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Integral and Supremal Operators on Weighted Function Spaces

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Prague, 12 December 2016 Martin Křepela

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Abstract: The common topic of this thesis is boundedness of integral and supremal operators between function spaces with weights. The results of this work have the form of characterizations of validity of weighted operator inequalities for appropriate cones of functions. The outcome can be divided into three categories according to the particular type of studied operators and function spaces.

The first part involves a convolution operator acting on general weighted Lorentz spaces of types Λ, Γ and S defined in terms of the nonincreasing rearrangement, Hardy-Littlewood maximal function and the difference of these two, respectively. It is characterized when a convolution-type operator with a fixed kernel is bounded between the aforementioned function spaces. Furthermore, weighted Young-type convolution inequalities are obtained and a certain optimality property of involved rearrangement-invariant domain spaces is proved. The additional provided information includes a comparison of the results to the previously known ones and an overview of basic properties of some new function spaces appearing in the proven inequalities.

The second type of investigated objects are bilinear and multilinear operators defined as a product of linear Hardy-type operators or in a similar way. It is determined when a bilinear Hardy operator inequality holds either for all nonnegative or all nonnegative and nonincreasing functions on the real semiaxis. The proof technique is based on a reduction of the bilinear problems to linear ones to which known weighted inequalities are applicable. The use of this method to solve other questions concerning more general multilinear operators is described as well.

In the third part, the focus is laid on iterated supremal and integral Hardy operators, a basic Hardy operator with a kernel and applications of these to more complicated weighted problems and embeddings of generalized Lorentz spaces. Several open problems related to missing cases of parameters are solved, therefore completing the theory of the involved fundamental Hardy-type operators. The results have a standard explicit form of integral or supremal conditions which are compatible with those known previously. It allows for a straightforward application in various situations involving more complicated weighted inequalities.

Keywords: integral operators, supremal operators, weighted function spaces, Hardy inequality

Název: Integrální a supremální operátory na váhových prostorech funkcí

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Abstrakt: Ústředním tématem této práce je omezenost integrálních a supremálních operátorů na prostorech funkcí s vahou. Získané výsledky mají podobu charakterizací váhových nerovností pro vhodné množiny funkcí a lze je rozdělit do tří skupin podle povahy studovaných operátorů a prostrorů funkcí.

První část se zabývá operátorem konvoluce na Lorentzových prostorech typů Λ, Γ a S s obecnou vahou. Výstupem je charakterizace omezenosti konvolučního operátoru s daným jádrem mezi různými prostory uvedeného typu. Výsledky mají podobu zobecněných Youngových nerovností a zahrnují důkaz optimality prostorů, jež v těchto nerovnostech vystupují. Dalšími získanými poznatky je srovnání s klasickými Youngovými nerovnostmi a souvisejícími výsledky a rovněž přehled základních vlastností jistých nových prostorů funkcí figurujících v dokázaných tvrzeních.

Předmětem druhé části jsou bilineární, případně multilineární operátory definované jakožto součin více lineárních operátorů Hardyho typu nebo podobným způsobem. Je dokázána charakterizace váhové bilineární Hardyho nerovnosti na množině nezáporných nebo nezáporných a nerostoucích funkcí definovaných na poloose kladných reálných čísel. Technika důkazů je zde založena na převedení studovaného problému na problém omezenosti jednodušších lineárních oprátorů na váhových prostorech funkcí a následném využití kombinací známých výsledků. Je rovněž ukázáno, jak stejnou myšlenku využít k získání odpovědí na další rozličné otázky týkající se obecných mulitilineárních operátorů.

Třetí část je zaměřena na základní i iterované supremální a integrální operátory Hardyho typu s jádrem a jejich použití k řešení složitějších problémů souvisejících s váhovými nerovnostmí a vnořeními zobecněných Lorentzových prostorů. Je vyřešeno několik otevřených problémů v podobě chybějících charakterizací omezenosti základních operátorů. Získané podmínky jsou vždy explictně vyjádřeny, svou formou odpovídají podmínkám ve dříve známých případech, a je je tudíž možné přímo použít v dalích složitějších situacích, zejména tehdy, kdy jsou využívány takzvané redukční metody pro práci s váhovými nerovnostmi.

Klíčová slova: integrální operátory, supremální operátory, váhové prostory funkcí, Hardyho nerovnost

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Contents

1
Introduction

The main topic of the research presented in this thesis are operator inequalities related to weighted function spaces. In many ways, this topic is connected to harmonic analysis, interpolation and approximation theory and other branches of functional analysis. Results from these fields are of enormous interest in the theory of partial differential equations on all its levels, from the investigation of existence and regularity of solutions to the more practical outcome involving explicit constructions of approximative solutions. This finds applications in numerical modelling of "real-life" problems from physics as well as other sciences. Operator inequalities and theory of function spaces also find use in other parts of approximation theory than just those concerned with differential equations. Theoretical results which involve approximation of functions have a great practical impact in fields like signal and image processing, data compression, electrical engineering and other.

The particular problems studied in this thesis can be summarily described as follows:

(i) proving weighted Young-type inequalities, related to boundedness of a convolution operator between weighted Lorentz spaces, by reducing the problem to weighted Hardy inequalities;

(ii) characterizing boundedness of various bilinear operators, in particular of Hardy type, between weighted Lorentz and Lebesgue spaces by employing an iteration technique;

(iii) proving weighted inequalities involving some fundamental Hardy-type operators acting on weighted Lebesgue spaces, and an application of the results to embeddings of generalized Lorentz spaces.

The motivation of part (i) was to improve the classical Young-O'Neil inequalities for $L^{p,q}$ spaces by proving their analogues in general weighted Lorentz spaces. These inequalities can be directly applied to get sufficient conditions of Lorentz-space boundedness of convolution operators with a fixed kernel. The obtained results are optimal in the sense that the boundedness conditions implied by the Young-type inequality are both sufficient and necessary, provided that the kernel in the operator is positive and radially decreasing. Many of the classical operators (for example, the Riesz fractional integral) satisfy this requirement.

Part (ii) is connected to the previous one by the fact that bilinear Hardy operators play a significant role in the proof technique used in (i). In part (ii), that technique, based on a certain iteration process, is further developed to be used in more situations. The iteration method is simpler than those used in some older papers on similar topics and the older results are simplified in most cases. Using the iteration method also demonstrates the importance of the theory of function spaces and inequalities with general weights since the knowledge of various general-weighted inequalities is an indispensable ingredient in the method.

Probably the most important achievement of part (iii) is the closing of several gaps in the theory of Hardy operators (even on its fundamental level) and proving boundedness results for certain new iterated supremal Hardy-type operators. The particular choice of investigated operators was motivated by a subsequent application of the developed theory in solving a complicate problem concerning embeddings of generalized Lorentz spaces. Solving the open problems concerning the missing cases of weighted Hardy-type inequalities also allowed to complete the results involving weighted Young-type convolution inequalities. This makes a link between parts (iii) and (i).

The thesis itself consists of two main parts. The first one is this introductory summary. The content of the second one are the following nine papers which were or are to be published independently. The presented publication status is as of December 2016.

[I] M. Křepela, *Convolution inequalities in weighted Lorentz spaces*, Math. Inequal. Appl. **17** (2014), 1201–1223.

[II] M. Křepela, *Convolution in rearrangement-invariant spaces defined in terms of oscillation and the maximal function*, Z. Anal. Anwend. **33** (2014), 369–383.

[III] M. Křepela, *Convolution in weighted Lorentz spaces of type* Γ, Math. Scand. **119** (2016), 113–132.

[IV] M. Křepela, *Bilinear weighted Hardy inequality for nonincreasing functions*, to appear in Publ. Mat.

[V] M. Křepela, *Iterating bilinear Hardy inequalities*, to appear in Proc. Edinb. Math. Soc.

[VI] M. Křepela, *Integral conditions for Hardy-type operators involving suprema*, to appear in Collect. Math.

[VII] A. Gogatishvili, M. Křepela, L. Pick and F. Soudský, *Embeddings of classical Lorentz spaces involving weighted integral means*, preprint.

[VIII] M. Křepela, *Boundedness of Hardy-type operators with a kernel: integral weighted conditions for the case* $0 < q < 1 \le p < \infty$, submitted.

[IX] M. Křepela, *Convolution inequalities in weighted Lorentz spaces, case* $0 < q < 1$, to appear in Math. Inequal. Appl.

These papers are summarily referred to as "the main papers" in the introduction. The reference marks [I–IX] are used to specify a particular paper from the list.

The introductory text is divided into several chapters, the first of which is this "introduction to the introduction". Chapter 2 contains an overview of the elementary theory of function spaces with focus on those spaces and classes which are relevant for the main papers. Above all, this means introducing weighted Lebesgue spaces, weighted Lorentz spaces Λ, Γ, S and some more related structures, listing their basic properties and summarizing various existing results. In a similar manner, Chapter 3 introduces the operators and inequalities which are the subject of investigation in the thesis. The text further continues with Chapter 4 where the contents of the main papers are briefly summarized. That chapter outlines the proof ideas and techniques of the papers, the relation of the obtained results to previous research and potential applications.

This doctoral thesis is linked to the author's licentiate thesis [74] which contained the results of papers [I–III]. Some larger portions of the text in Sections 3.2 and 4.1 and certain minor parts of Chapter 2 already appeared in the introductory summary in [74].

2
Weighted function spaces

All the research questions and results of this thesis have a connection to function spaces. In this chapter, relevant function spaces and some of their basic properties are introduced.

Very vaguely said, a function space is a set – or a "family" – of functions sharing a certain property. Such property may be, for instance, integrability, differentiability, boundedness or other. Grouping functions to vector-space structures is in many ways practical and it formed a base for a great amount of important research in functional analysis.

Suppose, for example, that $\mathscr{M}$ denotes the cone of real-valued μ-measurable functions defined on certain measure space $(\Omega, \mathfrak{M}, \mu)$. A usual way to define a function space X, consisting of functions from $\mathscr{M}$, is by using a mapping $\|\cdot\|_X : \mathscr{M} \to [0, \infty]$. One defines the set X, which may be called the *function space generated by* $\|\cdot\|_X$, by

$$X := \{f \in \mathscr{M};\ \|f\|_X < \infty\}.$$

The functional $\|\cdot\|_X$ might be a norm, which then justifies calling X a function *space*. However, $\|\cdot\|_X$ may as well satisfy only conditions which are much weaker than those defining a norm. Nevertheless, X is often called a function *space* even in such relaxed cases although the term may be in a strict sense incorrect (for example, X does not have to be a linear space). The term *function space* may, in such potentially problematic cases, be also replaced by the "safer" form *function class*.

The norm property of a functional generating a function space may be important. So it is, for instance, in some classical theorems of functional analysis requiring the involved structures to be Banach spaces. On the other hand, there are many other problems for which the norm property is irrelevant. The problems solved in this thesis fall, in fact, mostly in the second category.

The functional $\|\cdot\|_X$ may control various properties of functions. They can have a global character (e.g. the value of the integral of $|f|$ over a set Ω) or local ones (modulus of continuity, properties of ∇f, etc.). In what follows, the spaces of interest are mostly those based on the global behavior of functions, namely on various "integral properties" of those. The simplest, though probably the most important, example of such a space is the Lebesgue L^p space. It is defined as follows.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let $p \in (0, \infty]$. For any real-valued function f measurable on $(\Omega, \mathfrak{M}, \mu)$ define

$$\begin{aligned} \|f\|_p &:= \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}} && \text{if } 0 < p < \infty, \\ \|f\|_\infty &:= \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| && \text{if } p = \infty. \end{aligned}$$

In here, one uses the notation

$$\operatorname*{ess\,sup}_{x \in \Omega} |f(x)| := \inf\{c > 0;\ |f| \le c \ \mu\text{-a.e. on } \Omega\}.$$

Then the space L^p is defined by

$$L^p(\Omega, \mathfrak{M}, \mu) := \left\{ f : (\Omega, \mathfrak{M}, \mu) \to \mathbb{R} \ \mu\text{-measurable};\ \|f\|_p < \infty \right\}.$$

Often in this text, the underlying measure space $(\Omega, \mathfrak{M}, \mu)$ is $\mathbb{R}^d$ with the d-dimensional Lebesgue measure (and the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^d$). It is written only L^p instead of $L^p(\Omega, \mathfrak{M}, \mu)$ if there is no risk of confusion about the underlying measure space.

For every $p \in (0, \infty]$, the L^p space is indeed a linear space. The mapping $\|\cdot\|_p$ is a norm if and only if $p \in [1, \infty]$. If $p \in (0, 1)$, then $\|\cdot\|_p$ is merely a quasi-norm since the Minkowski inequality fails in this case.

The symbol $\mathscr{M}_\mu(\Omega)$ will be used to denote the cone of μ-measurable real-valued functions on $(\Omega, \mathfrak{M}, \mu)$, and $\mathscr{M}_\mu^+(\Omega)$ will stand for the cone of all $f \in \mathscr{M}_\mu(\Omega)$ such that $f \ge 0$ μ-a.e. If μ is the Lebesgue measure (and $\mathfrak{M}$ is the Lebesgue σ-algebra), one writes simply $\mathscr{M}(\Omega)$ and $\mathscr{M}^+(\Omega)$. The set Ω will be always specified, usually as $\mathbb{R}^d$ or an interval on the real axis.

A special case of an L^p space which is worth highlighting is the *weighted Lebesgue space* $L^p(v)$ over $(0, \infty)$. It consists of all functions $f \in \mathscr{M}(0, \infty)$ such that

$$\begin{aligned} \|f\|_{L^p(v)} &:= \left(\int_0^\infty |f(x)|^p v(x) \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty && \text{if } 0 < p < \infty, \\ \|f\|_{L^\infty(v)} &:= \operatorname*{ess\,sup}_{x > 0} |f(x)| v(x) < \infty && \text{if } p = \infty, \end{aligned}$$

where v is a given nonnegative measurable function on $(0, \infty)$. The essential supremum in the case $p = \infty$ is, naturally, taken with respect to the Lebesgue measure.

The L^p spaces have many useful properties. This motivated the introduction of the *Banach function spaces* by W. A. J. Luxemburg in [81]. Roughly speaking, the idea was to introduce a general type of spaces based on a set of properties inspired by the properties of L^p spaces. The proper definition reads as follows.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let $\varrho : \mathscr{M}_\mu^+(\Omega) \to [0,\infty]$ be a mapping. Then ϱ is called a *Banach function norm* if for all functions $f, g, f_n \in \mathscr{M}_\mu^+(\Omega)$ $(n \in \mathbb{N})$, for all constants $a \geq 0$ and all μ-measurable sets $E \subset \Omega$ the following conditions are satisfied:

(P1) $\varrho(f+g) \leq \varrho(f) + \varrho(g)$,

(P2) $\varrho(af) = a\varrho(f)$,

(P3) $\varrho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.,

(P4) $0 \leq g \leq f$ μ-a.e. $\Rightarrow$ $\varrho(g) \leq \varrho(f)$,

(P5) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow$ $\varrho(f_n) \uparrow \varrho(f)$,

(P6) $|E| < \infty \Rightarrow \varrho(\chi_E) < \infty$,

(P7) $|E| < \infty \Rightarrow \int_E f \,\mathrm{d}\mu \leq C_E \varrho(f)$ for some constant $C_E \in (0,\infty)$ depending on E and ϱ but independent of f.

If ϱ is a Banach function norm, the collection

$$X_\varrho := \left\{ f \in \mathscr{M}_\mu(\Omega),\ \varrho(|f|) < \infty \right\} \tag{1}$$

is called a *Banach function space*.

There is a particular subclass of Banach function spaces called the *rearrangement-invariant* spaces. They are based on the following definition.

Let $f \in \mathscr{M}_\mu(\Omega)$. The *nonincreasing rearrangement* of f, denoted by f^*, is defined by

$$f^*(t) := \inf\left\{ s \geq 0;\ \mu(\{x \in \mathbb{R}^d,\ |f(x)| > s\}) \leq t \right\}, \quad t \in (0, \mu(\Omega)).$$

A Banach function norm ϱ is called a *rearrangement-invariant* (shortly *r.i.*) *norm* if, for all functions $f, g \in \mathscr{M}_\mu^+(\Omega)$, it satisfies

(P8) $f^* = g^*$ on $(0, \mu(\Omega)) \Rightarrow \varrho(f) = \varrho(g)$.

As it was suggested before, being a norm might be a rather unnecessarily strong property of a functional. One may therefore introduce some additional terms for function classes based on weaker conditions.

A mapping $\varrho : \mathscr{M}_\mu^+(\Omega) \to [0,\infty]$ is said to be a *rearrangement-invariant quasi-norm* if conditions (P2)–(P8) and

(P1*) $\varrho(f+g) \leq B(\varrho(f) + \varrho(g))$ with a constant $B \in (1,\infty)$ independent of f, g

are satisfied for all functions $f, g, f_n \in \mathscr{M}_\mu^+(\Omega)$ $(n \in \mathbb{N})$, all constants $a \geq 0$ and all μ-measurable sets $E \subset \Omega$. In this case, the collection X_ϱ defined by (1) will be called a *rearrangement-invariant quasi-space*.

Furthermore, the collection X_ϱ is called a *rearrangement-invariant lattice* if the mapping ϱ satisfies the conditions (P2), (P4), (P6) and (P8) for all $f,\ g \in \mathscr{M}_\mu^+(\Omega)$, all $a \geq 0$ and all μ-measurable $E \subset \Omega$.

If X_ϱ is, at least, an r.i. lattice generated by a mapping ϱ, the notation $\|f\|_X := \varrho(|f|)$ and $X := X_\varrho$ may and will be used. In this way, the notation corresponds to the one used in the beginning of this chapter where the symbol $\|\cdot\|_X$ denoted the functional generating a function space.

The simplest example of an r.i. space is the L^p space over $\mathbb{R}^d$ with the Lebesgue measure and with $p \in [1,\infty]$. If $p \in (0,1)$, this structure becomes only an r.i. quasi-space. The weighted Lebesgue space $L^p(v)$ is not r.i. unless the weight function v is constant a.e. Other typical function spaces which are not r.i. are the Sobolev spaces. This is not surprising since the information about differentiability – which is by its nature a local property of a function – is lost when passing from a function to its nonincreasing rearrangement.

In what follows, we will always assume that $(\Omega, \mathfrak{M}, \mu)$ is a measure space such that $\mu(\Omega) = \infty$. It is, of course, possible to modify the definitions of the spaces introduced below so that they correspond to the case of functions defined on a measure space of finite measure.

By generalizing and refining the classical Lebesgue L^p spaces, it is possible to create a wider and finer scale of r.i. spaces (or lattices). The first step in such direction is made by defining the *Lorentz space* $L_{p,q}$. This structure is generated by the following functional:

$$\|f\|_{p,q} := \left(\int_0^\infty (f^*(t))^q t^{\frac{q}{p}-1} \,\mathrm{d}t \right)^{\frac{1}{q}}, \qquad 0 < p,q < \infty,$$
$$\|f\|_{p,\infty} := \sup_{t>0} f^*(t) t^{\frac{1}{p}}, \qquad 0 < p < q = \infty.$$

As usual, the mapping $\|\cdot\|_{p,q}$ is not necessarily a norm, but if $p \in (1,\infty)$ and $q \in (1,\infty]$, then $L_{p,q}$ is *normable*. This means that there exists a norm which is equivalent to the mapping $\|\cdot\|_{p,q}$. To show this, one introduces the following functionals:

$$\|f\|_{(p,q)} := \left(\int_0^\infty (f^{**}(t))^q t^{\frac{q}{p}-1} \,\mathrm{d}t \right)^{\frac{1}{q}}, \qquad 0 < p,q < \infty,$$
$$\|f\|_{(p,\infty)} := \sup_{t>0} f^{**}(t) t^{\frac{1}{p}}, \qquad 0 < p < q = \infty.$$

In here, the symbol f^{**} denotes the *Hardy-Littlewood maximal function* of f^* given by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \,\mathrm{d}s, \quad t > 0.$$

It can be proved (see [11]) that $\|\cdot\|_{p,q}$ is equivalent to $\|\cdot\|_{(p,q)}$ when $p \in (1,\infty)$ and $q \in (1,\infty]$.

The $L_{p,q}$ and $L_{(p,q)}$ spaces play a significant role in interpolation theory [11, 12] since the $L_{(p,q)}$ spaces are real interpolation spaces between L^1 and L^∞. Besides that, these spaces also appear in various refinings of classical inequalities. An example of these are the generalizations of the Young inequality presented later in Section 3.2.

Introducing and studying new function spaces is motivated by various reasons. One of them may be that such new spaces are interpolation spaces between other previously known ones. Another reason might be, for instance, the fact that a new space is the dual or associated space to a known one, it is the optimal domain or range space for an operator, etc. Further on in this text, some of those aspects will be discussed in connection to various particular cases.

At this point, it is time to show the definition of an *associate space*. Let ϱ be a Banach function norm for functions on $\mathscr{M}_\mu^+(\Omega)$. Then its *associate norm* ϱ' is defined by

$$\varrho'(g) := \sup_{\substack{f \in \mathscr{M}_\mu^+(\Omega) \\ \varrho(f) \leq 1}} \int_\Omega f\, g \,\mathrm{d}\mu$$

for all $g \in \mathscr{M}_\mu^+(\Omega)$. The Banach function space $X' := X_{\varrho'}$ is called the *associate space* of X_ϱ.

It should be noted that the associate norm and associate space are indeed a Banach function norm and a Banach function space, respectively. For proofs of these claims as well as for more details see [11]. The definition may be also extended to cover even r.i. lattices.

From now on, the notion of a weight will be used in a somewhat restricted sense. Indeed, a *weight* will always mean a function $v \in \mathscr{M}^+(0,\infty)$ such that

$$0 < \int_0^t v(s)\,\mathrm{d}s < \infty \quad \text{for all } t > 0.$$

In [79, 80], G. G. Lorentz defined a more general class of function spaces – the Λ spaces. They may be understood as general weighted variants of the $L_{p,q}$ spaces, and are defined as follows.

Let $p \in (0,\infty]$ and let v be a weight. The *weighted Lorentz space* $\Lambda^p(v)$ is the set

$$\left\{ f \in \mathscr{M}_\mu(\Omega);\ \|f\|_{\Lambda^p(v)} < \infty \right\},$$

where

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \qquad \text{if } 0 < p < \infty,$$

$$\|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^*(t) v(t) \qquad \text{if } p = \infty.$$

The spaces $\Lambda^p(v)$ with $p<\infty$ are usually called *classical weighted Lorentz spaces* and, as was said, they appeared first in [79, 80]. The *weak-type weighted Lorentz spaces*, that means those with $p=\infty$, were introduced in [28] and further treated, for instance, in [24, 27, 29, 30].

The class of $\Lambda^p(v)$-spaces encompasses a variety of function spaces which are obtained as special cases of $\Lambda^p(v)$ by a particular choice of the weight v. These include the aforementioned $L_{p,q}$ spaces, Lorentz-Zygmund spaces [10] and their generalizations [92], Lorentz-Karamata spaces [86] and other.

A Λ space, in spite of the name, is not necessarily a linear set. The main cause of this problem is the fact that the rearrangement mapping $f\mapsto f^*$ is, in general, not sublinear. To formulate this precisely, if the measure space $(\Omega,\mathfrak{M},\mu)$ contains at least two disjoint sets of positive μ-measure, then for each $n\in\mathbb{N}$ and $t\in(0,\mu(\Omega))$ there exist functions $f,\ g\in\mathscr{M}_\mu(\Omega)$ such that

$$(f+g)^*(t)\geq n\,(f^*(t)+g^*(t)).$$

Even though they do not have to be linear, let alone normed spaces, the name "weighted Lorentz spaces" is commonly used for the $\Lambda^p(v)$ spaces. The questions of linearity of $\Lambda^p(v)$ and of their (quasi-)normability were studied in [33].

E. Sawyer [104] first described the associate space to $\Lambda^p(v)$. This type of a function space is called the Γ space and is defined in the following way.

If $p\in(0,\infty]$ and v is a weight, the weighted Lorentz space $\Gamma^p(v)$ is the set

$$\left\{f\in\mathscr{M}_\mu(\Omega);\ \|f\|_{\Gamma^p(v)}<\infty\right\},$$

where

$$\begin{aligned}
\|f\|_{\Gamma^p(v)} &:= \left(\int_0^\infty (f^{**}(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, && \text{if } 0<p<\infty,\\
\|f\|_{\Gamma^\infty(v)} &:= \operatorname*{ess\,sup}_{t>0} f^{**}(t)v(t), && \text{if } p=\infty.
\end{aligned}$$

The classical $\Gamma^p(v)$ space with $p<\infty$ is the one introduced in [104], although a space with a norm involving f^{**} was explicitly presented already in A.-P. Calderón's paper [21]. The weak-type spaces $\Gamma^\infty(v)$ appeared in [28] and were, as well as their weak-Λ counterparts, further studied for example in [24, 27, 29, 30, 48].

The relation between Λ and Γ spaces is rather strong. The aforementioned associatedness property has the following form (cf. [104]): if $p\in(1,\infty)$, $p':=\frac{p}{p-1}$ and v is a weight, then the r.i. space (lattice) X generated by the functional

$$\|f\|_X := \left(\int_0^\infty (f^{**}(t))^{p'} t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} + \|f^*\|_1\|v\|_1^{-\frac{1}{p}}$$

is the associate space to $\Lambda^p(v)$. If $v\notin L^1$, then the second term is not present. For other cases of p, see [27, 104].

Another relation between Λ and Γ concerns the normability issue. Indeed, it holds (see [6, 104]) that with $p \in (1, \infty)$ the functional $\|\cdot\|_{\Lambda^p(v)}$ is equivalent to a norm if and only if $v \in B_p$, where

$$B_p := \left\{ v \in \mathcal{M}^+(0,\infty);\ \exists C \in (0,\infty)\ \forall t > 0:\ t^p \int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \le C \int_0^t v(s)\,\mathrm{d}s \right\}.$$

Notice that if $v \in B_p$, then $\Lambda^p(v) = \Gamma^p(v)$ with equivalent norms. (For more details see [29–31, 42, 110].)

The question of linearity and normability of $\Gamma^p(v)$ is considerably simpler than in the case of $\Lambda^p(v)$. The reason is that the Hardy-Littlewood maximal function does satisfy

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t) \tag{2}$$

for all $t > 0$ and all locally integrable functions f, g (see [11]). Thanks to (2), the functional $\|\cdot\|_{\Gamma^p(v)}$ is always at least an r.i. quasi-norm, for $p \ge 1$ it is an r.i. norm by the Minkowski inequality. Functional properties of Γ were studied in more detail in [64], for example.

There are further generalizations of Γ spaces [38, 39, 47, 51], based on generalized versions of the Hardy-Littlewood maximal operator. These are represented, for instance, by the $\Gamma_u^p(v)$ space (see [47]) generated by

$$\|f\|_{\Gamma_u^p(v)} := \left(\int_0^\infty \left(\int_0^t u(s)\,\mathrm{d}s \right)^{-p} \left(\int_0^t f^*(s)u(s)\,\mathrm{d}s \right)^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}},$$

and the *generalized* Γ *space* $\mathrm{G}\Gamma^{p,m}(u,v)$ (see [VII]) generated by

$$\|f\|_{\mathrm{G}\Gamma^{p,m}(u,v)} := \left(\int_0^\infty \left(\int_0^t (f^*(s))^m u(s)\,\mathrm{d}s \right)^{\frac{p}{m}} v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}. \tag{3}$$

In both cases, u and v are weights and $m, p \in (0, \infty)$, with further extensions to the weak-type cases $m = \infty$, $p = \infty$ possible in the standard way. These spaces are the subject of investigation in paper [VII] and are discussed further below in this introductory summary.

The last Lorentz-type "space" considered here is the class S, introduced in [25]. If $p \in (0, \infty]$ and v is a weight, the class $S^p(v)$ is defined as

$$\left\{ f \in \mathcal{M}_\mu(\Omega);\ \lim_{s\to\infty} f^*(s) = 0,\ \|f\|_{S^p(v)} < \infty \right\},$$

where

$$\|f\|_{S^p(v)} := \left(\int_0^\infty (f^{**}(t) - f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \qquad \text{if } 0 < p < \infty,$$

$$\|f\|_{S^\infty(v)} := \operatorname*{ess\,sup}_{t>0} (f^{**}(t) - f^*(t))\, v(t) \qquad \text{if } p = \infty.$$

Unlike the Λ and Γ spaces, the class $S^p(v)$ is not even an r.i. lattice (for a detailed study of this and related issues, see [25]). The functional $f^{**} - f^*$ is important in various parts of analysis [7–9, 11, 18, 25, 50, 66, 68, 69, 71, 73, 83, 84, 100] and represents a natural tool to measure oscillation of f, see [9, 11].

It might be reasonable to compare the class $S^p(v)$ to the $L_{\infty,q}$ spaces which consist of functions $f \in \mathscr{M}_\mu(\Omega)$ such that

$$\|f\|_{L_{\infty,q}} := \|f^*\|_1 + \left(\int_0^\infty (f^{**}(t) - f^*(t))^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty.$$

(In here, the case $q < \infty$ is considered.) This shows that the $S^p(v)$ spaces in a sense generalize the $L_{\infty,q}$ spaces. Details and applications of the $L_{\infty,q}$ spaces can be found in [9, 11].

3
Operators and inequalities

In general, all of the problems studied in this work involve finding conditions under which a certain operator is bounded between given function spaces. By definition, this means to determine when a certain functional inequality is valid. This chapter introduces relevant operators, their properties of interest as well as inequalities which are produced when those operators are studied. After a general summary relevant for all of the involved operators, the text is divided to two sections corresponding to the two important classes of operators investigated in the main papers.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and $\mathscr{X} \subset \mathscr{M}_\mu(\Omega)$. Then an *operator* T is any mapping $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$. Such a "toothless" general definition may be further specified in various ways.

Suppose that addition and scalar multiplication is defined on $\mathscr{M}_\mu(\Omega)$ pointwise, and that $\mathscr{X}$ is a linear set. Then, an operator $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$ is said to be *homogeneous* if for all $f \in \mathscr{X}$ and $a \in \mathbb{R}$ it satisfies

$$T(af) = aTf.$$

Next, T is said to be *linear* if it is homogeneous and for all f, $g \in \mathscr{X}$ it satisfies

$$T(f+g) = Tf + Tg.$$

An operator $T : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathscr{M}_\mu(\Omega)$ is called *bilinear* if the operators $T(\cdot, g)$ and $T(f, \cdot)$ are linear for any fixed $g \in \mathscr{X}_2$ and $f \in \mathscr{X}_1$, respectively. This definition may be clearly extended to *multilinear* operators.

Furthermore, $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$ is said to be *positive* if it maps nonnegative functions to nonnegative functions, i.e., if $T(\mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)) \subset \mathscr{M}_\mu^+(\Omega)$. A positive operator T is called *quasi-linear* if it is homogeneous and there exist constants C_1, $C_2 \in [0, \infty)$ such that for all f, $g \in \mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)$ there holds

$$C_1(Tf + Tg) \leq T(f+g) \leq C_2(Tf + Tg)$$

pointwise μ-a.e. on Ω. A positive homogeneous operator T is called *sublinear* if

$$T(f+g) \leq Tf + Tg$$

pointwise μ-a.e. on Ω for all f, $g \in \mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)$.

A standard question concerning operators and function spaces is whether an operator T maps a function space X into a function space Y, i.e., whether $T(X) \subset Y$. In fact, the simple set inclusion is preferably replaced by *boundedness* of $T : X \to Y$ in the sense of the following definition.

Let X, Y be two function spaces (classes) of functions from the cone $\mathscr{M}_\mu(\Omega)$. An operator $T : X \to \mathscr{M}_\mu(\Omega)$ is said to be *bounded between* X *and* Y if there exists a constant $C \in (0, \infty)$ such that for all $f \in X$ (or all $f \in \mathscr{M}_\mu(\Omega)$) the inequality

$$||Tf||_Y \leq C||f||_X \tag{4}$$

is satisfied. An important particular case is attained by the choice $T = I$, where I is the identity operator. Inequality (4) then has the form

$$||f||_Y \leq C||f||_X. \tag{5}$$

If there exits a constant $C \in (0, \infty)$ such that (5) holds for all $f \in X$, then it is said that X is *embedded in* Y, and one writes $X \hookrightarrow Y$.

Often, there is an interest in finding the *optimal constant* C in (4) or (5), i.e., the least C the respective inequality holds with for all $f \in X$. The optimal C in (4) can be expressed by

$$C = \sup_{f \in X} \frac{||Tf||_Y}{||f||_X}, \tag{6}$$

if the convention $\frac{0}{0} := 0$, $\frac{a}{0} := \infty$ for $a \in (0, \infty]$ is considered. Obviously, if the optimal constant is infinite, then T is not bounded between X and Y.

The definitions of boundedness, an embedding or the optimal constant do not require any special properties of X, Y and $||\cdot||_X$, $||\cdot||_Y$. The functionals $||\cdot||_X$ and $||\cdot||_Y$ generating X and Y, respectively, might be rather arbitrary and the definitions will still make sense. If T is linear and the functional $||\cdot||_X$ satisfies at least $||af||_X = |a|||f||_X$ for all $f \in X$, $a \in \mathbb{R}$ and $||\cdot||_Y$ has an analogous property, then (6) can be rewritten as

$$C = \sup_{\substack{f \in X \\ ||f||_X = 1}} ||Tf||_Y.$$

Finally, it might be worth noticing that if X and Y are Banach function spaces (in the sense of Luxemburg's definition), then the assertions $X \subset Y$ and $X \hookrightarrow Y$ coincide (see [11, p. 7]).

3.1 Hardy operators

Variants of the Hardy operator appear almost everywhere in this publication. A simple form of an operator from this family is the *Hardy average operator A* defined by

$$Af(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s, \qquad t > 0,$$

for $f \in \mathscr{M}^+(0,\infty)$ for which the integral makes sense (even as being infinite). The Hardy-Littlewood maximal function g^{**}, as presented in the previous chapter, is therefore the image of g^* under the mapping A, i.e., $g^{**} = A(g^*)$.

The research on Hardy operators and Hardy inequalities has a long and complex history and making a thorough exposition of it is definitely not an ambition of this introduction. Many publications about this topic exist (see [75, 76, 91]) and an interested reader may therefore consult them. In what follows, the focus is laid on those parts of the existing theory of Hardy operators which are relevant to the research presented in the main papers.

The Hardy average operator can be viewed as an one-dimensional relative of the *maximal operator*. The latter is defined as follows. For $f \in \mathscr{M}(\mathbb{R}^d)$ put

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(x)|\,\mathrm{d}x, \qquad x \in \mathbb{R}^d.$$

In here, $B(x,r)$ denotes the ball of radius r centered at the point x, and $|B(x,r)|$ the (Lebesgue) measure of the ball. This is not the only type of maximal operator in use. Other frequently used variants involve non-centered balls, cubes, weighted forms, etc., see [11, 57, 58, 70]. Such types of maximal operators also have different properties and each of them might be useful in different situations.

The maximal operator is indispensable in various areas of analysis. In particular, it is crucial in the theory of differentiability of functions, e.g., in proving the Lebesgue and Rademacher differentiability theorems [36, 114]. Besides that, it finds applications in interpolation theory, Fourier analysis, singular integral theory and other fields, especially those in which smoothness of functions plays a significant role (see [11, 36, 57, 58, 113]).

One of the reasons for the interest in the Hardy operator is its close relation to the maximal operator in the framework of r.i. spaces. Namely (see [11, p. 122]), there exist positive real constants C_1, C_2 such that for all $f \in \mathscr{M}(\mathbb{R}^d)$ and all $t > 0$ one has

$$C_1(Mf)^*(t) \le f^{**}(t) \le C_2(Mf)^*(t). \tag{7}$$

These two relations are sometimes called the *Herz estimates*. They greatly simplify problems concerning the maximal operator. In particular, this affects the investigation of boundedness of M between r.i. spaces since this problem reduces to the question of boundedness of the one-dimensional Hardy operator restricted to nonincreasing functions.

The research presented here features various forms of weighted Hardy inequalities, i.e., it focuses on boundedness of Hardy-type operators in weighted (Lebesgue and Lorentz) spaces. Whenever general weights are in play, it is more practical to work with an even simpler form of the basic Hardy operator. It is defined by

$$Hf(t) := \int_0^t f(s)\,\mathrm{d}s, \qquad t > 0,$$

for any $f \in \mathscr{M}(0,\infty)$ for which the integral makes sense. Omitting the factor $\frac{1}{t}$ (in comparison with the classical average operator A) makes no difference in the weighted settings since this factor may be always incorporated in the weight.

Similarly, one defines the *Copson operator* $\widetilde{H}$ by

$$\widetilde{H}f(t) := \int_t^\infty f(s)\,\mathrm{d}s, \qquad t > 0,$$

for any $f \in \mathscr{M}(0,\infty)$ for which the integral makes sense. This operator is sometimes nicknamed *dual Hardy* or *Hardy adjoint*, since it is adjoint to the operator H in the sense of the identity

$$\int_0^\infty Hf(t)g(t)\,\mathrm{d}t = \int_0^\infty f(t)\widetilde{H}g(t)\,\mathrm{d}t$$

being satisfied for all $f, g \in \mathscr{M}(0,\infty)$ for which both sides of the equation make sense. Both H and $\widetilde{H}$ are linear operators.

The Hardy operator H is bounded between $L^p(v)$ and $L^q(w)$, with $p, q \in (0,\infty)$, if and only if there exists a constant $C \in [0,\infty)$ such that the *weighted Hardy inequality*

$$\left(\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{8}$$

holds for all $f \in \mathscr{M}^+(0,\infty)$. Explicit conditions characterizing when this occurs are known. Namely, if $1 < p \le q < \infty$, then (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{A}_1 := \sup_{t>0}\left(\int_t^\infty w(s)\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_0^t v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty.$$

If $0 < q < p < \infty$ and $q \ne 1 < p$, then (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{A}_2 := \left(\int_0^\infty \left(\int_t^\infty w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}\left(\int_0^t v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{q'}} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty,$$

where $p' := \frac{p}{p-1}$ and $r := \frac{pq}{p-q}$. Moreover, these expressions even provide estimates of the optimal constant C in (8). Indeed, if $1 < p \le q < \infty$, then the optimal constant C in (8), i.e., the least $C \in [0, \infty]$ such that (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ satisfies

$$C \approx \mathscr{A}_1.$$

The equivalence symbol "$\approx$" means that there exist positive real numbers $D_1 = D_1(p,q)$, $D_2 = D_2(p,q)$ dependent on p, q and such that

$$D_1 C \le \mathscr{A}_1 \le D_2 C.$$

Analogously, in the case $0 < q < p < \infty$, $q \ne 1 < p$ one has $C \approx \mathscr{A}_2$ for the optimal C. The characterizations involving expressions $\mathscr{A}_1$, $\mathscr{A}_2$ were proved by B. Muckenhoupt [85], V. G. Mazja [82] and G. Sinnamon [106]. Their variants for the limit cases $p = 1$, $q = 1$ or the weak cases $p = \infty$, $q = \infty$ are also known, see [91].

Notice that if $0 < p < 1$, then inequality (8) with nontrivial weights v, w can never hold for all $f \in \mathscr{M}^+(0,\infty)$. It is caused by the fact that the $L^p(v)$ space with a parameter $0 < p < 1$ admits locally nonintegrable functions with a singularity possible at any point $t \in (0,\infty)$. For more details, see [78], [VIII].

By the change of variables $t \to \frac{1}{t}$, the inequality (8) involving the operator H is transformed to a corresponding inequality involving the adjoint operator $\widetilde{H}$. Hence, boundedness of H between weighted Lebesgue spaces is equivalent to boundedness of $\widetilde{H}$ between another two weighted Lebesgue spaces.

There is a generalization of the Hardy operator which is especially important in the theory of weighted inequalities and weighted function spaces. It has the following form.

Let $U : [0,\infty)^2 \to [0,\infty)$ be a measurable function satisfying:

(i) $U(x,y)$ is nonincreasing in x and nondecreasing in y;

(ii) there exists a constant $\vartheta \in (0,\infty)$ such that

$$U(x,z) \le \vartheta \left(U(x,y) + U(y,z)\right)$$

for all $0 \le x < y < z < \infty$;

(iii) $U(0,y) > 0$ for all $y > 0$.

Then define

$$H_U f(t) := \int_0^t f(s) U(s,t)\,\mathrm{d}s, \quad \widetilde{H}_U f(t) := \int_t^\infty f(s) U(t,s)\,\mathrm{d}s, \quad t > 0, \qquad (9)$$

for any $f \in \mathscr{M}(0,\infty)$ such that the involved integral makes sense. The *kernel* U satisfying the conditions (i)–(iii) is sometimes called the *Oinarov kernel*. In paper [VIII], the name *ϑ-regular kernel* is used instead to emphasize the exact value of the constant ϑ (and to hint that R. Oinarov was not the first to use such a kernel).

Boundedness of the operator H_U between the weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ was, with respect to various settings of parameters p, q, studied and characterized by S. Bloom and R. Kerman [14], R. Oinarov [88], V. D. Stepanov [115, 117], Q. Lai [77], D. V. Prokhorov [99] and by the author in paper [VIII]. The characterizing conditions obtained in these papers are not listed in this summary. Instead, the reader may find them in [VIII] and the references therein.

The operator H_U includes the ordinary Hardy operator H as a special case ($U \equiv 1$). Another typical example of a ϑ-regular kernel is the function $U(s,t) := \int_s^t u(x)\,\mathrm{d}x$, where u is a nonnegative locally integrable function (of one variable). Many complicated problems related to weighted inequalities, in particular those involving various kinds of iterated Hardy-type operators, may be approached by methods which in their final phase reduce the problem to dealing with a H_U operator (including the case $U \equiv 1$). In this sense, operators with ϑ-regular kernels can be viewed as a cornerstone of the theory of weighted Hardy-type inequalities and related function spaces.

As the example (7) showed, there is an interest in studying restricted Hardy-type inequalities stemming from their immediate application to problems concerning boundedness of maximal operators. By the term *restricted* it is meant here that a certain inequality holds for all functions from a given subset of $\mathscr{M}^+(0,\infty)$. Examples of such subsets may be the cones of all nonincreasing or nondecreasing functions from $\mathscr{M}^+(0,\infty)$, of all convex functions from there, etc. On the other hand, the term *nonrestricted* refers to an inequality being satisfied for all $f \in \mathscr{M}^+(0,\infty)$.

A simple example of a restricted inequality problem is characterizing when the weighted Hardy inequality (8) holds for all *nonincreasing* $f \in \mathscr{M}^+(0,\infty)$. The problem may be obviously rephrased as a question of finding conditions under which $\Lambda^p(v)$ is embedded to $\Gamma^q(\widetilde{w})$ with $\widetilde{w}(t) := w(t)t^{-q}$ for all $t > 0$. Clearly, the conditions $\mathscr{A}_1$, $\mathscr{A}_2$ are sufficient, in the respective settings of parameters p, q, for (8) to hold for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$. However, they are not necessary in this case. The validity of this restricted Hardy inequality was studied by many authors in numerous papers such as [26–30, 40, 55, 104, 109, 110, 116]. The corresponding characterizations are now known for the full range $p, q \in (0,\infty]$. To observe the difference between a nonrestricted and restricted inequality, the reader may compare the conditions $\mathscr{A}_1$, $\mathscr{A}_2$ with their counterparts related to the restricted problem (cf. [27]) which are shown below.

If $0 < p \le q < \infty$, then (8) holds for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{B}_1 := \sup_{t>0} \left(\int_0^t w(s)s^q \,\mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^t v(s)\,\mathrm{d}s \right)^{-\frac{1}{p}} < \infty$$

and

$$\mathscr{B}_2 := \sup_{t>0} \left(\int_t^\infty w(s)\,\mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^t V^{-p'}(s)v(s)s^{p'}\,\mathrm{d}s \right)^{\frac{1}{p'}} < \infty.$$

If $0 < q < p < \infty$, then (8) holds for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{B}_3 := \left(\int_0^\infty \left(\int_0^t w(s) s^q \, \mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t v(s) \, \mathrm{d}s \right)^{-\frac{r}{p}} w(t) t^q \, \mathrm{d}t \right)^{\frac{1}{r}} < \infty$$

and

$$\mathscr{B}_4 := \left(\int_0^\infty \left(\int_t^\infty w(s) \, \mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t V^{-p'}(s) v(s) s^{p'} \, \mathrm{d}s \right)^{\frac{r}{p'}} w(t) \, \mathrm{d}t \right)^{\frac{1}{r}} < \infty,$$

where $p' := \frac{p}{p-1}$ and $r := \frac{pq}{p-q}$. In the respective cases, the optimal constants C in the inequality (8) (restricted to nonnegative nonincreasing functions) satisfy $C \approx \mathscr{B}_1 + \mathscr{B}_2$ and $C \approx \mathscr{B}_3 + \mathscr{B}_4$.

The reader may also notice that the inequality (8) may hold in this restricted sense even for $0 < p < 1$ with nontrivial weights v, w. It contrasts with the nonrestricted case where this was impossible.

Another example of restricted inequalities are those restricted to the cone of quasi-concave functions. Naturally, this type represents embeddings and operator inequalities involving the Γ spaces. These were studied, for instance, in [41, 54, 107, 108].

In general, it can be said that working with restricted inequalities is more difficult than doing so with the nonrestricted ones. This observation led to the development of the so-called *reduction methods*. These have gained certain popularity since the 2000's [41, 43, 46, 49, 52–54, 107]. The idea behind these methods is to reduce a restricted weighted operator inequality to an equivalent nonrestricted one. The new inequality generally involves some new weights and probably a more complicated operator. However, in most cases the new problem becomes easier by the mere fact that the inequality is nonrestricted.

The slightly ambivalent term "more complicated operator" has been already used in this text several times. In the context of this thesis, it mostly means various variants of *iterated Hardy-type operators*. These are operators constructed by iterating or mixing the integral operators H, $\widetilde{H}$ and their *supremal* counterparts S, $\widetilde{S}$ defined by

$$Sf(t) := \operatorname*{ess\,sup}_{s \in (0,t)} f(s), \qquad t > 0,$$

and

$$\widetilde{S}f(t) := \operatorname*{ess\,sup}_{s \in (t,\infty)} f(s), \qquad t > 0,$$

for $f \in \mathscr{M}^+(0,\infty)$. Certain applications also require adding some "inner weights" to such iterated operators. Accordingly, some of the problems solved in the main papers involve iterated operators defined in the following way.

Let u be a weight and $m \in (0,\infty)$. Then for $f \in \mathscr{M}^+(0,\infty)$ define

$$\begin{aligned} G_I f(t) &:= H^{\frac{1}{m}}(u\widetilde{H}^m f)(t), & G_S s(t) &:= S(u\widetilde{H} f)(t), \\ A_I(t) &:= H^{\frac{1}{m}}(uH^m f)(t), & A_S(t) &:= S(uHf)(t) \end{aligned} \tag{10}$$

at each point $t > 0$. Similarly, the "adjoint" variants of these operators are defined by replacing each operator H and S by its respective "adjoint" version $\widetilde{H}$ and $\widetilde{S}$ and vice versa in the above definitions. For example,

$$\widetilde{G}_I f(t) := \widetilde{H}^{\frac{1}{m}}(uH^m f)(t).$$

The operators G_I, G_S, $\widetilde{G}_I$ and $\widetilde{G}_S$, each of which is composed of one "ordinary" operator H or I and one "adjoint" operator $\widetilde{H}$ or $\widetilde{S}$, will be summarily called *gop operators*. This name refers to the initials of the authors of the paper [44] where boundedness of G_S between weighted Lebesgue spaces was studied. Not surprisingly, operators A_I, A_S, $\widetilde{A}_I$ and $\widetilde{A}_S$ then bear the name *antigop operators*. The letters I and S in the lower indices stand for "integral" and "supremal".

Gop and antigop operators play a prominent role in interpolation theory [10–12, 17, 22]. Besides that, they frequently appear as an outcome of applying a reduction method to problems involving restricted weighted inequalities for (iterated) Hardy operators. This is the case in paper [VII], where a certain weighted double-operator inequality is studied and reduced into problems of boundedness of gop and antigop operators between $L^p(v)$ and $L^q(w)$.

3.2 Convolution

Investigation of various properties of the convolution operator was the main task in the original thesis topic proposal. In the final outcome, convolution is the topic of papers [I–III] and [IX]. A general background is provided by this section.

Given two functions $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, their *convolution* $f * g$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{11}$$

if the integral makes sense. The space $\mathbb{R}^d$ (with the d-dimensional Lebesgue measure) is considered to be the integration domain, unless specified else. Definitions with different underlying measure spaces are possible and some of them will be mentioned later.

The concept of convolution has a very broad use both in the theory and practical applications. On the theoretical level, it is, above all, prominent in Fourier analysis and approximation theory (see e.g. [34, 57, 58, 120]). As it was said in the beginning of this introductory summary, results from these fields of mathematics have direct practical applications. In the particular case of convolution, the fields in which the concept is applied are, for example, signal and image processing, electrical engineering, probability, statistics, etc.

The convolution, understood as an operator acting on $L^1_{\mathrm{loc}}(\mathbb{R}^d) \times L^1_{\mathrm{loc}}(\mathbb{R}^d)$, is a bilinear operator. One may also fix a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and consider the linear operator

$$T_g f(x) := (f * g)(x), \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{12}$$

The function g will be called a kernel, similarly to the previous use of the word. Numerous important operators can be expressed as T_g with a particularly chosen kernel g. Examples include the Riesz potential (fractional integral) operator or Riemann-Liouville integral [11, 58], Bessel [58] and Newton [35] potential operators, Hilbert and Riesz transforms [11, 57], Stieltjes transform [118], mollifiers [1, 35], etc. In Fourier analysis, convolution with Dirichlet, Fejér and Jackson kernels appears frequently in the theory (see [34, 57, 120]).

As it can be expected by a reader who has gone through the previous section, boundedness of convolution operators between function spaces is one of the main questions in this thesis. In case of the operator T_g, the problem may be stated as follows. Given the domain function space X and the range space Z, find conditions on the kernel g, under which there holds

$$\|f * g\|_Z \le C(g)\|f\|_X, \qquad f \in X. \tag{13}$$

This notation means that (for each fixed g) there exists a constant $C(g) \in (0, \infty)$ such that the inequality holds for all functions $f \in X$. Analogous notation is used from now on. The validity of (13) therefore defines boundedness of T_g between X and Z. The sought conditions should, as usual, characterize the boundedness, i.e., they should be both necessary and sufficient.

In the main papers which deal with convolution, the main focus is laid on inequalities of the form (13) in which the term $C(g)$ is equal or equivalent to a norm of the kernel g in a certain function space Y. In this case, the concerned convolution inequality gets the visually pleasant shape

$$\|f * g\|_Z \le C\|f\|_X \|g\|_Y, \qquad f \in X,\ g \in Y. \tag{14}$$

Naturally, the constant C is meant to be independent of both f and g.

The most famous and fundamental result of the above type is the classical *Young inequality*. It reads as follows. If $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, then

$$\|f * g\|_q \le \|f\|_p \|g\|_r, \qquad f \in L^p,\ g \in L^r.$$

The assumption $p \le q$ may not be dropped, see [62] for a proof that T_g, unless it is trivial, is not bounded between L^p and L^q when $q < p$.

The Young inequality is essential whenever convolution is used in connection with function spaces. Its classical applications are found in pure analysis and theory of partial differential equations (see [1, 11, 12, 57, 120]). A more peculiar example is given by the use of the Young inequality within the kinetic theory of gases (see [3, 4, 60] and the references therein).

The original Young inequality might be considered a model result for many further developments. Not surprisingly, convolution inequalities having the form (14) are often called *Young-type (convolution) inequalities*.

There has been an extensive research into more general Young-type inequalities (14) with spaces X, Y, Z other than L^p. In his fundamental paper [89], R. O'Neil proved a theorem which has since become known as the *Young-O'Neil inequality*. This theorem states that, if $1 < p, q, r < \infty$ and $1 \le a, b, c \le \infty$ are such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, then

$$\|f * g\|_{q,a} \le C\|f\|_{p,b}\|g\|_{r,c}, \qquad f \in L_{p,b},\ g \in L_{r,c}. \tag{15}$$

An essential contribution of the paper [89] is the proof of a particularly important pointwise inequality, which will be referred to as the *O'Neil inequality*. It states that, for any $f, g \in L^1_{\mathrm{loc}}$ and any $t > 0$, one has

$$(f * g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\, \mathrm{d}s. \tag{16}$$

The proof of this inequality, as presented in [89], works correctly for the ordinary convolution (as given by (11)) but it contains some flaws if used with O'Neil's more general definition of a convolution operator. This was observed and corrected by L. Y. H. Yap in [119] by adding certain assumptions into the definition of a general convolution operator.

The O'Neil inequality is sharp in the following sense. There exists a constant D depending on the dimension of $\mathbb{R}^d$ and such that for all radially decreasing functions f, $g \in \mathcal{M}^+(\mathbb{R}^d)$ and all $t > 0$ one has

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\, \mathrm{d}s \le D (f * g)^{**}(t). \tag{17}$$

A function $f \in \mathcal{M}^+(\mathbb{R}^d)$ is called *radially decreasing* if there exists a nonincreasing function $\varphi \in \mathcal{M}^+(0, \infty)$ such that $f(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^d$. The reverse inequality (17) for $d = 1$ was mentioned in [89] without proof. In paper [I], an elementary proof for that case is shown (cf. also [105]). The proof for a general dimension d may be found in [72].

In [15, 63, 119], inequality (15) was shown to hold even for an extended range of parameters $0 < a, b, c \le \infty$ (while the other conditions on a, b, c, p, q, r remain the same as above). Furthermore, a limiting case of (15) with $1 < p < \infty$, $1 \le b, c \le \infty$, $1 = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c} < 1$ was studied by H. Brézis and S. Wainger in [19], leading to the following result:

$$\|f * g\|_{BW_a} \le C\|f\|_{p,b}\big(\|g\|_{r,c} + \|g\|_1\big), \qquad f \in L_{p,b},\ g \in L_{r,c} \cap L_1,$$

where

$$\|f\|_{BW_a} := \left(\int_0^1 \left(\frac{f^*(t)}{1+|\log t|} \right)^a \frac{\mathrm{d}t}{t} \right)^{\frac{1}{a}}.$$

A. P. Blozinski [16] considered another limiting case of the parameters, namely such that $p = q > 1$, $r = 1$ and $0 < a, b \leq \infty$. He showed that, with these parameters, if $g \geq 0$ and $T_g : L_{p,b} \to L_{p,a}$, then necessarily $g = 0$ a.e. It is important to notice that in this case the Lebesgue spaces are defined over the whole $\mathbb{R}^d$, i.e., $L_{p,a} = L_{p,a}(\mathbb{R}^d)$ and $L_{p,b} = L_{p,b}(\mathbb{R}^d)$. If $\mathbb{R}^d$ is replaced by an underlying measure space with a finite measure, then the aforementioned result of [16] does not need to be true, as it is shown below.

E. Nursultanov and S. Tikhonov [87] investigated boundedness of convolution of 1-periodic functions in $L_{p,q}$ spaces. Such functions may be equivalently represented by functions on a torus. Naturally, the involved $L_{p,q}$ spaces are also defined so that the underlying measure space is the interval $(0,1)$ (or the torus) equipped with the Lebesgue measure. The authors of [87] showed that, in the setting $1 \leq p < \infty$, $1 \leq a, b, c \leq \infty$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, one has

$$\|f * g\|_{L_{(p,a)}(0,1)} \leq C \|f\|_{L_{(p,b)}(0,1)} \|g\|_{L_{(1,c)}(0,1)}, \qquad f \in L_{(p,b)}(0,1),\; g \in L_{(1,c)}(0,1).$$

This contrasts with the previous negative result of Blozinski (recall that $L_{p,q} = L_{(p,q)}$ if $1 < p < \infty$ and $1 \leq q \leq \infty$). Among other results of [87] is the Young-O'Neil inequality for spaces $L_{\infty,p}(0,1)$, which states that if $0 < \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, then

$$\|f * g\|_{L_{\infty,a}(0,1)} \leq 4 \|f\|_{L_{\infty,b}(0,1)} \|g\|_{L_{(1,c)}(0,1)}, \qquad f \in L_{\infty,b}(0,1),\; g \in L_{(1,c)}(0,1).$$

Young-type inequalities and boundedness of convolution operators were further studied in the framework of weighted Lebesgue spaces with power weights [20, 65], L^p spaces with general Borel measures [5] and Wiener amalgam spaces [63]. In [13, 37, 67], the authors investigated under which conditions the $L^p(w)$ space is a convolution algebra, i.e., when the inequality

$$\|f * g\|_{L^p(w)} \leq \|f\|_{L^p(w)} \|g\|_{L^p(w)}, \qquad f, g \in L^p(w),$$

is satisfied. The convolution algebra property of r.i. spaces and various general properties of the convolution operator acting on r.i. spaces were also investigated by E. A. Pavlov in [93–98].

Analogues of the Young inequality in the Lebesgue spaces with variable exponent $L^{p(x)}$ were obtained by S. Samko in [101, 102] (see also [23, 103] and the references given therein).

Moreover, in [90] R. O'Neil investigated the behavior of a convolution operator in Orlicz spaces, providing a corresponding Young-type convolution inequality for these spaces.

4
Content summary of the main papers

In this chapter, the reader may find an overview of the content of the main papers. It includes the research questions, provided answers and other contributions of the papers, an outline of used methods, relations of the obtained results to the previously existing ones, applications, etc.

4.1 Forever Young

The goal of papers [I–III, IX] is to provide conditions of boundedness of the convolution operator in the weighted Lorentz-type spaces/classes Γ, Λ and S, and related Young-type convolution inequalities. The three first papers, i.e., [I–III], were contained in the author's licentiate thesis [74] bearing the appropriate name “Forever Young”. The remaining paper [IX] was not a part of [74] but was finished later, complementing the results of [I].

The problems treated in the aforementioned papers were not investigated by other authors before, except for some special cases of weights such as those establishing the $L_{p,q}$ spaces. (Those older results were listed in the survey in Section 3.2.) The setting of papers [I–III, IX] offers a considerably greater generality, making little or no assumptions on the weights. Moreover, the technique implemented in these papers is different from those used in previously existing works.

4.1.1 Papers [I] and [IX]

Let two weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ be given. The research questions of papers [I, IX] are stated as follows.

(i) Characterize the conditions on the kernel g, the weights and exponents under which the convolution operator T_g (see (12)) is bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$.

(ii) Find the optimal r.i. lattice Y such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Lambda^p(v)}\|g\|_Y, \qquad f \in \Lambda^p(v),\ g \in Y, \tag{18}$$

holds (with C independent of f, g).

Optimality of Y has the following meaning: if there exists another r.i. lattice $\widetilde{Y}$ such that (18) is satisfied with $\widetilde{Y}$ in place of Y, then necessarily $\widetilde{Y} \hookrightarrow Y$. In this sense, the optimal r.i. lattice Y is the largest r.i. lattice for which (18) holds.

In order to provide answers to the questions, the following method is implemented. At first, the O'Neil inequality (16) is used, giving

$$\|f * g\|_{\Gamma^q(w)} \leq \left\| t \mapsto t f^{**}(t) g^{**}(t) + \int_t^\infty f^* g^* \right\|_{L^q(w)}. \tag{19}$$

If the right-hand side can be estimated by the term $\|f\|_{\Lambda^p(v)}$, then T_g is bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. Such estimates correspond to the inequalities

$$\|t \mapsto t f^{**}(t) g^{**}(t)\|_{L^q(w)} \leq C_1 \|f\|_{\Lambda^p(v)}, \qquad f \in \Lambda^p(v),$$

and

$$\left\| t \mapsto \int_t^\infty f^* g^* \right\|_{L^q(w)} \leq C_2 \|f\|_{\Lambda^p(v)}, \qquad f \in \Lambda^p(v). \tag{20}$$

Both of these are weighted Hardy-type inequalities restricted to nonnegative nonincreasing functions. They have been systematically studied (cf. Section 3.1) and the optimal constants $C_i = C_i(q, v, w, p, q)$, $i \in \{1, 2\}$ the inequalities hold with are known, see [25, 27, 54], [VIII]. (This knowledge had a certain gap leading to the split of articles [I] and [IX], see below.) One thus gets the condition

$$C_i < \infty, \qquad i \in \{1, 2\}, \tag{21}$$

which is obviously sufficient for boundedness of T_g between $\Lambda^p(v)$ and $\Gamma^q(w)$. To prove that it is also necessary, one uses the reverse O'Neil inequality (17). It yields that if $g \in \mathscr{M}^+(\mathbb{R}^d)$ is radially decreasing, then (21) is a necessary condition for boundedness of T_g from $\Lambda^p(v)$ to $\Gamma^q(w)$.

In the next step, it is observed that the sum $C_1 + C_2$ is equivalent to an r.i. norm (or quasi-norm) of g, denoted by $\|g\|_Y$. It gives the Young-type inequality (18). Optimality of Y is granted thanks to the necessity part (valid for nonnegative radially decreasing functions g) and thanks to the space Y being rearrangement-invariant.

The range of exponents covered by [I] is $0 < p \leq q \leq \infty$, $1 \leq q < p < \infty$ and $0 < q < p = \infty$. The range restriction (compared to the whole quadrant $p, q \in (0, \infty]$) is caused by the fact that, at the time of [I], the validity of (20) was not satisfactorily characterized in the notorious case $0 < q < 1$ & $q < p < \infty$. The latter problem is equivalent (see [54, Theorem 4.1]) to characterizing boundedness of a particular type of the operator $\widetilde{H}_U$ (see (9)) between weighted Lebesgue spaces. For the setting of parameters required by this particular situation, the solution was known only in form of a discrete condition due to Q. Lai [78]. It was, however, inappropriate for the intended application. This whole problem was later eliminated by paper [VIII]. Based on that improvement, paper [IX] could be written, completing the results of [I]. Hence, [IX] deals with the same problem as [I] in the originally missing case $0 < q < 1$ & $q < p < \infty$. The complete range of parameters provided by both [I] and [IX] becomes $p, q \in (0, \infty]$.

The article [I] is written in such way that the results cover convolution on both $\mathbb{R}^d$ and on a compact interval for periodic functions. It is shown that the "classical" results [63,87,89,119] follow as special cases of the presented theorems. In particular, it is pointed out that both the result of [16], stating that T_g with a nonnegative nontrivial g is not bounded between $L_{p,b}(\mathbb{R}^d)$ and $L_{p,a}(\mathbb{R}^d)$, $p > 1$, and the result of [87], stating that the same boundedness is possible in case of 1-periodic functions on $[0,1]$, are consequences of a single theorem of [I]. The proven optimality of the domain space Y is a key point for drawing such conclusions.

The optimality aspect together with the general-weight setting are the main advantages of the results in [I, IX]. Thanks to the proven necessity of the provided conditions in case of a nonnegative radially decreasing kernel g, the general results of these papers may be directly applied to particular operators which have a form of convolution with a symmetrical kernel. The Riesz fractional integral operator (convolution with the Riesz potential) is a typical example, other similar and plausible operators were named in Section 3.2. In this way, one obtains characterizations of boundedness of such operators between the concerned Lorentz spaces.

Furthermore, the last part of [I] deals with r.i. spaces which appear as the optimal domain Y. As a rule, the space Y may be expressed as an intersection of certain Γ spaces and another type of an r.i. space with a norm based on an iterated Hardy operator. The latter type of a function space is denoted by "K" in [I]. It is generated by the functional

$$\|f\|_{K^{p,q}(u,v)} := \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t))^p u(t)\,\mathrm{d}t \right)^{\frac{q}{p}} v(x)\,\mathrm{d}x \right)^{\frac{1}{q}}, \tag{22}$$

or the "weak" variants of it created by the standard replacement of one of the integrals by an essential supremum over the same domain. (Are both the integrals replaced in such manner, the space becomes a weak Γ space.)

A K-type space with a special choice of weights appeared, for example, in [32] in connection with Sobolev embeddings of Morrey spaces. Recently, this type of a space was identified in [51] as the associate space to a generalized Γ space. In [IV, V], embeddings of such spaces are used to handle bilinear Hardy operator inequalities. Above all, most relevant for [I] is the role of these spaces as the optimal domain Y in the investigated Young-type inequalities. The K spaces (with other weight and exponent settings) play the same role in [II, III, IX] as well. The final section of [I] contains a summary of their elementary properties.

Both papers [II] and [III] deal with questions which are analogous to those of [I] in other Lorentz-space settings. The same method is implemented, using the O'Neil inequality and reduction of the problem to weighted Hardy inequalities. The results also feature similar optimality properties. Corresponding details are therefore omitted in the content descriptions of [II] and [III] below.

4.1.2 Paper [II]

The investigated problem reads as follows. Given $p, q \in (0,\infty]$ and weights v, w, characterize when T_g is bounded between $S^p(v)(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\Gamma^q(w)(\mathbb{R})$. Again, the main goal is to give a result in the shape of a Young-type inequality, in this case of the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{S^p(v)}\|g\|_Y, \qquad f \in S^p(v),\ g \in Y \cap L^1.$$

The O'Neil-inequality-based method from [I] is used to solve this problem. Another ingredient is observing that the right-hand side of O'Neil's inequality (16) is equal to

$$\lim_{s\to\infty} s f^{**}(s) g^{**}(s) + \int_t^\infty (f^{**} - f^*)(g^{**} - g^*)$$

for any $t > 0$. If $f \in S^p(v)$ and $g \in L^1$, the first term is equal to zero, thus the whole problem is equivalent to characterizing the validity of the inequality

$$\left\| t \mapsto \int_t^\infty (f^{**} - f^*)(g^{**} - g^*) \right\|_{L^q(w)} \le C\|f\|_{S^p(v)}, \qquad f \in S^p(v). \tag{23}$$

For any $f \in \mathcal{M}(\mathbb{R})$, the function

$$t \mapsto t(f^{**}(t) - f^*(t)) \tag{24}$$

is nonnegative and nondecreasing on $(0,\infty)$, and any nonnegative nondecreasing functions may be approximated by functions having the form (24) for some $f \in \mathcal{M}(\mathbb{R})$ (cf. [108, Lemma 1.2]). Hence, (23) is equivalent to

$$\left\| t \mapsto \int_t^\infty \varphi(s) \frac{g^{**}(s) - g^*(s)}{s} \, \mathrm{d}s \right\|_{L^q(w)} \le C\|\varphi\|_{L^p(\varrho)}, \quad \varphi \in \mathcal{M}^+(0,\infty), \text{ nonincreasing},$$

where $\varrho(t) := v(t)t^{-p}$. Then, known characterizations of the validity of the last inequality are used to complete the work.

The results have similar properties (e.g., optimality) as their counterparts in [I]. The range of parameters covered by [II] is $p, q \in (0,\infty]$.

4.1.3 Paper [III]

This paper focuses on the problem of boundedness of T_g between spaces $\Gamma^p(v)$ and $\Gamma^q(w)$. Once again, the final result is the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \qquad f \in \Gamma^q(w),\ g \in Y,$$

with the r.i. space Y being optimal for the given pair of spaces $\Gamma^p(v)$ and $\Gamma^q(w)$ in the same sense as in the previous papers.

The method from [I] is applicable again. Similarly to [II], rewriting the right-hand side of the O'Neil inequality (16) in different terms proves to be advantageous. In particular, the following observation is made.

Consider $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If there exists a function $\gamma \in \mathscr{M}^+(0,\infty)$ with compact support and such that

$$g^*(t) = \int_t^\infty \frac{\gamma(s)}{s}\,\mathrm{d}s, \qquad t > 0, \tag{25}$$

then for all $t > 0$ there holds

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s = f^{**}(t)\int_0^t \gamma(s)\,\mathrm{d}s + \int_t^\infty \gamma(s)f^{**}(s)\,\mathrm{d}s =: T(f^{**})(t).$$

Hence, it is needed to characterize the validity of the inequality

$$\|T(f^{**})\|_{L^q(w)} \le C\|f^{**}\|_{L^p(v)}, \qquad f \in \mathscr{M}(\mathbb{R}^d).$$

To this end, a reduction theorem from [41] for linear operators acting on the cone of quasi-concave functions is applied. The standard observation confirming that the nonincreasing rearrangement of any function g^* can be approximated by functions of the form (25) is also used to extend the results to a general function g.

The described approach based on [41] is adopted in the case $p, q \in (1,\infty)$. In the other cases presented in the paper, different methods are used, involving other known results about Hardy-type inequalities.

The range of exponents covered by [III] is $1 \le p, q \le \infty$, $0 < p < 1$ & $q \in \{1,\infty\}$ and $p = \infty$ & $0 < q < 1$.

4.2 Bilinear Hardy operators

Numerous bilinear or multilinear operators can be produced by combining classical linear operators (such as Hardy or Copson) in form of a product or in various other ways.

An application of such bilinear mappings is illustrated by the central role of the operators

$$R_1(f,g)(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s, \; R_2(f,g)(t) := \int_t^\infty f(s)g(s)\,\mathrm{d}s, \; t > 0, \tag{26}$$

in the papers on convolution presented in the previous section (cf. the O'Neil inequality (16) and its subsequent use). The idea of characterizing boundedness of bilinear Hardy operators by further developing some techniques from [I] was suggested to the author by J. Soria and led to the creation of papers [IV] and [V].

4.2.1 Paper [IV]

A simple bilinear Hardy-type operator is defined by

$$T(f,g)(t) := \int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s, \qquad t>0, \tag{27}$$

for any $f, g \in L^1_{\mathrm{loc}}(0,\infty)$. With exception of the factor $\frac{1}{t}$, this operator is identical to the operator R_1 mentioned in the previous paragraph. The purpose of paper [IV] is to give characterizations of boundedness of T, restricted to nonnegative nonincreasing functions, between $\Lambda^{p_1}(v_1) \times \Lambda^{p_2}(v_2)$ and $L^q(w)$ (or, equivalently, between $L^{p_1}(v_1) \times L^{p_2}(v_2)$ and $L^q(w)$). In other words, the desired result is a characterization of the validity of the weighted bilinear Hardy inequality

$$\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_0^\infty (f^*)^{p_1} v_1\right)^{\frac{1}{p_1}} \left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}$$

for all functions $f, g \in \mathscr{M}(\mathbb{R})$, with obvious modifications for the weak spaces with p_1, p_2 or q equal to ∞. Notice that the real line $\mathbb{R}$ as the domain of the functions $f, g \in \mathscr{M}(\mathbb{R})$ can be replaced by any reasonable measure space $(\Omega, \mathfrak{M}, \mu)$, if needed.

The result is proved by a so-called *iteration method*. The idea of it is somewhat similar to the one used in the articles about convolution operators. In the first step, the function g is fixed and considered a part of the weight

$$\psi(t) := \left(\int_0^t g^*(s)\,\mathrm{d}s\right)^q w(t), \qquad t>0.$$

The problem is then approached as a standard weighted Hardy inequality for nonincreasing functions,

$$\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^q \psi(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le D \left(\int_0^\infty (f^*(t))^{p_1} v_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}, \qquad f \in \mathscr{M}(\mathbb{R}),$$

allowing the use of the known descriptions of the optimal constant D which can be written as

$$D = \sup_{\substack{f\in\mathscr{M}(\mathbb{R}) \\ \|f\|_{\Lambda^{p_1}(v_1)} \neq 0}} \frac{\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^q \psi(t)\,\mathrm{d}t\right)^{\frac{1}{q}}}{\left(\int_0^\infty (f^*(t))^{p_1} v_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}}.$$

This quantity depends on the function g (contained in the weight ψ) and can be in all cases expressed as $\|g\|_X$, where X is an r.i. lattice that can be described as an intersection of certain “K spaces” (see (22)) and “J spaces”. The latter type is an analogue to the K space and it is obtained by replacing the integral $\int_x^\infty$ by $\int_0^x$ in the (quasi-)norm (22). Details are, naturally, to be found in [IV].

The next step of the iteration method is to characterize, in terms of p_1, p_2, q, v_1, v_2 and w, when the inequality

$$\|g\|_X \leq C\left(\int_0^\infty (g^*(t))^{p_2} v_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}, \qquad g \in \mathscr{M}(\mathbb{R}),$$

is satisfied. Thanks to the construction, the optimal C in here is also the requested optimal C in the original bilinear Hardy inequality. Due to the nature of $\|\cdot\|_X$, the problem reduces to characterizing certain embeddings $\Lambda \hookrightarrow J$ and $\Lambda \hookrightarrow K$. Providing such characterizations makes a substantial part of the work in [IV]. In that paper however, those characterizations play a rather auxiliary role and are used as means of solving the main problem concerning the bilinear Hardy inequality. Nevertheless, the description of the involved embeddings is of independent interest exceeding the particular application in [IV].

One of the ambitions of [IV] was to provide a complete list of conditions for all possible cases of exponents $p_1, p_2, q \in (0, \infty]$. This was achieved indeed, with certain logical consequences for the final length of the paper (there are 23 different cases).

4.2.2 Paper [V]

The "point of departure" of the author's research on bilinear Hardy operator inequalities carried out in papers [IV, V] was the article by Aguilar, Ortega and Ramírez [2]. It contains a characterization of boundedness of the bilinear Hardy operator T from (27) between $L^{p_1}(v_1) \times L^{p_2}(v_2)$ and $L^q(w)$. This result motivated the question whether an analogy could be proved in the restricted case – that was the problem solved in [IV].

In the first part of [V], the original problem from [2] was revisited. Namely, it was shown that the results of [2] can be obtained in a significantly simpler way by the iteration method. Moreover, more equivalent forms of the characterizing conditions were found, in most cases reducing the number of terms required in the expressions. Existence of equivalent conditions is a common feature in problems concerning weighted inequalities (cf. [27, 42, 45]) but it was not observed in [2]. Knowledge of the equivalent expressions is rather practical, especially when it is needed to combine or compare various weighted conditions. Frequently, this was the case in papers [I–IV].

Paper [V] continues with another part, the purpose of which is to demonstrate the application of the iteration method to other problems related to bilinear and multilinear operators. Several variants of Hardy and similar bilinear operator inequalities are chosen as examples. The point was not to give full characterizations as in the first part of [V] or in [IV] but rather to show a universal way how to find these. Whenever there is interest in doing so, the reader should be able to apply the techniques described in [V] to get explicit solutions to the problems presented in the paper.

4.3 Iterated operators

Iterated operators, in particular those associated with the name Hardy, were introduced in Section 3.1. It might be useful to emphasize the difference between *iterated operators* and the *iteration method* of treating bilinear operators, since both notions appear frequently here. An *iterated operator* T is constructed by composition of two or more "known" operators T_i, i.e., $T = T_1 \circ T_2$. Above all, the name is used in here for iterated Hardy operators, for instance such as the gop and antigop operators from (10). In contrast, the *iteration method* is simply the technique used in papers [I–V] to treat bilinear operators.

Studying nonrestricted inequalities representing boundedness of Hardy operators, simple or iterated, between weighted Lebesgue spaces is a fundamental problem. It can be illustrated by the following observations.

The point of the reduction methods (see Section 3.1) is to represent a restricted weighted inequality by one or more nonrestricted weighted inequalities. The price to pay in the latter case is usually the presence of a more complicated (e.g., iterated) operator in the nonrestricted inequality. Reverting the process, i.e., representing a nonrestricted inequality by a restricted one, is possible for some weights but not in general.

Next, the reduction only says that one problem is equivalent to another, a direct solution of one of them thus still needs to be found. If one has to choose whether to aim for a direct proof of a nonrestricted problem or of a restricted one, the first option is usually preferable, even if it involves dealing with a more complicated operator.

Since the restriction is given in terms of monotonicity of functions, Hardy operators naturally appear when reduction methods are used – this stems from representing a nonincreasing nonnegative function f by $\int_{\cdot}^{\infty} h(s)\,\mathrm{d}s$, where $h \in \mathscr{M}^+(0,\infty)$. Nondecreasing functions are represented in an analogous way. All these aspects make nonrestricted inequalities with (iterated) Hardy operators a "root case".

Hardy operators which are fundamental, in the sense of the previous description, were researched in the papers presented below, in particular in [VI, VIII]. Paper [VII] deals with a more complicated problem by its systematic reduction to more of the "root-case" operator inequalities, therefore it represents an application of these fundamental results.

4.3.1 Paper [VI]

The first research problem of this paper is to characterize under which conditions the inequality

$$\left(\int_0^\infty \left(\sup_{s\in[t,\infty)} u(s) \int_s^\infty f(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \quad f \in \mathscr{M}^+(0,\infty),$$

holds, with $p \in [1,\infty)$ and $q \in (0,\infty)$ and weights u, v, w. In other words, one is asking for a characterization of boundedness of the supremal antigop operator

$$\widetilde{A}_S : f \mapsto \widetilde{S}(u\widetilde{H}f), \qquad f \in \mathscr{M}^+(0,\infty)$$

(see (10)) with the inner weight u, between $L^p(v)$ and $L^q(w)$.

The second question answered in [VI] is under which conditions the inequality

$$\left(\int_0^\infty \left(\sup_{s\in[t,\infty)} u(s)g(s) \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}$$

is satisfied for all nonincreasing functions $g \in \mathscr{M}^+(0,\infty)$. By a simple reduction argument (shown in Theorem 8 of [VI]), the answer to the second question follows from the answer to the first one.

The second problem was investigated already in [44] and an explicit characterization was given there for exponents satisfying $0 < p \leq q < \infty$. However, in the remaining case $0 < q < p < \infty$, the authors of [44] produced only a discrete condition that involves a supremum over all possible partitions of the interval $(0,\infty)$. Such conditions are unfortunately almost nonverifiable and this effectively prevents them from being used in any applications.

In paper [VI], both the first and the second problem were solved by providing explicit integral conditions for all cases of positive p and q. Finding the correct form of the explicit condition related to the $q < p$ case is the main achievement of [VI]. This had not been done before and it opened the door to completing the theory of related weighted inequalities by providing reasonable conditions for all plausible cases of exponents.

The proofs in [VI] are based on the method of *(dyadic) discretization*, also called the *blocking technique*. This method is an excellent means of dealing with Hardy-type inequalities in weighted settings. A classical introduction to the technique may be found in the book [59].

The core of the discretization method is a simple but extremely useful proposition which reads as follows.

Let $\alpha \in (0,\infty)$. Then there exists a positive constant $C = C(\alpha)$ such that for all $k_{\min}, k_{\max} \in \mathbb{Z} \cup \{\pm\infty\}$ such that $k_{\min} < k_{\max}$ and all nonnegative sequences $\{a_k\}_{k=k_{\min}}^{k_{\max}}$ one has

$$\sum_{k=k_{\min}}^{k_{\max}} 2^{-k} \left(\sum_{j=k_{\min}}^{k} a_j \right)^\alpha \leq C \sum_{k=k_{\min}}^{k_{\max}} 2^{-k} a_k^\alpha, \qquad \sum_{k=k_{\min}}^{k_{\max}} 2^{k} \left(\sum_{j=k}^{k_{\max}} a_j \right)^\alpha \leq C \sum_{k=k_{\min}}^{k_{\max}} 2^{k} a_k^\alpha.$$

These inequalities have more variants (see [54, 56] and [VI]). Namely, suprema can be used in place of sums, and the sequence $\{2^k\}$ may be replaced by any sequence of real numbers b_k such that $\beta := \inf_{k_{\min} \leq k < k_{\max}} \frac{b_{k+1}}{b_k} > 1$. In the latter case, the constant C also depends on the parameter β (C increases with decreasing β). In either case, by means of these inequalities one can eliminate the discrete Hardy operator (represented by the “inner sum”) in the expression.

The discrete inequalities from above are applied to the Hardy operators acting on functions in the following way. Consider, for example, the expression $\|\widetilde{H}f\|_{L^q(w)}$, i.e.,

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}. \tag{28}$$

For simplicity, suppose that $\int_0^\infty w(s)\,\mathrm{d}s = \infty$ and $0 < \int_0^t w(s)\,\mathrm{d}s < \infty$ for all $t \in (0,\infty)$. Then there exists a sequence $\{t_k\}_{k\in\mathbb{Z}}$ of points from $(0,\infty)$ such that $\int_0^{t_k} w(s)\,\mathrm{d}s = 2^k$ for each $k \in \mathbb{Z}$. Then one gets

$$\begin{aligned}
\left(\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} &= \left(\sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \\
&\approx \left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{t_k}^\infty f(s)\,\mathrm{d}s\right)^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{k\in\mathbb{Z}} 2^k \left(\sum_{j=k}^\infty \int_{t_j}^{t_{j+1}} f(s)\,\mathrm{d}s\right)^q\right)^{\frac{1}{q}} \\
&\approx \left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{t_k}^{t_{k+1}} f(s)\,\mathrm{d}s\right)^q\right)^{\frac{1}{q}}.
\end{aligned}$$

The symbol "$\approx$" has the usual meaning, in this particular case the equivalence constants may depend only on q. In the last expression there is no longer a Hardy operator present. If, for example, the goal is to compare this expression with the $L^p(v)$ norm of f, one can proceed just by using the Hölder inequality (in both its variants for functions and sequences). The term *discretization* refers to the pass from the integral expression at the beginning to the discrete sum at the end.

The discretization in its original form is able to eliminate one Hardy operator, either integral or supremal. However, the operator of interest in [VI] is the supremal antigop operator, so instead of (28), the initial expression is

$$\left(\int_0^\infty \left(\sup_{z\in[t,\infty)} u(z)\int_t^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Therefore, one needs to treat the inner operator as well.

The basic discretization method was improved in paper [VI] in order to meet this goal. The idea was to use a two-stage discretization constructed in the following way. The first stage is the same as in the simple discretization method, using a point sequence $\{t_k\}$ satisfying

$$\int_{t_k}^{t_{k+1}} w(s)\,\mathrm{d}s = 2\int_{t_{k-1}}^{t_k} w(s)\,\mathrm{d}s$$

for all relevant indices k. By this means, the outer supremal operator is eliminated in the way shown above in the example treating (28). In the second stage, a subsequence $\{t_{k_n}\}$ is constructed in such way that

$$\sum_{k=k_n}^{k_{n+1}} 2^k \sup_{t\in[t_k,t_{k+1}]} u^q(t) \geq 2\sum_{k=k_{n-1}}^{k_n} 2^k \sup_{t\in[t_k,t_{k+1}]} u^q(t)$$

for all relevant n. Clustering the "k-terms" into the "n-blocks" and using an appropriate elementary discrete proposition again, one can then eliminate the inner integral operator as well.

Naturally, what was shown in here is only a simplified description of the main idea of the technique. The reader may consult the complete version with all details in the text of paper [IV]. The discretization method of this kind works with no restrictions on the parameters $p, q \in (0,\infty)$ or on the weights. It can be applied to any kind of an "once-iterated" Hardy operator such as the gop and antigop operators.

4.3.2 Paper [VII]

The generalized Γ space $\mathrm{G}\Gamma^{p,m}(u,w)$ generated by the functional defined in (3) is the central object of interest in paper [VII]. The particular aspect which is studied in there is the existence of the embedding $\mathrm{G}\Gamma^{p_1,m_1}(u_1,w_1) \hookrightarrow \mathrm{G}\Gamma^{p_2,m_2}(u_2,w_2)$ between two different generalized Γ spaces. In other words, one wants to characterize, in terms of the exponents p_1, p_2, m_1, m_2 and the weights u_1, u_2, w_1, w_2, when the inequality

$$\left(\int_0^\infty \left(\int_0^t (f^*)^{m_2} u_2\right)^{\frac{p_2}{m_2}} w_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}} \leq C\left(\int_0^\infty \left(\int_0^t (f^*)^{m_1} u_1\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}} \tag{29}$$

holds for all $f \in \mathscr{M}(\mathbb{R}^d)$.

The inequality above is an example of a two-operator inequality with a different operator on each side. This type of an inequality is usually rather hard to deal with. A particular case of the presented problem with $u_1 = u_2$ and $m_1 = m_2$ was solved in [47]. However, adding the inner exponents and, especially, the different inner weights u_1, u_2 makes the problem significantly more difficult.

The motivation for investigating inequality (29) comes from certain problems in partial differential equations theory, (29) can be also used to provide a comparison between different weighted maximal operators in a Λ space setting via the Herz estimates (7). Besides that, inequalities of the form (29) also play an essential role in determining normability of the generalized Γ spaces by using the technique of [111].

The subject of [VII] is closely related to iterated Hardy-type operators. Indeed, if one substitutes f^* for $\int_{\cdot}^{\infty} h$, each side of (29) expresses a weighted-Lebesgue-space norm of a certain integral gop operator evaluated at h. More importantly, the proofs of the results in [VII] providing characterizations of the validity of (29) also rely strongly on reducing the problem into iterated Hardy operator inequalities. Paper [VII] makes use of a great amount of results concerning weighted Hardy-type inequalities. For instance, all the gop and antigop operators from (10) find their applications in [VII]. This justifies placing the paper in the section about iterated operators.

The proofs in [VII] are based on using the fact that, for $p \in (1,\infty)$, the space $L^{p'}(v^{1-p'})$ (with $p' = \frac{p}{p-1}$) is the associate space to $L^p(v)$. By definition, this means that for every $f \in \mathcal{M}(0,\infty)$, every weight v and exponent $p \in (1,\infty)$ one has

$$\|f\|_{L^p(v)} = \sup_{g \in \mathcal{M}^+(0,\infty)} \frac{\|fg\|_1}{\|g\|_{L^{p'}(v^{1-p'})}},$$

with the convention "$\frac{0}{0} = 0$", "$\frac{a}{0} = \infty$", "$\frac{a}{\infty} = 0$" ($a > 0$) applied. The argument is used to eliminate the inner integral in the expression on the left-hand side of (29). Namely, if $p_2 > m_2$, the left-hand side of (29) is equal to

$$\sup_{g \in \mathcal{M}^+} \frac{\left(\int_0^\infty g(t) \int_0^t (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2 m_2}}}$$
$$= \sup_{g \in \mathcal{M}^+} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2 m_2}}},$$

where $\mathcal{M}^+$ stands for $\mathcal{M}^+(0,\infty)$. The optimal constant C in (29) may be then written as follows.

$$
\begin{aligned}
C &= \sup_{f\in\mathcal{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s\right)^{\frac{p_2}{m_2}} w_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_1} u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}}\\
&= \sup_{f\in\mathcal{M}(\mathbb{R}^d)}\sup_{g\in\mathcal{M}^+} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}} w_2^{\frac{m_2}{m_2-p_2}}\right)^{\frac{p_2-m_2}{p_2m_2}} \left(\int_0^\infty \left(\int_0^t (f^*)^{m_1} u_1\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}}\\
&= \sup_{g\in\mathcal{M}^+} \frac{1}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}} w_2^{\frac{m_2}{m_2-p_2}}\right)^{\frac{p_2-m_2}{p_2m_2}}} \sup_{f\in\mathcal{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t (f^*)^{m_1} u_1\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}}\\
&= \sup_{g\in\mathcal{M}^+} \frac{1}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}} w_2^{\frac{m_2}{m_2-p_2}}\right)^{\frac{p_2-m_2}{p_2m_2}}} \left[\sup_{f\in\mathcal{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty (f^*(s))^{\frac{m_2}{m_1}} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{m_1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t f^* u_1\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{m_1}{p_1}}}\right]^{\frac{1}{m_1}}.
\end{aligned}
$$

The expression in the square bracket on the last line equals the optimal constant (cf. (5), (6)) of the embedding $\Gamma_{u_1}^{\frac{p_1}{m_1}}(\varphi) \hookrightarrow \Lambda^{\frac{m_2}{m_1}}(\psi)$, where $\varphi := w_1\left(\int_0^{\cdot} u_1\right)^{\frac{p_1}{m_1}}$ and $\psi := u_2 \int_{\cdot}^\infty g$. This constant can be expressed by the known characterizations from [47] as a term depending on the weights, exponents and the function g (but independent of f). It may have various forms depending on the exponents. Nevertheless, in all cases this form corresponds to a weighted-Lebesgue-space norm of the image of g under a certain gop or antigop operator, or to a sum of more such norms. Therefore, the whole problem reduces to a greater number of simpler problems concerning gop and antigop operators. These are handled by using appropriate known results, among them also those of paper [VI].

The used *duality method* relying on the expression of $L^p(v)$ as the associate space to $L^{p'}(v^{1-p'})$ can be, of course, applied to other questions, for example those concerning integral gop and antigop operators. However, its relative simplicity comes at the cost of the parameter restriction $p > 1$. In [VII], this condition is reflected by the restriction $p_2 > m_2$ which is present throughout the whole paper and cannot be lifted as long as the duality method is used. It is possible that the case $p_2 < m_2$ could be treated by a technique created by further improving the discretization method used in [47]. However, this is beyond the scope of paper [VII].

4.3.3 Paper [VIII]

The success in finding the missing explicit condition in paper [VI] clearly suggested how to solve another open problem. This problem has an even more fundamental character since it concerns the Hardy operator H_U with a ϑ-regular kernel U (see (9)). It was the last remaining case for which boundedness of H_U between $L^p(v)$ and $L^q(w)$ had not been characterized by an explicit integral condition. Namely, it involved the troublesome parameter setting $0 < q < 1 \le p < \infty$.

In the other cases, i.e., for $p, q \in [1, \infty]$, simple integral conditions have been known since the time the problem had gained interest in the 1990's. These results may be found in the articles [14, 88, 117]. In the setting $0 < q < 1 \le p < \infty$ however, only a discrete condition was known [77]. Recently, it was complemented by an integral condition [99] which though still contained a complicated implicit expression involving one of the weights. In [117] there was also shown that some conditions related to the case $1 < q < p < \infty$ were sufficient or necessary in the case $0 < q < 1 \le p < \infty$ but no combination of them gave the desired characterization (i.e., both necessity and sufficiency). Finding the correct integral conditions remained an open problem.

This problem was successfully solved in paper [VIII], therefore filling the gap and completing the theory concerning Hardy operators with ϑ-regular kernels. It should be noted that even though the parameter combination $0 < q < 1 \le p < \infty$ may seem rather obscure ($L^q(w)$ is not a Banach space then), the H_U-operator inequality with this setting is far from being useless. For example, using reduction methods to problems involving more complicated mappings (operators) between $L^q(w)$ to $L^p(v)$ with the setting $1 \le q < p < \infty$ often results in getting inequalities with the (left-hand-side) exponent q between 0 and 1, and the (right-hand-side) exponent p equal to 1. See, for instance, the reduction in [VI, Theorem 8] which is exactly the case when a certain characterization for $0 < q < 1 = p$ is necessary for solving the problem studied in there.

The proof technique employed in [VIII] is essentially the same as in [VI], thus it relies on a two-stage discretization method. A minor difference is taking the constant ϑ (related to the ϑ-regular kernel U) into account when constructing the sequence $\{t_k\}$. Obtained results are then applied to solve another open problem involving the Copson operator restricted to nonincreasing functions. This was later used in paper [IX] to complete the results concerning Young-O'Neil inequalities, as it was described in the section devoted to papers [I] and [IX].

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Paper I

Martin Křepela

Convolution inequalities in weighted Lorentz spaces

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CONVOLUTION INEQUALITIES IN WEIGHTED LORENTZ SPACES

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Abstract. We characterize boundedness of a convolution operator with a fixed kernel between the weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ for $0 < p \leq q \leq \infty$, $1 \leq q < p < \infty$ and $0 < q \leq p = \infty$. We provide corresponding weighted Young-type inequalities and also study basic properties of some new involved r.i. spaces.

1. Introduction

Methods involving convolution of a function f with a kernel function g, i.e.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x, \quad t \in \mathbb{R}, \tag{1}$$

have experienced a great attention and a widespread use in various important parts of analysis. By choosing a specific kernel in this general setting, we get many well-known operators, which themselves are of substantial importance. As examples here we can mention Newton, Riesz or Bessel potentials, Stieltjes and Hilbert transforms, mollifying operators, etc. One of the main questions in this field is the boundedness of the linear operator given by a fixed g and the formula

$$T_g : f \mapsto f * g$$

between certain function spaces. This problem is further related to convolution inequalities. The classic case is the well-known Young inequality stating that for $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ it holds

$$\|f * g\|_q \leq \|f\|_p \|g\|_r, \quad f \in L^p,\ g \in L^r.$$

Here $\|\cdot\|_p$ denotes the Lebesgue L^p-norm. The connection to the boundedness question is obvious: If X, Y, Z are given function spaces and the inequality

$$\|f * g\|_Z \leq C\|f\|_X \|g\|_Y, \quad f \in X,\ g \in Y, \tag{2}$$

we get the boundedness $T_g : X \to Z$ for any $g \in Y$. On the other hand, if we have the estimate $\|T_g\|_{X\to Z} \leq C\|g\|_Y$, then we retrieve (2). Notice here also that the assumption $p \leq q$ in the Young inequality cannot be avoided. Indeed, as shown

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by Hörmander in [9], a nontrivial convolution operator is never bounded from L^p to L^q if $q < p$.

The Young inequality was further developed for classical Lorentz spaces $L_{\alpha,\beta}$, $1 \le \alpha < \infty$, generated by

$$\|f\|_{L_{\alpha,\beta}} := \left(\int_0^\infty (f^*(x))^\beta x^{\frac{\beta}{\alpha}-1}\, \mathrm{d}x \right)^{\frac{1}{\beta}}, \quad 1 \le \beta < \infty,$$

$$\|f\|_{L_{\alpha,\infty}} := \sup_{x \in (0,\infty)} f^*(x) x^{\frac{1}{\alpha}},$$

and $L_{(\alpha,\beta)}$ generated by

$$\|f\|_{L_{(\alpha,\beta)}} := \|f^{**}\|_{L_{\alpha,\beta}}.$$

Here f^* stands for the nonincreasing rearrangement of f and f^{**} for the Hardy-Littlewood maximal function (see e.g. [1]).

O'Neil [16] proved that, for $1 < a, b, c < \infty$ and $1 \le q < p \le \infty$ such that $1 + \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, the inequality

$$\|f * g\|_{L_{a,q}} \le C \|f\|_{L_{b,p}} \|g\|_{L_{c,r}}, \quad f \in L_{b,p},\ g \in L_{c,r}, \tag{3}$$

is satisfied. This result was further improved in [10, 20] up to the range $0 < a, b, c < \infty$ and $1 \le q < p \le \infty$. Blozinski [2] showed that in a limit case of (3) with $a = b$ and $c = 1$, for an a.e. nonnegative g,

$$T_g : L_{p,b} \to L_{q,b}$$

holds if and only if $g = 0$ a.e. However, in a recent paper [14] Nursultanov and Tikhonov proved that the same problem has a nontrivial solution if we replace the interval of integration in (1) by $(0,1)$ and consider the convolution for 1-periodic functions. In that case the inequality

$$\|f * g\|_{L_{b,q}} \le C \|f\|_{L_{b,p}} \|g\|_{L_{(1,r)}}$$

was shown to be satisfied for all 1-periodic $f \in L_{b,p}$, $g \in L_{(1,r)}$. Here the functionals $\|\cdot\|_{L_{\alpha,\beta}}$, $\|\cdot\|_{L_{(\alpha,\beta)}}$ are naturally given just on $(0,1)$, as well.

In this paper, we provide necessary and sufficient conditions for the boundedness $T_g : \Lambda^p(v) \to \Gamma^q(w)$ for fixed weights v, w and various combinations of the parameters p, q. Moreover, we obtain Young-type inequalities (2) for $X = \Lambda^p(v)$, $Z = \Gamma^q(w)$ and characterize the largest rearrangement-invariant space Y for which these inequalities are valid.

To obtain these results we use the classical O'Neil inequality [16] and the weighted Hardy-type inequalities which have undergone a wide development in the last two decades. A survey of the classical cases may be found e.g. in [4], newer and more general results are developed and summarized in [8]. (For further related results see e.g. [12].) Our method enables us to obtain both the results for convolutions on $\mathbb{R}$ and on a finite interval.

Our paper proceeds in the following way: In Section 2 we present the definitions, state the problems and prove some preliminary results. Section 3 includes the main results, i.e. the weighted Young-type inequalities involving Λ and Γ spaces. In Section 4 we present some additional results and also verify that the results of [2, 14, 16] mentioned above follow as special cases of our theorems. Finally, Section 5 deals with some fundamental properties of function spaces which appear in the inequalities.

2. PRELIMINARIES

Throughout the text we use the following notation: If Ω is a measurable subset of $\mathbb{R}$, we write $\mathscr{M}(\Omega) := \{f : \Omega \to \mathbb{R}$ measurable$\}$ and $\mathscr{M}_+(\Omega) := \{f \in \mathscr{M}(\Omega);\ f \geq 0$ a.e.$\}$. If $p \in (1,\infty)$, we define the conjugate exponent p' by $p' := \frac{p}{p-1}$.

In what follows, we will consider $m \in (0,\infty]$, unless specified else. We denote

$$\mathscr{P}_m := \begin{cases} \{f \in \mathscr{M}(\mathbb{R});\ m\text{-periodic}\} & \text{if } m < \infty, \\ \mathscr{M}(\mathbb{R}) & \text{if } m = \infty, \end{cases}$$

and

$$\mathscr{E}_m := \left\{f \in \mathscr{P}_m;\ f \geq 0 \text{ on } \mathbb{R},\ f \text{ is even},\ f \text{ is nonincreasing on} \left(0, \tfrac{m}{2}\right)\right\}.$$

Notice that $\mathscr{P}_m, \mathscr{E}_m \subset \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ in the sense of the restriction of f to $\left(-\frac{m}{2}, \frac{m}{2}\right)$. We introduce these classes to be able to treat both the convolution on $\mathbb{R}$ (as in [16] etc.) and the convolution of m-periodic functions, $m < \infty$, (as in [14]) at once. In the case $m = \infty$, the description of the classes is rather simple: $\mathscr{P}_\infty = \mathscr{M}(\mathbb{R})$ and $\mathscr{E}_\infty$ consists of nonnegative "symmetrically decreasing" functions on $\mathbb{R}$.

The usual notation $F \lesssim G$ means that $F \leq CG$ where C is a constant independent of appropriate quantities in F and G. If $C^{-1}F \leq G \leq CF$ with such C, we write $F \simeq G$ and C is then called the *equivalence constant*. By L^1_{loc} we denote the set of all locally integrable functions on $\mathbb{R}$. Next, a *weight* w is a nonnegative function on $(0,m)$ such that for all $t \in (0,m)$ it holds $0 < W(t) < \infty$, where

$$W(t) := \int_0^t w(s)\,\mathrm{d}s, \quad t \in [0,m].$$

For a weight w, the $L^q(w)$-norm of $f \in \mathscr{M}(0,m)$ is given by

$$\|f\|_{L^q(w)} := \int_0^m |f(t)|^q w(t)\,\mathrm{d}t, \quad q < \infty,$$

$$\|f\|_{L^\infty(w)} := \operatorname*{ess\,sup}_{t \in (0,m)} |f(t)| w(t).$$

Let $f, g \in \mathscr{P}_m$. We define the *convolution* $f * g$ by

$$(f * g)(t) := \int_{-\frac{m}{2}}^{\frac{m}{2}} f(x) g(t-x)\,\mathrm{d}x, \tag{4}$$

if the right-hand side is well-defined for a.e. $t \in \left(-\frac{m}{2}, \frac{m}{2}\right)$. Notice that if $f * g$ is defined, then $f * g \in \mathscr{P}_m$.

For $f \in \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ we define the *nonincreasing rearrangement* of f by

$$f^*(t) := \inf\left\{s \geq 0;\ \left|\left\{\tau \in \left(-\tfrac{m}{2}, \tfrac{m}{2}\right),\ |f(\tau)| > s\right\}\right| \leq t\right\}, \quad t \in (0, m), \tag{5}$$

and the *maximal function* f^{**} by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t \in (0, m), \tag{6}$$

see e.g. [1]. Observe that, although the m-periodic function f (for $m < \infty$) is defined on $\mathbb{R}$, the above defined rearrangement of f represents just the rearrangement of f's restriction to the interval of periodicity. If $m = \infty$, we get the "standard" rearrangement and convolution on $\mathbb{R}$. Again, this approach will allow us to cover the results for both finite and infinite m by a single theorem. It may be also worth noticing that, if $f \in \mathscr{E}_m$, the properties of f yield $f(t) = f^*(2t)$ for all $t \in \left(0, \frac{m}{2}\right)$, a fact which will be useful later.

The following definition includes the standard definition of an r.i. norm (see [1]), modified for functions from the class $\mathscr{P}_m$.

Definition 2.1. Let $\varrho : \mathscr{P}_m \to [0, \infty]$ be a mapping. We call ϱ a *rearrangement-invariant (r.i.) Banach function norm* or just simply an *r.i. norm* if for all $f, g, f_n \in \mathscr{P}_m$, $(n \in \mathbb{N})$, for all constants $a \geq 0$ and all measurable subsets E of $\left(-\frac{m}{2}, \frac{m}{2}\right)$, the following properties hold:

- (P1) $\varrho(f + g) \leq \varrho(f) + \varrho(g)$,
- (P2) $\varrho(af) = a\varrho(f)$,
- (P3) $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e.,
- (P4) $0 \leq g \leq f$ a.e. $\Rightarrow \varrho(g) \leq \varrho(f)$,
- (P5) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \varrho(f_n) \uparrow \varrho(f)$,
- (P6) $|E| < \infty \Rightarrow \varrho(\chi_E) < \infty$,
- (P7) $|E| < \infty \Rightarrow \int_E f \leq C_E \varrho(f)$ for some constant $C_E \in (0, \infty)$ depending on E and ϱ but independent of f,
- (P8) $f^* = g^*$ on $(0, m) \Rightarrow \varrho(f) = \varrho(g)$.

If ϱ is an r.i. norm, the collection $X = X(\varrho)$ of all functions $f \in \mathscr{P}_m$ such that $\varrho(|f|) < \infty$ is called an *r.i. space*. For formal reasons, we will consider the set consisting only of the zero function to be also an r.i. space.

The mapping ϱ is called an *r.i. quasi-norm* if for all $f, g, f_n \in \mathscr{P}_m$, $(n \in \mathbb{N})$, all $a \geq 0$ and all measurable $E \subset \left(-\frac{m}{2}, \frac{m}{2}\right)$, the conditions

- (P1*) $\varrho(f + g) \leq B(\varrho(f) + \varrho(g))$ for some constant $B \in (1, \infty)$ independent of f, g,

and (P2)-(P8) are satisfied. In that case, $X(\varrho)$ is said to be a *quasi-normed r.i. space*. We call $X(\varrho)$ an *r.i. lattice* if for all $f, g \in \mathscr{P}_m$, all $a \geq 0$ and all measurable $E \subset \left(-\frac{m}{2}, \frac{m}{2}\right)$, the conditions (P2), (P4), (P6) and (P8) are satisfied.

If $X(\varrho)$ is an r.i. lattice, for every $f \in \mathscr{P}_m$ we define $\|f\|_X := \varrho(|f|)$. Notice that $\|\cdot\|_X$ is not necessarily a norm.

We say that an r.i. lattice X *is embedded* into an r.i. lattice Y and write $X \hookrightarrow Y$ if there exists a constant $C > 0$ such that $\|f\|_Y \le C\|f\|_X$ for all $f \in X$.

Let $g \in \mathscr{P}_m$. We consider the operator T_g defined by

$$T_g : f \mapsto f * g, \tag{7}$$

acting on all functions $f \in \mathscr{P}_m$ for which $f * g$ is defined. We will study the boundedness

$$T_g : \Lambda^p(v) \to \Gamma^q(w),$$

where v, w are weights on $(0, m)$ and $\Lambda^p(v)$, $\Gamma^q(w)$ are the *weighted Lorentz spaces* defined as

$$\Lambda^p(v) := \left\{ f \in \mathscr{P}_m;\ \|f\|_{\Lambda^p(v)} := \left(\int_0^m (f^*(x))^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\},$$

$$\Gamma^q(w) := \left\{ f \in \mathscr{P}_m;\ \|f\|_{\Gamma^q(w)} := \left(\int_0^m (f^{**}(x))^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} < \infty \right\}$$

for $p, q \in (0, \infty)$, and

$$\Lambda^\infty(v) := \left\{ f \in \mathscr{P}_m;\ \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{x \in (0,m)} f^*(x) v(x) < \infty \right\},$$

$$\Gamma^\infty(w) := \left\{ f \in \mathscr{P}_m;\ \|f\|_{\Gamma^\infty(w)} := \operatorname*{ess\,sup}_{x \in (0,m)} f^{**}(x) w(x) < \infty \right\}.$$

Of course, for $m < \infty$, the Λ or Γ norm of $f \in \mathscr{P}_m$ controls just the behavior of f on the periodical segment. Let us also point out that $\Lambda^p(v)$ with $p \in (0, \infty]$ is not necessarily a normed (not even quasi-normed) linear space (see e.g. [7] and the references therein). Since

$$(f + g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad t \in (0, m), \tag{8}$$

(see e.g. [1, p. 54]), the structure $\Gamma^q(w)$ is a normed linear space for $q \in [1, \infty]$ but only quasi-normed for $q \in (0, 1)$. However, we will still refer to $\Lambda^p(v)$ and $\Gamma^q(w)$ as to "spaces" and to $\|\cdot\|_{\Lambda^p(v)}$ and $\|\cdot\|_{\Gamma^q(w)}$ as to "norms". Notice also that the weighted Lorentz spaces are always at least r.i. lattices.

Our first aim is the following: Given weights v, w and exponents p, q, we want to find sufficient conditions on the kernel g under which $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded, i.e.

$$\|f * g\|_{\Gamma^q(w)} = \|T_g f\|_{\Gamma^q(w)} \le C\|f\|_{\Lambda^p(v)}, \quad f \in \Lambda^p(v), \tag{9}$$

and to obtain estimates for the optimal constant $C = \|T_g\|_{\Lambda^p(v) \to \Gamma^q(w)}$ in terms of g. Recall that the operator norm of T_g is given by

$$\|T_g\|_{X \to Z} := \sup_{\|f\|_X \le 1} \|T_g f\|_Z.$$

Let us formally put $\|T_g\|_{X\to Z} := \infty$ if there exists a function $f \in X$ such that $T_g f$ is not defined.

In addition to this, it will be shown that if $g \in \mathscr{E}_m$, then the sufficient conditions are also necessary for the boundedness $T_g : \Lambda^p(v) \to \Gamma^q(w)$.

Later on, we will see that $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)}$ is estimated from above by a norm of g in an r.i. space Y. (In case of $g \in \mathscr{E}_m$, it will even hold $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \simeq \|g\|_Y$.) This will allow us to write the result in the form of a Young-O'Neil inequality

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_Y, \quad f \in \Lambda^p(v),\ g \in Y. \tag{10}$$

Moreover, the space Y will be optimal in the following sense:

Definition 2.2. Let X, Y, Z be r.i. lattices. We say that Y is *optimal for the pair* (X, Z) if the inequality (2) holds and the following is satisfied: If $\widetilde{Y}$ is an r.i. lattice such that

$$\|f * g\|_Z \lesssim \|f\|_X\|g\|_{\widetilde{Y}}, \quad f \in X,\ g \in \widetilde{Y},$$

holds, then $\widetilde{Y} \hookrightarrow Y$.

In other words, the optimal lattice for (X, Z) is the essentially largest one for which (2) is satisfied.

The key result in our method is the O'Neil inequality [16, Lemma 2.5]:

Lemma 2.3. *Let $m \in (0, \infty]$ and $f, g \in \mathscr{P}_m \cap L^1_{\mathrm{loc}}$. Then, for every $t \in (0, m)$ it holds*

$$(f * g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^m f^*(s) g^*(s)\,\mathrm{d}s. \tag{11}$$

Observe that for convolutions both on a bounded and unbounded interval we get the same estimate (11) which allows us to treat the two cases at once, as mentioned before.

Furthermore, we are going to use the fact that the O'Neil inequality is sharp in the following way:

Lemma 2.4. *Let $m \in (0, \infty]$. Let $f, g \in \mathscr{E}_m \cap L^1_{\mathrm{loc}}$. Then for every $t \in (0, m)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^m f^*(y) g^*(y)\,\mathrm{d}y \le 12 (f * g)^{**}(t). \tag{12}$$

Proof. The result was mentioned in [16] without proof. A part of the proof is sketched e.g. in [18, Remark, p. 145]. For the convenience of the reader, we present the whole proof here.

Let $m \in (0, \infty]$ and $f, g \in \mathscr{E}_m \cap L^1_{\mathrm{loc}}$. According to the symmetry, we observe that $f(t) = f^*(2t)$ and $g(t) = g^*(2t)$ for all $t \in \left(0, \frac{m}{2}\right)$. Now let $t \in \left(0, \frac{m}{2}\right)$ be fixed. Then

$$\int_0^t f(t-x)g(x)\,\mathrm{d}x \geq g(t)\int_0^t f(t-x)\,\mathrm{d}x = g(t)\int_0^t f(x)\,\mathrm{d}x$$
$$= g^*(2t)\int_0^t f^*(2x)\,\mathrm{d}x = \frac{g^*(2t)}{2}\int_0^{2t} f^*(x)\,\mathrm{d}x.$$

Next,

$$\int_t^{\frac{m}{2}} f(t-x)g(x)\,\mathrm{d}x = \int_t^{\frac{m}{2}} f(x-t)g(x)\,\mathrm{d}x \geq \int_t^{\frac{m}{2}} f(x)g(x)\,\mathrm{d}x$$
$$= \int_t^{\frac{m}{2}} f^*(2x)g^*(2x)\,\mathrm{d}x = \frac{1}{2}\int_{2t}^{m} f^*(x)g^*(x)\,\mathrm{d}x.$$

Thus it holds

$$(f*g)(t) \geq \int_0^t f(t-x)g(x)\,\mathrm{d}x + \int_t^{\frac{m}{2}} f(t-x)g(x)\,\mathrm{d}x$$
$$\geq \frac{1}{2}\left(g^*(2t)\int_0^{2t} f^*(x)\,\mathrm{d}x + \int_{2t}^{m} f^*(x)g^*(x)\,\mathrm{d}x \right).$$

Hence, we get $g^*(2t)\int_0^{2t} f^*(x)\,\mathrm{d}x + \int_{2t}^m f^*(x)g^*(x)\,\mathrm{d}x \leq 2(f*g)(t)$. The left-hand side is equal to the expression $\int_0^m f^*(x)\min\{g^*(x), g^*(2t)\}\,\mathrm{d}x$ which is clearly nonincreasing in t. Thus, we obtain

$$g^*(2t)\int_0^{2t} f^*(x)\,\mathrm{d}x + \int_{2t}^{m} f^*(x)g^*(x)\,\mathrm{d}x \leq 2(f*g)^*(t). \tag{13}$$

Now, using Fubini theorem and the following part of (13):

$$g^*(2t)\int_0^{2t} f^*(x)\,\mathrm{d}x \leq 2(f*g)^*(t)$$

(once as it is and once with f and g having changed places), we write

$$\begin{aligned}
2t\, g^{**}(2t) f^{**}(2t) &= \frac{1}{2t}\int_0^{2t} g^*(y)\,\mathrm{d}y \int_0^{2t} f^*(x)\,\mathrm{d}x \\
&= \frac{1}{2t}\int_0^{2t} g^*(y) \int_0^{y} f^*(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2t}\int_0^{2t} g^*(y)\int_y^{2t} f^*(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2t}\int_0^{2t} g^*(y) \int_0^{y} f^*(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2t}\int_0^{2t} f^*(x)\int_0^{x} g^*(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \frac{2}{t}\int_0^{2t} (f*g)^*\left(\frac{y}{2}\right)\mathrm{d}y \\
&= \frac{4}{t}\int_0^{t} (f*g)^*(y)\,\mathrm{d}y \\
&= 4(f*g)^{**}(t).
\end{aligned}$$

Combining this and (13), we finally proceed to

$$\begin{aligned}
2t\, g^{**}(2t) f^{**}(2t) + \int_{2t}^{m} f^*(x) g^*(x)\,\mathrm{d}x &\le 4(f*g)^{**}(t) + 2(f*g)^*(t) \\
&\le 6(f*g)^{**}(t) \le 12(f*g)^{**}(2t).
\end{aligned}$$

Since $t \in \left(0, \frac{m}{2}\right)$, we have proved (12). □

Remark 2.5. Let $a, b \in \mathbb{R}$ and $\widetilde{f}, \widetilde{g} \in \mathscr{E}_m \cap L^1_{\text{loc}}$. Then the inequality (12) is actually satisfied for any $f, g \in L^1_{\text{loc}}$ such that $f(t) = \widetilde{f}(t+a)$ and $g(t) = \widetilde{g}(t+b)$ for all $t \in \mathbb{R}$. It follows from the fact that $(f*g)^* = (\widetilde{f} * \widetilde{g})^*$.

3. Main results

We start this section with the general theorem below. It treats the boundedness of the operator T_g between an r.i. lattice X and $\Gamma^q(w)$.

Theorem 3.1. *Let $m \in (0, \infty]$. Let X be an r.i. lattice over $\left(-\frac{m}{2}, \frac{m}{2}\right)$ and let $g \in \mathscr{P}_m$. Let w be a weight and $q \in (0, \infty]$. For $f \in \mathscr{P}_m$, $t \in (0, m)$ put*

$$R^1_g f(t) := t f^{**}(t) g^{**}(t),\; R^2_g f(t) := \int_t^m f^*(s) g^*(s)\,\mathrm{d}s,\; R_g f(t) := R^1_g f(t) + R^2_g f(t).$$

Then

(i) *If $R_g : X \to L^q(w)$ is bounded, then $T_g : X \to \Gamma^q(w)$ is bounded and*

$$\|T_g\|_{X\to\Gamma^q(w)} \lesssim \|R_g\|_{X\to L^q(w)} < \infty.$$

(ii) *Let $g \in \mathscr{E}_m$. If $T_g : X \to \Gamma^q(w)$ is bounded, then $R_g : X \to L^q(w)$ is bounded and*

$$\|R_g\|_{X\to L^q(w)} \lesssim \|T_g\|_{X\to\Gamma^q(w)} < \infty.$$

(iii) *If there exists an r.i. space Y over $\left(-\frac{m}{2}, \frac{m}{2}\right)$ such that, for all $g \in \mathscr{P}_m$, it holds $\|R_g\|_{X\to L^q(w)} \simeq \|g\|_Y$, then Y is optimal for $(X, \Gamma^q(w))$.*

Proof. (i) It holds $\|R_{|g|}\|_{X\to L^q(w)} = \|R_g\|_{X\to L^q(w)} < \infty$. Thus, for any $f \in X$, it holds $R_{|g|}|f|(t) < \infty$ for $t \in (0,m)$. From (11) we get $(T_{|g|}|f|)^{**}(t) \le R_{|g|}|f|(t) < \infty$ for $t \in (0,m)$, therefore $T_{|g|}|f|(t) < \infty$ for a.e. $t \in (0,m)$. Thus, $|T_g f(t)| \le T_{|g|}|f|(t)$ for a.e. $t \in (0,m)$, so T_g is well-defined on X. Next, we get

$$\|T_g\|_{X\to\Gamma^q(w)} = \sup_{\|f\|_X\le 1} \|(T_g f)^{**}\|_{L^q(w)} \le \sup_{\|f\|_X\le 1} \|R_g f\|_{L^q(w)} = \|R_g\|_{X\to L^q(w)}.$$

(ii) Let $g \in \mathscr{E}_m$ and $T_g : X \to \Gamma^q(w)$ be bounded. By definition of the operator norm, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions such that $\|f_n\|_X \le 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|R_g f_n\|_{L^q(w)} = \|R_g\|_{X\to L^q(w)}.$$

Since $R_g f = R_g \widetilde{f}$ if $f^* = \widetilde{f}^*$, we may assume that $f_n \in \mathscr{E}_m$, $n \in \mathbb{N}$. Thus, by Lemma 2.4 we obtain $\|R_g f_n\|_{L^q(w)} \le 12\|f_n * g\|_{\Gamma^q(w)}$, hence

$$\tfrac{1}{12}\|R_g\|_{X\to L^q(w)} = \tfrac{1}{12}\lim_{n\to\infty} \|R_g f_n\|_{L^q(w)} \le \liminf_{n\to\infty} \|f_n * g\|_{\Gamma^q(w)} \le \|T_g\|_{X\to\Gamma^q(w)},$$

so the proof of this part is finished.

(iii) If $g \in Y$, we get

$$\|f * g\|_{\Gamma^q(w)} = \|T_g f\|_{\Gamma^q(w)} \lesssim \|f\|_X \|T_g\|_{X\to\Gamma^q(w)} \simeq \|f\|_X \|R_g\|_{X\to L^q(w)} \lesssim \|f\|_X \|g\|_Y,$$

hence (2) holds with the given Y. Now let $\widetilde{Y}$ be an r.i. lattice such that

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_X \|g\|_{\widetilde{Y}}, \quad f \in X,\ g \in \widetilde{Y}. \tag{14}$$

Let $g \in \mathscr{E}_m$ and $\|g\|_{\widetilde{Y}} < \infty$. From (14) we get that $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}$. Hence, (ii) yields that $\|R_g\|_{X\to\Lambda^q(w)} \lesssim \|T_g\|_{X\to\Gamma^q(w)}$. Together we obtain

$$\|g\|_Y \simeq \|R_g\|_{X\to\Lambda^q(w)} \lesssim \|T_g\|_{X\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}.$$

Since Y, Y are r.i., it holds

$$\|g\|_Y \lesssim \|g\|_{\widetilde{Y}}, \quad g \in \widetilde{Y},$$

hence $\widetilde{Y} \hookrightarrow Y$. Therefore, we have proved that Y is optimal for the pair $(X, \Gamma^q(w))$. □

Now we are ready to bring the desired results about the convolution operator between $\Lambda^p(v)$ and $\Gamma^q(w)$. We are going to characterize the norm $\|\cdot\|_Y$ of the r.i. space $Y := \{h \in \mathscr{P}_m;\ \|h\|_Y < \infty\}$ which is optimal for $(\Lambda^p(v), \Gamma^q(w))$ in (10). The form of the results varies depending on the mutual relation of p

and q. We need to find estimates on $\|R^1_g\|_{\Lambda^p(v)\to L^q(w)}$, $\|R^2_g\|_{\Lambda^p(v)\to L^q(w)}$. The norm $\|R^1_g\|_{\Lambda^p(v)\to L^q(w)}$ equals the best constant C_1 such that

$$\left(\int_0^m (f^{**}(t))^q t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_1 \left(\int_0^m (f^*))^p v\right)^{\frac{1}{p}}, \quad f\in\mathscr{M}\left(-\tfrac{m}{2},\tfrac{m}{2}\right), \tag{15}$$

holds, while $\|R^2_g\|_{\Lambda^p(v)\to L^q(w)}$ equals the best C_2 in

$$\left(\int_0^m \left(\int_t^m f^*(s)g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_2 \left(\int_0^m (f^*)^p v\right)^{\frac{1}{p}}, \quad f\in\mathscr{M}\left(-\tfrac{m}{2},\tfrac{m}{2}\right). \tag{16}$$

Both (15) and (16) are Hardy-type inequalities for monotone functions and the optimal constants C_1, C_2 have been fully characterized. The inequality (15) represents the embedding $\Lambda \hookrightarrow \Gamma$ (see e.g. [3, 4]). A similar survey of (16) may be found e.g. in [8]. Direct references are given in the proof of Theorem 3.2 below.

In what follows, we will use the fact that for any $m\in(0,\infty]$ and any $\varphi,\psi\in\mathscr{M}_+(\mathbb{R})$ it holds

$$\sup_{x\in(0,m)}\varphi(x)+\sup_{x\in(0,m)}\psi(x)\simeq\sup_{x\in(0,m)}[\varphi(x)+\psi(x)].$$

We also apply the convention "$\frac{\infty}{\infty}:=0$".

Theorem 3.2. *Let $m\in(0,\infty]$ and let v,w be weights. For $g\in\mathscr{P}_m$ let $\|g\|_Y$ be given by the following:*

(i) *If $0<p\le 1$, $p\le q<\infty$, let*

$$\|g\|_Y := \sup_{x\in(0,m)} xV^{-\frac{1}{p}}(x)\left[(g^{**}(x))^q W(x)+\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}}.$$

(ii) *If $1<p\le q<\infty$, let*

$$\begin{aligned}\|g\|_Y := \sup_{x\in(0,m)} &\left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac{1}{q}}(x)\\ &+ g^{**}(x)xW^{\frac{1}{q}}(x)V^{-\frac{1}{p}}(x)\\ &+\left(\int_0^x t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

(iii) *If* $1 < q < p < \infty$, *let*

$$\|g\|_Y := \left[\int_0^m \left(\left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_0^x t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t \right)^{\frac{r}{q'}} x^{p'} V^{-p'}(x) v(x) \right.\right.$$

$$+ (g^{**}(x))^r x^r W^{\frac{r}{q}}(x) V^{-\frac{r}{q}}(x) v(x)$$

$$\left.\left. + W^{\frac{r}{p}}(x) w(x) \left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t \right)^{\frac{r}{p'}} \mathrm{d}x \right) \right]^{\frac{1}{r}}$$

$$+ \left(\int_0^m x^q (g^{**}(x))^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m).$$

(iv) *If* $1 = q < p < \infty$, *let*

$$\|g\|_Y := \left[\int_0^m \left(g^{**}(x) W(x) + \int_x^m g^{**}(t) w(t)\,\mathrm{d}t \right)^{p'} x^{p'} V^{-p'}(x) v(x)\,\mathrm{d}x \right]^{\frac{1}{p'}}$$

$$+ \int_0^m x g^{**}(x) w(x)\,\mathrm{d}x \; V^{-\frac{1}{p}}(m).$$

Then, for each choice of p, q *from the previous list, the inequality* (10) *is satisfied. If* $g \in \mathscr{E}_m$, *then* $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \simeq \|g\|_Y$. *The space* $(Y, \|\cdot\|_Y)$ *is optimal for the pair* $(\Lambda^p(v), \Gamma^q(w))$.

Proof. As for checking that Y generated by $\|\cdot\|_Y$ in each of the cases is a (quasi-)normed r.i. space, we refer to Proposition 5.6.

Now let us focus on the main part of the proof. At first, clearly it is $\|R_g\|_{\Lambda^p(v)\to L^q(w)} \simeq \|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$. In each case (i)-(iv), we will use the known equivalent estimates of $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}$, $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$. They have a form of certain functionals of g and we will show that, when added together, they actually form a norm of g in Y, i.e. $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$ for every $g \in \mathscr{P}_m$.

Then the results will follow from Theorem 3.1: By its (i) part, if $g \in Y$, then $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded and $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \lesssim \|g\|_Y$, hence (10) is satisfied. By Theorem 3.1(ii), if $g \in \mathscr{E}_m$, then we get even $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \simeq \|g\|_Y$. Theorem 3.1(iii) then implies the optimality of Y.

So, in each case we just need to check that $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$, obtained from the appropriate Hardy-type inequalities, are equivalent to $\|g\|_Y$ for any $g \in \mathscr{P}_m$.

(i) By [19, Theorem 3(b)] and [13, Theorem 2.1(a)] we get

$$\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} \simeq \sup_{x\in(0,m)} V^{-\frac{1}{p}}(x)\left[x\left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^x t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\right],$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \sup_{x\in(0,m)} V^{-\frac{1}{p}}(x)\left(\int_0^x\left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Obviously, $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \|g\|_Y$.

(ii) From [17, Theorem 2] and the dual version of [15, Theorem 1.1] it follows:

$$\begin{aligned}
\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)} \left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_0^x t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}\\
&\quad + \sup_{x\in(0,m)}\left(\int_0^x t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x)\\
&=: A_1 + A_2,\\
\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)}\left(\int_x^m\left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac{1}{q}}(x)\\
&\quad + \sup_{x\in(0,m)}\left(\int_0^x\left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x)\\
&=: A_3 + A_4.
\end{aligned}$$

Since for every $x\in(0,m)$ it holds

$$\begin{aligned}
V^{-\frac{1}{p}}(x) &\geq \left(V^{1-p'}(x) - V^{1-p'}(m)\right)^{\frac{1}{p'}}\\
&= \left(\int_x^m \left(-V^{1-p'}\right)'(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \simeq \left(\int_x^m V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}},
\end{aligned}\tag{17}$$

we get

$$\left(\int_0^x\left(\int_0^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^m V^{-p'}v\right)^{\frac{1}{p'}} \lesssim \left(\int_0^x\left(\int_0^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x)$$

and so

$$A_5 := \sup_{x\in(0,m)}\left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac{1}{q}}(x) \lesssim A_2 + A_3 + A_4.$$

Observe also that $A_3 \lesssim A_5$. Hence

$$\begin{aligned}\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}+\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\lesssim A_1+A_2+A_3+A_4 \le A_1+A_2+A_3+A_4+A_5\\ &\lesssim A_1+A_2+A_4+A_5\\ &\lesssim A_1+A_2+A_3+A_4\\ &\lesssim \|R_g^1\|_{\Lambda^p(v)\to L^q(w)}+\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}.\end{aligned}$$

Since $A_1+A_2+A_4+A_5 \simeq \|g\|_Y$, we have obtained $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}+\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \|g\|_Y$.

(iii) In this case [17, Theorem 2] and the dual version of [15, Theorem 1.2] (cf. also [4, Theorem 4.1] and [8, Theorem 5.1]) yield

$$\begin{aligned}&\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}\\ &\simeq\left(\int_0^m\left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{q}}\left(\int_0^t x^{p'}V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{q'}} t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{r}}\\ &+\left(\int_0^m\left(\int_0^x t^q(g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{r}}\\ &+\left(\int_0^m x^q(g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m)\\ &=: A_1+A_2+A_3,\end{aligned}$$

$$\begin{aligned}\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\simeq \left(\int_0^m\left(\int_0^x\left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{r}}\\ &+\left(\int_0^\infty\left(\int_x^m\left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{r}{p'}} W^{\frac{r}{p}}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{r}}\\ &=: A_4+A_5.\end{aligned}$$

Clearly it holds

$$A_2+A_4 \simeq \left(\int_0^m\left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r W^{\frac{r}{q}}(x)V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{r}} =: A_6.$$

Integration by parts and (17) provides that for all $t \in (0,m)$ we have

$$\int_t^m W^{\frac{r}{p}}(x)\left(\int_x^m V^{-p'}v\right)^{\frac{r}{p'}} w(x)\,\mathrm{d}x \lesssim \int_t^m W^{\frac{r}{q}}(x)V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x.$$

The function $x \mapsto \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r$ is nondecreasing, so by Hardy's lemma (an analogue of [1, Proposition 3.6, p. 56]) we obtain

$$A_7 := \left(\int_0^m \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r W^{\frac{r}{p}}(x)\left(\int_x^m V^{-p'}v\right)^{\frac{r}{p'}} w(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim A_6,$$

thus also $A_5 + A_7 \lesssim A_2 + A_4 + A_5$. Next, we can write

$$A_5 + A_7 \simeq \left(\int_0^m W^{\frac{r}{p}}(x)w(x)\left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{r}{p'}} \mathrm{d}x\right)^{\frac{1}{r}} =: A_8,$$

hence putting all the estimates together yields

$$A_2 + A_4 + A_5 \lesssim A_6 + A_8 \lesssim A_2 + A_4 + A_5$$

and so finally $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq A_1 + A_3 + A_6 + A_8 \simeq \|g\|_Y$.

(iv) By [4, Theorem 4.1(iv)] and [8, Theorem 5.1(v)] we have

$$\begin{aligned}
\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} \simeq {} & \left(\int_0^m \left(\int_0^x t\, g^{**}(t)w(t)\,\mathrm{d}t\right)^{p'} V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \\
& + \left(\int_0^m \left(\int_x^\infty g^{**}(t)w(t)\,\mathrm{d}t\right)^{p'} x^{p'} V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \\
& + \int_0^m x\, g^{**}(x)w(x)\,\mathrm{d}x\ V^{-\frac{1}{p}}(\infty) \\
& =: A_1 + A_2 + A_3, \\
\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq {} & \left(\int_0^m \left(\int_0^x \int_t^x g^*(y)\,\mathrm{d}y\ w(t)\,\mathrm{d}t\right)^{p'} V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}.
\end{aligned}$$

Clearly,

$$A_1 + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \left(\int_0^m (g^{**}(x))^{p'} x^{p'} W^{p'}(x)V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} =: A_4,$$

hence $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq A_2 + A_3 + A_4 \simeq \|g\|_Y$. $\square$

For a given combination of weights v, w and exponents p, q in Theorems 3.2-3.6 we got the optimal space $(Y, ||\cdot||_Y)$. However, this space may consist only of a.e. zero functions. In such case we have the following observation:

Corollary 3.3. *Let $m \in (0,\infty]$, $p, q \in (0,\infty]$, let v, w be weights. Let the optimal space Y for $(\Lambda^p(v), \Gamma^q(w))$ in (10) satisfy $Y = \{0\}$. Let $g \in \mathscr{P}_m$ be nonnegative a.e. and such that $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded. Then $g = 0$ a.e.*

Proof. Let $g \in \mathscr{P}_m$ be nonnegative and $g \not\equiv 0$ in measure. Then there exist $\epsilon > 0$, $a, b \in \left(-\frac{m}{2}, \frac{m}{2}\right)$ and $h = \epsilon\chi_{(a,b)}$ such that $h \le g$ a.e. Since $h \not\equiv 0$, it holds $||h||_Y = \infty$ and therefore, by Theorem 3.1(ii) and Remark 2.5, T_h is not bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. Since $0 \le h \le g$, for every nonnegative $f \in \Lambda^p(v)$ we get $0 \le T_h f \le T_g f$. Thus also $(T_h f)^* \le (T_g f)^*$ and it follows that T_g is not bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. □

Remark 3.4. In general, functions from $\Lambda^p(v)$ do not have to be locally integrable. In particular, for $p \in (0,\infty)$, we know that $\Lambda^p(v) \subset L^1_{\mathrm{loc}}$ if and only if one of the following conditions is satisfied (cf. [4, 17, 19]):

(a) $p \in (0,1]$ and $\limsup_{t\to 0+} t V^{-\frac{1}{p}}(t) < \infty$,
(b) $p \in (1,\infty)$ and there exists $\epsilon > 0$ such that $\int_0^\epsilon t^{p'-1} V^{1-p'}(t)\,\mathrm{d}t < \infty$.

Let $\Lambda^p(v) \not\subset L^1_{\mathrm{loc}}$. Then T_g is well-defined on $\Lambda^p(v)$ if and only if $g = 0$ a.e. One may directly check that $Y = \{0\}$ in all cases of Theorem 3.2(i)-(iv). Hence, this theorem (trivially) holds even for $\Lambda^p(v) \not\subset L^1_{\mathrm{loc}}$, thus we do not assume (a) or (b) in its statement.

Now we state the results for the weak-type spaces. The way of proving them is the same as in Theorem 3.2. Analogues of Corollary 3.3 and Remark 3.4 hold for these cases as well.

Theorem 3.5. *Let $m \in (0,\infty]$. Let v, w be weights. For $g \in \mathscr{P}_m$ let $||g||_Y$ be given by what follows:*

(i) *If $0 < p \le 1$, then*

$$||g||_Y := \operatorname*{ess\,sup}_{0<x<y<m} \left[g^{**}(y) w(y) x V^{-\frac{1}{p}}(x) + g^{**}(y) w(x) y V^{-\frac{1}{p}}(y) \right].$$

(ii) *If $1 < p < \infty$, then*

$$||g||_Y := \operatorname*{ess\,sup}_{0<x<m} w(x) \left[\left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t \right)^{\frac{1}{p'}} + g^{**}(x) \left(\int_0^x t^{p'-1} V^{1-p'}(t)\,\mathrm{d}t \right)^{\frac{1}{p'}} \right].$$

Then, for $p \in (0,\infty)$, it holds

$$||f * g||_{\Gamma^\infty(w)} \lesssim ||f||_{\Lambda^p(v)} ||g||_Y, \quad f \in \Lambda^p(v),\ g \in Y. \tag{18}$$

Moreover, if $g \in \mathscr{E}_m$, *then* $\|T_g\|_{\Lambda^p(v)\to\Gamma^\infty(w)} \simeq \|g\|_Y$. *The space* $(Y, \|\cdot\|_Y)$ *is optimal for the pair* $(\Lambda^p(v), \Gamma^\infty(w))$.

Proof. We will again show that $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}$ and apply Theorem 3.1. (For more details see the proof of Theorem 3.2.)

(i) From [5, Theorem 3.3] (see also [4, Theorem 4.2]) and [8, Theorem 5.3] it follows:

$$\|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{0<x<y<m} g^{**}(y)w(y)xV^{-\frac{1}{p}}(x),$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{0<x<y<m} \int_x^y g^*(s)\,\mathrm{d}s\; w(x)V^{-\frac{1}{p}}(y).$$

In the definition of $\|g\|_Y$ we observe that

$$\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} + B,$$

where

$$B := \operatorname*{ess\,sup}_{x\in(0,m)} xg^{**}(x)w(x)V^{-\frac{1}{p}}(x).$$

However, it is easy to see that $B \le \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)}$, therefore

$$\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}.$$

(ii) From the same sources as in (i) we obtain the following characterizations:

$$\|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{x\in(0,m)} g^{**}(x)w(x)\left(\int_0^x t^{p'-1}V^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}},$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{x\in(0,m)} w(x)\left(\int_x^m\left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}.$$

Since

$$\left(\int_x^m \frac{v(t)}{V^{p'}(t)}\,\mathrm{d}t\right)^{\frac{1}{p'}} \le V^{-\frac{1}{p}}(x) = \frac{V^{-\frac{1}{p}}(x)}{x}\left(\int_0^x t^{p'-1}\,\mathrm{d}t\right)^{\frac{1}{p'}} \le \frac{1}{x}\left(\int_0^x \frac{t^{p'-1}}{V^{p'-1}(t)}\,\mathrm{d}t\right)^{\frac{1}{p'}},$$

we get

$$B := \operatorname*{ess\,sup}_{x>0} xg^{**}(x)w(x)\left(\int_x^\infty V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \le \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)}.$$

Thus,

$$\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} + B \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}$$

and the proof is finished. ☐

Theorem 3.6. *Let $m \in (0, \infty]$. Let v, w be weights. For $g \in \mathscr{P}_m$ let $\|g\|_Y$ be given by what follows:*

(i) *If $0 < q < \infty$, then*

$$\|g\|_Y := \left(\int_0^m \left(g^{**}(x) \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} + \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

(ii) *If $q = \infty$, then*

$$\|g\|_Y := \operatorname*{ess\,sup}_{x\in(0,m)} \left(g^{**}(x) \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} + \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} \right) w(x).$$

Then, for $q \in (0, \infty]$, it holds

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^\infty(v)} \|g\|_Y, \quad f \in \Lambda^\infty(v),\ g \in Y. \tag{19}$$

Moreover, if $g \in \mathscr{E}_m$, then $\|T_g\|_{\Lambda^\infty(v)\to\Gamma^q(w)} \simeq \|g\|_Y$. The space $(Y, \|\cdot\|_Y)$ is optimal for the pair $(\Lambda^\infty(v), \Gamma^q(w))$.

Proof. Once again, let us show $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)}$ and apply Theorem 3.1. (See details in the analogous proof of Theorem 3.2.)

(i) From [8, Theorem 5.5] it follows

$$\|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} = \left(\int_0^m \left(g^{**}(x) \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}},$$

$$\|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)} = \left(\int_0^m \left(\int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

One clearly sees that $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)}$.

(ii) Here, by [8, Theorem 5.5] as well, we get

$$\|R_g^1\|_{\Lambda^\infty(v)\to L^\infty(w)} = \operatorname*{ess\,sup}_{x\in(0,m)} g^{**}(x) \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} w(x),$$

$$\|R_g^2\|_{\Lambda^\infty(v)\to L^\infty(w)} = \operatorname*{ess\,sup}_{x\in(0,m)} \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} w(x),$$

and thus obviously $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^\infty(w)}$. □

4. FURTHER RESULTS AND APPLICATIONS

At first, here we present two additional results of independent interest. The proposition below provides an alternative expression for the right-hand side of O'Neil inequality (11):

Proposition 4.1. *Let $m \in (0,\infty]$ and let $f,g \in \mathscr{P}_m \cap L^1_{\mathrm{loc}}$. Then for every $t \in (0,m)$ it holds:*

$$tf^{**}(t)g^{**}(t) + \int_t^m f^*g^* = \limsup_{s\to m-} sf^{**}(s)g^{**}(s) + \int_t^m (f^{**}-f^*)(g^{**}-g^*).$$

Proof. We may assume that $f^{**}, g^{**} < \infty$ on $(0,\infty)$, otherwise the identity holds trivially. Recall that $(g^{**})'(t) = \frac{g^*(t)-g^{**}(t)}{t}$ for all $t > 0$. Assume first $m < \infty$ and take a fixed $t \in (0,m)$. Then integration by parts yields

$$\int_t^m f^{**}(g^{**}-g^*) = \left[-sf^{**}(s)g^{**}(s)\right]_{s=t}^m + \int_t^m f^*g^{**}.$$

Subtracting $\int_t^m f^*(g^{**}-g^*)$ from both sides, we get

$$\int_t^m (f^{**}-f^*)(g^{**}-g^*) = \left[-sf^{**}(s)g^{**}(s)\right]_{s=t}^m + \int_t^m f^*g^*,$$

hence

$$tf^{**}(t)g^{**}(t) + \int_t^m f^*g^* = mf^{**}(m)g^{**}(m) + \int_t^m (f^{**}-f^*)(g^{**}-g^*). \tag{20}$$

Notice that since all integrals involved in the procedure exist and are finite, all performed steps were correct. Now, consider $f,g \in \mathscr{M}$ and suppose that f^*, f^{**}, g^*, g^{**} are rearrangements on $\mathbb{R}$ (given by (5) and (6) with $m=\infty$). By the previous part, (20) holds for any parameter $m \in (0,\infty)$, thus passing $m \to \infty$ on both sides and using the monotone convergence theorem gives the result for $m = \infty$. $\square$

We now get the following corollary:

Corollary 4.2. *Let $m \in (0,\infty)$, $f,g \in \mathscr{P}_m \cap L^1_{\mathrm{loc}}$ and let w be a weight. Denote $\|\cdot\|_1 := \|\cdot\|_{L^1(0,m)}$. Then*

$$\int_0^m (f*g)^{**}w \le \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))W(t)\,\mathrm{d}t. \tag{21}$$

Proof. Following Lemma 2.3 and Proposition 4.1 we get

$$
\begin{aligned}
&\int_0^m (f*g)^{**}(t)w(t)\,\mathrm{d}t \\
&\le \int_0^m \left[tf^{**}(t)g^{**}(t) + \int_t^m f^*(s)g^*(s))\,\mathrm{d}s \right] w(t)\,\mathrm{d}t \\
&\le \int_0^m \left[mf^{**}(m)g^{**}(m) + \int_t^m (f^{**}(s)-f^*(s))(g^{**}(s)-g^*(s))\,\mathrm{d}s \right] w(t)\,\mathrm{d}t \\
&\le \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m\int_t^m (f^{**}(s)-f^*(s))(g^{**}(s)-g^*(s))w(t)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))W(t)\,\mathrm{d}t.
\end{aligned}
$$

□

This improves the result of [14, Lemma 2.1], in which a weaker version of it is proved, namely with g^{**} instead of $g^{**}-g^*$ in the integrand on the right-hand side of (21).

Next, let us show that our theorems cover the classical convolution-related results which we thus can obtain by applying the inequalities from Section 2 to special choices of weights.

Remark 4.3. O'Neil's result [16, Theorem 2.6] says that for $1 < a, b, c < \infty$ and $1 \le q < p < \infty$ such that $1+\frac{1}{a} = \frac{1}{b}+\frac{1}{c}$ and $\frac{1}{r} = \frac{1}{q}-\frac{1}{p}$ the inequality (3) holds for all $f, g \in \mathscr{P}_m$, where m may be both finite or infinite and the functionals $\|\cdot\|_{L_{\alpha,\beta}}$ are defined on a corresponding interval $(0, m)$. Let us show that this result now follows as a special case of Theorem 3.2(iii)/(iv):

Consider $q > 1$. Recall that since $a, b, c > 1$, it holds $\|\cdot\|_{L_{a,q}} \simeq \|\cdot\|_{L_{(a,q)}}$ and analogously for $L_{c,r}$ (see e.g. [1, p. 219]). Hence, it suffices to confirm the inequality

$$
\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_{\Gamma^r(u)} \tag{22}
$$

with $v(x) := x^{\frac{p}{b}-1}$, $w(x) := x^{\frac{q}{a}-1}$ and $u(x) := x^{\frac{r}{c}-1}$. By application of Theorem 3.2(iii) and a direct calculation involving the given weights, we obtain that

$\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_Y$ holds with

$$\|g\|_Y \simeq \|g\|_{\Gamma^r(u)} + V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_0^m\left(\int_x^m (g^{**}(t))^q t^{\frac{q}{a}-1}\,\mathrm{d}t\right)^{\frac{r}{q}} x^{\frac{r(b-1)}{b}-1}\,\mathrm{d}x\right)^{\frac{1}{r}}$$

$$+\left(\int_0^m\left(\int_x^m (g^{**}(t))^{p'} t^{\frac{b-p}{b(p-1)}}\,\mathrm{d}t\right)^{\frac{r}{p'}} x^{\frac{r}{a}-1}\,\mathrm{d}x\right)^{\frac{1}{r}}.$$

Since g^{**} is nonincreasing, the Hardy-type inequality [8, Theorem 5.1(iii)] implies

$$\left(\int_0^m\left(\int_x^m (g^{**}(t))^q t^{\frac{q}{a}-1}\,\mathrm{d}t\right)^{\frac{r}{q}} x^{\frac{r(b-1)}{b}-1}\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim \|g\|_{\Gamma^r(u)},$$

$$\left(\int_0^m\left(\int_x^m (g^{**}(t))^{p'} t^{\frac{b-p}{b(p-1)}}\,\mathrm{d}t\right)^{\frac{r}{p'}} x^{\frac{r}{a}-1}\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim \|g\|_{\Gamma^r(u)}.$$

If $m = \infty$, we obtain that $V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} = 0$ since $V(\infty) = \infty$ (by the convention "$\frac{\infty}{\infty} = 0$"). For $m < \infty$, from [17, Remark (i), p. 148] it follows

$$V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le \|g\|_{\Gamma^r(u)}.$$

Verifying the requirements of all the used theorems is yet again done by a direct calculation of the weights. We got $\|\cdot\|_Y \simeq \|\cdot\|_{\Gamma^r(u)}$ and it shows that (22) holds.

The case $q = 1$ follows analogously using Theorem 3.2(iv) and the same sources. Therefore, we checked that from Theorem 3.2 it follows that the inequality (3) holds and $L_{c,r}$ is the optimal space for the pair $(L_{b,p}, L_{a,q})$.

Remark 4.4. Furthermore, we can investigate the limit case of (3) with $a = b$ and $c = 1$. Using exactly the same method as above, we reach the inequality

$$\|f*g\|_{L_{b,q}} \lesssim \|f\|_{L_{b,p}}\|g\|_{L_{(1,r)}}, \quad f \in L_{b,p},\ g \in L_{(1,r)}.$$

For $m < \infty$ we obtain the result of [14, Theorem 2.1(a)] so. Unlike the case of a finite m, for $m = \infty$ the space $L_{(1,r)}$, which we obtained as the optimal one, consists only of the a.e. zero function. Thus, Corollary 3.3 yields: If $g \in \mathscr{P}_m$ is nonnegative, then T_g is bounded from $L_{b,p}$ to $L_{b,q}$ if and only if $g = 0$ a.e. Hence, we recovered the result of [2, Theorem 2] for convolution operators.

5. Properties of related function spaces

In this part we introduce a new type of function spaces based on the optimal space Y we got in the previous and list some basic properties of these structures. We define them as systems of functions over the domain $\left(-\frac{m}{2}, \frac{m}{2}\right)$, where m is, without loss of generality, taken from $[1,\infty]$.

Definition 5.1. Let $m \in [1,\infty]$, $p,q \in (0,\infty)$ and let u,v be weights. For $g \in \mathscr{P}_m$ we define

$$\|g\|_{K^{p,q}(u,v)} := \left(\int_0^m \left(\int_x^m (g^{**}(t))^p u(t)\,\mathrm{d}t\right)^{\frac{q}{p}} v(x)\,\mathrm{d}x\right)^{\frac{1}{q}},$$

$$\|g\|_{K^{p,\infty}(u,v)} := \operatorname*{ess\,sup}_{x\in(0,m)} \left(\int_x^m (g^{**}(t))^p u(t)\,\mathrm{d}t\right)^{\frac{1}{p}} v(x),$$

$$\|g\|_{K^{\infty,q}(u,v)} := \left(\int_0^m \operatorname*{ess\,sup}_{t\in(x,m)} \left(g^{**}(t)u(t)\right)^q v(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Then we put $K^{p,q}(u,v) := \left\{f\in\mathscr{P}_m;\ \|f\|_{K^{p,q}(u,v)}<\infty\right\}$, analogously we define $K^{p,\infty}(u,v)$ and $K^{\infty,q}(u,v)$.

We could also consider the norm

$$\|g\|_{K^{\infty,\infty}(u,v)} := \operatorname*{ess\,sup}_{t>x>0} g^{**}(t)u(t)v(x).$$

However, this would bring no innovation since $\|\cdot\|_{K^{\infty,\infty}(u,v)}$ then coincides with $\|\cdot\|_{\Gamma^\infty(\omega)}$ for $\omega(t) := u(t)\operatorname{ess\,sup}_{x\in(0,t)} v(x)$.

Function spaces which actually are special cases of these have already been sporadically mentioned before. For example, in [6], the space $K^{1,\infty}(u,v)$ with a special choice of u,v appears as the optimal space for a certain Sobolev embedding into a Morrey-type space.

We start with showing the conditions under which a K space is nontrivial.

Proposition 5.2. *Let $m \in [1,\infty]$ and let u,v be weights. Then:*

(i) *If $0 < p,q < \infty$, then $K^{p,q}(u,v) \neq \{0\}$ if and only if*

$$\int_0^m \left(\int_x^m \frac{u(t)}{(t+1)^p}\,\mathrm{d}t\right)^{\frac{q}{p}} v(x)\,\mathrm{d}x < \infty. \tag{23}$$

(ii) *If $0 < p < \infty$, then $K^{p,\infty}(u,v) \neq \{0\}$ if and only if*

$$\operatorname*{ess\,sup}_{x\in(0,m)} \left(\int_x^m \frac{u(t)}{(t+1)^p}\,\mathrm{d}t\right)^{\frac{1}{p}} v(x) < \infty.$$

(iii) *If* $0 < q < \infty$, *then* $K^{\infty,q}(u,v) \neq \{0\}$ *if and only if*

$$\int_0^m \operatorname*{ess\,sup}_{t\in(x,m)} \frac{u^q(t)}{(t+1)^q} v(x)\,\mathrm{d}x < \infty.$$

Proof. (i) At first, one sees that for all $t > 0$ it holds

$$\frac{1}{2(t+1)^p} \leq \chi_{[0,1)}(t) + \frac{\chi_{[1,m)}(t)}{t^p} \leq \frac{2^p}{(t+1)^p}. \tag{24}$$

Assume that there exists $0 \neq f \in \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ such that $\|f\|_{K^{p,q}(u,v)} < \infty$. Then it holds $0 < f^{**}(1) < \infty$ and by (24) we get

$$\infty > \|f\|^q_{K^{p,q}(u,v)} \geq \|f^{**}(1)\chi_{[0,1]}\|^q_{K^{p,q}(u,v)} \geq \frac{(f^{**}(1))^q}{2^q} \int_0^m \left(\int_x^m \frac{u(t)}{(t+1)^p}\,\mathrm{d}t \right)^{\frac{q}{p}} v(x).$$

Now assume that (23) holds. Then by the other part of (24) we obtain that $\chi_{[0,1]} \in K^{p,q}(u,v)$. Cases (ii) and (iii) are proved analogously. $\square$

Recall (see e.g. [1, p. 73]) the spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ generated by the norms

$$\|f\|_{L^1+L^\infty} := \inf_{f=f_1+f_2} \{\|f_1\|_1 + \|f_2\|_\infty\}, \quad \|f\|_{L^1\cap L^\infty} := \max\{\|f\|_1, \|f\|_\infty\},$$

where $L^1 = L^1(0,m)$ and $L^\infty = L^\infty(0,m)$.

Proposition 5.3. *Let* $m \in [1,\infty]$. *Let* $0 < p,q \leq \infty$ *and let* u,v *be weights such that* $K^{p,q}(u,v) \neq \{0\}$. *Then*

$$L^1 \cap L^\infty \hookrightarrow K^{p,q}(u,v) \hookrightarrow L^1 + L^\infty.$$

Proof. This is proved directly by exactly the same method as in [11, Proposition 1.4(2)] where an analogous result for Γ spaces is shown. $\square$

From Proposition 5.3 we see that if $\|\cdot\|_{K^{p,q}(u,v)} \lesssim \|\cdot\|_{L^1+L^\infty}$, then $K^{p,q}(u,v) = L^1 + L^\infty$ in the sense of equivalence of norms. This is considered to be another type of triviality. We characterize it by what follows:

Proposition 5.4. *Let* $m \in [1,\infty]$ *and let* u,v *be weights. Then:*

(i) *If* $0 < p,q < \infty$, *then* $K^{p,q}(u,v) = L^1 + L^\infty$ *if and only if*

$$C := \int_0^m \left(\int_x^m \left(\frac{1}{t}+1\right)^p u(t)\,\mathrm{d}t \right)^{\frac{q}{p}} v(x)\,\mathrm{d}x < \infty.$$

(ii) *If* $0 < p < \infty$, *then* $K^{p,\infty}(u,v) = L^1 + L^\infty$ *if and only if*

$$\operatorname*{ess\,sup}_{x\in(0,m)} \left(\int_x^m \left(\frac{1}{t}+1\right)^p u(t)\,\mathrm{d}t \right)^{\frac{1}{p}} v(x) < \infty.$$

(iii) *If* $0<q<\infty$*, then* $K^{\infty,q}(u,v)=L^1+L^\infty$ *if and only if*

$$\int_0^m \operatorname*{ess\,sup}_{t\in(x,m)} \left(\frac{1}{t}+1\right)^q u^q(t)v(x)\,\mathrm{d}x<\infty.$$

Proof. (i) First let us suppose that $C=\infty$. For each $n\in\mathbb{N}$ we define the function $f_n := n\chi_{[0,\frac{1}{n}]}+1$. Then $\|f_n\|_{L^1+L^\infty}\leq 1$ for all $n\in\mathbb{N}$ but by the monotone convergence theorem it holds $\|f_n\|^q_{K^{p,q}(u,v)}\uparrow C=\infty$. Thus, $L^1+L^\infty\not\hookrightarrow K^{p,q}(u,v)$.

Now assume that $C<\infty$. Let $f\in L^1+L^\infty$ be arbitrary. Let $f_1\in L^1$ and $f_2\in L^\infty$ be functions such that $f=f_1+f_2$ and $\|f\|_{L^1+L^\infty}\geq\frac{1}{2}(\|f_1\|_1+\|f_2\|_\infty)$. Then $f^{**}(t)\leq\frac{\|f_1\|_1}{t}+\|f_2\|_\infty$, $t\in(0,m)$, and thus it holds

$$\begin{aligned}\|f\|^q_{K^{p,q}(u,v)}=\|f_1+f_2\|^q_{K^{p,q}(u,v)}&\leq\int_0^m\left(\int_x^m\left(\frac{\|f_1\|_1}{t}+\|f_2\|_\infty\right)^p u(t)\,\mathrm{d}t\right)^{\frac{q}{p}} v(x)\,\mathrm{d}x\\ &\leq 2^q C\|f\|^q_{L^1+L^\infty},\end{aligned}$$

hence $L^1+L^\infty\hookrightarrow K^{p,q}(u,v)$. Thus, $L^1+L^\infty=K^{p,q}(u,v)$ by Proposition 5.3. The proofs of (ii) and (iii) are analogous. □

Remark 5.5. Notice that if $m<\infty$, the conditions may be slightly simplified: In Proposition 5.2(i), the factor $\frac{u(t)}{(t+1)^p}$ in (23) may be replaced just by $u(t)$ and analogously in Proposition 5.2(ii),(iii). In Proposition 5.4(i) we may replace $\left(1+\frac{1}{t}\right)^p$ by $\frac{1}{t^p}$ and similarly in (ii) and (iii).

Finally, let us justify our use of the word “space” in connection with these structures.

Proposition 5.6. *Let* $m\in[1,\infty]$*. Let* $p,q\in(0,\infty]$ *and let* u,v *be weights such that* $K^{p,q}(u,v)\neq\{0\}$*. Then* $\|\cdot\|_{K^{p,q}(u,v)}$ *is an r.i. quasi-norm. If* $p,q\geq 1$*, then* $\|\cdot\|_{K^{p,q}(u,v)}$ *is an r.i. norm.*

Proof. We will check that the functional $\|\cdot\|_{K^{p,q}(u,v)}$ satisfies the P-properties from Definition 2.1. The (P1*) property follows from (8). In the case $p,q\geq 1$, Minkowski inequality is used to get (P1). Conditions (P2)-(P4) are easy to check using the properties of rearrangement (see [1, p. 41]). Property (P6) follows by the nontriviality conditions of Proposition 5.2. Next, let $E\subset\left(-\frac{m}{2},\frac{m}{2}\right)$ be measurable and $|E|<\infty$. It holds (see [1, p. 74]) that $\int_0^{|E|}f^*=\inf_{f=f_1+f_2}(\|f_1\|_1+|E|\|f_2\|_\infty)$ and, by Proposition 5.3, there exists a constant $C>0$ such that $\|f\|_{L^1+L^\infty}\leq C\|f\|_{K^{p,q}(u,v)}$ for all $f\in K^{p,q}(u,v)$. Hence, for all $f\in K^{p,q}(u,v)$ we get

$$\int_E f\leq\int_0^{|E|}f^*=\inf_{f=f_1+f_2}(\|f_1\|_1+|E|\|f_2\|_\infty)\leq(1+|E|)\|f\|_{L^1+L^\infty}\leq C(1+|E|)\|f\|_{K^{p,q}(u,v)}.$$

Thus, (P7) holds. The last condition (P8) is obvious. □

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Paper II

Martin Křepela

Convolution in rearrangement-invariant spaces defined in terms of oscillation and the maximal function

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CONVOLUTION IN REARRANGEMENT-INVARIANT SPACES DEFINED IN TERMS OF OSCILLATION AND THE MAXIMAL FUNCTION

MARTIN KŘEPELA

ABSTRACT. We characterize boundedness of a convolution operator with a fixed kernel between the classes $S^p(v)$, defined in terms of oscillation, and weighted Lorentz spaces $\Gamma^q(w)$, defined in terms of the maximal function, for $0 < p, q \le \infty$. We prove corresponding weighted Young-type inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{S^p(v)}\|g\|_Y$$

and characterize the optimal rearrangement-invariant space Y for which these inequalities hold.

1. INTRODUCTION

The classical Young inequality

$$\|f * g\|_q \le \|f\|_p\|g\|_r,$$

where $1 \le p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and $f * g$ is the convolution given by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t - x)\,\mathrm{d}x, \quad t \in \mathbb{R},$$

is one of the fundamental results related to the convolution and function spaces. It has been already modified and generalized for classes of function spaces that are wider than the Lebesgue spaces in the original Young inequality. O'Neil [14] extended the result for the two-parametric Lorentz spaces $L_{p,q}$. Precisely, he proved that, for $1 < p, q, r < \infty$ and $1 \le a, b, c \le \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, the inequality

$$\|f * g\|_{L_{q,a}} \le C\|f\|_{L_{p,b}}\|g\|_{L_{r,c}}, \quad f \in L_{p,b},\ g \in L_{r,c},$$

holds. This problem was further studied e.g. in [3, 10, 18] and the result was also improved up to the range $1 < p, q, r < \infty$ and $0 < a, b, c \le \infty$. Nursultanov and Tikhonov [13] recently studied the same question considering convolution of periodic functions.

In the preceding paper [11] the author studied the boundedness of the operator T_g given by

$$T_g f(t) := (f * g)(t)$$

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between weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ with given weights v, w and exponents p, q. It turned out that the result could be expressed by Young-type inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Lambda^p(v)}\|g\|_Y, \quad f \in \Lambda^p(v),\ g \in Y,$$

where the best r.i. space Y, such that this inequality holds, was characterized.

In this paper we deal with similar questions with $S^p(v)$ in place of $\Lambda^p(v)$. The class $S^p(v)$ is defined in terms of $f^{**} - f^*$, where f^* is the nonincreasing rearrangement of f and f^{**} is the maximal function of f (for precise definitions see Section 2 below). The quantity $f^{**} - f^*$ naturally represents the oscillation of f (see the fundamental paper of Bennett, DeVore and Sharpley [1]) and has appeared in numerous applications, particularly within the theory of Sobolev embeddings (see e.g. [4] and the references therein).

We are going to solve the following problems: At first, given exponents $p, q \in (0, \infty]$ and weights v, w, we provide conditions on the kernel $g \in L^1$ under which T_g is bounded between $S^p(v)$ and $\Gamma^q(w)$, written $T_g : S^p(v) \to \Gamma^q(w)$. Precisely, we will show that there exists an r.i. space Y such that $T_g : S^p(v) \to \Gamma^q(w)$ if (and in reasonable cases also only if) $g \in Y$ and characterize the norm of Y. Next, we write these results in the form of Young-type convolution inequalities

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{S^p(v)}\|g\|_Y, \quad f \in S^p(v),\ g \in L^1 \cap Y. \tag{1}$$

The constant C here in general depends on p, q but is independent of f, g, v, w. We will also show that the space Y we obtained is the essentially largest (optimal) r.i. space for which the inequality (1) is valid.

To get the desired results, we employ a similar technique as in [11]. We represent the investigated convolution-related inequalities by certain Hardy-type weighted inequalities and then treat the problem by working with the latter ones. This is done in Section 3. The final result shaped as the Young-type inequality (1) is presented in Section 4.

2. Preliminaries

Let us present some definitions and technical results we are going to use. The set of all measurable functions on $\mathbb{R}$ is denoted by $\mathscr{M}(\mathbb{R})$. The symbols $\mathscr{M}_+(0, \infty)$ and $\mathscr{M}_+(\mathbb{R})$ stand for the sets of all nonnegative measurable functions on $(0, \infty)$ and $\mathbb{R}$, respectively. If $p \in (1, \infty)$, we define $p' := \frac{p}{p-1}$. The notation $A \lesssim B$ means that $A \le CB$ where C is a positive constant independent of relevant quantities. Unless specified else, C actually depends only on the exponents p and q, if they are involved. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$. The *optimal constant* C in an inequality $A \le CB$ is the least C such that the inequality holds. By writing inequalities in the form

$$A(f) \lesssim B(f), \quad f \in X,$$

we always mean that $A(f) \lesssim B(f)$ is satisfied for all $f \in X$.

A *weight* is any nonnegative function on $(0, \infty)$. such that $0 < W(t) < \infty$ for all $t > 0$, where $W(t) := \int_0^t w(s)\,\mathrm{d}s$.

If $f \in \mathscr{M}(\mathbb{R})$, we define the *nonincreasing rearrangement* of f by

$$f^*(t) := \inf\{s > 0;\ |\{x \in \mathbb{R};\ |f(x)| > s\}| \le t\}, \quad t > 0,$$

and the *Hardy-Littlewood maximal function* of f by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0.$$

If u is a weight, then a generalized version of the maximal function is defined by

$$f_u^{**}(t) := \frac{1}{U(t)}\int_0^t f^*(s)u(s)\,\mathrm{d}s, \quad t > 0.$$

By L^1 we denote the Lebesgue-integrable functions on $\mathbb{R}$. The symbol L^1_{loc} stands for locally integrable functions on $\mathbb{R}$. If $q \in (0,\infty]$ and w is a weight, then $L^q(w)$ denotes the Lebesgue L^q-space over the interval $(0,\infty)$ with the measure $w(t)\,\mathrm{d}t$.

Let $\varrho : \mathscr{M}(\mathbb{R}) \to [0,\infty]$ be a functional with the following properties:

(i) $E \subset \mathbb{R}$, $|E| < \infty \Rightarrow \varrho(\chi_E) < \infty$,
(ii) $f \in \mathscr{M}(\mathbb{R})$, $c \ge 0 \Rightarrow \varrho(cf) = c\varrho(f)$ (positive homogeneity),
(iii) $f, g \in \mathscr{M}(\mathbb{R})$, $0 \le f \le g$ a.e. $\Rightarrow \varrho(f) \le \varrho(g)$ (lattice property),
(iv) $f, g \in \mathscr{M}(\mathbb{R})$, $f^* = g^* \Rightarrow \varrho(f) = \varrho(g)$ (r.i. property).

The set $X = X(\varrho) := \{f \in \mathscr{M}(\mathbb{R}),\ \varrho(f) < \infty\}$ is called a *rearrangement-invariant (r.i.) lattice*. For such X we define $\|f\|_X := \varrho(|f|)$ for all $f \in X$.

For the definition of a *rearrangement-invariant space* see [2, p. 59].

Let $p \in (0,\infty]$ and u, v be weights. The *weighted Lorentz spaces* are defined by what follows:

$$\Lambda^p(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$\Lambda^\infty(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^*(t)v(t) < \infty \right\}, \quad p = \infty,$$

$$\Gamma_u^p(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Gamma_u^p(v)} := \left(\int_0^\infty (f_u^{**}(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$\Gamma_u^\infty(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Gamma_u^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f_u^{**}(t)v(t) < \infty \right\}, \quad p = \infty.$$

If $u \equiv 1$, we write just $\Gamma^p(v)$, $\Gamma^\infty(v)$. Next, we denote

$$\mathbb{A} := \{f \in \mathscr{M}(\mathbb{R}); f^*(\infty) = 0\}.$$

Clearly, any function $f \in \mathbb{A}$ satisfies $f^{**}(\infty) = 0$.

The class $S^p(v)$ is given by

$$S^p(v) := \left\{ f \in \mathbb{A};\ \|f\|_{S^p(v)} := \left(\int_0^\infty (f^{**}(t) - f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$S^\infty(v) := \left\{ f \in \mathbb{A};\ \|f\|_{S^\infty(v)} := \operatorname*{ess\,sup}_{t>0} (f^{**}(t) - f^*(t)) v(t) < \infty \right\}, \qquad p = \infty.$$

The Γ-spaces with $u \equiv 1$ are linear and the functional $\|\cdot\|_{\Gamma^p(v)}$ is at least a quasinorm. In fact, for $p \in [1,\infty]$ it is a norm. The key property is the sublinearity of the maximal function (see e.g. [2, p. 54]), i.e.

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad t > 0.$$

On the other hand, the rearrangement itself is not sublinear and the Λ-"spaces" need not to be linear [7]. However, they are always at least r.i. lattices.

In contrast with that, $S^p(v)$ in general does not even have the lattice property. A detailed study of this and other functional properties of $S^p(v)$ was published in [4].

Obviously, $\Gamma^p(v) \subset S^p(v)$ for any $p \in (0,\infty]$ and any weight v. In case of $p \in (0,\infty)$, we will work with weights v satisfying the conditions

$$\int_\varepsilon^\infty \frac{v(t)}{t^p}\,\mathrm{d}t < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \int_0^\infty \frac{v(t)}{t^p}\,\mathrm{d}t = \infty. \tag{2}$$

It can be checked easily that if the first part of (2) is not satisfied, then $\Gamma^p(v) = S^p(v) = \{0\}$, while failing the other part implies that $L^1 \subset \Gamma^p(v) \subset S^p(v)$. By the symbol $\mathscr{V}_p$ we denote the set of all weights v satisfying (2) with $p \in (0,\infty)$. Similarly, $\mathscr{V}_\infty$ stands for the set of all weights satisfying

$$\operatorname*{ess\,sup}_{t>\varepsilon} \frac{v(t)}{t} < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{t>0} \frac{v(t)}{t} = \infty.$$

A useful tool for investigation of convolution inequalities is the O'Neil inequality [14, Lemma 2.5]:

Lemma 2.1. *Let $f, g \in L^1_{\mathrm{loc}}$. Then, for every $t \in (0,\infty)$ it holds*

$$(f*g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s.$$

We are going to use this inequality with an alternative expression of its right-hand side from [11, Proposition 4.1]:

Lemma 2.2. *Let $f, g \in L^1_{\mathrm{loc}}$. Then for every $t \in (0,\infty)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s$$
$$= \limsup_{s\to\infty} s f^{**}(s) g^{**}(s) + \int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s))\,\mathrm{d}s.$$

In particular, if $f \in \mathbb{A}$ and $g \in L^1$, then $\lim_{s\to\infty} s f^{**}(s) g^{**}(s) = 0$. Thus, Lemmas 2.1 and 2.2 together yield

$$(3) \qquad (f*g)^{**}(t) \le \int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s))\,\mathrm{d}s, \quad t>0.$$

As observed already in [14], O'Neil inequality has also a converse form (for the proof of the following statement see e.g. [11, Lemma 2.3]).

Lemma 2.3. *Let $f, g \in L^1_{\mathrm{loc}}$ be nonnegative even functions which are nonincreasing on $(0,\infty)$. Then for every $t \in (0,\infty)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(y) g^*(y)\,\mathrm{d}y \le 12 (f*g)^{**}(t).$$

From now on we denote the "positive symmetrically decreasing" functions by

$$PSD := \{f;\ f \in \mathscr{M}_+(\mathbb{R}),\ f \text{ is even},\ f \text{ is nonincreasing on } (0,\infty)\}.$$

Applying Lemmas 2.2, 2.3 and the observation (3), we reach the following conclusion: Let $f \in \mathbb{A}$, $g \in L^1$ and assume that both $f, g \in PSD$. Then

$$(4) \qquad \int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s))\,\mathrm{d}s \le 12 (f*g)^{**}(t), \quad t>0.$$

The last preliminary result is the proposition below (cf. e.g. [16, Lemma 1.2], [5, Proposition 7.2]).

Proposition 2.4. *Let h be a nonnegative and nonincreasing real-valued function on $(0,\infty)$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n \in \mathscr{M}(\mathbb{R})$ such that for a.e. $t>0$ it holds*

$$\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^*\left(\frac{1}{t}\right)}{t} \uparrow h(t), \quad n\to\infty.$$

Proof. There exists a nonnegative Radon measure ν on $(0,\infty)$ such that for a.e. $t>0$ it is

$$(5) \qquad h(t) = \int_{[t,\infty)} \frac{\mathrm{d}\nu(x)}{x}.$$

For any $n \in \mathbb{N}$ we can find a function $f_n \in \mathscr{M}(\mathbb{R})$ such that

$$f_n^*(t) = \int\limits_{(0,\frac{1}{t})} \chi_{(\frac{1}{n},\infty)}(x)\mathrm{d}\nu(x)$$

for all $t > 0$. Now choose any $t > 0$ such that (5) holds. By Fubini theorem,

$$\begin{aligned}\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^*\left(\frac{1}{t}\right)}{t} &= \int\limits_0^{\frac{1}{t}} \int\limits_{(0,\frac{1}{s})} \chi_{(\frac{1}{n},\infty)}(x)\mathrm{d}\nu(x)\,\mathrm{d}s - \frac{1}{t}\int\limits_{(0,t)} \chi_{(\frac{1}{n},\infty)}(x)\mathrm{d}\nu(x)\\ &= \int\limits_{(0,\infty)} \int\limits_0^{\min\left\{\frac{1}{x},\frac{1}{t}\right\}} \mathrm{d}s\, \chi_{(\frac{1}{n},\infty)}(x)\mathrm{d}\nu(x) - \frac{1}{t}\int\limits_{(0,t)} \chi_{(\frac{1}{n},\infty)}(x)\mathrm{d}\nu(x)\\ &= \int\limits_{[t,\infty)} \frac{\chi_{(\frac{1}{n},\infty)}(x)}{x}\mathrm{d}\nu(x) \uparrow h(t), \quad n\to\infty. \qquad \square\end{aligned}$$

3. Inequalities with $f^{**} - f^*$ and boundedness of the convolution operator

As mentioned in the introduction, we are going to describe when $T_g : S^p(v) \to \Gamma^q(w)$ is bounded and, above all, what is the optimal r.i. space Y such that the inequality $\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)}\|g\|_Y$ holds for all $f \in S^p(v)$ and $g \in L^1 \cap Y$. The problem is connected to inequalities involving the expression $f^{**} - f^*$ which are shown in the following lemma. It is a direct consequence of the O'Neil inequality (3).

Lemma 3.1. *Let $p, q \in (0,\infty]$. Let v, w be weights, $v \in \mathscr{V}_p$. Let $g \in L^1$.*

(i) *If $p, q \in (0,\infty)$ and*

$$\begin{aligned}&\left(\int\limits_0^\infty \left(\int\limits_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t))\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\le C_{(6)}\left(\int\limits_0^\infty (f^{**}(x) - f^*(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}, \quad f \in S^p(v),\end{aligned} \tag{6}$$

then $T_g : S^p(v) \to \Gamma^q(w)$ and, moreover, the optimal constant $C_{(6)}$ satisfies $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \le C_{(6)}$.

(ii) *If* $0 < p < \infty = q$ *and*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{x>0} \int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t \; w(x) \\
&\le C_{(7)} \left(\int_0^\infty (f^{**}(x)-f^*(x))^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\end{aligned} \tag{7}
$$

then $T_g : S^p(v) \to \Gamma^\infty(w)$ *and, moreover, the optimal constant* $C_{(8)}$ *satisfies* $\|T_g\|_{S^p(v)\to\Gamma^\infty(w)} \le C_{(8)}$.

(iii) *If* $0 < q < \infty = p$ *and*

$$
\begin{aligned}
&\left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\le C_{(8)} \operatorname*{ess\,sup}_{x>0} (f^{**}(x)-f^*(x))v(x), \quad f \in S^\infty(v),
\end{aligned} \tag{8}
$$

then $T_g : S^\infty(v) \to \Gamma^q(w)$ *and, moreover, the optimal constant* $C_{(7)}$ *satisfies* $\|T_g\|_{S^\infty(v)\to\Gamma^q(w)} \le C_{(7)}$.

(iv) *If* $p = q = \infty$ *and*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{x>0} \int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t \; w(x) \\
&\le C_{(9)} \operatorname*{ess\,sup}_{x>0} (f^{**}(x)-f^*(x))v(x), \quad f \in S^\infty(v),
\end{aligned} \tag{9}
$$

then $T_g : S^\infty(v) \to \Gamma^\infty(w)$ *and, moreover, the optimal constant* $C_{(9)}$ *satisfies* $\|T_g\|_{S^\infty(v)\to\Gamma^\infty(w)} \le C_{(9)}$.

The next result is inverse to the previous lemma, showing that the validity of the inequalities with $f^{**} - f^*$ from that lemma is also necessary for the boundedness of T_g, given that $g \in PSD$.

Lemma 3.2. *Let* $p, q \in (0, \infty]$. *Let* v, w *be weights,* $v \in \mathcal{V}_p$. *Let* $g \in L^1 \cap PSD$.

(i) *If* $p, q \in (0, \infty)$ *and* $T_g : S^p(v) \to \Gamma^q(w)$, *then* (6) *holds and the optimal constant* $C_{(6)}$ *satisfies* $C_{(6)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$.

(ii) *If* $0 < p < \infty = q$ *and* $T_g : S^p(v) \to \Gamma^\infty(w)$, *then* (7) *holds and the optimal constant* $C_{(7)}$ *satisfies* $C_{(7)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^\infty(w)}$.

(iii) *If* $0 < q < \infty = p$ *and* $T_g : S^\infty(v) \to \Gamma^q(w)$, *then* (8) *holds and the optimal constant* $C_{(8)}$ *satisfies* $C_{(8)} \lesssim \|T_g\|_{S^\infty(v)\to\Gamma^q(w)}$.

(iv) *If* $p = q = \infty$ *and* $T_g : S^\infty(v) \to \Gamma^\infty(w)$, *then* (9) *holds and the optimal constant* $C_{(9)}$ *satisfies* $C_{(9)} \lesssim \|T_g\|_{S^\infty(v)\to\Gamma^\infty(w)}$.

Proof. Let us show (i), the other cases are analogous. By (4), for the optimal constant $C_{(6)}$ we get

$$
\begin{aligned}
C_{(6)} &= \sup_{\|f\|_{S^p(v)}\le 1} \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \\
&= \sup_{\substack{\|f\|_{S^p(v)}\le 1 \\ f\in PSD}} \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t \right)^q w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\le 12 \sup_{\substack{\|f\|_{S^p(v)}\le 1 \\ f\in PSD}} \left(\int_0^\infty ((f*g)^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \\
&\le \|T_g\|_{S^p(v)\to\Gamma^q(w)}.
\end{aligned}
$$

□

Now we characterize under which conditions on weights and exponents the inequalities of Lemma 3.1 are satisfied.

Theorem 3.3. *Let $p,q\in(0,\infty)$. Let v,w be weights, $v\in\mathscr{V}_p$. Let $g\in L^1$.*

(i) *If $1<p\le q<\infty$, then (6) holds if and only if*

$$
A_{(10)} := \sup_{x>0} \left(\int_x^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \left(\int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \right)^{-\frac{1}{p}} < \infty \tag{10}
$$

and

$$
A_{(11)} := \sup_{x>0} W^{\frac{1}{q}}(x) \left(\int_x^\infty (g^{**}(t))^{p'} \left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p}\,\mathrm{d}t \right)^{\frac{1}{p'}} < \infty. \tag{11}
$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)}\simeq A_{(10)}+A_{(11)}$.

(ii) *If $0<p\le 1$, $0<p\le q<\infty$, then (6) holds if and only if $A_{(10)}<\infty$ and*

$$
A_{(12)} := \sup_{x>0} g^{**}(x) W^{\frac{1}{q}}(x) \left(\int_x^\infty v(t)\,\mathrm{d}t \right)^{-\frac{1}{p}} < \infty. \tag{12}
$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)}\simeq A_{(10)}+A_{(12)}$.

(iii) *If $1<p<\infty$, $0<q<p$, then (6) holds if and only if*

$$
A_{(13)} := \left(\int_0^\infty \left(\int_x^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_x^\infty \frac{v(t)}{t^p}\,\mathrm{d}t \right)^{-\frac{r}{q}} \frac{v(x)}{x^p}\,\mathrm{d}x \right)^{\frac{1}{r}} < \infty \tag{13}
$$

and

$$A_{(14)} := \Bigg(\int_0^\infty W^{\frac{r}{p}}(x) w(x) \times \left(\int_x^\infty (g^{**}(t))^{p'} \left(\int_t^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \, \mathrm{d}t \right)^{\frac{r}{p'}} \mathrm{d}x \Bigg)^{\frac{1}{r}} < \infty. \tag{14}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(13)} + A_{(14)}$.

(iv) *If* $0 < q < p \le 1$, *then* (6) *holds if and only if* $A_{(13)} < \infty$ *and*

$$A_{(15)} := \left(\int_0^\infty \sup_{x \le t < \infty} (g^{**}(t))^r \left(\int_0^t \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{r}{p}} W^{\frac{r}{p}}(x) w(x) \, \mathrm{d}x \right)^{\frac{1}{r}} < \infty. \tag{15}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(13)} + A_{(15)}$.

Proof. Let us show (i). After the change of variable $x \mapsto \frac{1}{x}$, inequality (6) is written as

$$\begin{aligned} &\left(\int_0^\infty \left(\int_0^x \frac{f^{**}\left(\frac{1}{t}\right) - f^{*}\left(\frac{1}{t}\right)}{t} \cdot \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \right)^q \frac{w\left(\frac{1}{x}\right)}{x^2} \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\le C_{(6)} \left(\int_0^\infty \left(\frac{f^{**}\left(\frac{1}{x}\right) - f^{*}\left(\frac{1}{x}\right)}{x} \right)^p v\left(\frac{1}{x}\right) x^{p-2} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad f \in \mathscr{M}(\mathbb{R}). \end{aligned} \tag{16}$$

Let us denote by $\mathscr{M}_+^{\downarrow}(0,\infty)$ the cone of nonnegative and nonincreasing functions on $(0,\infty)$. We claim that (16) is true if and only if

$$\begin{aligned} &\left(\int_0^\infty \left(\int_0^x \varphi(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \right)^q \frac{w\left(\frac{1}{x}\right)}{x^2} \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\le C_{(6)} \left(\int_0^\infty \varphi^p(x) \frac{v\left(\frac{1}{x}\right)}{x^{2-p}} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad \varphi \in \mathscr{M}_+^{\downarrow}(0,\infty). \end{aligned} \tag{17}$$

Indeed, every function $t \mapsto \frac{f^{**}\left(\frac{1}{t}\right) - f^{*}\left(\frac{1}{t}\right)}{t}$ is nonnegative and nonincreasing on $(0,\infty)$, hence (17) implies (16). On the other hand, if $\varphi \in \mathscr{M}_+^{\downarrow}(0,\infty)$ is given, by Proposition 2.4 we find $f_n \in \mathscr{M}(\mathbb{R})$ such that $\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^{*}\left(\frac{1}{t}\right)}{t} \uparrow \varphi(t)$ for a.e. $t \in (0,\infty)$. Since (16) holds for every f_n in place of f, by the monotone convergence theorem we get (17) for the given φ. Hence, (16) implies (17).

Inequality (17) defines the embedding

$$\Lambda^p(\widetilde{v}) \hookrightarrow \Gamma_u^q(\widetilde{w}) \tag{18}$$

with

$$\widetilde{v}(x) := v\left(\frac{1}{x}\right)x^{p-2}, \quad \widetilde{w}(x) := w\left(\frac{1}{x}\right)x^{q-2}, \quad u(x) := \frac{g^{**}\left(\frac{1}{x}\right) - g^{*}\left(\frac{1}{x}\right)}{x}.$$

By [8, Theorem 3.1(iii)] or a modified version of [6, Theorem 4.1(i)], (18) (as well as (17)) holds if and only if $A_{(10)} + A_{(11)} < \infty$ and the optimal $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(10)} + A_{(11)}$, which is the result.

In cases (ii)–(iv) we proceed in the same way, the only difference being the conditions characterizing (18) for different settings of p and q. These characterizations of (18) may be found in [8, Theorem 3.1] or, alternatively, in [6, Theorem 4.1] for (ii) and (iii) and [5, Theorem 3.1] for (iv). Note that in [5, 6] the results are given just for $u = 1$. □

Remark 3.4. For $1 \leq p < \infty$, Theorem 3.3 can be alternatively obtained using the reduction theorem [9, Theorem 2.2] and Hardy inequalities for nonnegative functions (see e.g. [12, 15]).

In the case $q = \infty$, i.e. for (7), we get

Theorem 3.5. *Let $p \in (0, \infty)$. Let v, w be weights, $v \in \mathscr{V}_p$. Let $g \in L^1$. Then*

(i) *If $0 < p \leq 1$, then (7) holds if and only if*

$$A_{(19)} := \operatorname*{ess\,sup}_{x>0} w(x) \sup_{t>x} g^{**}(t) \left(\int_t^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{1}{p}} < \infty. \tag{19}$$

Moreover, the optimal constant $C_{(7)}$ satisfies $C_{(7)} \simeq A_{(19)}$.

(ii) *If $1 < p < \infty$, then (7) holds if and only if*

$$\begin{aligned} A_{(20)} := \operatorname*{ess\,sup}_{x>0} w(x) \Bigg[&\left(\int_x^\infty (g^{**}(t))^{p'} \left(\int_t^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \, \mathrm{d}t \right)^{\frac{1}{p'}} \\ &+ g^{**}(x) \left(\int_x^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{1}{p}} \Bigg] < \infty. \end{aligned} \tag{20}$$

Moreover, the optimal constant $C_{(7)}$ satisfies $C_{(7)} \simeq A_{(20)}$.

Proof. Following the same reasoning as in the proof of Theorem 3.3, the inequality (7) is equivalent to

$$\begin{aligned} &\operatorname*{ess\,sup}_{x>0} \int_0^x \varphi(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \; w\left(\frac{1}{x}\right) \\ &\leq C_{(7)} \left(\int_0^\infty \varphi^p(x) v\left(\frac{1}{x}\right) x^{p-2} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad \varphi \in \mathscr{M}_+^{\downarrow}(0, \infty). \end{aligned}$$

Denote $v_p(x) := v\left(\frac{1}{x}\right)x^{p-2}$. The optimal $C_{(7)}$ satisfies

$$
\begin{aligned}
C_{(7)} &= \sup_{\|f\|_{\Lambda^p(v_p)}\le 1} \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\int_0^x f^*(t)\frac{g^{**}\left(\frac{1}{t}\right)-g^*\left(\frac{1}{t}\right)}{t}\,\mathrm{d}t \\
&= \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\sup_{\|f\|_{\Lambda^p(v_p)}\le 1}\int_0^x f^*(t)\frac{g^{**}\left(\frac{1}{t}\right)-g^*\left(\frac{1}{t}\right)}{t}\,\mathrm{d}t.
\end{aligned} \tag{21}
$$

In the following calculations, we are going to use the condition (2) without further comment.

(i) If $0 < p \le 1$, [6, Theorem 3.1(i)] gives

$$
\sup_{\|f\|_{\Lambda^p(v_p)}\le 1}\int_0^x f^*(t)\frac{g^{**}\left(\frac{1}{t}\right)-g^*\left(\frac{1}{t}\right)}{t}\,\mathrm{d}t \simeq \sup_{t\in(0,x)}\int_0^t \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-\frac{1}{p}}.
$$

Hence, we get

$$
\begin{aligned}
C_{(7)} &\simeq \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\sup_{t\in(0,x)}\int_0^t \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-\frac{1}{p}} \\
&= \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\sup_{t\in\left(\frac{1}{x},\infty\right)} g^{**}(t)\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-\frac{1}{p}} \\
&= A_{(19)}.
\end{aligned}
$$

(ii) If $1 < p < \infty$, by [6, Theorem 3.1(ii)] we have

$$\sup_{\|f\|_{\Lambda^p(v_p)}\le 1} \int_0^x f^*(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^*\left(\frac{1}{t}\right)}{t} \mathrm{d}t$$

$$\simeq \left(\int_0^x \left(\int_0^t \frac{g^{**}\left(\frac{1}{s}\right) - g^*\left(\frac{1}{s}\right)}{s} \mathrm{d}s \right)^{p'} \left(\int_0^t v_p(s) \mathrm{d}s \right)^{-p'} v_p(t) \mathrm{d}t \right)^{\frac{1}{p'}}$$

$$+ \int_0^x \frac{g^{**}\left(\frac{1}{s}\right) - g^*\left(\frac{1}{s}\right)}{s} \mathrm{d}s \left(\int_x^\infty \left(\int_0^t v_p(s) \mathrm{d}s \right)^{-p'} v_p(t) \mathrm{d}t \right)^{\frac{1}{p'}}$$

$$= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{p'} \left(\int_t^\infty \frac{v(s)}{s^p} \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \mathrm{d}t \right)^{\frac{1}{p'}}$$

$$+ g^{**}\left(\frac{1}{x}\right) \left(\int_0^{\frac{1}{x}} \left(\int_t^\infty \frac{v(s)}{s^p} \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \mathrm{d}t \right)^{\frac{1}{p'}}$$

$$= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{p'} \left(\int_t^\infty \frac{v(s)}{s^p} \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \mathrm{d}t \right)^{\frac{1}{p'}} + g^{**}\left(\frac{1}{x}\right) \left(\int_{\frac{1}{x}}^\infty \frac{v(s)}{s^p} \mathrm{d}s \right)^{-\frac{1}{p}}.$$

Hence, (21) implies $C_{(7)} \simeq A_{(20)}$ for the optimal $C_{(7)}$. □

For the last case, $p = \infty$, which covers the inequalities (8) and (9), we have the following theorem.

Theorem 3.6. *Let v, w be weights, $v \in \mathcal{V}_\infty$. Let $g \in L^1$. Then*

(i) *For $0 < q < \infty$, the inequality (8) holds and only if*

$$(22) \qquad A_{(22)} := \left(\int_0^\infty \left(\int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \mathrm{d}t \right)^q w(x) \mathrm{d}x \right)^{\frac{1}{q}} < \infty.$$

Moreover, the optimal constant $C_{(8)}$ satisfies $C_{(8)} \simeq A_{(22)}$.

(ii) *The inequality (9) holds if and only if*

$$(23) \qquad A_{(23)} := \operatorname*{ess\,sup}_{x>0} \int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \mathrm{d}t \ w(x) < \infty.$$

Moreover, the optimal constant $C_{(9)}$ satisfies $C_{(9)} \simeq A_{(23)}$.

Proof. Here we use the same technique as in Theorems 3.3 and 3.5. During the process we apply e.g. the result of [17, Proposition 2.7]. We omit the details. □

Remark 3.7. In each of the particular settings of the exponents p,q in Theorem 3.3(i)–(iv), the functionals $A_{(10)},\dots,A_{(15)}$ are r.i. norms of g, with the following exceptions: In (iii) and (iv), if $0<q<1$, then $A_{(13)}$ is in general just an r.i. quasi-norm, the same applies to $A_{(15)}$ in (iv) if $r<1$. Similarly, the functionals $A_{(19)}$ and $A_{(20)}$ in Theorem 3.5 are r.i. norms of g. For a detailed proof of this, see e.g. [11, Proposition 5.6].

In Theorem 3.6, the functional $A_{(23)}$ acting on $g\in L^1$ is an r.i. norm of g. The functional $A_{(22)}$ is, in general, an r.i. quasi-norm, for $q\geq 1$ an r.i. norm. Let us prove the claim about $A_{(22)}$. At first, since $t\mapsto \left(\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}\right)^{-1}$ is nondecreasing, its derivative, which we denote by

$$\delta(t):=\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}},$$

exists and is nonnegative for a.e. $t\in(0,\infty)$. Let $x\in(0,\infty)$. Suppose that

$$\int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t<\infty.$$

Then, by monotonicity of $\left(\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}\right)^{-1}$, we have

$$\begin{aligned}\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}&=\frac{1}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\int_t^\infty \frac{g^{**}(y)-g^*(y)}{y}\,\mathrm{d}y\\&\leq \int_t^\infty \frac{g^{**}(y)-g^*(y)}{y\ \operatorname{ess\,sup}_{s\in(y,\infty)} v(s)s^{-1}}\,\mathrm{d}y \xrightarrow{t\to\infty} 0.\end{aligned}$$

Hence, by partial integration and the previous, we get

$$\begin{aligned}\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t&=\left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\right]_{t=x}^\infty+\int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t\\&=\int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t-\frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}}\\&<\infty.\end{aligned}$$

Now assume, on the other hand, that $\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t<\infty$. Then,

$$\int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t=\frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}}+\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t<\infty.$$

Thus, we see that $A_{(22)}$ is equal to

$$\left(\int_0^\infty\left(\frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}}+\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

This expression is an r.i. quasi-norm of g, for $q \geq 1$ it is an r.i. norm. To check this, we refer again to [11].

In the same way as above, we may show that $A_{(23)}$ is an r.i. norm.

4. YOUNG-TYPE CONVOLUTION INEQUALITIES WITH THE CLASS S ON THE RIGHT-HAND SIDE

In the previous part we obtained the conditions for boundedness of T_g. Let us now summarize these results and apply them to get the desired convolution inequalities. Note that, in what follows, if we define $\|\cdot\|_Y$ first, then the space Y is naturally defined as $Y := \{f \in \mathscr{M}(\mathbb{R});\ \|f\|_Y < \infty\}$.

Theorem 4.1. *Let $p, q \in (0, \infty]$. Let v, w be weights, $v \in \mathscr{V}_p$. For $g \in L^1$ define $\|g\|_Y$ by what follows:*

$$\|g\|_Y := \begin{cases} A_{(10)} + A_{(11)} & \text{if } \ 1 < p \leq q < \infty; \\ A_{(10)} + A_{(12)} & \text{if } \ 0 < p \leq 1,\ 0 < p \leq q < \infty; \\ A_{(13)} + A_{(14)} & \text{if } \ 1 < p < \infty,\ 0 < q < p; \\ A_{(13)} + A_{(15)} & \text{if } \ 0 < q < p \leq 1; \\ A_{(19)} & \text{if } \ 0 < p \leq 1,\ q = \infty; \\ A_{(20)} & \text{if } \ 1 < p < \infty,\ q = \infty; \\ A_{(22)} & \text{if } \ p = \infty,\ 0 < q < \infty; \\ A_{(23)} & \text{if } \ p = q = \infty. \end{cases}$$

Then

(i) *If $g \in Y$, then $T_g : S^p(v) \to \Gamma^q(w)$ and*

$$\|T_g\|_{S^p(v) \to \Gamma^q(w)} \lesssim \|g\|_Y.$$

(ii) *If $g \in PSD$ and $T_g : S^p(v) \to \Gamma^q(w)$, then $g \in Y$ and*

$$\|g\|_Y \lesssim \|T_g\|_{S^p(v) \to \Gamma^q(w)}.$$

(iii) *The inequality*

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)} \|g\|_Y, \quad f \in S^p(v),\ g \in L^1 \cap Y, \tag{24}$$

is satisfied. Moreover, if $\widetilde{Y}$ is any r.i. lattice such that (24) *holds with $\widetilde{Y}$ in place of Y, then $L^1 \cap \widetilde{Y} \hookrightarrow L^1 \cap Y$.*

Proof. Let us prove the assertions for the case $1 < p \leq q < \infty$. In the other cases, the only difference is that we work with another appropriate functional $A_{(\dots)}$.

(i) Let $g \in Y$, thus $A_{(10)} + A_{(11)} < \infty$. Then, by Theorem 3.3(i), the inequality (6) holds. Thus, from Lemma 3.1(i) it follows that $T_g : S^p(v) \to \Gamma^q(w)$ and $\|T_g\|_{S^p(v) \to \Gamma^q(w)} \lesssim C_{(6)} \simeq \|g\|_Y$.

(ii) Assume that $g \in PSD$ and $T_g : S^p(v) \to \Gamma^q(w)$. By Lemma 3.2(i), (6) holds and the optimal $C_{(6)}$ satisfies $C_{(6)} \lesssim \|T_g\|_{S^p(v) \to \Gamma^q(w)}$. Theorem 3.1(i) now yields that $A_{(10)} + A_{(11)} < \infty$, i.e. $g \in Y$. Moreover, we also get $\|g\|_Y \simeq C_{(6)} \lesssim \|T_g\|_{S^p(v) \to \Gamma^q(w)}$.

(iii) The inequality (24) follows from (i) and the relation

$$\|T_g f\|_{\Gamma^q(w)} \leq \|T_g\|_{S^p(v) \to \Gamma^q(w)} \|f\|_{S^p(v)}.$$

Let us prove the optimality of Y. Assume that $\widetilde{Y}$ is an r.i. lattice such that

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)} \|g\|_{\widetilde{Y}}, \quad f \in S^p(v),\ g \in L^1 \cap \widetilde{Y}. \tag{25}$$

Let $h \in L^1 \cap \widetilde{Y}$. We can find a function $g \in L^1 \cap \widetilde{Y} \cap PSD$ such that $g^* = h^*$. The inequality (25) yields that $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}$. Thus, $T_g : S^p(v) \to \Gamma^q(w)$ and by (ii) it holds $\|g\|_Y \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$. Together we get

$$\|g\|_Y \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}.$$

The functionals $\|\cdot\|_Y$ and $\|\cdot\|_{\widetilde{Y}}$ are r.i., thus we obtain

$$\|h\|_Y \lesssim \|h\|_{\widetilde{Y}}.$$

Since h was chosen arbitrarily, we got the desired embedding $L^1 \cap \widetilde{Y} \hookrightarrow L^1 \cap Y$. $\square$

Remark 4.2. For given weights v, w and exponents p, q, the optimal space Y may equal $\{0\}$. (Let us formally consider $\{0\}$ to be an r.i. space.) In that case, the operator T_g with a nonnegative kernel g is bounded between $S^p(v)$ and $\Gamma^q(w)$ if and only if $g = 0$ a.e. (cf. [11, Corollary 3.3]).

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Paper III

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Convolution in rearrangement-invariant spaces defined in terms of oscillation and the maximal function

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CONVOLUTION IN WEIGHTED LORENTZ SPACES OF TYPE Γ

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Abstract. We characterize boundedness of the convolution operator between weighted Lorentz spaces $\Gamma^p(v)$ and $\Gamma^q(w)$ for the range of parameters $p,q \in [1,\infty]$, or $p \in (0,1)$ and $q \in \{1,\infty\}$, or $p=\infty$ and $q \in (0,1)$. We provide Young-type convolution inequalities of the form

$$\|f*g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v),\ g \in Y,$$

characterizing the optimal rearrangement-invariant space Y for which the inequality is satisfied.

1. Introduction

Let f and g be locally integrable functions on $\mathbb{R}^d$, $d \in \mathbb{N}$. The *convolution* $f*g$ is given by

$$(f*g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y \quad x \in \mathbb{R}^d.$$

If the function g is fixed, we define the convolution operator T_g by

$$T_g f := f*g. \tag{1}$$

This paper has the following purpose. First, given weights v,w and exponents p,q, to characterize when the operator T_g is bounded between the weighted Lorentz spaces $\Gamma^p(v)$ and $\Gamma^q(w)$, in terms of the kernel g. Second, to prove related Young-type inequalities in the form

$$\|f*g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v),\ g \in Y,$$

and to characterize the optimal (i.e. essentially largest) rearrangement-invariant space Y such that this inequality holds. (For definitions see Section 2.)

A variety of results can be labeled as Young-type convolution inequalities. Their common ancestor is the classical Young inequality reading

$$\|f*g\|_q \le \|f\|_p\|g\|_r, \quad f \in L^p,\ g \in L^r,$$

where $1 \le p,q,r \le \infty$ and $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$. Results in a similar shape have been obtained for many classes of function spaces other than the Lebesgue spaces in the original Young inequality. In [15, 8, 19, 2] the Lorentz spaces $L_{p,q}$ were

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considered and the following inequality was proved: For $1 < p, q, r < \infty$ and $0 < a, b, c \le \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, it holds

$$\|f * g\|_{L_{q,a}} \le C\|f\|_{L_{p,b}}\|g\|_{L_{r,c}}, \quad f \in L_{p,b},\ g \in L_{r,c}.$$

An analogous problem for convolution of periodic functions on the real line was studied in [14].

In the papers [11, 12], inequalities of the type

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_X\|g\|_Y, \quad f \in X,\ g \in Y,$$

were obtained for X being the weighted Lorentz space $\Lambda^p(v)$ or the Lorentz-type class $S^p(v)$, defined in terms of oscillation. The proof technique there was based on the use of the O'Neil convolution inequality

$$(f * g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s, \quad t > 0, \tag{2}$$

(see [15, Lemma 2.5]) and various weighted Hardy-type inequalities. This method also granted that the obtained rearrangement-invariant space Y was optimal.

An analogous technique will be used here. After presenting the definitions and auxiliary results in Section 2, in Section 3 we will characterize, in terms of g, v, w, p, q, validity of the inequality

$$\left\| t \mapsto \left(t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \right) \right\|_{L^q(w)} \le C\|f\|_{\Gamma^p(v)}, \quad f \in \Gamma^p(v),$$

with C being a constant independent of f. The conditions obtained in this way will be, by the O'Neil inequality (2), sufficient for boundedness $T_g : \Gamma^p(v) \to \Gamma^q(w)$. To show their necessity as well, we will make use of a reverse O'Neil inequality (see Lemma 2.1) holding for positive radially decreasing functions. This is included in Section 4, where the results are presented in the form of Young-type inequalities

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v),\ g \in Y.$$

The result may indeed be formulated so, since, as observed in Section 3, the conditions on g characterizing the optimal constant C in (2) have the form of a norm of g in a rearrangement-invariant space Y. Its optimality will be proved as well.

Let us note here that although we will consider just $\mathbb{R}^d$ as the underlying space in this paper, the results can be easily modified for periodic functions on the real line, as it was done e.g. in [11].

2. Preliminaries

Throughout the text, the following notation is used: The positive integer d will denote the dimension of the space $\mathbb{R}^d$. By $\mathscr{M}(\Omega)$ we denote the set of all measurable functions on Ω with values in $[-\infty, \infty]$. We will work with the choice $\Omega=\mathbb{R}^d$ or $\Omega=(0,\infty)$. Similarly, $\mathscr{M}_+(\Omega)$ stands for the set of all nonnegative functions from $\mathscr{M}(\Omega)$. Next, we denote by $\mathscr{M}_+^{\odot}(\mathbb{R}^d)$ the set of all functions $f\in\mathscr{M}_+(\mathbb{R}^d)$ such that there exists a nonincreasing $f_0\in\mathscr{M}_+(0,\infty)$ such that for a.e. $x\in\mathbb{R}^d$ it holds $f(x)=f_0(|x|)$, i.e. $\mathscr{M}_+^{\odot}(\mathbb{R}^d)$ is the set of nonnegative radially decreasing functions on $\mathbb{R}^d$.

The notation $A\lesssim B$ means that $A\le CB$ where C is a positive constant independent of relevant quantities. Unless specified else, this C in fact always depends only on exponents p and q, if they are involved. If $A\lesssim B$ and $B\lesssim A$, we write $A\simeq B$. The *optimal constant* C in an inequality $A\le CB$ is the least C such that the inequality holds. By writing inequalities in the form

$$A(f)\lesssim B(f),\quad f\in X,$$

we mean that $A(f)\lesssim B(f)$ is satisfied for all $f\in X$.

If $f\in\mathscr{M}(\mathbb{R}^d)$, we define the *nonincreasing rearrangement* of f by

$$f^*(t):=\inf\left\{s>0;\ |\{x\in\mathbb{R}^d;\ |f(x)|>s\}|\le t\right\},\qquad t>0,$$

and the *Hardy-Littlewood maximal function* of f by

$$f^{**}(t):=\frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s,\qquad t>0.$$

For the definition of a *rearangement-invariant (r.i.) norm* and an *r.i. space* see [1]. We will also use the terms *r.i. quasi-norm* and *r.i. lattice*, as defined e.g. in [11]. Here we consider $\mathbb{R}^d$ to be the underlying measure space, unless specified else.

A *weight* is a function from $\mathscr{M}_+(0,\infty)$. We write $W(t):=\int_0^t w(s)\,\mathrm{d}s$ for $t>0$. By $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we denote the locally integrable functions on $\mathbb{R}^d$. If $q\in(0,\infty]$ and w is a weight, then $L^q(w)$ denotes the Lebesgue L^q-space over $(0,\infty)$ with the measure $w(t)\,\mathrm{d}t$.

Let $p\in(0,\infty]$, and v be a weight. The *weighted Lorentz spaces* are defined in the following way:

$$\Lambda^p(v):=\left\{f\in\mathscr{M}(\mathbb{R}^d);\ \|f\|_{\Lambda^p(v)}:=\left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}<\infty\right\},\quad p\in(0,\infty),$$

$$\Lambda^\infty(v):=\left\{f\in\mathscr{M}(\mathbb{R}^d);\ \|f\|_{\Lambda^\infty(v)}:=\operatorname*{ess\,sup}_{t>0} f^*(t)v(t)<\infty\right\},\qquad p=\infty,$$

$$\Gamma^p(v):=\left\{f\in\mathscr{M}(\mathbb{R}^d);\ \|f\|_{\Gamma^p(v)}:=\left(\int_0^\infty (f^{**}(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}<\infty\right\},\quad p\in(0,\infty),$$

$$\Gamma^\infty(v):=\left\{f\in\mathscr{M}(\mathbb{R}^d);\ \|f\|_{\Gamma^\infty(v)}:=\operatorname*{ess\,sup}_{t>0} f^{**}(t)v(t)<\infty\right\},\qquad p=\infty.$$

If we assume that $V(t) > 0$ for all $t > 0$, the functional $\|\cdot\|_{\Gamma^p(v)}$ is at least a quasi-norm, for $p \in [1, \infty]$ it is a norm. The key property here is the sublinearity of the maximal function, i.e.

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad f, g \in \mathscr{M}(\mathbb{R}^d),\ t > 0. \tag{3}$$

(See e.g. [1, p. 54].) In contrast, the Λ-“spaces” are not even linear sets in general. Functional properties of Λ and Γ are discussed in detail e.g. in [4, 9].

Let us list several auxiliary results. First, the O’Neil inequality (2) has also a converse form, as shown in the following lemma. The proof of this multi-dimensional version may be found e.g. in [10], the corresponding one-dimensional result was mentioned already in [15], its proof is shown e.g. in [11].

Lemma 2.1. *Let $f, g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$. Then for every $t \in (0, \infty)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(y) g^*(y)\,\mathrm{d}y \leq C_d (f * g)^{**}(t),$$

where C_d is a constant depending on the dimension d of the underlying space $\mathbb{R}^d$ but independent of f, g and t.

To handle inequalities involving the maximal function on both sides, it is possible to use the result of [5, Theorem 4.4]. It reads as follows:

Lemma 2.2. *Let $p, q \in (1, \infty)$ and let v, w be weights. Define the weight ψ by*

$$\psi(t) := \frac{t^{p'+p-1} V(t) \int_t^\infty v(s) s^{-p}\,\mathrm{d}s}{\left(V(t) + t^p \int_t^\infty v(s) s^{-p}\,\mathrm{d}s\right)^{p'+1}}, \quad t > 0. \tag{4}$$

Let R be a positive linear operator on $\mathscr{M}_+(0, \infty)$ and S be the Stieltjes operator given by

$$Sh(t) := \int_0^\infty \frac{h(s)}{s+t}\,\mathrm{d}s, \quad t > 0. \tag{5}$$

Then

$$\left(\int_0^\infty (R(f^{**})(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq K_1 \left(\int_0^\infty (f^{**}(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad f \in \mathscr{M}(\mathbb{R}^d),$$

if and only if

$$\left(\int_0^\infty (RSh(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq K_2 \left(\int_0^\infty h^p(t) \psi^{1-p}(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad h \in \mathscr{M}_+(0, \infty).$$

Moreover, it holds $K_1 \simeq K_2$.

The proposition below is a particular case of [17, Lemma 1.2].

Proposition 2.3. *Let $h \in \mathscr{M}$. Then there exists a sequence of nonnegative measurable functions γ_n with compact support in $(0,\infty)$ such that for a.e. $t>0$ it holds*

$$\int_t^\infty \gamma_n(s)\,\mathrm{d}s \uparrow h^*(t), \quad n\to\infty.$$

The next result follows by integration by parts (cf. [18, Lemma, p. 176]).

Proposition 2.4. *Let $1<q<p<\infty$ and $r:=\frac{pq}{p-q}$. Let v,w be weights. Then it holds*

$$\begin{aligned}\int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_t^\infty v\right)^{\frac{r}{p'}}\mathrm{d}t &\le \frac{q}{p'}\int_0^\infty W^{\frac{r}{q}}(t)\left(\int_t^\infty v\right)^{\frac{r}{q'}} v(t)\,\mathrm{d}t\\ &\le \int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_t^\infty v\right)^{\frac{r}{p'}}\mathrm{d}t.\end{aligned}$$

3. INEQUALITIES RELATED TO THE BOUNDEDNESS OF THE CONVOLUTION OPERATOR

In this section we are going to characterize validity of the inequality

$$\left\| t\mapsto\left(tf^{**}(t)g^{**}(t)+\int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s\right)\right\|_{L^q(w)} \le C_{(6)}\|f\|_{\Gamma^p(v)}, \quad f\in\Gamma^p(v), \tag{6}$$

by certain conditions on the kernel function g, the weights v,w and exponents p,q. By doing this, we obtain sufficient conditions for the boundedness of T_g between $\Gamma^p(v)$ and $\Gamma^q(w)$. Indeed, thanks to the O'Neil inequality (2), if (6) holds, then $T_g:\Gamma^p(v)\to\Gamma^q(w)$.

We start with (6) with the parameters satisfying $1<p,q<\infty$. The lemma below shows that (6) is equivalent to two certain weighted Hardy inequalities.

Lemma 3.1. *Let $p,q\in(1,\infty)$ and let v,w be weights. Let the weight ψ be defined by* (4). *Let $g\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then the inequality* (6) *holds if and only if*

$$\left(\int_0^\infty\left(\int_0^t h(s)\,\mathrm{d}s\right)^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac1q} \le C_{(7)}\left(\int_0^\infty h^p\psi^{1-p}\right)^{\frac1p}, \quad h\in\mathscr{M}_+(0,\infty), \tag{7}$$

and

$$\left(\int_0^\infty\left(\int_t^\infty h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac1q} \le C_{(8)}\left(\int_0^\infty h^p(g^{**})^{-p}\psi^{1-p}\right)^{\frac1p}, \quad h\in\mathscr{M}_+(0,\infty). \tag{8}$$

Moreover, the optimal constants satisfy $C_{(6)}\simeq C_{(7)}+C_{(8)}$.

Proof. Assume that there exists a nonnegative measurable function γ compactly supported in $(0,\infty)$ and such that

$$g^*(t) = \int_t^\infty \frac{\gamma(s)}{s}\,\mathrm{d}s, \quad t>0. \tag{9}$$

By the Fubini theorem, for any $t>0$ we obtain

$$\begin{aligned}
&tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s \\
&= f^{**}(t)\int_0^t\int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s + \int_t^\infty f^*(s)\int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s \\
&= f^{**}(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_0^t f^*(s)\,\mathrm{d}s \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x + \int_t^\infty f^*(s)\int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s \\
&= f^{**}(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty \frac{\gamma(x)}{x}\int_t^x f^*(s)\,\mathrm{d}s\,\mathrm{d}x \\
&= f^{**}(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \gamma(x)f^{**}(x)\,\mathrm{d}x.
\end{aligned} \tag{10}$$

Now define the positive linear operator $R : \mathscr{M}_+(0,\infty) \to \mathscr{M}_+(0,\infty)$ by

$$Rf(t) := f(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \gamma(x)f(x)\,\mathrm{d}x.$$

By Lemma 2.2, the inequality (6) holds if and only if

$$\left(\int_0^\infty (RSh(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(11)}\left(\int_0^\infty h^p(t)\phi^{1-p}(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad h\in\mathscr{M}_+(0,\infty), \tag{11}$$

where S is the Stieltjes operator (5). Moreover, $C_{(6)} \simeq C_{(11)}$ for the optimal constants. Recall that for any $h \in \mathscr{M}_+$ one has

$$\int_0^\infty \frac{h(s)}{s+t}\,\mathrm{d}s \le \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \le 2\int_0^\infty \frac{h(s)}{s+t}\,\mathrm{d}s, \quad t>0.$$

Let $h \in \mathscr{M}_+(0,\infty)$ and $t > 0$. We express $RSh(t)$ using g^{**} instead of γ, as follows:

$$\begin{aligned}
RSh(t) &\simeq \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x \\
&\quad + \int_t^\infty \frac{\gamma(x)}{x}\int_0^x h(s)\,\mathrm{d}s\,\mathrm{d}x + \int_t^\infty \gamma(x)\int_x^\infty \frac{h(s)}{s}\,\mathrm{d}s\,\mathrm{d}x \\
&= \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \int_0^t h(s)\,\mathrm{d}s \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \\
&\quad + \int_t^\infty h(s)\int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s + \int_t^\infty \frac{h(s)}{s}\int_t^s \gamma(x)\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{t}\int_0^t h \int_0^t \gamma + \int_t^\infty \frac{h(s)}{s}\int_0^s \gamma(x)\,\mathrm{d}x\,\mathrm{d}s + g^*(t)\int_0^t h + \int_t^\infty h g^*.
\end{aligned}$$

Since $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, one has $0 \le x g^*(x) \le x g^{**}(x) = \int_0^x g^*(y)\,\mathrm{d}y \xrightarrow{x\to 0+} 0$. Next, in a.e. point $t > 0$ the derivative of g^* exists and is equal to $-\frac{\gamma(t)}{t}$. Hence, integration by parts gives, for a.e. $t > 0$,

$$\int_0^t \gamma(x)\,\mathrm{d}x = \big[-x g^*(x)\big]_{x=0}^t + \int_0^t g^*(x)\,\mathrm{d}x = -t g^*(t) + \int_0^t g^*(x)\,\mathrm{d}x. \tag{12}$$

Applying this on the equivalent expression of $RSh(t)$ we calculated above, we obtain that, for a.e. $t > 0$, it holds

$$RSh(t) \simeq \frac{1}{t}\int_0^t h \int_0^t g^* + \int_t^\infty \frac{h(s)}{s}\int_0^s g^*(x)\,\mathrm{d}x\,\mathrm{d}s = g^{**}(t)\int_0^t h + \int_t^\infty h g^{**}.$$

Using this expression in (11), we observe that (11) is equivalent to (7) and (8) and the optimal constants satisfy $C_{(11)} \simeq C_{(7)} + C_{(8)}$, i.e. $C_{(6)} \simeq C_{(7)} + C_{(8)}$.

So far we proved the lemma for g satisfying (9). Now consider an arbitrary $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. By Proposition 2.3 we find a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of measurable non-negative functions with compact support in $(0,\infty)$ such that for a.e. $t > 0$ it holds

$$g_n^*(t) := \int_t^\infty \frac{\gamma_n(x)}{x}\,\mathrm{d}x \uparrow g^*(t), \quad n \to \infty. \tag{13}$$

It also holds $g_n^{**}(t) \uparrow g^{**}(t)$ for all $t > 0$. Using these approximations and the fact that the lemma holds for every g_n^*, we get that $C_{(6)} \simeq C_{(7)} + C_{(8)}$ for the optimal constants in the case of general g. □

An a priori characterization of (6) for $p, q \in (1,\infty)$ hence reads as follows.

Theorem 3.2. *Let $1 < p < \infty$. Let v, w be weights. Let ψ be given by* (4) *and $\Psi(t) := \int_0^t \psi$ for $t > 0$.*

(i) *Let $1 < p \le q < \infty$. Then the inequality* (6) *holds if and only if*

$$A_{(14)} := \sup_{t>0} \left(\int_t^\infty (g^{**}(s))^q w(s)\,\mathrm{d}s \right)^{\frac{1}{q}} \Psi^{\frac{1}{p'}}(t) < \infty \tag{14}$$

and

$$A_{(15)} := \sup_{t>0} W^{\frac{1}{q}}(t) \left(\int_t^\infty (g^{**}(s))^{p'} \psi(s)\,\mathrm{d}s \right)^{\frac{1}{p'}} < \infty. \tag{15}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(14)} + A_{(15)}$.

(ii) *Let $1 < q < p < \infty$ and let $r := \frac{pq}{p-q}$. Then the inequality* (6) *holds if and only if*

$$A_{(16)} := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(s))^q w(s)\,\mathrm{d}s \right)^{\frac{r}{q}} \Psi^{\frac{r}{q'}}(t)\psi(t)\,\mathrm{d}t \right)^{\frac{1}{r}} < \infty \tag{16}$$

and

$$A_{(17)} := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(s))^{p'} \psi(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \right)^{\frac{1}{r}} < \infty. \tag{17}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(16)} + A_{(17)}$.

Proof. (i) By the weighted Hardy inequality and its dual version (see e.g. [13, 16]), the inequalities (7) and (8) hold if and only if $A_{(14)} < \infty$ and $A_{(15)} < \infty$, respectively. We also have $C_{(7)} \simeq A_{(14)}$ and $C_{(8)} \simeq A_{(15)}$ for the optimal constants. The result then follows from Lemma 3.1.

(ii) We proceed analogously as in the previous case. The Hardy inequalities give that (7) holds if and only if $A_{(16)} < \infty$ and (8) holds if and only if

$$\left(\int_0^\infty W^{\frac{r}{q}}(t) \left(\int_t^\infty (g^{**}(s))^{p'} \psi(s)\,\mathrm{d}s \right)^{\frac{r}{q'}} (g^{**}(t))^{p'} \psi(t)\,\mathrm{d}t \right)^{\frac{1}{r}} < \infty.$$

This expression is by Proposition 2.4 equivalent to $A_{(17)}$. Finally, Lemma 3.1 gives the result again. Estimates on the optimal constants also follow, just as in (i). □

Let us now turn our focus to the "limit cases" of the exponents p and q. First such case is the choice $q = \infty$.

Theorem 3.3. *Let v, w be weights and let $q = \infty$.*

(i) *Let $0 < p < 1$. Then the inequality* (6) *holds if and only if*

$$A_{(18)} := \sup_{x>0} g^{**}(x)x \left(V(x) + x^p \int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \right)^{-\frac{1}{p}} \operatorname*{ess\,sup}_{t\in(0,x)} w(t) < \infty. \tag{18}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(18)}$.

(ii) *Let $1 < p < \infty$. Let ψ be given by* (4). *Then the inequality* (6) *holds if and only if*

$$A_{(19)} := \operatorname*{ess\,sup}_{t>0} w(t) \left((g^{**}(t))^{p'}\Psi(t) + \int_t^\infty (g^{**}(s))^{p'}\psi(s)\,\mathrm{d}s \right)^{\frac{1}{p'}} < \infty. \tag{19}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(19)}$.

Proof. The optimal constant $C_{(6)}$ is expressed in the following way:

$$\begin{aligned} C_{(6)} &= \sup_{\|f\|_{\Gamma^p(v)}\le 1} \operatorname*{ess\,sup}_{t>0} w(t) \left(g^{**}(t) \int_0^t f^* + \int_t^\infty f^* g^* \right) \\ &= \operatorname*{ess\,sup}_{t>0} w(t) \left(g^{**}(t) \sup_{\|f\|_{\Gamma^p(v)}\le 1} \int_0^t f^* + \sup_{\|f\|_{\Gamma^p(v)}\le 1} \int_t^\infty f^* g^* \right). \end{aligned} \tag{20}$$

Observe also that, for any $p \in (0,\infty)$, the function $\widetilde{V}_p$ defined by

$$\widetilde{V}_p(x) := V(x) + x^p \int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s, \quad x > 0,$$

is increasing on $(0,\infty)$, while the function $x \mapsto \widetilde{V}_p(x)x^{-p}$ is decreasing on $(0,\infty)$.

(i) Let $0 < p < 1$. Then [6, Theorem 4.2(i)] gives

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1} \int_0^t f^* \simeq \sup_{x>0} \int_0^x \chi_{[0,t]}(y)\,\mathrm{d}y\ \widetilde{V}_p^{-\frac{1}{p}}(x) = \sup_{x\in(0,t]} x\,\widetilde{V}_p^{-\frac{1}{p}}(x) = t\,\widetilde{V}_p^{-\frac{1}{p}}(t).$$

By the same source, it holds

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1} \int_t^\infty f^* g^* \simeq \sup_{x>0} \int_0^x g^*(y)\chi_{[t,\infty)}(y)\,\mathrm{d}y\ \widetilde{V}_p^{-\frac{1}{p}}(x) = \sup_{x\ge t} \int_t^x g^*(y)\,\mathrm{d}y\ \widetilde{V}_p^{-\frac{1}{p}}(x).$$

Using these calculations and (20), we now get

$$C_{(6)} \simeq \operatorname*{ess\,sup}_{t>0} w(t)\left(\int_0^t g^*(y)\,\mathrm{d}y\ \widetilde{V}_p^{-\frac{1}{p}}(t)+\sup_{x\geq t}\int_t^x g^*(y)\,\mathrm{d}y\ \widetilde{V}_p^{-\frac{1}{p}}(x)\right)$$

$$\simeq \operatorname*{ess\,sup}_{t>0} w(t)\sup_{x\geq t}\widetilde{V}_p^{-\frac{1}{p}}(x)\left(\int_0^t g^*+\int_t^x g^*\right)=A_{(18)}.$$

(ii) Let $1<p<\infty$. We proceed similarly as in (i). From [6, Theorem 4.2(ii)] it follows that

$$\sup_{\|f\|_{\Gamma^p(v)}\leq 1}\int_0^t f^* \simeq \left(\int_0^\infty\left(\sup_{y\geq x}\frac{1}{y}\int_0^y \chi_{[0,t]}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}=\left(\Psi(t)+t^{p'}\int_t^\infty\frac{\psi(x)}{x^{p'}}\,\mathrm{d}x\right)^{\frac{1}{p'}}$$

$$=\left(\Psi(t)+t^{p'}\int_t^\infty\left(\sup_{y\geq x}\frac{1}{y}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$

and

$$\sup_{\|f\|_{\Gamma^p(v)}\leq 1}\int_t^\infty f^*g^* \simeq \left(\int_0^\infty\left(\sup_{y\geq x}\frac{1}{y}\int_0^y g^*\chi_{[t,\infty)}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$

$$=\left(\left(\sup_{y\geq t}\frac{1}{y}\int_t^y g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\geq x}\frac{1}{y}\int_t^y g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}.$$

Together with (20), this gives

$$C_{(6)} \simeq \operatorname*{ess\,sup}_{t>0} w(t)\left[\left(\left(\sup_{y\geq t}\frac{1}{y}\int_0^t g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\geq x}\frac{1}{y}\int_0^t g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}\right.$$

$$\left.+\left(\left(\sup_{y\geq t}\frac{1}{y}\int_t^y g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\geq x}\frac{1}{y}\int_t^y g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}\right].$$

The right-hand side of the equation is equivalent to $A_{(19)}$ and the proof is finished. $\square$

Next, we proceed with the case $q=1$, covered by the following theorem.

Theorem 3.4. *Let v, w be weights and $q=1$.*

(i) *Let $0<p\le 1$. Then the inequality* (6) *holds if and only if*

$$A_{(21)} := \sup_{t>0} \frac{g^{**}(t)tW(t)+t\int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x}{\left(V(t)+t^p\int_t^\infty v(s)s^{-p}\,\mathrm{d}s\right)^{\frac{1}{p}}} < \infty. \tag{21}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(21)}$.

(ii) *Let $1<p<\infty$. Let ψ be given by* (4). *Then the inequality* (6) *holds if and only if*

$$A_{(22)} := \left(\int_0^\infty \left(g^{**}(t)W(t)+\int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x\right)^{p'} \psi(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} < \infty. \tag{22}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(22)}$.

Proof. Fubini theorem yields that (6) with $q=1$ is equivalent to

$$\int_0^\infty f^*(t)\left(g^*(t)W(t)+\int_t^\infty g^{**}w\right)\mathrm{d}t \le C_{(6)}\left(\int_0^\infty (f^{**})^p v\right)^{\frac{1}{p}}, \quad f\in\Gamma^p(v). \tag{23}$$

(i) By [6, Theorem 4.2(i)], inequality (23) holds if and only if

$$B_1 := \sup_{t>0} \frac{\int_0^t \left(g^*(x)W(x)+\int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\right)\mathrm{d}x}{\left(V(t)+t^p\int_t^\infty v(s)s^{-p}\,\mathrm{d}s\right)^{\frac{1}{p}}} < \infty.$$

Moreover, $C_{(6)} \simeq B_1$ for the optimal constant. Using the Fubini theorem we obtain

$$g^{**}(t)tW(t)+t\int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x = \int_0^t \left(g^*(x)W(x)+\int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\right)\mathrm{d}x \tag{24}$$

for all $t>0$. Hence, we have $B_1 = A_{(21)}$.

(ii) In this case, [6, Theorem 4.2(ii)] yields that (23) is satisfied if and only if

$$B_2 := \left(\int_0^\infty \left(\sup_{y\ge t}\frac{1}{y}\left(\int_0^y \left(g^*(x)W(x)+\int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\right)\mathrm{d}x\right)\right)^{p'} \psi(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} < \infty.$$

It also holds $C_{(6)} \simeq B_2$ for the optimal constant. Observe that the function $x \mapsto g^*(x)W(x) + \int_x^\infty g^{**}w$ is nonincreasing, which together with (24) gives

$$\sup_{y\ge t} \frac{1}{y}\left(\int_0^y \left(g^*(x)W(x)+\int_x^\infty g^{**}w\right)\mathrm{d}x\right) = \frac{1}{t}\left(\int_0^t \left(g^*(x)W(x)+\int_x^\infty g^{**}w\right)\mathrm{d}x\right)$$
$$= \frac{1}{t}\left(g^{**}(t)tW(t)+t\int_t^\infty g^{**}w\right)$$

for any $t>0$. Hence, we obtain $B_2 = A_{(22)}$. □

To deal with the case $p=\infty$, we will make use of a more general lemma below. In its proof we follow a similar pattern as in [3, Theorem 6.4], where a particular case was treated.

Lemma 3.5. *Let v be a weight and let $\|\cdot\|_X$ be an r.i. quasi-norm on $\mathscr{M}(0,\infty)$. Let $S:\mathscr{M}_+(0,\infty)\to\mathscr{M}_+(0,\infty)$ be a quasi-linear operator which, for all $f, f_n, g \in \mathscr{M}_+(0,\infty)$, $n\in\mathbb{N}$, satisfies the following conditions:*
(i) *$f\le g$ a.e. implies $Sf \le Sg$ a.e.;*
(ii) *$f_n \uparrow f$ a.e. implies $Sf_n \uparrow Sf$ a.e.*
Then the inequality

$$\|S(f^{**})\|_X \le C_{(25)} \operatorname*{ess\,sup}_{t>0} f^{**}(t)v(t), \quad f\in\Gamma^\infty(v), \tag{25}$$

holds if and only if

$$A_{(26)} := \|S\varrho\|_X < \infty, \tag{26}$$

where

$$\varrho(t) := \left(\operatorname*{ess\,sup}_{s>0} \min\left\{1,\frac{t}{s}\right\} v(s)\right)^{-1}, \quad t>0. \tag{27}$$

The optimal constant $C_{(25)}$ satisfies $C_{(25)} \simeq A_{(26)}$.

Proof. At first, observe that, for any $f\in\mathscr{M}(\mathbb{R}^d)$,

$$\begin{aligned}\|f\|_{\Gamma^\infty(v)} &= \max\left\{\operatorname*{ess\,sup}_{s>0} v(s)\sup_{t>s} f^{**}(t),\ \operatorname*{ess\,sup}_{s>0}\frac{v(s)}{s}\sup_{t\in(0,s)} tf^{**}(t)\right\}\\ &= \operatorname*{ess\,sup}_{t>0} f^{**}(t)\max\left\{\operatorname*{ess\,sup}_{s\in(0,t)} v(s), t\operatorname*{ess\,sup}_{s>t}\frac{v(s)}{s}\right\}\\ &= \|f\|_{\Gamma^\infty(\varrho^{-1})}.\end{aligned}$$

Let us prove that (26) is sufficient for (25). Suppose that (26) holds. Thanks to the properties of S, we have the following estimate:

$$\|S(f^{**})\|_X = \left\|S\left(\frac{f^{**}\varrho}{\varrho}\right)\right\|_X \le \sup_{t>0}\frac{f^{**}(t)}{\varrho(t)}\|S\varrho\|_X = \|f\|_{\Gamma^\infty(\varrho^{-1})}A_{(26)} = \|f\|_{\Gamma^\infty(v)}A_{(26)}.$$

Hence, (25) is satisfied and $C_{(25)} \le A_{(26)}$ for the optimal $C_{(25)}$.

Now we turn to the necessity of (26). Assume that (25) holds. Since ϱ is quasi-concave, there exists a function $f \in \mathscr{M}(\mathbb{R}^d)$ and a constant $\lambda > 0$ such that

$$\frac{1}{2}\left(\lambda + \int_0^t f^*\right) \le t\varrho(t) \le \left(\lambda + \int_0^t f^*\right), \quad t > 0. \tag{28}$$

Indeed, if ω denotes the least concave majorant of the function $t \mapsto t\varrho(t)$, then we may choose $\lambda := \lim_{s\to 0+} \omega(s)$ and $f \in \mathscr{M}(\mathbb{R}^d)$ such that $\omega(t) = \lambda + \int_0^t f^*$, $t > 0$. The inequality then follows by [1, Proposition 5.10, p. 71], since $t \mapsto t\varrho(t)$ is quasi-concave. In particular, (28) yields

$$\|f\|_{\Gamma^\infty(v)} \le 2 \operatorname*{ess\,sup}_{t>0} v(t)\varrho(t) \le 2.$$

We obtain

$$\begin{aligned} A_{(26)} &\lesssim \left\| S\left(s \mapsto \tfrac{\lambda}{s} + f^{**}(s)\right)\right\|_X \lesssim \left\| S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + \|S(f^{**})\|_X \\ &\lesssim \left\| S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + C_{(25)}\|f\|_{\Gamma^\infty(v)} \lesssim \left\| S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + 2C_{(25)}. \end{aligned}$$

If $\lambda = 0$, we are done, since $S(0) = 0$. Now suppose that $\lambda > 0$. Choose $\varepsilon > 0$ arbitrarily and let $g \in \mathscr{M}(\mathbb{R}^d)$ be such that $g^* = \frac{\lambda}{\varepsilon}\chi_{[0,\varepsilon]}$. Then $\|g\|_1 = \lambda$. By (28) it holds $\frac{1}{t\varrho(t)} \le \frac{2}{\lambda}$ for all $t > 0$. Thus,

$$\|g\|_{\Gamma^\infty(v)} = \|g\|_{\Gamma^\infty(\varrho^{-1})} = \sup_{t>0} \frac{\int_0^t g^*}{t\varrho(t)} \le \|g\|_1 \sup_{t>0} \frac{1}{t\varrho(t)} \le 2.$$

Next, for all $s > \varepsilon$ one has $g^{**}(s) = \frac{\lambda}{s}$. Therefore it holds

$$\left\| S\left(s \mapsto \frac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s}\right)\right\|_X = \left\| S(\chi_{[\varepsilon,\infty)} g^{**})\right\|_X \le \|S(g^{**})\| \le C_{(25)}\|g\|_{\Gamma^\infty(v)} \le 2C_{(25)}. \tag{29}$$

Since $\frac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s} \uparrow \frac{\lambda}{s}$ as $\varepsilon \to 0+$ for every $s > 0$, we get $S\left(s \mapsto \frac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s}\right) \uparrow S\left(s \mapsto \frac{\lambda}{s}\right)$ a.e. on $(0,\infty)$ as $\varepsilon \to 0+$. Hence, the Fatou property of $\|\cdot\|_X$ used in (29) gives

$$\left\| S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X \le 2C_{(25)}.$$

We have shown that $A_{(26)} \lesssim C_{(25)}$ and the proof is complete. □

Making an appropriate choice of the operator S in Lemma 3.5, we obtain the following theorem.

Theorem 3.6. *Let v, w be weights. Let $p = \infty$.*

(i) *Let $q \in (0,\infty)$. Then the inequality* (6) *is satisfied if and only if*

$$A_{(30)} := \left(\int_0^\infty \left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s>0} \min\left\{\frac{1}{t}, \frac{1}{s}\right\} v(s)} + \int_t^\infty g^*(x)\, \mathrm{d}\left(\frac{1}{\operatorname{ess\,sup}_{s>0} \min\left\{\frac{1}{x}, \frac{1}{s}\right\} v(s)} \right) \right]^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} < \infty. \tag{30}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(30)}$.

(ii) *Let* $q = \infty$. *Then the inequality* (6) *is satisfied if and only if*

$$A_{(31)} := \operatorname*{ess\,sup}_{t>0} \left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s>0} \min\left\{\frac{1}{t}, \frac{1}{s}\right\} v(s)} + \int_t^\infty g^*(x)\,\mathrm{d}\left(\frac{1}{\operatorname{ess\,sup}_{s>0} \min\left\{\frac{1}{x}, \frac{1}{s}\right\} v(s)} \right) \right] w(t) < \infty. \tag{31}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(31)}$.

Proof. Let us prove (i). Define the function ϱ by (27) and the function ω by

$$\omega(t) := t\varrho(t) = \frac{1}{\operatorname{ess\,sup}_{s>0} \min\left\{\frac{1}{t}, \frac{1}{s}\right\} v(s)}, \quad t > 0. \tag{32}$$

The function ω is nondecreasing and continuous on $(0, \infty)$. Thus, its derivative ω' exists a.e. on $(0, \infty)$. We may assume that $\omega(0+) := \lim_{t\to 0+} \omega(t)$ is finite, otherwise ω is constantly infinite, thus $\|\cdot\|_{\Gamma^\infty(v)} = \|\cdot\|_{\Gamma^\infty(\varrho^{-1})} \equiv 0$. Hence, we may write

$$\varrho(t) = \frac{\omega(t)}{t} = \frac{1}{t} \int_0^t \omega'(x)\,\mathrm{d}x + \frac{\omega(0+)}{t}, \quad t > 0. \tag{33}$$

Now suppose that there exists $\gamma \in \mathscr{M}_+(0, \infty)$ with compact support in $(0, \infty)$ such that (9) holds. Define

$$Sh(t) := h(t) \int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty h(x)\gamma(x)\,\mathrm{d}x, \quad h \in \mathscr{M}_+(0, \infty).$$

Using (10), we observe that the inequality (6) is equivalent to the inequality (25) with $X := L^q(w)$ and $C_{(6)} = C_{(25)}$. Lemma 3.5 yields that (25) holds if and only if $\|S\varrho\|_{L^q(w)} < \infty$. By (12), (33) and Fubini theorem, for every $t > 0$ we get

$$\begin{aligned}
S\varrho(t) &= \varrho(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \varrho(x)\gamma(x)\,\mathrm{d}x \\
&= \frac{1}{t}\int_0^t \omega'(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \frac{\omega(0+)}{t}\int_0^t \gamma(x)\,\mathrm{d}x \\
&+ \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \int_0^t \omega'(s)\,\mathrm{d}s + \int_t^\infty \frac{\gamma(x)}{x}\int_t^x \omega'(s)\,\mathrm{d}s\,\mathrm{d}x + \omega(0+)\int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \\
&= g^{**}(t)\int_0^t \omega'(s)\,\mathrm{d}s + g^{**}(t)\omega(0+) + \int_t^\infty g^*(s)\omega'(s)\,\mathrm{d}s \\
&= g^{**}(t)\omega(t) + \int_t^\infty g^*(s)\omega'(s)\,\mathrm{d}s.
\end{aligned}$$

Thus, we obtain $\|S\varrho\|_{L^q(w)} = A_{(30)}$. This completes the proof of (i) for g satisfying (9). For a general $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we use Proposition 2.3 to approximate g by appropriate functions g_n as in (13) and then obtain the result by the limit pass $n \to \infty$. The case (ii) is proved in the same way, choosing $X := L^\infty(w)$ in Lemma 3.5. □

So far we have not yet covered the case $p = 1$, $q \in (1,\infty)$. However, since $\|\cdot\|_{\Gamma^1(v)} = \|\cdot\|_{\Lambda^1(\widetilde{v})}$ with $\widetilde{v}(t) := \int_t^\infty \frac{v(s)}{s}\,\mathrm{d}s$, validity of (6) is characterized by [11, Theorem 3.2(i)]. From there we get the following result which completes our list.

Proposition 3.7. *Let v, w be weights. Let $p = 1$ and $q \in (1,\infty)$. Then the inequality* (6) *holds if and only if*

$$A_{(34)} := \sup_{t>0} \frac{g^{**}(t)\,t\,W^{\frac{1}{q}}(t) + t\left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{V(t) + t\int_t^\infty v(x)x^{-1}\,\mathrm{d}x} < \infty. \tag{34}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(34)}$.

Remark 3.8. The expression $A_{(14)}$, with p, q set as in Theorem 3.2(i), defines a norm of $g \in \mathscr{M}(\mathbb{R}^d)$. Similarly, the following expressions are norms: $A_{(15)}$, $A_{(16)}$, $A_{(17)}$, $A_{(18)}$, $A_{(19)}$, $A_{(21)}$, $A_{(22)}$ and $A_{(34)}$. In each case, the values of p and q correspond with the setting of the particular theorem or proposition. The subadditivity of the functional follows here from the subadditivity of the maximal function (3). For more details about r.i. spaces generated by these norms see [11].

Moreover, the expressions $A_{(30)}$ with $q \in [1,\infty)$ and $A_{(31)}$ each are equivalent to a norm of $g \in \mathscr{M}(\mathbb{R}^d)$. The expression $A_{(30)}$ with $q \in (0,1)$ defines a quasi-norm of $g \in \mathscr{M}(\mathbb{R}^d)$. These claims may be proved by replacing the function ω from (32) by its least concave majorant (cf. [1, p. 71]) and then performing a similar procedure as in (10) to rewrite the expressions using only f^{**} and not f^{*}. Then it is possible to use (3) again.

4. Young-type convolution inequalities for Γ-spaces

In the previous section we obtained sufficient conditions for boundedness of T_g between $\Gamma^p(v)$ and $\Gamma^q(w)$. But more can be said. If $g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$, then these conditions are also necessary. Moreover, the result can be given the form of a Young-type inequality. All of this is summarized in the main theorem below. Recall that we say that an r.i. lattice X *is embedded into* an r.i. lattice Y and write $X \hookrightarrow Y$, if there exists a constant $C > 0$ such that $\|f\|_Y \le C\|f\|_X$ for all $f \in X$.

Theorem 4.1. *Let v, w be weights. Depending on the parameters p, q, for $g \in \mathscr{M}(\mathbb{R}^d)$ define $\|g\|_Y$ by what follows:*

$$\|g\|_Y := \begin{cases} A_{(14)} + A_{(15)} & \text{if } 1 < p \le q < \infty; \\ A_{(16)} + A_{(17)} & \text{if } 1 < q < p < \infty; \\ A_{(34)} & \text{if } 1 = p < q < \infty; \\ A_{(18)} & \text{if } 0 < p < 1,\ q = \infty; \\ A_{(19)} & \text{if } 1 < p < q = \infty; \\ A_{(21)} & \text{if } 0 < p \le q = 1; \\ A_{(22)} & \text{if } 1 = q < p < \infty; \\ A_{(30)} & \text{if } 0 < q < p = \infty; \\ A_{(31)} & \text{if } p = q = \infty. \end{cases}$$

For each choice of p, q from the previous list define $Y := \{g \in \mathscr{M}(\mathbb{R}^d);\ \|g\|_Y < \infty\}$. Then:

(i) *If $g \in Y$, then $T_g : \Gamma^p(v) \to \Gamma^q(w)$ and*

$$\|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)} \lesssim \|g\|_Y.$$

(ii) *If $g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$ and $T_g : \Gamma^p(v) \to \Gamma^q(w)$, then $g \in Y$ and*

$$\|g\|_Y \lesssim \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)}.$$

(iii) *The inequality*

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v),\ g \in Y, \tag{35}$$

is satisfied. Moreover, if $\widetilde{Y}$ is any r.i. lattice such that (35) is satisfied with $\widetilde{Y}$ in place of Y, then $\widetilde{Y} \hookrightarrow Y$.

Proof. Let us consider the case $1 < p \le q < \infty$, the other ones are analogous.

(i) Let us define

$$R_g f(t) := t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s$$

for $f \in \mathscr{M}(\mathbb{R}^d)$ and $t > 0$. If $g \in Y$, then, by Theorem 3.2(i), the inequality (6) holds, with $C_{(6)} \simeq \|g\|_Y$. The O'Neil inequality (2) then gives

$$\|f * g\|_{\Gamma^q(w)} = \|(f * g)^{**}\|_{L^q(w)} \le \|R_g f\|_{L^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_Y.$$

Hence, (i) holds and so does the inequality (35).

(ii) Since $g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$, the reverse O'Neil inequality (Lemma 2.1) implies $R_g f \lesssim (T_g f)^{**}$ on $(0,\infty)$. Observe also that $R_g f = R_g \widetilde{f}$ whenever $f^* = \widetilde{f}^*$. Using Theorem 3.2(i) we get

$$\begin{aligned} \|g\|_Y &\lesssim \sup_{\|f\|_{\Gamma^p(v)}\le 1} \|R_g f\|_{L^q(w)} = \sup_{\substack{\|f\|_{\Gamma^p(v)}\le 1 \\ f\in \mathscr{M}_+^{\odot}(\mathbb{R}^d)}} \|R_g f\|_{L^q(w)} \\ &\lesssim \sup_{\substack{\|f\|_{\Gamma^p(v)}\le 1 \\ f\in \mathscr{M}_+^{\odot}(\mathbb{R}^d)}} \|T_g f\|_{\Gamma^q(w)} \le \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)}. \end{aligned}$$

(iii) Let $\widetilde{Y}$ by an r.i. lattice such that

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)} \|g\|_{\widetilde{Y}}, \quad f \in \Gamma^p(v),\ g \in \widetilde{Y}. \tag{36}$$

Let $h \in \widetilde{Y}$. There exists $g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$ such that $g^* = h^*$. From (36) it follows that $T_g : \Gamma^p(v) \to \Gamma^q(w)$ and $\|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}$. Thus, (ii) gives

$$\|g\|_Y \lesssim \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}.$$

Since $\|g\|_Y = \|h\|_Y$ and $\|g\|_{\widetilde{Y}} = \|h\|_{\widetilde{Y}}$, we have $\|h\|_Y \lesssim \|h\|_{\widetilde{Y}}$. Hence, we get $\widetilde{Y} \hookrightarrow Y$. □

Remark 4.2. (i) For given p, q, v, w the optimal space Y from Theorem 4.1 may be trivial, i.e. $Y = \{0\}$. In that case, T_g is not bounded between $\Gamma^p(v)$ and $\Gamma^q(w)$ for any nonnegative nontrivial kernel g (see [11, Corollary 3.3] for an analogy with $\Lambda^p(v)$ as the domain space).

(ii) The spaces Y from Theorem 4.1 are of the same type as those obtained in [11, 12] in analogous situations (with Λ and S, respectively, as the domain). Their basic functional properties were studied in [11]. Recently, in [7] these spaces appeared as associate spaces to the "generalized Γ-spaces" $G\Gamma$.

(iii) In [14, Theorem 4.1], the authors obtained a sufficient condition for the boundedness $T_g : \Gamma^p(v) \to \Gamma^q(w)$ with the following assumptions: u, v, w are weights, $1 < q < \infty$, $1 \le p, r \le \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, $\|w\|_1 = \infty$, $w \in B_q$, i.e. there exists $C > 0$ such that $\int_x^\infty w(t)t^{-q}\,\mathrm{d}t \le C x^{-q} W(x)$ for all $x > 0$, and, moreover, there exists $D > 0$ such that the weights satisfy the pointwise inequality

$$W(t) \le D w^{\frac{1}{q'}}(t) v^{\frac{1}{p}}(t) u^{\frac{1}{r}}(t), \quad t > 0.$$

It was shown that under these conditions it holds $\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)} \|g\|_{\Gamma^r(u)}$. This statement was proved in [14] using the rather strong assumptions on the weights, and it does not follow from Theorem 4.1 immediately. However, Theorem 4.1 provides a different sufficient condition for $T_g : \Gamma^p(v) \to \Gamma^q(w)$ with no additional assumptions on the weights and for a wider range of p and q, including the case $1 < q < p < \infty$. Moreover, this condition is also necessary provided that $g \in \mathscr{M}_+^{\odot}(\mathbb{R}^d)$.

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Paper IV

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Bilinear weighted Hardy inequality for nonincreasing functions

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BILINEAR WEIGHTED HARDY INEQUALITY FOR NONINCREASING FUNCTIONS

MARTIN KŘEPELA

ABSTRACT. We characterize the validity of the bilinear Hardy inequality for nonincreasing functions

$$\|f^{**}g^{**}\|_{L^q(w)} \le C\|f\|_{\Lambda^{p_1}(v_1)}\|g\|_{\Lambda^{p_2}(v_2)},$$

in terms of the weights v_1, v_2, w, covering the complete range of exponents $p_1, p_2, q \in (0,\infty]$.

The problem is solved by reducing it into the iterated Hardy-type inequalities

$$\left(\int_0^\infty \left(\int_0^x (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}} \le C\left(\int_0^\infty (g^*(x))^\gamma \omega(x)\,\mathrm{d}x\right)^{\frac{1}{\gamma}},$$

$$\left(\int_0^\infty \left(\int_x^\infty (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}} \le C\left(\int_0^\infty (g^*(x))^\gamma \omega(x)\,\mathrm{d}x\right)^{\frac{1}{\gamma}}.$$

Validity of these inequalities is characterized here for $0 < \alpha \le \beta < \infty$ and $0 < \gamma < \infty$.

1. Introduction

Consider the *bilinear Hardy operator*

$$H_2(f,g)(t) := \frac{1}{t^2}\int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s,$$

defined for all nonnegative measurable functions f, g on $(0,\infty)$. In this article, we will find necessary and sufficient conditions for the boundedness

$$H_2 : L^{p_1}_{\mathrm{dec}}(v_1) \times L^{p_2}_{\mathrm{dec}}(v_2) \to L^q(w)$$

with $p_1, p_2, q \in (0,\infty]$. In other words, the goal is to provide equivalent estimates of the constant

$$C_{(1)} = \sup_{f,g\in\mathscr{M}} \frac{\|f^{**}g^{**}\|_{L^q(w)}}{\|f\|_{\Lambda^{p_1}(v_1)}\|g\|_{\Lambda^{p_2}(v_2)}} \tag{1}$$

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in terms of p_1, p_2, q, v_1, v_2, w.

Let us at first summarize the used notation and symbols. Let $(\mathscr{R}, \mu)$ be an arbitrary totally σ-finite measure space. Then $\mathscr{M}$ denotes the cone of all extended real-valued μ-measurable functions on $\mathscr{R}$. Next, $\mathscr{M}_+$ denotes the cone of all extended nonnegative Lebesgue-measurable functions on $(0, \infty)$.

If $p \in (0,1) \cup (1, \infty]$, then $p' := \frac{p}{p-1}$. If $p = 1$, then $p' := \infty$. Notice that for $p \in (0,1)$ the number p' is negative. Furthermore, the conventions "$\frac{0}{0} = 0.\infty := 0$" and "$\frac{a}{0} := \infty$" for $a \in (0, \infty]$ are used throughout the text.

A *weight* is any nonnegative measurable function v on $(0, \infty)$ such that for all $t \in (0, \infty)$ it holds $0 < V(t) < \infty$, where V is defined by $V(t) := \int_0^t v$. If the weight is denoted by another letter, the corresponding capital letter plays an analogous role.

We say that a function $u \in \mathscr{M}_+$ is *integrable near the origin* if there exists $\varepsilon > 0$ such that $\int_0^\varepsilon u < \infty$. Notice that weights are integrable near the origin by definition.

The symbol $A \lesssim B$ means that $A \leq CB$, where C is an absolute constant independent of relevant quantities in A, B. In fact, throughout this article such C depends only on the exponents (p, q, α, β, etc.), thus it does not even depend on the weights. If both $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$.

By $A_{(\dots)}$ we denote the characteristic condition which appears on the line denoted by the number in the brackets. Certain significant optimal constants $C_{(\dots)}$ are denoted in a similar way. These symbols have a unique meaning throughout the whole paper. Symbols B_0, B_1, etc. are used in the proofs as an auxiliary notation for various quantities, and their meaning may differ between the theorems. However, within the proof of a single theorem or lemma, each symbol B_i is uniquely defined.

The text deals with various function spaces. The *weighted Lebesgue space* $L^p(v)$ consists of all extended real-valued Lebesgue-measurable functions h on $(0, \infty)$ such that $\|h\|_{L^p(v)} < \infty$. The functional $\|\cdot\|_{L^p(v)}$ is defined by

$$
\begin{aligned}
\|h\|_{L^p(v)} &:= \left(\int_0^\infty |h(x)|^p v(x) \, \mathrm{d}x \right)^{\frac{1}{p}}, & p \in (0, \infty), \\
\|h\|_{L^\infty(v)} &:= \operatorname*{ess\,sup}_{x>0} |h(x)| v(x), & p = \infty.
\end{aligned}
$$

The symbol $L^p_{\mathrm{dec}}(v)$ stands for the set of all nonnegative and nonincreasing functions from $L^p(v)$.

If $f \in \mathscr{M}$, then f^* denotes its *nonincreasing rearrangement* and f^{**} the *Hardy-Littlewood maximal function of* f, i.e.

$$
f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s, \qquad t > 0.
$$

For details see [3]. For the definitions of *rearrangement-invariant* (abbreviated *r.i.*) *spaces* and *r.i. (quasi-)norms* see [3, 7, 18]. If X and Y are r.i. spaces (or just r.i. lattices), we say that X *is embedded into* Y and write $X \hookrightarrow Y$ if there exists $C \in (0,\infty)$ such that for all $f \in X$ it holds

$$\|f\|_Y \le C\|f\|_X.$$

The least possible constant C in this inequality is called *the optimal constant of the embedding* $X \hookrightarrow Y$ and is equal to the norm of the identity operator between X and Y, denoted $\|Id\|_{X\to Y}$.

Let v be a weight and $p \in (0,\infty]$. The *weighted Lorentz spaces* $\Lambda^p(v)$ and $\Gamma^p(v)$ consist of all functions $f \in \mathscr{M}$ for which $\|f\|_{\Lambda^p(v)} < \infty$ and $\|f\|_{\Gamma^p(v)} < \infty$, respectively. Here it is

$$\|f\|_{\Lambda^p(v)} := \|f^*\|_{L^p(v)} \text{ and } \|f\|_{\Gamma^p(v)} := \|f^{**}\|_{L^p(v)}.$$

For more information about the Lorentz Λ and Γ spaces see e.g. [7] and the references therein.

Let φ, ψ be weights. For $g \in \mathscr{M}$ define

$$\|g\|_{J^{\alpha,\beta}(\varphi,\psi)} := \left[\int_0^\infty \left(\int_0^x (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right]^{\frac{1}{\beta}}, \qquad \alpha,\beta \in (0,\infty),$$

$$\|g\|_{J^{\alpha,\infty}(\varphi,\psi)} := \operatorname*{ess\,sup}_{x>0} \left(\int_0^x (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \psi(x), \qquad \alpha \in (0,\infty),$$

$$\|g\|_{K^{\alpha,\beta}(\varphi,\psi)} := \left[\int_0^\infty \left(\int_x^\infty (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right]^{\frac{1}{\beta}}, \qquad \alpha,\beta \in (0,\infty),$$

$$\|g\|_{K^{\alpha,\infty}(\varphi,\psi)} := \operatorname*{ess\,sup}_{x>0} \left(\int_x^\infty (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \psi(x), \qquad \alpha \in (0,\infty).$$

Then, as usual, it is $J^{\alpha,\beta}(\varphi,\psi) := \{f \in \mathscr{M};\ \|f\|_{J^{\alpha,\beta}(\varphi,\psi)} < \infty\}$ and $K^{\alpha,\beta}(\varphi,\psi) := \{f \in \mathscr{M};\ \|f\|_{K^{\alpha,\beta}(\varphi,\psi)} < \infty\}$. The "$K$-spaces" were defined in [18], where they appeared as optimal spaces in certain Young-type convolution inequalities. Besides that, in [16] it was shown that the associate space to the generalized Γ space is also a "K-space".

Now, let us briefly present some background to the problems we are about to investigate. The aforementioned operator H_2 is a bilinear version of the classical Hardy operator H_1, which is defined by

$$H_1 f(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$$

for all $f \in \mathscr{M}_+$. Boundedness of H_1 between weighted Lebesgue spaces is equivalent to the validity of the *weighted Hardy inequality*

$$\left[\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)\,\mathrm{d}s\right)^q w(x)\,\mathrm{d}x\right]^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2}$$

for all $f \in \mathscr{M}_+$, with C being a constant independent of f. The weights v, w for which this inequality is valid, have been characterized by Muckenhoupt [23], Bradley [5] and Maz'ja [22]. The weighted Hardy inequality has a broad variety of applications and represents now a basic tool in many parts of mathematical analysis, namely in the study of weighted function inequalities. For the results, history and applications of this problem, see [21, 25, 20].

In the last decades, much attention has been drawn by the so-called *restricted* inequalities. By this term it is meant that an inequality is not supposed to be satisfied by the whole set of nonnegative functions, but rather only by a certain, restricted, subset. In this way, one may ask under which conditions the inequality (2) is satisfied for all nonincreasing $f \in \mathscr{M}_+$. This is equivalent to the validity of

$$\left[\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le C\left(\int_0^\infty (f^*(t))^p\, v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{3}$$

for all $f \in \mathscr{M}$, with an independent C. Moreover, this corresponds to the boundedness $H_1 : L^p_{\mathrm{dec}}(v) \to L^q(w)$, or, in yet different words, the existence of the embedding of the Lorentz spaces $\Lambda^p(v) \hookrightarrow \Gamma^q(w)$.

The first results on the case $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$, $1 < p < \infty$ were obtained by Boyd [4] and in an explicit form by Ariño and Muckenhoupt [2]. The problem with $v \neq w$ and $p \neq q$, $1 < p, q < \infty$ was first successfully solved by Sawyer [26]. Many articles on this topic followed, providing the results for a wider range of parameters, see [30, 8, 9, 28, 10, 7, 6]. In [7] the results available in 2000 were surveyed.

The restricted operator inequalities may often be handled by the so-called "reduction theorems". These, in general, reduce a restricted inequality into certain nonrestricted inequalities. For example, the restriction to nonincreasing or quasiconcave functions may be handled in this way, see e.g. [27, 15, 17, 12].

Let us however turn the focus to the bilinear variants of the Hardy-type inequalities. Recently, Aguilar, Ortega and Ramírez [1] found necessary and sufficient conditions for the boundedness $H_2 : L^{p_1}(v_1) \times L^{p_2}(v_2) \to L^q(\widetilde{w})$, where $\widetilde{w}(t) := t^{2q} w(t)$. In other words, they characterized the validity of the *weighted bilinear Hardy inequality*

$$\left[\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le C\left(\int_0^\infty f^{p_1} v_1\right)^{\frac{1}{p_1}}\left(\int_0^\infty g^{p_2} v_2\right)^{\frac{1}{p_2}} \tag{4}$$

for all $f, g \in \mathscr{M}_+$. The covered range of exponents in there was $1 < p, q < \infty$. For some related results see also the references in [1].

The paper [1] motivated the work presented here. Indeed, here we consider a restricted version of (4) which may be called the *bilinear Hardy inequality for nonincreasing functions* and written in the form

$$\left[\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s\right)^q \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right]^{\frac{1}{q}} \leq C \left(\int_0^\infty (f^*)^{p_1} v_1\right)^{\frac{1}{p_1}} \left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}.$$

Notice that $C_{(1)}$ is the least constant C for which the above inequality holds for all $f, g \in \mathscr{M}$.

The proofs in [1] are based on the standard technique of discretization. Here, however, we choose a different approach. The idea is as follows. In the first step, let g in (1) be fixed. Treating $C_{(1)}$ as the optimal constant in the embedding $\Lambda^{p_1}(v_1) \hookrightarrow \Gamma^q((g^{**})^q w)$, one gets

$$C_{(1)} = \sup_{g \in \mathscr{M}} \frac{\|Id\|_{\Lambda^{p_1}(v_1) \to \Gamma^q((g^{**})^q w)}}{\|g\|_{\Lambda^{p_2}(v_2)}}.$$

The two-side estimate of $\|Id\|_{\Lambda^{p_1}(v_1) \to \Gamma^q((g^{**})^q w)}$ is known for all $p_1, q \in (0, \infty]$ and it is equivalent to $\|g\|_X$, a certain rearrangement-invariant (quasi-)norm of g. Hence, in the next step, if we can find the optimal constant $\|Id\|_{\Lambda^{p_2}(v_2) \to X}$, the whole problem is solved.

It will be shown that $\|\cdot\|_X$ can be expressed as a sum of (quasi-)norms in the r.i. spaces $J^{\alpha,\beta}(\varphi, \psi)$ and $K^{\alpha,\beta}(\varphi, \psi)$ (see Section 2 for the definitions). In Section 3 we find characterizations of the embeddings $\Lambda^\gamma(\omega) \hookrightarrow J^{\alpha,\beta}(\varphi, \psi)$ and $\Lambda^\gamma(\omega) \hookrightarrow K^{\alpha,\beta}(\varphi, \psi)$ for $0 < \alpha \leq \beta < \infty$ and $0 < \gamma < \infty$. In other words, we characterize the weights and exponents such that the inequalities

$$\left(\int_0^\infty \left(\int_0^x (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}} \leq C \left(\int_0^\infty (g^*(x))^\gamma \omega(x)\,\mathrm{d}x\right)^{\frac{1}{\gamma}},$$

$$\left(\int_0^\infty \left(\int_x^\infty (g^{**}(t))^\alpha \varphi(t)\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}} \leq C \left(\int_0^\infty (g^*(x))^\gamma \omega(x)\,\mathrm{d}x\right)^{\frac{1}{\gamma}}$$

hold for all functions $g \in \mathscr{M}$. These results will be then used to find the desired estimates of the optimal constant $C_{(1)}$ in the bilinear Hardy inequality (this is the matter of Section 4). However, the description of the relation of the K-spaces to the other types of r.i. spaces, as well as the above weighted inequalities, are of independent interest.

2. Auxiliary results

Here we present various, usually known propositions which will be useful further on. First we may recall the following simple but useful principle. Let

$a, b \in [-\infty, \infty]$ and let f, g be nonnegative continuous functions on (a, b), f nondecreasing and g nonincreasing. Then the derivatives $f'(x)$, $g'(x)$ exist at a.e. $x \in (a, b)$. Denote $f(a+) := \lim_{x \to a+} f(x)$, $f(b-) := \lim_{x \to b-} f(x)$, similarly for g. Integration by parts then gives

$$\int_a^b f'(x)g(x)\,\mathrm{d}x + f(a+)g(a+) = f(b-)g(b-) - \int_a^b f(x)g'(x)\,\mathrm{d}x,$$

with the convention "$0.\infty := 0$" taking effect if needed. Thus, if we, for instance, consider $a := 0$, $b := \infty$, $f := W^\alpha$, $g := V^{-\beta}$ and $\alpha, \beta \in (0, \infty)$, we get

$$\int_0^\infty W^{\alpha-1}(x)w(x)V^{-\beta}(x)\,\mathrm{d}x \simeq W^\alpha(\infty)V^{-\beta}(\infty) + \int_0^\infty W^\alpha(x)V^{-\beta-1}(x)v(x)\,\mathrm{d}x. \tag{5}$$

Analogous situations arise if we take $f(x) := \left(\int_x^\infty w\right)^\alpha$, etc. However, if $\alpha < 1$, there might appear a certain problem related to the integrability of the involved functions (cf. [28, p. 93]). Observe that if we take $\alpha \in (0, 1)$ in (5) and a function $w \in \mathscr{M}_+$ which is not integrable near the origin, then the equivalence in (5) fails, as the left-hand side is equal to zero while the right-hand side is infinite. Since we originally assumed that w was a weight, which is by definition integrable near the origin, this problem, in fact, could not arise in (5). It may nevertheless do so in other situations when the involved function is not a weight in this sense and which thus require slightly more attention. We return to this issue in Proposition 2.3 below.

Anyway, combining or splitting weighted conditions using integration by parts in the described way is a common trick (see e.g. [30, Lemma, p. 176]). If there is no potential danger as described above (e.g. if the relevant exponents are grater than 1), we will use the technique throughout the text without detailed comments, and we will refer to it simply as to *integration by parts*.

Another well-known principle, to which we refer as to the *L^p-duality*, is expressed as follows. If $f \in \mathscr{M}_+$, $p \in (1, \infty)$ and v is a weight, then

$$\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} = \sup_{g \in \mathscr{M}_+} \frac{\int_0^\infty f(x)g(x)\,\mathrm{d}x}{\left(\int_0^\infty g^{p'}(x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}}.$$

We continue with other preliminary results.

Proposition 2.1. *Let $f, g \in \mathscr{M}_+$ and $0 < \lambda < \infty$. Then the identity*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \left(\int_s^x f(t)\,\mathrm{d}t\right)^\lambda g(s)\,\mathrm{d}s\right] = \lambda f(x) \int_0^x \left(\int_s^x f(t)\,\mathrm{d}t\right)^{\lambda-1} g(s)\,\mathrm{d}s$$

holds for a.e. $x>0$ for which the integral on the left-hand side is finite. Analogously, the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_x^\infty\left(\int_x^s f(t)\,\mathrm{d}t\right)^\lambda g(s)\,\mathrm{d}s\right]=-\lambda f(x)\int_x^\infty\left(\int_x^s f(t)\,\mathrm{d}t\right)^{\lambda-1} g(s)\,\mathrm{d}s$$

holds for a.e. $x>0$ for which the integral on the left-hand side is finite.

Proof. Let us prove the first statement, the second one is analogous. Let

$$x_0:=\sup\left\{x\in[0,\infty];\ \int_0^x\left(\int_s^x f(t)\,\mathrm{d}t\right)^\lambda g(s)\,\mathrm{d}s<\infty\right\}.$$

Then, for any $x\in[0,x_0)$, Fubini theorem yields

$$\begin{aligned}\int_0^x\left(\int_s^x f(t)\,\mathrm{d}t\right)^\lambda g(s)\,\mathrm{d}s&=\int_0^x\left[\int_s^x\lambda\left(\int_s^y f(t)\,\mathrm{d}t\right)^{\lambda-1} f(y)\,\mathrm{d}y\right]g(s)\,\mathrm{d}s\\&=\lambda\int_0^x f(y)\int_0^y\left(\int_s^y f(t)\,\mathrm{d}t\right)^{\lambda-1} g(s)\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

The expression on the second line is nondecreasing and continuous in x, therefore its derivative with respect to x exists and is equal to $\lambda f(x)\int_0^x\left(\int_s^x f(t)\,\mathrm{d}t\right)^{\lambda-1}g(s)\,\mathrm{d}s$ at a.e. point $x\in(0,x_0)$. □

Proposition 2.2. *Let $0<p\le q<\infty$ and let v,w be weights. Then it holds*

$$\sup_{\substack{\varphi\in\mathscr{M}_+\\ \varphi\text{ is nondecreasing}}}\frac{\left(\int_0^\infty\varphi^q(x)w(x)\,\mathrm{d}x\right)^{\frac1q}}{\left(\int_0^\infty\varphi^p(x)v(x)\,\mathrm{d}x\right)^{\frac1p}}\simeq\sup_{x>0}\left(\int_x^\infty w\right)^{\frac1q}\left(\int_x^\infty v\right)^{-\frac1p}.$$

Proof. This statement is analogous to a similar statement for nonincreasing functions (see [7, Theorem 3.1]). From there it can be also obtained directly by the change of variables $x\mapsto\frac1x$ in the integrals. □

Proposition 2.3. *Let $1<p<\infty$ and $0<q<p<\infty$. Let v,w be weights. Then*

$$C_{(6)}:=\sup_{f\in\mathscr{M}}\frac{\left(\int_0^\infty(f^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac1q}}{\left(\int_0^\infty(f^{*}(t))^p v(t)\,\mathrm{d}t\right)^{\frac1p}}\simeq A_{(7)}+A_{(8)},\tag{6}$$

where

$$A_{(7)}:=\left[\int_0^\infty\left(\frac{W(t)}{V(t)}\right)^{\frac{q}{p-q}}w(t)\,\mathrm{d}t\right]^{\frac{p-q}{pq}}\simeq\left[\int_0^\infty\left(\frac{W(t)}{V(t)}\right)^{\frac{p}{p-q}}v(t)\,\mathrm{d}t\right]^{\frac{p-q}{pq}}+W^{\frac1q}(\infty)V^{-\frac1p}(\infty)\tag{7}$$

and

$$(8)\qquad A_{(8)} := \left[\int_0^\infty \left(\int_t^\infty \frac{w(s)}{s^q}\, \mathrm{d}s \right)^{\frac{q}{p-q}} \left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}\, \mathrm{d}s \right)^{\frac{(p-1)q}{p-q}} \frac{w(t)}{t^q}\, \mathrm{d}t \right]^{\frac{p-q}{pq}}.$$

In particular, if $C_{(6)} < \infty$, then the function $s \mapsto v(s)s^{p'}V^{-p'}(s)$ is integrable near the origin.

Furthermore, if $q > 1$, or if $q < 1$ and the function $s \mapsto v(s)s^{p'}V^{-p'}(s)$ is integrable near the origin, then $A_{(8)} \simeq A_{(9)}$, where

$$(9)\qquad A_{(9)} := \left[\int_0^\infty \left(\int_t^\infty \frac{w(s)}{s^q}\, \mathrm{d}s \right)^{\frac{p}{p-q}} \left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}\, \mathrm{d}s \right)^{\frac{(q-1)p}{p-q}} \frac{v(t)t^{p'}}{V^{p'}(t)}\, \mathrm{d}t \right]^{\frac{p-q}{pq}}.$$

Proof. This assertion is stated in [7, Theorem 4.1(iii)] under the additional condition that $q \neq 1$. However, it is true even for $q = 1$, which may be checked using [11, Theorem 3.1(iv)] and [14, Theorem 3.1].

Let us say more on the equivalence $A_{(8)} \simeq A_{(9)}$. If $q > 1$ and the function u, defined by $u(s) := v(s)s^{p'}V^{-p'}(s)$ for $s > 0$, is not integrable near the origin (a simple example of such function u was given in [28, p. 93]), then both $A_{(8)}$ and $A_{(9)}$ are infinite. However, if $q < 1$ and u is not integrable near the origin, then $A_{(8)} = \infty$ but $A_{(9)} = 0$, since the exponent $\frac{(q-1)p}{p-q}$ is negative. □

Proposition 2.3 will be later used e.g. in the proofs of Lemmas 3.2 and 3.3 and Theorem 4.3. In the calculations within the proofs, we will need to use conditions in the form of $A_{(9)}$. The reason is that the function involving w appears only once in there and the resulting expression may be understood as the (quasi-)norm in a certain space. Nevertheless, for the final conditions which we state in the lemmas or theorems, we prefer the "safe" form in the style of $A_{(8)}$, i.e. avoiding the potentially negative exponents. In this way, the finiteness of the condition automatically implies the integrability of the "problematic" function near the origin.

The proposition below is a modification of [29, Proposition 2.7].

Proposition 2.4. *Let $\|\cdot\|_X$ be a functional acting on $\mathscr{M}_+$ such that for all $\lambda > 0$ and all $g, h \in \mathscr{M}_+$ such that $g \leq h$ a.e. it holds $\|g\|_X \leq \|h\|_X$ and $\|\lambda g\|_X \leq \lambda \|g\|_X$. Let v be a weight. Then*

$$(10)\qquad \sup_{f \in \mathscr{M}} \frac{\|f^*\|_X}{\|f\|_{\Lambda^\infty(v)}} = \left\| \left(\operatorname*{ess\,sup}_{y \in (0,\bullet)} v(y) \right)^{-1} \right\|_X.$$

Proof. Let $f^* \in \mathscr{M}$. Then, by the properties of $\|\cdot\|_X$, one has

$$\begin{aligned}\|f^*\|_X &\le \operatorname*{ess\,sup}_{x>0} f^*(x) \operatorname*{ess\,sup}_{y\in(0,x)} v(y) \left\| \left(\operatorname*{ess\,sup}_{y\in(0,\bullet)} v(y) \right)^{-1} \right\|_X \\ &= \operatorname*{ess\,sup}_{y>0} v(y) \operatorname*{ess\,sup}_{x\in(y,\infty)} f^*(x) \left\| \left(\operatorname*{ess\,sup}_{y\in(0,\bullet)} v(y) \right)^{-1} \right\|_X \\ &= \|f\|_{\Lambda^\infty(v)} \left\| \left(\operatorname*{ess\,sup}_{y\in(0,\bullet)} v(y) \right)^{-1} \right\|_X .\end{aligned}$$

Taking the supremum over $f \in \mathscr{M}$, we get the inequality "$\le$" in (10). Next, there exists $g \in \mathscr{M}$ such that $g^* = \left(\operatorname{ess\,sup}_{y\in(0,\bullet)} v(y) \right)^{-1}$ a.e. It is easy to observe that

$$\|g\|_{\Lambda^\infty(v)} = \operatorname*{ess\,sup}_{x>0} v(x) \left(\operatorname*{ess\,sup}_{y\in(0,x)} v(y) \right)^{-1} = 1.$$

Hence, it holds $\frac{\|g^*\|_X}{\|g\|_{\Lambda^\infty(v)}} = \|g^*\|_X = \left\| \left(\operatorname{ess\,sup}_{y\in(0,\bullet)} v(y) \right)^{-1} \right\|_X$ and thus the "$\ge$" inequality in (10) is satisfied. $\square$

3. EMBEDDINGS

In this section we characterize certain embeddings $\Lambda \hookrightarrow J$ and $\Lambda \hookrightarrow K$. These results will later form a crucial step in the proof of the bilinear Hardy inequality.

At first, observe that the embedding $\Lambda^\gamma(\omega) \to K^{\alpha,\infty}(\varphi,\psi)$ is characterized easily by rephrasing the problem as an embedding $\Lambda \hookrightarrow \Gamma$.

Proposition 3.1. *Let φ, ψ, ω be weights and $0 < \alpha, \beta, \gamma \le \infty$. Then*

$$\|Id\|_{\Lambda^\gamma(\omega)\to K^{\alpha,\infty}(\varphi,\psi)} = \operatorname*{ess\,sup}_{x>0} \psi(x) \|Id\|_{\Lambda^\gamma(\omega)\to\Gamma^\alpha(\varphi\chi_{[x,\infty)})}.$$

Proof. We have

$$\begin{aligned}\sup_{g\in\mathscr{M}} \operatorname*{ess\,sup}_{x>0} \frac{\left(\int_x^\infty (g^{**})^\alpha \varphi\right)^{\frac{1}{\alpha}} \psi(x)}{\left(\int_0^\infty (g^*)^\gamma \omega\right)^{\frac{1}{\gamma}}} &= \operatorname*{ess\,sup}_{x>0} \psi(x) \sup_{g\in\mathscr{M}} \frac{\left(\int_x^\infty (g^{**})^\alpha \varphi\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty (g^*)^\gamma \omega\right)^{\frac{1}{\gamma}}} \\ &= \operatorname*{ess\,sup}_{x>0} \psi(x) \|Id\|_{\Lambda^\gamma(\omega)\to\Gamma^\alpha(\varphi\chi_{[x,\infty)})}.\end{aligned}$$

$\square$

The embeddings $\Lambda \hookrightarrow \Gamma$ have been fully characterized (see [7], [6]). Similarly it can be dealt with the embedding $\Lambda^\gamma(\omega) \to J^{\alpha,\infty}(\varphi,\psi)$, where the problem reduces to a characterization the boundedness of the dual Hardy operator on the cone of nonincreasing functions. Results regarding the latter problem are also at our disposal, se e.g. [17].

Recall that if φ, ψ, ω are weights, then $\Phi(t) := \int_0^t \varphi$, $\Psi(t) := \int_0^t \psi$, $\Omega(t) := \int_0^t \omega$ for $t > 0$. In the couple of lemmas below there will appear a function σ, defined by

$$\sigma(x) := \sup_{t\in(0,x)} \left(t\Omega^{-\frac{1}{\gamma}}(t) \right)^{\frac{\gamma\alpha}{\gamma-\alpha}}, \quad x > 0, \tag{11}$$

where ω is a weight and $\alpha,\gamma\in(0,\infty)$ are exponents specified later. The function σ is continuous and nondecreasing on $(0,\infty)$, hence its derivative σ' exists at almost every point $x>0$ and, furthermore, for all $x>0$ it holds $\sigma(x)=\int_0^x\sigma'(t)\,\mathrm{d}t+\sigma(0+)$, where $\sigma(0+):=\limsup_{t\to0+}\left(t\Omega^{-\frac{1}{\gamma}}(t)\right)^{\frac{\gamma\alpha}{\gamma-\alpha}}$. This notation and properties of σ are used in the lemmas without further comment.

The lemma below brings a characterization of the embedding $\Lambda^\gamma(\omega)\hookrightarrow J^{\alpha,\beta}(\varphi,\psi)$ for $0<\alpha\le\beta<\infty$ and $\alpha<\gamma<\infty$.

Lemma 3.2. *Let φ, ψ, ω be weights. Denote*

$$C_{(12)}:=\sup_{g\in\mathscr{M}}\frac{\left(\int_0^\infty\left(\int_0^x(g^{**})^\alpha\varphi\right)^{\frac{\beta}{\alpha}}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}}}{\left(\int_0^\infty(g^*)^\gamma\omega\right)^{\frac{1}{\gamma}}}. \tag{12}$$

(i) *Let $0<\alpha<\gamma\le\beta<\infty$ and $1<\gamma$. Then $C_{(12)}\simeq A_{(13)}+A_{(14)}$, where*

$$A_{(13)}:=\sup_{x>0}\Omega^{-\frac{1}{\gamma}}(x)\left(\int_0^x\Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}}+\sup_{x>0}\left(\int_0^x\Phi^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty\psi\right)^{\frac{1}{\beta}} \tag{13}$$

and

$$A_{(14)}:=\sup_{x>0}\left[\int_0^x\left(\int_s^x\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}}\frac{\varphi(s)}{s^\alpha}\left(\int_0^s\frac{y^{\gamma'}\omega(y)}{\Omega^{\gamma'}(y)}\,\mathrm{d}y\right)^{\frac{\alpha(\gamma-1)}{\gamma-\alpha}}\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty\psi\right)^{\frac{1}{\beta}}$$
$$+\sup_{x>0}\left[\int_x^\infty\left(\int_x^s\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}\left(\int_0^x\frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}. \tag{14}$$

(ii) *Let $0<\alpha<\beta<\gamma<\infty$ and $1<\gamma$. Then $C_{(12)}\simeq A_{(15)}+A_{(16)}$, where*

$$A_{(15)}:=\left[\int_0^\infty\Omega^{\frac{\beta}{\beta-\gamma}}(x)\left(\int_0^x\Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{\beta}{\gamma-\beta}}\Phi^{\frac{\beta}{\alpha}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}$$
$$+\left[\int_0^\infty\left(\int_0^x\Phi^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma(\beta-\alpha)}{\alpha(\gamma-\beta)}}\Phi^{\frac{\gamma}{\gamma-\alpha}}(x)\Omega^{\frac{\gamma}{\alpha-\gamma}}(x)\omega(x)\left(\int_x^\infty\psi\right)^{\frac{\gamma}{\gamma-\beta}}\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}} \tag{15}$$

and

$$
\begin{aligned}
(16)\qquad A_{(16)} := {} & \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha} \left(\int_0^s \frac{y^{\gamma'}\omega(y)}{\Omega^{\gamma'}(y)}\,\mathrm{d}y \right)^{\frac{\alpha(\gamma-1)}{\gamma-\alpha}} \mathrm{d}s \right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \right. \\
& \left. \times \left(\int_x^\infty \psi \right)^{\frac{\beta}{\gamma-\beta}} \psi(x)\,\mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}} \\
& + \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s \right)^{\frac{\beta}{\gamma-\beta}} \frac{\varphi(x)}{x^\alpha} \right. \\
& \left. \times \int_x^\infty \left(\int_x^y \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\beta-\alpha}{\alpha}} \psi(y)\,\mathrm{d}y \left(\int_0^x \frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s \right)^{\frac{\beta(\gamma-1)}{\gamma-\beta}} \mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}}.
\end{aligned}
$$

(iii) *Let* $0 < \alpha < \gamma \le \beta < \infty$ *and* $\gamma \le 1$. *Let* σ *be given by* (11). *Then* $C_{(12)} \simeq A_{(13)} + A_{(17)}$, *where*

$$
\begin{aligned}
(17)\qquad A_{(17)} := {} & \sup_{x>0} \left[\int_0^x \left(\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha}\,\sigma(s)\,\mathrm{d}s \right]^{\frac{\gamma-\alpha}{\gamma\alpha}} \left(\int_x^\infty \psi \right)^{\frac{1}{\beta}} \\
& + \sup_{x>0} \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(x) \left[\int_x^\infty \left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s \right]^{\frac{1}{\beta}}.
\end{aligned}
$$

(iv) *Let* $0 < \alpha < \beta < \gamma \le 1$. *Let* σ *be given by* (11). *Then* $C_{(12)} \simeq A_{(15)} + A_{(18)} + A_{(19)}$, *where*

$$
(18)\quad A_{(18)} := \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t \right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha}\sigma(s)\,\mathrm{d}s \right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \left(\int_x^\infty \psi \right)^{\frac{\beta}{\gamma-\beta}} \psi(x)\,\mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}}
$$

and

$$(19)\qquad A_{(19)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s\right)^{\frac{\beta}{\gamma-\beta}}\right.$$

$$\left.\times \int_x^\infty \left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta-\alpha}{\alpha}} \psi(s)\,\mathrm{d}s\ \frac{\varphi(x)}{x^\alpha}\,\sigma^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}}(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}.$$

Proof. We have

$$(20)\quad C_{(12)} = \sup_{g\in\mathscr{M}}\sup_{h\in\mathscr{M}_+} \frac{1}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}\cdot\frac{\left(\int_0^\infty h(x)\int_0^x (g^{**}(t))^\alpha\varphi(t)\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty (g^*)^\gamma\omega\right)^{\frac{1}{\gamma}}}$$

$$(21)\qquad = \sup_{h\in\mathscr{M}_+}\frac{1}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}\sup_{g\in\mathscr{M}}\frac{\left(\int_0^\infty (g^{**}(t))^\alpha\varphi(t)\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty (g^*)^\gamma\omega\right)^{\frac{1}{\gamma}}}$$

$$=: B_0.$$

In step (20) we used duality of L^p-spaces and (21) follows by Fubini theorem and changing the order of the suprema.

To make the notation shorter, define the function u by

$$(22)\qquad u(s) := \frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)},\quad s>0.$$

Now suppose that $\gamma>1$. Assume that u is integrable near the origin. Then by Proposition 2.3 it holds

$$B_0 \simeq \sup_{h\in\mathscr{M}_+}\frac{\left(\int_0^\infty\left(\int_0^s\varphi(t)\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}(s)\omega(s)\,\mathrm{d}s\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$+\sup_{h\in\mathscr{M}_+}\frac{\left(\int_0^\infty\varphi(t)\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}\Omega^{\frac{1}{\gamma}}(\infty)}$$

$$+\sup_{h\in\mathscr{M}_+}\frac{\left(\int_0^\infty\left(\int_s^\infty\frac{\varphi(t)}{t^\alpha}\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\left(\int_0^s u(y)\,\mathrm{d}y\right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}}u(s)\,\mathrm{d}s\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$=: B_1+B_2+B_3.$$

Consider now the case (i). It holds

$$B_1 \simeq \sup_{h\in\mathcal{M}_+} \frac{\left(\int_0^\infty \left(\int_0^s h(x)\,\mathrm{d}x\right)^{\frac{\gamma}{\gamma-\alpha}} \Omega^{\frac{\gamma}{\alpha-\gamma}}(s)\omega(s)\,\mathrm{d}s\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\Phi^{\frac{\beta}{\alpha-\beta}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} \tag{23}$$

$$+ \sup_{h\in\mathcal{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty h(x)\,\mathrm{d}x\right)^{\frac{\gamma}{\gamma-\alpha}} \Phi^{\frac{\gamma}{\gamma-\alpha}}(s)\Omega^{\frac{\gamma}{\alpha-\gamma}}(s)\omega(s)\,\mathrm{d}s\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$\simeq \sup_{x>0}\left(\int_x^\infty \Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_0^x \Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}} + \sup_{x>0}\left(\int_0^x \Phi^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty \psi\right)^{\frac{1}{\beta}}, \tag{24}$$

where (23) follows by Fubini theorem and (24) by Hardy inequality (see [21, p. 3–4]). Next, Fubini theorem and L^p-duality yield

$$B_2 \simeq \left(\int_0^\infty \Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}} \Omega^{-\frac{1}{\gamma}}(\infty) = \sup_{x>0}\left(\int_0^x \Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}} \left(\Omega^{\frac{\alpha}{\alpha-\gamma}}(\infty)\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}. \tag{25}$$

Therefore, we have

$$\begin{aligned} B_2+B_1 &\simeq \sup_{x>0}\left(\Omega^{\frac{\alpha}{\alpha-\gamma}}(\infty)+\int_x^\infty \Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_0^x \Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}} + \sup_{x>0}\left(\int_0^x \Phi^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}\omega\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty \psi\right)^{\frac{1}{\beta}} \\ &\simeq A_{(13)}. \end{aligned}$$

Notice that this equivalence in fact does not involve the function u at all, hence it holds for any $u \in \mathcal{M}_+$. The assumption on u will be used only in the next part. By Fubini theorem, B_3 is equal to

$$\sup_{h\in\mathcal{M}_+} \frac{\left[\int_0^\infty \left(\int_s^\infty h(x)\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{\gamma}{\gamma-\alpha}} \left(\int_0^s u(y)\,\mathrm{d}y\right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}} u(s)\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}.$$

This expression is, by the dual version of [24, Theorem 1.1], equivalent to

$$\sup_{x>0}\left[\int_0^x\left(\int_s^x\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\left(\int_0^s u(y)\,\mathrm{d}y\right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}}u(s)\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty\psi\right)^{\frac{1}{\beta}}$$
$$+\sup_{x>0}\left[\int_x^\infty\left(\int_x^s\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}\left(\int_0^x u(s)\,\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}},$$

which is, in turn, equivalent to $A_{(14)}$ by Proposition 2.3, since u is integrable at the origin. Finally, observe that if u is not integrable at the origin, then necessarily both $B_0=\infty$ (see the proof sketch of Proposition 2.3) and $A_{(14)}=\infty$. On the other hand, if $A_{(14)}<\infty$, then u is integrable at the origin. Hence, $C_{(12)}=B_0<\infty$ holds if and only if $A_{(13)}+A_{(14)}<\infty$. Moreover, $C_{(12)}\simeq A_{(13)}+A_{(14)}$, all without any additional assumptions on the weight u.

In case (ii), using an appropriate version of Hardy inequality and L^p-duality (cf. the analogous situation in (23), (24) and (25)), we prove that $B_1+B_2\simeq A_{(15)}$. To estimate B_3, we use [24, Theorem 1.2]. Then we get

$$B_3\simeq\left[\int_0^\infty\left(\int_0^x\left(\int_s^x\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\left(\int_0^s\frac{y^{\gamma'}\omega(y)}{\Omega^{\gamma'}(y)}\,\mathrm{d}y\right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}}\frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}}\left(\int_x^\infty\psi\right)^{\frac{\beta}{\gamma-\beta}}\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}$$
$$+\left[\int_0^\infty\left(\int_x^\infty\left(\int_x^s\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right)^{\frac{\gamma}{\gamma-\beta}}\left(\int_0^x\frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s\right)^{\frac{\gamma(\beta-1)}{\gamma-\beta}}\frac{x^{\gamma'}\omega(x)}{\Omega^{\gamma'}(x)}\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}.$$

Using the assumption of integrability at the origin of u, one may show then by integration by parts that the above expression is equivalent to $A_{(16)}$. While handling the second term in the sum, one also needs to use Proposition 2.1. Finally, the additional assumption on u is removed in the same way as in case (i).

Now we assume $0<\gamma\le 1$. From [6, Theorem 3.1] it follows that $B_0=B_1+B_2+B_4$, where

$$B_4:=\sup_{h\in\mathscr{M}_+}\frac{\left[\int_0^\infty\sup_{0<t\le s}\left(\frac{t}{V^{\frac{1}{\gamma}}(t)}\right)^{\frac{\gamma\alpha}{\gamma-\alpha}}\left(\int_s^\infty\frac{\varphi(t)}{t^\alpha}\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}}\frac{\varphi(s)}{s^\alpha}\int_s^\infty h(x)\,\mathrm{d}x\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}.$$

Furthermore,

$$B_4 \simeq \sup_{h\in\mathscr{M}_+} \frac{\left[\int_0^\infty \sigma'(s)\left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} \tag{26}$$

$$+ \sup_{h\in\mathscr{M}_+} \frac{\sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+)\left(\int_0^\infty \frac{\varphi(s)}{s^\alpha}\int_s^\infty h(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$= \sup_{h\in\mathscr{M}_+} \frac{\left[\int_0^\infty \sigma'(s)\left(\int_s^\infty h(x)\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{\gamma}{\gamma-\alpha}}\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} \tag{27}$$

$$+ \sup_{h\in\mathscr{M}_+} \frac{\sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+)\left(\int_0^\infty h(x)\int_0^x \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\,\mathrm{d}x\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$=: B_5 + B_6.$$

For (26) one uses integration by parts and (27) follows by Fubini theorem. Next, by L^p-duality, we get

$$B_6 = \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+)\left[\int_0^\infty\left(\int_0^x \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}}\psi(x)\,\mathrm{d}x\right]^{\frac{1}{\beta}}. \tag{28}$$

Consider now the case (iii). From the dual version of [24, Theorem 1.1] it follows

$$B_5 \simeq \sup_{x>0}\left[\int_0^x\left(\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}}\sigma'(s)\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}\left(\int_x^\infty \psi\right)^{\frac{1}{\beta}}$$

$$+ \sup_{x>0}\left(\int_0^x \sigma'\right)^{\frac{\gamma-\alpha}{\gamma\alpha}}\left[\int_x^\infty\left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}.$$

Using this characterization, the expression of B_6 from (28) and integrating by parts, one obtains $B_5+B_6 \simeq A_{(17)}$. Earlier (when considering $\beta \geq \gamma > 1$) we proved that $B_1+B_2 \simeq A_{(13)}$. The same is true here, as the argument is correct even for $\beta \geq \gamma$ with $0<\gamma\leq 1$. Hence, it follows that $C_{(12)} \simeq B_1+B_2+B_5+B_6 \simeq A_{(13)}+A_{(17)}$ and the proof of this part is complete.

We proceed with (iv). Estimating B_1 and B_2 is done in the same way as in (ii). It remains to show that $B_5+B_6 \simeq A_{(18)}+A_{(19)}$. By the dual version of [24, Theorem 1.2], one has

$$
\begin{aligned}
(29)\qquad B_5 &\simeq \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}} \sigma'(s)\,\mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \left(\int_x^\infty \psi\right)^{\frac{\beta}{\gamma-\beta}} \psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}} \\
&+ \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s\right)^{\frac{\gamma}{\gamma-\beta}} \left(\int_0^x \sigma'\right)^{\frac{\gamma(\beta-\alpha)}{\alpha(\gamma-\beta)}} \sigma'(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}} \\
&=: B_7 + B_8.
\end{aligned}
$$

Now, integration by parts provides

$$
(30)\qquad A_{(18)} \simeq B_7 + \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+) \left[\int_0^\infty \left(\int_0^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \left(\int_x^\infty \psi\right)^{\frac{\beta}{\gamma-\beta}} \psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}.
$$

Next, it holds

$$
\sup_{x>0} \left[\int_x^\infty \left(\int_t^\infty \psi\right)^{\frac{\beta}{\gamma-\beta}} \psi(t)\,\mathrm{d}t\right]^{\frac{\alpha(\gamma-\beta)}{\gamma\beta}} \left(\int_x^\infty \psi\right)^{-\frac{\alpha}{\beta}} \simeq 1,
$$

thus, by Proposition 2.2, we get

$$
\left[\int_0^\infty \left(\int_0^x \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \left(\int_x^\infty \psi\right)^{\frac{\beta}{\gamma-\beta}} \psi(x)\,\mathrm{d}x\right]^{\frac{\alpha(\gamma-\beta)}{\gamma\beta}} \lesssim \left[\int_0^\infty \left(\int_0^x \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right]^{\frac{\alpha}{\beta}}.
$$

Applying this in (30) (and considering (28)) we obtain

$$
(31)\qquad B_7 \lesssim A_{(18)} \lesssim B_7 + B_6.
$$

Furthermore, from Proposition 2.1 and integration by parts it follows that $B_6 + B_8 \simeq A_{(19)}$. Combining this estimate with (31) and (29), we finally get $B_5 + B_6 \simeq B_6 + B_7 + B_8 \simeq A_{(18)} + A_{(19)}$, which we needed to prove. □

The next lemma characterizes the embedding $\Lambda^\gamma(\omega) \hookrightarrow K^{\alpha,\beta}(\varphi,\psi)$ for $0<\alpha\le\beta<\infty$ and $\alpha<\gamma<\infty$.

Lemma 3.3. *Let φ, ψ, ω be weights. Denote*

$$
(32)\qquad C_{(32)} := \sup_{g\in\mathscr{M}} \frac{\left(\int_0^\infty \left(\int_x^\infty (g^{**})^\alpha \varphi\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right)^{\frac{1}{\beta}}}{\left(\int_0^\infty (g^*)^\gamma \omega\right)^{\frac{1}{\gamma}}}.
$$

(i) *Let* $0<\alpha<\gamma\le\beta<\infty$ *and* $1<\gamma$. *Then* $C_{(32)}\simeq A_{(33)}+A_{(35)}+A_{(36)}$, *where*

$$A_{(33)}:=\sup_{x>0}\left[\int_x^\infty\left(\int_x^s\varphi\right)^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}(s)\omega(s)\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}\Psi^{\frac{1}{\beta}}(x) \tag{33}$$

$$+\sup_{x>0}\Omega^{-\frac{1}{\gamma}}(x)\left[\int_0^x\left(\int_s^x\varphi\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}, \tag{34}$$

$$A_{(35)}:=\sup_{x>0}\left[\int_x^\infty\left(\int_s^\infty\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}}\frac{\varphi(s)}{s^\alpha}\left(\int_0^s\frac{y^{\gamma'}\omega(y)}{\Omega^{\gamma'}(y)}\,\mathrm{d}y\right)^{\frac{\alpha(\gamma-1)}{\gamma-\alpha}}\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}}\Psi^{\frac{1}{\beta}}(x) \tag{35}$$

and

$$A_{(36)}:=\sup_{x>0}\left[\int_x^\infty\left(\int_s^\infty\frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}\left(\int_0^x\frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s\right)^{\frac{1}{\gamma'}}. \tag{36}$$

(ii) *Let* $0<\alpha<\beta<\gamma<\infty$ *and* $1<\gamma$. *Then* $C_{(32)}\simeq A_{(37)}+A_{(38)}+A_{(39)}$, *where*

$$A_{(37)}:=\left[\int_0^\infty\left(\int_x^\infty\left(\int_x^s\varphi\right)^{\frac{\gamma}{\gamma-\alpha}}\Omega^{\frac{\gamma}{\alpha-\gamma}}(s)\omega(s)\,\mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}}\Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}} \tag{37}$$

$$+\left[\int_0^\infty\left(\int_0^x\left(\int_s^x\varphi\right)^{\frac{\beta}{\alpha}}\psi(s)\,\mathrm{d}s\right)^{\frac{\beta}{\gamma-\beta}}\int_0^x\left(\int_s^x\varphi\right)^{\frac{\beta-\alpha}{\alpha}}\psi(s)\,\mathrm{d}s\right.$$

$$\left.\times\,\varphi(x)\Omega^{\frac{\beta}{\beta-\gamma}}(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}},$$

$$A_{(38)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha} \left(\int_0^s \frac{y^{\gamma'}\omega(y)}{\Omega^{\gamma'}(y)}\,\mathrm{d}y\right)^{\frac{\alpha(\gamma-1)}{\gamma-\alpha}} \mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \times \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}} \tag{38}$$

and

$$A_{(39)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s\right)^{\frac{\beta}{\gamma-\beta}} \left(\int_x^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(x) \times \left(\int_0^x \frac{s^{\gamma'}\omega(s)}{\Omega^{\gamma'}(s)}\,\mathrm{d}s\right)^{\frac{\beta(\gamma-1)}{\gamma-\beta}} \mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}. \tag{39}$$

(iii) *Let* $0 < \alpha < \gamma \le \beta < \infty$ *and* $\gamma \le 1$. *Let* σ *be given by* (11). *Then* $C_{(32)} \simeq A_{(33)} + A_{(40)} + A_{(41)}$, *where*

$$A_{(40)} := \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha}\sigma(s)\,\mathrm{d}s\right]^{\frac{\gamma-\alpha}{\gamma\alpha}} \Psi^{\frac{1}{\beta}}(x). \tag{40}$$

and

$$A_{(41)} := \sup_{x>0} \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(x) \left[\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\beta}{\alpha}} \psi(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}. \tag{41}$$

(iv) *Let* $0 < \alpha < \beta < \gamma \le 1$. *Let* σ *be given by* (11). *Then* $C_{(32)} \simeq A_{(37)} + A_{(42)} + A_{(43)}$, *where*

$$A_{(42)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha}\sigma(s)\,\mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}. \tag{42}$$

and

$$A_{(43)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \,\mathrm{d}t \right)^{\frac{\beta}{\alpha}} \psi(s) \,\mathrm{d}s \right)^{\frac{\beta}{\gamma-\beta}} \left(\int_x^\infty \frac{\varphi(y)}{y^\alpha} \,\mathrm{d}y \right)^{\frac{\beta}{\alpha}} \times \psi(x) \sigma^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}}(x) \,\mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}}. \tag{43}$$

Proof. The proof is to a great extent analogous to that of Lemma 3.2 but there are some additional steps which we show below.

Let u be defined by (22). If $1 > \gamma$, L^p-duality and Proposition 2.3 gives

$$\begin{aligned}
C_{(12)} &= \sup_{h \in \mathscr{M}_+} \frac{1}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}} \right)^{\frac{\beta-\alpha}{\beta\alpha}}} \sup_{g \in \mathscr{M}} \frac{\left(\int_0^\infty (g^{**}(t))^\alpha \varphi(t) \int_0^t h(x) \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty (g^*)^\gamma \omega \right)^{\frac{1}{\gamma}}} \\
&\simeq \sup_{h \in \mathscr{M}_+} \frac{\left(\int_0^\infty \left(\int_0^s \varphi(t) \int_0^t h(x) \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{\gamma}{\gamma-\alpha}} \Omega^{\frac{\gamma}{\alpha-\gamma}}(s) \omega(s) \,\mathrm{d}s \right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}} \right)^{\frac{\beta-\alpha}{\beta\alpha}}} \\
&\quad + \sup_{h \in \mathscr{M}_+} \frac{\left(\int_0^\infty \varphi(t) \int_0^t h(x) \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}} \right)^{\frac{\beta-\alpha}{\beta\alpha}} \Omega^{\frac{1}{\gamma}}(\infty)} \\
&\quad + \sup_{h \in \mathscr{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \int_0^t h(x) \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha} \int_0^s h(t) \,\mathrm{d}t \left(\int_0^s u \right)^{\frac{\alpha(\gamma-1)}{\gamma-\alpha}} \mathrm{d}s \right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}} \right)^{\frac{\beta-\alpha}{\beta\alpha}}}.
\end{aligned} \tag{44}$$

If u is integrable near the origin, then the term (44) is equivalent to

$$\sup_{h \in \mathscr{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \int_0^t h(x) \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{\gamma}{\gamma-\alpha}} \left(\int_0^s u(y) \,\mathrm{d}y \right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}} u(s) \,\mathrm{d}s \right)^{\frac{\gamma-\alpha}{\gamma\alpha}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}} \right)^{\frac{\beta-\alpha}{\beta\alpha}}}.$$

(i) Suppose that u is integrable near the origin. As in Lemma 3.2(i), using Hardy inequality, [24, Theorem 1.1] and the dual version of it one shows that $C_{(32)} \simeq A_{(33)} + B_1 + A_{(36)}$, where

$$B_1 := \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \,\mathrm{d}t \right)^{\frac{\gamma}{\gamma-\alpha}} \left(\int_0^s \frac{y^{\gamma'} \omega(y)}{\Omega^{\gamma'}(y)} \,\mathrm{d}y \right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}} \frac{s^{\gamma'} \omega(s)}{\Omega^{\gamma'}(s)} \,\mathrm{d}s \right]^{\frac{\gamma-\alpha}{\gamma\alpha}} \Psi^{\frac{1}{\beta}}(x).$$

Integration by parts gives $B_1 + B_2 \simeq A_{(35)}$ with

$$B_2 := \sup_{x>0} \left(\int_x^\infty \frac{\varphi(t)}{t^\alpha} \mathrm{d}t \right)^{\frac{1}{\alpha}} \left(\int_0^x \frac{y^{\gamma'} \omega(y)}{\Omega^{\gamma'}(y)} \mathrm{d}y \right)^{\frac{1}{\gamma'}} \Psi^{\frac{1}{\beta}}(x).$$

Using the proof idea of [13, Lemma 2.2] (a similar problem was also treated in [19, Proposition 3.2]), one checks that $B_2 \lesssim B_1 + A_{(36)}$. This implies that $B_1 + A_{(36)} \simeq A_{(35)} + A_{(36)}$, hence $C_{(32)} \simeq A_{(33)} + A_{(35)} + A_{(36)}$. Finally, we make the following observation, same as in Lemma 3.2. If u is not integrable near the origin, then $C_{(32)} = \infty$ (see (44)) and $A_{(36)} = \infty$. Hence, the equivalence $C_{(32)} \simeq A_{(33)} + B_1 + A_{(36)}$ holds even without additional assumptions on u.

(ii) Analogously to (i) we assume that u is integrable near the origin and get $C_{(32)} \simeq A_{(37)} + B_3 + A_{(39)}$, where

$$B_3 := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \mathrm{d}t \right)^{\frac{\gamma}{\gamma-\alpha}} \left(\int_0^s \frac{y^{\gamma'} \omega(y)}{\Omega^{\gamma'}(y)} \mathrm{d}y \right)^{\frac{\gamma(\alpha-1)}{\gamma-\alpha}} \frac{s^{\gamma'} \omega(s)}{\Omega^{\gamma'}(s)} \mathrm{d}s \right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x) \psi(x) \mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}}.$$

By integration by parts it follows that $B_3 + B_4 \simeq A_{(38)}$, where

$$B_4 := \left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(t)}{t^\alpha} \mathrm{d}t \right)^{\frac{\beta\gamma}{\alpha(\gamma-\beta)}} \left(\int_0^x \frac{y^{\gamma'} \omega(y)}{\Omega^{\gamma'}(y)} \mathrm{d}y \right)^{\frac{\beta(\gamma-1)}{\gamma-\beta}} \Psi^{\frac{\beta}{\gamma-\beta}}(x) \psi(x) \mathrm{d}x \right]^{\frac{\gamma-\beta}{\gamma\beta}}.$$

Following the idea of [14, Theorem 3.1] (cf. [19, Proposition 3.3]) one shows that $B_4 \lesssim B_3 + A_{(39)}$. Then $B_3 + A_{(39)} \simeq A_{(38)} + A_{(39)}$ and thus $C_{(32)} \simeq A_{(37)} + A_{(38)} + A_{(39)}$. The final dropping of the integrability assumption on u is performed in the same way as in (i).

In the remaining part of the proof we will assume that $\gamma \in (0,1]$, which is the case in (iii) and (iv).

(iii) Using the same ideas as in Lemma 3.2(iii), one shows that $C_{(32)} \simeq A_{(33)} + B_5 + A_{(41)}$, where

$$B_5 := \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha} \mathrm{d}t \right)^{\frac{\gamma}{\gamma-\alpha}} \sigma'(s) \mathrm{d}s \right]^{\frac{\gamma-\alpha}{\gamma\alpha}} \Psi^{\frac{1}{\beta}}(x).$$

Integration by parts yields

$$B_5 + \sup_{x>0} \left(\int_x^\infty \frac{\varphi(t)}{t^\alpha} \mathrm{d}t \right)^{\frac{1}{\alpha}} \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(x) \Psi^{\frac{1}{\beta}}(x) \simeq A_{(40)},$$

hence $B_5 \lesssim A_{(40)}$. Moreover, it also holds

$$\sup_{x>0}\left(\int_x^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(x)\Psi^{\frac{1}{\beta}}(x) \lesssim B_5 + A_{(41)},$$

which is proved by using the same argument from [13] as in (i). Combining the obtained relations, we conclude that $C_{(32)} \simeq A_{(33)} + A_{(40)} + A_{(41)}$.

(iv) In an analogy to Lemma 3.2(iv) it is proved that $C_{(32)} \simeq A_{(37)} + B_6 + A_{(43)}$, where

$$B_6 := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}} \sigma'(s)\,\mathrm{d}s\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}.$$

For any $x>0$, integration by parts gives

$$\int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\alpha}{\gamma-\alpha}} \frac{\varphi(s)}{s^\alpha}\sigma(s)\,\mathrm{d}s \simeq \int_x^\infty \left(\int_s^\infty \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\right)^{\frac{\gamma}{\gamma-\alpha}} \sigma'(s)\,\mathrm{d}s + \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\gamma}{\gamma-\alpha}}\sigma(x).$$

Hence, one gets

$$\begin{aligned}
A_{(42)} &\simeq B_6 + \left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \sigma^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}}(x)\Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}\\
&\simeq B_6 + \left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \left(\int_0^x \sigma'\right)^{\frac{\beta(\gamma-\alpha)}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}\\
&\quad + \sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+)\left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\gamma-\beta}{\gamma\beta}}\\
&=: B_6 + B_7 + B_8.
\end{aligned}$$

Using the same argument as in (ii) (based on [14]), we can show that $B_7 \lesssim B_6 + A_{(43)}$. Next, since the function $s \mapsto \frac{\varphi(s)}{s^\alpha}$ is nonincreasing, we obtain

$$\left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\gamma\beta}{\alpha(\gamma-\beta)}} \Psi^{\frac{\beta}{\gamma-\beta}}(x)\psi(x)\,\mathrm{d}x\right]^{\frac{\alpha(\gamma-\beta)}{\gamma\beta}} \lesssim \left[\int_0^\infty \left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}} \psi(x)\,\mathrm{d}x\right]^{\frac{\alpha}{\beta}}$$

by using the characterization of the embedding $\Lambda \hookrightarrow \Lambda$ [7, Theorem 3.1]. Thus, since

$$\sigma^{\frac{\gamma-\alpha}{\gamma\alpha}}(0+)\left[\int_0^\infty\left(\int_x^\infty \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}}\psi(x)\,\mathrm{d}x\right]^{\frac{1}{\beta}} \lesssim A_{(43)},$$

we get the inequality $B_8 \lesssim A_{(43)}$. Summarizing, we obtained $A_{(42)} + A_{(43)} \simeq B_6 + A_{(43)}$, hence $C_{(32)} \simeq A_{(37)} + A_{(42)} + A_{(43)}$ and the proof is completed. □

Although $\alpha < \gamma$ was assumed in the above statements, the proof method is not limited to this case. In fact, only the assumption $\alpha \le \beta$ is crucial for the duality approach. We may hence consider the case $0 < \gamma \le \alpha \le \beta < \infty$ and characterize the embedding $\Lambda^\gamma(\omega) \hookrightarrow J^{\alpha,\beta}(\varphi,\psi)$ using the same technique as before. The proof becomes actually considerably simpler in this case.

Proposition 3.4. *Let φ, ψ, ω be weights.*

(i) *Let $1 < \gamma \le \alpha \le \beta < \infty$. Then $C_{(12)} \simeq A_{(45)} + A_{(46)}$, where*

$$A_{(45)} := \sup_{x>0}\left(\int_0^x \Phi^{\frac{\beta}{\alpha}}\psi\right)^{\frac{1}{\beta}}\Omega^{-\frac{1}{\gamma}}(x) + \sup_{x>0}\left(\int_x^\infty \psi\right)^{\frac{1}{\beta}}\Phi^{\frac{1}{\alpha}}(x)\Omega^{-\frac{1}{\gamma}}(x) \tag{45}$$

and

$$A_{(46)} := \sup_{x>0}\left(\int_x^\infty\left(\int_x^t \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}}\psi(t)\,\mathrm{d}t\right)^{\frac{1}{\beta}}\left(\int_0^x \frac{t^{\gamma'}\omega(t)}{\Omega^{\gamma'}(t)}\,\mathrm{d}t\right)^{\frac{1}{\gamma'}}. \tag{46}$$

(ii) *Let $0 < \gamma \le 1$ and $\gamma \le \alpha \le \beta < \infty$. Then $C_{(12)} \simeq A_{(45)} + A_{(47)}$, where*

$$A_{(47)} := \sup_{x>0}\left(\int_x^\infty\left(\int_x^t \frac{\varphi(s)}{s^\alpha}\,\mathrm{d}s\right)^{\frac{\beta}{\alpha}}\psi(t)\,\mathrm{d}t\right)^{\frac{1}{\beta}} x\Omega^{-\frac{1}{\gamma}}(x). \tag{47}$$

Proof. Just as in (20) and (21), one has

$$C_{(12)} = \sup_{h\in\mathscr{M}_+}\frac{1}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}}\psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}\sup_{g\in\mathscr{M}}\frac{\left(\int_0^\infty (g^{**}(t))^\alpha\varphi(t)\int_t^\infty h(x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha}}}{\left(\int_0^\infty (g^*)^\gamma\omega\right)^{\frac{1}{\gamma}}} =: B.$$

Consider the case (i). Then

$$B \simeq \sup_{h\in\mathscr{M}_+} \sup_{x>0} \frac{\left(\int_0^x \varphi(t) \int_t^\infty h(s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \Omega^{-\frac{1}{\gamma}}(x)}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} \tag{48}$$

$$+ \sup_{h\in\mathscr{M}_+} \sup_{x>0} \frac{\left(\int_x^\infty \frac{\varphi(t)}{t^\alpha} \int_t^\infty h(s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \left(\int_0^x t^{\gamma'} \omega(t)\Omega^{-\gamma'}(t)\,\mathrm{d}t\right)^{\frac{1}{\gamma'}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$\simeq \sup_{x>0} \sup_{h\in\mathscr{M}_+} \frac{\left(\int_0^x h\Phi\right)^{\frac{1}{\alpha}} \Omega^{-\frac{1}{\gamma}}(x)}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} + \sup_{x>0} \sup_{h\in\mathscr{M}_+} \frac{\left(\int_x^\infty h\right)^{\frac{1}{\alpha}} \Phi^{\frac{1}{\alpha}}(x)\Omega^{-\frac{1}{\gamma}}(x)}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}} \tag{49}$$

$$+ \sup_{x>0} \sup_{h\in\mathscr{M}_+} \frac{\left(\int_x^\infty h(s) \int_x^s \frac{\varphi(t)}{t^\alpha}\,\mathrm{d}t\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \left(\int_0^x t^{\gamma'} \omega(t)\Omega^{-\gamma'}(t)\,\mathrm{d}t\right)^{\frac{1}{\gamma'}}}{\left(\int_0^\infty h^{\frac{\beta}{\beta-\alpha}} \psi^{\frac{\alpha}{\alpha-\beta}}\right)^{\frac{\beta-\alpha}{\beta\alpha}}}$$

$$= A_{(45)} + A_{(46)}. \tag{50}$$

Step (48) follows by [7, Theorem 4.1(i)], step (49) by Fubini theorem and changing the order of the suprema, and (50) is due to L^p-duality.

Case (ii) is proved analogously, using [7, Theorem 4.1(ii)] to estimate B. □

Proving an analogous proposition concerning the embedding $\Lambda^\gamma(\omega) \hookrightarrow K^{\alpha,\beta}(\varphi,\psi)$, $0 < \gamma \le \alpha \le \beta < \infty$, is left to an interested reader.

4. Bilinear Hardy inequality

At this point we have all the preliminary results needed to characterize the validity of the Hardy-type inequality (4) or, in other words, to provide equivalent estimates on $C_{(1)}$. The form of the results depends on the values of the exponents p_1, p_2 and q and their mutual relation. In fact, in this three-parameter setting, 23 different cases are possible and need separate treatment. For a better orientation, we present all the possible settings in the table below with references to the theorem in which each particular case is presented. Note that in some cases the roles of p_1 and p_2 may be switched in the corresponding theorem, compared with the entry in the table.

<table>
<tr><th colspan="4">Configuration of the exponents</th><th>Theorem</th></tr>
<tr><td rowspan="9">$0 < p_1, p_2 \leq q$</td><td rowspan="2">$0 < p_1, p_2 \leq 1$</td><td colspan="2">$q < \infty$</td><td>4.2(i)</td></tr>
<tr><td colspan="2">$q = \infty$</td><td>4.4(i)</td></tr>
<tr><td rowspan="3">$0 < p_1 \leq 1 < p_2$</td><td colspan="2">$q < \infty$</td><td>4.1(ii)</td></tr>
<tr><td rowspan="2">$q = \infty$</td><td>$p_2 < \infty$</td><td>4.4(ii)</td></tr>
<tr><td>$p_2 = \infty$</td><td>4.4(iii)</td></tr>
<tr><td rowspan="4">$1 < p_1, p_2$</td><td colspan="2">$q < \infty$</td><td>4.1(i)</td></tr>
<tr><td rowspan="3">$q = \infty$</td><td>$p_1, p_2 < \infty$</td><td>4.4(iv)</td></tr>
<tr><td>$p_1 < p_2 = \infty$</td><td>4.4(v)</td></tr>
<tr><td>$p_1 = p_2 = \infty$</td><td>4.4(vi)</td></tr>
<tr><td rowspan="5">$0 < p_1 \leq q < p_2$</td><td rowspan="3">$0 < p_1 \leq 1$</td><td colspan="2">$p_2 \leq 1$</td><td>4.2(iii)</td></tr>
<tr><td colspan="2">$1 < p_2 < \infty$</td><td>4.2(ii)</td></tr>
<tr><td colspan="2">$p_2 = \infty$</td><td>4.5(ii)</td></tr>
<tr><td rowspan="2">$1 < p_1$</td><td colspan="2">$p_2 < \infty$</td><td>4.1(iii)</td></tr>
<tr><td colspan="2">$p_2 = \infty$</td><td>4.5(i)</td></tr>
<tr><td rowspan="9">$0 < q < p_1, p_2$</td><td rowspan="2">$0 < p_1, p_2 \leq 1$</td><td colspan="2">$1/q \geq 1/p_1 + 1/p_2$</td><td>4.3(v)</td></tr>
<tr><td colspan="2">$1/q > 1/p_1 + 1/p_2$</td><td>4.3(vi)</td></tr>
<tr><td rowspan="3">$0 < p_2 \leq 1 < p_1$</td><td rowspan="2">$p_1 < \infty$</td><td>$1/q \geq 1/p_1 + 1/p_2$</td><td>4.3(iii)</td></tr>
<tr><td>$1/q > 1/p_1 + 1/p_2$</td><td>4.3(iv)</td></tr>
<tr><td colspan="2">$p_1 = \infty$</td><td>4.5(iv)</td></tr>
<tr><td rowspan="4">$1 < p_1, p_2$</td><td rowspan="2">$p_1, p_2 < \infty$</td><td>$1/q \geq 1/p_1 + 1/p_2$</td><td>4.3(i)</td></tr>
<tr><td>$1/q > 1/p_1 + 1/p_2$</td><td>4.3(ii)</td></tr>
<tr><td colspan="2">$p_1 < p_2 = \infty$</td><td>4.5(iii)</td></tr>
<tr><td colspan="2">$p_1 = p_2 = \infty$</td><td>4.5(v)</td></tr>
</table>

Let us now present and prove the results. We start with the configurations in which only the "classical" spaces appear, i.e. those where all the exponents are finite. First such case is $1 < p_1 \leq q < \infty$.

Theorem 4.1. *Let v_1, v_2, w be weights.*

(i) *Let $1 < p_1, p_2 \leq q$. Then $C_{(1)} \simeq A_{(51)} + A^{1,2}_{(52)} + A^{2,1}_{(52)} + A_{(53)}$, where*

$$A_{(51)} := \sup_{t>0} W^{\frac{1}{q}}(t) V_1^{-\frac{1}{p_1}}(t) V_2^{-\frac{1}{p_2}}(t), \tag{51}$$

$$A^{i,j}_{(52)} := \sup_{0<t<x<\infty} \left(\int_t^x \frac{w(s)}{s^q} \, ds \right)^{\frac{1}{q}} V_i^{-\frac{1}{p_i}}(x) \left(\int_0^t \frac{s^{p'_j} v_j(s)}{V_j^{p'_j}(s)} \, ds \right)^{\frac{1}{p'_j}} \tag{52}$$

and

$$A_{(53)} := \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^{2q}} \, ds \right)^{\frac{1}{q}} \left(\int_0^t \frac{s^{p'_1} v_1(s)}{V_1^{p'_1}(s)} \, ds \right)^{\frac{1}{p'_1}} \left(\int_0^t \frac{s^{p'_2} v_2(s)}{V_2^{p'_2}(s)} \, ds \right)^{\frac{1}{p'_2}}. \tag{53}$$

(ii) *Let* $0 < p_2 \le 1 < p_1 \le q$. *Then* $C_{(1)} \simeq A_{(51)} + A^{1,2}_{(54)} + A^{2,1}_{(52)} + A_{(55)}$, *where*

$$A^{i,j}_{(54)} := \sup_{0<t<x<\infty} \left(\int_t^x \frac{w(s)}{s^q} \, \mathrm{d}s \right)^{\frac{1}{q}} V_i^{-\frac{1}{p_i}}(x) t V_j^{-\frac{1}{p_j}}(t) \tag{54}$$

and

$$A_{(55)} := \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^{2q}} \, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^t \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \, \mathrm{d}s \right)^{\frac{1}{p_1'}} t V_2^{-\frac{1}{p_2}}(t). \tag{55}$$

(iii) *Let* $1 < p_1 \le q < p_2 < \infty$. *Define* $r_2 := \frac{p_2 q}{p_2 - q}$. *Then* $C_{(1)} \simeq A_{(56)} + A_{(57)} + A_{(58)}$, *where*

$$A_{(56)} := \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \left(\int_0^x W^{\frac{r_2}{p_2}}(t) w(t) V_2^{-\frac{r_2}{p_2}}(t) \, \mathrm{d}t \right)^{\frac{1}{r_2}}, \tag{56}$$

$$A_{(57)} := \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \left[\int_0^x \left(\int_t^x \frac{w(s)}{s^q} \, \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^q} \left(\int_0^t \frac{v_2(s) s^{p_2'}}{V_2^{p_2'}(s)} \, \mathrm{d}s \right)^{\frac{r_2}{p_2'}} \mathrm{d}t \right]^{\frac{1}{r_2}} \tag{57}$$

and

$$\begin{aligned} A_{(58)} := {} & \sup_{x>0} \left(\int_0^x \frac{v_1(s) s^{p_1'}}{V_1^{p_1'}(s)} \, \mathrm{d}s \right)^{\frac{1}{p_1'}} \left[\int_x^\infty \left(\int_x^t \frac{w(s)}{s^q} \, \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^q} V_2^{-\frac{r_2}{p_2}}(t) \, \mathrm{d}t \right]^{\frac{1}{r_2}} \\ & + \sup_{x>0} \left(\int_0^x \frac{v_1(s) s^{p_1'}}{V_1^{p_1'}(s)} \, \mathrm{d}s \right)^{\frac{1}{p_1'}} \left[\int_x^\infty \left(\int_t^\infty \frac{w(s)}{s^{2q}} \, \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^{2q}} \left(\int_0^t \frac{v_2(s) s^{p_2'}}{V_2^{p_2'}(s)} \, \mathrm{d}s \right)^{\frac{r_2}{p_2'}} \mathrm{d}t \right]^{\frac{1}{r_2}}. \end{aligned} \tag{58}$$

Proof. Since $1 < p_1 \le q < \infty$, by [7, Theorem 4.1(i)], we get

$$
\begin{aligned}
C_{(1)} &\simeq \sup_{g\in\Lambda^{p_2}(v_2)} \sup_{x>0} \left(\int_0^x (g^{**})^q w \right)^{\frac{1}{q}} V_1^{-\frac{1}{p_1}}(x) \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\
&\quad + \sup_{g\in\Lambda^{p_2}(v_2)} \sup_{x>0} \left(\int_x^\infty \frac{(g^{**}(s))^q w(s)}{s^q} \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \mathrm{d}s \right)^{\frac{1}{p_1'}} \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\
&= \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \sup_{g\in\Lambda^{p_2}(v_2)} \left(\int_0^x (g^{**})^q w \right)^{\frac{1}{q}} \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\
&\quad + \sup_{x>0} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \mathrm{d}s \right)^{\frac{1}{p_1'}} \sup_{g\in\Lambda^{p_2}(v_2)} \left(\int_x^\infty \frac{(g^{**}(s))^q w(s)}{s^q} \mathrm{d}s \right)^{\frac{1}{q}} \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\
&= \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \|Id\|_{\Lambda^{p_2}(v_2)\to\Gamma^q(w\chi_{[0,x]})} \\
&\quad + \sup_{x>0} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \mathrm{d}s \right)^{\frac{1}{p_1'}} \|Id\|_{\Lambda^{p_2}(v_2)\to\Gamma^q\left(s\mapsto w(s)s^{-q}\chi_{[x,\infty)}(s)\right)}. \\
&=: B_1 + B_2.
\end{aligned}
$$

Now we separate the different cases. In (i), [7, Theorem 4.1(i)] yields $B_1 + B_2 \simeq A_{(51)} + A^{1,2}_{(52)} + A^{2,1}_{(52)} + A_{(53)}$. In (ii), [7, Theorem 4.1(ii)] gives that $B_1 \simeq A_{(51)} + A^{1,2}_{(54)}$ and $B_2 \simeq A^{2,1}_{(52)} + A_{(55)}$. Finally, in (iii), Proposition 2.3 yields $B_1 + B_2 \simeq A_{(56)} + A_{(57)} + A_{(58)}$. $\square$

Now we consider the case $0 < p_1 \le 1$, $p_1 \le q$.

Theorem 4.2. *Let v_1, v_2, w be weights.*

(i) *Let $0 < p_1, p_2 \le 1$ and $0 < p_1, p_2 \le q$. Then $C_{(1)} \simeq A_{(51)} + A^{1,2}_{(54)} + A^{2,1}_{(54)} + A^{1,2}_{(59)} + A^{2,1}_{(59)}$, where*

$$
A^{i,j}_{(59)} := \sup_{0<x<t<\infty} \left(\int_t^\infty \frac{w(s)}{s^{2q}} \mathrm{d}s \right)^{\frac{1}{q}} t V_i^{-\frac{1}{p_i}}(t) x V_j^{-\frac{1}{p_j}}(x). \tag{59}
$$

(ii) *Let $0 < p_1 \le 1 < p_2 < \infty$ and $p_1 \le q < p_2$. Then $C_{(1)} \simeq A_{(56)} + A_{(57)} + A_{(60)} + A_{(61)}$, where*

$$
A_{(60)} := \sup_{x>0} x V_1^{-\frac{1}{p_1}}(x) \left[\int_x^\infty \left(\int_x^t \frac{w(s)}{s^q} \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^q} V_2^{-\frac{r_2}{p_2}}(t) \mathrm{d}t \right]^{\frac{1}{r_2}} \tag{60}
$$

and

$$A_{(61)} := \sup_{x>0} x V_1^{-\frac{1}{p_1}}(x) \left[\int_x^\infty \left(\int_t^\infty \frac{w(s)}{s^{2q}} \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^{2q}} \left(\int_0^t \frac{s^{p_2'} v_2(s)}{V_2^{p_2'}(s)} \mathrm{d}s \right)^{\frac{r_2}{p_2'}} \mathrm{d}t \right]^{\frac{1}{r_2}}. \tag{61}$$

(iii) *Let* $0 < p_1 \le q < p_2 \le 1$. *Then* $C_{(1)} \simeq A_{(56)} + A_{(60)} + A_{(62)}$, *where*

$$\begin{aligned} A_{(62)} := {} & \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \left[\int_0^x \left(\int_t^x \frac{w(s)}{s^q} \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^q} \sup_{s\in(0,t)} \frac{s^{r_2}}{V_2^{\frac{r_2}{p_2}}(s)} \mathrm{d}t \right]^{\frac{1}{r_2}} \\ & + \sup_{x>0} x V_1^{-\frac{1}{p_1}}(x) \left[\int_x^\infty \left(\int_t^\infty \frac{w(s)}{s^{2q}} \mathrm{d}s \right)^{\frac{r_2}{p_2}} \frac{w(t)}{t^{2q}} \sup_{s\in(0,t)} \frac{s^{r_2}}{V_2^{\frac{r_2}{p_2}}(s)} \mathrm{d}t \right]^{\frac{1}{r_2}}. \end{aligned} \tag{62}$$

Proof. Similarly as in Theorem 4.1, by [7, Theorem 4.1(ii)] (since $0 < p_1 \le 1$, $p_1 \le q < \infty$) we obtain

$$\begin{aligned} C_{(1)} \simeq {} & \sup_{g\in\Lambda^{p_2}(v_2)} \sup_{x>0} \left(\int_0^x (g^{**})^q w \right)^{\frac{1}{q}} V_1^{-\frac{1}{p_1}}(x) \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\ & + \sup_{g\in\Lambda^{p_2}(v_2)} \sup_{x>0} \left(\int_x^\infty \frac{(g^{**}(s))^q w(s)}{s^q} \mathrm{d}s \right)^{\frac{1}{q}} x V_1^{-\frac{1}{p_1}}(x) \|g\|^{-1}_{\Lambda^{p_2}(v_2)} \\ = {} & \sup_{x>0} V_1^{-\frac{1}{p_1}}(x) \|Id\|_{\Lambda^{p_2}(v_2)\to\Gamma^q(w\chi_{[0,x]})} \\ & + \sup_{x>0} x V_1^{-\frac{1}{p_1}}(x) \|Id\|_{\Lambda^{p_2}(v_2)\to\Gamma^q\left(s\mapsto w(s)s^{-q}\chi_{[x,\infty)}(s)\right)} \\ =: {} & B_1 + B_2. \end{aligned}$$

In (i), by [7, Theorem 4.1(ii)], we have $B_1 + B_2 \simeq A_{(51)} + A^{1,2}_{(54)} + A^{2,1}_{(54)} + A^{1,2}_{(59)} + A^{2,1}_{(59)}$. In (ii) it is $B_1 + B_2 \simeq A_{(56)} + A_{(57)} + A_{(60)} + A_{(61)}$ by Proposition 2.3 and finally in (iii) one gets $B_1 + B_2 \simeq A_{(56)} + A_{(60)} + A_{(62)}$ by [6, Theorem 3.1]. $\square$

We continue with the case $0 < q < p_1, p_2 < \infty$. This case is usually the most complicated one, especially if $p_1, p_2 \le 1$. Recall that if $q \in (0,1)\cup(1,\infty)$, then $q' := \frac{q}{q-1}$, while if $q = 1$, then $q' := \infty$.

Theorem 4.3. *Let* v_1, v_2, w *be weights. Let* $0 < q < p_1, p_2 < \infty$. *Define* $r_i := \frac{p_i q}{p_i - q}$, $i \in \{1,2\}$, *and* $R := \frac{p_1 p_2 q}{p_1 p_2 - p_1 q - p_2 q}$.

(i) *Let* $1 < p_1, p_2$ *and* $\frac{1}{q} \le \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(64)} + A^{2,1}_{(64)} + A^{1,2}_{(65)} + A^{2,1}_{(65)} + A^{1,2}_{(66)} + A^{2,1}_{(66)}$, *where*

$$(63) \quad A^{i,j}_{(63)} := \sup_{x>0} \left(\int_0^x W^{\frac{r_i}{p_i}} w V_i^{-\frac{r_i}{p_i}} \right)^{\frac{1}{r_i}} V_j^{-\frac{1}{p_j}}(x),$$

$$(64) \quad A^{i,j}_{(64)} := \sup_{x>0} \left[\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_j}{q}} V_j^{-\frac{r_j}{q}}(s) v_j(s) \mathrm{d}s \right]^{\frac{1}{r_j}} \left(\int_0^x \frac{t^{p_i'} v_i(t)}{V_i^{p_i'}(t)} \mathrm{d}t \right)^{\frac{1}{p_i'}},$$

$$(65) \quad A^{i,j}_{(65)} := \sup_{x>0} \left[\int_0^x \left(\int_s^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_i}{p_i}} \frac{w(s)}{s^q} \left(\int_0^s \frac{t^{p_i'} v_i(t)}{V_i^{p_i'}(t)} \mathrm{d}t \right)^{\frac{r_i}{p_i'}} \mathrm{d}s \right]^{\frac{1}{r_i}} V_j^{-\frac{1}{p_j}}(x)$$

and

$$(66) \; A^{i,j}_{(66)} := \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}} \mathrm{d}t \right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{t^{p_j'} v_j(t)}{V_j^{p_j'}(t)} \mathrm{d}t \right)^{\frac{r_j}{p_j'}} \mathrm{d}s \right]^{\frac{1}{r_j}} \left(\int_0^x \frac{t^{p_i'} v_i(t)}{V_i^{p_i'}(t)} \mathrm{d}t \right)^{\frac{1}{p_i'}}.$$

(ii) *Let* $1 < p_1, p_2$ *and* $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(68)} + A^{2,1}_{(68)} + A^{1,2}_{(69)} + A^{2,1}_{(69)}$, *where*

$$(67) \quad A^{i,j}_{(67)} := \left[\int_0^\infty \left(\int_0^x W^{\frac{r_j}{p_j}} w V_j^{-\frac{r_j}{p_j}} \right)^{\frac{r_j}{p_i - r_j}} W^{\frac{r_j}{p_j}}(x) w(x) V_j^{-\frac{r_j}{p_j}}(x) V_i^{\frac{r_j}{r_j - p_i}}(x) \mathrm{d}x \right]^{\frac{1}{R}},$$

$$\begin{aligned}
(68) \quad A^{i,j}_{(68)} := & \Bigg[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_j}{q}} V_j^{-\frac{r_j}{q}}(s) v_j(s) \mathrm{d}s \right)^{\frac{r_j}{p_i - r_j}} \\
& \times \frac{w(x)}{x^q} \int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_j}{p_j}} V_j^{-\frac{r_j}{q}}(s) v_j(s) \mathrm{d}s \left(\int_0^x \frac{s^{p_i'} v_i(s)}{V^{p_i'}(s)} \mathrm{d}s \right)^{\frac{p_i(r_j - 1)}{p_i - r_j}} \mathrm{d}x \Bigg]^{\frac{1}{R}} \\
& + \Bigg[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_i}{p_i}} \frac{w(s)}{s^q} \left(\int_0^s \frac{y^{p_i'} v_i(y)}{V_i^{p_i'}(y)} \mathrm{d}y \right)^{\frac{r_i}{p_i'}} \mathrm{d}s \right)^{\frac{r_i}{p_j - r_i}} \\
& \times \frac{w(x)}{x^q} \int_0^x \left(\int_s^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_i - p_i}{p_i}} \frac{w(s)}{s^q} \left(\int_0^s \frac{y^{p_i'} v_i(y)}{V_i^{p_i'}(y)} \mathrm{d}y \right)^{\frac{r_i}{p_i'}} \mathrm{d}s \, V_j^{\frac{r_i}{r_i - p_j}}(x) \mathrm{d}x \Bigg]^{\frac{1}{R}}
\end{aligned}$$

and

$$(69)\quad A^{i,j}_{(69)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t \right)^{\frac{r_i}{p_i}} \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{y^{p_i'} v_i(y)}{V_i^{p_i'}(y)}\,\mathrm{d}y \right)^{\frac{r_i}{p_i'}} \mathrm{d}s \right)^{\frac{r_i}{p_j - r_i}} \right.$$

$$\left. \times \left(\int_x^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t \right)^{\frac{r_i}{p_i}} \frac{w(x)}{x^{2q}} \left(\int_0^x \frac{y^{p_i'} v_i(y)}{V_i^{p_i'}(y)}\,\mathrm{d}y \right)^{\frac{r_i}{p_i'}} \left(\int_0^x \frac{s^{p_j'} v_j(s)}{V_j^{p_j'}(s)}\,\mathrm{d}s \right)^{\frac{r_i(p_j-1)}{p_j - r_i}} \mathrm{d}x \right]^{\frac{1}{R}}.$$

(iii) *Let* $p_2 \le 1 < p_1$ *and* $\frac{1}{q} \le \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(64)} + A^{1,2}_{(65)} + A_{(70)}$, *where*

$$(70)\quad A_{(70)} := \sup_{x>0} \left[\int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_2}{p_2}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y) \right]^{\frac{1}{r_2}} V_1^{-\frac{1}{p_1}}(x)$$

$$+ \sup_{x>0} \left[\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s \right]^{\frac{1}{r_1}} x V_2^{-\frac{1}{p_2}}(x)$$

$$+ \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t \right)^{\frac{r_2}{p_2}} \frac{w(s)}{s^{2q}} \sup_{y\in(0,s)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y)\,\mathrm{d}s \right]^{\frac{1}{r_2}} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s \right)^{\frac{1}{p_1'}}$$

$$+ \sup_{x>0} \left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t \right)^{\frac{r_1}{p_1}} \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)}\,\mathrm{d}y \right)^{\frac{r_1}{p_1'}} \mathrm{d}s \right]^{\frac{1}{r_1}} x V_2^{-\frac{1}{p_2}}(x).$$

(iv) *Let* $p_2 \le 1 < p_1$ *and* $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(68)} + A_{(71)}$, *where*

$$
\begin{aligned}
(71)\quad A_{(71)} := {} & \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2}{p_2}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y)\,\mathrm{d}s\right)^{\frac{r_2}{p_1-r_2}}\right.\\
& \left.\times \frac{w(x)}{x^q} \int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2-p_2}{p_2}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y)\,\mathrm{d}s\, V_1^{\frac{r_2}{r_2-p_1}}(x)\,\mathrm{d}x\right]^{\frac{1}{R}}\\
& + \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s\right)^{\frac{r_1}{p_2-r_1}}\right.\\
& \left.\times \frac{w(x)}{x^q} \int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{p_1}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s \sup_{y\in(0,x)} y^R V_2^{\frac{r_1}{r_1-p_2}}(y)\right]^{\frac{1}{R}}\\
& + \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_2}{p_2}} \frac{w(s)}{s^{2q}} \sup_{y\in(0,s)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y)\,\mathrm{d}s\right)^{\frac{p_1}{p_1-r_2}}\right.\\
& \left.\times \left(\int_x^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_2}{p_2}} \frac{w(x)}{x^{2q}} \sup_{y\in(0,x)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(y) \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right)^{\frac{r_2(p_1-1)}{p_1-r_2}} \mathrm{d}x\right]^{\frac{1}{R}}\\
& + \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_1}{p_1}} \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)}\,\mathrm{d}y\right)^{\frac{r_1}{p_1'}} \mathrm{d}s\right)^{\frac{r_1}{p_2-r_1}}\right.\\
& \left.\times \left(\int_x^\infty \frac{w(s)}{s^{2q}}\,\mathrm{d}s\right)^{\frac{r_1}{p_1}} \frac{w(x)}{x^{2q}} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right)^{\frac{r_1}{p_1'}} \sup_{y\in(0,x)} y^R V_2^{\frac{r_1}{r_1-p_2}}(y)\,\mathrm{d}x\right]^{\frac{1}{R}}.
\end{aligned}
$$

(v) *Let* $p_1, p_2 \le 1$ *and* $\frac{1}{q} \le \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(72)} + A^{2,1}_{(72)} + A^{1,2}_{(73)} + A^{2,1}_{(73)} + A^{1,2}_{(74)} + A^{2,1}_{(74)}$, *where*

$$A^{i,j}_{(72)} := \sup_{x>0} \left[\int_0^x \left(\int_s^x \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_i}{p_i}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_i} V_i^{-\frac{r_i}{p_i}}(y) \, \mathrm{d}s \right]^{\frac{1}{r_i}} V_j^{-\frac{1}{p_j}}(x), \tag{72}$$

$$A^{i,j}_{(73)} := \sup_{x>0} x V_i^{-\frac{1}{p_i}}(x) \left[\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^q} V_j^{-\frac{r_j}{p_j}}(s) \, \mathrm{d}s \right]^{\frac{1}{r_j}} \tag{73}$$

and

$$A^{i,j}_{(74)} := \sup_{x>0} x V_i^{-\frac{1}{p_i}}(x) \left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}} \, \mathrm{d}t \right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^{2q}} \sup_{y\in(0,s)} y^{r_j} V_j^{-\frac{r_j}{p_j}}(y) \, \mathrm{d}s \right]^{\frac{1}{r_j}}. \tag{74}$$

(vi) *Let* $p_1, p_2 \le 1$ *and* $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C_{(1)} \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(75)} + A^{2,1}_{(75)} + A^{1,2}_{(76)} + A^{2,1}_{(76)} + A^{1,2}_{(77)} + A^{2,1}_{(77)}$, *where*

$$\begin{aligned} A^{i,j}_{(75)} := \Bigg[\int_0^\infty & \left(\int_0^x \left(\int_s^x \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_j} V_j^{-\frac{r_j}{p_j}}(y) \right)^{\frac{r_j}{p_i - r_j}} \\ & \times \frac{w(x)}{x^q} \int_0^x \left(\int_s^x \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_j - p_j}{p_j}} \frac{w(s)}{s^q} \sup_{y\in(0,s)} y^{r_j} V_j^{-\frac{r_j}{p_j}}(y) \, \mathrm{d}s \, V_i^{-\frac{r_i}{p_i}}(x) \, \mathrm{d}x \Bigg]^{\frac{1}{R}}, \end{aligned} \tag{75}$$

$$\begin{aligned} A^{i,j}_{(76)} := \Bigg[\int_0^\infty & \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^q} V_j^{-\frac{r_j}{p_j}}(s) \, \mathrm{d}s \right)^{\frac{r_j}{p_i - r_j}} \\ & \times \frac{w(x)}{x^q} \int_x^\infty \left(\int_x^s \frac{w(t)}{t^q} \, \mathrm{d}t \right)^{\frac{r_j - p_j}{p_j}} \frac{w(s)}{s^q} V_j^{-\frac{r_j}{p_j}}(s) \, \mathrm{d}s \sup_{y\in(0,x)} y^R V_i^{\frac{r_j}{r_j - p_i}}(y) \, \mathrm{d}x \Bigg]^{\frac{1}{R}} \end{aligned} \tag{76}$$

and

$$(77)\quad A^{i,j}_{(77)} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_j}{p_j}} \frac{w(s)}{s^{2q}} \sup_{y\in(0,s)} y^{r_j} V_j^{-\frac{r_j}{p_j}}(y)\,\mathrm{d}s\right)^{\frac{r_j}{p_i-r_j}}\right.$$
$$\left.\times \left(\int_x^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_j}{p_j}} \frac{w(x)}{x^{2q}} \sup_{y\in(0,x)} y^{r_j} V_j^{-\frac{r_j}{p_j}}(y) \sup_{t\in(0,x)} t^R V_i^{\frac{r_j}{r_j-p_i}}(t)\,\mathrm{d}x\right]^{\frac{1}{R}}.$$

Proof. Consider first the case $1 < p_1$. Assume that the function u_1 defined by

$$u_1(x) := \int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s, \quad x > 0,$$

is integrable near the origin. Then, applying Proposition 2.3, we obtain

$$(78)\qquad C_{(1)} \simeq \sup_{g\in\mathcal{M}} \frac{\left(\int_0^\infty \left(\int_0^x (g^{**})^q w\right)^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}}(x) v_1(x)\,\mathrm{d}x\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}}$$
$$+ \sup_{g\in\mathcal{M}} \frac{\left(\int_0^\infty \left(\int_x^\infty \frac{((g^{**}(s))^q w(s)}{s^q}\,\mathrm{d}s\right)^{\frac{r_1}{q}} \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right)^{\frac{r_1}{q'}} \frac{x^{p_1'} v_1(x)}{V_1^{p_1'}(x)}\,\mathrm{d}x\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}}$$
$$+ \sup_{g\in\mathcal{M}} \frac{\left(\int_0^\infty (g^{**})^q w\right)^{\frac{1}{q}} V_1^{-\frac{1}{p_1}}(\infty)}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}}$$
$$=: B_1 + B_2 + B_3.$$

(i) We use Lemma 3.2(i) with the setting $\alpha := q$, $\beta := r_1$, $\gamma := p_2$, $\varphi := w$, $\psi(t) := V_1^{-\frac{r_1}{q}}(t) v_1(t)$, $\omega := v_2$, we obtain the characterization of B_1, and Proposition 2.3 to get the characterization of B_3. We obtain the equivalence

$$B_1 + B_3 \simeq B_4 + A^{2,1}_{(64)} + A^{2,1}_{(65)},$$

where

$$B_4 := \sup_{x>0} \left(\int_0^x W^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}} v_1\right)^{\frac{1}{r_1}} V_2^{-\frac{1}{p_2}}(x) + \sup_{x>0} \left(\int_0^x W^{\frac{r_2}{q}} V_2^{-\frac{r_2}{q}} v_2\right)^{\frac{1}{r_2}} V_1^{-\frac{1}{p_1}}(x).$$

Integration by parts yields

$$A^{1,2}_{(63)} + A^{2,1}_{(63)} \simeq B_4 + \sup_{x>0} W^{\frac{1}{q}}(x) V_1^{-\frac{1}{p_1}}(x) V_2^{-\frac{1}{p_2}}(x).$$

Moreover, the following series of inequalities holds true.

$$
\begin{aligned}
\sup_{x>0} W^{\frac{1}{q}}(x)V_1^{-\frac{1}{p_1}}(x)V_2^{-\frac{1}{p_2}}(x) &\simeq \sup_{g\in\mathscr{M}} \frac{\left(\int_0^\infty (g^*(t))^{r_1} W^{\frac{r_1}{p_1}}(t)w(t)V_1^{-\frac{r_1}{p_j}}(t)\,\mathrm{d}t\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*(t))^{p_2}v_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}} \\
&\lesssim \sup_{g\in\mathscr{M}} \frac{\left(\int_0^\infty (g^{**}(t))^{r_1} W^{\frac{r_1}{p_1}}(t)w(t)V_1^{-\frac{r_1}{p_j}}(t)\,\mathrm{d}t\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*(t))^{p_2}v_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}} \\
&\le \sup_{g\in\mathscr{M}} \frac{\left(\int_0^\infty \left(\int_0^t (g^{**})^q w\right)^{\frac{r_1}{p_1}} (g^{**}(t))^q w(t)V_1^{-\frac{r_1}{p_j}}(t)\,\mathrm{d}t\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*(t))^{p_2}v_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}} \\
&\simeq B_1 + B_3.
\end{aligned}
$$

The first step is due to the characterization of $\Lambda \hookrightarrow \Lambda$ [7, Theorem 3.1(ii)] and the last equivalence follows by integration by parts. Notice that the resulting relation

$$
\sup_{x>0} W^{\frac{1}{q}}(x)V_1^{-\frac{1}{p_1}}(x)V_2^{-\frac{1}{p_2}}(x) \lesssim B_1 + B_3 \tag{79}
$$

is established also if we consider the settings of cases (iii) and (v), i.e. if $p_1 \le 1$ or $p_2 \le 1$ and the other relations between the parameters remain unchanged. To continue, combining the obtained estimates we get

$$
B_1 + B_3 \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{2,1}_{(64)} + A^{2,1}_{(65)}. \tag{80}
$$

To deal with B_2, we use Lemma 3.3(i), setting $\alpha := q$, $\beta := r_1$, $\gamma := p_2$, $\varphi(t) := \frac{w(t)}{t^q}$, $\psi(t) := \left(\int_0^t s^{p_1'}v_1(s)V_1^{-p_1'}(s)\,\mathrm{d}s\right)^{\frac{r_1}{q'}} t^{p_1'}v_1(t)V_1^{-p_1'}(t)$, $\omega := v_2$. We obtain

$$
\begin{aligned}
B_2 \simeq A^{1,2}_{(64)} + A^{1,2}_{(66)} &+ \sup_{x>0}\left[\int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{q}} \left(\int_0^s \frac{t^{p_1'}v_1(t)}{V_1^{p_1'}(t)}\,\mathrm{d}t\right)^{\frac{r_1}{q'}} \frac{s^{p_1'}v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right]^{\frac{1}{r_1}} V_2^{-\frac{1}{p_2}}(x) \\
&+ \sup_{x>0}\left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t\right)^{\frac{r_1}{q}} \left(\int_0^s \frac{t^{p_1'}v_1(t)}{V_1^{p_1'}(t)}\,\mathrm{d}t\right)^{\frac{r_1}{q'}} \frac{s^{p_1'}v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right]^{\frac{1}{r_1}} \left(\int_0^x \frac{t^{p_2'}v_2(t)}{V_2^{p_2'}(t)}\,\mathrm{d}t\right)^{\frac{1}{p_2'}}.
\end{aligned}
$$

We now handle the third term in the sum by integration by parts and the fourth one in the same way as an analogous term in the proof of Lemma 3.3(i), concluding that $B_2 \simeq A^{1,2}_{(64)} + A^{1,2}_{(65)} + A^{1,2}_{(66)} + A^{2,1}_{(66)}$. Together we get

$$
C_{(1)} \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(64)} + A^{2,1}_{(64)} + A^{1,2}_{(65)} + A^{2,1}_{(65)} + A^{1,2}_{(66)} + A^{2,1}_{(66)}, \tag{81}
$$

still assuming the integrability of u_1 near the origin. Now we perform the usual final argument to drop the assumption on u_1. If u_1 is not integrable near the origin, then both $A^{1,2}_{(64)} = \infty$ and $B_2 = \infty$, the latter by Proposition 2.3. Since $B_2 = \infty$, it also holds $C_{(1)} = \infty$. Then the both sides of (81) are infinite, hence the equivalence holds trivially. The same argument may be repeated in cases (ii)–(iv), only replacing $A^{1,2}_{(64)}$ with another appropriate condition, when needed.

(ii) Here we use Lemmas 3.2(ii) and 3.3(ii) again, with the same respective settings of parameters as in the case (i), to estimate B_1 and B_2. Besides that, we also make use of Proposition 2.3 to estimate B_3. For B_1 and B_3 we so obtain

$$B_1 + B_3 \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{2,1}_{(68)}.$$

In order to get this equivalence, we in fact also need to prove the inequality

$$\left[\int_0^\infty \left(\int_0^x W^{\frac{r_1}{p_1}} w V_1^{-\frac{r_1}{p_1}}\right)^{\frac{r_1}{p_2-r_1}} W^{\frac{r_1}{p_1}}(x) w(x) V_1^{-\frac{r_1}{p_1}}(x) V_2^{\frac{r_1}{r_1-p_2}}(x)\,\mathrm{d}x\right]^{\frac{1}{R}} \lesssim B_1 + B_3.$$

It is done by reusing the argument used to establish (79) (notice the supremal condition from (79) being replaced by an integral condition this time, this is due to the different setting of parameters). The above inequality is also true in case (iv). Now we continue with B_2. We get

$$\begin{aligned}
B_2 \simeq & \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2}{q}} V_2^{-\frac{r_2}{q}}(s) v_2(s)\,\mathrm{d}s\right)^{\frac{r_2}{p_1-r_2}}\right. \\
& \left.\times \frac{w(x)}{x^q} \int_x^\infty \left(\int_x^y \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2}{p_2}} V_2^{-\frac{r_2}{q}}(y) v_2(y)\,\mathrm{d}y \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V^{p_1'}(s)}\,\mathrm{d}s\right)^{\frac{r_2(p_1-1)}{p_1-r_2}} \mathrm{d}x\right]^{\frac{1}{R}} \\
& + \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{q}} \left(\int_0^s \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)}\,\mathrm{d}y\right)^{\frac{r_1}{q'}} \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\right)^{\frac{r_1}{p_2-r_1}}\right. \\
& \left.\times \frac{w(x)}{x^q} \int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{p_1}} \left(\int_0^s \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)}\,\mathrm{d}y\right)^{\frac{r_1}{q'}} \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)}\,\mathrm{d}s\, V_2^{\frac{r_1}{r_1-p_2}}(x)\,\mathrm{d}x\right]^{\frac{1}{R}} \\
& + B_5 + B_6,
\end{aligned}$$

where

$$B_5 := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}} \,\mathrm{d}t \right)^{\frac{r_2}{p_2}} \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{y^{p_2'} v_2(y)}{V_2^{p_2'}(y)} \,\mathrm{d}y \right)^{\frac{r_2}{p_2'}} \mathrm{d}s \right)^{\frac{p_1}{p_1-r_2}} \right.$$

$$\left. \times \left(\int_0^x \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \,\mathrm{d}s \right)^{\frac{p_1(r_2-1)}{p_1-r_2}} \frac{x^{p_1'} v_1(x)}{V_1^{p_1'}(x)} \,\mathrm{d}x \right]^{\frac{1}{R}}$$

and

$$B_6 := \left[\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}} \,\mathrm{d}t \right)^{\frac{r_1}{q}} \left(\int_0^s \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)} \,\mathrm{d}y \right)^{\frac{r_1}{q'}} \frac{s^{p_1'} v_1(s)}{V_1^{p_1'}(s)} \,\mathrm{d}s \right)^{\frac{r_1}{p_2-r_1}} \right.$$

$$\left. \times \left(\int_x^\infty \frac{w(t)}{t^{2q}} \,\mathrm{d}t \right)^{\frac{r_1}{q}} \left(\int_0^x \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)} \,\mathrm{d}y \right)^{\frac{r_1}{q'}} \frac{x^{p_1'} v_1(x)}{V_1^{p_1'}(x)} \left(\int_0^x \frac{s^{p_2'} v_2(s)}{V_2^{p_2'}(s)} \,\mathrm{d}s \right)^{\frac{r_1(p_2-1)}{p_2-r_1}} \mathrm{d}x \right]^{\frac{1}{R}}.$$

Using integration by parts together with Proposition 2.1, one shows that the first two terms in B_2 are equivalent to $A^{1,2}_{(68)}$, hence $B_2 \simeq A^{1,2}_{(68)} + B_5 + B_6$. Similarly we prove that $B_5 \simeq A^{2,1}_{(69)}$. Next, again by integration by parts we get

$$\begin{aligned} A^{1,2}_{(69)} &\simeq B_6 + \left[\int_0^\infty \left(\int_x^\infty \frac{w(t)}{t^{2q}} \,\mathrm{d}t \right)^{\frac{r_1 p_2}{q(p_2-r_1)}} \left(\int_0^x \frac{y^{p_1'} v_1(y)}{V_1^{p_1'}(y)} \,\mathrm{d}y \right)^{\frac{r_1 p_2}{p_1'(p_2-r_1)}} \left(\int_0^x \frac{s^{p_2'} v_2(s)}{V_2^{p_2'}(s)} \,\mathrm{d}s \right)^{\frac{p_2(r_1-1)}{p_2-r_1}} \mathrm{d}x \right]^{\frac{1}{R}} \\ &\lesssim B_6 + B_5, \end{aligned}$$

hence $B_5 + B_6 \simeq A^{1,2}_{(69)} + A^{2,1}_{(69)}$ and therefore also $B_2 \simeq A^{1,2}_{(68)} + A^{1,2}_{(69)} + A^{2,1}_{(69)}$. Altogether, it holds

$$C_{(1)} \simeq B_1 + B_2 + B_3 \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(68)} + A^{2,1}_{(68)} + A^{1,2}_{(69)} + A^{2,1}_{(69)}.$$

Finally, the assumption of integrability of u_1 is removed in a similar way as in (i).

(iii) Using Lemmas 3.2(iii) and 3.3(iii) with the same setting as in (i) and then repeating the argument from (i) to show (80), we get

$$C_{(1)} \simeq B_1 + B_2 + B_3 \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(64)} + A^{1,2}_{(65)} + A_{(70)}.$$

Then we prove that this statement holds also if u_1 is not integrable near the origin, by imitating the argument from (i).

(iv) Here we use Lemmas 3.2(iv) and 3.3(iv) to get the estimate of $B_1 + B_2 + B_3$. Further adjustments of the conditions are made using the corresponding arguments from (ii). We omit the details.

Now suppose that $p_1 \le 1$, which is the case in (v) and (vi). For $i \in \{1,2\}$ denote

$$\sigma_i(x) := \sup_{0<y\le x} y^{r_i} V_i^{-\frac{r_i}{p_i}}(y), \quad x > 0.$$

Using [6, Theorem 3.1] and integration by parts, we obtain

$$\begin{aligned}
(82) \qquad C_{(1)} &\simeq B_1 + B_3 + \sup_{g\in\mathcal{M}} \frac{\left(\int_0^\infty \left(\int_x^\infty \frac{(g^{**}(t))^q w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{p_1}} \frac{(g^{**}(x))^q w(x)}{x^q}\sigma_1(x)\,\mathrm{d}x\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}} \\
&\simeq B_1 + B_3 + \sup_{g\in\mathcal{M}} \frac{\left(\int_0^\infty \left(\int_x^\infty \frac{(g^{**}(t))^q w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{q}} \sigma_1'(x)\,\mathrm{d}x\right)^{\frac{1}{r_1}}}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}} \\
&\quad + \sup_{g\in\mathcal{M}} \frac{\sigma_1^{\frac{1}{r_1}}(0+)\left(\int_0^\infty \frac{(g^{**}(x))^q w(x)}{x^q}\,\mathrm{d}x\right)^{\frac{1}{q}}}{\left(\int_0^\infty (g^*)^{p_2} v_2\right)^{\frac{1}{p_2}}} \\
&=: B_1 + B_3 + B_7 + B_8.
\end{aligned}$$

(v) We use Lemma 3.2(iii), setting $\alpha := q$, $\beta := r_1$, $\gamma := p_2$, $\varphi := w$, $\psi := V_1^{-\frac{r_1}{q}} v_1$, $\omega := v_2$, to obtain estimates of B_1; Lemma 3.3(iii), setting $\alpha := q$, $\beta := r_1$, $\gamma := p_2$, $\varphi(t) := \frac{w(t)}{t^q}$, $\psi := \sigma_1'$, $\omega := v_2$, to estimate B_7; and [6, Theorem 3.1] to estimate B_3 and B_8. Using the obtained expressions in (82) and applying also the argument used in (i) to show (79), we get

$$(83) \qquad C_{(1)} \simeq A^{1,2}_{(63)} + A^{2,1}_{(63)} + B_9 + B_{10} + B_{11} + B_{12} + B_{13} + A^{2,1}_{(72)} + A^{1,2}_{(74)},$$

where

$$B_9 := \sup_{x>0} \sigma_2^{\frac{1}{r_2}}(x) \left[\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s\right]^{\frac{1}{r_1}},$$

$$B_{10} := \sigma_1^{\frac{1}{r_1}}(0+) \left[\int_0^\infty \left(\int_0^x \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2}{p_2}} \frac{w(x)}{x^q} V_2^{-\frac{r_2}{p_2}}(x)\,\mathrm{d}x\right]^{\frac{1}{r_2}},$$

$$B_{11} := \sup_{x>0} \left(\int_0^x \sigma_1'\right)^{\frac{1}{r_1}} \left[\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{\frac{r_2}{q}} V_2^{-\frac{r_2}{q}}(s) v_2(s)\,\mathrm{d}s\right]^{\frac{1}{r_2}},$$

$$B_{12} := \sup_{x>0} \left[\int_0^x \left(\int_s^x \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_1}{q}} \sigma_1'(s)\,\mathrm{d}s \right]^{\frac{1}{r_1}} V_2^{-\frac{1}{p_2}}(x),$$

$$B_{13} := \sup_{x>0} \sigma_2^{\frac{1}{r_2}}(x) \left[\int_x^\infty \left(\int_s^\infty \frac{w(t)}{t^{2q}}\,\mathrm{d}t \right)^{\frac{r_1}{q}} \sigma_1'(s)\,\mathrm{d}s \right]^{\frac{1}{r_1}}.$$

By integration by parts one verifies the following inequalities: $B_9 \lesssim A^{2,1}_{(73)}$, $B_{10} + B_{11} \lesssim A^{1,2}_{(73)}$, $B_{12} \lesssim A^{1,2}_{(72)}$ and $B_{13} \lesssim A^{2,1}_{(74)}$. From these estimates and (83) it follows

$$C_{(1)} \lesssim A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(72)} + A^{2,1}_{(72)} + A^{1,2}_{(73)} + A^{2,1}_{(73)} + A^{1,2}_{(74)} + A^{2,1}_{(74)}.$$

Next, integration by parts yields the following: $A^{1,2}_{(72)} \lesssim B_{10} + B_{12}$, $A^{1,2}_{(73)} \lesssim B_{10} + B_{11} + B_{12}$, $A^{2,1}_{(73)} \lesssim B_9 + A^{2,1}_{(73)}$ and $A^{2,1}_{(74)} \lesssim B_{13} + A^{1,2}_{(74)}$. Using all these inequalities in (83), we get

$$A^{1,2}_{(63)} + A^{2,1}_{(63)} + A^{1,2}_{(72)} + A^{2,1}_{(72)} + A^{1,2}_{(73)} + A^{2,1}_{(73)} + A^{1,2}_{(74)} + A^{2,1}_{(74)} \lesssim C_{(1)}.$$

The proof of this part is then completed.

(vi) Analogously to the case (v) we use Lemma 3.2(iv) to estimate B_1, Lemma 3.3(iv) to estimate B_7, and [6, Theorem 3.1] to get an estimate of B_3 and B_8. Inserting these expressions into (82) and merging some of them by integration by parts (similarly to the case (ii)), we obtain

$$C_{(1)} \simeq A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(75)} + A^{1,2}_{(77)} + A^{1,2}_{(77)} + B_{10} + B_{14} + B_{15} + B_{16}, \tag{84}$$

where

$$B_{14} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_1}{q}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s \right)^{\frac{r_1}{p_2-r_1}} \right.$$
$$\left. \times \frac{w(x)}{x^q} \int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_1}{p_1}} V_1^{-\frac{r_1}{q}}(s) v_1(s)\,\mathrm{d}s\, \sigma_2^{\frac{p_1}{p_1-r_2}}(x)\,\mathrm{d}x \right]^{\frac{1}{R}},$$

$$B_{15} := \left[\int_0^\infty \left(\int_x^\infty \left(\int_x^s \frac{w(t)}{t^q}\,\mathrm{d}t \right)^{\frac{r_2}{q}} V_2^{-\frac{r_2}{q}}(s) v_2(s)\,\mathrm{d}s \right)^{\frac{p_1}{p_1-r_2}} \left(\int_0^x \sigma_1' \right)^{\frac{r_1}{p_2-r_1}} \sigma_1'(x)\,\mathrm{d}x \right]^{\frac{1}{R}},$$

$$B_{16} := \left[\int_0^\infty \left(\int_0^x \left(\int_s^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_1}{q}} \sigma_1'(s) \mathrm{d}s \right)^{\frac{r_1}{p_2 - r_1}} \right.$$

$$\left. \times \frac{w(x)}{x^q} \int_0^x \left(\int_s^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_1}{p_1}} \sigma_1'(s) \mathrm{d}s \, V_2^{\frac{r_1}{r_1 - p_2}}(x) \mathrm{d}x \right]^{\frac{1}{R}}.$$

Performing integration by parts, one gets $B_{10} \lesssim A^{2,1}_{(76)}$, $B_{14} \lesssim A^{2,1}_{(76)}$, $B_{15} \lesssim A^{1,2}_{(76)}$ and $B_{16} \lesssim A^{1,2}_{(75)}$. We apply these inequalities to replace the "B-parts" in (84), and so we obtain

$$C_{(1)} \lesssim A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(75)} + A^{2,1}_{(75)} + A^{1,2}_{(76)} + A^{2,1}_{(76)} + A^{1,2}_{(77)} + A^{2,1}_{(77)}.$$

Now observe that

$$A^{2,1}_{(75)} \simeq B_{16} + \sigma_1^{\frac{1}{r_1}}(0+) \left(\int_0^\infty \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{1}{q}} V_2^{-\frac{1}{p_2}}(\infty) \tag{85}$$

$$+ \sigma_1^{\frac{1}{r_1}}(0+) \left[\int_0^\infty \left(\int_0^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{R}{q}} V_2^{\frac{p_2}{r_1 - p_2}}(x) v_2(x) \mathrm{d}x \right]^{\frac{1}{R}}$$

$$\lesssim B_{16} + B_{10} + \sigma_1^{\frac{1}{r_1}}(0+) \left[\int_0^\infty \left(\int_0^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{R}{q}} V_2^{\frac{p_2}{r_1 - p_2}}(x) v_2(x) \mathrm{d}x \right]^{\frac{1}{R}}$$

$$\lesssim B_{16} + B_{10} \tag{86}$$

$$+ \sigma_1^{\frac{1}{r_1}}(0+) \left[\int_0^\infty \left(\int_0^x \frac{w(t)}{t^q} \mathrm{d}t \right)^{\frac{r_2}{q}} \left(\int_x^\infty V_2^{\frac{p_2}{r_1 - p_2}} v_2 \right)^{-\frac{r_2}{p_1}} V_2^{\frac{p_2}{r_1 - p_2}}(x) v_2(x) \mathrm{d}x \right]^{\frac{1}{R}}$$

$$\lesssim B_{16} + B_{10}. \tag{87}$$

Indeed, the estimates (85) and (87) follow by integration by parts, while (86) is granted by Proposition 2.2. We proved that $A^{2,1}_{(75)} \lesssim B_{16} + B_{10}$. By similar means it is shown that $A^{1,2}_{(76)} \lesssim B_{10} + B_{15} + B_{16}$ and $A^{2,1}_{(76)} \lesssim B_{14} + A^{1,2}_{(75)}$. Using these three estimates together with (84), we get

$$A^{1,2}_{(67)} + A^{2,1}_{(67)} + A^{1,2}_{(75)} + A^{2,1}_{(75)} + A^{1,2}_{(76)} + A^{2,1}_{(76)} + A^{1,2}_{(77)} + A^{2,1}_{(77)} \lesssim C_{(1)}.$$

This completes case (vi) and thus the whole proof. □

The next part deals with the "weak cases", i.e. such configurations of p_1, p_2, q that at least one of these exponents is infinite. The following theorem covers the case $q = \infty$.

Theorem 4.4. *Let v_1, v_2, w be weights. Let $q=\infty$.*

(i) *Let $0<p_1,p_2\le 1$. Then $C_{(1)}\simeq A_{(88)}$, where*

$$A_{(88)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \sup_{s\in(0,x)} s V_1^{-\frac{1}{p_1}}(s) \sup_{t\in(0,x)} t V_2^{-\frac{1}{p_2}}(t). \tag{88}$$

(ii) *Let $0<p_1\le 1<p_2<\infty$. Then $C_{(1)}\simeq A_{(89)}$, where*

$$A_{(89)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \sup_{s\in(0,x)} s V_1^{-\frac{1}{p_1}}(s) \left(\int_0^x t^{p_2'-1} V_2^{1-p_2'}(t)\,\mathrm{d}t\right)^{\frac{1}{p_2'}}. \tag{89}$$

(iii) *Let $0<p_1\le 1<p_2=\infty$. Then $C_{(1)}\simeq A_{(90)}$, where*

$$A_{(90)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \sup_{s\in(0,x)} s V_1^{-\frac{1}{p_1}}(s) \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_2(y)}. \tag{90}$$

(iv) *Let $1<p_1,p_2<\infty$. Then $C_{(1)}\simeq A_{(91)}$, where*

$$A_{(91)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \left(\int_0^x s^{p_1'-1} V_1^{1-p_1'}(s)\,\mathrm{d}s\right)^{\frac{1}{p_1'}} \left(\int_0^x t^{p_2'-1} V_2^{1-p_2'}(t)\,\mathrm{d}t\right)^{\frac{1}{p_2'}}. \tag{91}$$

(v) *Let $1<p_1<p_2=\infty$. Then $C_{(1)}\simeq A_{(92)}$, where*

$$A_{(92)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \left(\int_0^x s^{p_1'-1} V_1^{1-p_1'}(s)\,\mathrm{d}s\right)^{\frac{1}{p_1'}} \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_2(y)}. \tag{92}$$

(vi) *Let $p_1=p_2=\infty$. Then $C_{(1)}=A_{(93)}$, where*

$$A_{(93)} := \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \int_0^x \frac{\mathrm{d}s}{\operatorname{ess\,sup}_{y\in(0,s)} v_1(y)} \int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_2(y)}. \tag{93}$$

Proof. We have

$$\begin{aligned}
C_{(1)} &= \sup_{f\in\mathscr{M}} \sup_{g\in\mathscr{M}} \frac{\operatorname{ess\,sup}_{x>0} f^{**}(x) g^{**}(x) w(x)}{\|f\|_{\Lambda^{p_1}(v_1)} \|g\|_{\Lambda^{p_2}(v_2)}} \\
&= \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \sup_{f\in\mathscr{M}} \frac{\int_0^x f^*(t)\,\mathrm{d}t}{\|f\|_{\Lambda^{p_1}(v_1)}} \sup_{g\in\mathscr{M}} \frac{\int_0^x g^*(t)\,\mathrm{d}t}{\|g\|_{\Lambda^{p_2}(v_2)}} \\
&= \operatorname*{ess\,sup}_{x>0} \frac{w(x)}{x^2} \|Id\|_{\Lambda^{p_1}(v_1)\to\Lambda^1\left(\chi_{(0,x)}\right)} \|Id\|_{\Lambda^{p_2}\left(v_2\right)\to\Lambda^1\left(\chi_{(0,x)}\right)}.
\end{aligned}$$

Now, in all the cases we simply use the characterizations of the embedding $\Lambda^p(v)\hookrightarrow\Lambda^1\left(\chi_{(0,x)}\right)$ provided by [7, Theorem 3.1] and Proposition 2.4. $\square$

Finally, we complete the list with the last remaining case in which $0<q<\infty$ and $0<p_2\le p_1=\infty$.

Theorem 4.5. *Let v_1, v_2, w be weights. Let $p_1 = \infty$ and $0 < q < \infty$.*

(i) *Let $1 < p_2 \le q$. Then $C_{(1)} \simeq A_{(94)} + A_{(95)}$, where*

$$A_{(94)} := \sup_{x>0} \left[\int_0^x \frac{w(s)}{s^q} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right]^{\frac{1}{q}} V_2^{-\frac{1}{p_2}}(x) \tag{94}$$

and

$$A_{(95)} := \sup_{x>0} \left[\int_x^\infty \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right]^{\frac{1}{q}} \left(\int_0^x \frac{s^{p_2'} v_2(s)}{V_2^{p_2'}(s)} \mathrm{d}s \right)^{\frac{1}{p_2'}}. \tag{95}$$

(ii) *Let $0 < p_2 \le 1$ and $p_2 \le q$. Then $C_{(1)} \simeq A_{(94)} + A_{(96)}$, where*

$$A_{(96)} := \sup_{x>0} \left[\int_x^\infty \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right]^{\frac{1}{q}} x V_2^{-\frac{1}{p_2}}(x). \tag{96}$$

(iii) *Let $1 < p_2 < \infty$ and $0 < q < p_2$. Then $C_{(1)} \simeq A_{(97)} + A_{(98)}$, where*

$$A_{(97)} := \left[\int_0^\infty \left(\int_0^x \frac{w(s)}{s^q} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right)^{\frac{r_2}{p_2}} \times \frac{w(x)}{x^q} \left(\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q V_2^{-\frac{r_2}{p_2}}(x) \mathrm{d}x \right]^{\frac{1}{r_2}} \tag{97}$$

and

$$A_{(98)} := \left[\int_0^\infty \left(\int_x^\infty \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right)^{\frac{r_2}{p_2}} \times \frac{w(x)}{x^{2q}} \left(\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \left(\int_0^x \frac{s^{p_2'} v_2(s)}{V_2^{p_2'}(s)} \mathrm{d}s \right)^{\frac{r_2}{p_2'}} \mathrm{d}x \right]^{\frac{1}{r_2}}. \tag{98}$$

(iv) *Let $0 < q < p_2 \le 1$. Then $C_{(1)} \simeq A_{(97)} + A_{(99)}$, where*

$$A_{(99)} := \left[\int_0^\infty \left(\int_x^\infty \frac{w(s)}{s^{2q}} \left(\int_0^s \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \mathrm{d}t \right)^{\frac{r_2}{p_2}} \times \frac{w(x)}{x^{2q}} \left(\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \right)^q \sup_{y\in(0,x)} y^{r_2} V_2^{-\frac{r_2}{p_2}}(x) \mathrm{d}x \right]^{\frac{1}{r_2}}. \tag{99}$$

(v) *Let* $0 < q < p_2 = \infty$. *Then* $C_{(1)} \simeq A_{(100)}$, *where*

$$(100)\quad A_{(100)} := \left[\int_0^\infty \frac{w(x)}{x^{2q}} \left(\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{y\in(0,t)} v_1(y)} \int_0^x \frac{\mathrm{d}s}{\operatorname{ess\,sup}_{y\in(0,s)} v_2(y)} \right)^q \mathrm{d}x \right]^{\frac{1}{q}}.$$

Proof. From Proposition 2.4 it follows

$$\begin{aligned} C_{(1)} &= \sup_{g\in\mathscr{M}} \sup_{f\in\mathscr{M}} \frac{\left(\int_0^\infty (f^{**}(x))^q (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{\|f\|_{\Lambda^\infty(v_1)} \|g\|_{\Lambda^{p_2}(v_2)}} \\ &\simeq \sup_{g\in\mathscr{M}} \frac{\left[\int_0^\infty (g^{**}(x))^q \frac{w(x)}{x^q} \left(\int_0^x \frac{\mathrm{d}s}{\operatorname{ess\,sup}_{y\in(0,s)} v_1(y)} \right)^q \mathrm{d}x \right]^{\frac{1}{q}}}{\|g\|_{\Lambda^{p_2}(v_2)}} \\ &\simeq \|Id\|_{\Lambda^{p_2}(v_2)\to\Gamma^q\left(x\mapsto \frac{w(x)}{x^q}\left[\left(\operatorname{ess\,sup}_{y\in(0,s)} v_1(y)\right)^{-1}\right]^q\right)}. \end{aligned}$$

The rest is done by application of the characterization of the involved embedding $\Gamma \hookrightarrow \Lambda$, which can be found in [7, Theorem 4.1] (cases (i) and (ii)), Proposition 2.3 (for case (iii)), [6, Theorem 3.1] (case (iv)) and finally Proposition 2.4 for case (v). ☐

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Paper V

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Iterating bilinear Hardy inequalities

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ITERATING BILINEAR HARDY INEQUALITIES

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ABSTRACT. An iteration technique to characterize boundedness of certain types of multilinear operators is presented, reducing the problem into a corresponding linear-operator case. The method gives a simple proof of a characterization of validity of the weighted bilinear Hardy inequality

$$\left(\int_a^b \left(\int_a^t f \int_a^t g\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_a^b f^{p_1} v_1\right)^{\frac{1}{p_1}} \left(\int_a^b f^{p_2} v_2\right)^{\frac{1}{p_2}}$$

for all nonnegative f, g on (a,b), for $1 < p_1, p_2, q < \infty$. More equivalent characterizing conditions are presented. The same technique is applied to various further problems, in particular those involving multilinear integral operators of Hardy type.

1. INTRODUCTION

Let $-\infty \le a < b \le \infty$. Let the symbol $\mathscr{M}_+$ denote the cone of nonnegative Lebesgue-measurable functions on (a,b). The *Hardy operator* H_1 and the *"dual Hardy" operator* H_1' are operators acting on $\mathscr{M}_+$, defined by

$$H_1 f(t) := \int_a^t f(s)\,\mathrm{d}s, \quad H_1' f(t) := \int_t^b f(s)\,\mathrm{d}s, \qquad t \in (a,b).$$

Recall that the *weighted Lebesgue space* $L^\alpha(u)$ consists of all real-valued Lebesgue-measurable functions f on (a,b) such that

$$||f||_{L^\alpha(u)} := \left(\int_a^b |f(t)|^\alpha u(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} < \infty.$$

Here $1 \le \alpha < \infty$ and u is a *weight*, i.e. simply a fixed function $u \in \mathscr{M}_+$.

It is well known under which conditions the operator H_1 is bounded from $L^\alpha(u)$ to $L^\beta(z)$, or, in other words, when the *weighted Hardy inequality*

$$\left(\int_a^b \left(\int_a^t f\right)^\beta z(t)\,\mathrm{d}t\right)^{\frac{1}{\beta}} \le C \left(\int_a^b f^\alpha u\right)^{\frac{1}{\alpha}} \tag{1}$$

holds for all $f \in \mathscr{M}_+$. Namely, the following theorems hold (see [15, 2, 14, 13]):

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Theorem 1.1. *Let u, z be weights. For $\alpha, \beta \in (1,\infty)$ set*

$$C_{(2)} := \sup_{f \in \mathscr{M}_+} \left(\int_a^b \left(\int_a^t f \right)^{\beta} z(t)\,\mathrm{d}t \right)^{\frac{1}{\beta}} \left(\int_a^b f^{\alpha} u \right)^{-\frac{1}{\alpha}}. \tag{2}$$

Then

(i) *If $1 < \alpha \le \beta < \infty$, then*

$$C_{(2)} \simeq \sup_{a<x<b} \left(\int_x^b z \right)^{\frac{1}{\beta}} \left(\int_a^x u^{1-\alpha'} \right)^{\frac{1}{\alpha'}}.$$

(ii) *If $1 < \beta < \alpha < \infty$ and $\gamma := \frac{\alpha\beta}{\alpha-\beta}$, then*

$$C_{(2)} \simeq \left(\int_a^b \left(\int_x^b z \right)^{\frac{\gamma}{\beta}} \left(\int_a^x u^{1-\alpha'} \right)^{\frac{\gamma}{\beta'}} u^{1-\alpha'}(x)\,\mathrm{d}x \right)^{\frac{1}{\gamma}} \simeq \left(\int_a^b \left(\int_x^b z \right)^{\frac{\gamma}{\alpha}} \left(\int_a^x u^{1-\alpha'} \right)^{\frac{\gamma}{\alpha'}} z(x)\,\mathrm{d}x \right)^{\frac{1}{\gamma}}.$$

Theorem 1.2. *Let u, z be weights. For $\alpha, \beta \in (1,\infty)$ set*

$$C_{(3)} := \sup_{f \in \mathscr{M}_+} \left(\int_a^b \left(\int_t^b f \right)^{\beta} z(t)\,\mathrm{d}t \right)^{\frac{1}{\beta}} \left(\int_a^b f^{\alpha} u \right)^{-\frac{1}{\alpha}}. \tag{3}$$

Then

(i) *If $1 < \alpha \le \beta < \infty$, then*

$$C_{(3)} \simeq \sup_{a<x<b} \left(\int_a^x z \right)^{\frac{1}{\beta}} \left(\int_x^b u^{1-\alpha'} \right)^{\frac{1}{\alpha'}}.$$

(ii) *If $1 < \beta < \alpha < \infty$ and $\gamma := \frac{\alpha\beta}{\alpha-\beta}$, then*

$$C_{(3)} \simeq \left(\int_a^b \left(\int_a^x z \right)^{\frac{\gamma}{\beta}} \left(\int_x^b u^{1-\alpha'} \right)^{\frac{\gamma}{\beta'}} u^{1-\alpha'}(x)\,\mathrm{d}x \right)^{\frac{1}{\gamma}} \simeq \left(\int_a^b \left(\int_a^x z \right)^{\frac{\gamma}{\alpha}} \left(\int_x^b u^{1-\alpha'} \right)^{\frac{\gamma}{\alpha'}} z(x)\,\mathrm{d}x \right)^{\frac{1}{\gamma}}.$$

In both these cases, as well as further on, we will use the conventions "$\frac{1}{0} := \infty$", "$\frac{1}{\infty} := 0$", "$0.\infty := 0$". Observe that then the two preceding theorems are indeed true even for weights with zero value on a set of nonzero measure. In particular, we may use them for a weight w such that $w = w\chi_{(c,b)}$ for some $c \in (a,b)$. This formal detail will be used at a certain point.

Notice also the two equivalent conditions in each of the (ii)-cases. Existence of such alternative conditions is a common feature in weighted Hardy-type inequalities. Often it proves to be useful to find such equivalent expressions since each of them may be applicable in different particular situations.

Let us now consider the *bilinear Hardy operator* H_2, acting on $\mathscr{M}_+ \times \mathscr{M}_+$ and defined by

$$H_2(f,g)(t) := \int_a^t f(s)\,\mathrm{d}s \int_a^t g(s)\,\mathrm{d}s, \qquad t \in (a,b).$$

Recently, Aguilar, Ortega and Ramírez [1] characterized the boundedness $H_2 : L^{p_1}(v_1) \times L^{p_2}(v_2) \to L^q(w)$, or, equivalently, the validity of the *bilinear weighted Hardy inequality*

$$\text{(4)} \qquad \left(\int_a^b \left(\int_0^t f \right)^q \left(\int_0^t g \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \leq C \left(\int_a^b f^{p_1} v_1 \right)^{\frac{1}{p_1}} \left(\int_a^b f^{p_2} v_2 \right)^{\frac{1}{p_2}}$$

for all f, $g \in \mathscr{M}_+$. The range of exponents was $1 < p, q < \infty$. To prove these results, the authors used the discretization technique, a standard yet technical method which proves to be rather unnecessarily complicated in this case.

In this article, we first present a much easier proof of the characterization of (4). In most cases we also manage to reduce the number of conditions, compared to those of [1]. Our proof technique will be refered to as to the “iteration method”. The idea is to proceed simply in two steps, each time treating the problem as the ordinary Hardy inequality (1). Especially in the “easy case” p_1, $p_2 \leq q$ the proof becomes extremely simple. Let us note that the same idea was also used in [12] to characterize the bilinear Hardy inequality for decreasing functions.

Having proved the aforementioned characterizations of (4) in Section 2, we then continue by providing more alternative conditions. Existence of equivalent conditions is a common feature of weighted inequalities, although it was not observed in [1].

Fairly obviously, the iteration method is not limited just to the bilinear case and the case of Hardy operator. Hence, in the final part we present more applications of this technique to a variety of problems involving other operators.

As a final remark in this introduction, let us just recall the following duality property of the $L^p(v)$-spaces. Namely, if $p \in (1,\infty)$ and v is a weight, then for any $f \in \mathscr{M}_+$ it holds

$$\text{(5)} \qquad \left(\int_0^\infty f^p(x) v(x)\,\mathrm{d}x \right)^{\frac{1}{p}} = \sup_{h \in \mathscr{M}_+} \frac{\int_0^\infty f(x)h(x)\,\mathrm{d}x}{\left(\int_0^\infty h^{p'}(x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{1}{p'}}}.$$

2. BILINEAR WEIGHTED HARDY INEQUALITY

Using the iteration method, in this part we characterize the quantity

$$(6)\quad C_{(6)} := \sup_{f,g\in\mathscr{M}_+} \left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_a^b f^{p_1} v_1\right)^{-\frac{1}{p_1}} \left(\int_a^b g^{p_2} v_2\right)^{-\frac{1}{p_2}},$$

which is the optimal constant C in the inequality (4). The following notation we be used from now on: $F \lesssim G$ means that there exists a constant $C \in (0,\infty)$ such that $F \le CG$ and C is "independent of relevant quantities in F and G". More precisely, in this paper this constant C depends always only on the exponents p, p_1, p_2, q. If $F \lesssim G$ and $G \lesssim F$, we write $F \simeq G$.

We will provide such conditions A that $C_{(6)} \simeq A$, without explicit estimates on the constants D_1, D_2 such that $D_1 A \le C_{(6)} \le D_2 A$. An exact calculation of these constants is left to the interested reader.

Theorem 2.1. *Let v_1, v_2, w be weights, $1 < p_1, p_2, q < \infty$, $p_1 \le q$, $p_2 \le q$. Then $C_{(6)} \simeq A_{(7)}$, where*

$$(7)\qquad A_{(7)} := \sup_{a<x<b} \left(\int_x^b w\right)^{\frac{1}{q}} \left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_a^x v_2^{1-p_2'}\right)^{\frac{1}{p_2'}}.$$

Proof. It holds

$$\begin{aligned}
C_{(6)} &= \sup_{g\in\mathscr{M}_+} \sup_{f\in\mathscr{M}_+} \left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_a^b f^{p_1} v_1\right)^{-\frac{1}{p_1}} \left(\int_a^b g^{p_2} v_2\right)^{-\frac{1}{p_2}} \\
(8)\quad &\simeq \sup_{g\in\mathscr{M}_+} \sup_{a<x<b} \left(\int_x^b \left(\int_a^y g\right)^q w(y)\,\mathrm{d}y\right)^{\frac{1}{q}} \left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_a^b g^{p_2} v_2\right)^{-\frac{1}{p_2}} \\
&= \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_{g\in\mathscr{M}_+} \left(\int_x^b \left(\int_a^y g\right)^q w(y)\,\mathrm{d}y\right)^{\frac{1}{q}} \left(\int_a^b g^{p_2} v_2\right)^{-\frac{1}{p_2}} \\
(9)\quad &\simeq \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_{x<y<b} \left(\int_y^b w\right)^{\frac{1}{q}} \left(\int_a^y v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \\
&= A_{(7)}.
\end{aligned}$$

Step (8) follows from Theorem 1.1(i) with the setting $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := \left(\int_a^t g\right)^q w(t)$. Step (9) follows from the same theorem with the setting $\alpha := p_2$, $\beta := q$, $u := v_2$, $z := \chi_{(x,b)} w$. $\square$

Theorem 2.2. *Let v_1, v_2, w be weights, $1 < p_1 \le q < p_2 < \infty$ and $r_2 := \frac{p_2 q}{p_2 - q}$. Then $C_{(6)} \simeq A_{(10)}$, where*

$$(10)\qquad A_{(10)} := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{1}{p_1{}'}} \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{p_2}} \left(\int_a^y v_2^{1-p_2{}'} \right)^{\frac{r_2}{p_2{}'}} w(y)\,\mathrm{d}y \right)^{\frac{1}{r_2}}.$$

Proof. In the same way as in Theorem 2.1, using Theorem 1.1(i) (with $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := \left(\int_a^t g\right)^q w(t)$) in the first step and Theorem 1.1(ii) (with $\alpha := p_2$, $\beta := q$, $u := v_2$, $z := \chi_{(x,b)} w$) in the second one, we get

$$C_{(6)} \simeq \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{1}{p_1{}'}} \sup_{g \in \mathscr{M}_+} \left(\int_x^b \left(\int_a^y g \right)^q w(y)\,\mathrm{d}y \right)^{\frac{1}{q}} \left(\int_a^b g^{p_2} v_2 \right)^{-\frac{1}{p_2}} \simeq A_{(10)}.$$

□

Theorem 2.3. *Let v_1, v_2, w be weights, $1 < q < p_i < \infty$, $r_i := \frac{p_i q}{p_i - q}$ for $i \in \{1,2\}$ and let $\frac{1}{q} \le \frac{1}{p_1} + \frac{1}{p_2}$. Then $C_{(6)} \simeq A_{(11)} + A_{(12)}$, where*

$$(11)\quad A_{(11)} := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{1}{p_1{}'}} \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{q}} \left(\int_a^y v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{1}{r_2}},$$

$$(12)\quad A_{(12)} := \sup_{a<x<b} \left(\int_a^x v_2^{1-p_2{}'} \right)^{\frac{1}{p_2{}'}} \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_1}{q}} \left(\int_a^y v_1^{1-p_1{}'} \right)^{\frac{r_1}{q'}} v_1^{1-p_1{}'}(y)\,\mathrm{d}y \right)^{\frac{1}{r_1}}.$$

Proof. We have

$$
\begin{aligned}
(13)\quad C_{(6)} &\simeq \sup_{g\in\mathscr{M}_+} \frac{\left(\int_a^b \left(\int_x^b \left(\int_a^y g\right)^q w(y)\,\mathrm{d}y\right)^{\frac{r_1}{q}} \left(\int_a^x v_1^{1-p_1{}'}\right)^{\frac{r_1}{q'}} v_1^{1-p_1{}'}(x)\,\mathrm{d}x\right)^{\frac{1}{r_1}}}{\left(\int_a^b g^{p_2} v_2\right)^{\frac{1}{p_2}}} \\
(14)\quad &= \sup_{h\in\mathscr{M}_+}\sup_{g\in\mathscr{M}_+} \frac{\left(\int_a^b \left(\int_a^y g\right)^q w(y)\int_a^y h(t)\,\mathrm{d}t\,\mathrm{d}y\right)^{\frac{1}{q}}}{\left(\int_a^b g_2^{p_2} v_2\right)^{\frac{1}{p_2}} \left(\int_a^b h^{\frac{p_1}{q}}(y)\left(\int_a^y v_1^{1-p_1{}'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1{}'}{r_1}}(y)\,\mathrm{d}y\right)^{\frac{1}{p_1}}} \\
(15)\quad &\simeq \sup_{h\in\mathscr{M}_+} \frac{\left(\int_a^b \left(\int_x^b w(y)\int_a^y h(t)\,\mathrm{d}t\,\mathrm{d}y\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2{}'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}}(y)\left(\int_a^y v_1^{1-p_1{}'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1{}'}{r_1}}(y)\,\mathrm{d}y\right)^{\frac{1}{p_1}}} \\
(16)\quad &= \sup_{h\in\mathscr{M}_+} \frac{\left(\int_a^b \left(\int_a^x h\int_x^b w + \int_x^b h(t)\int_t^b w(y)\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2{}'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}}(y)\left(\int_a^x v_1^{1-p_1{}'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1{}'}{r_1}}(y)\,\mathrm{d}y\right)^{\frac{1}{p_1}}} \\
&\simeq \sup_{h\in\mathscr{M}_+} \left[\frac{\left(\int_a^b \left(\int_a^x h\right)^{\frac{r_2}{q}} \left(\int_x^b w\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2{}'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\right)^{\frac{q}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}}(y)\left(\int_a^y v_1^{1-p_1{}'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1{}'}{r_1}}(y)\,\mathrm{d}y\right)^{\frac{q}{p_1}}}\right]^{\frac{1}{q}} \\
&\quad + \sup_{h\in\mathscr{M}_+} \left[\frac{\left(\int_a^b \left(\int_x^b h\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2{}'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\right)^{\frac{q}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}}(y)\left(\int_y^b w\right)^{-\frac{p_1}{q}} \left(\int_a^y v_1^{1-p_1{}'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1{}'}{r_1}}(y)\,\mathrm{d}y\right)^{\frac{q}{p_1}}}\right]^{\frac{1}{q}} \\
&=: B_1 + B_2.
\end{aligned}
$$

Here, step (13) follows by Theorem 1.1(i), setting $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := \left(\int_a^t g\right)^q w(t)$. Step (14) is due to duality, see (5). In (15) we use Theorem 1.1(i) with $\alpha := p_2$, $\beta := q$, $u := v_2$, $z(y) := w(y)\int_a^y h$. Next, (16) holds by the Fubini theorem. Finally, by Theorem 1.1(i), setting $\alpha := \frac{p_1}{q}$, $\beta := \frac{r_2}{q}$, $u(y) := \left(\int_a^y v_1^{1-p_1'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1'}{r_1}}(y)$, $z(x) := \left(\int_x^b w\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(x)$ we get $B_1 \simeq A_{(11)}$. Similarly, Theorem 1.2(i) with $\alpha := \frac{p_1}{q}$, $\beta := \frac{r_2}{q}$, $u(y) := \left(\int_x^b w\right)^{-\frac{p_1}{q}} \left(\int_a^y v_1^{1-p_1'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1'}{r_1}}(y)$, $z(x) := \left(\int_a^x v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(x)$ yields $B_2 \simeq A_{(12)}$. $\square$

Theorem 2.4. *Let v_1, v_2, w be weights, $1 < q < p_i < \infty$, $r_i := \frac{p_i q}{p_i - q}$ for $i \in \{1,2\}$ and let $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Let $\frac{1}{s} = \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$. Then $C_{(6)} \simeq A_{(17)} + A_{(18)}$, where*

$$(17)\quad A_{(17)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{q}} \left(\int_a^y v_2^{1-p_2'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(y)\,\mathrm{d}y \right)^{\frac{s}{r_2}} \left(\int_a^x v_1^{1-p_1'} \right)^{\frac{s}{r_2'}} v_1^{1-p_1'}(x)\,\mathrm{d}x \right)^{\frac{1}{s}},$$

$$(18)\quad A_{(18)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_1}{q}} \left(\int_a^y v_1^{1-p_1'} \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,\mathrm{d}y \right)^{\frac{s}{r_1}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x)\,\mathrm{d}x \right)^{\frac{1}{s}}.$$

Proof. As in the proof of Theorem 2.3, one has $C_{(6)} \simeq B_1 + B_2$, where B_1 and B_2 are defined as in there. Next, Theorem 1.1(ii) with $\alpha := \frac{p_1}{q}$, $\beta := \frac{r_2}{q}$, $u(y) := \left(\int_a^y v_1^{1-p_1'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1'}{r_1}}(y)$, $z(x) := \left(\int_x^b w\right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(x)$ gives $B_1 \simeq A_{(17)}$, and Theorem 1.2(ii) with $\alpha := \frac{p_1}{q}$, $\beta := \frac{r_2}{q}$, $u(y) := \left(\int_x^b w\right)^{-\frac{p_1}{q}} \left(\int_a^y v_1^{1-p_1'}\right)^{-\frac{p_1}{q'}} v_1^{\frac{p_1'}{r_1}}(y)$, $z(x) := \left(\int_a^x v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(x)$ gives $B_2 \simeq A_{(18)}$. $\square$

3. EQUIVALENT CONDITIONS

The "A-conditions" from the previous section have more equivalent forms. This can be observed simply by comparing the conditions we obtained with those from [1]. We are going to make this comparison and even to prove the equivalences of the conditions directly.

Proposition 3.1. *In the setting from Theorem 2.2, it holds* $A_{(10)} \simeq A_{(7)} + A_{(12)}$.

Proof. For all $x \in (a,b)$ integration by parts (cf. [17, Lemma, p. 176]) yields

$$\left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{p_2}} \left(\int_a^y v_2^{1-p_2'} \right)^{\frac{r_2}{p_2'}} w(y)\,\mathrm{d}y \right)^{\frac{1}{r_2}}$$

$$\simeq \left(\int_x^b w \right)^{\frac{1}{q}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{1}{p_2'}} + \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{q}} \left(\int_a^y v_2^{1-p_2'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(y)\,\mathrm{d}y \right)^{\frac{1}{r_2}}.$$

Multiplying both sides by $\left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}$ we show that $A_{(10)} \simeq A_{(7)} + A_{(12)}$ holds even pointwise, i.e. without the supremum over x. □

Proposition 3.2. *In the setting from Theorem 2.3, it holds*

$$A_{(11)} + A_{(12)} \simeq A_{(7)} + A_{(11)} + A_{(12)} \simeq A_{(10)} + A^*_{(12)}, \tag{19}$$

where

$$A^*_{(12)} := \sup_{a<x<b} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{1}{p_2'}} \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_1}{p_1}} \left(\int_a^y v_1^{1-p_1'} \right)^{\frac{r_1}{p_1'}} w(y)\,\mathrm{d}y \right)^{\frac{1}{r_1}}.$$

Proof. The second equivalence in (19) holds pointwise for $x \in (a,b)$ by partial integration. The fact that we proved $C_{(6)} \simeq A_{(11)} + A_{(12)}$, while in [1, Theorem 3] it was proved that $C_{(6)} \simeq A_{(7)} + A_{(11)} + A_{(12)}$ gives an indirect proof of the first equivalence in (19).

A simple direct proof of the inequality $A_{(7)} \lesssim A_{(11)} + A_{(12)}$ can be obtained by employing the idea from [6, Lemma 2.2]. It goes as follows. For each $x \in (a,b)$ exists $y(x) \in (a,x)$ such that

$$\int_a^{y(x)} v_1^{1-p_1'} = \int_{y(x)}^x v_1^{1-p_1'} = \frac{1}{2} \int_a^x v_1^{1-p_1'}.$$

Now we get

$$
\begin{aligned}
&\left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_a^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_a^x v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \simeq \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_a^{y(x)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_a^x v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \\
&= \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_a^{y(x)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_a^{y(x)} v_2^{1-p_2'} + \int_{y(x)}^x v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \\
&\simeq \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_{y(x)}^x v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_a^{y(x)} v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} + \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_a^{y(x)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_{y(x)}^x v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \\
&\simeq \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_{y(x)}^x \left(\int_{y(x)}^t v_1^{1-p_1'}\right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(t)\,\mathrm{d}t\right)^{\frac{1}{r_1}}\left(\int_a^{y(x)} v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \\
&+ \left(\int_x^b w\right)^{\frac{1}{q}}\left(\int_a^{y(x)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_{y(x)}^x \left(\int_{y(x)}^t v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(t)\,\mathrm{d}t\right)^{\frac{1}{r_2}} \\
&\le \left(\int_a^{y(x)} v_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\left(\int_{y(x)}^x \left(\int_t^b w\right)^{\frac{r_1}{q}}\left(\int_a^t v_1^{1-p_1'}\right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(t)\,\mathrm{d}t\right)^{\frac{1}{r_1}} \\
&\le \left(\int_a^{y(x)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_{y(x)}^x \left(\int_t^b w\right)^{\frac{r_2}{q}}\left(\int_a^t v_2^{1-p_2'}\right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(t)\,\mathrm{d}t\right)^{\frac{1}{r_2}} \\
&\le A_{(12)} + A_{(11)}.
\end{aligned}
$$

Taking the supremum over $x \in (a,b)$, we obtain $A_{(7)} \lesssim A_{(11)} + A_{(12)}$. Observe that this inequality does not hold pointwise in x, rather only with the supremum. □

Proposition 3.3. *In the setting from Theorem 2.4, it holds*

$$
A_{(17)} + A_{(18)} \simeq A^* + A_{(17)} + A_{(18)}, \tag{20}
$$

where

$$
A^* := \left(\int_a^b \left(\int_x^b w\right)^{\frac{s}{p_1}+\frac{s}{p_2}} w(x)\left(\int_a^x v_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_a^x v_2^{1-p_2'}\right)^{\frac{s}{p_2'}}\,\mathrm{d}x\right)^{\frac{1}{s}}.
$$

Moreover, it holds $A_{(18)} \simeq A^*_{(18)}$, *where*

$$A^*_{(17)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w \right)^{\frac{r_2}{q}} \left(\int_a^y v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \right.$$
$$\left. \times \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}} \left(\int_x^b w \right)^{\frac{r_2}{q}} \left(\int_a^x v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \right)^{\frac{1}{s}},$$

and $A_{(18)} \simeq A^*_{(18)}$, *where* $A^*_{(18)}$ *is an analogy to* $A^*_{(17)}$ *with the indices* 1 *and* 2 *switched.*

Proof. The equivalence $A_{(17)} \simeq A^*_{(17)}$ follows directly by integration by parts. Theorem 2.4 yields $C_{(6)} \simeq A_{(17)} + A_{(18)}$, while [1, Theorem 4] gives $C_{(6)} \simeq A^* + A_{(17)} + A_{(18)}$, hence (20) is true.

However, we will as well provide a direct proof of (20). Obviously, we need just to prove that $A^* \lesssim A_{(17)} + A_{(18)}$. At first, integrating by parts we get

$$\begin{aligned}
(A^*)^s \simeq & \int_a^b \left(\int_x^b w \right)^{\frac{s}{q}} \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}} \left(\int_a^x v_2^{1-p_2{}'} \right)^{\frac{s}{r_1{}'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\
& + \int_a^b \left(\int_x^b w \right)^{\frac{s}{q}} \left(\int_a^x v_2^{1-p_2{}'} \right)^{\frac{s}{p_2{}'}} \left(\int_a^x v_1^{1-p_1{}'} \right)^{\frac{s}{r_2{}'}} v_1^{1-p_1{}'}(x)\,\mathrm{d}x \\
& =: B_3 + B_4.
\end{aligned}$$

Now we prove $B_3 \lesssim A_{(17)} + A^*_{(18)}$. The idea resembles the one of [7, Theorem 3.1]. We may suppose that for all $\varepsilon \in (0, b-a)$ it holds $\int_a^{a+\varepsilon} v_2^{1-p_2{}'} < \infty$, otherwise all the terms B_3, $A_{(17)}$, $A^*_{(18)}$ become infinite. We also assume that $\int_a^b v_2^{1-p_2{}'} = \infty$ (if this is not satisfied, then the following part of the proof needs only minor changes). Now, for $k \in \mathbb{Z}$ let $x_k \in (a,b)$ be such that $\int_a^{x_k} v_2^{1-p_2{}'} = 2^k$, and let $y_k \in [x_k, x_{k+1}]$ be such that

$$\sup_{y \in [x_k, x_{k+1}]} \left(\int_y^b w \right)^{\frac{s}{q}} \left(\int_a^y v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}} = \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_a^{y_k} v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}}.$$

Now we can write

$$\begin{aligned}
B_3 &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x) \left(\int_x^b w \right)^{\frac{s}{q}} \left(\int_a^x v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \mathrm{d}x \\
&\le \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x)\,\mathrm{d}x \sup_{y\in[x_k,x_{k+1}]} \left(\int_y^b w \right)^{\frac{s}{q}} \left(\int_a^y v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{ks}{p_2'}} \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_a^{y_k} v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \\
&\simeq \sum_{k\in\mathbb{Z}} 2^{\frac{ks}{p_2'}} \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_{y_{k-4}}^{y_k} v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} + \sum_{k\in\mathbb{Z}} 2^{\frac{ks}{p_2'}} \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_a^{y_{k-4}} v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \\
&=: B_5 + B_6.
\end{aligned}$$

Observe that for all $k\in\mathbb{Z}$ it holds

$$2^k \le \int_a^{y_k} v_2^{1-p_2'} \le 2^{k+1}, \quad 2^{k-1} \le \int_{y_{k-2}}^{y_k} v_2^{1-p_2'} \le 2^{k+1}.$$

Hence,

$$\begin{aligned}
B_5 &\lesssim \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x)\,\mathrm{d}x \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_{y_{k-4}}^{y_k} v_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \\
&\simeq \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x)\,\mathrm{d}x \left(\int_{y_k}^b w \right)^{\frac{s}{q}} \left(\int_{y_{k-4}}^{y_k} \left(\int_{y_{k-4}}^y v_1^{1-p_1'} \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,\mathrm{d}y \right)^{\frac{s}{r_1}} \\
&\le \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x)\,\mathrm{d}x \left(\int_{y_{k-4}}^{y_k} \left(\int_y^b w \right)^{\frac{r_1}{q}} \left(\int_a^y v_1^{1-p_1'} \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,\mathrm{d}y \right)^{\frac{s}{r_1}} \\
&\le \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \left(\int_a^x v_2^{1-p_2'} \right)^{\frac{s}{r_1'}} v_2^{1-p_2'}(x) \left(\int_x^{y_k} \left(\int_y^b w \right)^{\frac{r_1}{q}} \left(\int_a^y v_1^{1-p_1'} \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,\mathrm{d}y \right)^{\frac{s}{r_1}} \mathrm{d}x \\
&\le 2(A_{(18)})^s.
\end{aligned}$$

Next, we have to estimate B_6. At first, for any $k \in \mathbb{Z}$ it holds

$$
\begin{aligned}
2^{\frac{ks}{p_2{}'}} &\lesssim \int_{y_{k-4}}^{y_{k-2}} \left(\int_{y_{k-4}}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \left(\int_{y_{k-2}}^{y_k} \left(\int_{y_{k-2}}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \\
&\le \int_{y_{k-4}}^{y_{k-2}} \left(\int_{y_{k-2}}^{y_k} \left(\int_{a}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \left(\int_{a}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\
&\le \int_{y_{k-4}}^{y_{k-2}} \left(\int_{x}^{y_k} \left(\int_{a}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \left(\int_{a}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\
&\le \int_{y_{k-4}}^{y_k} \left(\int_{x}^{b} \left(\int_{a}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \left(\int_{a}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
B_6 &\lesssim \sum_{k\in\mathbb{Z}} \int_{y_{k-4}}^{y_k} \left(\int_{x}^{b} \left(\int_{a}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \left(\int_{a}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\
&\quad \times \left(\int_{y_k}^{b} w \right)^{\frac{s}{q}} \left(\int_{a}^{y_{k-4}} v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}} \\
&\le \sum_{k\in\mathbb{Z}} \int_{y_{k-4}}^{y_k} \left(\int_{x}^{b} \left(\int_{y}^{b} w \right)^{\frac{r_2}{q}} \left(\int_{a}^{y} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{\frac{s}{p_1}} \\
&\quad \times \left(\int_{x}^{b} w \right)^{\frac{r_2}{q}} \left(\int_{a}^{x} v_2^{1-p_2{}'} \right)^{\frac{r_2}{q'}} v_2^{1-p_2{}'}(x) \left(\int_{a}^{x} v_1^{1-p_1{}'} \right)^{\frac{s}{p_1{}'}} \mathrm{d}x \\
&\le 4(A^*_{(17)})^s.
\end{aligned}
$$

At this point we have proved $(B_3)^{\frac{1}{s}} \lesssim A_{(17)} + A^*_{(18)}$. Exactly in the same way, only switching the indices 1 and 2, one proves $(B_4)^{\frac{1}{s}} \lesssim A_{(18)} + A^*_{(17)}$. Using all the estimates we collected, we get $A^* \simeq (B_3)^{\frac{1}{s}} + (B_4)^{\frac{1}{s}} \lesssim A_{(17)} + A_{(18)} + A^*_{(17)} + A^*_{(18)} \simeq A_{(17)} + A_{(18)}$, which we wanted to show. □

4. FURTHER RESULTS

In this final part we show examples of various further problems, which may be successfully treated by the iteration method.

The following notation will be used: Unless specified otherwise, $\mathscr{M}$ denotes the cone of all (extended) real-valued measurable functions on a suitable measure space $(\mathscr{R},\mu)$. For $f\in\mathscr{M}$, the symbol f^* denotes the *nonincreasing rearrangement* of f, and $f^{**}(t):=\frac{1}{t}\int_0^t f^*$ for $t\in(0,\mu(\mathscr{R}))$ (see [3] for details). If u is a weight on $(0,\mu(\mathscr{R}))$, then we define $f_u^{**}(t):=\left(\int_0^t u\right)^{-1}\int_0^t f^*u$. For definitions of *rearrangement-invariant (r.i.) spaces* and *r.i. lattices*, see e.g. [3, 4, 9].

If $0<p<\infty$ and u,v are weights on $(0,\mu(\mathscr{R}))$, the *weighted Lorentz "spaces"* $\Lambda^p(v)$, $\Gamma^p(v)$ and $\Gamma_u^p(v)$ are defined as follows.

$$\Lambda^p(v):=\left\{f\in\mathscr{M};\ \|f\|_{\Lambda^p(v)}:=\|f^*\|_{L^p(v)}<\infty\right\},$$
$$\Gamma^p(v):=\left\{f\in\mathscr{M};\ \|f\|_{\Gamma^p(v)}:=\|f^{**}\|_{L^p(v)}<\infty\right\},$$
$$\Gamma_u^p(v):=\left\{f\in\mathscr{M};\ \|f\|_{\Gamma_u^p(v)}:=\|f_u^{**}\|_{L^p(v)}<\infty\right\}.$$

In here, of course, the $L^p(v)$-space consists of functions over $(0,\mu(\mathscr{R}))$.

If X,Y are r.i. spaces (lattices), we say that X *is embedded into* Y and write $X\hookrightarrow Y$, if there exists $C\in(0,\infty)$ such that for all $f\in X$ it holds $\|f\|_Y\le C\|f\|_X$.

4.1. **Multilinear Hardy operator.** The iteration method may be obviously extended for a multilinear Hardy operator H_n defined by

$$H_n(f_1,\dots,f_n)(t):=\prod_{i=1}^n H_1f_i(t)$$

for $f_i\in\mathscr{M}_+$, $i=1,\dots,n$, and $t\in(a,b)$. In this case we obtain the following recursive formula for the norm of H_n:

$$\begin{aligned}
&\|H_n\|_{L^{p_1}(v_1)\times\cdots\times L^{p_n}(v_n)\to L^q(w)}\\
&=\sup_{\substack{f_i\in\mathscr{M}_+\\ i=1,\dots,n}}\frac{\left(\int_a^b\left(H_{n-1}(f_1,\dots,f_{n-1})(t)\right)^q\left(H_1f_n(t)\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}}{\prod_{i=1}^{n-1}\|f_i\|_{L^{p_i}(v_i)}\|f_n\|_{L^{p_n}(v_n)}}\\
&=\sup_{f_n\in\mathscr{M}_+}\frac{\|H_{n-1}\|_{L^{p_1}(v_1)\times\cdots\times L^{p_{n-1}}(v_{n-1})\to L^q(w(H_1f_n)^q)}}{\|f_n\|_{L^{p_n}(v_n)}}.
\end{aligned}$$

In this way one can deduce the conditions on the weights and exponents under which $H_n:L^{p_1}(v_1)\times\cdots\times L^{p_n}(v_n)\to L^q(w)$, using only the knowledge of the conditions for $H_1:L^p(v)\to L^q(w)$. During the process there is no need for a method harder than changing the order of suprema, Fubini theorem and L^p-duality.

4.2. **Other product-based operators.** Clearly, the idea above applies to any operator T such that

$$T(f_1,\dots,f_n)=\prod_{i=1}^{n} T_i f_i, \tag{21}$$

where T_i are certain other operators. Using the iteration method, we might be able to get conditions for boundedness $T: X_1\times\cdots\times X_n \to X$ from the conditions for $T_i : Y_i \to Z_i$, where X, X_i, Y_i, Z_i are some suitable spaces (or even more general structures, e.g. r.i. lattices). Simple examples of such operators T include products of the "dual Hardy" operators, or products of a mixture of Hardy, "dual Hardy" operators, Hardy-type integral or supremal operators with kernels etc.

4.3. **"Multidimensional" Hardy operators involving nonincreasing rearrangement.** Let K be a weight (kernel). Define the Hardy-type operator $\mathscr{H}_{1,K}$ and its "dual version" $\mathscr{H}'_{1,K}$ by

$$\mathscr{H}_{1,K}f(t):=\int_0^t f^*(s)K(s)\,\mathrm{d}s, \qquad \mathscr{H}'_{1,K}f(t):=\int_t^\infty f^*(s)K(s)\,\mathrm{d}s$$

for any $f\in\mathscr{M}$. If $K\equiv 1$, we write just $\mathscr{H}_1 := \mathscr{H}_{1,K}$ and $\mathscr{H}'_1 := \mathscr{H}'_{1,K}$. Let us note that these operators are in general not linear.

Consider the operator $\mathscr{H}_2$ constructed as

$$\mathscr{H}_2(f,g)(t):=\mathscr{H}_1 f(t)\,\mathscr{H}_1 g(t)=\int_0^t f^*(s)\,\mathrm{d}s\int_0^t g^*(s)\,\mathrm{d}s.$$

This operator is obviously a special case of T from (21). In [12], the iteration method was used to characterize boundedness $\mathscr{H}_2 : \Lambda^{p_1}(v_1)\times\Lambda^{p_2}(v_2)\to L^q(w)$, i.e. to produce weighted bilinear Hardy inequalities for nonincreasing functions.

Let us take yet another Hardy-type operator $\widetilde{\mathscr{H}_2}$, defined by

$$\widetilde{\mathscr{H}_2}(f,g)(t):=\int_0^t f^*(s)g^*(s)\,\mathrm{d}s,$$

and study its boundedness $\widetilde{\mathscr{H}_2} : \Lambda^{p_1}(v_1)\times\Lambda^{p_2}(v_2)\to L^q(w)$. (The same idea may be used if the Λ-spaces are replaced by other appropriate structures.) Observe that $\widetilde{\mathscr{H}_2}(f,g)(t)=\mathscr{H}_{1,g^*}(f)$. We get

$$\begin{aligned} ||\widetilde{\mathscr{H}_2}||_{\Lambda^{p_1}(v_1)\times\Lambda^{p_2}(v_2)\to L^q(w)} &= \sup_{g\in\mathscr{M}}\frac{1}{||g||_{\Lambda^{p_2}(v_2)}}\sup_{f\in\mathscr{M}}\frac{\left\|\int_0^\bullet f^*g^*\right\|_{L^q(w)}}{||f||_{\Lambda^{p_1}(v_1)}} \\ &= \sup_{g\in\mathscr{M}}\frac{||id||_{\Lambda^{p_1}(v_1)\to\Gamma^q_{g^*}(\psi)}}{||g||_{\Lambda^{p_2}(v_2)}}. \end{aligned} \tag{22}$$

Here $\psi(t):=w(t)\left(\int_0^t g^*\right)^q$. We may now use the known characterization of the embedding $\Lambda^{p_1}(v_1)\hookrightarrow\Gamma^q_{g^*}(\psi)$ (see e.g. [5]). This embedding is also, in other words, equivalent to the $\Lambda^{p_1}(v_1)\to L^q(w)$ boundedness of the operator $\mathscr{H}_{1,g^*}$.

Anyway, the optimal constant $\|id\|_{\Lambda^{p_1}(v_1)\to\Gamma^q_{g^*}(\psi)}$ usually takes a form of a sum of the $L^\alpha(\varphi)$-norms of $\mathscr{H}_{1,K}(g)$, $\mathscr{H}'_{1,K}(g)$ or supremal variants of these operators. Here K, α and φ depend on the original parameters p, q, v_1, v_2, w. Hence, in the next phase, (22) will dissolve into a sum of factors

$$\sup_{g\in\mathscr{M}} \frac{\left\|\mathscr{H}_{1,K}(g)\right\|_{L^\alpha(\varphi)}}{\|g\|_{\Lambda^{p_2}(v_2)}},$$

or similar ones. Then we again use suitable existing characterizations of boundedness of $\mathscr{H}_{1,K}$, $\mathscr{H}'_{1,K}$ or, if needed, some supremal variants of those operators. In this way, the desired estimate on $\|\widetilde{\mathscr{H}_2}\|_{\Lambda^{p_1}(v_1)\times\Lambda^{p_2}(v_2)\to L^q(w)}$ will be obtained. The required boundedness characterizations for $\mathscr{H}'_{1,K}$ may be found in [8]. Corresponding conditions for other Hardy-type operators (e.g. the supremal ones) may be derived using the reduction theorems presented in [8]. The boundedness conditions for $\mathscr{H}_{1,K}$ are, as we already mentioned once, listed in [5].

In a similar way, higher-order operators like $\mathscr{H}_n$, $\widetilde{\mathscr{H}_n}$, etc., constructed analogously to their $n=2$ cases, may be treated. It is, however, worth noting that the complexity of the involved expressions grows rapidly with increasing n. Proofs involving general-weight cases using the iteration method may thus become very technical.

4.4. **General product-type operator in a Γ-space.** Let, for simplicity, $\mathscr{M}$ denote the cone of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$. Motivated by [16], we now consider an arbitrary operator P mapping $\mathscr{M}\times\mathscr{M}$ into $\mathscr{M}$ and such that the inequality

$$\int_0^t (P(f,g))^*(s)\,\mathrm{d}s \le \int_0^t f^*(s)g^*(s)\,\mathrm{d}s \tag{23}$$

holds for all f, $g\in\mathscr{M}$ and $t>0$. The simplest example of such operator is the ordinary product operator $P(f,g):=fg$ (see [3, p. 88]).

Let X_1, X_2 be r.i. spaces (or lattices) of functions defined over $\mathbb{R}^n$. It is now easy to find conditions for the boundedness $P: X_1\times X_2\to\Gamma^q(w)$. By (23), one gets

$$C_{(24)} := \sup_{f,g\in\mathscr{M}} \frac{\|P(f,g)\|_{\Gamma^q(w)}}{\|f\|_{X_1}\|g\|_{X_2}} \le \sup_{f,g\in\mathscr{M}} \frac{\|\widetilde{\mathscr{H}_2}(f,g)\|_{L^q(t\mapsto t^{-q}w(t))}}{\|f\|_{X_1}\|g\|_{X_2}}. \tag{24}$$

The problem of finding an upper bound for $C_{(24)}$ hence reduces into a certain boundedness question regarding the operator $\widetilde{\mathscr{H}_2}$, which was treated in the previous section.

The possibility of providing a lower bound for $C_{(24)}$ depends to a great extent on the “sharpness” of (23). Let us here, for example, consider the simple operator $P(f,g):=fg$. It may be checked easily that if both f and g are positive and radially decreasing, then $\int_0^t (fg)^* = \int_0^t f^*g^*$, and therefore equality in (23) is attained for these functions. This in turn implies that the two suprema in (24) are equal. The substantial facts here are that X_1 and X_2 are r.i., and that every $f \in \mathscr{M}_i$ may be rearranged into a positive (nonnegative) radially decreasing (non-increasing) function $h \in \mathscr{M}_i$ such that $f^* \equiv h^*$. For details of these ideas we refer to [9, 10, 11].

A general product operator may be also defined in another way, as suggested by O'Neil in [16]. See the final remark in the section below for more details.

4.5. **Convolution in a Γ-space.** Again, let $\mathscr{M}$ stand for the cone of Lebesgue-measurable real-valued functions on $\mathbb{R}^n$. The convolution of $f \in \mathscr{M}$ and $g \in \mathscr{M}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x-y)\,\mathrm{d}y. \tag{25}$$

As shown in [16], the bilinear operator $T(f,g) := f * g$ satisfies the O'Neil convolution inequality

$$(T(f,g))^{**}(t) \le \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \tag{26}$$

for all f, $g \in \mathscr{M}$ and all $t > 0$. Moreover, in case of both f and g being positive and radially decreasing, the reverse inequality holds with a constant depending only on the dimension n (see [16, 9, 11]). Observe that the right-hand side of (26) is again composed of certain Hardy-type operators acting on f, g.

In the papers [9, 10, 11], the following problem was studied: Given that X is one of the spaces $\Lambda^p(v)$, $\Gamma^p(v)$ or the class $S^p(v)$ (see [10]), characterize the largest r.i. space Y such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \le C \|f\|_X \|g\|_Y$$

holds for all f, $g \in \mathscr{M}$. In particular, an r.i. space Y was found such that for every positive radially decreasing g it holds

$$\sup_{f \in \mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_X} \simeq \|g\|_Y. \tag{27}$$

In all the cases $X = \Lambda^p(v)$, $\Gamma^p(v)$, $S^p(v)$ it turns out that this (quasi-)norm $\|\cdot\|_Y$ may be expressed as $\|\cdot\|_Y \simeq \|\cdot\|_{Y_1} + \|\cdot\|_{Y_2}$ with Y_1 being a Γ-type space and Y_2 a K-type space. The latter type was defined in [9].

A related problem, which may be successfully approached using the iteration method and the above results, is stated as follows. Under which conditions does the inequality $\|f * g\|_{\Gamma^q(w)} \le C \|f\|_{\Lambda^{p_1}(v_1)} \|g\|_{\Lambda^{p_2}(v_2)}$ hold for all f, $g \in \mathscr{M}$? In other words, one is being asked for a characterization of

$$\sup_{f,g \in \mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_{X_1} \|g\|_{X_2}}, \tag{28}$$

where $X_1 = \Lambda^{p_1}(v_1)$ and $X_2 = \Lambda^{p_2}(v_2)$. In view of (27), we proceed as follows:

$$\sup_{g\in\mathscr{M}} \frac{\|g\|_Y}{\|g\|_{X_2}} = \sup_{\substack{g\in\mathscr{M} \\ g \text{ pos. rad. dec.}}} \frac{\|g\|_Y}{\|g\|_{X_2}} \simeq \sup_{f,g\in\mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_{X_1}\|g\|_{X_2}}.$$

(Notice that X_2, Y are r.i., thus the first two terms are indeed equal.) Since we know that in this case "$\|\cdot\|_Y \simeq \|\cdot\|_\Gamma + \|\cdot\|_K$", the problem is reduced into finding the optimal constants for certain embeddings $\Lambda \hookrightarrow \Gamma$ and $\Lambda \hookrightarrow K$. Characterizations of $\Lambda \hookrightarrow \Gamma$ are well known (see e.g. [4, 5]), the problem of $\Lambda \hookrightarrow K$ was studied in [12].

The same strategy may be used if we choose X_1, X_2 in (28) as any other combination of Λ, Γ or S, or even as other r.i. spaces.

Moreover, in [16] O'Neil proposed a fairly general definition of a convolution operator as a bilinear operator T satisfying

$$\begin{aligned} \|T(f,g)\|_1 &\le \|f\|_1\|g\|_1, \\ \|T(f,g)\|_\infty &\le \|f\|_\infty\|g\|_1, \\ \|T(f,g)\|_\infty &\le \|f\|_1\|g\|_\infty. \end{aligned} \tag{29}$$

He then attempted to prove that a bilinear operator is a convolution operator in this sense if and only if it satisfies (26) for all f, g. However, as pointed out by Yap [18], O'Neil's proof of this statement contains a minor flaw and it seems that it cannot be fixed without some additional assumptions on T. For example, assuming that

$$\begin{aligned} &T \text{ maps pairs of positive functions into a positive function,} \\ &\forall f, f_n, g \ge 0:\ [f_n \uparrow f \text{ a.e.} \Rightarrow T(f_n,g) \uparrow T(f,g) \text{ a.e.}], \end{aligned} \tag{30}$$

should overcome the problem. Despite these problems with technical details, O'Neil's proof idea is correct for the ordinary convolution operator (25), which indeed satisfies (26).

Anyway, our technique of estimating (28) works for any bilinear operator satisfying the inequality (26). Thus, it also applies to the class of operators satisfying the interpolation inequalities (29) and the additional conditions (30).

Besides this, O'Neil as well suggested a definition of a general product operator P by means of conditions analogous to (29) (see [16]). For such operators the inequality (23) plays a similar role as (26) does for the general convolution operators. Again it seems that assuming conditions like (30) is necessary to prove that this general product operator satisfies (23). That is why we defined the "product operator" in the previous section by (23) and not in O'Neil's style by some interpolation-type inequalities. As in the case of convolution operators, we may still choose the latter approach with some careful corrections.

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Paper VI

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Integral conditions for Hardy-type operators involving suprema

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INTEGRAL CONDITIONS FOR HARDY-TYPE OPERATORS INVOLVING SUPREMA

MARTIN KŘEPELA

ABSTRACT. We characterize the validity of the weighted inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)\int_s^\infty g(x)\,\mathrm{d}x\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

for all nonnegative functions g on $(0,\infty)$, with exponents in the range $1\le p<\infty$ and $0<q<\infty$.

Moreover, we give an integral characterization of the inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)f(s)\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

being satisfied for all nonnegative nonincreasing functions f on $(0,\infty)$ in the case
$0<q<p<\infty$, for which an integral condition was previously unknown.

1. INTRODUCTION

In this paper we study the *supremal Hardy-type operators* R_u and S_u defined, for a nonnegative measurable function f on $(0,\infty)$, by

$$R_uf(t) := \sup_{s\in[t,\infty)} u(s)f(s), \qquad t>0,$$

and

$$S_uf(t) := \sup_{s\in[t,\infty)} u(s)\int_s^\infty f(x)\,\mathrm{d}x, \qquad t>0,$$

where u is a fixed continuous weight on $(0,\infty)$. The first goal is to characterize boundedness of the operator S_u between weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ (see Section 2 for the definitions). That is, to provide necessary and sufficient conditions for the inequality

$$\|S_u g\|_{L^q(w)} \le C\|g\|_{L^p(v)} \tag{1}$$

to hold for all nonnegative measurable functions g on $(0,\infty)$. We do this for the range of exponents $p\in[1,\infty)$ and $q\in(0,\infty)$.

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Our second goal is to determine when an analogous inequality holds for the operator R_u restricted to nonincreasing functions. Precisely, we characterize the validity of

$$\|R_u f\|_{L^q(w)} \leq C\|f\|_{L^p(v)} \tag{2}$$

for all nonnegative and nonincreasing functions f on $(0,\infty)$, in the range $p,q \in (0,\infty)$.

The second question was studied in [5, Theorem 3.2], and a characterization was found. However, the authors succeeded to find a simple supremal/integral condition only for the case $0 < p \leq q < \infty$. (This result is listed here as Theorem 3.3(i).)

In the case $1 \leq p < \infty$, $0 < q < p$, [5] provides only a discrete condition involving a supremum of all "covering sequences" of points partitioning the half-axis $(0,\infty)$. Such condition is unfortunately only hardly verifiable and therefore of little practical use in further applications. In such situations there is always a strong interest in finding a simpler and more explicit condition. We solve this particular problem here in Theorem 3.3(ii) and provide a condition having a standard integral form.

There is actually more than one way to solve this problem. In a recent and not yet published paper [4] the authors present a certain reduction method, applying which an integral condition for validity of (2) on nonnegative nonincreasing functions may be derived as well. The resulting characterization is, however, more complicated than the one we derive in here and, in a certain sense, it does not match the condition for $0 < p \leq q < \infty$ proved in [5]. More details on this issue are mentioned in Section 4. Reduction methods for weighted inequalities were investigated in more papers, as e.g. [8, 9, 10].

Besides the treatment of R_u, the paper [5] offered a complete characterization of the $L^q(w)$–$L^p(v)$ boundedness of another supremal operator

$$T_u f(t) := \sup_{s\in[t,\infty)} u(s) \int_0^s f(x)\,\mathrm{d}x, \quad t > 0,$$

where u is a fixed continuous weight and the operator T_u is defined for non-negative functions f. The interest in studying this operator stems, among other things, from its relation to the fractional maximal operator. For details, see [5] and the references given therein.

The operator S_u, which we are focusing on in this paper, appears often when *iterated Hardy-type inequalities* and *iterated Hardy-type operators* are studied. It is in fact itself an example of an iterated Hardy-type operator, as it is composed of the dual Hardy operator $H'f(t) := \int_t^\infty f$ and the supremal Hardy-type operator R_u. In a recent work [2], finding a characterization of the $L^q(w)$–$L^p(v)$ boundedness of S_u turns out to be necessary for proving certain embeddings between generalized Lorentz-type spaces with norms based on weighted integral means. This application is the main motivation of this paper.

Another one is, as mentioned before, the goal of finding the missing integral condition for the operator R_u acting on nonincreasing functions in the case $q < p$. It is reached easily once the results regarding S_u are established, since the inequality (2) can be reformulated as a particular case of the inequality (1). It may be worth noting that the process can be also reversed, allowing to characterize (1) for nonnegative functions when knowing the conditions for validity of (2) for nonincreasing functions. In this way, however, some additional assumptions on the weights might be required and they do not seem to be easily removable. Hence, treating S_u first is the preferred choice of action.

The proof technique used here is based on the dyadic discretization of weights, also called the *blocking technique*, which is a common tool for handling weighted inequalities. A comprehensive introduction into this technique is found for example in [12].

To fit the problems investigated in this article, the method needed to be modified and improved in a certain way. Roughly speaking, the key feature is the simultaneous control of both the weights w and u. It seems likely that the same method may be applied to obtain integral conditions in other problems where only discrete conditions or none at all have been known so far.

Let us also briefly describe the structure of the paper. In Section 2 below, we present the definitions and summarize auxiliary results. The main results together with their proofs are included in Section 3. Finally, in the last part, Section 4, we briefly compare the obtained conditions to the alternative characterizations which can be reached by the reduction methods of [4].

2. Definitions and preliminaries

The standard notation $A \lesssim B$ means that there exists a constant C "independent of relevant quantities in A and B" such that $A \le CB$. In this paper, the exact translation of this folklore phrase is that the constant C may depend only on exponents p and q. We write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$.

The symbol $\mathscr{M}_+$ denotes the cone of all nonnegative Lebesgue-measurable functions on $(0,\infty)$. By $\mathscr{M}_+^{\downarrow}$ we denote the cone of all nonincreasing functions from $\mathscr{M}_+$.

A *weight* is a function $w \in \mathscr{M}_+$ such that for all $t \in (0,\infty)$ there holds $0 < W(t) < \infty$, where

$$W(t) := \int_0^t w(s)\,\mathrm{d}s.$$

The symbol V has an analogous relation to the weight v.

Let v be a weight and $p \in (0,\infty)$. The *weighted Lebesgue space* $L^p(v) = L^p(v)(0,\infty)$ consists of all real-valued Lebesgue-measurable functions f on $(0,\infty)$ such that

$$\|f\|_{L^p(v)} := \left(\int_0^\infty |f(t)|^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

If $p \in (1,\infty)$, then the conjugate exponent p' is defined by $p' := \frac{p}{p-1}$.

We say that $\mathbb{I} \subseteq \mathbb{Z} \cup \{\pm\infty\}$ is an *index set* if there exist $k_{\min}, k_{\max} \in \mathbb{Z} \cup \{\pm\infty\}$ such that $k_{\min} < k_{\max}$ and

$$\mathbb{I} = \{k \in \mathbb{Z},\, k_{\min} \le k \le k_{\max}\},$$

where the respective inequality is replaced by a strict one if $k_{\min} = \infty$ or $k_{\max} = \infty$.

Let $\mathbb{I}$ be an index set. A positive sequence $\{b_k\}_{k\in\mathbb{I}}$ is called *strongly increasing*, denoted $b_k \upuparrows$, if

$$\sigma := \inf\left\{\frac{b_{k+1}}{b_k},\ k \in \mathbb{I} \setminus \{k_{\max}\}\right\} > 1. \tag{3}$$

Finally, let $n, k \in \mathbb{N}$, $z \in \mathbb{N} \cup \{0\}$, $0 \le k < n$. We write $z \bmod n = k$ if there exists $j \in \mathbb{N} \cup \{0\}$ such that $z = jn + k$. In other words, k is the remainder after division of the number z by the number n.

The proposition below was proved in [11, Proposition 2.1] (although there is a minor error in the estimate of the constant in the original article). It is in fact a key element in the discretization method.

Proposition 2.1. *Let $\mathbb{I}$ be an index set and let $0 < \alpha < \infty$. Let $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$ be two nonnegative sequences such that $b_k \upuparrows$. Then there exists $C \in (1,\infty)$ such that*

$$\left(\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} a_m\right)^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}} \le C \left(\sum_{k=k_{\min}}^{k_{\max}} a_k^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}}.$$

The constant C satisfies

$$C \le \begin{cases} 1 + \dfrac{1}{\sigma^{\alpha} - 1} & \text{if } \alpha \le 1, \\[2ex] \left(1 + \dfrac{1}{\sigma^{\frac{1}{\alpha-1}}}\right)^{\alpha-1} \left(1 + \dfrac{1}{\sigma^{\alpha-1} - 1}\right) & \text{if } \alpha > 1, \end{cases} \tag{4}$$

where σ is defined by (3).

Observe that the value of the estimates in (4) decreases with increasing σ. Hence, it suffices to know a lower bound for σ to get a usable constant C. This leads to the following corollary.

Corollary 2.2. *Let $0 < \alpha < \infty$ and $1 < D < \infty$. Then there exists a constant $C_{\alpha,D} \in (0,\infty)$ such that, for any index set $\mathbb{I}$ and any two nonnegative sequences $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$, satisfying $b_{k+1} \ge D b_k$ for all $k \in \mathbb{I} \setminus \{k_{\max}\}$, there holds*

$$\left(\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} a_m\right)^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}} \le C_{\alpha,D} \left(\sum_{k=k_{\min}}^{k_{\max}} a_k^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Moreover, since $\sup_{k\le m\le k_{\max}} a_m \le \sum_{m=k}^{k_{\max}} a_m$, we obtain another corollary.

Corollary 2.3. *Let* $0 < \alpha < \infty$ *and* $1 < D < \infty$. *Then there exists a constant* $C_{\alpha,D} \in (0,\infty)$ *such that, for any index set* $\mathbb{I}$ *and any two nonnegative sequences* $\{a_k\}_{k\in\mathbb{I}}$ *and* $\{b_k\}_{k\in\mathbb{I}}$, *satisfying* $b_{k+1} \geq D b_k$ *for all* $k \in \mathbb{I}\setminus\{k_{\max}\}$, *there holds*

$$\left(\sum_{k=k_{\min}}^{k_{\max}}\left(\sup_{k\leq m\leq k_{\max}} a_m\right)^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}} \leq C_{\alpha,D}\left(\sum_{k=k_{\min}}^{k_{\max}} a_k^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Now we recall a useful property of $L^p(v)$-spaces. If v is a weight, $p \in (1,\infty)$ and $0 \leq x < y \leq \infty$, Hölder inequality yields

$$\int_x^y h(s)\,\mathrm{d}s \leq \left(\int_x^y h^p(s)v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}$$

for any nonnegative measurable function h on (x,y). Moreover, the well-known description of the dual space to an L^p-space gives the following saturation property

$$\left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} = \sup_{\substack{h\in L^p(v)\\ \|h\|_{L^p(v)}\neq 0}} \frac{\int_x^y |h(s)|\,\mathrm{d}s}{\left(\int_x^y |h(s)|^p v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}}.$$

In particular, if $\int_x^y v^{1-p'}(s)\,\mathrm{d}s < \infty$, there exists a nonnegative function $g \in L^p(v)\cap L^1$ such that

$$2\frac{\int_x^y g(s)\,\mathrm{d}s}{\left(\int_x^y g^p(s)v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} \geq \sup_{\substack{h\in L^p(v)\\ \|h\|_{L^p(v)}\neq 0}} \frac{\int_x^y |h(s)|\,\mathrm{d}s}{\left(\int_x^y |h(s)|^p v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} = \left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}.$$

Moreover, the function g may be taken such that $\|g\|_{L^p(v)} = 1$, in which case we get

$$\left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} \leq 2\int_x^y g(s)\,\mathrm{d}s < \infty.$$

This property is used throughout the text and referred to as the *duality of L^p-spaces*. Similar results, of course, exist for l^p-spaces consisting of sequences. We summarize them in the next two propositions.

Proposition 2.4. *Let $\mathbb{I}$ be an index set and let $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$ be two nonnegative sequences.*

(i) *Let $0 < p \le q < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} \le \left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}} \sup_{j\in\mathbb{I}} b_j^{\frac{1}{q}}.$$

(ii) *Let $0 < q < p < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} \le \left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}}.$$

Proof. Case (i) is proved using convexity of the $\frac{q}{p}$-th power (with $p \le q$) and the Jensen inequality. Case (ii) follows from the Hölder inequality with the pair of exponents $\frac{p}{q}$ and $\frac{p}{p-q}$. □

Proposition 2.5. *Let $\mathbb{I}$ be an index set and $\{b_k\}_{k\in\mathbb{I}}$ be a nonnegative sequence. Let $0 < q < p < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}} = \sup_{\{a_k\}_{k\in\mathbb{I}}} \frac{\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}}}{\left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}}},$$

where the supremum is taken over all positive sequences $\{a_k\}_{k\in\mathbb{I}}$. In particular, if $\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}} < \infty$, then there exists a nonnegative sequence $\{a_k\}_{k\in\mathbb{I}}$ such that $\sum_{k\in\mathbb{I}} a_k^p = 1$ and

$$\left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}} \le 2\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} < \infty.$$

Obviously, in accordance with the other terminology of ours, Proposition 2.5 could be called "duality of l^p-spaces".

3. Main results

Theorem 3.1. *Let v, w be weights and let u be a continuous weight. Consider the inequality*

$$\left(\int_0^\infty \left[\sup_{x\in[t,\infty)} u(x)\int_x^\infty g(s)\,\mathrm{d}s\right]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(5)} \left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{5}$$

(i) *Let $1 < p \le q < \infty$. Then the inequality* (5) *holds for all $g \in \mathscr{M}_+$ if and only if*

$$A_{(6)} := \sup_{t\in(0,\infty)} \left(\int_0^t w(x) \sup_{z\in[x,t]} u^q(z)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty. \tag{6}$$

Moreover, the least constant $C_{(5)}$ such that (5) *holds for all $g \in \mathscr{M}_+$ satisfies $C_{(5)} \approx A_{(6)}$.*

(ii) *Let $1 < p < \infty$ and $0 < q < p < \infty$. Set $r := \frac{pq}{p-q}$. Then the inequality* (5) *holds for all $g \in \mathcal{M}_+$ if and only if*

$$A_{(7)} = \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{z \in [t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty \tag{7}$$

and

(8)

$$A_{(8)} = \left(\int_0^\infty \left(\int_0^t w(x) \sup_{y \in [x,t]} u^q(y)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z \in [t,\infty)} u^q(z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty.$$

Moreover, the least constant $C_{(5)}$ such that (5) *holds for all $g \in \mathcal{M}_+$ satisfies $C_{(5)} \approx A_{(7)} + A_{(8)}$.*

Proof. For the start, let us assume that there exists a finite $K \in \mathbb{Z}$ such that $\int_0^\infty w = 2^K$. It is possible to find a sequence of points $\{t_k\}_{k=-\infty}^K$ satisfying $t_k \in (0,\infty)$, $t_k > t_{k-1}$ and $\int_0^{t_k} w = 2^k$ for every $k \in \mathbb{Z}$, $k < K$. We also define $t_K := \infty$. For every $k \in \mathbb{Z}$ such that $k \le K-1$ define the *k-th segment*

$$\Delta_k := [t_k, t_{k+1}).$$

Then we have

$$2^k = \int_0^{t_k} w(s)\,\mathrm{d}s = \int_{\Delta_k} w(s)\,\mathrm{d}s = 2\int_{\Delta_{k-1}} w(s)\,\mathrm{d}s. \tag{9}$$

Throughout the proof, we use the notation

$$U(x,y) := \sup_{z \in [x,y)} u(z)$$

for any $0 \le x < y \le \infty$. If the interval $[x,y)$ is the k-th segment, we write shortly

$$U(\Delta_k) := U(t_k, t_{k+1}).$$

Observe that

$$U(x,z) \le U(x,y) + U(y,z) \quad \text{whenever} \quad 0 \le x \le y \le z \le \infty. \tag{10}$$

Choose a fixed $\mu \in \mathbb{Z}$ such that $\mu \le K-2$. Define the finite set $\mathbb{Z}_\mu := \{k \in \mathbb{Z}, \mu \le k \le K-1\}$. Now we construct a subset of indices in the following way: At first, set $k_0 := \mu$ and $k_1 := \mu + 1$. We continue inductively.

(S) Let $k_0, \dots, k_n$ be already defined. Then:

(a) If $k_n = K$, define $N := n-1$ and stop the procedure.

(b) If $k_n < K$, proceed as follows. If there exists any index $j \in \mathbb{Z}$ such that $k_n < j \le K$ and

$$\sum_{k=k_n}^{j-1} 2^k U^q(\Delta_k) \ge 2 \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k),$$

then define k_{n+1} as the smallest such index j and proceed again with step (S). If no such j exists, set $N := n$, define $k_{N+1} := K$ and so finish the construction.

In this way, we obtain a set of indices $\{k_0, \dots, k_N\} \subseteq \mathbb{Z}_\mu$ and $k_{N+1} = K$.

To continue, we may call the interval $[t_{k_n}, t_{k_n+1})$ the *n-th block*. For every $n \in \mathbb{N}$ such that $n \leq N$ there holds either

$$k_{n+1} = k_n + 1,$$

which means that the n-th block consists only of one segment (the k_n-th one), or

$$k_{n+1} > k_n + 1,$$

which means that the n-th block consists of more than one segment. If the latter is the case, we will say that $n \in \mathbb{A}$. Precisely, we put

$$\mathbb{A} := \{n \in \mathbb{N},\, n \leq N,\, k_{n+1} > k_n + 1\}.$$

Notice that this set may be empty but it is always satisfied

$$\mathbb{Z}_\mu = \{k_{n+1} - 1\}_{n=0}^{N} \cup \big\{k \in \mathbb{Z},\, k_n \leq k \leq k_{n+1} - 2\big\}_{n \in \mathbb{A}}. \tag{11}$$

In plain words, each segment is either the last segment (i.e. the one with the highest index k) in a block, or it lies in a block which contains multiple segments but this particular segment is not the last one of them.

From the way it was constructed it follows that the system has the following properties. At first, for every $n \in \mathbb{N}$ such that $n < N$ one gets

$$\sum_{k=k_n}^{k_{n+1}-1} 2^k U^q(\Delta_k) \geq 2 \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k). \tag{12}$$

This is not necessarily true for the last, N-th block, but it will not be an issue. Next, for all $n \in \mathbb{A}$ we have

$$\sum_{k=k_n}^{k_{n+1}-2} 2^k U^q(\Delta_k) < 2 \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k). \tag{13}$$

Furthermore, by iterating (12) it is shown that, for every $n \in \mathbb{N}$, $n \leq N$,

$$\begin{aligned}
\sum_{k=\mu}^{k_n - 1} 2^k U^q(\Delta_k) = \sum_{i=0}^{n-1} \sum_{k=k_i}^{k_{i+1}-1} 2^k U^q(\Delta_k) \leq \sum_{i=0}^{n-1} 2^{i-n+1} \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k) \\
\leq 2 \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k),
\end{aligned}$$

hence

$$\sum_{k=\mu}^{k_n - 1} 2^k U^q(\Delta_k) \leq 2 \sum_{k=k_{n-1}}^{k_n - 1} 2^k U^q(\Delta_k). \tag{14}$$

Now suppose that $n \in \mathbb{N}$, $n \le N$, $k \in \mathbb{Z}$ is such that $k < k_{n+1}$ and $t \in \Delta_k$. Then we have

$$\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x = \int_{t_\mu}^{t_k} w(x) U^q(x,t)\,\mathrm{d}x + \int_{t_k}^{t} w(x) U^q(x,t)\,\mathrm{d}x$$

$$\lesssim \int_{t_\mu}^{t_k} w(x) U^q(x,t_k)\,\mathrm{d}x + \int_{t_\mu}^{t_k} w(x)\,\mathrm{d}x\, U^q(t_k,t) + \int_{t_k}^{t} w(x) U^q(x,t)\,\mathrm{d}x \tag{15}$$

$$\le \sum_{j=\mu}^{k-1} \int_{\Delta_j} w(x)\,\mathrm{d}x\; U^q(t_j,t_k) + \int_{t_\mu}^{t_{k+1}} w(x)\,\mathrm{d}x\; U^q(t_k,t)$$

$$\lesssim \sum_{j=\mu}^{k-1} 2^j U^q(t_j,t_k) + 2^k U^q(t_k,t) \tag{16}$$

$$= \sum_{j=\mu}^{k-1} 2^j \left(\sum_{i=j}^{k-1} U(\Delta_i) \right)^q + 2^k U^q(t_k,t)$$

$$\lesssim \sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) + 2^k U^q(t_k,t). \tag{17}$$

Step (15) follows by (10), step (16) is due to (9) and (17) holds by Corollary 2.2. Next, if $k \le k_n$, then

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \le \sum_{j=\mu}^{k_n-1} 2^j U^q(\Delta_j) \lesssim \sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j),$$

where the second inequality follows by (14). If $k > k_n$, then $n \in \mathbb{A}$, $k_n + 1 \le k \le k_{n+1} - 1$ and we get

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \le \sum_{j=\mu}^{k_{n+1}-2} 2^j U^q(\Delta_j) = \sum_{j=\mu}^{k_n-1} 2^j U^q(\Delta_j) + \sum_{j=k_n}^{k_{n+1}-2} 2^j U^q(\Delta_j)$$

$$\lesssim \sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j).$$

The last inequality is granted by (13) and (14). We have proved that

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \lesssim \sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j).$$

Inserting this into the inequality obtained at (17), we finally receive

$$\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x \lesssim \sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) + 2^k U^q(t_k,t) \tag{18}$$

for any $n \in \mathbb{N}$, $n \le N$, $k \in \mathbb{Z}$, $k < k_{n+1}$ and $t \in \Delta_k$.

Yet another useful inequality reads

$$\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \lesssim \int_{t_{k_{n-1}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \tag{19}$$

for any $n \in \mathbb{N}$ such that $n \le N$. Indeed, this follows from the following observation:

$$\begin{aligned}\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) &\lesssim \sum_{j=k_{n-1}}^{k_n-1} \int_{\Delta_{j-1}} w(t)\,\mathrm{d}t\ U^q(\Delta_j) \le \sum_{j=k_{n-1}}^{k_n-1} \int_{\Delta_{j-1}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \\ &= \int_{t_{k_{n-1}-1}}^{t_{k_n-1}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \le \int_{t_{k_{n-1}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t,\end{aligned}$$

in which we also used (9) to establish the first inequality.

We have prepared the core of the proof method now and may begin with the main part, which is split into proving sufficiency of the respective A-conditions for the validity of (5), and their necessity.

Sufficiency. Choose a function $g \in L^p(v)$. We start by estimating

$$\begin{aligned}
&\left[\int_{t_\mu}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^{\infty} g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&= \left[\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} w(t)\left(\sup_{x\in[t,\infty)} u(x)\int_x^{\infty} g\right)^q \mathrm{d}t\right]^{\frac{1}{q}}\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} w(t)\,\mathrm{d}t\left(\sup_{x\in[t_k,\infty)} u(x)\int_x^{\infty} g\right)^q\right]^{\frac{1}{q}}\\
&= \left[\sum_{k\in\mathbb{Z}_\mu} 2^k\left(\sup_{x\in[t_k,\infty)} u(x)\int_x^{\infty} g\right)^q\right]^{\frac{1}{q}} \qquad (20)\\
&\approx \left[\sum_{k\in\mathbb{Z}_\mu} 2^k\left(\sup_{x\in\Delta_k} u(x)\int_x^{\infty} g\right)^q\right]^{\frac{1}{q}} \qquad (21)\\
&\approx \left[\sum_{k\in\mathbb{Z}_\mu} 2^k\left(\sup_{x\in\Delta_k} u(x)\int_x^{t_{k+1}} g\right)^q\right]^{\frac{1}{q}} + \left[\sum_{k\in\mathbb{Z}_\mu} 2^k U^q(\Delta_k)\left(\int_{t_{k+1}}^{\infty} g\right)^q\right]^{\frac{1}{q}}\\
&=: B_1 + B_2.
\end{aligned}$$

Step (20) follows from (9), and step (21) from Corollary 2.3.

Moreover, B_2 can be further estimated as follows.

$$
\begin{aligned}
(22)\quad B_2 &\approx \left[\sum_{n=1}^{N} 2^{k_n-1} U^q(\Delta_{k_n-1}) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{n+1}-2} 2^k U^q(\Delta_k) \left(\int_{t_{k+1}}^{\infty} g\right)^q\right]^{\frac{1}{q}} \\
&\le \left[\sum_{n=1}^{N} 2^{k_n-1} U^q(\Delta_{k_n-1}) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{n+1}-2} 2^k U^q(\Delta_k) \left(\int_{t_{k_n+1}}^{\infty} g\right)^q\right]^{\frac{1}{q}} \\
(23)\quad &\lesssim \left[\sum_{n=1}^{N} 2^{k_n-1} U^q(\Delta_{k_n-1}) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n+1}}^{\infty} g\right)^q\right]^{\frac{1}{q}} \\
&\le \left[\sum_{n=1}^{N} 2^{k_n-1} U^q(\Delta_{k_n-1}) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} \\
&\lesssim \left[\sum_{n=1}^{N} \sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{\infty} g\right)^q\right]^{\frac{1}{q}} \\
(24)\quad &\lesssim \left[\sum_{n=1}^{N} \sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{n+1}}} g\right)^q\right]^{\frac{1}{q}} \\
(25)\quad &\le \left[\sum_{n=1}^{N} \sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{q}{p'}} \left(\int_{t_{k_n}}^{t_{k_{n+1}}} g^p v\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
&=: B_3.
\end{aligned}
$$

In here, step (22) follows from (11), and step (23) from (13). In (24) we used Corollary 2.2, considering also (12). Step (25) follows by Hölder inequality.

The above estimates resulting in B_1 and B_3 are valid independently of the relation between p and q. The rest will be split into the cases (i) and (ii).

(i) Let $1 < p \le q < \infty$. Suppose that $A_{(6)} < \infty$. The goal is to show that $C_{(5)} \lesssim A_{(6)}$. First, we get

$$
\begin{aligned}
(26)\qquad B_1 &\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{k+1}} v^{1-p'}\right)^{\frac{q}{p'}} \left(\int_x^{t_{k+1}} g^p v\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{k+1}} v^{1-p'}\right)^{\frac{q}{p'}} \left(\int_{\Delta_k} g^p v\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
(27)\qquad &\le \sup_{k\in\mathbb{Z}_\mu} 2^{\frac{k}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \left(\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} g^p v\right)^{\frac{1}{p}} \\
&\le \sup_{k\in\mathbb{Z}_\mu} 2^{\frac{k}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
(28)\qquad &= \sup_{k\in\mathbb{Z}_\mu} \left(\int_0^{t_k} w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&\le \sup_{k\in\mathbb{Z}_\mu} \sup_{x\in\Delta_k} \left(\int_0^{x} w(t)U^q(t,x)\,\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_x^{\infty} v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&= A_{(6)}\|g\|_{L^p(v)}.
\end{aligned}
$$

Step (26) follows from Hölder inequality, step (27) from Proposition 2.4(i). In (28) we used (9).

We proceed with B_3.

$$
\begin{aligned}
(29)\qquad B_3 &\le \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{1}{p'}} \left(\sum_{n=1}^{N}\int_{t_{k_n}}^{t_{k_{n+1}}} g^p v\right)^{\frac{1}{p}} \\
&\le \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
(30)\qquad &\lesssim \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\int_0^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&\le A_{(6)}\|g\|_{L^p(v)}.
\end{aligned}
$$

Step (29) follows by Proposition 2.4(i), and (30) is due to (19).

At this point we have proved that for an arbitrary $\mu \in \mathbb{Z}$ such that $\mu \le K-2$ and an arbitrarily chosen $g \in L^p(v)$ there holds

$$\left[\int_{t_\mu}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \lesssim A_{(6)}\|g\|_{L^p(v)},$$

where the constant contained in the symbol "$\lesssim$" is independent of g, u, v, w and μ. If needed, the reader may verify the independence of μ by re-checking all the estimates above. Now let $\mu \to -\infty$, then $t_\mu \downarrow 0$ and the monotone convergence theorem yields

$$\left[\int_{0}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \lesssim A_{(6)}\|g\|_{L^p(v)}.$$

Recall that until now we have assumed that $\int_0^\infty w = 2^K$ with $K \in \mathbb{Z}$, and therefore for all weights w such that $\int_0^\infty w < \infty$ (one may multiply w by a constant and use homogeneity). To prove the statement for a general weight w, suppose that $\int_0^\infty w = \infty$ and $A_{(6)} < \infty$. Find, e.g. by truncation, a sequence of weights $\{w_K\}_{K=1}^\infty$ such that $\int_0^\infty w_K = 2^K$ and $w_K \uparrow w$ pointwise as $K \to \infty$. By the previous part of the proof, for all $K \in \mathbb{N}$ we have

$$\begin{aligned}
&\left[\int_{0}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w_K(t)\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&\lesssim \sup_{x>0}\left(\int_0^x \sup_{x\in[t,x]} u^q(y) w_K(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}}\|g\|_{L^p(v)}\\
&\le \sup_{x>0}\left(\int_0^x \sup_{x\in[t,x]} u^q(y) w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}}\|g\|_{L^p(v)}\\
&= A_{(6)}\|g\|_{L^p(v)}.
\end{aligned}$$

Letting $K \to \infty$, by the monotone convergence theorem it follows

$$\left[\int_{0}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \lesssim A_{(6)}\|g\|_{L^p(v)}.$$

The function $g \in L^p(v)$ is arbitrary and the constant in "$\lesssim$" does not depend on g, hence (5) holds and the optimal $C_{(5)}$ must satisfy $C_{(5)} \lesssim A_{(6)}$ in the case $1 < p \le q < \infty$.

(ii) Let $1 < p < \infty$ and $0 < q < p$. Assume $A_{(7)} + A_{(8)} < \infty$. Then for B_1 we have

$$
\begin{aligned}
B_1 &\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{k+1}} v^{1-p'} \right)^{\frac{q}{p'}} \left(\int_x^{t_{k+1}} g^p v \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} && (31)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{k+1}} v^{1-p'} \right)^{\frac{q}{p'}} \left(\int_{\Delta_k} g^p v \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{x\in\Delta_k} u^r(x) \left(\int_x^{t_{k+1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \left(\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} g^p v \right)^{\frac{1}{p}} && (32)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{x\in\Delta_k} u^r(x) \left(\int_x^{t_{k+1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \sup_{x\in\Delta_k} u^r(x) \left(\int_x^{t_{k+1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)} && (33)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} W^{\frac{r}{p}}(t) w(t) \sup_{x\in[t,\infty)} u^r(x) \left(\int_x^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)} \\
&= A_{(7)} \|g\|_{L^p(v)}.
\end{aligned}
$$

Here, (31) follows from the Hölder inequality, step (32) makes use of Proposition 2.4(ii) and in (33) one applies property (9).

Before we continue with B_3, let us notice that for any $t \in (0,\infty)$ we have

$$
\begin{aligned}
\sup_{y\in[t,\infty)} \sup_{z\in[t,y]} u(z) \left(\int_y^{\infty} v^{1-p'} \right)^{\frac{1}{p'}} &= \sup_{z\in[t,\infty]} u(z) \sup_{y\in[z,\infty)} \left(\int_y^{\infty} v^{1-p'} \right)^{\frac{1}{p'}} \\
&= \sup_{z\in[t,\infty]} u(z) \left(\int_z^{\infty} v^{1-p'} \right)^{\frac{1}{p'}}. && (34)
\end{aligned}
$$

Define $k_{-1} := k_0 - 1 = \mu - 1$. Now it is possible to write

(35)
$$B_3 \le \left[\sum_{n=1}^{N}\left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\left(\sum_{n=1}^{N}\int_{t_{k_n}}^{t_{k_{n+1}}} g^p v\right)^{\frac{1}{p}}$$

$$\le \left[\sum_{n=1}^{N}\left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

(36)
$$\le \left[\sum_{n=1}^{N}\left(\int_{t_{k_{n-2}}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$\lesssim \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}}\left(\int_{t_{k_{n-2}}}^{t} w(x)U^q(x,t_{k_n})\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$\lesssim \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}}\left(\int_{t_{k_{n-2}}}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$+ \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}}\left(\int_{t_{k_{n-2}}}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^r(t,t_{k_n})\,\mathrm{d}t\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$\le \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}}\left(\int_{0}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} U^q(t,z)\left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$+ \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}} W^{\frac{r}{p}}(t)w(t)\sup_{z\in[t,\infty)} U^r(t,z)\,\mathrm{d}t\left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

(37)
$$= \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}}\left(\int_{0}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$+ \left[\sum_{n=1}^{N}\int_{t_{k_{n-2}}}^{t_{k_n}} W^{\frac{r}{p}}(t)w(t)\sup_{z\in[t,\infty)} u^r(z)\left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} ||g||_{L^p(v)}$$

$$(38)\quad \lesssim \sum_{i=0}^{1}\left[\sum_{\substack{1\le n\le N\\ n \bmod 2=i}} \int_{t_{k_{n-2}}}^{t_{k_n}} \left(\int_0^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \times \|g\|_{L^p(v)}$$

$$+\sum_{i=0}^{1}\left[\sum_{\substack{1\le n\le N\\ n \bmod 2=i}} \int_{t_{k_{n-2}}}^{t_{k_n}} W^{\frac{r}{p}}(t)w(t) \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \|g\|_{L^p(v)}$$

$$\lesssim \left(A_{(8)}+A_{(7)}\right)\|g\|_{L^p(v)}.$$

On line (35) we applied Proposition 2.4(ii). Step (36) is based on (19) and the inequality $t_{k_{n-1}-1} \ge t_{k_{n-2}}$ which is valid for all $n \in \{1,\dots,N\}$. The identity on (37) follows from (34). On line (38) we split the sums into sums over even and odd numbers n so that the intervals $[t_{k_{n-2}}, t_{k_n}]$ become disjoint. This manoeuver will be commonly used in the rest of the paper.

Omitting the details, now we let $\mu \to -\infty$ and $K \to \infty$ as in the final part of the proof of sufficiency in case (i). We obtain

$$\left[\int_0^\infty \left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \lesssim \left(A_{(7)}+A_{(8)}\right)\|g\|_{L^p(v)}$$

for our arbitrarily chosen $g \in L^p(v)$. Hence, (5) is valid for all $g \in \mathscr{M}_+$ and the optimal $C_{(5)}$ satisfies $C_{(5)} \lesssim A_{(7)}+A_{(8)}$. This completes the sufficiency part.

Necessity. Suppose that (5) holds for all $g \in \mathscr{M}_+$. Let $1 < p < \infty$ and let $q \in (0,\infty)$ be arbitrary. Let $x > 0$. By the duality of L^p-spaces there exists a function $\varphi \in L^p(v)$ such that $\varphi(t) = 0$ for all $t < x$,

$$\int_x^\infty \varphi^p v = \int_0^\infty \varphi^p v = 1 \qquad \text{and} \qquad \left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \le 2\int_x^\infty \varphi.$$

Then

$$\left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \lesssim \left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{\frac{1}{q}}\int_x^\infty \varphi$$

$$\le \left[\int_0^x \left(\sup_{y\in[t,x]} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le \left[\int_0^x \left(\sup_{y\in[t,\infty)} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}}$$

$$= \left[\int_0^x \left(\sup_{y\in[t,\infty)} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le C_{(5)}\|\varphi\|_{L^p(v)} = C_{(5)}.$$

Taking the supremum over $x > 0$, we obtain

$$A_{(6)} \lesssim C_{(5)}. \tag{39}$$

This proves that the condition $A_{(6)}$ is in fact necessary in both cases (i) and (ii). The proof of case (i) is therefore complete.

In the rest of the proof we will deal with case (ii). Thus, from now on assume that $1 < p < \infty$ and $0 < q < p$.

Since we assumed $C_{(5)} < \infty$, inequality (39) implies

$$\int_x^\infty v^{1-p'}(s)\,\mathrm{d}s < \infty \quad \text{for all } x \in (0, \infty). \tag{40}$$

It may be checked as follows. Let $x > 0$. By the definition of a weight, we have $\int_0^s w > 0$ and $\int_0^s u > 0$ for any $s > 0$. Hence, both u and w are positive a.e. on an interval $(0, \varepsilon)$ with $\varepsilon > 0$, which implies that $\int_0^x w(t)u^q(t)\,\mathrm{d}t > 0$. Using (39), we now get

$$\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \lesssim C_{(5)} \left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{-\frac{1}{q}} \leq C_{(5)} \left(\int_0^x w(t)u^q(t)\,\mathrm{d}t\right)^{-\frac{1}{q}} < \infty.$$

Now assume again that $\int_0^\infty w = 2^K$, define the k-segments, choose $\mu \in \mathbb{Z}$ such that $\mu \leq K-2$ and construct the n-blocks. Then we have

$$\begin{aligned}
&\left[\int_{t_\mu}^\infty \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
&= \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} && (41) \\
&= \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{k\leq j\leq N} \sup_{z\in\Delta_j} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} && (42)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^{t_{k+1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&+ \left[\sum_{k=\mu}^{K-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{k+1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&=: B_4 + B_5.
\end{aligned}
$$

In (41) we applied (9) and step (42) follows from Corollary 2.3.

Using (9) and (39), we continue by a preliminary estimate.

$$
\begin{aligned}
B_4 &\leq \left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{k=\mu}^{K-1} \left(\int_0^{t_k} w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\leq \left[\sum_{k=\mu}^{K-1} \sup_{z\in\Delta_k} \left(\int_0^{z} w(t) U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\leq (K-\mu)^{\frac{1}{r}} A_{(6)} \\
&\lesssim (K-\mu)^{\frac{1}{r}} C_{(5)} < \infty.
\end{aligned}
$$

An attentive reader could now rightfully accuse the author of cheating. Indeed, the previous chain of inequalities provides an estimate of B_4 by $C_{(5)}$ and may thus seem to be what we want, but the estimate is not uniform. The problem is the term $K-\mu$ which depends on the auxiliary sum. To get the proper uniform bound we therefore need to do more work. However, by the previous estimate we managed to show that $B_4 < \infty$, which was the true reason why we made it. The information about the finiteness is needed in what follows.

For each $k \in \mathbb{Z}_\mu$ let $z_k \in \Delta_k$ be such that

$$
2u^r(z_k)\left(\int_{z_k}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \geq \sup_{z\in\Delta_k} u^r(z) \left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}. \tag{43}
$$

Both sides of the inequality are finite, which follows from the finiteness of B_4.

Now, since by (40) one has $\int_{z_k}^{t_{k+1}} v^{1-p'} < \infty$ for all $k \in \mathbb{Z}_\mu$, duality of L^p-spaces yields that for each $k \in \mathbb{Z}_\mu$ there exists a nonnegative function h_k with support in the interval $[z_k, t_{k+1}]$ and such that

$$(44)\qquad \int\limits_{\Delta_k} h_k^p v = \int\limits_{z_k}^{t_{k+1}} h_k^p v = 1 \qquad \text{and} \qquad \left(\int\limits_{z_k}^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \leq 2\int\limits_{z_k}^{t_{k+1}} h_k.$$

Then

$$(45)\quad \sup_{z\in\Delta_k} u(z)\left(\int\limits_{z}^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \lesssim u(z_k)\left(\int\limits_{z_k}^{t_{k+1}} v^{1-p'}\right)^{\frac{1}{p'}} \lesssim u(z_k)\int\limits_{z_k}^{t_{k+1}} h_k \leq \sup_{z\in\Delta_k} u(z)\int\limits_{z}^{t_{k+1}} h_k.$$

Furthermore, since $B_4 < \infty$, by Proposition 2.5 there exists a nonnegative sequence $\{a_k\}_{k\in\mathbb{Z}_\mu}$ such that $\sum_{k\in\mathbb{Z}_\mu} a_k^p = 1$ and

$$(46)\quad \left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z)\left(\int\limits_{z}^{t_{k+1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}$$
$$\leq 2\left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z)\left(\int\limits_{z}^{t_{k+1}} v^{1-p'}\right)^{\frac{q}{p'}} a_k^q\right]^{\frac{1}{q}}.$$

Define the function $h := \sum_{k=1}^{K-1} a_k h_k$. Then it satisfies

$$\|h\|_{L^p(v)} = \left(\sum_{k\in\mathbb{Z}_\mu}\int\limits_{\Delta_k} h^p v\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}_\mu} a_k^p \int\limits_{\Delta_k} h_k^p v\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}_\mu} a_k^p\right)^{\frac{1}{p}} = 1.$$

We may finally derive the following estimate on B_4.

$$B_4 \lesssim \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z) \left(\int_z^{t_{k+1}} v^{1-p'} \right)^{\frac{q}{p'}} a_k^q \right]^{\frac{1}{q}} \tag{47}$$

$$\lesssim \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z) \left(\int_z^{t_{k+1}} h_k \right)^{q} a_k^q \right]^{\frac{1}{q}} \tag{48}$$

$$= \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z) \left(\int_z^{t_{k+1}} h \right)^{q} \right]^{\frac{1}{q}}$$

$$\lesssim \left[\sum_{k=\mu}^{K-1} \int_{\Delta_{k-1}} w(t)\,\mathrm{d}t \sup_{z\in\Delta_k} u^q(z) \left(\int_z^{t_{k+1}} h \right)^{q} \right]^{\frac{1}{q}} \tag{49}$$

$$\leq \left[\sum_{k=\mu}^{K-1} \int_{\Delta_{k-1}} w(t) \left(\sup_{z\in[t,\infty)} u(z) \int_z^{\infty} h(s)\,\mathrm{d}s \right)^{q} \mathrm{d}t \right]^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\infty} w(t) \left(\sup_{z\in[t,\infty)} u(z) \int_z^{\infty} h(s)\,\mathrm{d}s \right)^{q} \mathrm{d}t \right)^{\frac{1}{q}}$$

$$\leq C_{(5)} \|h\|_{L^p(v)} = C_{(5)}.$$

Here in (47) we used (46) and in (48) we used (45). The inequality on (49) is, as usual, due to (9). Only now we obtained the "proper" estimate

$$B_4 \lesssim C_{(5)},$$

in which the constant behind the symbol "$\lesssim$" really depends only on p and q.

We proceed with B_5 as follows.

$$B_5 = \left[\sum_{n=1}^{N} \sum_{k=k_n-1}^{k_{n+1}-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{k+1}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}}$$

$$\leq \left[\sum_{n=1}^{N} \sum_{k=k_n-1}^{k_{n+1}-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]$$

$$\leq \left[\sum_{n=1}^{N}\left(\sum_{k=k_n-1}^{k_{n+1}-2} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \tag{50}$$

$$\lesssim \left[\sum_{n=1}^{N}\left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \tag{51}$$

$$\lesssim \left[\sum_{n=1}^{N}\left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \tag{52}$$

$$=: B_6.$$

Step (50) follows from Jensen inequality since $\frac{r}{q} > 1$. Step (51) follows from (13). In (52) one uses Corollary 2.2, considering also (12).

Before estimating further, let us first prove finiteness of B_6, as we did in case of B_4. By (19) and (39) we obtain

$$B_6 \leq \left[\sum_{n=1}^{N}\left(\int_0^{t_{k_n}} w(t)U^q(t,t_{k_n})\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \leq N^{\frac{1}{r}}A_{(6)} \lesssim N^{\frac{1}{r}}C_{(5)} < \infty.$$

Considering (40) and the L^p-duality, for each $n \in \mathbb{N}$ such that $n \leq N$ we can find a function g_n supported in the interval $[t_{k_n}, t_{k_{n+1}}]$ and such that

$$\int_{t_{k_n}}^{t_{k_{n+1}}} g_n^p v = 1 \qquad \text{and} \qquad \left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{1}{p'}} \leq 2\int_{t_{k_n}}^{t_{k_{n+1}}} g_n. \tag{53}$$

Furthermore, since we know that $B_6 < \infty$, by Proposition 2.5 we find a nonnegative sequence $\{c_n\}_{n=1}^N$ such that $\sum_{n=1}^N c_n^p = 1$ and

$$B_6 = \left[\sum_{n=1}^{N}\left(\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \tag{54}$$

$$\leq 2\left[\sum_{n=1}^{N}\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{q}{p'}} c_n^q\right]^{\frac{1}{q}}.$$

Define the function $g := \sum_{n=1}^{N} c_n g_n$. It is easy to verify that $\|g\|_{L^p(v)} = 1$. Moreover, we obtain

$$
\begin{aligned}
B_6 &\lesssim \left[\sum_{n=1}^{N}\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{q}{p'}} c_n^q\right]^{\frac{1}{q}} \qquad (55)\\
&\lesssim \left[\sum_{n=1}^{N}\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{n+1}}} g_n\right)^{q} c_n^q\right]^{\frac{1}{q}} \qquad (56)\\
&= \left[\sum_{n=1}^{N}\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{n+1}}} g\right)^{q}\right]^{\frac{1}{q}}\\
&\lesssim \left[\sum_{n=1}^{N}\sum_{k=k_{n-1}}^{k_n-1} 2^k U^q(\Delta_k)\left(\int_{t_{k+1}}^{\infty} g\right)^{q}\right]^{\frac{1}{q}}\\
&= \left[\sum_{k=\mu}^{k_N-1} 2^k U^q(\Delta_k)\left(\int_{t_{k+1}}^{\infty} g\right)^{q}\right]^{\frac{1}{q}}\\
&\lesssim \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{k-1}} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} U^q(t_k,z)\left(\int_{z}^{\infty} g(s)\,\mathrm{d}s\right)^{q}\right]^{\frac{1}{q}} \qquad (57)\\
&= \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{k-1}} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^q(z)\left(\int_{z}^{\infty} g(s)\,\mathrm{d}s\right)^{q}\right]^{\frac{1}{q}} \qquad (58)\\
&\le \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{k-1}} w(t)\left(\sup_{z\in[t,\infty)} u(z)\int_{z}^{\infty} g(s)\,\mathrm{d}s\right)^{q}\mathrm{d}t\right]^{\frac{1}{q}}\\
&\le \left(\int_{0}^{\infty} w(t)\left(\sup_{z\in[t,\infty)} u(z)\int_{z}^{\infty} g(s)\,\mathrm{d}s\right)^{q}\right]^{\frac{1}{q}}\\
&\le C_{(5)}\|g\|_{L^p(v)} = C_{(5)}.
\end{aligned}
$$

Here, (55) is the same as (54), inequality (56) follows from (53) and inequality (57) from (9). An argument analogous to (34) is used to establish the identity (58).

We have shown

$$B_5 \lesssim B_6 \lesssim C_{(5)},$$

hence, combining this with the other estimates, we get

$$\left[\int_{t_\mu}^{\infty}\left(\int_{t_\mu}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \lesssim B_4 + B_5 \tag{59}$$

$$\lesssim B_4 + B_6 \lesssim C_{(5)}.$$

Letting $\mu \to -\infty$ and then $K \to \infty$ analogously as we did before, we obtain

$$A_{(7)} \lesssim C_{(5)} \tag{60}$$

for a general weight w.

In the rest we will focus on the condition $A_{(8)}$. At first, observe that for any $0 < a < t < \infty$ the inequality

$$U^{\frac{rq}{p}}(a,t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \le \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \tag{61}$$

holds true. Indeed, one has

$$\sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \le \sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{z\in[t,\infty)} \sup_{\tau\in[t,z)} u^q(\tau)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}$$

$$= \sup_{z\in[t,\infty)} \sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{\tau\in[t,z)} u^q(\tau)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \le \sup_{z\in[t,\infty)} \sup_{s\in[a,z)} u^r(s)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}$$

$$\le \sup_{z\in[a,\infty)} \sup_{s\in[a,z)} u^r(s)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} = \sup_{z\in[a,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}.$$

Identity (34) implies the last step.

The starting point for estimating $A_{(8)}$ is the following decomposition.

$$\begin{aligned}
&\left[\int_{t_\mu}^{\infty}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
(62)\quad &\approx \left[\int_{\Delta_\mu}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{n=1}^{N}\int_{\Delta_{k_{n+1}-1}}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{n+1}-2}\int_{\Delta_k}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&=: B_7+B_8+B_9.
\end{aligned}$$

For B_7 one has

$$\begin{aligned}
B_7 &\le \left[\int_{\Delta_\mu}\left(\int_{t_\mu}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^{\frac{rq}{p}}(t_\mu,t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
(63)\quad &\le \left[\int_{\Delta_\mu}\left(\int_{t_\mu}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\,\mathrm{d}t\sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
(64)\quad &\lesssim \left[2^{\frac{\mu r}{q}}\sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
(65)\quad &\lesssim \left[\int_0^{t_\mu} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t\sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\le A_{(7)}\lesssim C_{(5)}.
\end{aligned}$$

We used (61) to get (63), and (9) was used for (64) and (65). The very last inequality was obtained in (60).

Next, for B_8 we get

$$
\begin{aligned}
B_8 \lesssim & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \int\limits_{\Delta_{k_{n+1}-1}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int\limits_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
& + \left[\sum_{n=1}^{N} 2^{k_{n+1}\frac{r}{p}} \int\limits_{\Delta_{k_{n+1}-1}} w(t) U^{\frac{rq}{p}}(t_{k_{n+1}-1}, t) \sup_{z\in[t,\infty)} u^q(z)\left(\int\limits_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
\lesssim & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{n+1}} \sup_{z\in\left[t_{k_{n+1}-1},\infty\right)} u^q(z)\left(\int\limits_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
& + \left[\sum_{n=1}^{N} 2^{k_{n+1}\frac{r}{q}} \sup_{z\in\left[t_{k_{n+1}-1},\infty\right)} u^r(z)\left(\int\limits_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
=: & \, B_{10} + B_{11}.
\end{aligned}
$$

The first step follows by (18). In the second step we used (9) to estimate the first summand, and (61) and (9) to estimate the second one.

Now formally define $k_{-1} := k_0 - 1 = \mu - 1$ and $t_{k_{N+2}-1} := \infty$. Furthermore, observe that, by (12), one has

$$
\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{n+1}} \geq 2^{\frac{r}{p}} \left(\sum_{j=k_{n-2}}^{k_{n-1}-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{n+1}} \geq 2^{\frac{r}{q}} \left(\sum_{j=k_{n-2}}^{k_{n-1}-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_n}
$$

for every $n \in \mathbb{N}$ which satisfies $2 \leq n \leq N$. Therefore, since $2^{\frac{r}{q}} > 1$, the sequence $\{b_n\}_{n=1}^N$ with $b_n := \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{n+1}}$ is strongly increasing.

For B_{10} we then obtain

$$
\begin{aligned}
B_{10} &= \left[\sum_{n=1}^{N} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} 2^{k_{n+1}} \sup_{n+1 \le i \le N+1} \sup_{z \in \left[t_{k_i-1}, t_{k_{i+1}-1} \right)} u^q(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} 2^{k_{n+1}} \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)} u^q(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} 2^{k_{n+1}} \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)} u^q(z) \left(\int_z^{t_{k_{N+2}-1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\quad + \left[\sum_{n=1}^{N-1} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} 2^{k_{n+1}} U^q(t_{k_{n+1}-1}, t_{k_{N+2}-1}) \left(\int_{t_{k_{N+2}-1}}^\infty v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&=: B_{12} + B_{13}.
\end{aligned}
$$

The second step follows from Corollary 2.3.

Let us proceed with B_{12}. We get

$$
\begin{aligned}
B_{12} &\le \left[\sum_{n=1}^{N} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} 2^{k_{n+1}} \right. \\
&\qquad \left. \times \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)} U^q(t_{k_{n+1}-1}, z) \left(\int_z^{t_{k_{N+2}-1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{k_{n+1}-2}}^{t_{k_{n+1}-1}} w(t)\,\mathrm{d}t \right. \\
&\qquad \left. \times \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{n+2}-1} \right)} U^q(t_{k_{n+1}-1}, z) \left(\int_z^{t_{k_{n+2}-1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}}
\end{aligned} \tag{66}
$$

$$\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\right.$$

$$\left.\times \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \int_{t_{k_{n+1}-2}}^{z} w(t)U^q(t,z)\,\mathrm{d}t \left(\int_{z}^{t_{k_{N+2}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}$$

$$\lesssim \left[\sum_{n=1}^{N}\left(\int_{t_{k_{n-1}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\right)^{\frac{r}{p}}\right. \tag{67}$$

$$\left.\times \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{n+2}-1}\right)} \int_{t_{k_{n+1}-2}}^{z} w(t)U^q(t,z)\,\mathrm{d}t \left(\int_{z}^{t_{k_{n+2}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}$$

$$\leq \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \left(\int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_{z}^{t_{k_{N+2}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}$$

$$=: B_{14}.$$

In (66) one uses (9) and (67) follows from (19) and the relation $t_{k_{n+1}-1} \geq t_{k_n}$.

Let us check finiteness of B_{14}. We have

$$B_{14} \leq \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \left(\int_{0}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}$$

$$\leq N^{\frac{1}{r}} A_{(6)} \lesssim N^{\frac{1}{r}} C_{(5)} < \infty.$$

Again we made use of the already proved estimate (39).

Now, for each $n \in \mathbb{N}$ such that $2 \leq n \leq N+1$ find a number $z_n' \in [t_{k_n-1}, t_{k_{n+1}-1})$ such that

$$2\left(\int_{t_{k_{n-1}-1}}^{z_n'} w(t)U^q(t,z_n')\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_{z_n'}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}} \tag{68}$$

$$\geq \sup_{z\in\left[t_{k_n-1},t_{k_{n+1}-1}\right)} \left(\int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_{z}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}.$$

This is possible since the right hand side is finite, which in turn follows from the finiteness of B_{14}. Following (40) and the L^p-duality, for each $n \in \mathbb{N}$ such that

$2 \leq n \leq N+1$ there exists a nonnegative function f_n supported in $[z_n', t_{k_{n+1}-1}]$ and such that

$$\int_{t_{k_n-1}}^{t_{k_{n+1}-1}} f_n^p v = \int_{z_n'}^{t_{k_{n+1}-1}} f_n^p v = 1 \qquad \text{and} \qquad \left(\int_{t_{k_n-1}}^{t_{k_{n+1}-1}} v^{1-p'} \right)^{\frac{1}{p'}} \leq 2 \int_{t_{k_n-1}}^{t_{k_{n+1}-1}} f_n.$$

An argument analogous to that of (45) then yields

$$\sup_{z \in \left[t_{k_n-1}, t_{k_{n+1}-1} \right)} \left(\int_{t_{k_{n-1}-1}}^{z} w(t) U^q(t,z) \, \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_z^{\infty} v^{1-p'} \right)^{\frac{1}{p'}} \tag{69}$$

$$\lesssim \sup_{z \in \left[t_{k_n-1}, t_{k_{n+1}-1} \right)} \left(\int_{t_{k_{n-1}-1}}^{z} w(t) U^q(t,z) \, \mathrm{d}t \right)^{\frac{1}{q}} \int_z^{\infty} f_n.$$

Next, since $B_{14} < \infty$, by Proposition 2.5 there exists a nonnegative sequence $\{d_n\}_{n=2}^{N+1}$ such that $\sum_{n=2}^{N+1} d_n^p = 1$ and

$$B_{14} = \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)} \left(\int_{t_{k_{n-1}-1}}^{z} w(t) U^q(t,z) \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_z^{t_{k_{N+2}-1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}}$$

$$\leq 2 \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)} \int_{t_{k_{n-1}-1}}^{z} w(t) U^q(t,z) \, \mathrm{d}t \left(\int_z^{t_{k_{N+2}-1}} v^{1-p'} \right)^{\frac{q}{p'}} d_n^q \right]^{\frac{1}{q}}. \tag{70}$$

As expected, now we define the function $f := \sum_{n=2}^{N+1} d_n f_n$. An easy check confirms that $\|f\|_{L^p(v)} = 1$. Before continuing, let us make one more observation. Let $n \in \mathbb{N}$ be such that $2 \leq n \leq N+1$ and let $z \in \left[t_{k_{n+1}-1}, t_{k_{N+2}-1} \right)$. Then

$$\begin{aligned}
\int_{t_{k_{n-1}-1}}^{z} w(t) U^q(t,z) \left(\int_z^{\infty} f(s) \, \mathrm{d}s \right)^q \mathrm{d}t &\leq \int_{t_{k_{n-1}-1}}^{z} w(t) \sup_{x \in [t,\infty)} U^q(t,x) \left(\int_x^{\infty} f(s) \, \mathrm{d}s \right)^q \mathrm{d}t \\
&= \int_{t_{k_{n-1}-1}}^{z} w(t) \sup_{x \in [t,\infty)} u^q(x) \left(\int_x^{\infty} f(s) \, \mathrm{d}s \right)^q \mathrm{d}t \\
&\leq \int_{t_{k_{n-1}-1}}^{t_{k_{N+2}-1}} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f(s) \, \mathrm{d}s \right)^q \mathrm{d}t.
\end{aligned}$$

The second step is an analogy to (34). Taking supremum over $z \in \big[t_{k_{n+1}-1}, t_{k_{N+2}-1}\big)$, we get

$$\text{(71)} \qquad \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{n+2}-1}\right)} \int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\left(\int_z^\infty f\right)^q \mathrm{d}t \le \int_{t_{k_{n-1}-1}}^{t_{k_{n+2}-1}} w(t)\left(\sup_{x\in[t,\infty)} u^q(x)\int_x^\infty f\right)^q \mathrm{d}t.$$

Now we estimate

$$\text{(72)} \qquad B_{14} \lesssim \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\mathrm{d}t \left(\int_z^{t_{k_{N+2}-1}} v^{1-p'}\right)^{\frac{q}{p'}} d_n^q\right]^{\frac{1}{q}}$$

$$\text{(73)} \qquad \lesssim \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\mathrm{d}t \left(\int_z^{t_{k_{N+2}-1}} f_n\right)^{q} d_n^q\right]^{\frac{1}{q}}$$

$$= \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\mathrm{d}t \left(\int_z^{t_{k_{N+2}-1}} f(s)\mathrm{d}s\right)^{q}\right]^{\frac{1}{q}}$$

$$= \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{n+1}-1},t_{k_{N+2}-1}\right)} \int_{t_{k_{n-1}-1}}^{z} w(t)U^q(t,z)\mathrm{d}t \left(\int_z^{\infty} f(s)\mathrm{d}s\right)^{q}\right]^{\frac{1}{q}}$$

$$\text{(74)} \qquad \le \left[\sum_{n=1}^{N} \int_{t_{k_{n-1}-1}}^{t_{k_{N+2}-1}} w(t)\left(\sup_{x\in[t,\infty)} u^q(x)\int_x^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}t\right]^{\frac{1}{q}}$$

$$\lesssim \sum_{i=0}^{2}\left[\sum_{\substack{1\le n\le N\\ n \bmod 3 = i}} \int_{t_{k_{n-1}-1}}^{t_{k_{N+2}-1}} w(t)\left(\sup_{x\in[t,\infty)} u^q(x)\int_x^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}t\right]^{\frac{1}{q}}$$

$$\lesssim \left[\int_0^\infty w(t)\left(\sup_{x\in[t,\infty)} u^q(x)\int_x^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}t\right]^{\frac{1}{q}}$$

$$\le C_{(5)}\|f\|_{L^p(v)} = C_{(5)}.$$

The inequality (72) is taken from (70), and step (73) follows from (69). In (74) we used (71).

Let us now return to B_{13}. We have

$$
\begin{aligned}
B_{13} &\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{n+1}-1} U^q(t_{k_{n+1}-1}, t_{k_{N+2}-1})\left(\int_{t_{k_{N+2}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_{n+1}-1}^{k_{N+2}-2} 2^k U^q(t_k, t_{k_{N+2}-1})\left(\int_{t_{k_{N+2}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_{n+1}-1}^{k_{N+2}-2} 2^k U^q(\Delta_k)\left(\int_{t_{k_{N+2}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (75) \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{n+1}-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_{N+2}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (76) \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{k=k_n}^{k_{n+1}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_{N+2}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (77) \\
&\le \left[\sum_{n=1}^{N}\left(\sum_{k=k_n}^{k_{n+1}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{k=k_n}^{k_{n+1}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (78) \\
&= B_6 \lesssim C_{(5)}.
\end{aligned}
$$

The estimate (75) follows from Corollary 2.2, step (76) is due to (13) and step (77) due to (12). Inequality (78) is implied by Corollary 2.2. The final estimate $B_6 \lesssim C_{(5)}$ was obtained in an earlier stage of the proof.

Now we have

$$B_{10} \lesssim B_{12} + B_{13} \lesssim B_{14} + B_{13} \lesssim C_{(5)}.$$

Next term to proceed with is B_{11}. We have

$$
\begin{aligned}
B_{11} &\lesssim \left[\sum_{n=1}^{N} \int_{t_{k_{n+1}-2}}^{t_{k_{n+1}-1}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{n+1}-1},\infty)} u^r(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{n=1}^{N} \int_{t_{k_{n+1}-2}}^{t_{k_{n+1}-1}} W^{\frac{r}{p}}(t) w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \\
&\le A_{(7)} \lesssim C_{(5)}.
\end{aligned} \tag{79}
$$

In (79) we used (9). Recall also the earlier result (60).

At this point we have completed the estimate

$$
B_8 \lesssim B_{10} + B_{11} \lesssim C_{(5)}.
$$

We return even deeper to the term B_9. By (18), we obtain

$$
\begin{aligned}
B_9 \lesssim & \left[\sum_{n\in\mathbb{A}} \left(\sum_{j=k_{n-1}}^{k_n-1} 2^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{n+1}-2} \int_{\Delta_k} w(t) \sup_{z\in[t,\infty)} u^q(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \\
& + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{n+1}-2} 2^{\frac{kr}{p}} \int_{\Delta_k} w(t) U^{\frac{rq}{p}}(t_k,t) \sup_{z\in[t,\infty)} u^q(z) \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \\
=: & \, B_{15} + B_{16}.
\end{aligned}
$$

Next, one has

$$
\begin{aligned}
B_{15} \lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{k=k_n}^{k_{n+1}-2}\int_{\Delta_k} w(t)\sup_{z\in[t,t_{k_{n+1}-1})}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&+\left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{k=k_n}^{k_{n+1}-2}\int_{\Delta_k} w(t)\,\mathrm{d}t\sup_{z\in[t_{k_{n+1}-1},\infty)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
\text{(80)}\quad\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{k=k_n}^{k_{n+1}-2}2^k U^q(t_k,t_{k_{n+1}-1})\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&+\left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{n+1}}\sup_{z\in[t_{k_{n+1}-1},\infty)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
\le & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{k=k_n}^{k_{n+1}-2}2^k U^q(t_k,t_{k_{n+1}-1})\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}+B_{10}\\
\text{(81)}\quad\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{k=k_n}^{k_{n+1}-2}2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}+B_{10}\\
\text{(82)}\quad\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}+B_{10}\\
\le & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}+B_{10}\\
\text{(83)}\quad\lesssim & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{n-1}}^{k_n-1}2^j U^q(\Delta_j)\right)^{\frac{r}{q}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}+B_{10}\\
= & B_6+B_{10}\lesssim C_{(5)}.
\end{aligned}
$$

In step (80) we used (9) and step (81) follows from Corollary 2.2. Step (82) is due to (13). To get (83) recall (12) and use Corollary 2.2. The estimate $B_6+B_{10}\lesssim C_{(5)}$ was obtained earlier.

The term B_{16} is the last remaining one. We get

$$
\begin{aligned}
(84)\qquad B_{16} &\leq \left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{n+1}-2} 2^{\frac{kr}{p}}\int_{\Delta_k} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{n+1}-2} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\leq \left[\sum_{k=1}^{K} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
(85)\qquad &\lesssim \left[\sum_{k=1}^{K} \int_{\Delta_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\leq \left[\sum_{k=1}^{K} \int_{\Delta_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\leq A_{(7)} \lesssim C_{(5)}.
\end{aligned}
$$

To get the inequality (84) we used (61). Step (85) follows from (9). For the final estimate see (60).

We have shown

$$B_9 \lesssim B_{15} + B_{16} \lesssim C_{(5)}.$$

Now, collecting all our estimates and returning all the way back to the initial decomposition (62), we check that we have proved

$$\left[\int_{t_\mu}^{\infty}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} \lesssim C_{(5)}.$$

Letting $\mu\to-\infty$ and $K\to\infty$ as previously done finally yields

$$A_{(8)} \lesssim C_{(5)}.$$

Therefore, necessity of conditions $A_{(7)}$ and $A_{(8)}$ in case (ii) is verified and the proof is finished. □

The previous theorem has, not surprisingly, its analogue for $p=1$. It may be proved by a similar technique as Theorem 3.1. Given the length of the previous proof, the reader will hopefully excuse omitting of the proof the theorem below which is the aforementioned version for $p=1$.

Theorem 3.2. *Let v, w be weights and let u be a continuous weight. Consider the inequality*

$$\left(\int_0^\infty \Big[\sup_{x\in[t,\infty)} u(x)\int_x^\infty g(s)\,\mathrm{d}s\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(86)}\int_0^\infty g(t)v(t)\,\mathrm{d}t. \tag{86}$$

(i) *Let $1\le q<\infty$. Then* (86) *holds for all $g\in\mathscr{M}_+$ if and only if*

$$A_{(87)} := \sup_{t\in(0,\infty)}\left(\int_0^t w(x)\sup_{z\in[x,t]} u^q(z)\,\mathrm{d}x\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{s\in[t,\infty)}\frac{1}{v(s)} < \infty. \tag{87}$$

Moreover, the least constant $C_{(86)}$ such that (86) *holds for all $g\in\mathscr{M}_+$ satisfies $C_{(86)}\approx A_{(87)}$.*

(ii) *Let $0<q<1$. Then* (86) *holds for all $g\in\mathscr{M}_+$ if and only if*

$$A_{(88)} := \left(\int_0^\infty W^{\frac{q}{1-q}}(t)w(t)\sup_{z\in[t,\infty)} u^{\frac{q}{1-q}}(z)\left(\operatorname*{ess\,sup}_{s\in[z,\infty)}\frac{1}{v(s)}\right)^{\frac{q}{1-q}}\mathrm{d}t\right)^{\frac{1-q}{q}} < \infty \tag{88}$$

and

(89)

$$A_{(89)} := \left(\int_0^\infty\left(\int_0^t w(x)\sup_{y\in[x,t]} u^q(y)\,\mathrm{d}x\right)^{\frac{q}{1-q}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\operatorname*{ess\,sup}_{s\in[z,\infty)}\frac{1}{v(s)}\right)^{\frac{q}{1-q}}\mathrm{d}t\right)^{\frac{1-q}{q}} < \infty.$$

Moreover, the least constant $C_{(86)}$ such that (86) *holds for all $g\in\mathscr{M}_+$ satisfies $C_{(86)}\approx A_{(88)}+A_{(89)}$.*

As it was forecast in the introduction, the results which are now at our disposal, namely those of Theorem 3.2, allow us to find the missing integral condition characterizing boundedness of the supremal operator R_u acting on $\mathscr{M}_+^{\downarrow}$. Case (i) in the theorem below was proved in [5, Theorem 3.2(i)] and is listed here for the sake of completeness. Case (ii) is the new result containing the integral condition for $0<q<p<\infty$. The proof in fact covers both cases.

Theorem 3.3. *Let v, w be weights and let u be a continuous weight.*

(i) *Let $0<p\le q<\infty$. Then the inequality*

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)f(s)\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(90)}\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{90}$$

holds for all $f\in\mathscr{M}_+^{\downarrow}$ if and only if

$$A_{(91)} := \sup_{t\in(0,\infty)}\left(\int_0^t w(x)\sup_{y\in[x,t)} u^q(y)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t) < \infty. \tag{91}$$

Moreover, the least constant $C_{(90)}$ *such that* (90) *holds for all* $f \in \mathscr{M}_+^{\downarrow}$ *satisfies*

$$C_{(90)} \approx A_{(91)}.$$

(ii) *Let* $0 < q < p < \infty$ *and* $r = \frac{pq}{p-q}$. *Then* (90) *holds for all* $f \in \mathscr{M}_+^{\downarrow}$ *if and only if*

$$A_{(92)} := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_0^z v(s)\,\mathrm{d}s \right)^{-\frac{r}{p}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty \tag{92}$$

and

$$A_{(93)} := \left(\int_0^\infty \left(\int_0^t w(x) \sup_{y\in[x,t]} u^q(y)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z) \left(\int_0^z v(s)\,\mathrm{d}s \right)^{-\frac{r}{p}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty. \tag{93}$$

Moreover, the least constant $C_{(90)}$ *such that* (90) *holds for all* $f \in \mathscr{M}_+^{\downarrow}$ *satisfies*

$$C_{(90)} \approx A_{(92)} + A_{(93)}.$$

Proof. Since $p > 0$, the function $f \in \mathscr{M}$ is nonincreasing if and only if the function $g := f^{\frac{1}{p}}$ is nonincreasing. Hence, (90) holds for all $f \in \mathscr{M}_+^{\downarrow}$ if and only if

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s) g^{\frac{1}{p}}(s) \Big]^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C_{(90)} \left(\int_0^\infty g(t) v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}$$

holds for all $g \in \mathscr{M}_+^{\downarrow}$. By a standard argument (see e.g. [13, Lemma 1.2]), this is equivalent to the inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s) \left(\int_s^\infty h(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \Big]^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C_{(90)} \left(\int_0^\infty \int_t^\infty h(x)\,\mathrm{d}x\; v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}$$

being satisfied for all $h \in \mathscr{M}_+$. By taking the p-th power and applying Fubini theorem, this is true if and only if

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u^p(s) \int_s^\infty h(x)\,\mathrm{d}x \Big]^{\frac{q}{p}} w(t)\,\mathrm{d}t \right)^{\frac{p}{q}} \le C_{(90)}^p \int_0^\infty h(t) V(t)\,\mathrm{d}t$$

holds for all $h \in \mathscr{M}_+$. The result now follows from Theorem 3.2. □

4. Comparison of the conditions

The paper [4] lists a variety of reduction theorems for weighted inequalities. These results, in general, allow for an equivalent reformulation of a weighted inequality in the form of another weighted inequality, often on a different cone of functions. A particular case [4, Corollary 3.5] then offers an equivalent representation of inequality (1), involving the operator S_u, by an analogous inequality with the operator $T_{\tilde{u}}$ (and with different weights). Hence, by using [4, Corollary 3.5], [5, Theorems 4.1 and 4.4] and after a careful recalculation of exponents, one can show that the validity of (5) for all $g \in \mathscr{M}_+$ is characterized by the following conditions.

(i) If $1 < p \le q < \infty$, then (5) holds for all $g \in \mathscr{M}_+$ if and only if

$$A_{(94)} := \sup_{t\in(0,\infty)} u(t) W^{\frac{1}{q}}(t) \left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{1}{p'}} + \sup_{t\in(0,\infty)} \left(\int_t^\infty w(x) \sup_{y\in[x,\infty)} u^q(y) \left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{2q}{p'+1}} \mathrm{d}x \right)^{\frac{1}{q}} \times \left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{-1}{p(p'+1)}} < \infty. \tag{94}$$

(ii) If $1 < p < \infty$ and $0 < q < p$, then (5) holds for all $g \in \mathscr{M}_+$ if and only if

$$A_{(95)} := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{y\in[t,\infty)} u^r(y) \left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}x \right)^{\frac{1}{r}} + \left(\int_0^\infty \left(\int_t^\infty w(x) \sup_{y\in[x,\infty)} u^q(y) \left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{2q}{p'+1}} \mathrm{d}x \right)^{\frac{r}{q}} \times \left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}-\frac{2r}{p'+1}-1} v^{1-p'}(t)\,\mathrm{d}t \right)^{\frac{1}{r}} < \infty. \tag{95}$$

Observe that these conditions are different from those presented in Theorem 3.1. In case (i), it is easily shown that the first term in $A_{(94)}$ is dominated by $A_{(6)}$. In (ii), the first half of $A_{(95)}$ is in fact $A_{(7)}$, but the second term in $A_{(95)}$ is different from the condition $A_{(8)}$. Notice, in particular, the "flipped" interval of integration in the term involving w in the second part of the condition $A_{(95)}$ (and the same in $A_{(94)}$). This difference can be traced back to the "flip" from S_u to $T_{\tilde{u}}$ in the reduction technique of [4].

It can be said that conditions $A_{(6)}$, $A_{(7)}$ and $A_{(8)}$ belong to one "class" (that may be called "classical conditions"), and $A_{(94)}$, $A_{(95)}$ belong to another one ("flipped conditions"). Existence of such equivalent classes of conditions is a rather common phenomenon, see e.g. [3, 6, 7].

The "classical" conditions are simpler than their "flipped" counterparts and, moreover, are compatible with older results, as these mostly have the "classical" form as well. Such matching issues are important in situations when combining conditions is needed. That is often the case in problems concerning the iterated inequalities and more complicated function spaces based on them.

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Paper VII

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Embeddings of Lorentz-type spaces involving weighted integral means

Preprint

EMBEDDINGS OF LORENTZ-TYPE SPACES INVOLVING WEIGHTED INTEGRAL MEANS

AMIRAN GOGATISHVILI, MARTIN KŘEPELA, LUBOŠ PICK AND FILIP SOUDSKÝ

ABSTRACT. We characterize embeddings between Lorentz-type spaces defined with respect to two different weighted means. In particular, we establish two-sided estimates of the optimal constant C in the inequality

$$\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_2}u_2(s)\,ds\right)^{\frac{p_2}{m_2}}w_2(t)\,dt\right)^{\frac{1}{p_2}}\leq C\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_1}u_1(s)\,ds\right)^{\frac{p_1}{m_1}}w_1(t)\,dt\right)^{\frac{1}{p_1}},$$

where $p_1,p_2,m_1,m_2\in(0,\infty)$, u_1,u_2,w_1,w_2 are weights on $(0,\infty)$ and $p_2>m_2$. The most innovative part consists of the fact that possibly different general inner weights u_1 and u_2 are allowed. Proofs are based on a combination of duality techniques with weighted inequalities for iterated operators involving integrals and suprema.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we study weighted inequalities of the form

$$\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_2}u_2(s)\,ds\right)^{\frac{p_2}{m_2}}w_2(t)\,dt\right)^{\frac{1}{p_2}}\leq C\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_1}u_1(s)\,ds\right)^{\frac{p_1}{m_1}}w_1(t)\,dt\right)^{\frac{1}{p_1}}, \tag{1}$$

where p_1,p_2,m_1,m_2 are positive real numbers and u_1,u_2,w_1,w_2 are *weights*, that is, measurable non-negative functions on $(0,\infty)$ and $p_2>m_2$. The inequality is required to hold with some positive constant C for all scalar measurable functions f defined on a σ-finite measure space $(\mathcal{R},\mu)$. By f^* we denote the *non-increasing rearrangement* of f, given by

$$f^*(t)=\inf\{\lambda\in\mathbb{R}\colon \mu(\{x\in\mathcal{R}:|f(x)|>\lambda\})\leq t\}\quad \text{for } t\in(0,\infty).$$

Our main goal is to establish easily verifiable necessary and sufficient conditions on the parameters $p_1,p_2,m_1,m_2\in(0,\infty)$ and the weights u_1,u_2,w_1,w_2 for which (1) holds and to give two-sided estimates of the optimal constant C.

We denote by $\mathfrak{M}(\mathcal{R},\mu)$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty,\infty]$. We also define $\mathfrak{M}_+(\mathcal{R},\mu)=\{g\in\mathfrak{M}(\mathcal{R},\mu)\colon g\geq 0\}$.

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The inequality (1) can be viewed as a continuous embedding between appropriate function spaces. As usual, we say that a (quasi-)normed space X is *embedded* into another such space, Y, if $X \subset Y$ and the identity operator is continuous from X to Y. We denote by $\mathrm{G}\Gamma_{u,w}^{m,p}$ the collection of all functions $f \in \mathfrak{M}(\mathscr{R}, \mu)$ such that

$$\|f\|_{\mathrm{G}\Gamma_{u,w}^{m,p}} := \left(\int_0^\infty \left(\int_0^t f^*(s)^p u(s)\, ds \right)^{\frac{m}{p}} w(t)\, dt \right)^{\frac{1}{m}} < \infty,$$

where $m, p \in (0, \infty)$ and w, u are weights (on $(0, \infty)$. Under this notation, (1) is equivalent to the continuous embedding

$$\mathrm{G}\Gamma_{u_1,w_1}^{p_1,m_1} \hookrightarrow \mathrm{G}\Gamma_{u_2,w_2}^{p_2,m_2}. \tag{2}$$

Moreover, the norm of the embedding (2) coincides with the optimal (smallest) constant C that renders (1) true.

The study of function spaces involving weights and rearrangements goes back to early 1950's, when the fundamental paper of Lorentz [41] appeared, followed later by [42]. In [41], the space $\Lambda^p(v)$ was defined as the set of all $f \in \mathfrak{M}(\mathscr{R}, \mu)$ for which the functional

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty f^*(t)^p v(t)\, dt \right)^{\frac{1}{p}}$$

is finite, where $p \in (0, \infty)$ and v is a weight on $(0, \infty)$. These spaces proved to be indispensable in a wide range of disciplines of mathematical analysis, in particular in theory of interpolation, theory of operators of harmonic analysis and theory of partial differential equations. A major breakthrough in the theory was seen in 1990, when Ariño and Muckenhoupt in [2] characterized those parameters $p \in (1, \infty)$ and weights v for which the Hardy–Littlewood maximal operator is bounded on $\Lambda^p(v)$, and Sawyer in [47] developed a duality concept for spaces $\Lambda^p(v)$. Among other results, Sawyer obtained a generalization of the theorem of Ariño and Muckenhoupt to the situation in which two possibly different exponents and two possibly different weights are allowed. He also reformulated the action of the maximal operator on weighted Lebesgue spaces restricted to the cone of non-decreasing functions in terms of embeddings between function spaces by introducing the space $\Gamma^p(v)$ as the family of all $f \in \mathfrak{M}(\mathscr{R}, \mu)$ for which the functional

$$\|f\|_{\Gamma^p(v)} := \left(\int_0^\infty f^{**}(t)^p v(t)\, dt \right)^{\frac{1}{p}}$$

is finite, where f^{**} is the *maximal non-increasing rearrangement* of f, defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds \quad \text{for } t \in (0, \infty). \tag{3}$$

For every $f \in \mathfrak{M}(\mathcal{R},\mu)$ and every $t\in(0,\infty)$, the estimate $f^*(t)\le f^{**}(t)$ holds. As a consequence, one trivially has $\Gamma^p(v)\hookrightarrow\Lambda^p(v)$ for any p and v.

During the 1990's, the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ were put under a serious scrutiny under the common label *classical Lorentz spaces*. Their basic functional properties as well as embedding relations between them were characterized. It would be next to impossible to give a complete account of the literature which is available to this subject nowadays. Let us quote at least the efforts of M. Carro, A. García del Amo, M. Gol'dman, H. Heinig, L. Maligranda, J. Martín, C. Neugebauer, R. Oinarov, J. Soria, G. Sinnamon, V.D. Stepanov that resulted in a long series of papers, see [4, 7, 8, 9, 10, 24, 31, 32, 33, 34, 40, 44, 45, 48, 51, 52, 53, 54]. The first attempt to survey the situation in the field was given in [7] where the contemporary state of the art was described. Since then, however, important new results have been obtained and things have changed essentially again.

A significant progress in the study of classical Lorentz spaces was made in the early 2000's due to the efforts of Sinnamon [49, 50] and to the development of a new approach based on discretization and anti-discretization techniques in [25]. Using these new techniques, embeddings of classical Lorentz spaces in cases that had resisted for years were finally characterized, the notable last missing case being added in [6]. This rounded off one particular level of results.

As a consequence of these advances, the field could have been explored deeper (see e.g. [5, 6, 26, 27]). One of the most important innovations was the involvement of function spaces involving *inner weighted means*. In order to describe such function spaces, let us first consider the weighted version of (3), namely

$$f_u^{**}(t)=\frac{1}{U(t)}\int_0^t f^*(s)u(s)\,ds \quad \text{for } t\in(0,\infty), \tag{4}$$

where u is a given weight on $(0,\infty)$ and

$$U(t):=\int_0^t u(s)\,ds \quad \text{for } t\in(0,\infty).$$

Given $p\in(0,\infty)$ and another weight, v, on $(0,\infty)$, we define the space $\Gamma_u^p(v)$ as the collection of all functions $f\in\mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{\Gamma_u^p(v)}:=\left(\int_0^\infty f_u^{**}(t)^p v(t)\,dt\right)^{\frac{1}{p}}<\infty.$$

Some effort was spent in order to recover general embedding results for classical Lorentz spaces by methods that would avoid the powerful but technically complicated discretization-antidiscretization scheme, but only with a partial success (see e.g. [29, 30, 19]). a recent overview of the field of embeddings of classical Lorentz spaces can be found in [46, Chapter 10].

There exists plenty of motivation for studying relations between classical Lorentz spaces in great detail. For example, in the recent work [1], information about classical Lorentz spaces is used in order to investigate the continuity properties of local solutions to the n-Laplace equation

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = f(x) \quad \text{in } \Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^n$.

Recently, new spaces came into play, for a good reason. Given two parameters $m, p \in (0,\infty)$ and a weight v, on $(0,\infty)$, the space $G\Gamma(p,m,v)$ is defined as the the collection of all functions $f \in \mathfrak{M}(\mathscr{R},\mu)$ such that

$$\|f\|_{G\Gamma(p,m,v)} := \left(\int_0^b \left(\int_0^t f^*(s)^p \, ds \right)^{\frac{m}{p}} v(t)\, dt \right)^{\frac{1}{m}} < \infty.$$

These spaces turn out to be important among other reasons because of their intimate connection to the so-called *grand Lebesgue spaces* and their slightly younger relatives called *small Lebesgue spaces*. The grand Lebesgue space was introduced by Iwaniec and Sbordone in [35] in connection with integrability properties of Jacobians. Since it is a relatively complicated structure, it took some time before its dual was characterized. This was done by Fiorenza in [14]. In that paper also the small Lebesgue spaces were introduced. It was shown later by Fiorenza and Karadzhov in [15] that the norm in the small Lebesgue space can be equivalently expressed in terms of the functional governing the $G\Gamma(p,m,w)$ space with appropriate parameters and weights. Further results in this direction were obtained e.g. in [16, 17, 18]. The associate space of $G\Gamma(p,m,w)$ was then completely characterized in [28].

The techniques in the background of many of the results mentioned inevitably involve weighted inequalities involving Hardy-type integral operators. However, we also witness a still growing importance of weighted inequalities involving *supremum operators*. These operators have been studied recently (see e.g. [11], [23] or [21]) in connection with several problems in analysis including action of fractional maximal operators, optimality of function spaces in Sobolev embeddings, or the interpolation theory, but the available results are far from being complete.

In [25], the characterization of the embeddings of the form

$$\Gamma_u^q(w) \hookrightarrow \Gamma_u^p(v), \tag{5}$$

where $p, q \in (0,\infty)$ and u, v, w are weights on $(0,\infty)$, was completed. It was an important step ahead and applications followed instantly, but it still suffered from the principal restriction that the inner weight u had to be the same on both sides of the embedding.

On the side of applications, there exists a significant desire for two-sided estimates of optimal constants in embeddings of the type (5) with *two possibly different inner weights*. The motivation arises usually in tasks that involve, in a way, two possibly different integral mean operators. To give at least one example, let us recall the long-time extensive research of the optimality of function spaces in Sobolev-type embeddings, carried out e.g. in [13, 36, 37, 38, 12]. For instance, the considerations in [38, Theorem 3.1], where the explicit formula for the optimal rearrangement-invariant function norm in a Sobolev inequality is sought and the known implicit one is reduced to a formula involving an integral mean with respect to another weight function, show that characterizations of embeddings of the form (1) are useful.

Most of the functions which we shall deal with will be defined on $(0,\infty)$. If this is the case, then $(\mathscr{R},\mu)$ is the interval $(0,\infty)$ endowed with the one-dimensional Lebesgue measure λ_1, and we shall write just $\mathfrak{M}$ and $\mathfrak{M}_+$ instead of $\mathfrak{M}((0,\infty),\lambda_1)$ and $\mathfrak{M}_+((0,\infty),\lambda_1)$ respectively.

Let u_1, u_2, w_1 and w_2 be weights on $(0,\infty)$ and $t\in(0,\infty)$. We will use the following notation:

$$U_1(t)=\int_0^t u_1(s)\,ds,\ U_2(t)=\int_0^t u_2(s)\,ds,\ W_1(t)=\int_0^t w_1(s)\,ds,\ W_2(t)=\int_0^t w_2(s)\,ds.$$

Further, let $m_1,m_2,p_1,p_2\in(1,\infty)$. We define

$$\varphi(t)=\int_0^t U_1(s)^{\frac{p_1}{m_1}}w_1(s)\,ds+U_1(t)^{\frac{p_1}{m_1}}\int_t^\infty w_1(s)\,ds\quad\text{for } t\in(0,\infty).$$

Note that, for every $t\in(0,\infty)$, one has $\varphi(t)=\|\chi_{(0,t)}\|_{G\Gamma^{p_1,m_1}_{u_1,w_1}(0,\infty)}$. We also set

$$\sigma(t)=\frac{U_1(t)^{\frac{p_1^2}{m_1(p_1-m_2)}-1}u_1(t)\int_0^t U_1(s)^{\frac{p_1}{m_1}}w_1(s)\,ds\int_t^\infty w_1(s)\,ds}{\varphi(t)^{\frac{p_1}{p_1-m_2}+1}},\quad t\in(0,\infty).$$

Throughout the paper, the expressions of the form $0\cdot\infty$ or $\frac{0}{0}$ are taken as zero. For $p\in(1,\infty)$, we define $p'=\frac{p}{p-1}$. We write $A\approx B$ when the ratio A/B is bounded from below and from above by positive constants independent of appropriate quantities appearing in expressions A and B.

We shall now state the principal result of the paper.

Theorem 1.1. *Let $m_1,m_2,p_1,p_2\in(1,\infty)$. Assume that $p_2>m_2$. Let u_1, u_2, w_1 and w_2 be weights. Assume that*

- *u_1 is strictly positive,* $\int_0^t u_1(s)\,ds<\infty$ *for all* $t\in(0,\infty)$, $\int_0^\infty u_1(t)\,dt=\infty$,
- $\int_0^t w_1(s)U_1(s)^{\frac{p_1}{m_1}}\,ds<\infty$, $\int_t^\infty w_1(s)U_1(s)^{\frac{p_1}{m_1}}\,ds=\infty$ *for all* $t\in(0,\infty)$,
- $\int_0^t w_1(s)\,ds=\infty$, $\int_t^\infty w_1(s)\,ds<\infty$ *for all* $t\in(0,\infty)$.

Let

$$C=\sup_{f\in\mathfrak{M}}\frac{\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_2}u_2(s)\,ds\right)^{\frac{p_2}{m_2}}w_2(t)\,dt\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty\left(\int_0^t f^*(s)^{m_1}u_1(s)\,ds\right)^{\frac{p_1}{m_1}}w_1(t)\,dt\right)^{\frac{1}{p_1}}}. \tag{6}$$

(a) *Let $m_1\le m_2$ and $p_1\le m_2$. Then*

$$C\approx B_1,$$

where

$$B_1=\sup_{t\in(0,\infty)}\frac{\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds+U_2(t)^{\frac{p_2}{m_2}}\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{p_2}}}{\varphi(t)^{\frac{1}{p_1}}}.$$

(b-i) *Let $m_1\le m_2$, $p_1>m_2$ and $p_1\le p_2$. Then*

$$C\approx B_2+B_3,$$

where

$$B_2=\sup_{t\in(0,\infty)}\left(U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}\int_0^t\sigma(s)\,ds+\int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds\right)^{\frac{1}{p_2}}$$

and

$$B_3=\sup_{t\in(0,\infty)}\left(\int_0^t\sup_{y\in(s,t)}U_2(y)^{\frac{p_1}{p_1-m_2}}U_1(y)^{-\frac{m_2p_1}{m_1(p_1-m_2)}}\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}\left(\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{p_2}}.$$

(b-ii) *Let $m_1\le m_2$, $p_1>m_2$ and $p_1>p_2$. Then*

$$C\approx B_4+B_5+B_6+B_7,$$

where

$$B_4=\left(\int_0^\infty\left(\int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}\sigma(s)\,ds\right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}}U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}\right.$$
$$\left.\times\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds\right)^{\frac{p_1}{p_1-m_2}}\sigma(t)\,dt\right)^{\frac{p_1-p_2}{p_1p_2}},$$

$$B_5=\left(\int_0^\infty\sup_{s\in(t,\infty)}U_1(s)^{-\frac{p_1p_2}{m_1(p_1-p_2)}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{p_1}{p_1-p_2}}\left(\int_0^t\sigma(s)\,ds\right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}}\sigma(t)\,dt\right)^{\frac{p_1-p_2}{p_1p_2}},$$

$$B_6 = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{p_1p_2}{m_2(p_1-p_2)}} U_1(s)^{-\frac{p_1p_2}{m_1(p_1-p_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{p_1-p_2}{p_1p_2}}$$

and

$$B_7 = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{p_1}{p_1-m_2}} U_1(s)^{-\frac{p_1m_2}{m_1(p_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} \times \left(\int_0^t \sup_{y\in(s,t)} U_2(y)^{\frac{p_1}{p_1-m_2}} U_1(y)^{-\frac{p_1m_2}{m_1(p_1-m_2)}} \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{p_1-p_2}{p_1p_2}}.$$

(c-i) *Let $m_1 > m_2$, $p_1 \le m_2$ and $m_1 \le p_2$. Then*

$$C \approx B_8 + B_9,$$

where

$$B_8 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{m_1}}}{\varphi(t)^{\frac{1}{p_1}}} \sup_{s\in(t,\infty)} U_1(s)^{-\frac{1}{m_1}} \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)\,dy \right)^{\frac{1}{p_2}}$$

and

$$B_9 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{m_1}}}{\varphi(t)^{\frac{1}{p_1}}} \sup_{s\in(t,\infty)} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{1}{p_2}} \left(\int_t^s U_2(y)^{\frac{m_1}{m_1-m_2}} U_1(y)^{-\frac{m_1}{m_1-m_2}} u_1(y)\,dy \right)^{\frac{m_1-m_2}{m_1m_2}}.$$

(c-ii) *Let $m_1 > m_2$, $p_1 \le m_2$ and $m_1 > p_2$. Then*

$$C \approx B_{10} + B_{11} + B_{12},$$

where

$$B_{10} = \sup_{t\in(0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{1}{p_2}}}{\varphi(t)^{\frac{1}{p_1}}},$$

$$B_{11} = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{m_1}} \left(\int_t^\infty \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)\,dy \right)^{\frac{p_2}{m_1-p_2}} U_2(s)^{\frac{p_2}{m_2}} w_2(s) U_1(s)^{-\frac{p_2}{m_1-p_2}}\,ds \right)^{\frac{m_1-p_2}{m_1p_2}}}{\varphi(t)^{\frac{1}{p_1}}}$$

and

$$B_{12}=\sup_{t\in(0,\infty)}\frac{U_1(t)^{\frac{1}{m_1}}}{\varphi(t)^{\frac{1}{p_1}}}\left(\int_t^\infty\left(\int_t^s U_2(y)^{\frac{m_1}{m_1-m_2}}U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{p_2(m_1-m_2)}{m_2(m_1-p_2)}}\right.$$
$$\left.\times\left(\int_s^\infty w_2(y)\,dy\right)^{\frac{p_2}{m_1-p_2}}w_2(t)\,dt\right)^{\frac{m_1-p_2}{m_1p_2}}.$$

(d-i) *Let $m_2<p_1<m_1\le p_2$. Then*

$$C\approx B_{13}+B_{14}+B_{15},$$

where

$$B_{13}=\sup_{t\in(0,\infty)}\left(\int_0^t\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}U_1(t)^{-\frac{1}{m_1}}\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds\right)^{\frac{1}{p_2}},$$

$$B_{14}=\sup_{t\in(0,\infty)}\left(\int_t^\infty U_1(s)^{-\frac{p_1m_2}{m_1(p_1-m_2)}}\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds\right)^{\frac{1}{p_2}}$$

and

$$B_{15}=\sup_{t\in(0,\infty)}\left(\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{p_2}}\left(\int_0^t\left(\int_s^t U_1(y)^{-\frac{m_1}{m_1-m_2}}U_2(y)^{\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{p_1(m_1-m_2)}{m_1(p_1-m_2)}}\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}.$$

(d-ii) *Let $m_2<p_1\le p_2<m_1$. Then*

$$C\approx B_{14}+B_{15}+B_{16},$$

where

$$B_{16}=\sup_{t\in(0,\infty)}\left(\int_0^t\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}\left(\int_t^\infty U_1(s)^{-\frac{p_2}{m_1-p_2}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{p_2}{m_1-p_2}}\right.$$
$$\left.\times U_2(s)^{\frac{p_2}{m_2}}w_2(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^t\sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}\left(\int_t^\infty\left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}U_2(y)^{\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{p_2(m_1-m_2)}{m_2(m_1-p_2)}}\right.$$
$$\left.\times\left(\int_s^\infty w_2(y)\,dy\right)^{\frac{p_2}{m_1-p_2}}w_2(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}}.$$

(d-iii) *Let $m_2 < p_2 < p_1 < m_1$. Then*

$$C \approx B_{17},$$

where

$$B_{17} = \left(\int_0^\infty \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{p_1}{p_1-p_2}} \left(\int_t^\infty U_1(s)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \times U_1(t)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \sigma(t)\,dt \right)^{\frac{p_1-p_2}{p_1 p_2}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{p_2}{m_1-p_2}} \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)\,dy \right)^{\frac{p_2}{m_1-p_2}} U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{p_1(m_1-p_2)}{m_1(p_1-p_2)}} \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{p_1-p_2}{p_1 p_2}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}} U_2(y)^{\frac{m_1}{m_1-m_2}} u_1(y)\,dy \right)^{\frac{p_2(m_1-m_2)}{m_2(m_1-p_2)}} \times \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_2}{m_1-p_2}} w_2(s)\,ds \right)^{\frac{p_1(m_1-p_2)}{m_1(p_1-p_2)}} \left(\int_0^t \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{p_1-p_2}{p_1 p_2}}$$

$$+ \left(\int_0^\infty \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{m_1}{m_1-m_2}} U_2(y)^{\frac{m_1}{m_1-m_2}} u_1(y)\,dy \right)^{\frac{m_1-m_2}{m_1}} \sigma(s)\,ds \right)^{\frac{p_2(p_1-m_2)}{m_2(p_1-p_2)}} \times \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{p_2}{p_1-p_2}} w_2(t)\,dt \right)^{\frac{p_1-p_2}{p_1 p_2}}.$$

The cases when either $p_2 < m_2$ or $p_2 > m_2$, $p_1 > m_2$, $m_1 > m_2$ and $p_1 \geq m_1$ remain open. In the case when $p_2 = m_2$, the space $G\Gamma_{u_2,w_2}^{p_2,m_2}$ degenerates to a classical Lorentz space of type Λ for which everything is known ([25]).

The key ingredient of the proof of Theorem 1.1 is a combination of duality techniques with embedding results for classical Lorentz spaces and estimates of optimal constants in weighted inequalities involving iterated integral and supremum operators. Detailed analysis of separate cases leads to the need of necessary and sufficient conditions for various, quite different in nature, inequalities, of which only some are known. Interestingly, some of these results have been obtained only quite recently, such as [20], for instance. Even more interestingly, some are not known at all and will appear here for the first time.

The proof can be naturally expected to be quite technical and to involve plenty of computation. There is hardly any way to avoid it. We shall therefore do our best to simplify the notation, shorten the formulas, and make the exposition as reader-friendly as possible.

The paper is organized as follows. In the next section we collect the necessary background material. We intend to save the reader plenty of tedious work since the relevant results are scattered over literature with inconsistent notation. We also characterize several inequalities involving iterated integral and supremum operators which are not available and will also be needed in the proofs. In the last section we present the proof of Theorem 1.1.

2. Background material

In this section we collect background results that will be used in the proof of the main theorem.

We begin with the well-known duality principle in weighted Lebesgue spaces. If
$p \in (1,\infty)$, $f \in \mathfrak{M}_+$ and v is a weight on $(0,\infty)$, then

$$\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}} = \sup_{h\in\mathfrak{M}_+} \frac{\int_0^\infty f(t)h(t)\,dt}{\left(\int_0^\infty h(t)^{p'} v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}}}. \tag{7}$$

Let us now recall a quantified version of classical Hardy inequalities.

Theorem 2.1 ([3, Theorem 1] and [43, Theorem 1.3.1]). *Let $1<p,q<\infty$ and let u,v,w be weights on $(0,\infty)$. Let*

$$K = \sup_{f\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t f(s)u(s)\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}}}.$$

(a) *Let $1<p\le q<\infty$. Then $K \approx A_1$, where*

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_t^\infty w(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^t u(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *Let* $1 < q < p < \infty$. *Then* $K \approx A_2$, *where*

$$A_2 = \left(\int_0^\infty \left(\int_t^\infty w(s)\,ds \right)^{\frac{p}{p-q}} \left(\int_0^t u(s)^{p'} v(s)^{1-p'}\,ds \right)^{\frac{p(q-1)}{p-q}} u(t)^{p'} v(t)^{1-p'}\,dt \right)^{\frac{p-q}{pq}}.$$

Theorem 2.2 ([3, Theorem 2] and [43, Theorem 1.3.2]). *Let* $1 < p, q < \infty$ *and let* v *and* w *be weights on* $(0, \infty)$. *Let*

$$K = \sup_{f \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty f(s)\,ds \right)^q w(t)\,dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)\,dt \right)^{\frac{1}{p}}}.$$

(a) *Let* $1 < p \le q < \infty$. *Then* $K \approx A_1$, *where*

$$A_1 = \sup_{t \in (0,\infty)} \left(\int_0^t w(s)\,ds \right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\,ds \right)^{\frac{1}{p'}}.$$

(b) *Let* $1 < q < p < \infty$. *Then* $K \approx A_2$, *where*

$$A_2 = \left(\int_0^\infty \left(\int_0^t w(s)\,ds \right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\,ds \right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\,dt \right)^{\frac{p-q}{pq}}.$$

We now turn our attention to inequalities involving supremum operators.

Theorem 2.3 ([23, Theorem 4.1(i) and Theorem 4.4]). *Let* $0 < p, q < \infty$. *Let* u *be a continuous weight and let* v, w *and* ϱ *be weights such that* $0 < \int_0^t v(s)\,ds < \infty$ *and* $0 < \int_0^t w(s)\,ds < \infty$ *for every* $t \in (0, \infty)$. *Let*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \sup\limits_{s \in (t,\infty)} u(s)^q \left(\int_0^s g(y) \varrho(y)\,dy \right)^q w(t)\,dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty g(t)^p v(t)\,dt \right)^{\frac{1}{p}}}.$$

(a) *Let* $1 < p \le q < \infty$. *Then* $K \approx A_1$, *where*

$$A_1 = \sup_{t \in (0,\infty)} \left(\sup_{s \in (t,\infty)} u(s)^q \int_0^s w(y)\,dy + \int_t^\infty \sup_{y \in (s,\infty)} u(y)^q w(s)\,ds \right)^{\frac{1}{q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'} ds \right)^{\frac{1}{p'}}.$$

(b) *Let* $1 \le p < \infty$ *and* $0 < q < p$. *Then* $K \approx A_2 + A_3$, *where*

$$A_2 = \left(\int_0^\infty \sup_{s \in (t,\infty)} u(s)^q \left(\int_t^\infty \sup_{y \in (s,\infty)} u(y)^q w(s)\,ds \right)^{\frac{q}{p-q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds \right)^{\frac{q(p-1)}{p-q}} w(t)\,dt \right)^{\frac{p-q}{pq}}$$

and

$$A_3 = \left(\int_0^\infty \sup_{s \in (t,\infty)} u(s)^{\frac{pq}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\, dy \right)^{\frac{q(p-1)}{p-q}} \left(\int_0^t w(s)\, ds \right)^{\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{pq}}.$$

One of the most important ingredients of the proof of the main theorem will be the following quantified version of an embedding between classical Lorentz spaces in a certain particular case.

Theorem 2.4 ([25, Theorem 4.2]). *Let u, v, w be weights on $[0,\infty)$. Let $p, q \in (0,\infty)$. Assume that the following conditions are satisfied:*

- $\lim_{t\to\infty} U(t) = \infty$,
- $\int_0^\infty \frac{v(s)}{U(s)^p + U(t)^p}\, ds < \infty$ *for every* $t \in (0,\infty)$,
- $\int_0^1 \frac{v(s)}{U(s)^p}\, ds = \infty$,
- $\int_1^\infty v(s)\, ds = \infty$.

Let

$$K = \sup_{f \in \mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(t)^q w(t)\, dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty f_u^{**}(t)^p v(t)\, dt\right)^{\frac{1}{p}}}.$$

(a) *If $0 < p \le q < \infty$ and $1 \le q < \infty$, then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{t \in (0,\infty)} \frac{W(t)^{\frac{1}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{1}{p}}}.$$

(b) *If $1 \le q < p < \infty$, then*

$$K \approx A_2,$$

where

$$A_2 = \left(\int_0^\infty \frac{\sup_{y \in (t,\infty)} U(y)^{-\frac{pq}{p-q}} W(y)^{\frac{p}{p-q}} V(t) U(t)^{\frac{pq}{p-q}+p-1} u(t) \int_t^\infty U(s)^{-p} v(s)\, ds}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{p}{p-q}+1}}\, dt \right)^{\frac{p-q}{pq}}.$$

(c) *If $0 < p \le q < 1$, then*

$$K \approx A_3,$$

where

$$A_3 = \sup_{t \in (0,\infty)} \frac{W(t)^{\frac{1}{q}} + U(t) \left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds \right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{1}{p}}}.$$

(d) *If $0 < q < 1$ and $0 < q < p$, then*

$$K \approx A_4,$$

where

$$A_4 = \left(\int_0^\infty \frac{\left(W(t)^{\frac{1}{1-q}} + U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}} \, ds \right)^{\frac{p(1-q)}{p-q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds \right)^{\frac{p}{p-q}+1}} \right.$$
$$\left. \times V(t) U(t)^{p-1} u(t) \int_t^\infty U(s)^{-p} v(s)\, ds \, dt \right)^{\frac{p-q}{pq}}.$$

We now recall characterization of a weighted inequality involving a kernel operator.

Theorem 2.5 ([45, Theorems 1.1 and 1.2])**.** *Let* $1 < p, q < \infty$ *and let* v *and* w *be weights. Let*

$$K = \sup_{f \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t h(s) \int_s^t u(y)\, dy \, ds \right)^q w(t)\, dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty (f(t))^p v(t)\, dt \right)^{\frac{1}{p}}}.$$

(a) *Let* $1 < p \le q < \infty$*. Then* $K \approx A_1 + A_2$*, where*

$$A_1 = \sup_{t \in (0,\infty)} \left(\int_t^\infty \left(\int_s^t u(y)\, dy \right)^q w(s)\, ds \right)^{\frac{1}{q}} \left(\int_0^t v(s)^{1-p'} \, ds \right)^{\frac{1}{p'}}$$

and

$$A_2 = \sup_{t \in (0,\infty)} \left(\int_t^\infty w(s)\, ds \right)^{\frac{1}{q}} \left(\int_0^t \left(\int_s^t u(y)\, dy \right)^{p'} v(s)^{1-p'} \, ds \right)^{\frac{1}{p'}}.$$

(b) *Let* $1 < q < p < \infty$*. Then* $K \approx A_3 + A_4$*, where*

$$A_3 = \left(\int_0^\infty \left(\left(\int_s^\infty \left(\int_s^t u(y)\, dy \right)^q w(s)\, ds \right) \left(\int_0^t v(s)^{1-p'} \, ds \right)^{q-1} \right)^{\frac{p}{p-q}} v(t)^{1-p'} \, dt \right)^{\frac{p-q}{pq}}$$

and

$$A_4 = \left(\int_0^\infty \left(\left(\int_t^\infty w(s)\, ds \right) \left(\int_0^t \left(\int_s^t u(y)\, dy \right)^{p'} v(s)^{1-p'} \, ds \right)^{p-1} \right)^{\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{pq}}.$$

Now we shall present a quantified version of a weighted inequality involving a specific combination of a supremum operator and an integral operator.

Theorem 2.6 ([39, Theorem 6]). *Let v and w be weights on $(0,\infty)$ and let u be a continuous weight on $(0,\infty)$. Let*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \sup\limits_{s\in(t,\infty)} u(s)^q \left(\int_s^\infty g(y)\,dy\right)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}}.$$

(a) *Assume that $1<p\le q<\infty$. Then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_0^t \sup_{y\in(s,t)} u(y)^q w(s)\,ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *Assume that $1<p<\infty$ and $0<q<p<\infty$. Then*

$$K \approx A_2 + A_3,$$

where

$$A_2 = \left(\int_0^\infty \sup_{s\in(t,\infty)} u(s)^{\frac{pq}{p-q}} W(t)^{\frac{q}{p-q}} w(t) \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{q(p-1)}{p-q}} dt\right)^{\frac{p-q}{pq}}$$

and

$$A_3 = \left(\int_0^\infty \sup_{s\in(t,\infty)} u(s)^q \left(\int_s^\infty v(y)^{1-p'}\,dy\right)^{\frac{q(p-1)}{p-q}} \left(\int_0^t \sup_{y\in(s,t)} u(y)^q w(s)\,ds\right)^{\frac{q}{p-q}} w(t)\,dt\right)^{\frac{p-q}{pq}}.$$

At one stage of the proof of the main result, a reformulation of conditions on weights will be required. This will be done through the following elementary lemma.

Lemma 2.7. *Let w, u be weights. Assume that*

$$\int_0^\infty u(t)\,dt = \infty.$$

Let $0<q<1$. Then, for every $t\in(0,\infty)$, one has

$$W(t)^{\frac{1}{q}} + U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} \approx U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}},$$

in which the constants of equivalence depend only on q.

Proof. Fix $t \in (0,\infty)$. Integration by parts yields

$$\begin{aligned}(8)\qquad &\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\, ds \\ &= q\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds + (1-q)\left(\lim_{y\to\infty} \frac{W(y)^{\frac{1}{1-q}}}{U(y)^{\frac{q}{1-q}}} - \frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}}\right).\end{aligned}$$

Therefore, we immediately have

$$\begin{aligned}&\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\, ds \\ &\le q\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds + (1-q)\lim_{y\to\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}}.\end{aligned}$$

Next,

$$\begin{aligned}\lim_{y\to\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}} &\le \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}} \\ &= \tfrac{q}{1-q} \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} \int_y^\infty U(s)^{-\frac{1}{1-q}} u(s)\, ds \\ &\le \tfrac{q}{1-q} \sup_{t\le y<\infty} \int_y^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds \\ &= \tfrac{q}{1-q} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.\end{aligned}$$

Altogether, we obtain

$$(9)\qquad \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\, ds \le 2q \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.$$

We also have

$$\begin{aligned}W(t)^{\frac{1}{1-q}} &= W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} U(t)^{-\frac{q}{1-q}} \\ &= \tfrac{1-q}{q} W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} \int_t^\infty U(s)^{-\frac{1}{1-q}} u(s)\, ds \\ &\le \tfrac{1-q}{q} U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.\end{aligned}$$

Raising the inequality to $\frac{1-q}{q}$, we get

$$W(t)^{\frac{1}{q}} \leq (\tfrac{1-q}{q})^{\frac{1-q}{q}} U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}}. \tag{10}$$

Altogether, (9) and (10) imply

$$\begin{aligned} &W(t)^{\frac{1}{q}} + U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} \\ &\leq C_q U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}} \end{aligned}$$

in which

$$C_q = \left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}} + (2q)^{\frac{1-q}{q}}.$$

Conversely, by (8) again, we have

$$\begin{aligned} &\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds \\ &\leq \tfrac{1}{q}\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds + \left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}} \frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}}. \end{aligned}$$

Raising this estimate to $\frac{1-q}{q}$ and multiplying it by $U(t)$, we obtain

$$\begin{aligned} &U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}} \\ &\leq \left(\tfrac{1}{q}\right)^{\frac{1-q}{q}} U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} + \left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}} W(t)^{\frac{1}{q}}. \end{aligned}$$

The proof is complete. □

We finish this section with two theorems in which we characterize weighted inequalities involving iteration of two integral operators.

Theorem 2.8. *Assume that $p, q, m \in (1,\infty)$ and $q < m$. Let u, v, w be weights on $(0,\infty)$. Let*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty g(y)\,dy\right)^q u(s)\,ds\right)^{\frac{m}{q}} w(t)\,dt\right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}}.$$

(a) *Let $1 < p \leq q < \infty$. Then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_t^\infty v(s)^{1-p'}\,ds \right)^{\frac{1}{p'}} \left(\int_0^t \left(\int_s^t u(y)\,dy \right)^{\frac{m}{q}} w(s)\,ds \right)^{\frac{1}{m}}.$$

(b) *Let* $1 < q < p < \infty$ *and* $p \le m$. *Then*

$$K \approx A_1 + A_2,$$

where

$$A_2 = \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy \right)^{\frac{p}{p-q}} \left(\int_s^\infty v(y)^{1-p'}\,dy \right)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\,ds \right)^{\frac{p-q}{pq}} \left(\int_0^t w(s)\,ds \right)^{\frac{1}{m}}.$$

(c) *Let* $1 < q < p < \infty$ *and* $m < p$. *Then*

$$K \approx A_3 + A_4,$$

where

$$A_3 = \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy \right)^{\frac{p}{p-q}} \left(\int_s^\infty v(y)^{1-p'}\,dy \right)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\,ds \right)^{\frac{m(p-q)}{q(p-m)}} W(s)^{\frac{m}{p-m}} w(s)\,ds \right)^{\frac{p-m}{pm}}$$

and

$$A_4 = \left(\int_0^\infty \left(\int_t^\infty v(s)^{1-p'}\,ds \right)^{\frac{p(m-1)}{p-m}} \left(\int_0^t \left(\int_s^t u(y)\,dy \right)^{\frac{m}{q}} w(s)\,ds \right)^{\frac{p}{p-m}} v(t)^{1-p'}\,dt \right)^{\frac{p-m}{pm}}.$$

Proof. We first observe that, by (7), one has

$$K = \sup_{g\in\mathfrak{M}_+} \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty h(t) \int_t^\infty \left(\int_s^\infty g(y)\,dy\right)^q u(s)\,ds\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}} \left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{mq}}}.$$

Interchanging suprema and using the Fubini theorem, we obtain

$$K = \sup_{h\in\mathfrak{M}_+} \frac{1}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{mq}}} \times \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty g(y)\,dy\right)^q \int_0^s h(t)\,dt\, u(s)\,ds\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}}. \tag{11}$$

Let $1 < p \le q < \infty$. Then, by Theorem 2.2(a), we get

$$\sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty g(y)\,dy\right)^q \int_0^s h(t)\,dt\, u(s)\,ds\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}} \approx \sup_{t\in(0,\infty)} \left(\int_0^t u(s) \int_0^s h(y)\,dy\,ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

Plugging this to (11), we get

$$K \approx \sup_{h\in\mathfrak{M}_+} \frac{\sup_{t\in(0,\infty)} \left(\int_0^t u(s)\int_0^s h(y)\,dy\,ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{mq}}}.$$

Now we interchange the suprema again, apply the Fubini theorem and raise all the expressions to q. We obtain

$$K^q \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{q}{p'}} \sup_{h\in\mathfrak{M}_+} \frac{\int_0^t h(s)\int_s^t u(y)\,dy\,ds}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{m}}}.$$

By (7), this yields $K \approx A_1$, proving the assertion in the case (a).

Let now $1 < q < p < \infty$. Then, by Theorem 2.1(b), we have

$$K^q \approx \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t u(s)\int_0^s h(y)\,dy\,ds\right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\,dt\right)^{\frac{p-q}{p}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{m}}}.$$

Consequently, by the Fubini theorem,

$$K^q \approx \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t h(y)\int_y^t u(s)\,ds\,dy\right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\,dt\right)^{\frac{p-q}{p}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{m}}}.$$

Now, in the case (b) the assertion follows from Theorem 2.5(a) and in the case (c) from Theorem 2.5(b). ☐

Theorem 2.9. *Assume that $m,p,q \in (1,\infty)$ and let u,v,w and ϱ be weights on $(0,\infty)$. Assume that $q < m$. Let*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y)\varrho(y)\,dy\right)^q u(s)\,ds\right)^{\frac{m}{q}} w(t)\,dt\right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}}.$$

(a) *If $p \le q < m$, then*

$$K \approx \sup_{t\in(0,\infty)} W(t)^{\frac{1}{m}} \left(\int_t^\infty u(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{m}{q}} w(s)\,ds\right)^{\frac{1}{m}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *If $q < p \le m$, then*

$$K \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{m}{q}} w(s)\,ds\right)^{\frac{1}{m}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}$$

$$+ \sup_{t\in(0,\infty)} W(t)^{\frac{1}{m}} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\,dy\right)^{\frac{p(q-1)}{p-q}} \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{p-q}{pq}}.$$

(c) *If $q < m < p$, then*

$$K \approx \left(\int_0^\infty \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{m(p-1)}{p-m}} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{m}{q}} w(s)\,ds\right)^{\frac{m}{p-m}}\right.$$

$$\left.\times \left(\int_t^\infty u(y)\,dy\right)^{\frac{m}{q}} w(t)\,dt\right)^{\frac{p-m}{mp}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\right)^{\frac{p(q-1)}{p-q}} \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{m(p-q)}{q(p-m)}}\right.$$

$$\left.\times W(t)^{\frac{m}{p-m}} w(t)\,dt\right)^{\frac{p-m}{mp}}.$$

Proof. The proof can be done in the same way as that of Theorem 2.8. □

We note that the assertion of Theorem 2.9 can be also extracted from [22], where however the characterizing conditions are formulated in modified way and where a completely different proof is presented.

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. As the first step of our analysis we will express the value of C in a modified way. For every fixed $g \in \mathfrak{M}_+$, set

$$A(g) = \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty h^*(t)^{\frac{m_2}{m_1}} u_2(t) \int_t^\infty g(s)\,ds\,dt\right)^{\frac{m_1}{m_2}}}{\left(\int_0^\infty h^{**}_{u_1}(t)^{\frac{p_1}{m_1}} w_1(t) U_1(t)^{\frac{p_1}{m_1}}\,dt\right)^{\frac{m_1}{p_1}}},$$

where we apply the notation introduced in (4). We claim that

$$C = \sup_{g \in \mathfrak{M}_+} \frac{A(g)^{\frac{1}{m_1}}}{\left(\int_0^\infty g(t)^{\frac{p_2}{p_2-m_2}} w_2(t)^{-\frac{m_2}{p_2-m_2}}\,dt\right)^{\frac{p_2-m_2}{p_2 m_2}}}. \tag{12}$$

Indeed, fix $f \in \mathfrak{M}$. Since $\frac{p_2}{m_2} > 1$, we can apply (7) to $p = \frac{p_2}{m_2}$ and $v = w_2$. Then $p' = \frac{p_2}{p_2-m_2}$ and $1 - p' = -\frac{m_2}{p_2-m_2}$, and so we get

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{m_2} u_2(s)\,ds\right)^{\frac{p_2}{m_2}} w_2(t)\,dt\right)^{\frac{1}{p_2}} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty g(t) \int_0^t f^*(s)^{m_2} u_2(s)\,ds\,dt\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g(s)^{\frac{p_2}{p_2-m_2}} w_2(s)^{-\frac{m_2}{p_2-m_2}}\,ds\right)^{\frac{p_2-m_2}{p_2 m_2}}}.$$

By the Fubini theorem, this turns into

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{m_2} u_2(s)\,ds\right)^{\frac{p_2}{m_2}} w_2(t)\,dt\right)^{\frac{1}{p_2}} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{m_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g(s)^{\frac{p_2}{p_2-m_2}} w_2(s)^{-\frac{m_2}{p_2-m_2}}\,ds\right)^{\frac{p_2-m_2}{p_2 m_2}}}.$$

Plugging this into (6), we get

$$C = \sup_{f \in \mathfrak{M}} \frac{1}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{m_1} u_1(s)\,ds\right)^{\frac{p_1}{m_1}} w_1(t)\,dt\right)^{\frac{1}{p_1}}} \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{m_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g(s)^{\frac{p_2}{p_2-m_2}} w_2(s)^{-\frac{m_2}{p_2-m_2}}\,ds\right)^{\frac{p_2-m_2}{p_2 m_2}}}.$$

On interchanging suprema, this yields

$$C = \sup_{g \in \mathfrak{M}_+} \frac{1}{\left(\int_0^\infty g(s)^{\frac{p_2}{p_2-m_2}} w_2(s)^{-\frac{m_2}{p_2-m_2}}\,ds\right)^{\frac{p_2-m_2}{p_2 m_2}}} \sup_{f \in \mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{m_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{m_1} u_1(s)\,ds\right)^{\frac{p_1}{m_1}} w_1(t)\,dt\right)^{\frac{1}{p_1}}}.$$

Now, for a change, fix $g \in \mathfrak{M}_+$. Given $f \in \mathfrak{M}$, set $h = |f|^{m_1}$. Then $f^* = (h^*)^{\frac{1}{m_1}}$, and we have

$$\sup_{f \in \mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{m_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{m_1} u_1(s)\,ds\right)^{\frac{p_1}{m_1}} w_1(t)\,dt\right)^{\frac{1}{p_1}}} = \sup_{h \in \mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^{\frac{m_2}{m_1}} u_2(t) \int_t^\infty g(s)\,ds\,dt\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty h^{**}_{u_1}(t)^{\frac{p_1}{m_1}} w_1(t) U_1(t)^{\frac{p_1}{m_1}}\,dt\right)^{\frac{1}{p_1}}}.$$

The quantity on the right-hand side now equals $A(g)^{\frac{1}{m_1}}$. This establishes (12).

We next observe that, for every fixed $g \in \mathfrak{M}_+$, one has

$$A(g) = \sup_{h \in \mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty h_u^{**}(t)^p v(t)\,dt\right)^{\frac{1}{p}}}$$

with

(13) $$p = \frac{p_1}{m_1},\ q = \frac{m_2}{m_1}$$

and

(14)

$$w(t) = u_2(t)\int_t^\infty g(s)\,ds, \quad v(t) = U_1(t)^{\frac{p_1}{m_1}} w_1(t), \quad u(t) = u_1(t), \qquad t \in (0,\infty).$$

Now, the quantity $A(g)$ can be equivalently evaluated in terms of parameters p, q and weights u, v, w via Theorem 2.4 (we note that the assumptions of that theorem are fulfilled). However, the expressions in cases (c) and (d) are not in a satisfactory form and we have to modify them through Lemma 2.7. The reason will become apparent soon – roughly speaking, we need to get rid of all the expressions that involve w and have to replace them by those involving W instead. Thus, by Lemma 2.7, we get

(c) if $0 < p \le q < 1$, then

$$A(g) \approx \sup_{t \in (0,\infty)} \frac{U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\,ds\right)^{\frac{1}{p}}},$$

and

(d) if $0 < q < 1$ and $0 < q < p$, then

$$A(g) \approx \left(\int_0^\infty \frac{U(t)^{\frac{pq}{p-q}+p-1} V(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{-\frac{p(q-1)}{p-q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\,ds\right)^{\frac{p}{p-q}+1}} \times \int_t^\infty U(s)^{-p} v(s)\,ds\,dt\right)^{\frac{p-q}{pq}}.$$

Our next step is "translation" of expressions characterizing $A(g)$ in cases (a)-(d) into the language of the parameters and weights occurring in Theorem 1.1 via (13) and (14). These expressions depend on g in a somewhat concealed way, namely through the weight w. It will be useful to note that

$$\varphi(t) = V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\,ds$$

and

$$W(t) = \int_0^t g(s)U_2(s)\,ds + U_2(t)\int_t^\infty g(s)\,ds.$$

We obtain the following reformulations of $A(g)$:

(a) if $m_1 \le m_2$ and $p_1 \le m_2$, then

$$A(g) \approx \sup_{t\in(0,\infty)} \frac{\left(\int_0^t g(s)U_2(s)\,ds + U_2(t)\int_t^\infty g(s)\,ds\right)^{\frac{m_1}{m_2}}}{\varphi(t)^{\frac{m_1}{p_1}}},$$

(b) if $m_1 \le m_2$ and $p_1 > m_2$, then

$$A(g) \approx \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{p_1}{p_1-m_2}} \sigma(t)\,dt\right)^{\frac{m_1(p_1-m_2)}{p_1 m_2}},$$

(c) if $m_1 > m_2$ and $p_1 \le m_2$, then

$$A(g) \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\,ds\right)^{\frac{m_1-m_2}{m_2}} \times \frac{U_1(t)}{\varphi(t)^{\frac{m_1}{p_1}}},$$

(d) if $m_1 > m_2$ and $p_1 > m_2$, then

$$A(g) \approx \left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{m_1}{m_1-m_2}} \times U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\,ds\right)^{\frac{p_1(m_1-m_2)}{m_1(p_1-m_2)}} \sigma(t)\,dt\right)^{\frac{m_1(p_1-m_2)}{p_1 m_2}}.$$

Now, let us introduce an abbreviated notation. We will write, for $g \in \mathfrak{M}$,

$$\|g\| = \left(\int_0^\infty g(t)^{\frac{p_2}{p_2-m_2}} w_2(t)^{-\frac{m_2}{p_2-m_2}}\,dt\right)^{\frac{p_2-m_2}{p_2}},$$

and set

$$D = \sup_{g \in \mathfrak{M}_+} \frac{A(g)^{\frac{m_2}{m_1}}}{\|g\|}.$$

Then, by (12),

$$C \approx D^{\frac{1}{m_2}}.$$

It follows from the above estimates that

(a) if $m_1 \le m_2$ and $p_1 \le m_2$, then $D \approx D_1 + D_2$, where

$$D_1 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t \in (0,\infty)} \frac{\int_0^t g(s) U_2(s)\, ds}{\varphi(t)^{\frac{m_2}{p_1}}}$$

and

$$D_2 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t \in (0,\infty)} \frac{U_2(t) \int_t^\infty g(s)\, ds}{\varphi(t)^{\frac{m_2}{p_1}}},$$

(b) if $m_1 \le m_2$ and $p_1 > m_2$, then $D \approx D_3 + D_4$, where

$$D_3 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \sup_{s \in (t,\infty)} U_1(s)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \left(\int_0^s g(y) U_2(y)\, dy \right)^{\frac{p_1}{p_1-m_2}} \sigma(t)\, dt \right)^{\frac{p_1-m_2}{p_1}},$$

and

$$D_4 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \sup_{s \in (t,\infty)} U_1(s)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} U_2(s)^{\frac{p_1}{p_1-m_2}} \left(\int_s^\infty g(y)\, dy \right)^{\frac{p_1}{p_1-m_2}} \sigma(t)\, dt \right)^{\frac{p_1-m_2}{p_1}},$$

(c) if $m_1 > m_2$ and $p_1 \le m_2$, then $D \approx D_5 + D_6$, where

$$D_5 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}} \left(\int_t^\infty \left(\int_0^s g(y) U_2(y)\, dy \right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\, ds \right)^{\frac{m_1-m_2}{m_1}}}{\varphi(t)^{\frac{m_1}{p_1}}},$$

and

$$D_6 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}} \left(\int_t^\infty \left(U_2(s) \int_s^\infty g(y)\, dy \right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\, ds \right)^{\frac{m_1-m_2}{m_1}}}{\varphi(t)^{\frac{m_1}{p_1}}},$$

(d) if $m_1 > m_2$ and $p_1 > m_2$, then $D \approx D_7 + D_8$, where

$$D_7 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y) U_2(y)\, dy \right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\, ds \right)^{\frac{p_1(m_1-m_2)}{m_1(p_1-m_2)}} \sigma(t)\, dt \right)^{\frac{p_1-m_2}{p_1}}$$

and

$$D_8 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \left(\int_t^\infty \left(U_2(s) \int_s^\infty g(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s)\,ds \right)^{\frac{p_1(m_1-m_2)}{m_1(p_1-m_2)}} \sigma(t)\,dt \right)^{\frac{p_1-m_2}{p_1}}.$$

Our final task is to establish two-sided estimates for D_1–D_8. We shall treat each case separately.

Case (a). Assume that $p_1 \le m_2$ and $m_1 \le m_2$. Interchanging the suprema, we have

$$D_1 = \sup_{t\in(0,\infty)} \frac{1}{\varphi(t)^{\frac{m_2}{p_1}}} \sup_{g\in\mathfrak{M}_+} \frac{\int_0^t g(s)U_2(s)\,ds}{\|g\|}.$$

We now fix $t \in (0,\infty)$ and apply (7) to

$$p = \frac{p_2}{m_2},\ f = U_2\chi_{(0,t)} \text{ and } v = w_2.$$

We then arrive at

$$D_1 = \sup_{t\in(0,\infty)} \frac{1}{\varphi(t)^{\frac{m_2}{p_1}}} \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

Similarly,

$$D_2 = \sup_{t\in(0,\infty)} \frac{U_2(t)}{\varphi(t)^{\frac{m_2}{p_1}}} \sup_{g\in\mathfrak{M}_+} \frac{\int_t^\infty g(s)\,ds}{\|g\|}.$$

Using (7) with a fixed $t \in (0,\infty)$ once again, this time to

$$p = \frac{p_2}{m_2},\ f = \chi_{(t,\infty)} \text{ and } v = w_2,$$

we get

$$D_2 = \sup_{t\in(0,\infty)} \frac{U_2(t)}{\varphi(t)^{\frac{m_2}{p_1}}} \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

Taking the m_2-roots, we get the assertion of the theorem in case (a).

Case (b). Assume that $p_1 > m_2$ and $m_1 \le m_2$. To characterize D_3 and D_4, we have to distinguish two subcases depending on the comparison of p_1 and p_2.

Case (b-i). Assume that $p_1 \le p_2$. Then, by Theorem 2.3(a), applied to

$$p = \frac{p_2}{p_2 - m_2},\ q = \frac{p_1}{p_1 - m_2},\ u = U_1^{-\frac{m_2}{m_1}},\ v = U_2^{-\frac{p_2}{p_2-m_2}} w_2^{-\frac{m_2}{p_2-m_2}},\ \varrho = U_2 \text{ and } w = \sigma,$$

we arrive at

$$D_3 \approx \sup_{t\in(0,\infty)} \left(\sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty \sup_{y\in(s,\infty)} U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(s)\,ds \right)^{\frac{p_1-m_2}{p_1}} \times \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

By monotonicity of U_1, we get

$$D_3 \approx \sup_{t\in(0,\infty)} \left(\sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(s)\,ds \right)^{\frac{p_1-m_2}{p_1}} \times \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

By the subadditivity of the supremum, one has, for a fixed $t \in (0,\infty)$,

$$\begin{aligned}
&\sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(y)\,dy \\
&\approx \sup_{s\in(t,\infty)} \left(U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^s \sigma(y)\,dy + \int_s^\infty U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(y)\,dy \right) \\
&= \sup_{s\in(t,\infty)} \int_0^\infty \min\left\{ U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}, U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \right\} \sigma(y)\,dy.
\end{aligned}$$

Using the monotonicity of U_1 once again, we conclude that the last expression is decreasing in $s \in (0,\infty)$. Hence,

$$\begin{aligned}
&\sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(y)\,dy \\
&\approx \int_0^\infty \min\left\{ U_1(y)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}}, U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \right\} \sigma(y)\,dy \\
&= U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^t \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(s)\,ds.
\end{aligned}$$

Altogether,

$$D_3 \approx \sup_{t\in(0,\infty)} \left(U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \int_0^t \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(s)\,ds \right)^{\frac{p_1-m_2}{p_1}} \times \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

Further, by Theorem 2.6(a), applied to

$$p = \frac{p_2}{p_2-m_2},\ q = \frac{p_1}{p_1-m_2},\ u = U_2 U_1^{-\frac{m_2}{m_1}},\ v = w_2^{-\frac{m_2}{p_2-m_2}} \text{ and } w = \sigma,$$

we get

$$D_4 \approx \sup_{t\in(0,\infty)} \left(\int_0^t \sup_{y\in(s,t)} U_2(y)^{\frac{p_1}{p_1-m_2}} U_1(y)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \sigma(s)\,ds \right)^{\frac{p_1-m_2}{p_1}} \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{m_2}{p_2}}.$$

Combining all the estimates obtained and taking the roots we establish the assertion of the theorem in the case (b-i).

Case (b-ii). Assume now that $p_1 > p_2$ (while still $p_1 > m_2$ and $m_1 \le m_2$). By Theorem 2.3(b), applied to

$$p = \frac{p_2}{p_2-m_2},\ q = \frac{p_1}{p_1-m_2},\ u = U_1^{-\frac{m_2}{m_1}},\ v = U_2^{-\frac{p_2}{p_2-m_2}} w_2^{-\frac{m_2}{p_2-m_2}},\ \varrho = U_2 \text{ and } w = \sigma,$$

and observing that this time $1 < q < p < \infty$, we get

$$D_3 \approx D_{31} + D_{32},$$

where

$$D_{31} = \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} U_1(t)^{-\frac{m_2}{m_1}\frac{p_1}{p_1-m_2}} \times \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds \right)^{\frac{p_1}{p_1-m_2}} \sigma(t)\,dt \right)^{\frac{m_2(p_1-p_2)}{p_1 p_2}}$$

and

$$D_{32} = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_1 p_2}{m_1(p_1-p_2)}} \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} \left(\int_0^t \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_2(p_1-p_2)}{p_1 p_2}}.$$

By Theorem 2.6(b), applied to

$$p = \frac{p_2}{p_2-m_2},\ q = \frac{p_1}{p_1-m_2},\ u = U_2 U_1^{-\frac{m_2}{m_1}},\ v = w_2^{-\frac{m_2}{p_2-m_2}} \text{ and } w = \sigma$$

we obtain

$$D_4 \approx D_{41} + D_{42},$$

where

$$D_{41} = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{p_1 p_2}{m_2(p_1-p_2)}} U_1(s)^{-\frac{p_1 p_2}{m_1(p_1-p_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} \right.$$
$$\left. \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_2(p_1-p_2)}{p_1 p_2}}$$

and

$$D_{42} = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{p_1}{p_1-m_2}} U_1(s)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} \right.$$
$$\left. \times \left(\int_0^t \sup_{y\in(s,t)} U_2(y)^{\frac{p_1}{p_1-m_2}} U_1(y)^{-\frac{p_1 m_2}{m_1(p_1-m_2)}} \sigma(s)\,ds \right)^{\frac{p_1(p_2-m_2)}{m_2(p_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_2(p_1-p_2)}{p_1 p_2}}.$$

Combining the estimates and taking the roots, we obtain the assertion of the theorem in the case (b-ii).

Case (c). Assume that $p_1 \le m_2$ and $m_1 > m_2$. We start by interchanging the suprema in the definition of D_5 and D_6. We get

$$D_5 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}}}{\varphi(t)^{\frac{m_1}{p_1}}} \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_t^\infty \left(\int_0^s g(y) U_2(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_2}{m_1-m_2}} u_1(s)\,ds \right)^{\frac{m_1-m_2}{m_1}}}{\|g\|},$$

and

$$D_6 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}}}{\varphi(t)^{\frac{m_1}{p_1}}} \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_t^\infty \left(\int_s^\infty g(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} U_2(s)^{\frac{m_1}{m_1-m_2}} U_1(s)^{-\frac{m_2}{m_1-m_2}} u_1(s)\,ds \right)^{\frac{m_1-m_2}{m_1}}}{\|g\|}.$$

We will distinguish two subcases. This time, the decisive factor is the comparison between m_1 and m_2.

Case (c-i). Assume that $m_1 \le p_2$ (while still $p_1 \le m_2$ and $m_1 > m_2$). Fix $t \in (0,\infty)$. Applying Theorem 2.1(a) to the parameters

$$p = \frac{p_2}{p_2 - m_2},\ q = \frac{m_1}{m_1 - m_2}$$

and the weights

$$u = U_2,\ v = w_2^{-\frac{m_2}{p_2-m_2}} \quad \text{and} \quad w(s) = U_1(s)^{-\frac{m_1}{m_1-m_2}} u_1(s) \chi_{(t,\infty)(s)},\ s \in (0,\infty),$$

we get

$$\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_t^\infty\left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}U_1(s)^{-\frac{m_1}{m_1-m_2}}u_1(s)\,ds\right)^{\frac{m_1-m_2}{m_1}}}{\|g\|}$$

$$\approx\sup_{s\in(0,\infty)}\left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\chi_{(t,\infty)}(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}}.$$

Since

$$\sup_{s\in(0,\infty)}\left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\chi_{(t,\infty)}(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}}$$

$$=\max\left\{\sup_{s\in(0,t)}\left(\int_t^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}};\right.$$

$$\left.\sup_{s\in(t,\infty)}\left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}}\right\}$$

$$=\sup_{s\in(t,\infty)}\left(\int_s^\infty U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}},$$

calculating the first integral we finally arrive at

$$\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_t^\infty\left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}U_1(s)^{-\frac{m_1}{m_1-m_2}}u_1(s)\,ds\right)^{\frac{m_1-m_2}{m_1}}}{\|g\|}$$

$$\approx\sup_{s\in(t,\infty)}U_1(s)^{-\frac{m_2}{m_1}}\left(\int_0^s U_2(y)^{\frac{p_2}{m_2}}w_2(y)\,dy\right)^{\frac{m_2}{p_2}}.$$

Similarly, by Theorem 2.2(a), applied to

$$p=\frac{p_2}{p_2-m_2},\ q=\frac{m_1}{m_1-m_2},\ v=w_2^{-\frac{m_2}{p_2-m_2}}\ \text{and}\ w=U_2^{\frac{m_1}{m_1-m_2}}U_1^{-\frac{m_1}{m_1-m_2}}u_1\chi_{(t,\infty)},$$

we get

$$\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_t^\infty\left(\int_s^\infty g(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}U_2(s)^{\frac{m_1}{m_1-m_2}}U_1(s)^{-\frac{m_2}{m_1-m_2}}u_1(s)\,ds\right)^{\frac{m_1-m_2}{m_1}}}{\|g\|}$$

$$\approx\sup_{s\in(t,\infty)}\left(\int_t^s U_2(y)^{\frac{m_1}{m_1-m_2}}U_1(y)^{-\frac{m_1}{m_1-m_2}}u_1(y)\,dy\right)^{\frac{m_1-m_2}{m_1}}\left(\int_s^\infty w_2(y)\,dy\right)^{\frac{m_2}{p_2}}.$$

The obtained estimates hold for every fixed $t \in (0,\infty)$. Hence, plugging them into the definitions of D_5 and D_6, we get

$$D_5 \approx \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}}}{\varphi(t)^{\frac{m_2}{p_1}}} \sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_2}{m_1}} \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)\,dy \right)^{\frac{m_2}{p_2}}$$

and

$$D_6 \approx \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}}}{\varphi(t)^{\frac{m_2}{p_1}}} \sup_{s\in(t,\infty)} \left(\int_t^s U_2(y)^{\frac{m_1}{m_1-m_2}} U_1(y)^{-\frac{m_1}{m_1-m_2}} u_1(y)\,dy \right)^{\frac{m_1-m_2}{m_1}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{m_2}{p_2}}.$$

Combining the estimates and taking the roots, we get the assertions of the theorem in case (c-i).

Case (c-ii). Assume that $m_1 \le p_2$ (and $p_1 \le m_2$ and $m_1 > m_2$ remain in power). By Theorem 2.1(b), applied to the same set of parameters as in the case (c-i), we obtain

$$\begin{aligned} D_5 \approx{}& \sup_{t\in(0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds\right)^{\frac{m_2}{p_2}}}{\varphi(t)^{\frac{m_2}{p_1}}} \\ &+ \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{m_2}{m_1}} \left(\int_t^\infty \left(\int_0^s U_2(y)^{\frac{p_2}{m_2}} w_2(y)dy\right)^{\frac{p_2}{m_1-p_2}} U_2(s)^{\frac{p_2}{m_2}} w_2(s) U_1(s)^{-\frac{p_2}{m_1-p_2}}\,ds\right)^{\frac{m_2(m_1-p_2)}{m_1p_2}}}{\varphi(t)^{\frac{m_2}{p_1}}}. \end{aligned}$$

By Theorem 2.2(b), again applied to the same array of parameters as in the case (c-i), we get

$$\begin{aligned} D_6 \approx \sup_{t\in(0,\infty)} &\left(\int_t^\infty \left(\int_t^s U_2(y)^{\frac{m_1}{m_1-m_2}} U_1(y)^{-\frac{m_1}{m_1-m_2}} u_1(y)\,dy \right)^{\frac{p_2(m_1-m_2)}{m_2(m_1-p_2)}} \right. \\ &\times \left. \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{p_2}{m_1-p_2}} w_2(t)\,dt \right)^{\frac{m_2(m_1-p_2)}{m_1p_2}} \frac{U_1(t)^{\frac{m_2}{m_1}}}{\varphi(t)^{\frac{m_2}{p_1}}}. \end{aligned}$$

Case (d). Assume that $p_1 > m_2$ and $m_1 > m_2$. Here we shall distinguish three subcases.

Case (d-i). Assume that $m_2 < p_1 < m_1 \le p_2$.

By Theorem 2.9(a), applied to

$p = \frac{p_2}{p_2-m_2}$, $q = \frac{m_1}{m_1-m_2}$, $m = \frac{p_1}{p_1-m_2}$, $\varrho = U_2$, $w = \sigma$, $u = U_1^{-\frac{m_1}{m_1-m_2}} u_1$ and $v = w_2^{-\frac{m_2}{p_2-m_2}}$, we get

$$D_7 \approx \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1}} U_1(t)^{-\frac{m_2}{m_1}} \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds\right)^{\frac{m_2}{p_2}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_t^\infty U_1(s)^{-\frac{p_1m_2}{m_1(p_1-m_2)}} \sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1}} \left(\int_0^t U_2(s)^{\frac{p_2}{m_2}} w_2(s)\,ds\right)^{\frac{m_2}{p_2}}$$

By Theorem 2.8(a), applied to

$$p = \tfrac{p_2}{p_2-m_2},\ q = \tfrac{m_1}{m_1-m_2},\ m = \tfrac{p_1}{p_1-m_2},\ w = \sigma,\ u = U_2^{\frac{m_1}{m_1-m_2}} U_1^{-\frac{m_1}{m_1-m_2}} u_1,\ v = w_2^{-\frac{m_2}{p_2-m_2}},$$

we get

$$D_8 \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty w_2(s)\,ds\right)^{\frac{m_2}{p_2}} \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{m_1}{m_1-m_2}} U_2(y)^{\frac{m_1}{m_1-m_2}} u_1(y)\,dy\right)^{\frac{p_1(m_1-m_2)}{m_1(p_1-m_2)}} \sigma(s)\,ds\right)^{\frac{p_1-m_2}{p_1}}.$$

The assertion of the theorem in the case (d-i) now follows by the usual combination of estimates and taking the roots.

Case (d-ii). Assume that $m_2 < p_1 \le p_2 < m_1$.

We follow the same line of argument as in case (d-i), applying this time Theorem 2.9(b) to evaluate D_7 and Theorem 2.8(b) to evaluate D_8.

Case (d-iii). Assume that $m_2 < p_2 < p_1 < m_1$.

Again, the assertion can be proved as in the case (d-i). This time we use Theorem 2.9(c) for D_7 and Theorem 2.8(c) for D_8.

The proof of the theorem is complete. □

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Paper VIII

Martin Křepela

Boundedness of Hardy-type operators with a kernel: integral weighted conditions for the case $0 < q < 1 \leq p < \infty$

Submitted

BOUNDEDNESS OF HARDY-TYPE OPERATORS WITH A KERNEL: INTEGRAL WEIGHTED CONDITIONS FOR THE CASE $0<q<1\le p<\infty$

MARTIN KŘEPELA

ABSTRACT. Let $U:[0,\infty)^2\to[0,\infty)$ be a measurable kernel satisfying:
(i) $U(x,y)$ is nonincreasing in x and nondecreasing in y;
(ii) there exists a constant $\vartheta>0$ such that

$$U(x,z)\le\vartheta\,(U(x,y)+U(y,z))$$

for all $0\le x<y<z<\infty$;
(iii) $U(0,y)>0$ for all $y>0$.
Let $0<q<1<p<\infty$. We prove that the weighted inequality

$$\left(\int_0^\infty\left(\int_0^t f(x)U(x,t)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

holds for all nonnegative measurable functions f on $(0,\infty)$ if and only if

$$\left(\int_0^\infty\left(\int_t^\infty w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_0^t U^{p'}(z,t)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}}<\infty$$

and

$$\left(\int_0^\infty\left(\int_t^\infty w(x)U^q(t,x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in(0,t)}U^q(z,t)\left(\int_0^z v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}}<\infty,$$

where $p':=\frac{p}{p-1}$ and $r:=\frac{pq}{p-q}$. Analogous conditions for the case $p=1$ and for the dual version of the inequality are also presented.

1. INTRODUCTION

Operators of the general form

$$Tf(x)=\int_0^\infty f(y)U(x,y)\,\mathrm{d}y,$$

where U is a *kernel*, play an indispensable role in various areas of analysis. The means of their investigation, naturally, greatly depend on additional properties of the kernel U.

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In the present article, we study the so-called *Hardy-type* operators

$$(1) \qquad Hf(x) = \int_0^x f(y)U(y,x)\,\mathrm{d}y, \qquad \text{and} \qquad H^*f(x) = \int_x^\infty f(y)U(x,y)\,\mathrm{d}y,$$

where the kernel $U : [0,\infty)^2 \to [0,\infty)$ is a measurable function which has the following properties:

(i) $U(x,y)$ is nonincreasing in x and nondecreasing in y;
(ii) there exists a constant $\vartheta > 0$ such that for all $0 \le x < y < z < \infty$ it holds

$$U(x,z) \le \vartheta\,(U(x,y) + U(y,z));$$

(iii) $U(0,y) > 0$ for all $y > 0$.

If $\vartheta > 0$ and U is a function satisfying the conditions above with the given parameter ϑ in point (ii), then we, for the sake of simplicity, call U a *ϑ-regular kernel*.

The simplest case of a ϑ-regular kernel U is the constant $U \equiv 1$, with which H and H^* become the ordinary Hardy and Copson ("dual Hardy") operators, respectively. Other examples of ϑ-regular kernels include the Riemann-Liouville kernel

$$U(x,y) = (y-x)^\alpha, \quad \alpha > 0,$$

the logarithmic kernel

$$U(x,y) = \log^\alpha\left(\frac{y}{x}\right), \quad \alpha > 0,$$

and the kernels

$$U(x,y) = \int_x^y u(t)\,\mathrm{d}t \quad \text{and} \quad U(x,y) = \operatorname*{ess\,sup}_{t\in(x,y)} u(t),$$

where u is a given nonnegative measurable function. These operators find applications, for instance, in the theory of differentiability of functions, interpolation theory and more topics involving function spaces. The two last-named examples of ϑ-regular kernels prove to be particularly useful in research of the so-called iterated Hardy operators [2, 3], for example.

The particular aspect we investigate in this paper is boundedness of the operators H and H^* with a ϑ-regular kernel U between weighted Lebesgue spaces. In order to define these spaces, we need to introduce several auxiliary terms first.

Throughout the text, by a *measurable function* we always mean a Lebesgue measurable function (on an appropriate subset of $\mathbb{R}$). The symbol $\mathscr{M}_+$ denotes the cone of all nonnegative measurable functions on $(0,\infty)$. A *weight* is a function $w \in \mathscr{M}_+$ on $(0,\infty)$ such that

$$0 < \int_0^t w(s)\,\mathrm{d}s < \infty \quad \text{for all} \quad t > 0.$$

Finally, if v is a weight and $p \in (0,\infty]$, then the *weighted Lebesgue space* $L^p(v) = L^p(v)(0,\infty)$ is defined as the set of all real-valued measurable functions f on

$(0,\infty)$ such that

$$\|f\|_{L^p(v)} := \left(\int_0^\infty |f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty \qquad \text{if } p<\infty,$$
$$\|f\|_{L^\infty(v)} := \operatorname*{ess\,sup}_{t\in(0,\infty)} |f(t)|v(t) < \infty \qquad \text{if } p=\infty.$$

Note that if $p \in (0,1)$, then $(L^p(v), \|\cdot\|_{L^p(v)})$ is in general not a normed linear space because of the absence of the Minkowski inequality in this case. However, as we deal only with the case $1 \le p < \infty$ anyway, this detail is not of our concern here.

Throughout the text, if $p \in (0,1)\cup(1,\infty)$, then p' is defined by $p' = \frac{p}{p-1}$. Analogous notation is used for q'.

In the following, assume that $\vartheta \in (0,\infty)$, U is a ϑ-regular kernel, H is the corresponding operator from (1) and v, w are weights. Boundedness of H between $L^p(v)$ and $L^q(w)$ corresponds, by definition, to validity of the inequality

$$\left(\int_0^\infty \left(\int_0^t f(x)U(x,t)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

for all functions $f \in \mathscr{M}_+$, and it was completely characterized for $p,q \in [1,\infty]$. The authors credited for this work are Bloom and Kerman [1], Oinarov [12] and Stepanov [17]. The results of [12], for instance, have the following form.

Theorem ([12, Theorem 1.1])**.** *Let $1<p\le q<\infty$. Then $H: L^p(v)\to L^q(w)$ is bounded if and only if*

$$E_1 := \sup_{t\in(0,\infty)} \left(\int_t^\infty U^q(t,x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} < \infty$$

and

$$E_2 := \sup_{t\in(0,\infty)} \left(\int_t^\infty w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t U^{p'}(x,t)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the least constant C such that the inequality

$$\|Hf\|_{L^q(w)} \le C\|f\|_{L^p(v)} \tag{2}$$

holds for all $f \in \mathscr{M}_+$ satisfies $C \approx E_1+E_2$.

Theorem ([12, Theorem 1.2])**.** *Let $1<q<p<\infty$ and $r := \frac{pq}{p-q}$. Then $H: L^p(v)\to L^q(w)$ is bounded if and only if*

$$E_3 := \left(\int_0^\infty \left(\int_t^\infty U^q(t,x)w(x)\,\mathrm{d}x\right)^{\frac{r}{q}} \left(\int_0^t v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{r}{q'}} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty$$

and

$$E_4 := \left(\int_0^\infty \left(\int_t^\infty w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \left(\int_0^t U^{p'}(x,t) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty.$$

Moreover, the least constant C such that (2) *holds for all $f \in L^p(v)$ satisfies $C \approx E_3 + E_4$.*

The conditions obtained in [1, 17] have a slightly different form, a more detailed comparison between them is found in [17].

As for the "limit cases", conditions for the case $p = \infty$ and $q \in (0, \infty]$ are obtained very easily, the same applies to the case $q = 1$ and $p \in [1, \infty)$ in which one simply uses the Fubini theorem. Yet another possible choice of parameters is $p = 1$ and $q \in (1, \infty]$. It was (at least for $q < \infty$) included in [12, Theorem 1.2] and the conditions may be recovered from that article by correctly interpreting the expressions involving the symbol p' in there. Another option is to follow the more general theorem [6, Chapter XI, Theorem 4].

If $0 < p < 1$, then the operator H can never be bounded (provided that U, v, w are nontrivial, which is always assumed here). The problem in here lies in the fact that for each $t > 0$ there exists $f_t \in L^p(v)$ which is not locally integrable at the point t. For more details, see e.g. [10].

No such difficulty arises if $0 < q < 1 \le p < \infty$. In this case, H may indeed be bounded between $L^p(v)$ and $L^q(w)$ and it is perfectly justified to ask for the conditions under which this occurs. As for the known answers to this question, the situation is however much worse than in the other cases.

When assumed $U \equiv 1$, i.e. for the ordinary Hardy operator, the boundedness characterization was found by Sinnamon [14] and it corresponds to the condition $E_3 < \infty$ (with $U \equiv 1$, of course). In the general case, in [17] it was shown that the condition $E_3 < \infty$ is sufficient but not necessary for $H : L^p(v) \to L^q(w)$ to be bounded, while the condition

$$E_5 := \left(\int_0^\infty \left(\int_t^\infty U^q(t,x) w(x)\,\mathrm{d}x \right)^{\frac{p'}{q}} v^{1-p'}(t)\,\mathrm{d}t \right)^{\frac{1}{p'}} < \infty$$

is necessary but not sufficient. For related counterexamples, see [16]. The fact that the two conditions do not meet is a significant drawback. An equivalent description of the optimal constant C in (2) is usually substantial for the result to be applicable in any way.

Lai [9] found equivalent conditions by proving that, with $0 < q < 1 < p < \infty$, the operator H is bounded from $L^p(v)$ to $L^q(w)$ if and only if

$$\widetilde{D}_1 := \sup_{\{t_k\}} \sum_k \left(\int_{t_k}^{t_{(k+1)}} w(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{(k-1)}}^{t_k} U^{p'}(x,t_k) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} < \infty$$

as well as

$$\widetilde{D}_2 := \sup_{\{t_k\}} \sum_k \left(\int_{t_k}^{t_{(k+1)}} w(t) U^q(t_k, t) \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{(k-1)}}^{t_k} v^{1-p'}(x) \, \mathrm{d}x \right)^{\frac{r}{p'}} < \infty.$$

The suprema in here are taken over all covering sequences, i.e. partitions of $(0, \infty)$ (see [9] or Section 2 for the definitions), and $r := \frac{pq}{p-q}$, as usual. Moreover, these conditions satisfy $\widetilde{D}_1 + \widetilde{D}_2 \approx C^r$ with the least C such that (2) holds for all $f \in \mathscr{M}_+$. Corresponding variants for $p = 1$ are also provided in [9]. The earlier use of similar partitioning techniques in the paper [11] of Martín-Reyes and Sawyer should be also credited.

Unfortunately, even though the $\widetilde{D}$-conditions are both sufficient and necessary, they are only hardly verifiable due to their discrete form involving all possible covering sequences. This fact has hindered their use in various applications (see e.g. [3]). In contrast, in the case $1 < q < p < \infty$ it is known (see [16, 9]) that $\widetilde{D}_1 + \widetilde{D}_2 \approx A_3^r + A_4^r$. This does not apply when $0 < q < 1 \le p < \infty$, as shown by the results of [17] mentioned earlier.

Rather recently, Prokhorov [13] found conditions for $0 < q < 1 \le p < \infty$ which have an integral form but involve a function ζ defined by

$$\zeta(x) := \sup \left\{ y \in (0, \infty); \int_y^\infty w(t) \, \mathrm{d}t \ge (\vartheta^q + 1) \int_x^\infty w(t) \, \mathrm{d}t \right\}, \quad x > 0.$$

The conditions presented in [13] even involve this function iterated three times. The presence of such an implicit expression involving the weight w virtually prevents any use of these conditions in applications which require further manipulation w (see Section 4 for an example). Finding explicit integral conditions for the case $0 < q < 1 \le p < \infty$, which would have a form comparable e.g. to E_3 and E_4, hence remained an open problem.

In this paper, we solve this problem and provide the missing integral conditions. No additional assumptions on the weights v, w and the ϑ-regular kernel U are required here, neither are any implicit expressions. The results are presented in Theorems 3.1, 3.2 and Corollaries 3.3, 3.4. The proofs are based on the well-known method of *dyadic discretization* (or *blocking technique*, see [5] for a basic introduction into this method). The particular variant of the technique employed here is essentially the same as the one used in [8].

Concerning the structure of this paper, this introduction is followed by Section 2 where additional definitions and various auxiliary results are presented. Section 3 consists of the main results, their proofs and some related remarks. In the final Section 4 we present certain examples of applications of the results.

2. Definitions and preliminaries

Let us first introduce the remaining notation and terminology used in the paper. We say that $\mathbb{I} \subseteq \mathbb{Z}$ is an *index set* if there exist $k_{\min}, k_{\max} \in \mathbb{Z}$ such that $k_{\min} \leq k_{\max}$ and

$$\mathbb{I} = \{k \in \mathbb{Z}, k_{\min} \leq k \leq k_{\max}\}.$$

Moreover, we denote

$$\mathbb{I}_0 := \mathbb{I} \setminus \{k_{\min}, k_{\max}\}.$$

Let $\mathbb{I}$ be an index set containing at least three indices. Then a sequence of points $\{t_k\}_{k\in\mathbb{I}}$ is called a *covering sequence* if $t_{k_{\min}} = 0$, $t_{k_{\max}} = \infty$ and $t_k < t_{(k+1)}$ whenever $k \in \mathbb{I} \setminus \{k_{\max}\}$.

Next, let $z \in \mathbb{N} \cup \{0\}$ and $n, k \in \mathbb{N}$ are such that $0 \leq k < n$. We write $z \bmod n = k$ if there exists $j \in \mathbb{N} \cup \{0\}$ such that $z = jn + k$. In other words, k is the remainder after division of the number z by the number n.

In the next part, we present various auxiliary results which will be needed later.

Proposition 2.1. *Let v be a weight and $0 \leq x < y \leq \infty$. Let f be a nonnegative measurable function on (x, y) and φ be a positive locally integrable function on (x, y). If $p \in (1, \infty)$, then*

$$\int_x^y f(s)\varphi(s)\,\mathrm{d}s \leq \left(\int_x^y f^p(s)v(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}. \tag{3}$$

Moreover, there exists a nonnegative measurable function g supported in $[x, y]$ and such that $\int_x^y g^p(s)v(s)\,\mathrm{d}s = 1$ and

$$\left(\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} = \int_x^y g(s)\varphi(s)\,\mathrm{d}s. \tag{4}$$

In the case $p = 1$ the statement holds with the expression $\left(\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}$ replaced by $\operatorname{ess\,sup}_{s\in(x,y)} \varphi(s)v^{-1}(s)$.

Proof. Assume that $p > 1$, the case $p = 1$ is treated analogously. Estimate (3) follows from the Hölder inequality. If $\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s < \infty$, then the choice $g := \varphi^{p'-1}v^{1-p'}\left(\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s\right)^{-\frac{1}{p}}$ gives (4). If $\int_x^y \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s = \infty$ and $v > 0$ a.e. on (x, y), then there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint measurable subsets of $(0, \infty)$ such that $\left(\int_{E_n} \varphi^{p'}(s)v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} = 2^n$ for all $n \in \mathbb{N}$. Then, by the previous part, for each $n \in \mathbb{N}$ there exists a measurable function g_n such that $g_n = 0$ on $(0, \infty) \setminus E_n$, $\int_{E_n} g_n^p(s)v(s)\,\mathrm{d}s = 2^{-n}$ and $\int_{E_n} g_n(s)\varphi(s)\,\mathrm{d}s =$

1. Define $g := \sum_{n\in\mathbb{N}} g_n$. Then it holds

$$\int_0^\infty g^p(s)v(s)\,\mathrm{d}s = \int_0^\infty \left(\sum_{n\in\mathbb{N}} g_n(s)\right)^p v(s)\,\mathrm{d}s = \sum_{n\in\mathbb{N}} \int_{E_n} g_n^p(s)v(s)\,\mathrm{d}s = 1$$

and $\int_0^\infty g(s)\varphi(s) = \sum_{n\in\mathbb{N}} \int_{E_n} g_n(s)\varphi(s) = \infty$. This gives (4). Finally, if there exists a set $E \subset (x,y)$ of finite positive measure and such that $v = 0$ on E, then (4) is obtained by choosing $g := v^{-\frac{1}{p}} \varphi^{p'-1} \chi_E \left(\int_E \varphi^{p'}(s)\,\mathrm{d}s\right)^{-\frac{1}{p}}$, applying the convention "$\frac{0}{0} = 0$". □

A discrete variant of the previous result reads as follows.

Proposition 2.2. *Let $\mathbb{I}$ be an index set. Let $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$ be two nonnegative sequences. Assume that $0 < q < p < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} \le \left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}}.$$

Moreover, there exists a nonnegative sequence $\{c_k\}_{k\in\mathbb{I}}$ such that $\sum_{k\in\mathbb{I}} c_k^p = 1$ and

$$\left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}} = \left(\sum_{k\in\mathbb{I}} c_k^q b_k\right)^{\frac{1}{q}}.$$

The next proposition was proved in [4, Proposition 2.1], more comments may be found e.g. in [8]. It is a fundamental part of the discretization method.

Proposition 2.3. *Let $0<\alpha<\infty$ and $1<D<\infty$. Then there exists a constant $C_{\alpha,D} \in (0,\infty)$ such that for any index set $\mathbb{I}$ and any two nonnegative sequences $\{b_k\}_{k\in\mathbb{I}}$ and $\{c_k\}_{k\in\mathbb{I}}$, satisfying*

$$b_{(k+1)} \ge D\, b_k \textit{ for all } k \in \mathbb{I}\setminus\{k_{\max}\},$$

it holds

$$\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} c_m\right)^\alpha b_k \le C_{\alpha,D} \sum_{k=k_{\min}}^{k_{\max}} c_k^\alpha b_k$$

and

$$\sum_{k=k_{\min}}^{k_{\max}} \left(\sup_{k\le m\le k_{\max}} c_m\right)^\alpha b_k \le C_{\alpha,D} \sum_{k=k_{\min}}^{k_{\max}} c_k^\alpha b_k.$$

The following result is an analogy to the previous proposition. We present a simple proof, although the result is also well known.

Proposition 2.4. *Let $0<\alpha<\infty$ and $1<D<\infty$. Then there exists a constant $C_{\alpha,D} \in (0,\infty)$ such that for any index set $\mathbb{I}$ and any two nonnegative sequences $\{b_k\}_{k\in\mathbb{I}}$ and $\{c_k\}_{k\in\mathbb{I}}$, satisfying*

$$b_{(k+1)} \ge D\, b_k \textit{ for all } k \in \mathbb{I}\setminus\{k_{\max}\},$$

it holds

$$\sup_{k_{\min}\le k\le k_{\max}} \left(\sum_{m=k}^{k_{\max}} c_m\right)^\alpha b_k \le C_{\alpha,D} \sup_{k_{\min}\le k\le k_{\max}} c_k^\alpha b_k.$$

Proof. It holds

$$\begin{aligned}
\sup_{k_{\min}\le k\le k_{\max}} \left(\sum_{m=k}^{k_{\max}} c_m\right)^{\alpha} b_k &= \sup_{k_{\min}\le k\le k_{\max}} \left(\sum_{m=k}^{k_{\max}} c_m b_m^{-\frac{1}{\alpha}} b_m^{\frac{1}{\alpha}}\right)^{\alpha} b_k \\
&\le \sup_{k_{\min}\le k\le k_{\max}} \left(\sum_{m=k}^{k_{\max}} b_m^{-\frac{1}{\alpha}}\right)^{\alpha} b_k \sup_{k\le i\le k_{\max}} c_i^{\alpha} b_i \\
&\le \sup_{k_{\min}\le k\le k_{\max}} \left(b_k^{-\frac{1}{\alpha}} \sum_{m=0}^{k_{\max}-k} D^{-\frac{m}{\alpha}}\right)^{\alpha} b_k \sup_{k\le i\le k_{\max}} c_i^{\alpha} b_i^{\alpha} \\
&\le \left(\sum_{m=0}^{\infty} D^{-\frac{m}{\alpha}}\right)^{\alpha} \sup_{k_{\min}\le i\le k_{\max}} c_i^{\alpha} b_i^{\alpha} \\
&= \frac{D}{(D^{\frac{1}{\alpha}}-1)^{\alpha}} \sup_{k_{\min}\le i\le k_{\max}} c_i^{\alpha} b_i^{\alpha}.
\end{aligned}$$

□

Applying Proposition 2.3, one obtains the next result. It is useful to handle inequalities involving a ϑ-regular kernel.

Proposition 2.5. *Let* $0<\alpha<\infty$ *and* $\vartheta\in[1,\infty)$. *Let* U *be a* ϑ*-regular kernel. Then there exists a constant* $C_{\alpha,\vartheta}\in(0,\infty)$ *such that, for any index set* $\mathbb{I}$, *any increasing sequence* $\{t_k\}_{k\in\mathbb{I}}$ *of points from* $(0,\infty]$ *and any nonnegative sequence* $\{a_k\}_{k\in\mathbb{I}\setminus\{k_{\max}\}}$ *satisfying*

$$a_{(k+1)} \ge 2\vartheta^{\alpha} a_k \text{ for all } k\in\mathbb{I}\setminus\{k_{\max}, k_{\max}-1\}, \tag{5}$$

it holds

$$\sum_{k=k_{\min}}^{k_{\max}-1} a_k U^{\alpha}(t_k, t_{k_{\max}}) \le C_{\alpha,\vartheta} \sum_{k=k_{\min}}^{k_{\max}-1} a_k U^{\alpha}(t_k, t_{(k+1)}).$$

Proof. Naturally, we may assume that $\mathbb{I}$ contains at least three indices. Let $k\in\mathbb{I}\setminus\{k_{\max}\}$. By iterating the inequality

$$U(x,z)\le \vartheta U(x,y)+\vartheta U(z,y) \qquad (x<y<z) \tag{6}$$

from the definition of the ϑ-regular kernel, we get

$$\begin{aligned}
U(t_k, t_{k_{\max}}) &\le \vartheta U(t_k, t_{(k+1)}) + \vartheta U(t_{(k+1)}, t_{k_{\max}}) \\
&\le \vartheta U(t_k, t_{(k+1)}) + \vartheta^2 U(t_{(k+1)}, t_{(k+2)}) + \vartheta^2 U(t_{(k+2)}, t_{k_{\max}}) \\
&\ \ \vdots \\
&\le \sum_{m=k}^{k_{\max}-1} \vartheta^{m-k+1} U(t_m, t_{(m+1)}) \\
&= \vartheta^{-k} \sum_{m=k}^{k_{\max}-1} \vartheta^{m+1} U(t_m, t_{(m+1)}).
\end{aligned}$$

Set $b_k := \vartheta^{-\alpha k} a_k$ for $k \in \mathbb{I} \setminus \{k_{\max}\}$. Then, by (5), for all $k \in \mathbb{I} \setminus \{k_{\max}, k_{\max} - 1\}$ it holds $b_{(k+1)} \geq 2b_k$. We obtain

$$\begin{aligned}
\sum_{k=k_{\min}}^{k_{\max}-1} a_k U^\alpha(t_k, t_{k_{\max}}) &\leq \sum_{k=k_{\min}}^{k_{\max}-1} \vartheta^{-\alpha k} a_k \left(\sum_{m=k}^{k_{\max}-1} \vartheta^{m+1} U(t_m, t_{(m+1)}) \right)^\alpha \\
&= \sum_{k=k_{\min}}^{k_{\max}-1} b_k \left(\sum_{m=k}^{k_{\max}-1} \vartheta^{m+1} U(t_m, t_{(m+1)}) \right)^\alpha \\
&\leq C_\alpha \sum_{k=k_{\min}}^{k_{\max}-1} b_k \vartheta^{\alpha(k+1)} U^\alpha(t_k, t_{(k+1)}) \qquad (7) \\
&= C_\alpha \vartheta^\alpha \sum_{k=k_{\min}}^{k_{\max}-1} a_k U^\alpha(t_k, t_{(k+1)}).
\end{aligned}$$

To get the inequality (7), we used Proposition 2.3, setting $D := 2$ and $c_m := U(t_m, t_{(m+1)})$ for the relevant indices m. This proves the statement. □

Notice that, by the definitions at the beginning of this section, we consider only finite index sets (and therefore also finite covering sequences later on). However, all the results of this section hold for infinite sequences as well. This may be easily shown by using a limit argument. We will nevertheless continue working with finite index sets and covering sequences only. The notion of supremum is used regularly even where it relates to a finite set and where it therefore could be replaced by a maximum. For further remarks see the last part of Section 3.

The final basic result concerns ϑ-regular kernels and reads as follows.

Proposition 2.6. *Let $0 \leq a < b < c \leq \infty$, $0 < \alpha < \infty$ and $1 \leq \vartheta < \infty$. Let U be a ϑ-regular kernel and ψ be a nonincreasing nonnegative function defined on $(0, \infty)$. Then*

$$\sup_{z \in [a,c)} U^\alpha(a,z)\psi(z) \leq (1+\vartheta) \left(\sup_{z \in [a,b]} U^\alpha(a,z)\psi(z) + \sup_{z \in [b,c)} U^\alpha(b,z)\psi(z) \right).$$

If $c < \infty$, the result is unchanged if the intervals $[a,c)$ and $[b,c)$ in the suprema are replaced by $[a,c]$ and $[b,c]$, respectively.

Proof. The result is a consequence to the following simple observation.

$$\begin{aligned}
\sup_{z \in [a,c)} U^\alpha(a,z)\psi(z) &\leq \sup_{z \in [a,b]} U^\alpha(a,z)\psi(z) + \sup_{z \in [b,c)} U^\alpha(a,z)\psi(z) \\
&\leq \sup_{z \in [a,b]} U^\alpha(a,z)\psi(z) + \vartheta U^\alpha(a,b) \sup_{z \in [b,c)} \psi(z) + \vartheta \sup_{z \in [b,c)} U^\alpha(b,z)\psi(z) \\
&= \sup_{z \in [a,b]} U^\alpha(a,z)\psi(z) + \vartheta U^\alpha(a,b)\psi(b) + \vartheta \sup_{z \in [b,c)} U^\alpha(b,z)\psi(z) \\
&\leq (1+\vartheta) \left(\sup_{z \in [a,b]} U^\alpha(a,z)\psi(z) + \sup_{z \in [b,c)} U^\alpha(b,z)\psi(z) \right). \qquad \square
\end{aligned}$$

3. MAIN RESULTS

This section contains the main theorems and their proofs. Remarks to the results and proof techniques can be found at the end of the section.

The notation $A \lesssim B$ means that $A \le CB$, where the constant C may depend *only* on the exponents p, q and the parameter ϑ. In particular, this C is always independent on the weights w, v, on certain indices (such as k, n, j, K, N, J, $\mu, \dots$), on the number of summands involved in sums etc. We write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$.

Theorem 3.1. *Let $0 < q < 1 < p < \infty$, $r := \frac{pq}{p-q}$ and $0 < \vartheta < \infty$. Let v, w be weights. Let U be a ϑ-regular kernel. Then the following assertions are equivalent:*

(i) *There exists a constant $C \in (0, \infty)$ such that the inequality*

$$\left(\int_0^\infty \left(\int_t^\infty f(x) U(t,x) \, \mathrm{d}x \right)^q w(t) \, \mathrm{d}t \right)^{\frac{1}{q}} \le C \left(\int_0^\infty f^p(t) v(t) \, \mathrm{d}t \right)^{\frac{1}{p}} \tag{8}$$

holds for all functions $f \in \mathscr{M}_+$.

(ii) *Both the conditions*

$$D_1 := \sup_{\substack{\{t_k\}_{k \in \mathbb{I}} \\ \text{covering} \\ \text{sequence}}} \sum_{k \in \mathbb{I}_0} \left(\int_{t_{(k-1)}}^{t_k} w(t) \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k, x) v^{1-p'}(x) \, \mathrm{d}x \right)^{\frac{r}{p'}} < \infty$$

and

$$D_2 := \sup_{\substack{\{t_k\}_{k \in \mathbb{I}} \\ \text{covering} \\ \text{sequence}}} \sum_{k \in \mathbb{I}_0} \left(\int_{t_{(k-1)}}^{t_k} w(t) U^q(t, t_k) \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x) \, \mathrm{d}x \right)^{\frac{r}{p'}} < \infty$$

are satisfied.

(iii) *Both the conditions*

$$A_1 := \int_0^\infty \left(\int_0^t w(x) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) \left(\int_t^\infty U^{p'}(t,z) v^{1-p'}(z) \, \mathrm{d}z \right)^{\frac{r}{p'}} \mathrm{d}t < \infty$$

and

$$A_2 := \int_0^\infty \left(\int_0^t w(x) U^q(x,t) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z \in [t, \infty)} U^q(t,z) \left(\int_z^\infty v^{1-p'}(s) \, \mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t < \infty$$

are satisfied.

Moreover, if C is the least constant such that (8) *holds for all functions $f \in \mathscr{M}_+$, then*

$$C^r \approx D_1 + D_2 \approx A_1 + A_2.$$

The variant of the previous theorem for $p = 1$ reads as follows.

Theorem 3.2. *Let $0 < q < 1 = p$ and $0 < \vartheta < \infty$. Let v, w be weights. Let U be a ϑ-regular kernel. Then the following assertions are equivalent:*

(i) *There exists a constant $C \in (0,\infty)$ such that the inequality* (8) *holds for all functions $f \in \mathscr{M}_+$.*

(ii) *Both the conditions*

$$D_3 := \sup_{\substack{\{t_k\}_{k\in\mathbb{I}} \\ \text{covering} \\ \text{sequence}}} \sum_{k\in\mathbb{I}_0} \left(\int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t \right)^{1-q'} \operatorname*{ess\,sup}_{x\in(t_k,t_{(k+1)})} U^{-q'}(t_k,x)\, v^{q'}(x)\,\mathrm{d}x < \infty$$

and

$$D_4 := \sup_{\substack{\{t_k\}_{k\in\mathbb{I}} \\ \text{covering} \\ \text{sequence}}} \sum_{k\in\mathbb{I}_0} \left(\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t \right)^{1-q'} \operatorname*{ess\,sup}_{x\in(t_k,t_{(k+1)})} v^{q'}(x)\,\mathrm{d}x < \infty$$

are satisfied.

(iii) *Both the conditions*

$$A_3 := \int_0^\infty \left(\int_0^t w(x)\,\mathrm{d}x \right)^{-q'} w(t) \operatorname*{ess\,sup}_{z\in(t,\infty)} U^{-q'}(t,z)\, v^{q'}(z)\,\mathrm{d}t < \infty$$

and

$$A_4 := \int_0^\infty \left(\int_0^t w(x)U^q(x,t)\,\mathrm{d}x \right)^{-q'} w(t) \operatorname*{ess\,sup}_{z\in(t,\infty)} U^{q}(t,z)\, v^{q'}(z)\,\mathrm{d}t < \infty$$

are satisfied.

Moreover, if C is the least constant such that (8) *holds for all functions $f \in \mathscr{M}_+$, then*

$$C^{-q'} \approx D_3 + D_4 \approx A_3 + A_4.$$

By performing a simple change of variables $t \to \frac{1}{t}$, one gets the two corollaries below. They are formulated without the discrete conditions, those corresponding to Corollary 3.3 were presented in Section 1. An interested reader may also derive all the discrete conditions easily from their respective counterparts in Theorems 3.1 and 3.2.

Corollary 3.3. *Let $0 < q < 1 < p < \infty$, $r := \frac{pq}{p-q}$ and $0 < \vartheta < \infty$. Let v, w be weights. Let U be a ϑ-regular kernel. Then the following assertions are equivalent:*

(i) *There exists a constant $C \in (0,\infty)$ such that the inequality*

$$\left(\int_0^\infty \left(\int_0^t f(x)U(x,t)\,\mathrm{d}x \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \tag{9}$$

holds for all functions $f \in \mathscr{M}_+$.

(ii) *Both the conditions*

$$A_1^* := \int_0^\infty \left(\int_t^\infty w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \left(\int_0^t U^{p'}(z,t) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \mathrm{d}t < \infty$$

and

$$A_2^* := \int_0^\infty \left(\int_t^\infty w(x) U^q(t,x)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in(0,t]} U^q(z,t) \left(\int_0^z v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t < \infty$$

are satisfied.

Moreover, if C *is the least constant such that* (9) *holds for all functions* $f \in \mathscr{M}_+$, *then*

$$C^r \approx A_1^* + A_2^*.$$

Corollary 3.4. *Let* $0 < q < 1 = p$ *and* $0 < \vartheta < \infty$. *Let* v, w *be weights. Let* U *be a* ϑ*-regular kernel. Then the following assertions are equivalent:*

(i) *There exists a constant* $C \in (0, \infty)$ *such that the inequality* (9) *holds for all functions* $f \in \mathscr{M}_+$.

(ii) *Both the conditions*

$$A_3^* := \int_0^\infty \left(\int_t^\infty w(x)\,\mathrm{d}x \right)^{-q'} w(t) \operatorname*{ess\,sup}_{z\in(0,t)} U^{-q'}(z,t) v^{q'}(z)\,\mathrm{d}t < \infty$$

and

$$A_4^* := \int_0^\infty \left(\int_t^\infty w(x) U^q(t,x)\,\mathrm{d}x \right)^{-q'} w(t) \operatorname*{ess\,sup}_{z\in(0,t)} U^q(z,t) v^{q'}(z)\,\mathrm{d}t < \infty$$

are satisfied.

Moreover, if C *is the least constant such that* (9) *holds for all functions* $f \in \mathscr{M}_+$, *then*

$$C^{-q'} \approx A_3^* + A_4^*.$$

The next part contains the proofs. The core components of the discretization method used in this article are summarized in Theorem 3.5 below. It is presented separately for the purpose of possible future reference since this particular variant of discretization may be used even in other problems (cf. [8]).

Throughout the text, parentheses are used in expressions that involve indices, producing symbols such as $t_{(k+1)}$, $t_{k_{(n+1)}}$, etc. The parentheses do not have a special meaning, i.e. $t_{(k+1)}$ simply means t with the index $k+1$. They are used to make it easier to distinguish between objects as $t_{k_{(n+1)}}$ and $t_{(k_n+1)}$, which, in general, are different and both of them appear frequently in the formulas.

Theorem 3.5. *Let $0 < q < \infty$ and $1 \le \vartheta < \infty$. Define*

$$\Theta := 2\vartheta^q.$$

Let U be a ϑ-regular kernel. Let $K \in \mathbb{Z}$ and $\mu \in \mathbb{Z}$ be such that $\mu \le K-2$. Define the index set

$$\mathbb{Z}_\mu := \{k \in \mathbb{Z};\ \mu \le k \le K-1\}. \tag{10}$$

Let w be a weight such that $\int_0^\infty w = \Theta^K$. Let $\{t_k\}_{k=-\infty}^K \subset (0,\infty]$ be a sequence of points such that

$$\int_0^{t_k} w(x)\,\mathrm{d}x = \Theta^k \tag{11}$$

for all $k \in \mathbb{Z}$ such that $k \le K$ and $t_K = \infty$. For all $k \in \mathbb{Z}$ such that $k \le K-1$, denote

$$\Delta_k := [t_k, t_{(k+1)})$$

and

$$U(\Delta_k) := U\left(t_k, t_{(k+1)}\right).$$

Then there exist a number $N \in \mathbb{N}$ and an index set $\{k_n\}_{n=0}^N \subset \mathbb{Z}_\mu$ with the following properties.

(i) *It holds $k_0 = \mu$ and $k_{(n+1)} = K$. Whenever $n \in \{0,\dots,N\}$, then $k_n + 1 \le k_{(n+1)}$ and therefore also*

$$t_{(k_n+1)} \le t_{k_{(n+1)}}. \tag{12}$$

If we define

$$\mathbb{A} := \{n \in \mathbb{N};\ n \le N,\ k_n + 1 < k_{(n+1)}\}, \tag{13}$$

then

$$\mathbb{Z}_\mu = \{k_{(n+1)} - 1;\ n \in \mathbb{N} \cup \{0\},\ n \le N\} \cup \{k;\ k \in \mathbb{Z},\ n \in \mathbb{A},\ k_n \le k \le k_{(n+1)} - 2\}. \tag{14}$$

(ii) *For every $n \in \mathbb{N}$ such that $n \le N-1$ it holds*

$$\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \ge \Theta \sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k) \tag{15}$$

and

$$\sum_{k=\mu}^{k_n-1} \Theta^k U^q(\Delta_k) \le \frac{\Theta}{\Theta-1} \sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k). \tag{16}$$

(iii) *For every $n \in \mathbb{A}$ it holds*

$$\sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(\Delta_k) < \Theta \sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k). \tag{17}$$

(iv) *For every $n \in \mathbb{N}$, $k \in \mathbb{Z}_\mu$ and $t \in (0, \infty]$ such that $n \le N$, $k \le k_{(n+1)} - 1$ and $t \in (t_k, t_{(k+1)}]$ it holds*

$$\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \lesssim \sum_{j=k_{(n-1)}}^{k_n - 1} \Theta^j U^q(\Delta_j) + \Theta^k U^q(t_k, t). \tag{18}$$

If the same conditions hold and it is even satisfied that $k \le k_{(n+1)} - 2$, then

$$\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \lesssim \sum_{j=k_{(n-1)}}^{k_n - 1} \Theta^j U^q(\Delta_j). \tag{19}$$

(v) *Define $k_{(-1)} := \mu - 1$. Then for every $n \in \mathbb{N}$ such that $n \le N$ it holds*

$$\sum_{j=k_{(n-1)}}^{k_n - 1} \Theta^j U^q(\Delta_j) \lesssim \int_{t_{k_{(n-2)}}}^{t_{k_n}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t. \tag{20}$$

Proof. At first, observe that it is indeed possible to choose the sequence $\{t_k\}$ with the required properties because the weight w is locally integrable. Since w may take zero values, the sequence $\{t_k\}$ need not be unique. In that case, we choose one fixed $\{t_k\}$ satisfying the requirements. From (11) we deduce that

$$\Theta^k = \int_0^{t_k} w(s)\,\mathrm{d}s = \frac{1}{\Theta - 1} \int_{\Delta_k} w(s)\,\mathrm{d}s = \frac{\Theta}{\Theta - 1} \int_{\Delta_{(k-1)}} w(s)\,\mathrm{d}s \tag{21}$$

for all $k \in \mathbb{Z}$ such that $k \le K - 1$.

We proceed with the construction of the index subset $\{k_n\}$. Define $k_0 := \mu$ and $k_1 := \mu + 1$ and continue inductively as follows.

($*$) Let $k_0, \dots, k_n$ be already defined. Then

(a) If $k_n = K$, define $N := n - 1$ and stop the procedure.

(b) If $k_n < K$ and there exists an index j such that $k_n < j \le K$ and

$$\sum_{k=k_n}^{j-1} \Theta^k U^q(\Delta_k) \ge \Theta \sum_{k=k_{(n-1)}}^{k_n - 1} \Theta^k U^q(\Delta_k), \tag{22}$$

then define $k_{(n+1)}$ as the smallest index j for which (22) holds. Then proceed again with step ($*$) with $n+1$ in place of n.

(c) If $k_n < K$ and and (22) holds for no index j such that $k_n < j \le K$, then define $N := n$, $k_{(n+1)} := K$ and stop the procedure.

In this manner, one obtains a finite sequence of indices $\{k_0, \dots, k_N\} \subseteq \mathbb{Z}_\mu$ and the final index $k_{(n+1)} = K$.

We will call each interval Δ_k the *k-th segment*, and each interval $[t_{k_n}, t_{(k_n+1)})$ the *n-th block*. If $n \in \mathbb{N}$ is such that $n \le N$, then the n-th block either consists of the single k_n-th segment, in which case it holds

$$k_{(n+1)} = k_n + 1,$$

or the n-th segment contains more than one segment and then

$$k_{(n+1)} > k_n + 1,$$

If the n-th block is of the second type, then $n \in \mathbb{A}$, according to the definition (13). Hence, (14) is satisfied, even though the set $\mathbb{A}$ may be empty. The relation (14) in plain words says that each segment is either the last one (i.e., with the highest index k) in a block, or it belongs to a block consisting of more than one segment and the investigated segment is not the last one of those. We have now proved (i).

The property (15) follows directly from the construction. If $n \in \mathbb{N}$ is such that $n \leq N$, then by iterating (15) one gets

$$\sum_{k=\mu}^{k_n-1} \Theta^k U^q(\Delta_k) = \sum_{i=0}^{n-1} \sum_{k=k_i}^{k_{(i+1)}-1} \Theta^k U^q(\Delta_k) \leq \sum_{i=0}^{n-1} \Theta^{i-n+1} \sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k)$$
$$\leq \frac{\Theta}{\Theta-1} \sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k).$$

Hence, (16) holds and (ii) is then proved.

Property (iii) is again a direct consequence of the way the blocks were constructed. We proceed with proving (iv). Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_\mu$ and $t \in (0, \infty]$ be such that $n \leq N$, $k \leq k_{(n+1)} - 1$ and $t \in (t_k, t_{(k+1)}]$. Then the following sequence of inequalities is valid:

$$\begin{aligned}
\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x &= \int_{t_\mu}^{t_k} w(x) U^q(x,t)\,\mathrm{d}x + \int_{t_k}^{t} w(x) U^q(x,t)\,\mathrm{d}x \\
&\lesssim \int_{t_\mu}^{t_k} w(x) U^q(x,t_k)\,\mathrm{d}x + \int_{t_\mu}^{t_k} w(x)\,\mathrm{d}x\, U^q(t_k,t) + \int_{t_k}^{t} w(x) U^q(x,t)\,\mathrm{d}x \\
&\leq \sum_{j=\mu}^{k-1} \int_{\Delta_j} w(x)\,\mathrm{d}x\ U^q(t_j,t_k) + \int_{t_\mu}^{t_{(k+1)}} w(x)\,\mathrm{d}x\ U^q(t_k,t) \\
&\lesssim \sum_{j=\mu}^{k-1} \Theta^j U^q(t_j,t_k) + \Theta^k U^q(t_k,t) \qquad (23) \\
&\lesssim \sum_{j=\mu}^{k-1} \Theta^j U^q(\Delta_j) + \Theta^k U^q(t_k,t). \qquad (24)
\end{aligned}$$

In here, step (23) follows by (21), and step (24) by Proposition 2.5. If $k \leq k_n$, then

$$\sum_{j=\mu}^{k-1} \Theta^j U^q(\Delta_j) \leq \sum_{j=\mu}^{k_n-1} \Theta^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j).$$

The second inequality here follows by (16). If $k > k_n$, then $n \in \mathbb{A}$, $k_n + 1 \le k \le k_{(n+1)} - 1$ and it holds

$$\sum_{j=\mu}^{k-1} \Theta^j U^q(\Delta_j) \le \sum_{j=\mu}^{k_{(n+1)}-2} \Theta^j U^q(\Delta_j) = \sum_{j=\mu}^{k_n-1} \Theta^j U^q(\Delta_j) + \sum_{j=k_n}^{k_{(n+1)}-2} \Theta^j U^q(\Delta_j)$$
$$\lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j).$$

The last inequality is granted by (16) and (17). We have proved that

$$\sum_{j=\mu}^{k-1} \Theta^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j).$$

Applying this in the inequality obtained at (24), we get the estimate (18). If we now add the assumption $k \le k_{(n+1)} - 2$, then (18) still holds and, in addition to that, we get

$$\Theta^k U^q(t_k, t) \le \Theta^k U^q(\Delta_k) \le \sum_{j=\mu}^{k_{(n+1)}-2} \Theta^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j).$$

In here, the last inequality follows from (16) and (17). Applying this result to (18), we obtain (19) and (iv) is thus proved.

To prove (v), let $n \in \mathbb{N}$ be such that $n \le N$ and observe the following:

$$\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \int_{\Delta_{j-1}} w(t)\,\mathrm{d}t\; U^q(\Delta_j) \le \sum_{j=k_{(n-1)}}^{k_n-1} \int_{\Delta_{j-1}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t$$
$$= \int_{t_{(k_{(n-1)}-1)}}^{t_{(k_n-1)}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t \le \int_{t_{k_{(n-2)}}}^{t_{k_n}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t.$$

In the first step, (21) was used. In the last one, we used the inequality $t_{k_{(n-2)}} \le t_{(k_{(n-1)}-1)}$ which follows from (12). □

Proof of Theorem 3.1. Without loss of generality, we may assume that $\vartheta \in [1, \infty)$. Indeed, if the kernel U is ϑ-regular with $\vartheta \in (0, 1)$, then U is obviously also 1-regular.

"(ii)⇒(i)". Assume that $D_1 < \infty$ and $D_2 < \infty$. Let us prove that (8) holds for all $f \in \mathscr{M}_+$ with the least constant C satisfying $C^r \lesssim D_1 + D_2$.

At first, let us assume that there exists $K \in \mathbb{Z}$ such that $\int_0^\infty w = 2^K$. Let $\mu \in \mathbb{Z}$ be such that $\mu \le K-2$ and define $\mathbb{Z}_\mu$ by (10). Let $\{t_k\}_{k=-\infty}^{K} \subset (0,\infty]$ be a sequence of points such that $t_K = \infty$ and (21) holds for all $k \in \mathbb{Z}$ such that $k \le K$. Let $\{k_n\}_{n=0}^{N} \subset \mathbb{Z}_\mu$ be the subsequence of indices granted by Theorem 3.5. Related notation from Theorem 3.5 will be used in what follows as well. Suppose that $f \in \mathscr{M}_+ \cap L^p(v)$. Then

$$
\begin{aligned}
&\int_{t_\mu}^{\infty}\left(\int_t^\infty f(x)U(t,x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t \\
&= \sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k}\left(\int_t^\infty f(x)U(t,x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t \\
&\lesssim \sum_{k\in\mathbb{Z}_\mu}\Theta^k\left(\int_{t_k}^\infty f(x)U(t_k,x)\,\mathrm{d}x\right)^q \\
&\lesssim \sum_{n=0}^{N}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^k\left(\int_{t_k}^{t_{k_{(n+1)}}} f(x)U(t_k,x)\,\mathrm{d}x\right)^q \\
&\quad + \sum_{n=0}^{N-1}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^k\left(\int_{t_{k_{(n+1)}}}^\infty f(x)U(t_k,x)\,\mathrm{d}x\right)^q \\
&\lesssim \sum_{n=0}^{N}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^k\left(\int_{t_k}^{t_{k_{(n+1)}}} f(x)U(t_k,x)\,\mathrm{d}x\right)^q \\
&\quad + \sum_{n=0}^{N-1}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^k U^q(t_k,t_{k_{(n+1)}})\left(\int_{t_{k_{(n+1)}}}^\infty f(x)\,\mathrm{d}x\right)^q \\
&\quad + \sum_{n=0}^{N-1}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^k\left(\int_{t_{k_{(n+1)}}}^\infty f(x)U(t_{k_{(n+1)}},x)\,\mathrm{d}x\right)^q \\
&=: B_1+B_2+B_3.
\end{aligned}
\tag{25}
$$

Inequality (25) follows from (21). Furthermore, we have

$$
\begin{aligned}
B_1 &= \sum_{n=0}^{N} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k \left(\int_{t_k}^{t_{k_{(n+1)}}} f(x) U(t_k, x)\, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}-1} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\, \mathrm{d}x \right)^q + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k \left(\int_{t_k}^{t_{k_{(n+1)}}} f(x) U(t_k, x)\, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}-1} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\, \mathrm{d}x \right)^q + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_k, x)\, \mathrm{d}x \right)^q \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k \left(\int_{t_k}^{t_{(k_{(n+1)}-1)}} f(x) U(t_k, x)\, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}-1} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\, \mathrm{d}x \right)^q \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\, \mathrm{d}x \right)^q \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(t_k, t_{(k_{(n+1)}-1)}) \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x)\, \mathrm{d}x \right)^q \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k \left(\int_{t_k}^{t_{(k_{(n+1)}-1)}} f(x) U(t_k, x)\, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\, \mathrm{d}x \right)^q \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(t_k, t_{(k_{(n+1)}-1)}) \left(\int_{t_k}^{t_{k_{(n+1)}}} f(x)\, \mathrm{d}x \right)^q \\
&=: B_4 + B_5.
\end{aligned}
$$

For the role of the symbol $\mathbb{A}$, see (13). In the next step, for formal reasons define $t_{(k_{(N+2)}-1)} := \infty$. Then we get

$$
\begin{aligned}
B_4 &= \sum_{n=0}^{N} \Theta^{k_{(n+1)}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f(x) U(t_{(k_{(n+1)}-1)}, x)\,\mathrm{d}x \right)^{q} \\
(26)\quad &\le \sum_{n=0}^{N} \Theta^{k_{(n+1)}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{q}{p'}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} f^{p}(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
(27)\quad &\le \left(\sum_{n=0}^{N} \Theta^{\frac{r}{q} k_{(n+1)}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \left(\sum_{n=0}^{N} \int_{\Delta_{(k_{(n+1)}-1)}} f^{p}(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
&\le \left(\sum_{n=0}^{N} \Theta^{\frac{r}{q} k_{(n+1)}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|_{L^p(v)}^{q} \\
(28)\quad &\lesssim \left(\sum_{n=0}^{N} \left(\int_{\Delta_{(k_{(n+1)}-2)}} w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|_{L^p(v)}^{q} \\
(29)\quad &\le \left(\sum_{n=0}^{N} \left(\int_{t_{(k_n-1)}}^{t_{(k_{(n+1)}-1)}} w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{(k_{(n+1)}-1)}}^{t_{(k_{(n+2)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, x) v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|_{L^p(v)}^{q} \\
&\le D_1^{\frac{q}{r}} \|f\|_{L^p(v)}^{q}.
\end{aligned}
$$

The Hölder inequality for functions was used in (26), and its discrete version (see Proposition 2.2) was used in (27). Step (28) follows from (21). In (29) we used the inequalities $t_{(k_n-1)} \le t_{(k_{(n+1)}-2)}$ and $t_{k_{(n+1)}} \le t_{(k_{(n+2)}-1)}$ which hold for all $n \in \{0, \ldots, N\}$ and both follow from (12) or the additional formal definition in the case $n = N$. Step (29) ensures that the sequence $\{t_{(k_n-1)}\}_{n=0}^{N}$ can be extended into a covering sequence (formally, $\{t_{(k_n-1)}\}_{n=0}^{N}$ itself is not a covering sequence since $t_{(k_0-1)} = t_{(\mu-1)} > 0$).

Regarding the term B_5, one has

$$
\begin{aligned}
B_5 &= \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(t_k, t_{(k_{(n+1)}-1)}) \left(\int_{t_k}^{t_{k_{(n+1)}}} f(x)\,\mathrm{d}x \right)^q \\
&\le \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(t_k, t_{(k_{(n+1)}-1)}) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} f(x)\,\mathrm{d}x \right)^q \\
&\lesssim \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} f(x)\,\mathrm{d}x \right)^q \qquad (30) \\
&\le \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{q}{p'}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} f^p(x)v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \qquad (31) \\
&\le \left(\sum_{n\in\mathbb{A}} \left(\sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \left(\sum_{n\in\mathbb{A}} \int_{t_{k_n}}^{t_{k_{(n+1)}}} f^p(x)v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \qquad (32) \\
&\le \left(\sum_{n\in\mathbb{A}} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(t)U^q(t, t_{k_n})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|^q_{L^p(v)} \qquad (33) \\
&\le D_2^{\frac{q}{r}} \|f\|^q_{L^p(v)}.
\end{aligned}
$$

Inequality (30) follows from Proposition 2.5. In steps (31) and (32) we used the appropriate versions of the Hölder inequality, cf. Propositions 2.1 and 2.2. Inequalities (17) and (20) give the estimate (33). We proved

$$B_1 \lesssim B_4 + B_5 \lesssim (D_1 + D_2)^{\frac{q}{r}} \|f\|^q_{L^p(v)}.$$

We continue with the term B_2.

$$
\begin{aligned}
B_2 &= \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(t_k, t_{k_{(n+1)}}) \left(\int_{t_{k_{(n+1)}}}^{\infty} f(x)\,\mathrm{d}x \right)^q \\
\text{(34)} \quad &\lesssim \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \left(\int_{t_{k_{(n+1)}}}^{\infty} f(x)\,\mathrm{d}x \right)^q \\
&= \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \left(\sum_{i=n+1}^{N} \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
\text{(35)} \quad &\lesssim \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f(x)\,\mathrm{d}x \right)^q \\
\text{(36)} \quad &\leq \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{q}{p'}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f^p(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
\text{(37)} \quad &\leq \left(\sum_{n=0}^{N-1} \left(\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \left(\sum_{n=0}^{N-1} \int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f^p(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
\text{(38)} \quad &\lesssim \left(\sum_{n=0}^{N-1} \left(\int_{t_{k_{(n-1)}}}^{t_{k_{(n+1)}}} w(t) U^q(t, t_{k_{(n+1)}})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|_{L^p(v)}^q \\
&\leq D_2^{\frac{q}{r}} \|f\|_{L^p(v)}^q.
\end{aligned}
$$

Step (34) follows from Proposition 2.5. Proposition 2.3 supplied with (15) gives

(35). In (36) and (37) we used the Hölder inequality (see Propositions 2.1 and 2.2). To get (38), one uses (20). We obtained

$$B_2 \lesssim D_2^{\frac{q}{r}} \|f\|_{L^p(v)}^q.$$

In what follows, without loss of generality we will assume that $N \geq 2$. If $N = 1$, then the terms involving $\sum_{j=0}^{N-2}$ (or similar) are simply not present in the calculations below.

The term B_3 is treated as follows.

$$\begin{aligned}
B_3 &= \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k \left(\int_{t_{k_{(n+1)}}}^{\infty} f(x) U(t_{k_{(n+1)}}, x) \, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\int_{t_{k_{(n+1)}}}^{\infty} f(x) U(t_{k_{(n+1)}}, x) \, \mathrm{d}x \right)^q \\
&= \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\sum_{i=n+1}^{N} \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x) U(t_{k_{(n+1)}}, x) \, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\sum_{i=n+1}^{N} \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x) U(t_{k_i}, x) \, \mathrm{d}x \right)^q \\
&\quad + \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}} \left(\sum_{i=n+2}^{N} U(t_{k_{(n+1)}}, t_{k_i}) \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x) \, \mathrm{d}x \right)^q \\
&=: B_6 + B_7.
\end{aligned}$$

Furthermore, it holds

$$
\begin{aligned}
B_6 &= \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\sum_{i=n+1}^{N} \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x) U(t_{k_i}, x)\, \mathrm{d}x \right)^q \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f(x) U(t_{k_{(n+1)}}, x)\, \mathrm{d}x \right)^q \qquad (39) \\
&\le \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U(t_{k_{(n+1)}}, x) v^{1-p'}(x)\, \mathrm{d}x \right)^{\frac{q}{p'}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f^p(x) v(x)\, \mathrm{d}x \right)^{\frac{q}{p}} \qquad (40) \\
&\le \left(\sum_{n=0}^{N-1} \Theta^{\frac{r}{q} k_{(n+1)}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U(t_{k_{(n+1)}}, x) v^{1-p'}(x)\, \mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \qquad (41) \\
&\quad \times \left(\sum_{n=0}^{N-1} \int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} f^p(x) v(x)\, \mathrm{d}x \right)^{\frac{q}{p}} \\
&\lesssim \left(\sum_{n=0}^{N-1} \left(\int_{t_{(k_{(n+1)}-1)}}^{t_{k_{(n+1)}}} w(x)\, \mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U(t_{k_{(n+1)}}, x) v^{1-p'}(x)\, \mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} ||f||^q_{L^p(v)} \qquad (42) \\
&\le \left(\sum_{n=0}^{N-1} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} w(x)\, \mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U(t_{k_{(n+1)}}, x) v^{1-p'}(x)\, \mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} ||f||^q_{L^p(v)} \qquad (43) \\
&\le D_1^{\frac{q}{r}} ||f||^q_{L^p(v)}.
\end{aligned}
$$

Step (39) follows by Proposition 2.3. As usual, in (40) and (41) we used the Hölder inequality. The inequality (42) is granted by (21), and (43) is a consequence of (12).

Next, for the term B_7 we have

$$
\begin{aligned}
B_7 &= \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}} \left(\sum_{i=n+2}^{N} U(t_{k_{(n+1)}}, t_{k_i}) \int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
\text{(44)} \quad &\le \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}} \sum_{i=n+2}^{N} U^q(t_{k_{(n+1)}}, t_{k_i}) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
&\le \sum_{i=2}^{N} \sum_{n=0}^{i-2} \Theta^{k_{(n+1)}} U^q(t_{k_{(n+1)}}, t_{k_i}) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
&\le \sum_{i=2}^{N} \sum_{k=\mu}^{k_i-1} \Theta^{k} U^q(t_k, t_{k_i}) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
\text{(45)} \quad &\lesssim \sum_{i=2}^{N} \sum_{k=\mu}^{k_i-1} \Theta^{k} U^q(\Delta_k) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} f(x)\,\mathrm{d}x \right)^q \\
\text{(46)} \quad &\le \sum_{i=2}^{N} \sum_{k=\mu}^{k_i-1} \Theta^{k} U^q(\Delta_k) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{q}{p'}} \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} f^p(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
\text{(47)} \quad &\le \left(\sum_{i=2}^{N} \left(\sum_{k=\mu}^{k_i-1} \Theta^{k} U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \left(\sum_{n=0}^{N-1} \int_{t_{k_i}}^{t_{k_{(i+1)}}} f^p(x) v(x)\,\mathrm{d}x \right)^{\frac{q}{p}} \\
\text{(48)} \quad &\lesssim \left(\sum_{i=2}^{N} \left(\int_{t_{k_{(i-2)}}}^{t_{k_i}} w(t) U^q(t, t_{k_i})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{q}{r}} \|f\|^q_{L^p(v)} \\
&\le D_2^{\frac{q}{r}} \|f\|^q_{L^p(v)}.
\end{aligned}
$$

Inequality (44) follows from concavity of the q-th power for $q < 1$. In (45) one uses Proposition 2.5. The Hölder inequality gives (46) and (47). Estimate (48) follows from (16) and (20). We proved

$$
B_3 \lesssim B_6 + B_7 \lesssim (D_1 + D_2)^{\frac{q}{r}} \|f\|^q_{L^p(v)}.
$$

Combined with the other estimates of B_1 and B_2, this yields

$$\int_{t_\mu}^{\infty}\left(\int_t^{\infty} f(x)U(t,x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t \lesssim (D_1+D_2)^{\frac{q}{r}}\|f\|^q_{L^p(v)}.$$

Observe that the constant related to the symbol "$\lesssim$" in here does not depend on the choice of μ. The reader may nevertheless notice that the construction of the n-blocks in fact depends on μ. However, the constants in the "$\lesssim$"-estimates proved with help of that construction are indeed independent of μ. Hence, we may perform the limit pass $\mu \to -\infty$. Since $t_\mu \to 0$ as $\mu \to -\infty$, the monotone convergence theorem (and taking the q-th root) yields

$$\left(\int_0^{\infty}\left(\int_t^{\infty} f(x)U(t,x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim (D_1+D_2)^{\frac{1}{r}}\|f\|_{L^p(v)}$$

for the fixed function $f \in \mathscr{M}_+ \cap L^p(v)$. Since the function f was chosen arbitrarily and the constant represented in "$\lesssim$" does not depend on f, the inequality (8) holds with $C = (D_1+D_2)^{\frac{1}{r}}$ for all functions $f \in \mathscr{M}_+$. Clearly, if C is the least constant such that (8) holds for all $f \in \mathscr{M}_+$, then

$$C^r \lesssim D_1 + D_2. \tag{49}$$

At this point, recall that so far we have assumed that $\int_0^\infty w(x)\,\mathrm{d}x = \Theta^K$ for a $K \in \mathbb{Z}$. Let us now complete the proof of this part for a general weight w.

At first, if $\int_0^\infty w(x)\,\mathrm{d}x$ is finite but not equal to any integer power of the parameter Θ, the result is simply obtained by multiplying w by a constant $c \in (1,2)$ such that $\int_0^\infty c\,w(x)\,\mathrm{d}x = \Theta^K$ for a $K \in \mathbb{Z}$, and then using homogeneity of the expressions $\int_0^\infty\left(\int_t^\infty f(x)U(t,x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t$, $D_1^{\frac{q}{r}}$ and $D_2^{\frac{q}{r}}$ with respect to w.

Finally, let us assume $\int_0^\infty w(x)\,\mathrm{d}x = \infty$. Choose an arbitrary function $f \in \mathscr{M}_+ \cap L^p(v)$. For each $m \in \mathbb{N}$ define $w_m := w\chi_{[0,m]}$ and denote by $D_{1,m}$ the expression D_1 with w replaced by w_m. Similarly we define $D_{2,m}$. Since the weight w is locally integrable, for each $m \in \mathbb{N}$ it holds $\int_0^\infty w_m(x)\,\mathrm{d}x < \infty$. Hence, by the previous part of the proof we get

$$\left(\int_0^{\infty}\left(\int_t^{\infty} f(x)U(t,x)\,\mathrm{d}x\right)^q w_m(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim (D_{1,m}+D_{2,m})^{\frac{1}{r}}\|f\|_{L^p(v)}.$$

Obviously, for all $m \in \mathbb{N}$ it holds $w_m \le w$ pointwise, hence $D_{1,m} \le D_1$ and $D_{2,m} \le D_2$. Thus, we get

$$\left(\int_0^{\infty}\left(\int_t^{\infty} f(x)U(t,x)\,\mathrm{d}x\right)^q w_m(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim (D_1+D_2)^{\frac{1}{r}}\|f\|_{L^p(v)}.$$

The constant in "$\lesssim$" does not depend on m or f and the latter was arbitrarily chosen. Since $w_m \uparrow w$ pointwise as $m \to \infty$, the monotone convergence theorem (for $m \to \infty$) yields that (8) holds for all functions $f \in \mathscr{M}_+$ and the best constant C in (8) satisfies (49). The proof of this part is now complete.

"(i)⇒(ii)". Suppose that (8) holds for all $f \in \mathscr{M}_+$ and $C \in (0,\infty)$ is the least constant such that this is true. We need to show that $D_1 + D_2 \lesssim C^r$.

Let $\{t_k\}_{k\in\mathbb{I}}$ be a covering sequence indexed by a set $\mathbb{I} = \{k_{\min},\dots,k_{\max}\} \subset \mathbb{Z}$. By Proposition 2.1, for each $k \in \mathbb{I}_0$ there exists a measurable function g_k supported in $[t_k, t_{(k+1)}]$ and such that $\|g_k\|_{L^p(v)} = 1$ as well as

$$\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} = \int_{t_k}^{t_{(k+1)}} g_k(x)U(t_k,x)\,\mathrm{d}x. \tag{50}$$

By Proposition 2.2 we can find a nonnegative sequence $\{c_k\}_{k\in\mathbb{I}_0}$ such that $\sum_{k\in\mathbb{I}_0} c_k^p = 1$ and

$$\left(\sum_{k\in\mathbb{I}_0}\left(\int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}} \tag{51}$$
$$= \left(\sum_{k\in\mathbb{I}_0} c_k^q \int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}}.$$

Define a function $g := \sum_{k\in\mathbb{I}_0} c_k g_k$ and recall that each g_k is supported in $[t_k, t_{(k+1)}]$. Hence,

$$\|g\|_{L^p(v)} = \left(\sum_{k\in\mathbb{I}_0} c_k^p \|g_k\|_{L^p(v)}^p\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{I}_0} c_k^p\right)^{\frac{1}{p}} = 1. \tag{52}$$

Finally, we get the following estimate.

$$\sum_{k\in\mathbb{I}_0}\left(\int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{r}{p'}}$$
$$= \left(\sum_{k\in\mathbb{I}_0} c_k^q \int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{q}{p'}}\right)^{\frac{r}{q}} \tag{53}$$
$$= \left(\sum_{k\in\mathbb{I}_0} c_k^q \int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} U(t_k,x)g_k(x)\,\mathrm{d}x\right)^{q}\right)^{\frac{r}{q}} \tag{54}$$

$$= \left(\sum_{k\in\mathbb{I}_0} \int_{t_{(k-1)}}^{t_k} w(t)\,\mathrm{d}t \left(\int_{t_k}^{t_{(k+1)}} U(t_k,x)g(x)\,\mathrm{d}x \right)^q \right)^{\frac{r}{q}}$$
$$\leq \left(\sum_{k\in\mathbb{I}_0} \int_{t_{(k-1)}}^{t_k} w(t) \left(\int_{t}^{t_{(k+1)}} U(t,x)g(x)\,\mathrm{d}x \right)^q \mathrm{d}t \right)^{\frac{r}{q}}$$
$$\leq \left(\int_0^\infty w(t) \left(\int_t^\infty U(t,x)g(x)\,\mathrm{d}x \right)^q \mathrm{d}t \right)^{\frac{r}{q}}$$
$$\leq C^r \|g\|^r_{L^p(v)} \tag{55}$$
$$= C^r. \tag{56}$$

In steps (53), (54), (55) and (56) we used (51), (50), (8) and (52), respectively. Since the covering sequence $\{t_k\}_{k\in\mathbb{I}}$ was chosen arbitrarily, by taking supremum over all covering sequences we obtain

$$D_1 \lesssim C^r.$$

In what follows, we are going to prove a similar estimate for D_2. Again, let $\{t_k\}_{k\in\mathbb{I}}$ be a covering sequence indexed by a set $\mathbb{I} = \{k_{\min},\dots,k_{\max}\} \subset \mathbb{Z}$. Proposition 2.1 yields that for every $k \in \mathbb{I}_0$ we can find a function h_k supported in $[t_k, t_{(k+1)}]$ and such that $\int_{t_k}^{t_{(k+1)}} h_k^p(x)v(x)\,\mathrm{d}x = 1$ and

$$\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} = \int_{t_k}^{t_{(k+1)}} h_k(x)\,\mathrm{d}x.$$

By Proposition 2.2, we may find a nonnegative sequence $\{d_k\}_{k\in\mathbb{I}_0}$ such that $\sum_{k\in\mathbb{I}_0} d_k^p = 1$ and

$$\left(\sum_{k\in\mathbb{I}_0} \left(\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}}$$
$$= \left(\sum_{k\in\mathbb{I}_0} d_k^q \int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t \left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x)\,\mathrm{d}x \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}}.$$

Define the function $h := \sum_{k\in\mathbb{I}_0} d_k h_k$. Then it is easy to verify that $\|h\|_{L^p(v)} = 1$. Moreover, we get the following estimate.

$$
\begin{aligned}
&\sum_{k\in\mathbb{I}_0}\left(\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{r}{p'}}\\
&=\left(\sum_{k\in\mathbb{I}_0} d_k^q\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{q}{p'}}\right)^{\frac{r}{q}}\\
&=\left(\sum_{k\in\mathbb{I}_0} d_k^q\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} h_k(x)\,\mathrm{d}x\right)^{q}\right)^{\frac{r}{q}}\\
&=\left(\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} h(x)\,\mathrm{d}x\right)^{q}\right)^{\frac{r}{q}}\\
&\le\left(\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k} w(t)\left(\int_{t_k}^{t_{(k+1)}} h(x)U(t,x)\,\mathrm{d}x\right)^{q}\mathrm{d}t\right)^{\frac{r}{q}}\\
&\le\left(\sum_{k\in\mathbb{I}_0}\int_{0}^{\infty} w(t)\left(\int_{t}^{\infty} h(x)U(t,x)\,\mathrm{d}x\right)^{q}\mathrm{d}t\right)^{\frac{r}{q}}\\
&\le C^r\|h\|_{L^p(v)}=C^r.
\end{aligned}
$$

The covering sequence $\{t_k\}_{k\in\mathbb{I}}$ was arbitrarily chosen in the beginning, hence we may take the supremum over all covering sequences, obtaining the relation

$$D_2\lesssim C^r.$$

The proof of the implication "(i)⇒(ii)" and of the related estimates is finished.

"(iii)⇒(ii)". Assume that $A_1<\infty$ and $A_2<\infty$. We will prove that $D_1+D_2\lesssim A_1+A_2$. Let $\{t_k\}_{k\in\mathbb{I}}$ be an arbitrary covering sequence indexed by a set $\mathbb{I}$. Then

$$
\begin{aligned}
&\sum_{k\in\mathbb{I}_0}\left(\int_{t_{(k-1)}}^{t_k} w(x)\,\mathrm{d}x\right)^{\frac{r}{q}}\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,t)v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{r}{p'}}\\
&\approx\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{t_{(k-1)}}^{x} w(s)\,\mathrm{d}s\right)^{\frac{r}{p}} w(x)\,\mathrm{d}x\left(\int_{t_k}^{t_{(k+1)}} U^{p'}(t_k,t)v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{r}{p'}}\\
&\le\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{0}^{x} w(s)\,\mathrm{d}s\right)^{\frac{r}{p}} w(x)\,\mathrm{d}x\left(\int_{x}^{\infty} U^{p'}(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{r}{p'}}=A_1.
\end{aligned}
$$

Taking the supremum over all covering sequences, we obtain $D_1 \lesssim A_1$. Similarly, for any fixed covering sequence $\{t_k\}_{k\in\mathbb{I}}$ we get

$$
\begin{aligned}
&\sum_{k\in\mathbb{I}_0}\left(\int_{t_{(k-1)}}^{t_k} w(t)U^q(t,t_k)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\approx\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{t_{(k-1)}}^{t} w(x)U^q(x,t_k)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\lesssim\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{t_{(k-1)}}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\quad+\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{t_{(k-1)}}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^r(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{t_{(k+1)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\leq\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{0}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^q(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\quad+\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{0}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^r(t,t_k)\,\mathrm{d}t\left(\int_{t_k}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\leq\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{0}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)}U^q(t,z)\left(\int_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\\
&\quad+\sum_{k\in\mathbb{I}_0}\int_{t_{(k-1)}}^{t_k}\left(\int_{0}^{t} w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_{t}^{\infty} U^{p'}(t,s)v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\\
&=A_2+A_1.
\end{aligned}
$$

Once again, taking the supremum over all covering sequences, we get $D_2 \lesssim A_2 + A_1$. Hence, we have shown that $D_1 + D_2 \lesssim A_1 + A_2$ and the implication "(iii)⇒(ii)" is proved.

"(ii) ⇒ (iii)". Suppose that $D_1 < \infty$ and $D_2 < \infty$ and let us show that $A_1 + A_2 \lesssim D_1 + D_2$ then.

Similarly as in the proof of "(ii) ⇒ (i)", let us first assume that $\int_0^\infty w = 2^K$ for some $K \in \mathbb{Z}$. Let $\mu \in \mathbb{Z}$ be such that $\mu \leq K-2$ and define $\mathbb{Z}_\mu$ by (10). Let $\{t_k\}_{k=-\infty}^K \subset (0,\infty]$ be the sequence of points from Theorem 3.5 and $\{k_n\}_{n=0}^N \subset$

$\mathbb{Z}_\mu$ be the subsequence of indices granted by the same theorem. Then

$$
\begin{aligned}
&\int_{t_\mu}^{\infty}\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_t^\infty U^{p'}(t,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}}\mathrm{d}t \\
&=\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k}\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_t^\infty U^{p'}(t,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}}\mathrm{d}t \\
&\le\sum_{k\in\mathbb{Z}_\mu}\int_0^{t_{(k+1)}}\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\,\mathrm{d}t\left(\int_{t_k}^\infty U^{p'}(t_k,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \\
&\lesssim\sum_{k\in\mathbb{Z}_\mu}\Theta^{\frac{kr}{q}}\left(\int_{t_k}^\infty U^{p'}(t_k,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \qquad (57)\\
&\approx\sum_{n=0}^{N}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^{\frac{kr}{q}}\left(\int_{t_k}^{t_{k_{(n+1)}}} U^{p'}(t_k,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \\
&\qquad+\sum_{n=0}^{N-1}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^{\frac{kr}{q}}U^r(t_k,t_{k_{(n+1)}})\left(\int_{t_{k_{(n+1)}}}^\infty v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \\
&\qquad+\sum_{n=0}^{N-1}\sum_{k=k_n}^{k_{(n+1)}-1}\Theta^{\frac{kr}{q}}\left(\int_{t_{k_{(n+1)}}}^\infty U^{p'}(t_{k_{(n+1)}},z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \\
&=: B_8+B_9+B_{10}.
\end{aligned}
$$

In step (57) we used (21). We continue by estimating each of the separate terms.

$$
\begin{aligned}
B_8 = & \sum_{n=0}^{N} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^{\frac{kr}{q}} \left(\int_{t_k}^{t_{k_{(n+1)}}} U^{p'}(t_k,z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
= & \sum_{n=0}^{N} \Theta^{(k_{(n+1)}-1)\frac{r}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)},z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} \left(\int_{t_k}^{t_{k_{(n+1)}}} U^{p'}(t_k,z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
\lesssim & \sum_{n=0}^{N} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)},z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} \left(\int_{t_k}^{t_{(k_{(n+1)}-1)}} U^{p'}(t_k,z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_k,z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
\lesssim & \sum_{n=0}^{N} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)},z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} U^{r}(t_k,t_{(k_{(n+1)}-1)}) \left(\int_{\Delta_{(k_{(n+1)}-1)}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} U^{r}(t_k,t_{(k_{(n+1)}-1)}) \left(\int_{t_k}^{t_{(k_{(n+1)}-1)}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
& + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} \left(\int_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)},z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} U^r(t_k, t_{(k_{(n+1)}-1)}) \left(\int\limits_{t_k}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\quad + \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} U^r(t_k, t_{(k_{(n+1)}-1)}) \left(\int\limits_{t_k}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&:= B_{11} + B_{12}.
\end{aligned}
$$

For B_{11} we have

$$
\begin{aligned}
B_{11} &= \sum_{n=0}^{N} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N} \left(\int\limits_{\Delta_{(k_{(n+1)}-2)}} w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (58) \\
&\le \sum_{k\in\mathbb{Z}_\mu} \left(\int\limits_{\Delta_{(k-1)}} w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int\limits_{\Delta_k} U^{p'}(t_{(k-1)}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\le D_1.
\end{aligned}
$$

In step (58) we used (21). Let us formally define $k_{(-1)} := \mu - 1$ and proceed with estimating B_{12}.

$$B_{12} = \sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \Theta^{\frac{kr}{q}} U^r(t_k, t_{(k_{(n+1)}-1)}) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(59)} \qquad \leq \sum_{n\in\mathbb{A}} \left(\sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(t_k, t_{(k_{(n+1)}-1)}) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(60)} \qquad \lesssim \sum_{n\in\mathbb{A}} \left(\sum_{k=k_n}^{k_{(n+1)}-2} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(61)} \qquad \lesssim \sum_{n\in\mathbb{A}} \left(\sum_{k=k_{(n-1)}}^{k_n-1} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(62)} \qquad \lesssim \sum_{n\in\mathbb{A}} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(x) U^q(x, t_{k_n})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\leq \sum_{n=1}^{N} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(x) U^q(x, t_{k_n})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(63)} \qquad = \sum_{i=0}^{1} \sum_{\substack{1\leq n\leq N \\ n \bmod 2 = i}} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(x) U^q(x, t_{k_n})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\lesssim D_2.$$

Since $\frac{r}{q} > 1$, the estimate (59) follows by convexity of the $\frac{r}{q}$-th power. Step (60) is due to Proposition 2.5. Step (61) then follows by (17), and step (62) by (20). Finally, in (63) we split the even and odd indices n, so that the intervals $(t_{k_{(n-2)}}, t_{k_n})$ involved in each n-indexed sum do not overlap. This standard step will be also used in other estimates further on.

So far we have proved

$$B_8 \lesssim B_{11} + B_{12} \lesssim D_1 + D_2.$$

The term B_9 is estimated as follows.

$$
\begin{aligned}
B_9 &= \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^{\frac{kr}{q}} U^r(t_k, t_{k_{(n+1)}}) \left(\int_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\le \sum_{n=0}^{N-1} \left(\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(t_k, t_{k_{(n+1)}}) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (64) \\
&\lesssim \sum_{n=0}^{N-1} \left(\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (65) \\
&= \sum_{n=0}^{N-1} \left(\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\sum_{j=n+1}^{N} \int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N-1} \left(\sum_{k=k_n}^{k_{(n+1)}-1} \Theta^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (66) \\
&\lesssim \sum_{n=0}^{N-1} \left(\int_{t_{k_{(n-1)}}}^{t_{k_{(n+1)}}} w(x) U^q(x, t_{k_{(n+1)}})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (67) \\
&= \sum_{i=0}^{1} \sum_{\substack{1 \le n \le N \\ n \bmod 2 = i}} \left(\int_{t_{k_{(n-1)}}}^{t_{k_{(n+1)}}} w(x) U^q(x, t_{k_{(n+1)}})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim D_2.
\end{aligned}
$$

We used convexity of the $\frac{r}{q}$-th power to get (64). Step (65) follows by Proposition 2.5. Inequality (66) is granted by Proposition 2.3 equipped with (15). Step (67) follows by (20). We proved

$$B_9 \lesssim D_2.$$

The term B_{10} is first handled in the following way.

$$
\begin{aligned}
B_{10} &= \sum_{n=0}^{N-1} \sum_{k=k_n}^{k_{(n+1)}-1} \Theta^{\frac{kr}{q}} \left(\int_{t_{k_{(n+1)}}}^{\infty} U^{p'}(t_{k_{(n+1)}}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{\infty} U^{p'}(t_{k_{(n+1)}}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&= \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\sum_{j=n+1}^{N} \int_{t_{k_j}}^{t_{k_{(j+1)}}} U^{p'}(t_{k_{(n+1)}}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\sum_{j=n+2}^{N} U^{p'}(t_{k_{(n+1)}}, t_{k_j}) \int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\quad + \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\sum_{j=n+1}^{N} \int_{t_{k_j}}^{t_{k_{(j+1)}}} U^{p'}(t_{k_j}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&=: B_{13} + B_{14}.
\end{aligned}
$$

Then, for B_{13} we have

$$
\begin{aligned}
B_{13} &= \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\sum_{j=n+2}^{N} U^{p'}(t_{k_{(n+1)}}, t_{k_j}) \int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\le \sum_{n=0}^{N-2} \Theta^{k_{(n+1)}\frac{r}{q}} \sum_{j=n+2}^{N} U^{r}(t_{k_{(n+1)}}, t_{k_j}) \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=2}^{N} \sum_{n=0}^{j-2} \Theta^{k_{(n+1)} \frac{r}{q}} U^{r}(t_{k_{(n+1)}}, t_{k_j}) \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\le \sum_{j=2}^{N} \left(\sum_{n=0}^{j-2} \Theta^{k_{(n+1)}} U^{q}(t_{k_{(n+1)}}, t_{k_j}) \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (69) \\
&\le \sum_{j=2}^{N} \left(\sum_{k=\mu}^{k_{(j-1)}} \Theta^{k} U^{q}(t_k, t_{k_j}) \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{j=2}^{N} \left(\sum_{k=\mu}^{k_{(j-1)}} \Theta^{k} U^{q}(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (70) \\
&\le \sum_{j=2}^{N} \left(\sum_{k=\mu}^{k_j-1} \Theta^{k} U^{q}(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{j=2}^{N} \left(\sum_{k=k_{(j-1)}}^{k_j-1} \Theta^{k} U^{q}(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (71) \\
&\lesssim \sum_{j=2}^{N} \left(\int_{t_{k_{(j-2)}}}^{t_{k_j}} w(x) U^{q}(x, t_{k_j})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \qquad (72) \\
&= \sum_{i=0}^{1} \sum_{\substack{2 \le j \le N \\ j \bmod 2 = i}} \left(\int_{t_{k_{(j-2)}}}^{t_{k_j}} w(x) U^{q}(x, t_{k_j})\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_j}}^{t_{k_{(j+1)}}} v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \\
&\lesssim D_2.
\end{aligned}
$$

Inequality (68) follows from concavity of the $\frac{r}{p'}$-th power since $\frac{r}{p'} < 1$. Similarly, convexity of the $\frac{r}{q}$-th power yields (69). Step (70) is due to Proposition 2.5, step

(71) follows by (16), and in step (72) we used (20). We continue as follows.

$$B_{14} = \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\sum_{j=n+1}^{N} \int_{t_{k_j}}^{t_{k_{(j+1)}}} U^{p'}(t_{k_j}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(73)} \qquad \lesssim \sum_{n=0}^{N-1} \Theta^{k_{(n+1)}\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U^{p'}(t_{k_j}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\text{(74)} \qquad \lesssim \sum_{n=0}^{N-1} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} U^{p'}(t_{k_j}, z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}}$$

$$\leq D_1.$$

To get (73), we used Proposition 2.3, and in (74) we applied (21). We have proved

$$B_{10} \lesssim B_{13} + B_{14} \lesssim D_1 + D_2.$$

Combining all the estimates we have obtained so far, we get

$$\text{(75)} \qquad \int_{t_\mu}^{\infty} \left(\int_0^t w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \left(\int_t^{\infty} U^{p'}(t,z) v^{1-p'}(z)\,\mathrm{d}z \right)^{\frac{r}{p'}} \mathrm{d}t \lesssim D_1 + D_2.$$

In the following part, we are going to perform estimates related to the term A_2. We have

$$\int_{t_\mu}^{\infty} \left(\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t$$

$$= \sum_{n=0}^{N} \int_{\Delta_{(k_{(n+1)}-1)}} \left(\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t$$

$$+ \sum_{n\in\mathbb{A}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} \left(\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t$$

$$=: B_{15} + B_{16}.$$

By (18), the term B_{15} is further estimated as follows.

$$\begin{aligned}
B_{15} &= \sum_{n=0}^{N} \int\limits_{\Delta_{(k_{(n+1)}-1)}} \left(\int\limits_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int\limits_{\Delta_{(k_{(n+1)}-1)}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\quad + \sum_{n=0}^{N} \int\limits_{\Delta_{(k_{(n+1)}-1)}} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)},t) w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&=: B_{17} + B_{18}.
\end{aligned}$$

Notice that, in B_{17}, the term corresponding to $n=0$ is indeed omitted, since for any $t\in\Delta_\mu$ it holds $\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x \lesssim \Theta^\mu U^q(t_\mu,t)$ and the right-hand side is thus already represented by the 0-th term in B_{18}.

Let us note that in what follows, expressions such as $\sup_{x\in(y,\infty]}\varphi(x)$ appear even where the argument $\varphi(x)$ is undefined for $x=\infty$. To fix this formal detail, suppose that, in such cases, $\sup_{x\in(y,\infty]}\varphi(x)$ is simply redefined as $\sup_{x\in(y,\infty)}\varphi(x)$. This will make expressions such as $\sum_{n=1}^{N}\sup_{x\in[t_{k_n},t_{k_{(n+1)}}]}\varphi(x)$ formally correct without need of treating the $(N+1)$-st summand separately. Besides this, the standard notation $\overline{\Delta}_k$ is used to denote the closure of Δ_k, i.e. the interval $[t_k,t_{(k+1)}]$.

We then estimate B_{17}.

$$\begin{aligned}
B_{17} &= \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int\limits_{\Delta_{(k_{(n+1)}-1)}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{(k_{(n+1)}-1)},\infty)} U^q(t_{(k_{(n+1)}-1)},z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \qquad (76) \\
&\lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}} U^q(t_{(k_{(n+1)}-1)},z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \qquad (77) \\
&\quad + \sum_{n=1}^{N-1} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}},z) \left(\int\limits_{z}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}
\end{aligned}$$

$$(78)\quad \lesssim \sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}\sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}}U^q(t_{(k_{(n+1)}-1)},z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$+\sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}\sup_{z\in[t_{k_{(n+1)}},\infty)}U^q(t_{k_{(n+1)}},z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$=: B_{19}+B_{20}.$$

Inequality (76) holds by (21), and (77) is due to Proposition 2.6. In (78) we used (15). Next, we have

$$B_{19}=\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}\sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}}U^q(t_{(k_{(n+1)}-1)},z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$\lesssim\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}\sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}}U^q(t_{(k_{(n+1)}-1)},z)\left(\int_z^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$+\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}U^q(\Delta_{(k_{(n+1)}-1)})\left(\int_{t_{k_{(n+1)}}}^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

(79)

$$\lesssim\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}-1}\sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}}U^q(t_{(k_{(n+1)}-1)},z)\left(\int_z^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$+\sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{q}}\left(\int_{t_{k_{(n+1)}}}^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$
$$=: B_{21}+B_{22}.$$

Step (79) is based on (15). For each $n\in\{1,\dots,N\}$ there exists a point $z_{(n+1)}\in\overline{\Delta}_{(k_{(n+1)}-1)}$ such that

$$(80)\quad \sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}}U^q(t_{(k_{(n+1)}-1)},z)\left(\int_z^{t_{k_{(n+1)}}}v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\le 2U^q(t_{(k_{(n+1)}-1)},z_{(n+1)})\left(\int_{z_{(n+1)}}^{t_{k_{(n+1)}}}v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}.$$

Define also $z_{(-1)} := 0$ and $z_{(N+2)} := \infty$. One then gets

$$
\begin{aligned}
B_{21} &= \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}} U^q(t_{(k_{(n+1)}-1)}, z) \left(\int_{z}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
\text{(81)} \quad &\lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} U^q(t_{(k_{(n+1)}-1)}, z_{(n+1)}) \left(\int_{z_{(n+1)}}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
\text{(82)} \quad &\lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{\Delta_{(k_{(n+1)}-2)}} w(t)\,\mathrm{d}t\, U^q(t_{(k_{(n+1)}-1)}, z_{(n+1)}) \left(\int_{z_{(n+1)}}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\le \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{(k_{(n+1)}-2)}}^{z_{(n+1)}} w(t) U^q(t, z_{(n+1)})\,\mathrm{d}t \left(\int_{z_{(n+1)}}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
\text{(83)} \quad &\le \sum_{n=1}^{N} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t \right)^{\frac{r}{p}} \int_{t_{(k_{(n+1)}-2)}}^{z_{(n+1)}} w(t) U^q(t, z_{(n+1)})\,\mathrm{d}t \left(\int_{z_{(n+1)}}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
\text{(84)} \quad &\le \sum_{n=1}^{N} \left(\int_{z_{(n-2)}}^{z_{(n+1)}} w(t) U^q(t, z_{(n+1)})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{z_{(n+1)}}^{z_{(n+2)}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&= \sum_{i=0}^{3} \sum_{\substack{1\le n\le N \\ n \bmod 4 = i}} \left(\int_{z_{(n-2)}}^{z_{(n+1)}} w(t) U^q(t, z_{(n+1)})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{z_{(n+1)}}^{z_{(n+2)}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim D_2.
\end{aligned}
$$

We used (80) in (81), and (21) in (82). Estimate (83) follows from (20). To get (84), we used the relation $z_{(n-1)} \le t_{k_{(n-1)}} \le t_{(k_{(n+1)}-2)}$ which holds for all relevant indices n. The second inequality $t_{k_{(n-1)}} \le t_{(k_{(n+1)}-2)}$ follows from (12).

Concerning B_{22}, we obtain

$$B_{22} = \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$= \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\sum_{i=n+1}^{N-1} \int_{t_{k_i}}^{t_{k_{(i+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$\lesssim \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \tag{85}$$

$$\lesssim \sum_{n=1}^{N-1} \left(\int_{t_{k_{(n-1)}}}^{t_{k_{(n+1)}}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \tag{86}$$

$$= \sum_{i=0}^{2} \sum_{\substack{1 \le n \le N-1 \\ n \bmod 3 = i}} \left(\int_{t_{k_{(n-1)}}}^{t_{k_{(n+1)}}} w(t) U^q(t, t_{k_n})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+1)}}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$\lesssim D_2.$$

Proposition 2.3 together with (15) yields (85). Estimate (86) follows from (20). We have proved

$$B_{19} \lesssim B_{21} + B_{22} \lesssim D_2.$$

We proceed with the term B_{20}.

$$B_{20} = \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z \in [t_{k_{(n+1)}}, \infty)} U^q(t_{k_{(n+1)}}, z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$\le \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}} \sup_{i \in \{n+1,\dots,N\}} \sup_{z \in [t_{k_i}, t_{k_{(i+1)}}]} U^q(t_{k_{(n+1)}}, z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$\lesssim \sum_{n=1}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}} \sup_{i \in \{n+1,\dots,N\}} \sup_{z \in [t_{k_i}, t_{k_{(i+1)}}]} U^q(t_{k_i}, z) \left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$+ \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}} \sup_{i \in \{n+2,\dots,N\}} U^q(t_{k_{(n+1)}}, t_{k_i}) \left(\int_{t_{k_i}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}}$$

$$=: B_{23} + B_{24}.$$

For B_{23} we have

$$
\begin{aligned}
B_{23} &= \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}\sup_{i\in\{n+1,\ldots,N\}}\sup_{z\in[t_{k_i},t_{k_{(i+1)}}]}U^q(t_{k_i},z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\lesssim \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}\sup_{z\in[t_{k_{(n+1)}},t_{k_{(n+2)}}]}U^q(t_{k_{(n+1)}},z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \qquad (87)\\
&\lesssim \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}\sup_{z\in[t_{k_{(n+1)}},t_{k_{(n+2)}})}U^q(t_{k_{(n+1)}},z)\left(\int_z^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\quad+\sum_{n=1}^{N-2}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}U^q(t_{k_{(n+1)}},t_{k_{(n+2)}})\left(\int_{t_{k_{(n+2)}}}^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&=: B_{25}+B_{26}.
\end{aligned}
$$

In step (87) we used Proposition 2.3, considering also (15). For each $n\in\{0,\ldots,N-1\}$ there exists a point $y_{(n+1)}\in[t_{k_{(n+1)}},t_{k_{(n+2)}}]$ such that

$$
\sup_{z\in[t_{k_{(n+1)}},t_{k_{(n+2)}}]}U^q(t_{k_{(n+1)}},z)\left(\int_z^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \le 2U^q(t_{k_{(n+1)}},y_{(n+1)})\left(\int_{y_{(n+1)}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}. \qquad (88)
$$

Define also $y_{(-1)}:=0$ and $y_{(N+2)}:=\infty$.

$$
\begin{aligned}
B_{25} &= \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}\sup_{z\in[t_{k_{(n+1)}},t_{k_{(n+2)}}]}U^q(t_{k_{(n+1)}},z)\left(\int_z^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\\
&\lesssim \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}U^q(t_{k_{(n+1)}},y_{(n+1)})\left(\int_{y_{(n+1)}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \qquad (89)\\
&\lesssim \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\int_{\Delta_{(k_{(n+1)}-1)}} w(t)\,\mathrm{d}t\; U^q(t_{k_{(n+1)}},y_{(n+1)})\left(\int_{y_{(n+1)}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \qquad (90)
\end{aligned}
$$

$$\leq \sum_{n=1}^{N-1}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\int_{y_{(n-2)}}^{y_{(n+1)}} w(t)U^q(t,y_{(n+1)})\,\mathrm{d}t\left(\int_{y_{(n+1)}}^{t_{k_{(n+2)}}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{91}$$

$$\lesssim \sum_{n=1}^{N-1}\left(\int_{y_{(n-2)}}^{y_{(n+1)}} w(t)U^q(t,y_{(n+1)})\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{y_{(n+1)}}^{y_{(n+2)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{92}$$

$$= \sum_{i=0}^{3}\sum_{\substack{1\leq n\leq N-1\\ n \bmod 4=i}}\left(\int_{y_{(n-2)}}^{y_{(n+1)}} w(t)U^q(t,y_{(n+1)})\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{y_{(n+1)}}^{y_{(n+2)}} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim D_2.$$

In (89) we used (88). Inequality (90) follows from (21). To get (91), we used the inequality $y_{(n-2)} \leq t_{k_{(n-1)}} \leq t_{(k_{(n+1)}-1)}$ (cf. (12)) satisfied for all relevant indices n. This inequality, together with (20), also yields (92).

Next, the term B_{26} is treated as follows.

$$B_{26} = \sum_{n=1}^{N-2}\left(\sum_{j=k_n}^{k_{(n+1)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}U^q(t_{k_{(n+1)}},t_{k_{(n+2)}})\left(\int_{t_{k_{(n+2)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim \sum_{n=1}^{N-2}\left(\sum_{j=k_{(n+1)}}^{k_{(n+2)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\Theta^{k_{(n+1)}}U^q(t_{k_{(n+1)}},t_{k_{(n+2)}})\left(\int_{t_{k_{(n+2)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{93}$$

$$\leq \sum_{n=1}^{N-2}\left(\sum_{j=k_{(n+1)}}^{k_{(n+2)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}}\sum_{j=k_{(n+1)}}^{k_{(n+2)}-1}\Theta^j U^q(t_j,t_{k_{(n+2)}})\left(\int_{t_{k_{(n+2)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim \sum_{n=1}^{N-2}\left(\sum_{j=k_{(n+1)}}^{k_{(n+2)}-1}\Theta^j U^q(\Delta_j)\right)^{\frac{r}{q}}\left(\int_{t_{k_{(n+2)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{94}$$

$$\leq B_{22}$$

$$\lesssim D_2.$$

Inequality (93) is obtained by using (15), and inequality (94) by Proposition 2.5. The final estimate $B_{22} \lesssim D_2$ was already proved before. We have obtained

$$B_{23} \lesssim B_{25} + B_{26} \lesssim D_2.$$

Let us now return to the term B_{24}. It holds

$$
\begin{aligned}
B_{24} &= \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}} \sup_{i\in\{n+2,\dots,N\}} U^q(t_{k_{(n+1)}}, t_{k_i}) \left(\int_{t_{k_i}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\leq \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sup_{i\in\{n+2,\dots,N\}} \sum_{j=\mu}^{k_i-1} \Theta^j U^q(t_j, t_{k_i}) \left(\int_{t_{k_i}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sup_{i\in\{n+2,\dots,N\}} \sum_{j=\mu}^{k_i-1} \Theta^j U^q(\Delta_j) \left(\int_{t_{k_i}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (95) \\
&\lesssim \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sup_{i\in\{n+2,\dots,N\}} \sum_{j=k_{(i-1)}}^{k_i-1} \Theta^j U^q(\Delta_j) \left(\int_{t_{k_i}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (96) \\
&= \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sup_{i\in\{n+2,\dots,N\}} \sum_{j=k_{(i-1)}}^{k_i-1} \Theta^j U^q(\Delta_j) \left(\sum_{m=i}^{N} \int_{t_{k_m}}^{t_{k_{(m+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sup_{i\in\{n+2,\dots,N\}} \sum_{j=k_{(i-1)}}^{k_i-1} \Theta^j U^q(\Delta_j) \left(\int_{t_{k_i}}^{t_{k_{(i+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (97) \\
&\lesssim \sum_{n=1}^{N-2} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \sum_{j=k_{(n+1)}}^{k_{(n+2)}-1} \Theta^j U^q(\Delta_j) \left(\int_{t_{k_{(n+2)}}}^{t_{k_{(n+3)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (98) \\
&\lesssim \sum_{n=1}^{N-2} \left(\sum_{j=k_{(n+1)}}^{k_{(n+2)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+2)}}}^{t_{k_{(n+3)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (99) \\
&\lesssim \sum_{n=1}^{N-2} \left(\int_{t_{k_n}}^{t_{k_{(n+2)}}} w(t) U^q(t, t_{k_{(n+2)}})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+2)}}}^{t_{k_{(n+3)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \quad (100) \\
&= \sum_{i=0}^{2} \sum_{\substack{1\leq n\leq N-2 \\ n \bmod 3 = i}} \left(\int_{t_{k_n}}^{t_{k_{(n+2)}}} w(t) U^q(t, t_{k_{(n+2)}})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+2)}}}^{t_{k_{(n+3)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim D_2.
\end{aligned}
$$

Inequality (95) follows from Proposition 2.5, and inequality (96) from (16). To get (97), one uses Proposition 2.4, considering also (15). Proposition 2.3, again with (15), yields (98). Step (99) follows from (15). In (100) we applied (20). Having proved $B_{24} \lesssim D_2$, we may now complete several more estimates, namely

$$B_{20} \lesssim B_{23} + B_{24} \lesssim D_2,$$

which, combined with the earlier results, gives

$$B_{17} \lesssim B_{19} + B_{20} \lesssim D_2.$$

The next untreated expression is B_{18}. It is estimated in the following way.

$$\begin{aligned}
B_{18} &= \sum_{n=0}^{N} \int\limits_{\Delta_{(k_{(n+1)}-1)}} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int\limits_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\lesssim \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) \sup_{z\in[t,t_{k_{(n+1)}}]} U^q(t,z) \left(\int\limits_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \qquad (101) \\
&\quad + \sum_{n=0}^{N-1} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}}, z) \left(\int\limits_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) \sup_{z\in[t,t_{k_{(n+1)}}]} U^q(t,z) \left(\int\limits_z^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\quad + \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) U^q(t, t_{k_{(n+1)}}) \left(\int\limits_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\quad + \sum_{n=0}^{N-1} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}}, z) \left(\int\limits_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&=: B_{27} + B_{28} + B_{29}.
\end{aligned}$$

Inequality (101) follows from Proposition 2.6. Define $t_{(k_{(N+2)}-1)} := \infty$. Then we have

$$
\begin{aligned}
B_{27} &= \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) \sup_{z\in[t, t_{k_{(n+1)}}]} U^{q}(t,z) \left(\int\limits_{z}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\le \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} w(t)\,\mathrm{d}t \sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}} U^{r}(t_{(k_{(n+1)}-1)}, z) \left(\int\limits_{z}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N} \Theta^{\frac{r}{q}(k_{(n+1)}-1)} \sup_{z\in\overline{\Delta}_{(k_{(n+1)}-1)}} U^{r}(t_{(k_{(n+1)}-1)}, z) \left(\int\limits_{z}^{t_{k_{(n+1)}}} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \qquad (102) \\
&\le \sum_{n=0}^{N} \Theta^{\frac{r}{q}(k_{(n+1)}-1)} \left(\int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, s) v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N} \left(\int\limits_{t_{(k_n-1)}}^{t_{(k_{(n+1)}-1)}} w(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int\limits_{t_{(k_{(n+1)}-1)}}^{t_{(k_{(n+2)}-1)}} U^{p'}(t_{(k_{(n+1)}-1)}, s) v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \qquad (103) \\
&\le D_1.
\end{aligned}
$$

Step (102) follows from (21). In (103) we used (21) and the inequalities $t_{(k_n-1)} \le t_{(k_{(n+1)}-2)}$ and $t_{k_{(n+1)}} \le t_{(k_{(n+2)}-1)}$ which hold for all $n \in \{0, \ldots, N\}$ thanks to (12) and the definition of $t_{(k_{(N+2)}-1)}$.

We continue with the term B_{28}, for which we get

$$
\begin{aligned}
B_{28} &= \sum_{n=0}^{N} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t) U^q(t, t_{k_{(n+1)}}) \left(\int\limits_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\le \sum_{n=0}^{N-1} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} w(t)\,\mathrm{d}t \;\; U^r(\Delta_{(k_{(n+1)}-1)}) \left(\int\limits_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{\frac{r}{q}(k_{(n+1)}-1)} U^r(\Delta_{(k_{(n+1)}-1)}) \left(\int\limits_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \qquad (104) \\
&\le \sum_{n=0}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\int\limits_{t_{k_{(n+1)}}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&= \left(\Theta^{\mu} U^q(\Delta_{\mu}) \right)^{\frac{r}{q}} \left(\int\limits_{t_{k_1}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} \qquad (105) \\
&\lesssim \left(\int\limits_{0}^{t_{\mu}} w(t)\,\mathrm{d}t \; U^q(\Delta_{\mu}) \right)^{\frac{r}{q}} \left(\int\limits_{t_{k_1}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} \qquad (106) \\
&\lesssim \left(\int\limits_{0}^{t_{(\mu+1)}} w(t) U^q(t, t_{(\mu+1)})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int\limits_{t_{k_1}}^{\infty} v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} \\
&\lesssim D_2. \qquad (107)
\end{aligned}
$$

To get (104), we made use of (21). In (105) we used the fact

$$
\sum_{j=k_0}^{k_1-1} \Theta^j U^q(\Delta_j) = \Theta^{\mu} U^q(\Delta_{\mu}) \qquad (108)
$$

(recall that $k_0 = \mu$ and $k_1 = \mu + 1$). Inequality (106) is a consequence of(21). The final estimate (107) follows from the relation $B_{22} \lesssim D_2$ which was proved earlier.

Concerning B_{29}, we may write

$$
\begin{aligned}
B_{29} &= \sum_{n=0}^{N-1} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} \int\limits_{\Delta_{(k_{(n+1)}-1)}} U^{\frac{rq}{p}}(t_{(k_{(n+1)}-1)}, t)\, w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}}, z)\left(\int\limits_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&\lesssim \sum_{n=0}^{N-1} \Theta^{\frac{r}{p}(k_{(n+1)}-1)} U^{\frac{rq}{p}}(\Delta_{(k_{(n+1)}-1)})\, \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}}, z)\left(\int\limits_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \qquad (109) \\
&\le \sum_{n=0}^{N-1} \left(\sum_{j=k_n}^{k_{(n+1)}-1} \Theta^j U^q(\Delta_j)\right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}}, z)\left(\int\limits_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&= \Theta^{\frac{r\mu}{q}} U^{\frac{rq}{p}}(\Delta_\mu) \sup_{z\in[t_{k_1},\infty)} U^q(t_{k_1}, z)\left(\int\limits_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} + B_{20} \qquad (110) \\
&\le \Theta^{\frac{r\mu}{q}} \left(\int\limits_{t_\mu}^\infty U^{p'}(t_\mu, s) v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} + B_{20} \\
&\lesssim \left(\int\limits_0^{t_\mu} w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int\limits_{t_\mu}^\infty U^{p'}(t_\mu, s) v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} + B_{20} \qquad (111) \\
&\lesssim D_1 + D_2. \qquad (112)
\end{aligned}
$$

Step (109) follows from (21), step (110) from (108), and step (111) from (21). To obtain (112), we used the estimate $B_{20} \lesssim D_2$ which was proved earlier. We have proved

$$B_{18} \lesssim B_{27} + B_{28} + B_{29} \lesssim D_1 + D_2.$$

Together with the estimate of B_{17} we obtained earlier, this also yields

$$B_{15} \lesssim B_{17} + B_{18} \lesssim D_2.$$

In the next part, we return to the expression B_{16}. It holds

$$B_{16} = \sum_{n\in\mathbb{A}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} \left(\int_{t_\mu}^{t} w(x) U^q(x,t)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t$$

$$\begin{aligned}
&(113)\quad \lesssim \sum_{n\in\mathbb{A}} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} w(t) \sup_{z\in[t,\infty)} U^q(t,z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&(114)\quad \lesssim \sum_{n\in\mathbb{A}} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} w(t) \sup_{z\in[t,t_{k_{(n+1)}}]} U^q(t,z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \\
&\qquad + \sum_{n\in\mathbb{A}} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}},z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&(115)\quad \lesssim \sum_{n\in\mathbb{A}} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \int_{t_{k_n}}^{t_{(k_{(n+1)}-1)}} w(t) U^q(t, t_{(k_{(n+1)}-1)})\,\mathrm{d}t \left(\int_{t_{k_n}}^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\qquad + \sum_{n\in\mathbb{A}} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}},z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&(116)\quad \lesssim \sum_{n=1}^{N} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&\qquad + \sum_{n=1}^{N-1} \left(\sum_{j=k_{(n-1)}}^{k_n-1} \Theta^j U^q(\Delta_j) \right)^{\frac{r}{p}} \Theta^{k_{(n+1)}-1} \sup_{z\in[t_{k_{(n+1)}},\infty)} U^q(t_{k_{(n+1)}},z) \left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \\
&(117)\quad = \left(\Theta^\mu U^q(\Delta_\mu) \right)^{\frac{r}{q}} \left(\int_{t_{k_1}}^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} + B_{20} \\
&(118)\quad \lesssim \left(\int_0^{t_\mu} w(t)\,\mathrm{d}t\, U^q(\Delta_\mu) \right)^{\frac{r}{q}} \left(\int_{t_{k_1}}^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} + B_{20} \\
&\qquad \lesssim \left(\int_0^{t_{(\mu+1)}} w(t) U^q(t, t_{(\mu+1)})\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_1}}^\infty v^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} + B_{22} + B_{20} \\
&(119)\quad \lesssim D_2.
\end{aligned}$$

Estimate (113) is granted by (19), and estimate (114) by Proposition 2.6. Step (115) is based on (21). In (116) we again applied (19). To get the relations (117) and (118), we used (108) and (21), respectively. The final inequality (119) follows from the already known relations $B_{22} \lesssim D_2$ and $B_{20} \lesssim D_2$. We have shown

$$B_{16} \lesssim D_2,$$

and thus also

$$\int_{t_\mu}^{\infty}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z)\left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \mathrm{d}t \lesssim B_{15}+B_{16} \lesssim D_1+D_2.$$

If we combine this inequality with (75), we reach

$$\begin{aligned}
&\int_{t_\mu}^{\infty}\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_t^{\infty} U^{p'}(t,z)v^{1-p'}(z)\,\mathrm{d}z\right)^{\frac{r}{p'}} \mathrm{d}t\\
&\quad+\int_{t_\mu}^{\infty}\left(\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} U^q(t,z)\left(\int_z^{\infty} v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \mathrm{d}t\\
&\lesssim D_1+D_2.
\end{aligned}$$

The constant related to the symbol "$\lesssim$" in here does not depend on the choice of μ, thus passing $\mu \to -\infty$ (notice $t_\mu \to 0$ as $\mu \to -\infty$) and applying the monotone convergence theorem yields

$$A_1 + A_2 \lesssim D_1 + D_2.$$

We have so far assumed that $\int_0^\infty w(x)\,\mathrm{d}x = \Theta^K$ for a $K \in \mathbb{Z}$. The result is extended to general weights w by the same procedure as the one used at the end of the proof of the implication "(ii)⇒(i)". The proof of the whole theorem is now complete. □

Proof of Theorem 3.2. Theorem 3.2 is proved in almost exactly the same way as Theorem 3.1. The difference is just in the use of appropriate "limit variants" of certain expressions for $p = 1$. Namely,

$$\left(\int_y^z U^{p'}(y,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \quad \text{is replaced by} \quad \operatorname*{ess\,sup}_{x\in(y,z)} U(y,x)v^{-1}(x)$$

and

$$\left(\int_y^z v^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \quad \text{is replaced by} \quad \operatorname*{ess\,sup}_{x\in(y,z)} v^{-1}(x),$$

whenever these expressions appear with some $0 \le y < z \le \infty$. To clarify the correspondence between A_2 and A_4, let us note that

$$\sup_{z\in[t,\infty)} U^q(t,z) \operatorname*{ess\,sup}_{s\in(z,\infty)} v^{q'}(s) = \operatorname*{ess\,sup}_{s\in(t,\infty)} v^{q'}(s) \sup_{z\in[t,s)} U^q(t,z) = \operatorname*{ess\,sup}_{s\in(t,\infty)} U^q(t,s) v^{q'}(s)$$

is true for all $t > 0$. Naturally, the limit variant of Proposition 2.1 for $p = 1$ is used in the proof as well. All the estimates are then analogous to their counterparts in the proof of Theorem 3.1. Therefore, we do not repeat them in here. □

Remark 3.6. (i) Theorem 3.1, which relates to the inequality (8), i.e. to the operator H^*, is the one proved here, while the result for H (i.e. for (9)) is presented as Corollary 3.3. Of course, the opposite order could have been chosen, since the version with H instead of H^* can be proved in an exactly analogous way. As mentioned before, the variants for H and H^* are equivalent by a change of variables in the integrals. The reason why the proof of the "dual" version is shown here is that the discretization-related notation is then the same as in [8].

(ii) Discretization based on finite covering sequences is used here, although the double-infinite (indexed by $\mathbb{Z}$) variant is far more usual in the literature (cf. [3, 9, 16]). The advantage of the finite version is that the proof works for L^1-weights w and then it is easily extrapolated for the non-L^1 weights by the final approximation argument. In order to work with infinite partitions, one needs to assume $w \notin L^1$. The pass to the L^1-weights then cannot be done in such an easy way as in the opposite order. The authors usually omit the case $w \in L^1$ (see e.g. [3]). Besides that, there is no essential difference between in the techniques based on finite and infinite partitions.

(iii) In Theorems 3.1 and 3.2, the equivalence "(i)$\Leftrightarrow$(ii)" was known before [9] and it is reproved here using another method than in [9]. The main achievement is the equivalence "(i)$\Leftrightarrow$(iii)" which can also be proved directly, by the same technique and without need for the discrete D-conditions (cf. [8]). Doing so would however require constructing more different special functions (such as g and h in the "(i)$\Rightarrow$(ii)" part of Theorem 3.1) and therefore also introducing additional notation.

(iv) The kernel U is not assumed to be continuous. However, for every $t > 0$ the function $U(t,\cdot)$ is nondecreasing, hence continuous almost everywhere on $(0,\infty)$. Thus, so is the function $U^q(t,\cdot)\left(\int_\cdot^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$. Therefore, the value of the expression A_2 remains unchanged if "$\sup_{z\in[t,\infty)}$" in there is replaced by "$\operatorname{ess\,sup}_{z\in[t,\infty)}$". Although the latter variant may seem to be the "proper" one, both are correct in this case. Besides that, the range $z \in [t,\infty)$ in the supremum or essential supremum may obviously be replaced by $z \in (t,\infty)$ without changing the value of A_2.

4. Applications

The integral conditions for the boundedness $H : L^p(v) \to L^q(w)$ with $0 < q < 1 \le p < \infty$ may be used to complete [3, Theorem 5.1] with two missing cases. (These cases are in fact included in [3] but covered there only by discrete conditions.)

Denote by $\mathscr{M}^{\downarrow}$ the cone of all nonnegative nonincreasing functions on $(0,\infty)$. The result then reads as follows.

Theorem 4.1. *Let u, v, w be weights, $0<q<p<\infty$, $q<1$ and $r=\frac{pq}{p-q}$.*

(i) *Let $0<p\le 1$. Then the inequality*

$$\left(\int_0^\infty\left(\int_t^\infty f(s)u(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{120}$$

holds for all $f\in\mathscr{M}^{\downarrow}$ if and only if

$$A_5:=\left(\int_0^\infty\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in(t,\infty)}\left(\int_t^z u(s)\,\mathrm{d}s\right)^r\left(\int_0^z v(y)\,\mathrm{d}y\right)^{-\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}<\infty$$

and

$$A_6:=\left(\int_0^\infty\left(\int_0^t w(x)\left(\int_x^t u(s)\,\mathrm{d}s\right)\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in(t,\infty)}\left(\int_t^z u(s)\,\mathrm{d}s\right)^q\left(\int_0^z v(y)\,\mathrm{d}y\right)^{-\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}<\infty.$$

Moreover, the least constant C such that (120) *holds for all $f\in\mathscr{M}^{\downarrow}$ satisfies $C\approx A_5+A_6$.*

(ii) *Let $p>1$. Then* (120) *holds for all $f\in\mathscr{M}^{\downarrow}$ if and only if $A_6<\infty$,*

$$A_7:=\left(\int_0^\infty\left(\int_0^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\left(\int_t^\infty\left(\int_t^z u(s)\,\mathrm{d}s\right)^{p'}\left(\int_0^z v(y)\,\mathrm{d}y\right)^{-p'} v(z)\,\mathrm{d}z\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}}<\infty$$

and $A_8<\infty$, where

$$A_8:=\begin{cases}\left(\int_0^\infty w(t)\left(\int_0^t u(s)\,\mathrm{d}s\right)^q\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_0^\infty v(y)\,\mathrm{d}y\right)^{-\frac{1}{p}}<\infty & \text{if } \int_0^\infty v(y)\,\mathrm{d}y<\infty,\\ 0 & \text{if } \int_0^\infty v(y)\,\mathrm{d}y=\infty.\end{cases}$$

Moreover, the least constant C such that (120) *holds for all $f\in\mathscr{M}^{\downarrow}$ satisfies $C\approx A_6+A_7+A_8$.*

Proof. (i) By [3, Theorem 4.1], (120) holds for all $f\in\mathscr{M}^{\downarrow}$ if and only if

$$\left(\int_0^\infty\left(\int_t^\infty\left(\int_t^x u(s)\,\mathrm{d}s\right)^p h(x)\,\mathrm{d}x\right)^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}}\le C^p\int_0^\infty h(s)\int_0^s v(y)\,\mathrm{d}y\,\mathrm{d}s \tag{121}$$

holds for all $h\in\mathscr{M}_+$. In fact, [3, Theorem 4.1] is stated with the assumption $\int_0^\infty v(y)\,\mathrm{d}y=\infty$ which is, however, not used in the proof in [3]. Validity of

(121) for all $h \in \mathscr{M}_+$ is equivalent to the condition $A_5 + A_6 < \infty$ by Theorem 3.2, since $U(x,y) = \left(\int_x^y u(s)\,\mathrm{d}s\right)^p$ is a ϑ-regular kernel (with $\vartheta = 2^p$).

(ii) By [3, Theorem 2.1], (120) holds for all $f \in \mathscr{M}^{\downarrow}$ if and only if $A_8 \le \infty$ and

$$\left(\int_0^\infty \left(\int_t^\infty \int_t^x u(s)\,\mathrm{d}s\; h(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_0^\infty h^p(s) \left(\int_0^s v(y)\,\mathrm{d}y\right)^p v^{1-p}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}$$

holds for all $h \in \mathscr{M}_+$. The latter is, by Theorem 3.1, equivalent to the condition $A_6^* + A_7 < \infty$, where

$$A_6^* := \left(\int_0^\infty \left(\int_0^t w(x) \left(\int_x^t u(s)\,\mathrm{d}s\right)^q \mathrm{d}x\right)^{\frac{r}{p}} w(t)\right.$$
$$\left.\times \sup_{z\in(t,\infty)} \left(\int_t^z u(s)\,\mathrm{d}s\right)^q \left(\int_z^\infty \left(\int_0^x v(y)\,\mathrm{d}y\right)^{-p'} v(x)\,\mathrm{d}x\right)^{\frac{r}{p'}} \mathrm{d}t\right)^{\frac{1}{r}}.$$

Since

$$\int_z^\infty \left(\int_0^s v(y)\,\mathrm{d}y\right)^{-p'} v(s)\,\mathrm{d}s + \left(\int_0^\infty v(y)\,\mathrm{d}y\right)^{1-p'} \approx \left(\int_0^z v(y)\,\mathrm{d}y\right)^{1-p'}$$

is satisfied for all $z > 0$, it is easy to verify that $A_6^* \lesssim A_6$ and $A_6 \lesssim A_6^* + A_8$.

In both cases (i) and (ii), the estimates on the optimal constant C also follow from [3, Theorem 2.1, Theorem 4.1] and Theorems 3.1 and 3.2. $\square$

In the case $0 < q < p \le 1$, in [3, Theorem 4.1] it was shown that (120) holds for all $f \in \mathscr{M}^{\downarrow}$ if and only if

$$\left(\int_0^\infty \left(\sup_{x\in[t,\infty)} f(x) \int_t^x u(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_0^\infty f^p(t) v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

holds for all $f \in \mathscr{M}^{\downarrow}$. Theorem 4.1 hence applies to this supremal operator inequality as well.

Theorem 4.1 may be further applied to prove certain weighted Young-type convolution inequalities (cf. [7]) in parameter settings which could not be reached so far. For this particular application, it is important that the weight w is not involved in any implicit conditions. For more details see [7].

As shown e.g. in [15, Theorem 4.4], certain weighted inequalities restricted to convex functions are equivalently represented by weighted inequalities involving a Hardy-type operator with the 1-regular Riemann-Liouville kernel $U(x,y) = (y-x)$. Hence, the results of this paper also provide characterizations of validity of those convex-function inequalities in the case $0 < q < 1 \le p < \infty$.

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Paper IX

Martin Křepela

Convolution inequalities in weighted Lorentz spaces: case $0 < q < 1$

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CONVOLUTION INEQUALITIES IN WEIGHTED LORENTZ SPACES: CASE $0 < q < 1$

MARTIN KŘEPELA

Abstract. Let g be a fixed nonnegative radially decreasing kernel g. In this paper, boundedness of the convolution operator $T_g f := f * g$ between the weighted Lorentz spaces $\Gamma^q(w)$ and $\Lambda^p(v)$ is characterized in the case $0 < q < 1$. The conditions are sufficient if the kernel g is just a general measurable function.

Furthermore, the largest rearrangement-invariant (quasi-)space Y is found such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Lambda^p(v)}\|g\|_Y$$

holds for all $f \in \Lambda^p(v)$ and $g \in Y$.

1. Introduction

Denote by $\mathscr{M}$ the cone of all measurable functions on $\mathbb{R}^n$. If $f, g \in \mathscr{M}$, the *convolution* of f and g is given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y$$

for any $x \in \mathbb{R}^n$ for which the integral is defined. If $g \in \mathscr{M}$ is fixed, it is possible to define the convolution operator T_g by

$$T_g f(x) := (f * g)(x)$$

for $f \in \mathscr{M}$ and $x \in \mathbb{R}^n$, provided that the right-hand side is well defined.

In [10], the author characterized boundedness of the operator T_g between weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ (see the definitions below) in the cases $0 < p < \infty$, $1 \le q < \infty$ and $p = \infty$, $0 < q \le \infty$. In the present article, the case $0 < q < 1$, $0 < p < \infty$ is treated, completing the results for the whole range $p, q \in (0, \infty]$.

Let $f \in \mathscr{M}$. The symbol f^* stands for the *nonincreasing rearrangement* of f, and f^{**} is the *Hardy-Littlewood maximal function* given by $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$ for $t > 0$ (see [2] for details).

A *weight* is a nonnegative measurable function w defined on $(0, \infty)$ and such that $0 < W(t) < \infty$ for all $t \in (0, \infty)$, where $W(t) := \int_0^t w(s)\,\mathrm{d}s$. The notation $V(t)$ has an analogous meaning for a weight v.

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Let v be a weight and $p \in (0,\infty)$. The *weighted Lebesgue space* $L^p(v)$ is the set of all measurable functions h on $(0,\infty)$ such that

$$\|h\|_{L^p(v)} := \left(\int_0^\infty |h(t)|^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Naturally, an analogy for $p = \infty$ also exists. By L^1 one denotes the space $L^1(\mathbb{R}^n)$, and L^1_{loc} stands for the space of all locally integrable functions on $\mathbb{R}^n$.

The *weighted Lorentz spaces* $\Lambda^p(v)$ and $\Gamma^p(v)$ are defined by

$$\Lambda^p(v) := \left\{ f \in \mathscr{M};\ \|f\|_{\Lambda^p(v)} := \|f^*\|_{L^p(v)} < \infty \right\},$$
$$\Gamma^p(v) := \left\{ f \in \mathscr{M};\ \|f\|_{\Gamma^p(v)} := \|f^{**}\|_{L^p(v)} < \infty \right\}.$$

For definitions of *rearrangement-invariant (r.i.) spaces, quasi-spaces* and *lattices*, see e.g. [2, 10]. The $\Lambda^p(v)$ and $\Gamma^p(v)$ "spaces" are always at least r.i. lattices, questions of their linearity and (quasi-)normability are treated e.g. in [6, 16] and articles referred therein.

An r.i. lattice X is said to be *essentially larger* than an r.i. lattice Y if $Y \subset X$ and for every $k \in \mathbb{N}$ there exists a function $f_k \in X$ such that $k\|f_k\|_X \le \|f_k\|_Y$. In other words, X is essentially larger than Y if $Y \subset X$ and X is not embedded in Y.

The notation $A \lesssim B$ means that for every $p, q \in (0,\infty)$ there exists a constant $C = C(p,q) \in [0,\infty)$ such that $A \le CB$. The constant C hence depends only on the parameters p and q. If both $A \lesssim B$ and $B \lesssim A$, one writes $A \approx B$.

The problem of boundedness of convolution-type operators between various function spaces was studied in a great number of articles, see e.g. [1, 3, 10, 11, 12, 8, 15, 14, 18] and the references therein. The technique employed in [10], which is also relevant for this paper, was based on using the O'Neil inequality

$$(f*g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s, \qquad f, g \in \mathscr{M}, \quad t > 0,$$

proved in [15], and its reverse version

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \le C(n)(f*g)^{**}(t), \qquad t > 0,$$

which holds for all nonnegative radially decreasing functions f, g on $\mathbb{R}^n$ with the constant $C(n)$ depending only on the dimension of $\mathbb{R}^n$. The reverse variant for functions from $\mathbb{R}^n$ was proved e.g. in [9]. The O'Neil inequalities were used in [10] to prove the following lemma.

Lemma 1.1. *Let X be an r.i. lattice, w be a weight, $g \in \mathscr{M}$ and $q \in (0, \infty]$. For $f \in \mathscr{M}$ and $t > 0$ define*

$$R_g^1 f(t) := t f^{**}(t) g^{**}(t),\ R_g^2 f(t) := \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s,\ R_g f(t) := R_g^1 f(t) + R_g^2 f(t).$$

Then

(i) *If $R_g : X \to L^q(w)$ is bounded, then $T_g : X \to \Gamma^q(w)$ is bounded and*

$$\|T_g\|_{X \to \Gamma^q(w)} \lesssim \|R_g\|_{X \to L^q(w)} < \infty.$$

(ii) *Let g be nonnegative and radially decreasing. If $T_g : X \to \Gamma^q(w)$ is bounded, then $R_g : X \to L^q(w)$ is bounded and*

$$\|R_g\|_{X \to L^q(w)} \lesssim \|T_g\|_{X \to \Gamma^q(w)} < \infty.$$

(iii) *Suppose there exists an r.i. lattice Y such that $\|R_g\|_{X \to L^q(w)} \approx \|g\|_Y$ for all $g \in \mathscr{M}$. Then Y is the essentially largest r.i. lattice such that the inequality*

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_X \|g\|_Y$$

holds for all $f \in X$ and $g \in Y$.

This result was proved as [10, Theorem 3.1] for both ordinary and periodic functions on $\mathbb{R}$ but it holds even for functions on $\mathbb{R}^n$. Further results in here will have the $\mathbb{R}^n$-form but they may be simply modified to cover periodic functions on $\mathbb{R}$. Besides this, the statement of [10, Theorem 3.1] contains the term "r.i. space" in place of the more general "r.i. lattice" used in Lemma 1.1(iii). However, both versions are correct as the space structure is not important to prove the result.

Thanks to Lemma 1.1, the problem of boundedness of T_g between $\Lambda^p(v)$ and $\Gamma^q(w)$ reduces to characterizing boundedness of R_g^1 and R_g^2 between $\Lambda^p(v)$ and $L^q(w)$. The problem for R_g^1 was completely solved for the whole range $p, q \in (0, \infty]$ (see [5, 4]). Similar characterizations for R_g^2 were known as well [7], but only for $q \geq 1$, at the time of publishing of [10]. Although [7] contains conditions even for $0 < q < 1$, in this case they have a discrete form which could not be applied. The case $0 < q < 1$ was therefore missing in [10]. However, recent progress in the required characterizations of Hardy-type inequalities [13] allows for completing the missing cases. Hence, this paper together with [10] cover the $\Lambda^p(v) \to \Gamma^q(w)$ convolution-operator boundedness for the whole range $p, q \in (0, \infty]$.

Before stating and proving the main result, it is useful to state the following technical lemma based on partial integration. Results of this type are well known (cf. [17, p. 176]) and are frequently used whenever weighted Hardy inequalities on monotone functions are studied.

Lemma 1.2. *Let φ, $\psi \in \mathscr{M}_+$ and φ is locally integrable. Let $0 < q < p < \infty$ and $r = \frac{pq}{p-q}$. Then*

$$\int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x \right)^{\frac{r}{p}} \varphi(t) \sup_{s\in(0,t)} \psi(s)\,\mathrm{d}t$$

$$\approx \int_0^\infty \varphi(t)\,\mathrm{d}t \limsup_{s\to 0+} \psi(s) + \int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)} \psi(s) \right) \mathrm{d}t.$$

Proof. Integration by parts yields

$$\int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x \right)^{\frac{r}{p}} \varphi(t) \sup_{s\in(0,t)} \psi(s)\,\mathrm{d}t + \frac{q}{r} \lim_{s\to\infty} \left(\int_s^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x)$$

$$= \frac{q}{r} \lim_{s\to 0+} \left(\int_s^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) + \frac{q}{r} \int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)} \psi(s) \right) \mathrm{d}t.$$

By monotonicity, one has

$$\lim_{s\to 0+} \left(\int_s^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) = \left(\int_0^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \limsup_{s\to 0+} \psi(s).$$

Furthermore, for every $s > 0$ it holds

$$\frac{q}{r} \left(\int_s^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) = \frac{q}{r} \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y \right)^{\frac{r}{q}} \varphi(t)\,\mathrm{d}t \sup_{x\in(0,s)} \psi(x)$$

$$\le \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y \right)^{\frac{r}{q}} \varphi(t) \sup_{x\in(0,t)} \psi(x)\,\mathrm{d}t,$$

hence

$$\frac{q}{r} \lim_{s\to\infty} \left(\int_s^\infty \varphi(t)\,\mathrm{d}t \right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) \le \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y \right)^{\frac{r}{q}} \varphi(t) \sup_{x\in(0,t)} \psi(x)\,\mathrm{d}t.$$

Combining all these observations gives the result. □

2. Results

In all what follows, the convention $0.\infty := 0$ is strictly enforced. For example, any expression of the form $CV^{-\frac{1}{p}}(\infty)$ is equal to zero whenever $V(\infty) = \infty$, even if $C = \infty$.

The theorem below is formulated for convolution of functions from $\mathbb{R}^n$. It might be easily modified to the case of periodic functions on $\mathbb{R}$ in spirit of [10].

Theorem 2.1. *Let v, w be weights.*

(i) *Let $0<q<p<1$ and $r=\frac{pq}{p-q}$. For any $g\in\mathscr{M}$ define*

$$A_1(g):=\left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\sup_{s\in(t,\infty)}(g^{**}(s))^r s^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_2(g):=\left(\int_0^\infty\left(\int_t^\infty(g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{q}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)}s^r V^{-\frac{r}{p}}(s)\right)\mathrm{d}t\right)^{\frac{1}{r}}$$

$$A_3(g):=\left(\int_0^\infty(g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\limsup_{s\to 0+}sV^{-\frac{1}{p}}(s).$$

and

$$\|g\|_Y:=A_1(g)+A_2(g)+A_3(g).$$

Then $(Y,\|g\|_Y)$ is the essentially largest r.i. lattice such that the inequality

$$\|f*g\|_{\Gamma^q(w)}\lesssim\|f\|_{\Lambda^p(v)}\|g\|_Y \tag{1}$$

holds for all $f\in\Lambda^p(v)$ and $g\in Y$. Moreover, if g is nonnegative and radially decreasing, then

$$\sup_{f\in\Lambda^p(v)}\frac{\|f*g\|_{\Gamma^q(w)}}{\|f\|_{\Lambda^p(v)}}\approx\|g\|_Y. \tag{2}$$

(ii) *Let $0<q<1<p<\infty$ and $r=\frac{pq}{p-q}$. For any $g\in\mathscr{M}$ define*

$$A_4(g):=\left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_t^\infty(g^{**}(s))^{p'}s^{p'}V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_5(g):=\left(\int_0^\infty\left(\int_t^\infty(g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{q}}\left(\int_0^t V^{-p'}(s)v(s)s^{p'}\,\mathrm{d}s\right)^{\frac{r}{q'}}V^{-p'}(t)v(t)t^{p'}\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_6(g):=\left(\int_0^\infty(g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\lim_{s\to 0+}\left(\int_0^s V^{-p'}(x)v(x)x^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}},$$

and

$$\|g\|_Y:=A_4(g)+A_5(g)+A_6(g)+\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty).$$

Then $(Y,\|g\|_Y)$ is the essentially largest r.i. lattice such that (1) *holds for all $f\in\Lambda^p(v)$ and $g\in Y$. Moreover, if g is nonnegative and radially decreasing, then* (2) *is satisfied.*

Proof. For the definitions of the operators R_g^1 and R_g^2 see Lemma 1.2.

(i) Fix $g \in \mathscr{M}$. By [4, Theorem 3.1] one gets $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} \approx B_1 + B_2$, where

$$B_1 := \left(\int_0^\infty \left(\int_0^t \left(\int_0^s g^*(x)\,\mathrm{d}x \right)^q w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t g^*(y)\,\mathrm{d}y \right)^q w(t) V^{-\frac{r}{p}}(t)\,\mathrm{d}t \right)^{\frac{1}{r}},$$

$$B_2 := \left(\int_0^\infty \sup_{s\in(0,t)} s^r V^{-\frac{r}{p}}(s) \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

Next, [13, Theorem 13(i)] gives $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \approx B_3 + B_4$, where

$$B_3 := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t \right)^{\frac{1}{r}},$$

$$B_4 := \left(\int_0^\infty \left(\int_0^t w(x) \left(\int_x^t g^*(y)\,\mathrm{d}y \right)^q \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{s\in(t,\infty)} \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^q V^{-\frac{r}{p}}(s)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

In view of Lemma 1.1, it suffices to prove that

$$B_1 + B_2 + B_3 + B_4 \approx A_1(g) + A_2(g) + A_3(g). \tag{3}$$

Lemma 1.2 implies that $B_2 \approx A_2(g) + A_3(g)$. Next, it is easy to see that $B_1 + B_3 + B_4 \lesssim A_1(g)$. Hence, the "$\lesssim$" inequality in (3) is verified.

The following part is aimed at proving the opposite estimate. Observe that $A_1(g) \approx B_3 + B_5$, where

$$B_5 := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

Assume that $W(\infty) = \infty$. There exists a (not necessarily unique) sequence $\{t_k\}_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$ it holds

$$2^k = \int_0^{t_k} w(x)\,\mathrm{d}x = \int_{t_k}^{t_{k+1}} w(x)\,\mathrm{d}x. \tag{4}$$

One gets

$$\begin{aligned}
B_5^r &= \int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_0^t g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t\\
&= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t)w(t)\left(\int_0^t g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t\\
&\le \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{t\in[t_k,t_{k+1}]}\left(\int_0^t g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t)\\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t\in[t_k,t_{k+1}]}\left(\int_0^t g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t) \qquad (5)\\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t_k) + \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t\in[t_k,t_{k+1}]}\left(\int_{t_{k-1}}^t g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t)\\
&=: B_6^r + B_7^r.
\end{aligned}$$

Inequality (5) follows from (4). The estimate then continues as follows.

$$\begin{aligned}
B_6^r &= \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t_k)\\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^t w(s)\,\mathrm{d}s\right)^{\frac{r}{p}} w(t)\,\mathrm{d}t\left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(t_k) \qquad (6)\\
&= \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^t w(s)\,\mathrm{d}s\right)^{\frac{r}{p}} w(t)\,\mathrm{d}t\left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^{\frac{rq}{p}}\left(\int_0^{t_{k-1}} g^*(y)\,\mathrm{d}y\right)^q V^{-\frac{r}{p}}(t_k)\\
&\le \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^t\left(\int_0^s g^*(x)\,\mathrm{d}x\right)^q w(s)\,\mathrm{d}s\right)^{\frac{r}{p}}\left(\int_0^t g^*(y)\,\mathrm{d}y\right)^q w(t)\,\mathrm{d}t\,V^{-\frac{r}{p}}(t_k)\\
&\le \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k}\left(\int_0^t\left(\int_0^s g^*(x)\,\mathrm{d}x\right)^q w(s)\,\mathrm{d}s\right)^{\frac{r}{p}}\left(\int_0^t g^*(y)\,\mathrm{d}y\right)^q w(t)V^{-\frac{r}{p}}(t)\,\mathrm{d}t\\
&= B_1^r.
\end{aligned}$$

In the step (6) one uses (4). For each $k \in \mathbb{Z}$ there exists $z_k \in [t_k, t_{k+1}]$ such that

$$(7) \qquad \left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) = \sup_{t \in [t_k, t_{k+1}]} \left(\int_{t_{k-1}}^{t} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t),$$

since the argument of the supremum is a continuous function. The term B_7 is then estimated by

$$\begin{aligned}
B_7^r &= \sum_{k \in \mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t \in [t_k, t_{k+1}]} \left(\int_{t_{k-1}}^{t} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t) \\
(8) \qquad &= \sum_{k \in \mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) \\
(9) \qquad &\lesssim \sum_{k \in \mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) \\
&\leq \sum_{k \in \mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t) w(t) \sup_{s \in (t,\infty)} \left(\int_{t}^{s} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \\
&= B_3^r.
\end{aligned}$$

Relation (7) implies (8), and (9) follows from (4). The obtained estimates yield the equivalence

$$A_1(g) \approx B_3 + B_5 \lesssim B_3 + B_6 + B_7 \lesssim B_1 + B_3,$$

which together with the known relation $B_2 \approx A_2(g) + A_3(g)$ gives the "$\gtrsim$" inequality in (3). Hence, (3) is proved. If $W(\infty) < \infty$, the proof is carried out analogously with appropriate minor modifications. Part (i) is now complete.

(ii) Fix $g \in \mathscr{M}$. By [5, Theorem 4.1(ii)] it holds $\|R_g^1\|_{\Lambda^p(v) \to L^q(w)} \approx B_1 + B_8$, where

$$B_8 := \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) s^{p'}\,\mathrm{d}s \right)^{\frac{r}{p'}} \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

Furthermore, from [13, Theorem 13(ii)] it follows that $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \approx B_4 + B_9 + B_{10}$, where

$$B_9 := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_t^\infty \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \right)^{\frac{1}{r}},$$

$$B_{10} := \left(\int_0^\infty \left(\int_t^\infty g^*(x)\,\mathrm{d}x \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(\infty).$$

By Lemma 1.1, the proof will be complete once the equivalence

$$(10)\quad B_1 + B_4 + B_8 + B_9 + B_{10} \approx A_4(g) + A_5(g) + A_6(g) + \|g\|_1 W^{\frac{1}{q}}(\infty) V^{-\frac{1}{p}}(\infty)$$

is established. Lemma 1.2 gives $A_5(g) + A_6(g) \approx B_8$. Next, it holds

$$\begin{aligned} B_1^r + B_4^r &\lesssim \int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_0^s g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t \\ &\lesssim \int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_0^s g^*(x)\,\mathrm{d}x \right)^r \left(\int_s^\infty V^{-p'}(y) v(y) y^{p'}\,\mathrm{d}y \right)^{\frac{r}{p'}} \mathrm{d}t \\ &\qquad + \int_0^\infty W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \left(\int_0^\infty g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(\infty) \\ &\lesssim A_4^r(g) + \|g\|_1^r W^{\frac{r}{q}}(\infty) V^{-\frac{r}{p}}(\infty). \end{aligned}$$

Obviously, the inequalities $B_9 \le A_4(g)$ and $B_{10} \le \|g\|_1 W^{\frac{1}{q}}(\infty) V^{-\frac{1}{p}}(\infty)$ are also valid. This proves the "$\lesssim$" inequality in (10).

To prove the converse part of (10), the same approach as in (i) is used. Suppose that $W(\infty) = \infty$ and let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of points such that (4) hold in each of them. Then it holds

$$A_4^r(g) := \int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_t^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \mathrm{d}t$$

$$\leq \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \left(\int_{t_k}^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_k}^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{11}$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^r \left(\int_{t_k}^\infty V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$+\sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_k}^\infty \left(\int_{t_{k-1}}^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim B_6^r + \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_{k-1}}^\infty \left(\int_{t_{k-1}}^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}$$

$$\lesssim B_6^r + \sum_{k\in\mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \left(\int_{t_{k-1}}^\infty \left(\int_{t_{k-1}}^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \tag{12}$$

$$\lesssim B_6^r + B_9^r.$$

Both the steps (11) and (12) are based on (4). Moreover, in part (i) it was proved that $B_6 \lesssim B_1$ and this estimate holds even this case, i.e. for $p > 1$. Hence, one obtains $A_4(g) \lesssim B_6 + B_9 \lesssim B_1 + B_9$. Next, it holds

$$\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)$$

$$= \left(\int_0^\infty \left(\int_0^t g^*(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(\infty) + B_{10}$$

$$\approx \left(\int_0^\infty \left(\int_0^t \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^q w(s)\,\mathrm{d}s\right)^{\frac{r}{p}} \left(\int_0^t g^*(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{r}} V^{-\frac{1}{p}}(\infty) + B_{10}$$

$$\leq B_1 + B_{10}.$$

The relation $A_5(g)+A_6(g)\approx B_8$ was mentioned earlier. The obtained estimates of $A_4(g)$, $A_5(g)$, $A_6(g)$ and $\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)$ together yield the "$\gtrsim$" inequality in (10). Hence, (10) is proved and so is the whole theorem. $\square$

Remark 2.2. (i) In both cases of Theorem 2.1, the functional $\|\cdot\|_Y$ is equivalent to an r.i. quasi-norm. Indeed, each of the expressions $A_i(g)$, $i=1,\dots,6$ itself is an r.i. quasi-norm. Some of the properties of the r.i. quasi-spaces generated by such quasi-norms are described in [10].

(ii) The "space" $\Lambda^p(v)$ may admit functions which are not locally integrable. Namely, it holds (see e.g. [10, Remark 3.4]) that $\Lambda^p(v)\subset L^1_{\mathrm{loc}}$ if and only if

(a) $\limsup_{s\to 0+} sV^{-\frac{1}{p}}(s)<\infty$ in the case $0<p\le 1$,

(b) there exists $\varepsilon>0$ such that $\int_0^\varepsilon V^{-p'}(s)v(s)s^{p'}\mathrm{d}s<\infty$ in the case $1<p<\infty$.

If $\Lambda^p(v)$ contains any $f\in L^1_{\mathrm{loc}}$, then the operator T_g cannot be bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$ unless $g=0$ a.e. This is reflected by the presence of the conditions $A_3(g)$ and $A_6(g)$ in the respective expressions $\|g\|_Y$ for $0<p\le 1$ and $1<p$. If (a) is not satisfied, then $A_3(g)=\infty$ unless $g=0$ a.e. An analogy holds for (b) and $A_6(g)$. Moreover, the term $\lim_{s\to 0+}\left(\int_0^s V^{-p'}(x)v(x)x^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}$ can attain only the values 0 or ∞ and thus so does $A_6(g)$. Hence, the term $A_6(g)$ is not present if $\Lambda^p(v)\subset L^1_{\mathrm{loc}}$.

(iii) If $V(\infty)<\infty$, the constant function $f\equiv 1$ belongs to $\Lambda^p(v)$. This f and any $g\in\mathscr{M}$ satisfy $T_g f\equiv\|g\|_1$. Hence, for T_g to be bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$ it is necessary that $g\in L^1$ and $W(\infty)<\infty$. This corresponds to the fact that

$$\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)\lesssim\|g\|_Y$$

in both cases (i) and (ii) of Theorem 2.1. This inequality is obvious in case (ii). In (i), it follows from the estimate

$$\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)\approx\left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t\left(\int_0^\infty g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(\infty)\right)^{\frac{1}{r}}\le A_1(g).$$

(iv) In view of the previous remark, the expressions of $\|g\|_Y$ in [10, Theorem 3.2] should be slightly corrected. Namely, in cases (iii) and (iv) thereof, the expression $\left(\int_0^m x^q(g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}V^{-\frac{1}{p}}(m)$ should be replaced by $\|g\|_1 W^{\frac{1}{q}}(m)V^{-\frac{1}{p}}(m)$. This mistake in [10] seems to be caused by using [7, Theorem 5.1], which assumes $V(\infty)=\infty$, in the proof. Using [7, Theorem 2.1] instead would lead to the correct appearance of the term $\left(\int_0^m\left(\int_x^m g^*(y)\,\mathrm{d}y\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m)$ in the affected formulas. This term is not covered by other parts of $\|g\|_Y$ in cases (iii) and (iv) of [10, Theorem 3.2], unlike the cases (i) and (ii) thereof, which are correct.

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