th one gets mapped to the corresponding block of σ, and the j-th block is mapped using the embedding g'_1 to the j-th block of σ. Then define mapping g_2 simply by mapping τ to the j-th block of σ using g'_2. It is easy to see that both g_1 and g_2 are in fact embeddings of $\rho[\pi_1, \ldots, \pi_k]$ and τ into σ. Furthermore, the images of $g_1 \circ f_1$ and $g_2 \circ f_2$ both lie in the j-th block of σ. Their equality then follows from the construction since $g'_1 \circ h_2 = g'_2 \circ f_2$.

It remains to show that if C is inflation-closed then $A_C = C$. But if C is inflation-closed, then it is closed under $\cdot[C]$, so A_C is also closed under $\cdot[C]$. And since A_C trivially contains the singleton permutation, for every $\pi \in C$ we have that $\pi = 1[\pi]$ also belongs to A_C. □

Furthermore, we can also prove claim analogous to Lemma 2.5.

Lemma 2.13. *If C is a permutation class and D is an atomic permutation class, then $A_C[A_D] \subseteq A_{C[D]}$. Consequently, if C and D are any 1-amalgamable permutation classes, then so is $C[D]$.*

Proof. The second claim follows from the first due to Observation 2.11 as any 1-amalgamable class is trivially atomic.

Let π be an 1-amalgamable permutation in C of length k and $\sigma_1, \ldots, \sigma_k$ 1-amalgamable permutations in D. We aim to show that $\pi[\sigma_1, \ldots, \sigma_k]$ is 1-amalgamable in $C[D]$. Let γ be any permutation in $C[D]$. We can express γ as $\alpha[\beta_1, \ldots, \beta_l]$ for some $\alpha \in C$ and $\beta_1, \ldots, \beta_l \in D$. Since D is atomic, there is a permutation $\beta \in D$ that contains all the $\beta_1, \ldots, \beta_l$.

It suffices to show that for any two embeddings f_1 and f_2 of the singleton permutation into $\pi[\sigma_1, \ldots, \sigma_k]$ and $\alpha[\beta, \ldots, \beta]$ there is a permutation ψ with embeddings g_1 and g_2 of $\pi[\sigma_1, \ldots, \sigma_k]$ and $\alpha[\beta, \ldots, \beta]$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Fix some embeddings f_1 and f_2. As before, view $\pi[\sigma_1, \ldots, \sigma_k]$ as a concatenation of k blocks, the i-th block being order-isomorphic to σ_i. Suppose that the image of f_1 lies in the j-th block. We can decompose f_1 naturally into an embedding h_1 of the singleton permutation into π whose image is π_j and an embedding h_2 of the singleton permutation into σ_j. Similarly, suppose that the image of f_2 lies in the t-th block of $\alpha[\beta, \ldots, \beta]$. Analogously, decompose f_2 into an embedding b_1 of the singleton permutation into α whose image is α_t and an embedding b_2 of the singleton permutation into β.

Since $\sigma_j \in A_D$, there is a permutation $\rho \in D$ with embeddings c_2 and d_2 of σ_j and β such that $c_2 \circ h_2 = d_2 \circ b_2$. And since $\pi \in A_C$, there is a permutation $\tau \in C$ with embeddings c_1 and d_1 of π and α such that $c_1 \circ h_1 = d_1 \circ b_1$. Our goal is to inflate τ with ρ, all the σ_i for $i \in [k] \setminus \{j\}$ and $l-1$ copies of β at the right places and obtain the desired 1-amalgamation ψ.

First, inflate the element of τ in the image of $c_1 \circ h_1$ with ρ. Now we just need to correctly inflate the remaining elements of τ. Essentially, for every $i \in [k] \setminus \{j\}$ we know which element of τ to inflate with σ_i by looking at the embedding c_1. Similarly, we know which elements to inflate with a copy of β by looking at the embedding d_1. We are however not guaranteed that these are all different elements of τ, since c_1 and d_1 can share more than one common point. In this case, we can again use the atomicity of D to obtain permutation containing both β and σ_i for any i. We omit constructing the embeddings g_1 and g_2 of $\pi[\sigma_1, \ldots, \sigma_k]$ and $\alpha[\beta, \ldots, \beta]$ into ψ as it straightforwardly follows our construction of ψ itself. □

As noted by Jelínek and Valtr [10, Lemma 1.5], it follows from the results of Nešetřil [11] that if a permutation class C is unsplittable then C is also 1-amalgamable. Using the same argument, we get the following stronger proposition relating the classes U_C and A_C.

Proposition 2.14. *Let C be a permutation class, then $U_C \subseteq A_C$.*

Proof. Let π be an unavoidable permutation in C and let τ be a permutation from C. By the definition of U_C, there is a permutation $\sigma \in C$ such that any red-blue coloring of σ has a red copy of τ or a blue copy of π. We claim that σ contains every 1-amalgamation of π and τ. Suppose for a contradiction that there are two embeddings f_1 and f_2 of the singleton permutation 1 into π and τ such that are no embeddings g_1 and g_2 of π and τ into σ that would satisfy $g_1 \circ f_1 = g_2 \circ f_2$.

Let $f_1(1) = a$ and $f_2(1) = b$. We aim to color the elements of σ to avoid both a red copy of τ and a blue copy of π. We color an element σ_i red if and only if there is an embedding of π which maps π_a to σ_i. Trivially, we cannot obtain a blue copy of π, since we must have colored the image of π_a red. On the other hand, suppose we obtained a red copy of τ. Then the image of τ_b was painted red which means that there is an embedding of π which maps π_a to the same element. We assumed that such a pair of embeddings does not exist, therefore we defined a coloring of σ that contains neither a red copy of τ nor a blue copy of π. □

Using this result we obtain a large number of 1-amalgamable classes simply because we can prove that they are unsplittable. Recalling Theorem 2.7 and Theorem 2.8, we can state the following corollaries.

Corollary 2.15. *If a permutation σ is simple, then $Av(\sigma)$ is 1-amalgamable.*

Corollary 2.16. *If a permutation class C can be obtained by iterated inflations of $Av(12)$, $Av(132)$ and their symmetries, then C is 1-amalgamable.*

Previously, no other way of showing that class is 1-amalgamable was known. We aim to develop different methods in the following chapters to show that there is a 1-amalgamable class that is also splittable.

2.2.2 2-Amalgamability

We have seen that 1-amalgamability is a weaker property than unsplittability and thus surely than amalgamability. It is natural to ask where does 2-amalgamability fit in this landscape. We proceed to answer this by showing that there are classes that are 2-amalgamable but not 3-amalgamable. On the other hand, we will show that there are only finitely many 2-amalgamable subclasses of separable permutations, which proves that 2-amalgamability is stronger property than 1-amalgamability since there are infinitely many 1-amalgamable such subclasses (that follows from Corollary 2.16).

Note that it was remarked by Jelínek and Valtr [10] that the class Av(132) and the class of separable permutations are 2-amalgamable. As far as we know, the proofs of these claims do not appear anywhere in the literature so we include them here.

Proposition 2.17. *The class Av(213) and its symmetries Av(312), Av(231) and Av(132) are 2-amalgamable.*

Proof. It suffices to show that Av(213) is 2-amalgamable since 2-amalgamability is trivially preserved when considering the classes of reverses or complements.

Consider a non-empty permutation $\pi \in \text{Av}(213)$ of length n and suppose that $\pi_k = 1$. Observe that all the elements $\pi_1, \ldots, \pi_{k-1}$ are larger than all the elements $\pi_{k+1}, \ldots, \pi_n$ since otherwise we would obtain an occurrence of 213. This way we obtained a well-defined and unique decomposition of π into two shorter 213-avoiding permutations. Let us denote $L(\pi)$ the permutation corresponding to $\pi_1, \ldots, \pi_{k-1}$ and similarly $R(\pi)$ the permutation corresponding to $\pi_{k+1}, \ldots, \pi_n$. Note that we will use this notation for both the parts of the original permutation and the permutations order-isomorphic to them. On the other hand for any two 213-avoiding permutations π and τ, the permutation $312[\pi, 1, \tau]$ is also 213-avoiding.

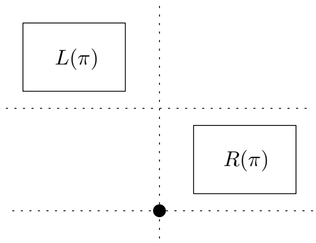


Figure 2.2: Decomposition of a non-empty 213-avoiding permutation π.

First, let us use this decomposition to show by induction that Av(213) is 1-amalgamable, which we already know from Corollary 2.16. Let π_1 and π_2 be two 213-avoiding permutations and f_1 and f_2 the mappings of the singleton permutation into π_1 and π_2. If at least one of the images of f_1 and f_2 does not coincide with the respective permutation's minimum we can use induction. We will discuss only one of the possible cases as they all use the very same argument. Suppose that the image of f_1 lies within $L(\pi_1)$. Let σ be the correct 1-amalgamation of $L(\pi_1)$ and π_2 which is guaranteed by the induction. It is easy to check that $312[\sigma, 1, \pi_2]$ is the desired 1-amalgamation of π_1 and π_2. And if both images of f_1 and f_2 coincide with the respective permutation's minima, we can get the desired 1-amalgamation as $54132[L(\pi_1), L(\pi_2), 1, R(\pi_1), R(\pi_2)]$.

Now we will extend this argument to show that Av(213) is indeed also 2-amalgamable. Again let π_1 and π_2 be two 213-avoiding permutations and f_1 and f_2 the mappings of either 12 or 21 into π_1 and π_2.

Case 1. If the whole image of one of the embeddings lies in the same part of the decomposition we can use induction similarly as before. Again we will consider only the case when the image of f_1 lies in $L(\pi_1)$. Let σ be the correct 2-amalgamation of $L(\pi_1)$ and π_2. And we obtain the desired 2-amalgamation as $312[\sigma, 1, R(\pi_1)]$. See Figure 2.3.

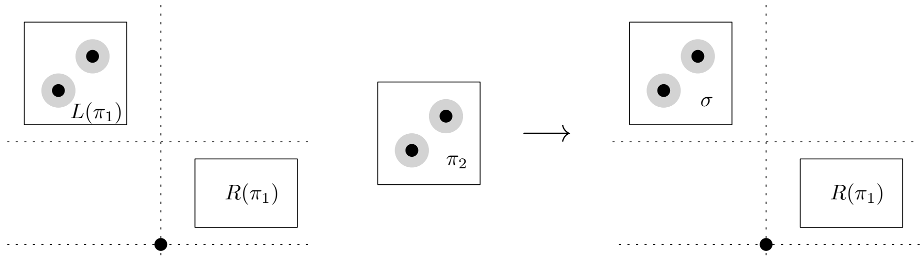


Figure 2.3: The 2-amalgamation of π_1 and π_2 when the whole image of f_1 lies inside $L(\pi_1)$.

Case 2. Now suppose that the images of both f_1 and f_2 intersect both parts of the respective decompositions. We can use 1-amalgamation to glue together $L(\pi_1)$ and $L(\pi_2)$, and similarly for $R(\pi_1)$ and $R(\pi_2)$. Let σ_1 be the correct 1-amalgamation of $L(\pi_1)$ and $L(\pi_2)$, analogously σ_2 for $R(\pi_1)$ and $R(\pi_2)$. We obtain the desired 2-amalgamation as $312[\sigma_1, 1, \sigma_2]$. Notice that this is in fact a 3-amalgamation that also identifies the minima of π_1 and π_2. See Figure 2.4.

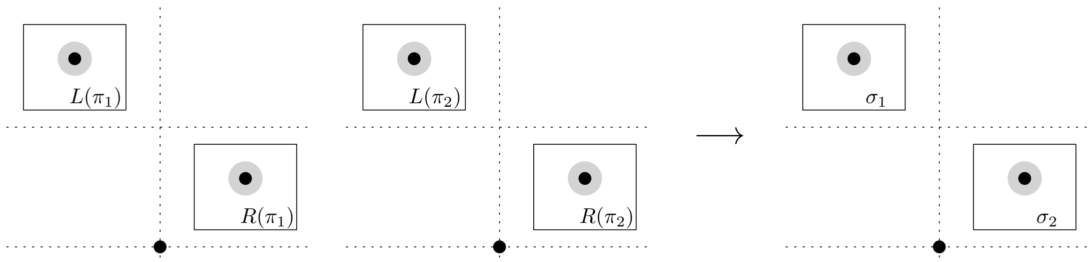


Figure 2.4: The 2-amalgamation of π_1 and π_2 when the images of both embeddings intersect both parts of the respective decompositions.

Case 3. Finally, we are left with the case when the images of both f_1 and f_2 contain the respective permutation's minimum. We are left with two possibilities. If these are embeddings of the permutation 21 then the other elements in their images lie in $L(\pi_1)$ and $L(\pi_2)$. Conversely, if these are embeddings of the permutation 12 then the other elements in their images lie in $R(\pi_1)$ and $R(\pi_2)$. As before, let us discuss only the first case. We know that there is a correct 1-amalgamation σ of $L(\pi_1)$ and $L(\pi_2)$. We obtain the desired 2-amalgamation as $4132[\sigma, 1, R(\pi_1), R(\pi_2)]$. See Figure 2.5.

□

Recall, that the class of separable permutations Sep is defined as the smallest non-empty class closed under taking both direct and skew sums. We can show that the class of all separable permutations is also 2-amalgamable. First, let us introduce some useful notation.

Definition 2.18. For every permutation $\pi \in \mathsf{Sep}$, we define a *decomposition tree* $T(\pi)$. A decomposition tree $T(\pi)$ is an ordered tree, i.e. rooted tree with an

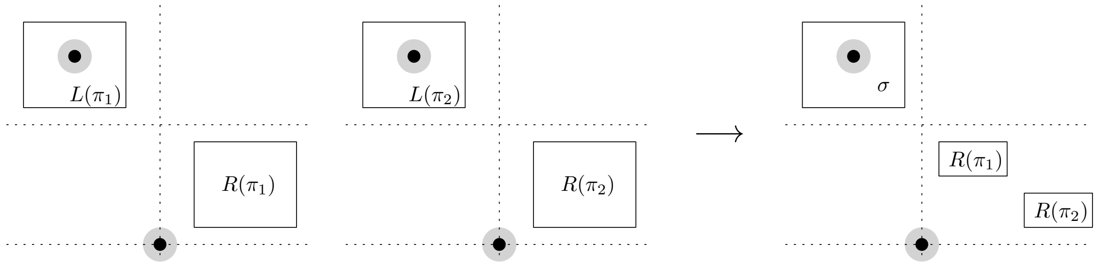


Figure 2.5: The 2-amalgamation of π_1 and π_2 when the images of both embeddings contain the respective permutation's minimum.

ordering specified for the children of each node, with two types of internal nodes: $\oplus$-nodes and $\ominus$-nodes, and leaves corresponding to the elements of π. Moreover, every child of a $\oplus$-node is a $\ominus$-node or a leaf, and every child of a $\ominus$-node is a $\oplus$-node or a leaf. We define the the tree $T(\pi)$ recursively: the singleton permutation is represented by a tree consisting of single leaf. For a sum-decomposable permutation of the form $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ where π_i is sum-indecomposable, $T(\pi)$ has an $\oplus$-node as its root with k children, whose subtrees are $T(\pi_1), \ldots, T(\pi_k)$. We construct the decomposition trees of sum-indecomposable permutations analogously. Note that ordered tree naturally induces a total order on its leaves. We will sometimes refer to the i-th leaf of a decomposition tree T.

It will be sometimes useful to consider similar trees without the condition on alternating node types on any path to root. A *relaxed decomposition tree* is an ordered tree consisting of internal $\oplus$-nodes, $\ominus$-nodes and leaf nodes. We say that such tree T *represents* π, if we obtain π by applying recursively the operations determined by node labels. Note that while for every $\pi \in \mathsf{Sep}$ there is a unique decomposition tree $T(\pi)$, there can be infinitely many relaxed decomposition trees that represent π since we allow internal nodes with a single child. We say that two relaxed decomposition trees T_1 and T_2 are *equivalent* if they represent the same separable permutation.

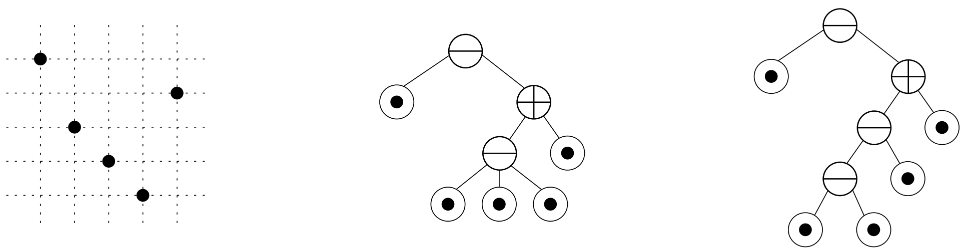


Figure 2.6: Separable permutation 53214, its decomposition tree and one of its possible relaxed decomposition trees.

The *least common ancestor* of two distinct leaves in a tree T is the unique internal node of maximum depth which is a common ancestor of the two leaves.

Observation 2.19. *Let π be a separable permutation of length n and T a relaxed decomposition representing π. Let π_i, π_j, where $i < j$, be its two elements with corresponding leaves v_i and v_j in T. We have $\pi_i < \pi_j$ if and only if the least common ancestor of v_i and v_j is a $\oplus$-node.*

Using this simple observation we can derive simple but very useful tool for proving the equivalence of two relaxed decomposition trees.

Lemma 2.20. *Two relaxed decomposition trees T_1 and T_2 are equivalent if and only if they have the same number of leaves and for every $i < j$ the least common ancestor of the i-th and j-th leaf in T_1 and T_2 are of the same type.*

Proof. Two permutations π and τ are order-isomorphic if and only if they are of the same length and for every $i < j$ we have $\tau_i < \tau_j$ if and only if $\pi_i < \pi_j$. Thus, we obtain this result by applying the previous observation to every such pair of indices. □

Proposition 2.21. *The class* `Sep` *is 2-amalgamable.*

Proof. First recall that `Sep` is 1-amalgamable since it is wreath-closed. As with other wreath-closed classes we can obtain one amalgamation of π and τ simply by inflating one with the other. It is useful to observe what that means in the language of decomposition trees. If we inflate an element of π with τ, we simply replace the corresponding leaf in $T(\pi)$ with $T(\tau)$. Note that the result does not have to be a decomposition tree as we may have introduced a parent-child node pair of the same type. However, it is a relaxed decomposition tree that represents the desired inflation.

Let π and τ be separable permutations and f_1 and f_2 the mappings of either 12 or 21 into π and τ. Let π_i and π_j be the elements of π in the image of f_1 and similarly, let τ_k and τ_l be the elements of τ in the image of f_2. Furthermore, let v_i and v_j be the corresponding leaves in $T(\pi)$ and u_k and u_l be the corresponding leaves in $T(\tau)$.

Case 1. First, consider the case when both the least common ancestor of v_i, v_j and the least common ancestor of u_k, u_l are roots of their respective decomposition trees. Let s be the child of root in $T(\tau)$ such that u_k lies in the subtree T_s rooted in s. Analogously, let t be the child of root in $T(\tau)$ such that u_l lies in the subtree T_t rooted in t. Note that since the root is the least common ancestor of u_k and u_l, s and t are different nodes.

We define a new relaxed decomposition tree T' as follows: replace the leaf v_i with the tree T_s and the leaf v_j with the tree T_t. Furthermore, we need to add the remaining children of the root in $T(\tau)$. However, they can be added with some care as the children of root in T'. Essentially, we only need them to retain their relative order and relative order to the nodes, that contain the subtrees T_s and T_t. For example see Figure 2.7.

Let σ be the permutation represented by T'. We define the embeddings of π and τ into σ by mapping the leaves of their respective decomposition trees to the leaves of T'. The leaves of $T(\tau)$ map trivially to the corresponding leaves added during the construction of T'. All the leaves of $T(\pi)$ except for v_i and v_j map to themselves as the are retained during the construction of T'. And v_i from $T(\pi)$ maps to the leaf corresponding to u_k in T' and analogously for v_j and u_l.

Clearly, the mappings identify the images of u_i, u_j with the images of v_k, v_l. It remains to show, that they are in fact embeddings of π and τ. We will consider the restrictions of T' to the images of the respective mappings and check that they are equivalent to the original decomposition trees using Lemma 2.20. Consider, the tree T' restricted to the leaves in the image of the map from $T(\pi)$. This tree

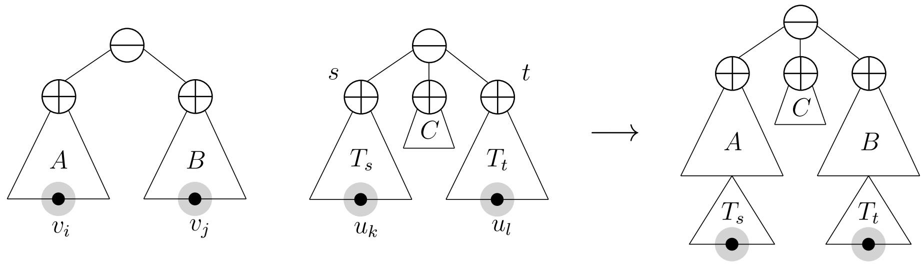


Figure 2.7: Example of 2-amalgamation of separable π and τ when the least common ancestors of both v_i, v_j and u_k, u_l are the roots of the respective decomposition trees.

trivially preserves the types of the least common ancestor nodes for every leaf pair. Now, consider the restriction of T' to the leaves in the image of the map from $T(\tau)$. We preserved the least common ancestor node type for any leaf pair whose least common ancestor in $T(\tau)$ is not its root. For all the remaining leaf pairs, their least common ancestor in T' is also its root which is of the same type as the root of $T(\pi)$. However, it follows from the definition of amalgamation that $\pi_i < \pi_j$ if and only if $\tau_k < \tau_l$ which means that the roots of $T(\pi)$, $T(\tau)$ and T' are necessarily of the same type.

Case 2. Now, consider the case that either the least common ancestor of v_i, v_j or the least common ancestor of u_k, u_l is not a root of the respective decomposition tree. Let s be the least common ancestor of v_i, v_j and t the least common ancestor of u_i, u_j. Furthermore, let T_s be the subtree rooted in s and T_t the subtree rooted in t. Finally, let T_1 be the tree obtained from $T(\pi)$ by replacing the whole T_s with a leaf node w_1, analogously T_2 the tree obtained from $T(\tau)$ by replacing the whole T_t with a leaf node w_2.

The main idea is to take 1-amalgamation of the permutations represented by T_1 and T_2 through the new leaf nodes w_1 and w_2 and inflate it with 2-amalgamation of T_s and T_t. Let T_3 be a decomposition tree representing 1-amalgamation of the permutations represented by T_1 and T_2 through w_1 and w_2 and let w be its leaf which both of them map to. We can construct 2-amalgamation of the permutations represented by T_s and T_t with natural restrictions of the embeddings due to the previous case. Let T_4 be the relaxed decomposition tree of this 2-amalgamation. Finally, we construct a relaxed decomposition tree T by replacing the leave w in T_3 with T_4.

We claim that T represents a desired 2-amalgamation of π and τ. This time we omit definitions of the mappings from $T(\pi)$ and $T(\tau)$ as they are very straightforward. As before, it is easy to check that the types of least ancestor nodes are preserved for all leaf pairs. And applying Lemma 2.20 we again infer that the mappings are in fact embeddings.

□

Finally, we are ready to list all the 2-amalgamable subclasses of separable permutations.

Theorem 2.22. *A subclass of separable permutations is 2-amalgamable if and only if it is amalgamable or it is one of the following: class Av*(213)*, its symmetries Av*(312)*, Av*(231) *and Av*(132)*, or the class* `Sep` *itself.*

Proof. Let C be a non-empty 2-amalgamable subclass of separable permutations. Clearly, C must contain the singleton permutation. Now consider the permutations of length 2. If C does not contain the permutation 12 then it must contain all decreasing permutations since it is 1-amalgamable. And the class Av(12) is trivially k-amalgamable for any k. Analogously, if C does not contain the permutation 21 the it must be equal to Av(21), which is again amalgamable.

Suppose that C contains both permutations of length 2 and let us continue with permutations of length 3. Since C is 1-amalgamable it must contain all the possible 1-amalgamations of 12 and 21. For each possible combination of embeddings of the singleton permutation into 12 and 21 we have two possible 1-amalgamations. See Figure 2.8.

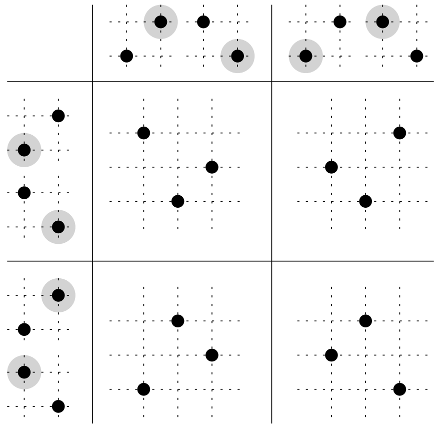


Figure 2.8: All possible 1-amalgamations of 12 and 21, each row and column is indexed by the combination of embeddings that may create permutations in that row or column.

One permutation from each row and column must be contained in C. Suppose first that C contains only 2 permutations from 312, 213, 231 and 132. These two patterns must cover all rows and columns which implies that C contains one of the diagonal pairs. Let us suppose that C contains 132, 213 and avoids both 312 and 231. We claim that since 132 belongs to C, for every $\pi \in C$ we also have $1 \oplus \pi \in C$. To see this consider taking a 2-amalgamation of 132 and π through its first and minimal element. Analogously, for every π contained in C we have $\pi \oplus 1 \in C$ because 213 belongs to C. It follows that C is sum-closed as for any $\pi, \sigma \in C$, we can take 1-amalgamation of $\pi \oplus 1$ and $1 \oplus \sigma$. As we argued before, C is a superclass of Av(12) and the smallest such class that is also sum-closed is the class of layered permutations, which is an amalgamable class. Symmetrically, if C contains 312 and 231 then it must be a superclass of all co-layered permutations.

Now, suppose that C contains 3 permutations from Figure 2.8. Let us suppose that C avoids 213 and contains 312, 231 and 132. Our previous arguments imply that C is skew-closed and moreover for every $\pi \in C$ it also contains $1 \oplus \pi$. Therefore, for any π and σ contained in C, we know that also $\pi \ominus (1 \oplus \sigma)$ belongs to C. Recalling the decomposition of 213-avoiding permutations used in the proof of Proposition 2.17, we see that the smallest class with these properties is Av(213) which we already proved to be 2-amalgamable. Symmetrically, if C contains 213 but avoids exactly one of its symmetries σ then C must contain the whole class Av(σ).

And finally, suppose that C contains all permutations of length 3. Using the same arguments we see that C must be both sum-closed and skew-closed. The only such non-empty subclass of separable permutations is `Sep` itself. □

It is unclear if there is any 2-amalgamable class that is not contained in `Sep`, apart from the class of all permutations. And if the answer is positive then the natural question is whether there are infinitely many such classes. Notice that this would also imply that infinitely many classes exist that are 2-amalgamable but not 3-amalgamable.

3. Left-to-right minima

In this chapter, we will consider the operation of inflating not all the elements of a permutation but only a specific subset, in our case the left-to-right minima. Our goal is to understand how splittability and 1-amalgamability behaves when looking at classes that are closed under these inflations.

We say that the element π_i *covers the element* π_j if $i < j$ and simultaneously $\pi_i < \pi_j$. The i-th element of a permutation π is then a *left-to-right minimum*, or shortly LR-minimum, if it is not covered by any other element. We denote the set of left-to-right minima indices of π as $LR\text{-}min(\pi)$.

Similarly we could define LR-maxima, RL-minima and RL-maxima. However we can easily translate between right-to-left and left-to-right orientation by looking at the reverses of the permutations, and similarly between maxima and minima by looking at the complements of the permutations. Therefore we restrict ourselves to dealing only with LR-minima from now on.

Definition 3.1. Suppose that $\pi \in S_n$ is a permutation with k LR-minima and let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of non-empty permutations. The *LR-inflation* of π by the sequence $\sigma_1, \ldots, \sigma_k$ is the inflation of LR-minima of π by $\sigma_1, \ldots, \sigma_k$. We denote this by $\pi\langle\sigma_1, \ldots, \sigma_k\rangle$.

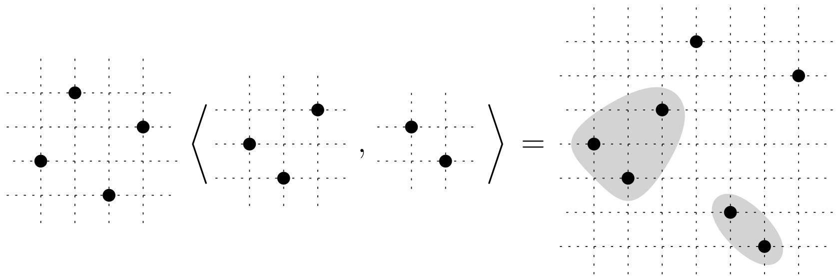


Figure 3.1: An example of LR-inflation: $2413\langle 213, 21\rangle = 4357216$.

Definition 3.2. We say that a permutation class C is *closed under LR-inflations* if for every $\pi \in C$ with k LR-minima, and for every k-tuple $\sigma_1, \ldots, \sigma_k$ of permutations from C, the LR-inflation $\pi\langle\sigma_1, \ldots, \sigma_k\rangle$ belongs to C. The *closure of* C *under LR-inflations* (or just *LR-closure* of C), denoted C^{LR}, is the smallest class which contains C and is closed under LR-inflations.

Recall that one can characterize inflation-closed classes by basis that consists of simple permutations. We can derive a similar characterization in the case of classes closed under LR-inflations. We say that a permutation is *LR-simple* if it cannot be obtained by LR-inflations except for the trivial ones. Using the same arguments, it is easy to see that a permutation class is closed under LR-inflations if and only every permutation in its basis is LR-simple. We can actually state a stronger proposition. For a permutation class C we can characterize the basis of its LR-closure given the basis of C.

Proposition 3.3. *For a set of permutations F, the LR-closure of $Av(F)$ is equal to $Av(F')$, where*

$$F' = \{\sigma \mid \sigma \text{ is LR-simple and } \exists \tau \in F : \sigma \text{ contains } \tau\}.$$

Proof. As we have already observed, the class $\text{Av}(F')$ is LR-closed since every permutation in its basis is LR-simple. Moreover, $\text{Av}(F')$ contains the whole class $\text{Av}(F)$ because every permutation in F' contains some permutation from F. This implies that the LR-closure of $\text{Av}(F)$ lies within $\text{Av}(F')$.

For the second inclusion, observe that every LR-simple permutation from $\text{Av}(F')$ cannot contain any permutation from F and thus all the LR-simple permutations of $\text{Av}(F')$ belong to $\text{Av}(F)^{\text{LR}}$ as well. And since $\text{Av}(F)^{\text{LR}}$ is closed under LR-inflations, $\text{Av}(F)^{\text{LR}}$ must also contain all the remaining permutations from $\text{Av}(F')$. □

3.1 LR-splittability

We aim to define a stronger version of splittability that would help us connect the properties of permutation classes and their LR-closures. A natural way to do that is to consider an operation similar to the regular merge, with LR-minima being shared between both parts.

Definition 3.4. We say that a permutation π is a *LR-merge* of permutations τ and σ, if its non LR-minimal elements can be partitioned into two disjoint sequences, such that one of them is, together with the sequence of LR-minima of π, an occurrence of τ, and the other is, together with the sequence of LR-minima of π, an occurrence of σ. For two permutation classes A and B, we write $A \odot_{\text{LR}} B$ for the class of all LR-merges of a permutation from A with a compatible permutation from B. Trivially, $A \odot_{\text{LR}} B$ is again a permutation class.

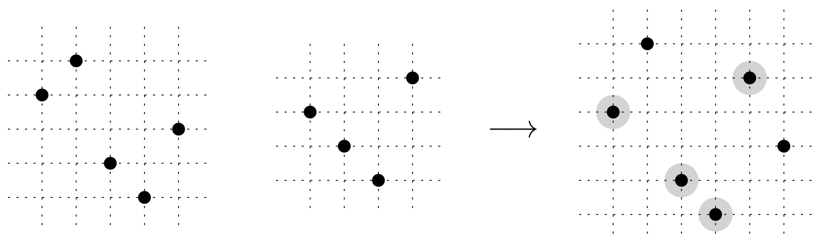


Figure 3.2: For example one possible LR-merge of 45213 and 3214 is the permutation 462153. The corresponding embedding of 3214 is indicated.

Note that we can also look at LR-merges as a special red-blue coloring of the permutation π in which the LR-minima are both blue and red at the same time. Naturally we can use this definition of LR-merge to define LR-splittability in the same way regular merge gives rise to the definition of splittability.

Definition 3.5. We say that a multiset of permutation classes $\{P_1, \ldots, P_m\}$ forms a *LR-splitting* of a permutation class C if $C \subseteq P_1 \odot_{\text{LR}} \cdots \odot_{\text{LR}} P_m$. We call P_i the *parts* of the LR-splitting. The LR-splitting is *nontrivial* if none of its parts is a superset of C, and the LR-splitting is *irredundant* if no proper submultiset of $\{P_1, \ldots, P_m\}$ forms a LR-splitting of C. A permutation class C is then *LR-splittable* if C admits a nontrivial LR-splitting.

Clearly, every LR-splittable class is splittable. Moreover, some properties of LR-splittability are analogous to the properties of splittability, as shown by the following lemma. We omit the proof as it uses the very same (and easy) arguments as the proof of Lemma 2.1.

Lemma 3.6. *For a class C of permutations, the following properties are equivalent:*

(a) C is LR-splittable.

(b) C has a nontrivial LR-splitting into two parts.

(c) C has a LR-splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial LR-splitting into two parts, in which each part is a principal class.

Now we can state some of the results that connect splittability and LR-splittability of permutation classes and their LR-closures.

Proposition 3.7. *Let C be a permutation class that is closed under LR inflations. Then C is splittable if and only if C is LR-splittable.*

Proof. Trivially, LR-splittability implies splittability since we can take the corresponding red-blue coloring and simply assign arbitrary color to each of the LR-minima. Now suppose that C admits splitting $\{D, E\}$ for some proper subclasses D and E. We aim to prove that also $C \subseteq D \odot_{\text{LR}} E$. Let us first show that C contains a permutation τ that belongs neither to D nor to E. From the definition of splittability, there are permutations $\tau_D \in C \setminus D$ and $\tau_E \in C \setminus E$. Define τ as LR-inflation of τ_D with τ_E, which clearly lies outside both subclasses D and E.

Let us suppose that there is $\pi \in C$ not belonging to $D \odot_{\text{LR}} E$, i.e., there is no red-blue coloring of π which proves it is a LR-merge of a permutation $\alpha \in D$ and a permutation $\beta \in E$. Let π' be the permutation created by inflating each LR-minimum of π with τ. Since π' belongs to C, it has a regular red-blue coloring with the permutation corresponding to the red elements $\pi'_R \in D$ and the permutation corresponding to the blue elements $\pi'_B \in E$. However there must be both colors in each block created by inflating a LR-minimum of π with τ, and therefore there is a valid red-blue coloring of π that assigns both colors to the LR-minima. □

Finally, we want to show that, under modest assumptions, the LR-splittability of a permutation class implies the LR-splittability (and thus the splittability) of its LR-closure. Notice the similarity of the following proposition (and its proof) to Lemma 2.6.

Proposition 3.8. *If C, D and E are permutation classes satisfying $C \subseteq D \odot_{\text{LR}} E$, then $C^{\text{LR}} \subseteq D^{\text{LR}} \odot_{\text{LR}} E^{\text{LR}}$. Consequently, if D^{LR} and E^{LR} do not contain the whole class C, then its closure C^{LR} is LR-splittable into parts D^{LR} and E^{LR}.*

Proof. We will inductively construct a valid red-blue coloring which proves that $C^{\text{LR}} \subseteq D^{\text{LR}} \odot_{\text{LR}} E^{\text{LR}}$. First, any $\pi \in C^{\text{LR}}$ that cannot be obtained from shorter permutations using LR-inflations must belong to C and we simply use the red-blue coloring that witnesses the inclusion $C \subseteq D \odot_{\text{LR}} E$.

Now take $\pi \in C^{\text{LR}}$ that can be obtained by LR-inflation from shorter permutations as $\pi = \alpha\langle\beta_1, \ldots, \beta_k\rangle$. We can already color the permutation α and all the permutations β_i and we construct a coloring of π in the following way: color the inflated blocks β_i according to the coloring of β_i and the remaining uninflated elements of α get the color according to the coloring of α. It remains to show that the permutation π_R corresponding to the red elements of π belongs to D^{LR} and the permutation π_B corresponding to the blue elements of π belongs to E^{LR}. Since the LR-minima of α are both red and blue, the permutation π_R is an LR-inflation of the red elements of α by the red elements of the permutations β_i. All these permutations belong to D^{LR} and thus their LR-inflation also belongs to D^{LR}. Using the very same argument we can show that π_B belongs to E^{LR}.

It remains to prove the second part of the statement by showing that in such case the splitting is nontrivial. However that follows easily from the assumption that both D^{LR} and E^{LR} do not contain the whole class C. □

3.2 LR-amalgamability

Similarly to the situation with LR-splittability we want to describe a property of permutation classes which would imply 1-amalgamability of their respective LR-closures.

Definition 3.9. We say that a permutation class C is *LR-amalgamable* if for any two permutations $\tau_1, \tau_2 \in C$ and any two mappings f_1 and f_2, where f_i is an embedding of the singleton permutation into τ_i and its image is not a LR-minimum of τ_i, there is a permutation $\sigma \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of τ_i into σ, $g_1 \circ f_1 = g_2 \circ f_2$ and moreover g_i preserves the property of being a LR-minimum.

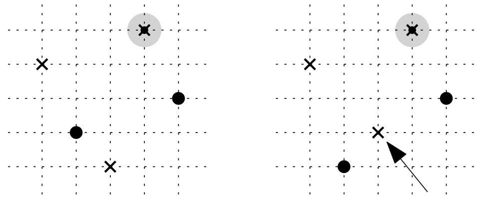


Figure 3.3: Two of the possible 1-amalgamations of 213 and 132 through the letter 3 are 42153 and 41253. Only 42153 is a LR-amalgamation since the letter 1 of 213 is no longer a LR-minimum of 41253.

Observe that LR-amalgamability does not imply 1-amalgamability since it does not guarantee 1-amalgamation over LR-minima and 1-amalgamability does not imply LR-amalgamability because it may not preserve the property of being a LR-minimum. Recall that we derived equivalence between LR-splittability and splittability for LR-closed classes in Proposition 3.7. In this case, we can still prove that LR-amalgamability implies 1-amalgamability for classes that are closed under LR-inflations. However, we can show the converse only by introducing further assumptions about the permutation class itself.

Lemma 3.10. *Let C be a permutation class that is closed under LR-inflations. If C is LR-amalgamable then C is also 1-amalgamable.*

Proof. Let π_1 and π_2 be arbitrary permutations from C and f_1, f_2 embeddings of the singleton permutation, f_i into the permutation π_i. If neither of the images of f_1 and f_2 is a LR-minimum of the respective permutation we obtain their 1-amalgamation directly since C is LR-amalgamable.

Now we can assume without loss of generality that the single element in the image of f_1 is a LR-minimum of π_1. We can create the resulting 1-amalgamation by simply inflating this LR-minimum by the permutation π_2. It is then easy to derive the mappings g_1 and g_2 that show it is the desired 1-amalgamation. □

Lemma 3.11. *Let C be a permutation class that is closed under LR-inflations and such that there is a permutation $\sigma \in C$ with $1 \oplus \sigma \notin C$. If C is 1-amalgamable then C is also LR-amalgamable.*

Proof. Let $\pi_1, \pi_2 \in C$ be permutations and f_1, f_2 embeddings of the singleton permutation, f_i into π_i such that the image of f_i avoids the LR-minima of π_i. First, let us show that 321 belongs to C. Suppose C does not contain 21. Then C is either an empty class or just $\{1\}$, both of them trivially 1-amalgamable and LR-amalgamable, or it contains 12. However, this means that C is equal to Av(21) due to 1-amalgamability and Av(21) is again both 1-amalgamable and LR-amalgamable. We are left with the case when C contains 21 and we can create 321 either by 1-amalgamation or LR-inflation.

Since C contains 321, it must also contain $\tau = 321\langle\sigma, 1, \sigma\rangle$ as it closed under LR-inflations. Let π_1' and π_2' be the permutations created in the following way: π_i' is obtained from π_i by inflating all its LR-minima with τ. Let h_1 and h_2 be embeddings of π_1 and π_2 into π_1' and π_2' such that the elements of π_i that are not LR-minima get mapped to the respective elements of π_i' and the LR-minima of π_i get mapped to the middle element of the corresponding copy of τ. Finally, let f_1' and f_2' be an embeddings of the singleton permutation into π_1' and π_2' defined as $f_i' = h_i \circ f_i$.

As C is 1-amalgamable, there is a permutation $\rho \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of π_i' into ρ, and $g_1 \circ f_1' = g_2 \circ f_2'$. Notice that C is trivially also 1-amalgamation of π_1 and π_2 with embeddings $g_i \circ h_i$ since $g_1 \circ h_1 \circ f_1 = g_2 \circ h_2 \circ f_2$. Our goal is to modify the embeddings $g_i \circ h_i$ such that they preserve the property of being LR-minimum.

We define mappings g_1' and g_2' in the following way: if j-th element of π_i is one of its LR-minima that does not get preserved under $g_i \circ h_i$ then we map it on an arbitrary LR-minimum that covers its image under $g_i \circ h_i$, other elements simply map according to $g_i \circ h_i$. Note that we did not break the amalgamation

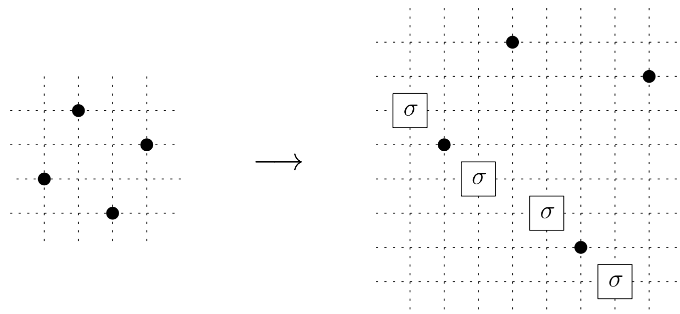


Figure 3.4: Example of inflating the LR-minima of 2413 with $321\langle\sigma, 1, \sigma\rangle$.

property, i.e. $g'_1 \circ f_1 = g'_2 \circ f_2$, as we remapped only LR-minima. While these mappings clearly preserve the property of being LR-minima it is not clear that they are still embeddings of π_1 and π_2.

As the cases are symmetric, let us assume that g'_1 is no longer an embedding of π_1. Observe that the images of the remapped LR-minima can be only to the left and down from their images under $g_1 \circ h_1$. Since g'_1 is not an embedding, there must be a LR-minimum of π_1 that got remapped too far to the left or down, in particular further then its left neighbor in π_1 or the largest of the elements that are smaller. Since this LR-minimum was inflated by $321\langle\sigma, 1, \sigma\rangle$ in π'_1, its image under g'_1 must form an occurrence of $1 \oplus \sigma$ with at least one of these two copies of σ which is clearly a contradiction. □

We conclude this section by relating LR-amalgamability of a permutation class and 1-amalgamability of its LR-closure.

Proposition 3.12. *If a permutation class C is LR-amalgamable then its LR-closure C^{LR} is LR-amalgamable and thus also 1-amalgamable.*

Proof. Let $\pi_1, \pi_2 \in C^{\text{LR}}$ be permutations and f_1, f_2 embeddings of the singleton permutation, f_i into π_i such that the image of f_i avoids the LR-minima of π_i. We aim to prove by induction on the length of π_1 and π_2 that there is a corresponding LR-amalgamation of π_1 and π_2. Consider two cases. If neither of the two permutations π_1 and π_2 can be obtained as a LR-inflation of a shorter permutation then they both belong to C. And since C itself is LR-amalgamable they have a desired LR-amalgamation that belongs to C.

Without loss of generality we can now assume that π_1 can be obtained by LR-inflations as $\pi_1 = \alpha\langle\beta_1, \dots, \beta_k\rangle$ where the permutations $\alpha, \beta_1, \dots, \beta_k$ are all strictly shorter than π_1. Again we consider two separate cases. First, assume that the image of the embedding f_1 lies inside the block corresponding to the j-th inflated LR-minimum of α, which is order-isomorphic to β_j. From induction we get a LR-amalgamation σ of β_j and π_2 for the embeddings f'_1 and f_2, where f'_1 is the embedding f_1 restricted to the inflated block of β_j. Observe that the permutation $\alpha\langle\beta_1, \dots, \beta_{j-1}, \sigma, \beta_{j+1}, \dots, \beta_k\rangle$ is precisely the LR-amalgamation of π_1 and π_2 we were looking for.

Finally we have to deal with the situation when the image of the embedding f_1 lies outside of the blocks corresponding to the inflated LR-minima of π_1. We can obtain from induction a LR-amalgamation σ of α and π_2 for the embeddings

f_1'' and f_2, where f_1'' is the embedding f_1 restricted to the permutation α. Let g_1 be the corresponding embedding of α into σ that preserves the LR-minima. We construct the desired LR-amalgamation of π_1 and π_2 in the following way: take σ and for every LR-minimum of α inflate its image under g_1 with the corresponding permutation β_i. The resulting permutation is clearly a 1-amalgamation of π_1 and π_2, and it also preserves the LR-minima.

Lemma 3.10 implies that C^{LR} is also 1-amalgamable. □

4. Connecting splittability and 1-amalgamability

In this chapter, we are finally ready to prove that 1-amalgamability and splittability are not equivalent. First we show that the class Av(1423, 1342) is both 1-amalgamable and splittable, and later we use this class to construct infinitely many classes that are also 1-amalgamable and splittable.

4.1 Single class

We shall first show that Av(1423, 1342) is a LR-closure of a simple principal class, which allows us to use the tools derived in Chapter 3 to prove its splittability and 1-amalgamability.

Proposition 4.1. *The class Av(1423, 1342) is the closure of Av(123) under LR-inflation.*

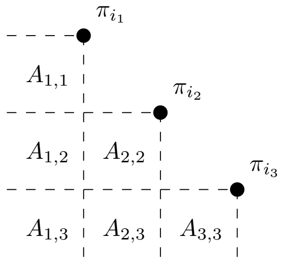
Proof. First, let us show that any permutation from the LR-closure of Av(123) avoids both 1423, 1342. Because both of these patterns contain 123, they would have to be created by the LR-inflations. However, that is not possible since there is no nontrivial interval in either 1423 or 1342 which contains the minimum element.

Now, let π be a permutation from Av(1423, 1342). We will show by induction that this permutation can be obtained by a repeated LR-inflation of permutations from Av(123). If π does not contain 123 the statement is trivially true. Otherwise, consider the set of the right-to-left maxima of π. We want to show that the remaining elements of π can be split into a descending sequence of intervals. If this holds then we can get π as an LR-inflation of 123-avoiding permutation by permutations order-isomorphic to the intervals. And by induction these shorter permutations can be obtained as repeated LR-inflations of 123-avoiding permutations.

Let us show that there is no occurrence of the pattern 132 that maps only the letter 2 on an RL-maximum. For a contradiction suppose we have such an occurrence and a corresponding embedding f of 132 into π. Then there must be an element covered by $\pi_{f(3)}$ since it is not an RL-maximum, i.e., an element π_k such that $k > f(3)$ and $\pi_k > \pi_{f(3)}$. However, π restricted to these four indices would form the pattern 1342. Using the same argument, we can also show that there is no occurrence of the pattern 132 which maps only the letter 3 on an RL-maximum as we would get an occurrence of the pattern 1423 together with the RL-maximum covered by the image of 2.

And finally, we conclude by showing that the elements of π that are not RL-maxima can indeed be split into a descending sequence of intervals. Let $I = \{i_1, \dots, i_m\}$ be the index set of the RL-maxima of π and furthermore define $i_0 = 0$ and $\pi_0 = n + 1$. Let us represent the remaining elements of π as a set A of $n - m$ points on a plane

$$A = \{(i, \pi_i) \mid \pi_i \text{ is not a RL-maximum of } \pi\}.$$



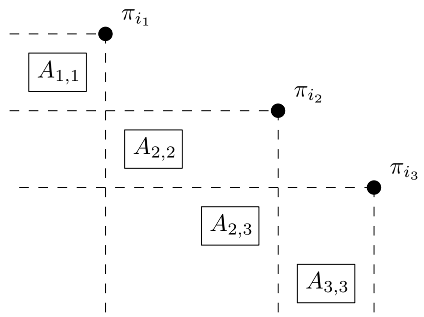


Figure 4.1: Partition of a general permutation with 3 RL-maxima into the sets $A_{j,k}$ and an example how the non-empty sets might look for some $\pi \in \mathrm{Av}(1423, 1342)$.

We define a partition of A into sets $A_{j,k}$ for any $1 \leq j < k \leq m$

$$A_{j,k} = \{(x, y) \mid (x, y) \in A \text{ and } i_{j-1} < x < i_j \text{ and } \pi_{i_k} < y < \pi_{i_{k-1}}\}.$$

For any j, k and l, every element of $A_{j,k}$ is larger than all the elements of $A_{j+1,l}$ in the second coordinate since otherwise we would get a 132 occurrence with the letter 3 mapped to π_{i_j}. Similarly for any j, k and l, every element of $A_{j,k}$ is to the left of all the elements of $A_{l,k+1}$ as otherwise we would get a 132 occurrence with the letter 2 mapped to π_{i_k}. This transitively implies that all non-empty sets $A_{j,k}$ correspond to a sequence of descending intervals. $\square$

Note that we could alternatively use Proposition 3.3 to derive a shorter proof. However, the provided proof gives more insight into the structure of this class.

In order to show that Av(1423, 1342) is splittable, we shall first prove the LR-splittability of Av(123) and then apply the results we have obtained in Section 3.1.

Lemma 4.2. *The class Av(123) is LR-splittable, and more precisely, it satisfies*

$$Av(123) \subseteq Av(463152) \odot_{\text{LR}} Av(463152).$$

Proof. Let π be a permutation from Av(123). Clearly π is a merge of two descending sequences, its LR-minima and the remaining elements. The idea is to decompose the non-minimal elements into runs such that for every run there is a LR-minimum covering each element of the run but none from the following run. This can be done easily by the following greedy algorithm. In one step of the algorithm, let π_i be the first non-minimal element which was not used yet and let j be the maximum integer such that π_j is a LR-minimum covering π_i. The next run then consists of all non-minimal elements starting from π_i that are covered by π_j.

We color each run blue or red such that adjacent runs have different colors. We obtained a red-blue coloring of the non-minimal elements and it only remains to check whether the monochromatic permutations form a proper subclass of Av(123). Observe that the first elements of two adjacent runs cannot be covered by a single LR-minimum, which implies that two elements from different non-adjacent runs cannot be covered by a single LR-minimum. By this observation, in the monochromatic permutations π_B and π_R any two elements covered by the same LR-minimum must belong to the same run.

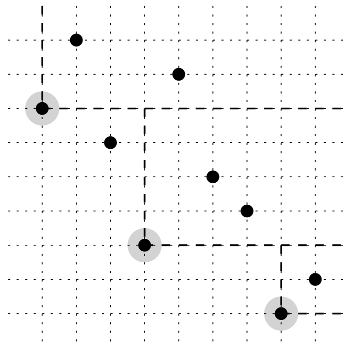


Figure 4.2: For example the 123-avoiding permutation 796385412 with the non-minimal elements split into three different runs.

We claim that a monochromatic copy of the pattern 463152 $\in$ Av(123) can never be created this way. Assume for contradiction that there is a permutation $\pi \in$ Av(123) on which the algorithm creates a monochromatic copy of 463152 and let f be the corresponding embedding of 463152 into π. Observe that every LR-minimum of 463152 is covering some other element and therefore f must preserve the property of being a LR-minimum, otherwise we would get an occurrence of the pattern 123. Following our earlier observations, the elements $\pi_{f(6)}$, $\pi_{f(5)}$ and $\pi_{f(2)}$ must fall into the same run since $\pi_{f(5)}$ shares LR-minima with both of the other two elements. And because elements of the same run are covered by a single LR-minimum, there is a LR-minimum π_i covering $\pi_{f(6)}$ and $\pi_{f(2)}$. However, π_i must then also cover $\pi_{f(3)}$ which contradicts the fact that $\pi_{f(3)}$ itself is a LR-minimum of π. □

Corollary 4.3. *The class Av*(1423, 1342) *is splittable.*

Proof. In the previous Lemma 4.2 we showed that Av(123) is LR-splittable, more precisely that Av(123) $\subseteq$ Av(463152) $\odot_{\text{LR}}$ Av(463152) . Since the permutation 463152 is LR-simple, we get the splittability of Av(123)$^{\text{LR}}$ from Proposition 3.8. Finally, owing to Proposition 4.1, we know that Av(123)$^{\text{LR}}$ and Av(1423, 1342) are in fact identical. □

Our final task is to show that Av(1423, 1342) is 1-amalgamable by proving the LR-amalgamability of Av(123). In order to do that we will use the following result which is due to Waton [12]. Note that Waton in fact proved the equivalent claim for parallel lines of positive slope and the permutation class Av(321).

Proposition 4.4 (Waton [12])**.** *The class of permutations that can be drawn on any two parallel lines of negative slope is Av*(123)*.*

Lemma 4.5. *The class Av*(123) *is LR-amalgamable.*

Proof. Fix arbitrary two parallel lines of negative slope in the plane. Let π_1 and π_2 be permutations avoiding 123 and mappings f_1 and f_2 where f_i is an embedding of the singleton permutation into π_i and its image is not a LR-minimum of π_i. According to Proposition 4.4 both π_1 and π_2 can be drawn from our fixed parallel lines. Fix sets of points A_1 and A_2 which lie on these lines and their corresponding respective permutations are π_1 and π_2. Moreover, we can choose the sets such that the elements in the images of f_1 and f_2 share the same coordinates. Otherwise we could translate one of the sets in the direction of the lines to align these two points. Finally, if a point $x \in A_1$ and a point $y \in A_2$ share one identical coordinate we can move x a little bit in the direction of the lines without changing the permutation corresponding to the set A_1.

We may easily see that the permutation corresponding to the union $A_1 \cup A_2$ with the natural mappings of π_1 and π_2 is the desired LR-amalgamation of π_1 and π_2. □

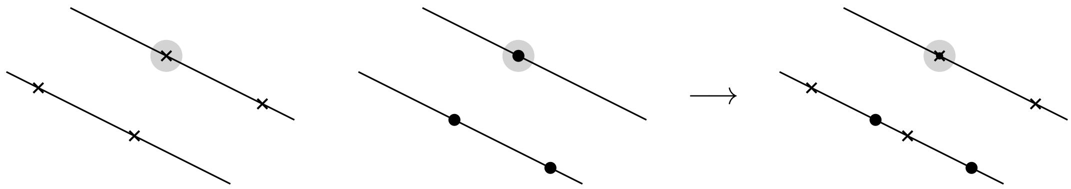


Figure 4.3: Example of two permutations 3142 and 231 drawn from two parallel lines with highlighted embeddings of the singleton permutation and their LR-amalgamation 532614.

Finally, we get the desired result by applying Proposition 3.12.

Corollary 4.6. *The class* $Av(1423, 1342)$ *is 1-amalgamable.*

4.2 Infinitely many classes

Our goal is to manufacture infinitely many 1-amalgamable and splittable permutation classes by iteratively inflating $\mathrm{Av}(1423, 1342)$. There might possible be different ways to achieve this goal, we chose to iterate inflating by co-layered permutations.

Let CL denote the class of co-layered permutations, defined as inflations of decreasing permutations by increasing ones, i.e. $\mathsf{CL} = \mathrm{Av}(12)[\mathrm{Av}(21)]$. Alternatively, it is not hard to see that $\mathsf{CL} = \mathrm{Av}(213, 132)$. Let us define the sequence of permutation classes $(C_n)_{n\geq 0}$ in the following way:

- $C_0 = \mathrm{Av}(1423, 1342)$ and
- $C_n = C_{n-1}[\mathsf{CL}]$.

We shall proceed by showing that C_n are all splittable, 1-amalgamable and pairwise different.

Lemma 4.7. *For every n, the class C_n is 1-amalgamable and splittable.*

Proof. We will show 1-amalgamability by induction on n. We have already proven that C_0 is 1-amalgamable in Section 4.1 (Corollary 4.6). Suppose that $n \geq 1$ and C_{n-1} is 1-amalgamable class. Recall that CL is 1-amalgamable due to Theorem 2.9. Finally, CL is atomic and thus, Lemma 2.13 implies that the inflation $C_{n-1}[\mathsf{CL}]$ must also be 1-amalgamable.

We shall prove a stronger statement that $C_n \subseteq \mathrm{Av}(463152) \odot \mathrm{Av}(463152)$ again by induction on n. This implies the splittability of C_n since $463152 \in C_n$. First, observe that 463152 is a simple permutation which means that Av(463152) is inflation-closed. It follows from Lemma 4.2 that $C_0 \subseteq \text{Av}(463152) \odot \text{Av}(463152)$ since 463152 is also LR-simple.

Now suppose that $n \geq 1$ and the statement holds for C_{n-1}. We know that $C_n \subseteq \text{Av}(463152)[\mathsf{CL}] \odot \text{Av}(463152)[\mathsf{CL}]$ by applying Lemma 2.6. And since CL is a subclass of Av(463152) that is itself inflation-closed, we obtain the desired splitting of C_n into parts Av(463152) and Av(463152). □

For the final lemma, we will actually need a result about separable permutations and their decomposition trees which we defined in Subsection 2.2.2 (Definition 2.18). The lemma essentially states that in a separable permutation the smallest interval containing two elements also contains most of the subtree rooted in their least common ancestor.

Lemma 4.8. *Let π be a separable permutation and i, j two different indices such that $i < j$. Let v be the least common ancestor of the leaves v_i and v_j corresponding to π_i and π_j in $T(\pi)$. The smallest interval in π that contains both π_i and π_j is represented by decomposition tree T that is obtained by taking the subtree of $T(\pi)$ rooted in v and deleting all its subtrees that are to the left of the tree containing v_i and to the right of the tree containing v_j.*

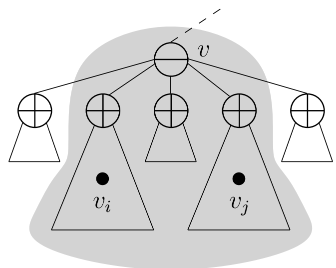


Figure 4.4: Decomposition tree of a separable permutation π with leaves v_i, v_j corresponding to π_i and π_j and their least common ancestor v. The highlighted subtree corresponds to the smallest interval that contains both π_i and π_j.

Proof. We will prove this by induction on the sum of the distance from v_i to v and the distance from v_j to v. If both v_i and v_j are children of v then it can be seen that the smallest interval must contain any subtree of v that lies between v_i and v_j.

Now without loss of generality suppose that v_i is not a child of v. Let w be the father of v_i and T' the decomposition tree obtained from $T(\pi)$ by replacing w with a new leaf w'. Observe that all the elements in the smallest interval containing w' and v_j in the permutation represented by T' must also lie in the smallest interval containing v_i and v_j in π. By applying induction on T', w and v_j we see that we are left with showing that the other subtrees of w lie in the smallest interval.

Notice that the subtrees of w to the right of v_i are contained in the interval trivially since they contain leaves representing elements that lie between π_i and π_j. Consider two cases depending on the type of nodes.

Case 1. The nodes w and v are of different types. Due to symmetry we can assume that w is a $\oplus$-node and v is a $\ominus$-node. Then all the elements represented by leaves in the subtrees of w to the left of v_i are larger than π_i and smaller than π_j (recall Observation 2.19) and thus must be contained in the interval.

Case 2. The nodes w and v are of the same types. Due to symmetry we can assume that both w and v are $\oplus$-nodes. Since v_i is not a child of v there is a node x on the path from w to v that is a $\ominus$-node. Let y be the child of x that lies on the path from w to x. We already know from induction that all children of x must belong to the the interval except for (possibly) the tree rooted in y. Therefore, if x has any children to the left of y they force the whole tree rooted in y to be in the interval as well. Otherwise, x must have a subtree to the right of y and there is k such that π_k corresponds to some leaf in this tree. Then all the elements represented by leaves in the subtrees of w to the left of v_i are larger than π_k and smaller than π_j and thus must belong to the interval.

□

Lemma 4.9. *For every i and j such that $i < j$, we have $C_i \subsetneq C_j$.*

Proof. It holds trivially that $C_i \subseteq C_j$ for $i < j$. In order to prove the strict inclusions, we will show that for every n there is a permutation $\gamma_n \in C_n$ that does not belong to C_i for any $i < n$.

We define permutations γ_i in the following way: $\gamma_0 = 12$ and γ_n is created by inflating all elements of γ_{n-1} with a copy of 3412, i.e. it is equal to $\gamma_{n-1}[3412, \ldots, 3412]$ for $n \geq 1$. We will use induction on n to show that $\gamma_n \in C_n$ but $\gamma_{n+1} \notin C_n$. This is equivalent to our original statement since γ_i is contained in γ_j for any i and j such that $i < j$.

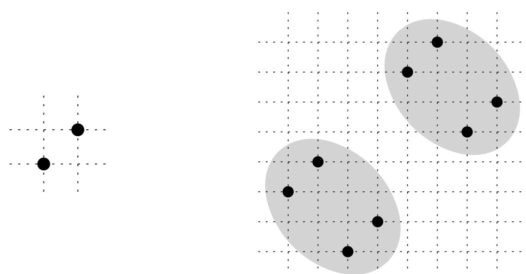


Figure 4.5: Permutations $\gamma_0 = 12$ and $\gamma_1 = 34127856$.

First, observe that γ_0 is trivially contained in $C_0 = \mathrm{Av}(1423, 1342)$ while γ_1 is not since γ_1 contains both 1423 and 1342. Now, suppose that $n \geq 1$. We obtain that $\gamma_n \in C_n$ from induction as γ_n is an inflation of γ_{n-1} with a co-layered permutation and γ_{n-1} belongs to C_{n-1}.

For a contradiction, let us assume that γ_{n+1} also belongs to C_n. We know that γ_{n+1} is not contained in C_{n-1} and thus it must be an inflation of $\pi \in C_{n-1}$ with $\sigma_1, \ldots, \sigma_k \in \mathsf{CL}$. Now, let us look at γ_{n+1} as a concatenation of blocks order-isomorphic to 3412. If we take an embedding f of π into γ_{n+1} then there must be at least one of the blocks that does not contain any element from the image of f since $\gamma_n \notin C_{n-1}$. It follows that there is $l \in [k]$ such that σ_l contains elements from two different blocks. Therefore, σ_l must contain all elements from these blocks due to Lemma 4.8 since σ_l forms an interval in γ_{n+1}. These blocks cannot create a copy of $3412 \oplus 3412$ since that is not a co-layered permutation, so they must form a copy of $3412 \ominus 3412$. Furthermore, observe that γ_i does not contain any interval order-isomorphic to 21 for any i. Thus when we apply again Lemma 4.8 on the corresponding elements of γ_n we get that σ_l must contain $3412[3412, 3412, 3412, 3412]$. And that is clearly a contradiction since it is not a co-layered permutation. □

Corollary 4.10. *There are infinitely many permutation classes that are both 1-amalgamable and splittable.*

5. Hereditary 2-colorings

In Chapter 3, we were considering the left-to-right minima of a permutation and derived interesting tools for working with LR-closures of classes. We would like to further generalize this concept. The main idea is to define a special subset of elements in every permutation (essentially playing the same role as LR-minima did) and use this set to define properties analogous to LR-splittability and LR-amalgamability. The important property of LR-minima is that for any permutations π, σ and an embedding f of σ into π, if f maps σ_i to a LR-minimum of π then σ_i must be a LR-minimum of σ. It is only natural to require this property from our subsets of elements. However, we prefer to view these subsets as a 2-colorings of every permutation of every length.

Definition 5.1. Let $c = (c_n)_{n\geq 0}$ be a sequence of mappings such that $c_n : S_n \to \{0,1\}^n$. For a permutation $\pi \in S_n$, an element π_i is a *0-element of* π (with respect to c) if $c_n(\pi)_i = 0$, otherwise π_i is a *1-element of* π. We say that c is a *hereditary 2-coloring* if for any permutations π, σ and an embedding f of σ into π, it holds that if $f(\sigma_i)$ is a 0-element of π then σ_i is a 0-element of σ.

The most natural way to define a hereditary 2-coloring is through pattern avoidance. Fix a permutation σ and its element σ_i. We define a coloring of π in the following way: an element π_j is a 0-element if and only if there is no embedding of σ into π that maps σ_i on π_j. We denote this type of coloring as $c^{\sigma,i}$. It is easy to see that such coloring has the desired property. See Figure 5.1.

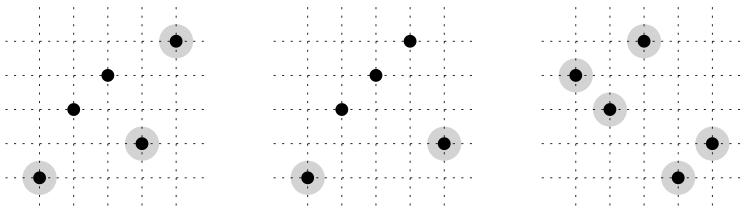


Figure 5.1: Example of a hereditary 2-coloring $c^{132,2}$ on permutations 13425, 13452 and 43512 with highlighted 0-elements.

Observe that an element π_i is a LR-minimum of π precisely when there is no embedding of 12 that maps 2 on π_i. Therefore, we have obtained a generalization of LR-minima as we promised.

Observation 5.2. *For $\pi \in S_n$, an element π_i is a LR-minimum if and only if it is a 0-element of π with respect to $c^{12,2}$.*

Definition 5.3. For two hereditary 2-colorings c and d we say that $c \preceq d$ if and only if for every n, permutation $\pi \in S_n$ and $i \in [n]$ it holds that $c_n(\pi)_i \geq d_n(\pi)_i$. In other words, for every permutation π the 0-elements of π with respect to c are a subset of the 0-elements with respect to d.

For example consider two permutations $\pi \in S_n$, $\sigma \in S_m$ and indices $i \in [n]$, $j \in [m]$ such that there is an embedding of σ into π that maps σ_j on π_i. It is easy to see that $c^{\sigma,j} \preceq c^{\pi,i}$. Furthermore, observe that we obtained a partially order set with both the least element that does not induce any 0-elements (denoted by $c^{1,1}$ in our notation) and the greatest element that does not induce any 1-elements.

Now we can define inflations restricted to 0-elements in the same way we defined LR-inflations.

Definition 5.4. Let c be a hereditary 2-coloring and suppose that $\pi \in S_n$ is a permutation with k 0-elements with respect to c and let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of non-empty permutations. The *c-inflation* of π by the sequence $\sigma_1, \ldots, \sigma_k$ is the inflation of 0-elements of π by $\sigma_1, \ldots, \sigma_k$. We denote this by $\pi\langle\sigma_1, \ldots, \sigma_k\rangle_c$.

Furthermore, for permutation classes A and B we use $A\langle B\rangle_c$ to denote the set of all c-inflations of a permutation $\pi \in A$ by the sequence $\sigma_1, \ldots, \sigma_k$ of permutations from B.

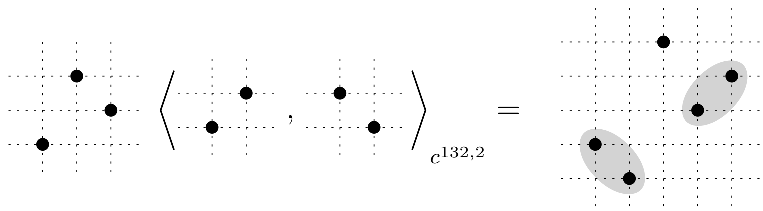


Figure 5.2: An example of $c^{132,2}$-inflation: $132\langle 21, 12\rangle_{c^{132,2}} = 21534$.

It is not hard to see that for permutation classes A and B, $A\langle B\rangle_c$ is again a permutation class. However, it is useful to include a proof here as it illustrates why we defined hereditary 2-coloring with the requirement about 0-elements.

Lemma 5.5. *For permutation classes A and B, the set of their c-inflations $A\langle B\rangle_c$ is a permutation class.*

Proof. Let π be a permutation from $A\langle B\rangle_c$, i.e. there is a permutation $\tau \in A$ and permutations $\sigma_1, \ldots, \sigma_k \in B$ such that $\pi = \tau\langle\sigma_1, \ldots, \sigma_k\rangle_c$. Let π' be permutation contained in π with a respective embedding f of π' into π. We can look at π as a regular inflation of τ by $\sigma_1, \ldots, \sigma_k$ and additional copies of the singleton permutation. It follows due to the embedding f that π' is an inflation of $\tau' \in A$ by $\sigma'_1, \ldots, \sigma'_k \in B$ and additional copies of the singleton permutation, where τ' is contained in τ and σ'_i is contained in σ_i for every i.

Since τ' is contained in τ we have a corresponding embedding g that maps the element inflated by σ'_i in π' to the element inflated by σ_i in π for every i. Due to the properties of hereditary 2-colorings, each of these elements must be a 0-element of τ' and thus, π' is in fact a c-inflation of τ' by $\sigma'_1, \ldots, \sigma'_k$. □

Definition 5.6. Let c be a hereditary 2-coloring. We say that a permutation class C is *closed under c-inflations* if for every $\pi \in C$ with k 0-elements, and for every k-tuple $\sigma_1, \ldots, \sigma_k$ of permutations from C, the c-inflation $\pi\langle\sigma_1, \ldots, \sigma_k\rangle_c$ belongs to C. The *closure of C under c-inflations* (or just *c-closure* of C), denoted C^c, is the smallest class which contains C and is closed under c-inflations.

Recall that we could characterize inflation-closed classes by basis that consists of simple permutations and LR-closed classes by basis that consists of of LR-simple permutations. It should come as no surprise that we can derive a more general result. We say that a permutation is *c-simple* if it cannot be obtained by c-inflations except for the trivial ones. Once again, it is easy to see that a permutation class is closed under c-inflations if and only every permutation in its basis is c-simple. Moreover, claim analogous to Proposition 3.3 holds as well. We however omit the proof here as the argument goes exactly the same.

Proposition 5.7. *Let c be a hereditary 2-coloring. For a set of permutations F, the c-closure of $Av(F)$ is equal to $Av(F')$, where*

$$F' = \{\sigma \mid \sigma \text{ is } c\text{-simple and } \exists \tau \in F : \sigma \text{ contains } \tau\}.$$

We note that this is a reoccurring situation in this chapter. While the hereditary 2-colorings seem like substantial generalization of LR-minima, the proofs in this chapter mostly follow the proofs from Chapter 3. Usually, not much more is needed than substituting 0-elements for LR-minima, c-inflations for LR-inflations, c-closures for LR-closures etc.

5.1 c-splittability

We can define splittability with respect to a hereditary 2-coloring analogously to the way we defined LR-splittability using LR-minima in Section 3.1. Note that from now on we will assume a fixed hereditary 2-coloring c.

Definition 5.8. We say that a permutation π is a *c-merge* of permutations τ and σ, if its 1-elements can be partitioned into two disjoint sequences, such that one of them is, together with the sequence of 0-elements of π, an occurrence of τ, and the other is, together with the sequence of 0-elements of π, an occurrence of σ. For two permutation classes A and B, we write $A \odot_c B$ for the set of all c-merges of a permutation from A with a compatible permutation from B.

As before, we can also look at c-merges as a special red-blue coloring of the permutation π in which the 0-elements are both blue and red at the same time.

In this general case, the set $A \odot_c B$ will often not be a permutation class. For example, let $A = \mathrm{Av}(3412)$, B be the class of all permutations and consider the set $C = A \odot_{c^{132,2}} B$. The permutation 35142 contains 1-elements at positions 2 and 4, therefore it can be expressed as a $c^{132,2}$-merge of $3142 \in A$ and $3412 \in B$. However, the permutation 3412, which is contained in 35142, does not have any 1-element and thus cannot be a $c^{132,2}$-merge of $\tau \in A$ and $\sigma \in B$ as A does not contain 3412. See Figure 5.3.

Let us remark that if we set c to be the greatest hereditary 2-coloring, i.e. the one that does not induce any 1-elements, then $A \odot_c B$ is simply the set $A \cap B$.

Definition 5.9. We say that a multiset of permutation classes $\{P_1, \dots, P_m\}$ forms a *c-splitting* of a permutation class C if $C \subseteq P_1 \odot_c \cdots \odot_c P_m$. We call P_i the *parts* of the c-splitting. The c-splitting is *nontrivial* if none of its parts is a superset of C, and the c-splitting is *irredundant* if no proper submultiset of $\{P_1, \dots, P_m\}$ forms a c-splitting of C. A permutation class C is then *c-splittable* if C admits a nontrivial c-splitting.

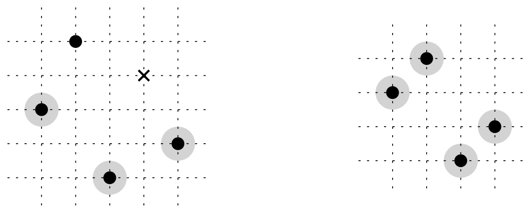


Figure 5.3: A hereditary 2-coloring $c^{132,2}$ on permutations 35142 and 3412 with highlighted 0-elements. The permutation 35142 is $c^{132,2}$-merge of 3412 and 3142, while 3412 can only be $c^{132,2}$-merge of 3412 and 3412.

We can state the following lemma, which is analogous to Lemma 3.6. In fact, we could again use almost the same proof as Lemma 2.1 since it does not really depend on the structure of the merges. Therefore, let us omit the proof here as well.

Lemma 5.10. *For a class C of permutations, the following properties are equivalent:*

(a) C is c-splittable.

(b) C has a nontrivial c-splitting into two parts.

(c) C has a c-splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial c-splitting into two parts, in which each part is a principal class.

In Section 3.1, we noticed that LR-splittability trivially implies splittability. We can show a nice generalization of this statement. Observe that $c^{1,1}$-splittability is in fact the regular splittability as $c^{1,1}$ does not induce any 0-elements.

Lemma 5.11. *Let c and d be hereditary 2-colorings such that $c \preceq d$ and let C be a permutation class. If C is d-splittable then C is also c-splittable.*

Proof. Suppose that C is d-splittable into classes D and E. We aim to show that C is also c-splittable into D and E. Let π be a permutation that belongs to C with a red-blue coloring of its 1-elements with respect to d that witnesses it being a d-merge of $\tau \in D$ and $\sigma \in E$. We can color the elements of π that are 0-elements with respect to d but not with respect to c in an arbitrary way. We obtained a red-blue coloring of 1-elements of π with respect to c that show it is a c-merge of permutation $\tau' \in D$ and $\sigma' \in E$. $\square$

By this point, it is perhaps not very surprising that we can state results connecting splittability and c-splittability of permutation classes and their c-closures in the same way we connected LR-splittability and splittability in Section 3.1.

Proposition 5.12. *Let C be a permutation class that is closed under c-inflations. Then C is splittable if and only if C is c-splittable.*

Proof. Trivially, c-splittability implies $c^{1,1}$-splittability (and thus splittability) due to Lemma 5.11. Now suppose that C admits splitting $\{D, E\}$ for some proper subclasses D and E. We aim to prove that also $C \subseteq D \odot_c E$. We would like to use similar argument as in the proof of Proposition 3.7 to show that C contains a permutation τ that belongs neither to D nor to E. However, we are not guaranteed that there is any permutation outside $D \cap E$ that has at least one 0-element.

Let us first assume that all permutations containing at least one 0-element belong to $D \cap E$ and let $\pi \in D \cap E$. In that case, we can color the 1-elements of π in an arbitrary way since π itself belongs to both classes D and E. On the other hand, any $\pi \in C$ that belongs only to one of the classes D and E has no 0-elements and thus we can simply use the coloring proving that C is splittable.

Otherwise, there must be a permutation $\tau_1 \in C$ outside $D \cap E$ which has at least one 0-element. Without loss of generality let us assume that $\tau_1 \in D$. From the definition of splittability, there is also a permutation $\tau_2 \in C \setminus D$. Define τ as c-inflation of τ_1 with τ_2, which lies outside both subclasses D and E.

Now, we can continue arguing in the same way as in the proof of Proposition 3.7. Let us suppose that there is $\pi \in C$ not belonging to $D \odot_c E$, i.e., there is no red-blue coloring of π which proves it is a c-merge of a permutation $\alpha \in D$ and a permutation $\beta \in E$. Let π' be the permutation created by inflating each 0-element of π with τ. Since π' belongs to C, it has a regular red-blue coloring with the permutation corresponding to the red elements $\pi'_R \in D$ and the permutation corresponding to the blue elements $\pi'_B \in E$. However there must be both colors in each block created by inflating a 0-element of π with τ, and therefore there is a valid red-blue coloring of π that assigns both colors to the 0-elements. □

Corollary 5.13. *Let c and d be hereditary 2-colorings such that $c \preceq d$ and let C be a permutation class that is closed under d-inflations. Then C is d-splittable if and only if C is c-splittable.*

Proof. We showed in Proposition 5.12 that if C is splittable then C is d-splittable. Using order of the colorings and Lemma 5.11, we see that d-splittability of C implies its c-splittability. And analogously it follows that c-splittability of C implies its $c^{1,1}$-splittability and thus splittability. □

We omit the proof of the following proposition as it is analogous to the proof Proposition 3.8 without any additional tricks.

Proposition 5.14. *If C, D and E are permutation classes satisfying $C \subseteq D \odot_c E$, then $C^c \subseteq D^c \odot_c E^c$. Consequently, if D^c and E^c do not contain the whole class C, then its closure C^c is c-splittable into parts D^c and E^c.*

5.2 c-amalgamability

In this section, we will define a more general version of LR-amalgamability and show that again, almost all results proved in Section 3.2 extend naturally to the more general setting.

Definition 5.15. We say that a permutation class C is *c-amalgamable* if for any two permutations $\tau_1, \tau_2 \in C$ and any two mappings f_1 and f_2, where f_i is an embedding of the singleton permutation into τ_i and its image is not a 0-element of τ_i, there is a permutation $\sigma \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of τ_i into σ, $g_1 \circ f_1 = g_2 \circ f_2$ and moreover g_i preserves the property of being a 0-element.

As before, observe that $c^{12,2}$-amalgamability is in fact LR-amalgamability and $c^{1,1}$-amalgamability is regular 1-amalgamability. We state the following lemma since we would only repeat the proof of Lemma 3.10.

Lemma 5.16. *Let C be a permutation class that is closed under c-inflations. If C is c-amalgamable then C is also 1-amalgamable.*

Interestingly, we are not able to prove statement analogous to the first part of Proposition 3.12, i.e. that c-closure of c-amalgamable class is also c-amalgamable. The difference in this general case is that the c-inflations may not behave as nicely as LR-inflations. If we look at the LR-inflation $\rho = \pi\langle\sigma_1, \ldots, \sigma_k\rangle$ and the corresponding embeddings $f_1, \ldots, f_k$ where f_i is an embedding of σ_i then $f_i(\sigma_{i\,j})$ is a LR-minimum of ρ if and only if $\sigma_{i\,j}$ is a LR-minimum of σ_i. On the other hand, in the same situation with c-inflations we are only guaranteed that if $f_i(\sigma_{i\,j})$ is 0-element with respect to c then $\sigma_{i\,j}$ is also 0-element. It can be seen that for a hereditary 2-coloring defined through pattern avoidance as $c^{\sigma,i}$ the stronger property holds if and only if σ is a simple permutation.

Notice that the proof of the following proposition is similar to the proof of Proposition 3.12. However we need to be very careful about the details here due to the reason described above.

Proposition 5.17. *If a permutation class C is c-amalgamable then its c-closure C^c is 1-amalgamable.*

Proof. Let $\pi_1, \pi_2 \in C^c$ be permutations and f_1, f_2 embeddings of the singleton permutation into π_1 and π_2. We aim to prove by induction on the length of π_1 and π_2 that there is a corresponding 1-amalgamation of π_1 and π_2. First, if any of π_1 and π_2 is a singleton permutation then their 1-amalgamation is simply the second permutation.

Without loss of generality, we can assume that π_1 can be obtained by c-inflations as $\pi_1 = \alpha_1\langle\beta_1, \ldots, \beta_k\rangle_c$ and π_2 by c-inflations as $\pi_2 = \alpha_2\langle\delta_1, \ldots, \delta_l\rangle_c$ where α and γ belong to C. We consider two separate cases. First, assume that the image of one of the embeddings lies inside the block corresponding to a c-inflated 0-element. Without loss of generality, suppose that the image of f_1 lies inside the block corresponding to the j-th inflated element order-isomorphic to β_j. Note that β_j is necessarily shorter than π_1 therefore we get from induction a 1-amalgamation σ of β_j and π_2 for the embeddings f_1' and f_2, where f_1' is the embedding f_1 restricted to the inflated block of β_j. Observe that the permutation $\alpha_1\langle\beta_1, \ldots, \beta_{j-1}, \sigma, \beta_{j+1}, \ldots, \beta_k\rangle_c$ is precisely the 1-amalgamation of π_1 and π_2 we were looking for.

Finally we have to deal with the situation when the images of both embeddings lie outside of the blocks corresponding to the inflated 0-elements of the respective permutations. We know there is a c-amalgamation σ of α_1 and α_2 for the embeddings g_1 and g_2, where g_i is the embedding f_i restricted to the permutation α_i. Let h_i be the corresponding embedding of α_i into σ that preserves the 0-elements. We construct the desired 1-amalgamation of π_1 and π_2 in the following way: take σ and for every 0-element of α_1 inflate its image under g_1 with the corresponding permutation β_i, analogously for 0-elements of α_2.

Note that the images of g_1 and g_2 might share more than one elements and in particular they can map two 0-elements on a single element of σ. For any i and j such that the i-th 0-element of π_1 and the j-th 0-element of π_2 get mapped to the same 0-element σ_m we can simply use induction to obtain an arbitrary 1-amalgamation ρ of β_i and δ_j and inflate σ_m with ρ. The resulting permutation is clearly a 1-amalgamation of π_1 and π_2. □

Conclusion and further directions

In Chapter 2, we studied, among other things, 1-amalgamability. Even though we derived some general conclusions we still do not understand what influences whether a given class is 1-amalgamable. Moreover, we do not really have any solid idea how one would go about proving a 1-amalgamability of any class that does not have a rather simple structure. Recall that the only class for which we are able to prove 1-amalgamability was $\mathrm{Av}(1423, 1342)$ and that was only since it is a LR-closure of $\mathrm{Av}(123)$.

Question 1. Is there a way to characterize which (principal) classes are 1-amalgamable?

We were able to characterize all 2-amalgamable classes of separable permutations in Theorem 2.22. Naturally, we would like to know whether there are any other 2-amalgamable classes.

Question 2. Are there any 2-amalgamable permutation classes outside of Sep and if so, are there infinitely many such classes?

Using our results about LR-inflations from Chapter 3, we were able to generate infinitely many 1-amalgamable and splittable classes in Chapter 4. However, all of them are quite similar since they are inflations of $\mathrm{Av}(1423, 1342)$. Can we find different 1-amalgamable and splittable class using the same method? Intuitively, LR-closed classes might be a good place to look for 1-amalgamable and splittable classes since they lie in between inflation-closed classes, that are 1-amalgamable but not splittable, and classes avoiding decomposable permutations, that are splittable but not 1-amalgamable.

Question 3. Is it possible to use our results concerning LR-inflations to infer 1-amalgamability and splittability of any permutation class different from $\mathrm{Av}(1423, 1342)$?

Finally, in Chapter 5, we introduced hereditary 2-colorings as a generalization of LR-minima and consequently, c-splittability and c-amalgamability that connect together the respective LR- and regular versions of splittability and 1-amalgamability. Originally, we hoped that we can use these tools to prove 1-amalgamability and splittability of different classes via the same method as in Chapter 4. However, we were not able to come up with a class for which we could show its c-splittability and c-amalgamability for arbitrary hereditary 2-coloring c (other than $\mathrm{Av}(123)$). Recall that if we had such a class with some additional assumptions about its c-splitting then its c-closure would be 1-amalgamable and splittable.

Question 4. Is there any c-splittable and c-amalgamable permutation class for arbitrary hereditary 2-coloring c that is different from $\mathrm{Av}(123)$ for $c = c^{12,2}$.

Finally as we mentioned in the introduction, we believe that hereditary 2-colorings may be of independent interest. Let us provide one reason for this belief. Jelínek and Valtr [10] proved splittability of classes avoiding $1 \oplus \sigma$ where σ is sum-indecomposable through arguments involving matchings. However, the general idea behind their proof can be easily translated into the language of LR-inflations and LR-simple permutations. Notice that we can write $1 \oplus \sigma$ as $12[1, \sigma]$ and LR-inflations are in fact $c^{12,2}$-inflations. Could we possibly use similar framework to prove (un)splittability of classes avoiding $\tau[1, \ldots, 1, \sigma, 1, \ldots, 1]$ where τ is a simple permutation?

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