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Topological entropy

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Abstract: In this thesis we study topological entropy as an invariant of topological dynamical systems. The first chapter contains basic definitions and examples of topological dynamical systems. In the second chapter we introduce the definition of topological entropy on a compact metric space. We study its properties, in particular the fact that it is invariant under conjugacy. The chapter concludes with calculation of topological entropy for the examples introduced in the first chapter. The last chapter deals with generalizing the notion of topological entropy to noncompact metric spaces. The case of piecewise affine maps on the real line is studied in more detail.

Keywords: dynamical system, topology, metric space, entropy

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Contents

Introduction

Dynamical systems are a broad topic in mathematics. In its most general form, a discrete-time dynamical system consists of a set X (sometimes referred to as "phase space") and a map $T\colon X \to X$ (sometimes called "time-evolution law"). There are also continuous-time dynamical systems where instead of a single map T we in fact have a family of maps $\{T^t : t \in \mathbb{R}^+\}$ which forms a semigroup, but we shall not pursue this direction.

The set X is often assumed to have some additional structure and the map T to be "compatible" with this structure. For example, in topological dynamics, X is a topological space and T is a continuous map on X. There are also other branches, such as ergodic theory, or theory of smooth dynamical systems. In this thesis, we will deal only with topological dynamics.

Topological entropy is a nonnegative real number measuring the complexity of a topological dynamical system. It was originally defined by Adler et al. [1965]. Their definition is valid for any compact Hausdorff space, as it defines complexity in terms of sizes of covers and refinements. In compact metric spaces, a different definition was introduced by Bowen [1971] and independently by Dinaburg [1970]. This definition uses (n, ε)-separated sets and relies on the metric. However, this notion depends only on the topology and it turns out that both definitions of topological entropy are equivalent. We will restrict ourselves only to metric spaces and use the latter definition for the purpose of this thesis.

There have been several attempts to generalize the notion of topological entropy to noncompact spaces. In our thesis, we will use the definition from Cánovas and Rodríguez [2005].

The structure of this thesis is as follows. Chapter 1 contains an introduction to basic notions in topological dynamics and presents several examples. Chapter 2 starts with the definition of topological entropy for a compact metric space, lists some of its properties, and concludes with calculation of topological entropy for the examples shown in Chapter 1. Chapter 3 shows the definition of topological entropy for noncompact spaces, and discusses the topological entropy of piecewise affine maps on the real line.

1. Topological dynamical systems

1.1 Basic definitions

Here we introduce basic definitions and notions in topological dynamical systems.

Definition 1.1. A *topological dynamical system* is a pair (X, T), where X is a topological space and $T\colon X \to X$ is a continuous map. We denote $T^0 = \mathrm{id}$ and for $n \in \mathbb{N}$, the *n-th iterate of T* is $T^n = T \circ T^{n-1}$. If T is an invertible homeomorphism, we also put $T^{-n} = (T^{-1})^n$. If T is not invertible, then $T^{-n}(A)$ denotes the preimage of the set $A \subset X$ under the map T^n.

Throughout this thesis, (X, T) will denote a topological dynamical system, unless stated otherwise.

Definition 1.2. A point $x \in X$ is called a *fixed point*, if $T(x) = x$. A point $x \in X$ is called a *periodic point*, if there exists a $P \in \mathbb{N}$ such that $T^P(x) = x$, P is called a *period* of x and x is said to be *P-periodic*. The smallest such P is called the *minimal period* of x.

Definition 1.3. Let $x \in X$. Then the *positive semiorbit* (sometimes just called the *orbit*) of x is $\mathcal{O}_T^+(x) = \bigcup_{n \in \mathbb{N}_0} T^n(x)$. In the invertible case we define the *negative semiorbit* of x as $\mathcal{O}_T^-(x) = \bigcup_{n \in \mathbb{N}_0} T^{-n}(x)$ and the *full orbit* $\mathcal{O}_T(x) = \mathcal{O}_T^+(x) \cup \mathcal{O}_T^-(x) = \bigcup_{n \in \mathbb{Z}} T^n(x)$.

Notice that if $A \subset X$ and $T(A) \subset A$, then $(A, T|_A)$ is a topological dynamical system. This motivates the next definition.

Definition 1.4. A set $A \subset X$ is called *forward T-invariant* (when clear from the context, we may just say *invariant*) if $T(A) \subset A$. If $T(A) = A$, then A is said to be *strictly invariant*.

Definition 1.5. Let (X, T) and (Y, S) be topological dynamical systems. Then a continuous map $\varphi\colon X \to Y$ is called a *topological semiconjugacy* if it is surjective and $S \circ \varphi = \varphi \circ T$. We may also write $\varphi\colon (X, T) \to (Y, S)$. If there exists a topological semiconjugacy $\varphi\colon (X, T) \to (Y, S)$, we say that (Y, S) is a *factor* of (X, T), or that (X, T) is an *extension* of (Y, S).

   If a semiconjugacy φ is also a homeomorphism, then φ is called a *topological conjugacy*. If there exists a topological conjugacy $\varphi\colon (X, T) \to (Y, S)$, we say that (X, T) and (Y, S) are *conjugate*.

Remark. It is easy to see that the conjugacy defines an equivalence relation on topological dynamical systems. Also, if $\varphi\colon (X, T) \to (Y, S)$ is a semiconjugacy and $x \in X$ is a periodic point, then $\varphi(x) \in Y$ is also periodic. The minimal period of x is greater or equal to the minimal period of $\varphi(x)$. This inequality can be strict, consider a point x with minimal period $P > 1$ and space $Y = \{y\}$. Then $\varphi(x) = y$ has minimal period 1.

Definition 1.6. A continuous map $T\colon X \to X$ is said to be *transitive*, if for every two nonempty open sets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $U \cap T^n(V) \neq \emptyset$.

1.2 Examples

Now we will present several examples of topological dynamical systems.

Example 1.7. Let $A \in \mathbb{R}^{n \times n}$ be a matrix. Then $(\mathbb{R}^n, A)$ is a *linear discrete topological dynamical system.* If A is regular, then the system is invertible. Orbit of a point $x_0 \in \mathbb{R}^n$ is $\{x_i : i \in \mathbb{N}_0\}$, where $x_{i+1} = Ax_i$ for $i \in \mathbb{N}_0$, so $x_i = A^i x$.

Let us take a look at qualitative behavior of orbits in this example. If x_0 is a vector that has nonzero projection to an eigenspace corresponding to (generalized) eigenvalue $\lambda \in \mathbb{C}$ with $\Re(\lambda) \neq 0$, then x_i either converges to 0 or diverges to infinity. In either case, this is quite simple dynamical behavior in the sense that no point "comes back" close to its initial position (except the fixed point 0).

Quite interesting is the behavior in the case when x_0 lies in the sum of planes corresponding to purely imaginary nonzero eigenvalues of A. For simplicity suppose that x_0 lies in one of these planes, namely in the plane corresponding to eigenvalue $\lambda \in \mathbb{C}$, $\Re(\lambda) = 0$. Then A acts as a rotation about angle $\Im(\lambda)$ on this plane (with respect to the basis $(v + \overline{v}, i(v - \overline{v}))$, where $v \neq 0$ is a complex eigenvector corresponding to λ). To understand the behavior of these points, we may look at the next example of *rotation on circle*, which is our first example of (nontrivial) recurrence in dynamical systems.

Example 1.8. We will call the factor space $S^1 = \mathbb{R}/\mathbb{Z}$ a *circle.* Useful way to look at this is to see S^1 as the interval $[0, 1)$ with arithmetic operations modulo 1. To see that this is indeed a "circle", notice that the unit circle in the complex plane $C = \{z \in \mathbb{C} : |z| = 1\}$ is isomorphic to S^1. The map $x \mapsto e^{2\pi i x}$ establishes an isomorphism between $(S^1, +)$ and $(C, \cdot)$. Then *rotation* about $\alpha > 0$ is the map $R_\alpha \colon S^1 \mapsto S^1$ defined by $R_\alpha(x) = x + \alpha \mod 1$. (This corresponds to rotation about angle $2\pi\alpha$ in the complex plane.)

Claim 1.9. (a) *Let α be rational. Then every point of S^1 is periodic.*

(b) *Let α be irrational. Then the orbit of every point in S^1 is dense.*

Proof. (a) Write $\alpha = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Then for any $x \in S^1$ we have $R_\alpha^q(x) = x + q\alpha = x + p \mod 1 = x$, so x is q-periodic.

(b) Let $x \in X$. All its images are distinct, because otherwise if $R_\alpha^m(x) = R_\alpha^k(x)$ for $m < k$, then $(k - m)\alpha \in \mathbb{Z}$ and α is rational. For $n \in \mathbb{N}$ consider the partition of S^1 to intervals $[0, 1/n)$, $[1/n, 2/n)$, $\ldots$, $[(n - 1)/n, 1)$. Then there exist $1 \leq m < k \leq n$ such that $R_\alpha^m(x)$ and $R_\alpha^k(x)$ belong to the same interval, because there are only finitely many intervals. Thus $d(R_\alpha^m(x), R_\alpha^k(x)) \leq 1/n$ and R_α^{k-m} is a rotation about nonzero angle smaller (in absolute value) than $1/n$. Then $\mathcal{O}_{R_\alpha}^+(x)$ is $1/n$-dense, which means that every point $y \in S^1$ has distance from $\mathcal{O}_{R_\alpha}^+(x)$ less than $1/n$. Since this is true for every n, we have that $\mathcal{O}_{R_\alpha}^+(x)$ is dense. $\square$

In the irrational case of rotation on the circle we have rather complicated behavior of each point – a dense orbit. But we still may say that the behavior is simple in the sense that *every* point behaves in the same way. It is time to present an example where this is not the case.

Example 1.10. For $m \in \mathbb{N}$, $m > 1$, let $\mathcal{A}_m = \{0, 1, \ldots, m - 1\}$ and define the *symbolic space* $\Sigma_m = \mathcal{A}_m^{\mathbb{Z}}$ to be the space of two–sided sequences of symbols in $\mathcal{A}_m$, and $\Sigma_m^+ = \mathcal{A}_m^{\mathbb{N}}$ to be the space of corresponding one–sided sequences. We define a map $\sigma \colon \Sigma_m \to \Sigma_m$ (or $\sigma \colon \Sigma_m^+ \to \Sigma_m^+$) by putting $\sigma(x)_i = x_{i+1}$ for $i \in \mathbb{Z}$ (or $i \in \mathbb{N}$). This map is called the *shift*. For two–sided sequences, the shift is a bijection, whereas for one–sided sequences, σ omits the first symbol and every sequence has m preimages.

Both spaces are compact in the product topology (because they are products of compact finite sets). The topology of Σ_m has a basis consisting of *cylinder* sets

$$C_{j_1, \ldots, j_k}^{n_1, \ldots, n_k} = \{x \in \Sigma_m : x_{n_i} = j_i, i = 1, \ldots, k\},$$

where n_i are pairwise distinct indices in $\mathbb{Z}$ and $j_i \in \mathcal{A}_m$. The basis for topology of Σ_m^+ is defined in the same way except that the indices are from $\mathbb{N}$. This topology is also generated by the metric

$$d(x, y) = \frac{1}{m^{\min\{|i| : x_i \neq y_i\}}}, \quad x, y \in \Sigma_m \text{ (or } x, y \in \Sigma_m^+).$$

Then open balls are the cylinders $B(x, m^{-\ell}) = C_{x_{-\ell}, \ldots, x_\ell}^{-\ell, \ldots, \ell}$ in Σ_m and $B(x, m^{-\ell}) = C_{x_1, \ldots, x_\ell}^{1, \ldots, \ell}$ in Σ_m^+. These balls form a subbase of the topology.

Claim 1.11. *In the symbolic space defined above*

(a) *the set of periodic points is dense,*

(b) *the set of points with dense orbit is dense.*

Proof.　(a) Let $x \in \Sigma_m$ be any sequence and let $\varepsilon > 0$. Then we find $\ell \in \mathbb{N}$ such that $m^{-\ell} < \varepsilon$ and define a periodic point $y \in \Sigma_m$ by setting $y_i = x_i$ for $|i| \leq \ell$ and then repeating the finite sequence $x_{-\ell}, \ldots, x_\ell$ to ensure periodicity with period $2\ell + 1$. Then $d(x, y) < \varepsilon$, so periodic points form a dense set. The construction for Σ_m^+ is analogous.

(b) Let $\mathcal{F}_m = \bigcup_{k=1}^{\infty} \mathcal{A}_m^k$ be the set of finite sequences with symbols from $\mathcal{A}_m$. Clearly $\mathcal{F}_m$ is countable, so we enumerate it by $\mathcal{F}_m = \{F_i : i \in \mathbb{N}\}$. Then we construct a sequence $a \in \Sigma_m^+$ by concatenating all finite sequences, that is $a = F_1 F_2 F_3 \ldots$. We will show that the orbit of a is dense. Let $B(x, m^{-\ell})$ be any ball. Then $x_1 x_2 \ldots x_\ell = F_i$ for some i. Then F_i is contained in a starting at some position j, so first ℓ elements of $\sigma^j(a)$ are F_i and $\sigma^j(a) \in B(x, m^{-\ell})$, so the orbit of a intersects every ball. Thus $\mathcal{O}_\sigma^+$ is dense in Σ_m^+. Moreover, when enumerating $\mathcal{F}_n$ we could choose F_1 to be arbitrary, that is, there exists a point with dense orbit beginning with arbitrary finite sequence. This means that the set of such points is dense, since it intersects every ball.

The construction for Σ_m is analogous, just not discussing the first ℓ elements but the elements at indices from $-\ell$ to ℓ and defining the left side of the sequence a by zeroes.

$\square$

Example 1.12. For $m \in \mathbb{N}$, $m \geq 2$, define the map $E_m \colon S^1 \to S^1$ by putting $E_m(x) = mx \mod 1$. Then (S^1, E_m) is a topological dynamical system and the map E_m is called the *times-m map* or *expanding map on the circle*.

Claim 1.13. *There exists a topological semiconjugacy from (Σ_m^+, σ) to (S^1, E_m).*

Proof. Define $\varphi \colon \Sigma_m^+ \to S^1$ by setting $\varphi(x) = \sum_{i=1}^{\infty} \frac{x_i}{m^i}$. Then $\varphi(x)$ is the number whose decimal representation in base m is $0.x$. This is a well–defined map since $|x_i| < m$ and the sum converges. Further, φ is surjective, since every point in $[0, 1)$ has a base m expansion. It is not hard to see that φ is continuous. Indeed, for $x, y \in \Sigma_m^+$ with $d(x, y) = m^{-\ell}$ we have

$$|\varphi(x) - \varphi(y)| \le \sum_{i=0}^{\infty} \left| \frac{x_i - y_i}{m^i} \right| = \sum_{i=\ell}^{\infty} \left| \frac{x_i - y_i}{m^i} \right| \le \sum_{i=\ell}^{\infty} \frac{m-1}{m^i} = m^{-\ell+1} = m \cdot d(x, y).$$

The condition for semiconjugacy is also satisfied:

$$E_m \circ \varphi(x) = m \cdot \sum_{i=1}^{\infty} \frac{x_i}{m^i} \quad \mod 1 = \sum_{i=1}^{\infty} \frac{x_i}{m^{i-1}} \quad \mod 1$$

$$= \sum_{i=2}^{\infty} \frac{x_i}{m^{i-1}} = \sum_{i=1}^{\infty} \frac{x_{i+1}}{m^i} = \sum_{i=1}^{\infty} \frac{(\sigma(x))_i}{m^i} = \varphi \circ \sigma(x).$$

Note that map φ is injective except at a countable collection of points where $\varphi(w_1, \dots, w_n, m-1, m-1, \dots) = \varphi(w_1, \dots, w_n + 1, 0, 0, \dots)$. $\square$

Corollary 1.14. *In (S^1, E_m), the set of periodic points is dense and also the set of points with dense orbit is dense.*

Proof. This is a consequence of Claim 1.11, using the remark below Definition 1.5. $\square$

Following are examples of maps on a closed interval.

Example 1.15. The map $S \colon [0, 1] \to [0, 1]$, defined by $S(x) = \min\{2x, 2 - 2x\}$ is called the *tent map*. We can write S as

$$S(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2 - 2x & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$
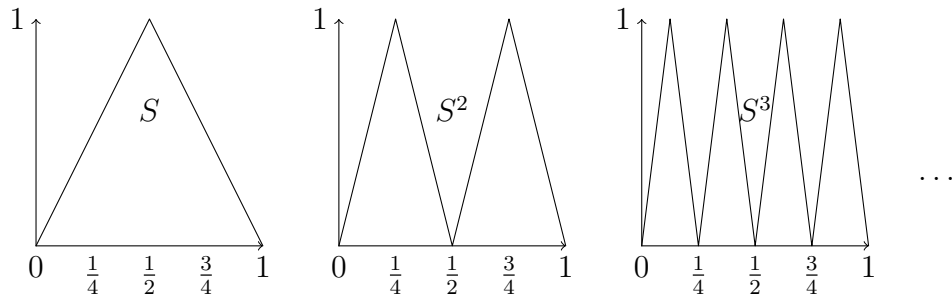

Figure 1.1: Iterations of the tent map.

Claim 1.16. *The tent map S is topologically transitive.*

Proof. By induction we can show that for $n \in \mathbb{N}$, S^n is defined on the interval $[0, 2^{-n+1}]$ by

$$S(x) = \begin{cases} 2^n x & \text{if } x \in \left[0, \frac{1}{2^n}\right], \\ 2^n - 2^n x & \text{if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right], \end{cases}$$

and that S^n is 2^{-n+1}-periodic. Thus for any $n \in \mathbb{N}$, $k \in \{0, \ldots, 2^n - 1\}$ we have $S\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) = [0, 1]$. Therefore if $U, V \subset [0, 1]$ are nonempty open sets, we find $n \in \mathbb{N}$, $k \in \{0, \ldots, 2^n - 1\}$, such that $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \subset V$. Then $U \cap S^n(V) = U \neq \emptyset$, showing that S is topologically transitive. $\square$

Example 1.17. The family of maps $T_\lambda \colon [0, 1] \to [0, 1]$, where as $T_\lambda(x) = \lambda x(1-x)$ is called the family of *quadratic maps*.

Claim 1.18. *The tent map S is conjugate to the quadratic map T_4.*

Proof. The map $\varphi \colon [0, 1] \to [0, 1]$ defined as $\varphi(x) = \sin^2\left(\frac{\pi x}{2}\right)$ is a conjugacy of $([0, 1], S)$ to $([0, 1], T_4)$. Indeed, it is a surjective homeomorphism and

$$T_4 \circ \varphi(x) = 4\sin^2\left(\frac{\pi x}{2}\right)\left(1 - \sin^2\left(\frac{\pi x}{2}\right)\right) = \left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right)^2 = \sin^2(\pi x),$$

$$\varphi \circ S(x) = \begin{cases} \sin^2(\pi x) & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \sin^2\left(\frac{\pi}{2}(2 - 2x)\right) = \sin^2(\pi - \pi x) = \sin^2(\pi x) & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$
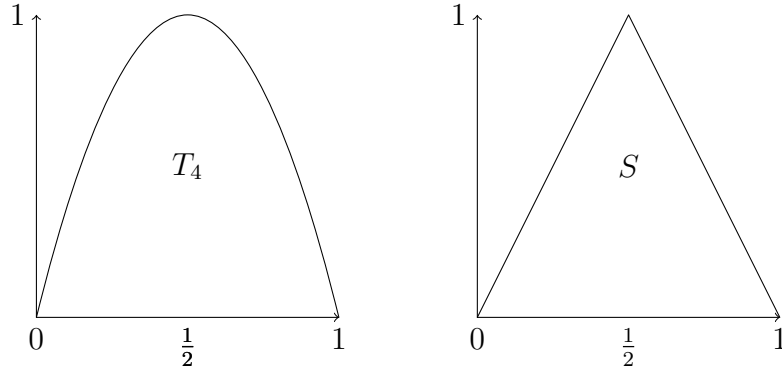
$\square$

Figure 1.2: The maps T_4 and S.

2. Topological entropy

2.1 Definition of topological entropy

Now our aim is to define topological entropy as a nonnegative real number representing the asymptotic average exponential growth of the number of distinguishable orbit segments.

In this chapter, we will only consider topological dynamical systems (X, T) such that X is a compact metric space with infinitely many points. The metric on X will be denoted by d (or d^X if needed). The notation used in this chapter has been adopted from Brin and Stuck [2002].

Definition 2.1. For every $n \in \mathbb{N}$ we define a map $d_n \colon X \times X \to \mathbb{R}$ by setting $d_n(x, y) = \max\limits_{0 \le i < n} d\left(T^i(x), T^i(y)\right)$ for $x, y \in X$.

Remark. Every d_n is a metric on X. We will denote by $\operatorname{diam}_n A$ the diameter of the set A under the metric d_n, and by $B_n(x, r)$ the ball centered at x with radius r in this metric.

Theorem 2.2. *The metrics d_n are pairwise topologically equivalent.*

Proof. Let $n \in \mathbb{N}$, $x \in X$ and $\delta > 0$. Then $B_n(x, \delta) \subset B(x, \delta)$. Now for the converse. The maps $T^1, T^2, \ldots, T^{n-1}$ are uniformly continuous (because they are continuous and X is compact). Therefore there are constants $\delta_1, \delta_2, \ldots, \delta_{n-1} > 0$ such that $d(x, y) < \delta_i$ implies $d(T^i(x), T^i(y)) < \delta$ for all $i = 1, 2, \ldots, n-1$. Then if we put $\delta_{\min} = \min\{\delta, \delta_1, \delta_2, \ldots, \delta_{n-1}\}$, we have $\delta_{\min} > 0$ and $B(x, \delta_{\min}) \subset B_n(x, \delta)$. This shows that the metrics d and d_n are topologically equivalent, which finishes the proof. $\qquad\square$

Definition 2.3. Let $n \in \mathbb{N}$ and $\varepsilon > 0$.

(a) A set $A \subset X$ is (n, ε)-*spanning* if for every $x \in X$ there is $a \in A$ such that $d_n(x, a) < \varepsilon$. Let $\operatorname{span}(n, \varepsilon, T) = \min\{\#A : A \subset X, A \text{ is } (n, \varepsilon)\text{-spanning}\}$.

(b) A set $A \subset X$ is (n, ε)-*separated* if $d_n(x, y) \ge \varepsilon$ for all $x, y \in A$. Let $\operatorname{sep}(n, \varepsilon, T) = \max\{\#A : A \subset X, A \text{ is } (n, \varepsilon)\text{-separated}\}$.

(c) Let $\operatorname{cov}(n, \varepsilon, T) = \min\{\#\mathcal{A} : X = \bigcup \mathcal{A}, \operatorname{diam}_n A < \varepsilon \text{ for all } A \in \mathcal{A}\}$.

Remark. Since we assume X to be compact, all numbers in Definition 2.3 are well–defined and finite.

Theorem 2.4. *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ we have*

$$\operatorname{cov}(n, 2\varepsilon, T) \le \operatorname{span}(n, \varepsilon, T) \le \operatorname{sep}(n, \varepsilon, T) \le \operatorname{cov}(n, \varepsilon, T).$$

Proof. Let $A \subset X$ be an (n, ε)-spanning set such that $\#A = \operatorname{span}(n, \varepsilon, T)$. Then the system $\{B_n(a, \varepsilon) : a \in A\}$ covers X and its elements have d_n-diameter 2ε. Thus we have proved that $\operatorname{cov}(n, 2\varepsilon, T) \le \operatorname{span}(n, \varepsilon, T)$.

Now let $B \subset X$ be an (n, ε)-separated set such that $\#B = \operatorname{sep}(n, \varepsilon, T)$. Then it is also (n, ε)-spanning. Otherwise there would be a point $y \in X$ with

$d_n(y, B) \geq \varepsilon$ and $B \cup \{y\}$ would be (n, ε)-separated, contradicting the maximality of $\#B$. Thus span $(n, \varepsilon, T) \leq$ sep (n, ε, T).

Finally, let $\mathcal{A}$ be a cover of X with $\mathrm{diam}_n A < \varepsilon$, for every $A \in \mathcal{A}$, such that $\#\mathcal{A} = \mathrm{cov}\,(n, \varepsilon, T)$. Then if B is (n, ε)-separated, then for each $A \in \mathcal{A}$, the intersection $B \cap A$ contains at most one element (because $\mathrm{diam}_n A < \varepsilon$). Therefore we have $\#B \leq \#\mathcal{A}$, giving sep $(n, \varepsilon, T) \leq \mathrm{cov}\,(n, \varepsilon, T)$. $\qquad\square$

Now we can define topological entropy.

Definition 2.5. Put $h_\varepsilon(T) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}\,(n, \varepsilon, T)$. Then the *topological entropy* of the map T is defined as $h_{\mathrm{top}}(T) = \sup\limits_{\varepsilon > 0} h_\varepsilon(T)$.

Remark. Because $\varepsilon \mapsto h_\varepsilon(T)$ is nondecreasing, we have $h_{\mathrm{top}}(T) = \lim_{\varepsilon \to 0^+} h_\varepsilon(t)$.

We can interpret the definition as follows. Having a fixed $\varepsilon > 0$, assume that we cannot distinguish between two points if their distance is smaller than ε. Then for an $n \in \mathbb{N}$, the number sep (n, ε, T) is the maximal number of points distinguishable by looking at their first n iterates. This can be said because an (n, ε)-separated set is such a set that any of its two points get at least ε far away from each other in one of the first n iterations of T. Taking the logarithm and dividing by n, we obtain the average growth of "ε-distinguishable" orbit segments in the first n iterations of T. Considering the limit superior as $n \to \infty$, we have the average asymptotic growth. Finally, by taking supremum for $\varepsilon > 0$ we obtain that *topological entropy is the asymptotic average growth of the number of orbit segments distinguishable with arbitrarily fine (but finite) precision.*

Remark. From Theorem 2.4 it follows that it is possible to use any of the expressions cov (n, ε, T), span (n, ε, T) or sep (n, ε, T) in the definition of topological entropy.

When we use the version of the definition with spanning number span (n, ε, T), we can give a different interpretation of topological entropy. Fix an $\varepsilon > 0$. If we have a $(1, \varepsilon)$-spanning set $A \subset X$, then a position of an arbitrary point $x \in X$ can be described with ε precision by assigning to it a point $a \in A$ such that $d(x, a) < \varepsilon$. Thus, for $n \in \mathbb{N}$, the number span (n, ε, T) is the minimal number of points such that we can describe a position of the first n iterations of an arbitrary point $x \in X$ with ε precision. If we think of points in a minimal (n, ε)-spanning set as being labeled by finite words consisting of 0's and 1's, and of log as being the base-2 logarithm, then $\log \mathrm{span}\,(n, \varepsilon, T)$ is the length of the word needed to describe the position of the first n iterates of a point $x \in X$ with ε precision. Dividing by n, we obtain the average increment of word length per iteration. Therefore, by taking supremum over $\varepsilon > 0$, we may say that *topological entropy is the asymptotic average amount of information per iteration needed to describe the position of a point with arbitrarily fine but finite precision.*

One may ask whether we can use limit instead of limit superior in the definition of h_ε. It turns out we can when using the covering number definition, because in this case the limit always exists. To prove this, the following lemma will be useful.

Lemma 2.6. *Let a sequence of real numbers $(a_n)_{n=1}^{\infty}$ be* subadditive, *that is, for every $m, n \in \mathbb{N}$, $a_{m+n} \leq a_m + a_n$. Then the limit $\lim_{n \to \infty} \frac{a_n}{n}$ exists.*

Proof. Let $n, p \in \mathbb{N}$, $p < n$. We divide n by p and obtain $n = pq + r$, where $q > 0$ and $0 \le r < p$ are integers. Then, using the subadditivity of $(a_n)_{n=1}^{\infty}$,

$$\frac{a_n}{n} \le \frac{a_{pq}}{n} + \frac{a_r}{n} \le \frac{a_{pq}}{pq} + \frac{a_r}{n} \le \frac{q \cdot a_p}{pq} + \frac{a_r}{n} = \frac{a_p}{p} + \frac{a_r}{n}.$$

Now applying limit superior as n tends to infinity and then taking the infimum over $p \in \mathbb{N}$ yields $\limsup_{n \to \infty} \frac{a_n}{n} \le \inf_{p \in \mathbb{N}} \frac{a_p}{p}$. Since clearly $\inf_{p \in \mathbb{N}} \frac{a_p}{p} \le \liminf_{n \to \infty} \frac{a_n}{n}$, we have $\limsup_{n \to \infty} \frac{a_n}{n} \le \liminf_{n \to \infty} \frac{a_n}{n}$, and consequently the existence of the limit in question. $\square$

Theorem 2.7. *For every $\varepsilon > 0$, the limit $\lim_{n \to \infty} \frac{1}{n} \log \mathrm{cov}\,(n, \varepsilon, T)$ exists.*

Proof. We will show that the sequence $(\log \mathrm{cov}\,(n, \varepsilon, T))_{n=1}^{\infty}$ is subadditive. Let $m, n \in \mathbb{N}$. Let $\mathcal{A}$, $\mathcal{B}$ be covers of X with $\mathrm{diam}_m\, A < \varepsilon$, $\mathrm{diam}_n\, B < \varepsilon$ for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\#\mathcal{A} = \mathrm{cov}\,(m, \varepsilon, T)$, $\#\mathcal{B} = \mathrm{cov}\,(n, \varepsilon, T)$. Put

$$\mathcal{C} = \{A \cap T^{-m}(B) : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Then $\mathcal{C}$ is a cover of X with sets of d_{n+m}-diameter less than ε. Therefore

$$\mathrm{cov}\,(n + m, \varepsilon, T) \le \#\mathcal{C} \le \#\mathcal{A} \cdot \#\mathcal{B} = \mathrm{cov}\,(m, \varepsilon, T) \cdot \mathrm{cov}\,(n, \varepsilon, T).$$

Taking logarithms we obtain desired subadditivity. Therefore by the preceding lemma, the limit $\lim_{n \to \infty} \frac{1}{n} \log \mathrm{cov}\,(n, \varepsilon, T)$ exists. $\square$

Remark. The limit $\lim_{n \to \infty} \frac{1}{n} \log \mathrm{span}\,(n, \varepsilon, T)$ may not exist (Katok and Hasselblatt [1996], p. 109).

2.2   Properties of topological entropy

It may seem odd that topological entropy is defined using a metric but it is called *topological* entropy. This is justified by the fact that topological entropy depends only on the topology in the following sense.

Theorem 2.8. *Let d, d' be two topologically equivalent metrics on X and let $h_{top}(T)$, $h'_{top}(T)$ be the topological entropies of T calculated with respect to d, d' respectively. Then $h_{top}(T) = h'_{top}(T)$.*

Proof. Let $\varepsilon > 0$ and put $D_\varepsilon = \{(x_1, x_2) \in X \times X : d(x_1, x_2) \ge \varepsilon\}$. The set D_ε is closed in the compact space $X \times X$ and therefore it is compact. Since d and d' induce the same topology on X, we have that d' is continuous on $X \times X$. Consequently, d' attains a minimum $\delta(\varepsilon)$ on the set D_ε. Necessarily $\delta(\varepsilon) > 0$, since $x_1 \ne x_2$ for all $(x_1, x_2) \in D_\varepsilon$. Thus $d'(x_1, x_2) < \delta(\varepsilon)$ implies $d(x_1, x_2) < \varepsilon$. This property immediately extends to the pair of metrics d_n, d'_n for all $n \in \mathbb{N}$. We have thus obtained a mapping $\varepsilon \mapsto \delta(\varepsilon)$ which is nondecreasing. Therefore the limit $\lim_{\varepsilon \to 0^+} \delta(\varepsilon)$ exists and is equal to $\inf \{d'(x, y) : (x, y) \in \bigcup_{\varepsilon > 0} D_\varepsilon\} = \inf \{d'(x, y) : x, y \in X, x \ne y\}$. Since X is compact and infinite, it has a limit point $x \in X$, so $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$. Therefore every covering of X with sets of d_n-diameter at most ε is a covering of X with sets of d'_n-diameter at most $\delta(\varepsilon)$ and thus

$$h_{\mathrm{top}}(T) = \lim_{\varepsilon \to 0^+} h_\varepsilon(T) \ge \lim_{\varepsilon \to 0^+} h'_{\delta(\varepsilon)}(T) = \lim_{\delta \to 0^+} h'_\delta(T) = h'_{\mathrm{top}}(T).$$

$\square$

Lemma 2.9. *Let (X, T), (Y, S) be conjugate topological dynamical systems such that the conjugacy $\varphi \colon (X, T) \to (Y, S)$ is an isometry. Then $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(S)$.*

Proof. Let $x, y \in X$. Then, using that φ is an isometry and a conjugacy, we have

$$d_n^X(x, y) = \max_{0 \leq i < n} d^X(T^i(x), T^i(y)) = \max_{0 \leq i < n} d^Y(\varphi(T^i(x)), \varphi(T^i(y)))$$

$$= \max_{0 \leq i < n} d^Y(S^i(\varphi(x)), S^i(\varphi(y))) = d_n^Y(\varphi(x), \varphi(y)).$$

Therefore $A \subset X$ is (n, ε)-separated in X if and only if $\varphi(A)$ is (n, ε)-separated in Y. Since $\#A = \#\varphi(A)$, this immediately gives $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(S)$. $\square$

Theorem 2.10. *Topological entropy is a topological invariant. That is, if the systems (X, T) and (Y, S) are conjugate, then $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(S)$.*

Proof. Let $\varphi \colon (X, T) \to (Y, S)$ be a topological conjugacy. Define a metric d' on X by putting $d'(x, y) = d^Y(\varphi(x), \varphi(y))$ for $x, y \in X$. Then, since φ is a homeomorphism, d' generates the same topology on X as d^X does. We know from Theorem 2.8 that $h_{\mathrm{top}}(T)$ does not depend on the choice of metric on X. Using the previous Lemma 2.9 and the fact that $\varphi \colon (X, d') \to (Y, d^y)$ is an isometry we get $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(S)$. $\square$

Corollary 2.11. *Let (X, T), (Y, S) be topological dynamical systems with entropies $h_{\mathrm{top}}(T) \neq h_{\mathrm{top}}(S)$. Then the systems are not conjugate.*

Theorem 2.12. *Let (Y, S) be a factor of (X, T). Then $h_{\mathrm{top}}(T) \geq h_{\mathrm{top}}(S)$.*

Proof. Denote by $\varphi \colon (X, T) \to (Y, S)$ a semiconjugacy. The map φ is uniformly continuous, because X is compact. So for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $d^X(x, y) < \delta(\varepsilon)$ implies $d^Y(\varphi(x), \varphi(y)) < \varepsilon$. Therefore for $n \in \mathbb{N}$, if $A \subset X$ is $(n, \delta(\varepsilon))$-spanning in X, then $\varphi(A)$ is (n, ε)-spanning in Y. Since $\#\varphi(A) \leq \#A$, we have

$$\mathrm{span}\,(n, \delta(\varepsilon), T) \geq \mathrm{span}\,(n, \varepsilon, S),$$

which after taking logarithms and limits gives the desired result:

$$h_{\mathrm{top}}(T) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \mathrm{span}\,(n, \delta(\varepsilon), T) \geq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \mathrm{span}\,(n, \varepsilon, S) = h_{\mathrm{top}}(S).$$

$\square$

Theorem 2.13. *Let (X, T), (Y, S) be topological dynamical systems. Then the following properties are satisfied.*

(a) *For every $m \in \mathbb{N}$, $h_{\mathrm{top}}(T^m) = m \cdot h_{\mathrm{top}}(T)$.*

(b) *If T is a homeomorphism, then for every $m \in \mathbb{Z}$, $h_{\mathrm{top}}(T^m) = |m| \cdot h_{\mathrm{top}}(T)$.*

(c) *Define $T \times S \colon X \times Y \to X \times Y$ by $(T \times S)(x, y) = (T(x), S(y))$. Then $h_{\mathrm{top}}(T \times S) = h_{\mathrm{top}}(T) + h_{\mathrm{top}}(S)$.*

Proof. (a) Let $m \in \mathbb{N}$. Then for all $x, y \in X$ and $n \in \mathbb{N}$ we have

$$\max_{0 \leq i < n} d\left(T^{mi}(x), T^{mi}(y)\right) \leq \max_{0 \leq j < mn} d\left(T^{j}(x), T^{j}(y)\right),$$

so $\operatorname{sep}(n, \varepsilon, T^m) \leq \operatorname{sep}(mn, \varepsilon, T)$. Thus

$$\limsup_{n \to \infty} \frac{1}{n} \log \operatorname{sep}(n, \varepsilon, T^m) \leq m \cdot \limsup_{n \to \infty} \frac{1}{mn} \log \operatorname{sep}(mn, \varepsilon, T)$$

$$\leq m \cdot \limsup_{\ell \to \infty} \frac{1}{\ell} \log \operatorname{sep}(\ell, \varepsilon, T).$$

Therefore $h_{\text{top}}(T^m) \leq m \cdot h_{\text{top}}(T)$.

Now for the other inequality. We claim that for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $d(x, y) < \delta(\varepsilon)$ implies $d_n(x, y) < \varepsilon$. To see this, simply repeat the construction of $\delta(\varepsilon)$ as in the proof of Theorem 2.8, using the fact that d and d_n are topologically equivalent metrics by Theorem 2.2. Thus $\operatorname{span}(n, \delta(\varepsilon), T^m) \geq \operatorname{span}(mn, \varepsilon, T)$, giving

$$\limsup_{n \to \infty} \frac{1}{n} \log \operatorname{span}(n, \delta(\varepsilon), T^m) \geq m \cdot \limsup_{n \to \infty} \frac{1}{mn} \log \operatorname{span}(mn, \varepsilon, T)$$

$$\geq m \cdot \limsup_{\ell \to \infty} \frac{1}{\ell} \log \operatorname{span}(\ell, \varepsilon, T).$$

Therefore we have $h_{\text{top}}(T^m) \geq m \cdot h_{\text{top}}(T)$.

(b) Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $A \subset X$ be (n, ε)-separated for T. It means that for all $x, y \in A$

$$\varepsilon \leq d_n(x, y) = \max_{i=0,\ldots,n-1} d\left(T^i(x), T^i(y)\right)$$

$$= \max_{j=n-1,\ldots,0} d\left(T^{-j}\left(T^{n-1}(x)\right), T^{-j}\left(T^{n-1}(y)\right)\right).$$

Thus $T^{n-1}(A)$ is (n, ε)-separated for T^{-1}. Also $\#A = \#T^{n-1}(A)$, since T is a bijection. Conversely, if $B \subset X$ is (n, ε)-separated for T^{-1}, then $T^{-(n-1)}(B)$ is (n, ε)-separated for T. Therefore $\operatorname{sep}(n, \varepsilon, T) = \operatorname{sep}(n, \varepsilon, T^{-1})$ and $h_{\text{top}}(T) = h_{\text{top}}(T^{-1})$. The rest of the claim follows from the fact $h_{\text{top}}(T^0) = h_{\text{top}}(\operatorname{id}) = 0$ by Lemma 2.9, and from part (a): for $m \in \mathbb{N}$ we have

$$h_{\text{top}}(T^{-m}) = h_{\text{top}}((T^m)^{-1}) = h_{\text{top}}(T^m) = m \cdot h_{\text{top}}(T).$$

(c) Define a metric d on $X \times Y$ as the maximum metric, that is, for every two points $(x_1, y_1), (x_2, y_2) \in X \times Y$, let

$$d((x_1, y_1), (x_2, y_2)) = \max\left\{d^X(x_1, x_2), d^Y(y_1, y_2)\right\}.$$

It is well–known that this metric generates the product topology on $X \times Y$. We also immediately have

$$d_n((x_1, y_1), (x_2, y_2)) = \max\left\{d_n^X(x_1, x_2), d_n^Y(y_1, y_2)\right\}$$

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for every $n \in \mathbb{N}$. If $U \subset X$, $B \subset Y$ satisfy $\mathrm{diam}_n A < \varepsilon$, $\mathrm{diam}_n B < \varepsilon$, then also $A \times B$ satisfies $\mathrm{diam}_n(A \times B) < \varepsilon$. This means that if $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{P}(Y)$ are covers with sets of d_n^X-diameter (d_n^Y-diameter) at most ε, then $\mathcal{A} \otimes \mathcal{B} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ is a cover of $X \times Y$ with sets of d_n-diameter at most ε. Thus

$$\mathrm{cov}\,(n, \varepsilon, T \times S) \leq \mathrm{cov}\,(n, \varepsilon, T) \cdot \mathrm{cov}\,(n, \varepsilon, S),$$

which after taking logarithms and limits gives

$$h_{\mathrm{top}}(T \times S) \leq h_{\mathrm{top}}(T) + h_{\mathrm{top}}(S).$$

On the other hand, if $M \subset X$, $N \subset Y$ are (n, ε)-separated, then $M \times N$ is (n, ε)-separated in $X \times Y$. Thus

$$\mathrm{sep}\,(n, \varepsilon, T \times S) \geq \mathrm{sep}\,(n, \varepsilon, T) \cdot \mathrm{sep}\,(n, \varepsilon, S),$$

after taking logarithms and limits we have $h_{\mathrm{top}}(T \times S) \geq h_{\mathrm{top}}(T) + h_{\mathrm{top}}(S)$. $\square$

Theorem 2.14. *Suppose that $X_1, \ldots, X_k \subset X$ are closed forward invariant sets such that $X = \bigcup_{i=1}^k X_i$. Then $h_{\mathrm{top}}(T) = \max\limits_{i=1,\ldots,k} h_{\mathrm{top}}(T|_{X_i})$.*

Proof. Every X_i is compact, because X is compact. Since every (n, ε)-separated set in X_i is (n, ε)-separated in X, we have $h_{\mathrm{top}}(T|_{X_i}) \leq h_{\mathrm{top}}(T)$. On the other hand, if B_i is (n, ε)-spanning in X_i, then $\bigcup_{i=1}^k B_i$ is (n, ε)-spanning in X. Thus $\mathrm{span}\,(n, \varepsilon, T) \leq \sum_{i=1}^k \mathrm{span}\,(n, \varepsilon, T|_{X_i}) \leq k \cdot \max\limits_{i=1,\ldots,k} \mathrm{span}\,(n, \varepsilon, T|_{X_i})$. Therefore

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathrm{span}\,(n, \varepsilon, T) \leq \limsup_{n \to \infty} \frac{\log k}{n} + \limsup_{n \to \infty} \frac{1}{n} \log \left(\max_{i=1,\ldots,k} \mathrm{span}\,(n, \varepsilon, T|_{X_i}) \right)$$

$$= \max_{i=1,\ldots,k} \limsup_{n \to \infty} \frac{1}{n} \log \left(\mathrm{span}\,(n, \varepsilon, T|_{X_i}) \right).$$

From this we obtain the desired inequality $h_{\mathrm{top}}(T) \leq \max\limits_{i=1,\ldots,k} h_{\mathrm{top}}(T|_{X_i})$. $\square$

We conclude this section with a statement of a theorem which we are not going to prove here (see Block and Coppel [1995], page 196).

Theorem 2.15. *Let (X, T) be a topological dynamical system and denote $X_\infty = \bigcap_{n=0}^\infty T^n(X)$. Then $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(T|_{X_\infty})$.*

2.3 Topological entropy for some examples

Theorem 2.16. *Let (X, T) be a topological dynamical system such that T is an isometry. Then $h_{\mathrm{top}}(T) = 0$.*

Proof. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Since for $x, y \in X$, $d(x, y) = d(T^n(x), T^n(y))$, it follows that $d_n = d$. Thus, $\mathrm{cov}\,(n, \varepsilon, T) = \mathrm{cov}\,(1, \varepsilon, T)$. This implies, since the value $\log \mathrm{cov}\,(1, \varepsilon, T) \in \mathbb{R}$ does not depend on n, that

$$h_\varepsilon(T) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{cov}\,(n, \varepsilon, T) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{cov}\,(1, \varepsilon, T) = 0.$$

Therefore $h_{\mathrm{top}}(T) = 0$. $\square$

Corollary 2.17. *The topological entropy of the rotation* $R_\alpha \colon S^1 \to S^1$ *from Example 1.8 is equal to* 0 *for any* $\alpha \in \mathbb{R}$.

Proof. The rotation is an isometry, so by the previous theorem, $h_{\text{top}}(R_\alpha) = 0$. $\square$

Claim 2.18. *Topological entropy of the shift* $\sigma \colon \Sigma_m \to \Sigma_m$ *from Example 1.10 is* $h_{\text{top}}(\sigma) = \log m$.

Proof. Let $l \in \mathbb{N}$ and $x \in \Sigma_m$. Then

$$B_n(x, m^{-l}) = \{ y \in \Sigma_m : x_i = y_i \text{ for } i = -l, -l+1, \ldots, l+n \}.$$

So if for every $\alpha = (\alpha_{-l}, \ldots, \alpha_{l+n}) \in \{0, \ldots, m-1\}^{2l+n+1}$ we define $x^\alpha \in \Sigma_m$ by

$$x_i^\alpha = \begin{cases} \alpha_i & \text{if } i \in \{-l, \ldots, l+n\}, \\ 0 & \text{otherwise}, \end{cases}$$

then $A = \{ x^\alpha : \alpha \in \mathcal{A}_m^{2l+n+1} \}$ is an (n, m^{-l})-separated set, $\#A = m^{2l+n+1}$. Therefore $\text{sep}\,(n, m^{-l}, \sigma) \geq m^{2l+n+1}$. Conversely, the set $\mathcal{B} = \{ B_n(x^\alpha, m^{-l}) : x^\alpha \in A \}$ covers Σ_m and $\text{diam}_n B_n(x^\alpha, m^{-l}) = m^{-l-1}$ for every $x^\alpha \in A$. Also $\#\mathcal{B} = m^{2l+n+1}$, therefore $\text{cov}\,(n, m^{-l}, \sigma) \leq m^{2l+n+1}$. It follows from Theorem 2.4 that in fact $\text{cov}\,(n, m^{-l}, \sigma) = \text{sep}\,(n, m^{-l}, \sigma) = m^{2l+n+1}$. Now we can see that

$$h_{\text{top}}(\sigma) = \lim_{l \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \text{cov}\,(n, m^{-l}, \sigma) = \lim_{l \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log m^{2l+n+1}$$

$$= \lim_{l \to \infty} \limsup_{n \to \infty} \frac{2l+n+1}{n} \log m = \lim_{l \to \infty} \log m = \log m.$$

$\square$

Corollary 2.19. *Topological entropy of the times-m map* $E_m \colon S^1 \to S^1$ *from Example 1.12 is* $h_{\text{top}}(E_m) = \log m$.

Proof. By Theorem 2.12 and Claim 1.13 we have $\log m = h_{\text{top}}(\sigma) \geq h_{\text{top}}(E_m)$.

On the other hand, for $0 < \varepsilon < 1/m$ and $n \in \mathbb{N}$, the ball $B_n(x, \varepsilon)$ is an interval of length $2\varepsilon/m^n$ centered at $x \in S^1$. Therefore if $A \subset S^1$ is (n, ε)-spanning, in other words if $\bigcup \{ B_n(a, \varepsilon) : a \in A \} = S^1$, then necessarily $\#A \geq m^n/(2\varepsilon)$, because S^1 has length 1. So

$$h_\varepsilon(E_m) = \limsup_{n \to \infty} \frac{1}{n} \log \text{span}\,(n, \varepsilon, E_m) \geq \limsup_{n \to \infty} \frac{-\log(2\varepsilon)}{n} + \log m = \log m$$

and $h_{\text{top}}(E_m) \geq \log m$. $\square$

Claim 2.20. *The topological entropy of the tent map* $S \colon [0, 1] \to [0, 1]$ *from Example 1.15 is* $h_{\text{top}}(S) = \log 2$.

Proof. This is a consequence of a more general Theorem 3.6 and the fact that S is topologically transitive by Claim 1.16. $\square$

Corollary 2.21. *The topological entropy of the map* $T_4 \colon [0, 1] \to [0, 1]$, $T_4(x) = 4x(x-1)$ *from Example 1.17 is* $h_{\text{top}}(T_4) = \log 2$.

Proof. This follows from the fact that T_4 is conjugate to the tent map S by Claim 1.18. $\square$

3. Topological entropy for noncompact metric spaces

3.1 Definition and properties

The definition of topological entropy we have used so far relies on the finiteness of the number $\operatorname{sep}(n, \varepsilon, T)$. This is guaranteed in the case of a compact metric space. Otherwise the number might be infinite even in very simple cases (consider identity on a space with infinite diameter). An approach that can be taken in this case is the following. For a noncompact space X, we consider its compact invariant subset $K \subset X$, and calculate the entropy of the original map restricted to this set. This will be a lower estimate of the entropy. Taking Theorem 2.15 into consideration, we can take only *strictly* invariant K and arrive at the following definition.

Definition 3.1. Let (X, T) be a topological dynamical system, where X is a metric space (not necessarily compact). Denote

$$\mathcal{K}(X, T) = \{K \subset X : K \text{ is compact and strictly invariant}\}.$$

Then the *topological entropy* of T is defined as

$$\operatorname{ent}(T) = \sup\{h_{\text{top}}(T|_K) : K \in \mathcal{K}(X, T)\}.$$

It is clear that for a compact space X, this definition coincides with the previous one. Indeed, if $K_1, K_2 \in \mathcal{K}(X, T)$, $K_1 \subset K_2$, then $h_{\text{top}}(K_1) \leq h_{\text{top}}(K_2)$, and since X_∞ is the largest element of $\mathcal{K}(X, T)$, the supremum is attained at X_∞, so $\operatorname{ent}(T) = h_{\text{top}}(T|_{X_\infty}) = h_{\text{top}}(T)$.

Remark. It is true that all the properties of topological entropy in Theorem 2.13 are also valid for the new definition. More precisely, if (X, T), (Y, S) are topological dynamical systems, then the following properties are satisfied (see Cánovas and Rodríguez [2005] for the proof):

(a) For every $m \in \mathbb{N}$, $\operatorname{ent}(T^m) = m \cdot \operatorname{ent}(T)$.

(b) If T is a homeomorphism, then for every $m \in \mathbb{Z}$, $\operatorname{ent}(T^m) = |m| \cdot \operatorname{ent}(T)$.

(c) Define $T \times S \colon X \times Y \to X \times Y$ as $(T \times S)(x, y) = (T(x), S(y))$. Then $\operatorname{ent}(T \times S) = \operatorname{ent}(T) + \operatorname{ent}(S)$.

Analogue of the property in Theorem 2.14 is also held. We recall the notation from Theorem 2.15: For $A \subset X$ compact invariant, denote $A_\infty = \bigcap_{n=1}^\infty T^n(A)$.

Theorem 3.2. *Suppose that $X_1, \ldots, X_k \subset X$ are closed forward invariant sets such that $X = \bigcup_{i=1}^k X_i$. Then $\operatorname{ent}(T) = \max_{i=1,\ldots,k} \operatorname{ent}(T|_{X_i})$.*

Proof. Since $\mathcal{K}(X_i, T|_{X_i}) \subset \mathcal{K}(X, T)$, we immediately have $\operatorname{ent}(T) \geq \operatorname{ent}(T|_{X_i})$. Conversely, for $K \in \mathcal{K}(X, T)$, we have $(K \cap X_i)_\infty \in \mathcal{K}(X_i, T|_{X_i})$, because $K \cap X_i$

is compact forward invariant. Thus from Theorem 2.14 we have that $h_{\text{top}}(T|_K) = \max_{i=1,\dots,k} h_{\text{top}}(T|_{K \cap X_i})$. Therefore, using Theorem 2.15 we see

$$
\begin{aligned}
\text{ent}(T) = \sup \left\{ \max_{i=1,\dots,k} h_{\text{top}}(T|_{K \cap X_i}) : K \in \mathcal{K}(X,T) \right\} = \\
\max_{i=1,\dots,k} \sup \left\{ h_{\text{top}}(T|_{(K \cap X_i)_\infty}) : K \in \mathcal{K}(X,T) \right\} \leq \\
\max_{i=1,\dots,k} \sup \left\{ h_{\text{top}}(T|_{K_i}) : K_i \in \mathcal{K}(X_i, T|_{X_i}) \right\} = \max_{i=1,\dots,k} \text{ent}(T|_{X_i}).
\end{aligned}
$$

$\square$

Remark. In the previous version of this theorem, that is in Theorem 2.14, the assumption that X_i be closed was necessary. Otherwise $h_{\text{top}}(f|_{X_i})$ would not be defined (because X_i would not be compact). In the new definition of entropy, $\text{ent}(T|_{X_i})$ is defined even when X_i is not closed in X. One thus may ask whether the previous theorem holds if we do not require X_i to be closed. In general this is not true, counterexample was given in Cánovas and Rodríguez [2005], Theorem 2.1 (b).

However, we can accomplish this at least for the case of uniformly continuous maps on a complete metric space by using an alternate definition of entropy. For a uniformly continuous map f on a (not necessarily complete) metric space X we have the continuous extension $\widehat{f} \colon \widehat{X} \to \widehat{X}$ of f, where $\widehat{X}$ is the completion of X. Then we define the new entropy of f as $\widehat{\text{ent}}(f) = \text{ent}(\widehat{f})$. Now for arbitrary forward invariant $X_1, \dots, X_k \subset X$ such that $X_1 \cup \dots \cup X_k = X$, we have $\widehat{\text{ent}}(f) = \max_{i=1,\dots,k} \widehat{\text{ent}}(f|_{X_i})$ by Theorem 3.2, because $\widehat{X_i}$ is the closure of X_i in $\widehat{X}$. In general $\widehat{\text{ent}}(X) \geq \text{ent}(X)$, where the inequality may be strict.

Follows a lemma that will be useful in the next section.

Lemma 3.3. *Let $K \in \mathcal{K}(X,T)$ and let $L \subset K$ be compact and invariant such that T is identity on $K \setminus L$. Then $h_{\text{top}}(T|_K) = h_{\text{top}}(T|_L)$*

Proof. Fix an $\varepsilon > 0$. Since $K \setminus L$ is totally bounded, there exists its finite cover with balls of radius ε, call C_ε the set of the centers of these balls. Then, since T is identity on $K \setminus L$, for every (n, ε)-spanning set A in L, the set $A \cup C_\varepsilon$ is (n, ε)-spanning in K. Therefore $\text{span}\,(n, \varepsilon, T|_K) \leq \text{span}\,(n, \varepsilon, T|_L) + \#C_\varepsilon$. Because C_ε does not depend on n, we have $h_\varepsilon(T|_K) \leq h_\varepsilon(T|_L)$.

The other inequality follows from the fact that if $B \subset L$ is (n, ε)-separated in L, then B is also (n, ε)-separated in K. Thus $\text{span}\,(n, \varepsilon, T|_K) \geq \text{span}\,(n, \varepsilon, T|_L)$, and consequently $h_{\text{top}}(f|_K) \geq h_{\text{top}}(f|_L)$. $\square$

Remark. The conclusion of the previous lemma also holds when $K \setminus L$ contains only 2-periodic points. This can be proven by a simple modification of the proof.

3.2 Piecewise affine maps on the real line

In this section, we take a look at generalizing known results about topological entropy of piecewise affine maps of an interval.

Definition 3.4. A map $f\colon [a,b] \to [a,b]$ is called *piecewise affine* if there exists a finite sequence $x_1, \ldots, x_n \in [a,b]$ such that $a = x_1 < \cdots < x_n = b$ and each of the maps $f|_{[x_i, x_{i+1}]}$, $i = 1, \ldots, n-1$ is affine.

A map $f\colon \mathbb{R} \to \mathbb{R}$ is called *piecewise affine* if there exists a finite sequence $x_1, \ldots, x_n \in \mathbb{R}$ such that $x_1 < \cdots < x_n$ and each of the maps $f|_{(-\infty, x_1]}$, $f|_{[x_n, \infty)}$, $f|_{[x_i, x_{i+1}]}$, $i = 1, \ldots, n-1$ is affine.

For a piecewise affine map f and point x distinct from all x_i, we will call $f'(x)$ the *slope* of f at x. The map f is said to be a *map of a constant slope* if the absolute value of the slope is the same at every point distinct from all x_i.

The composition of two piecewise affine maps is piecewise affine. A piecewise affine map $f\colon \mathbb{R} \to \mathbb{R}$ can be described by the numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and s_-, s_+, where x_i are as above, $f(x_i) = y_i$ for all $i = 1, \ldots, n$, $f'(t) = s_-$ for $t \in (-\infty, x_1)$ and $f'(t) = s_+$ for $t \in (x_n, \infty)$. Then the map f is

$$
f(t) = \begin{cases}
y_i & \text{if } t = x_i,\ i = 1 \ldots, n, \\
y_i + \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}(t - x_i) & \text{if } t \in (x_i, x_{i+1}),\ i = 1, \ldots, n-1, \\
y_1 + s_-(t - x_1) & \text{if } t \in (-\infty, x_1), \\
y_n + s_+(t - x_n) & \text{if } t \in (x_n, \infty).
\end{cases}
$$

Denote $y_{\max} = \max_{i=1,\ldots,n} y_i$, $y_{\min} = \min_{i=1,\ldots,n} y_i$. To f we assign the intervals $I_f = [a,b]$, $I_f^- = (-\infty, b]$, $I_f^+ = [a, \infty)$, where $a = \min\{x_1, y_{\min}\}$, $b = \max\{x_n, y_{\max}\}$. Notice that I_f is the smallest closed interval such that all (x_i, y_i) are in $I_f \times I_f$.
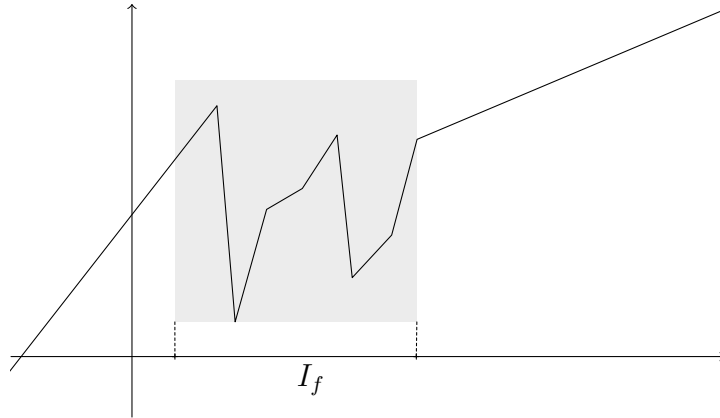

Figure 3.1: A piecewise affine map f and its interval I_f

Without loss of generality we may assume that $I_f = [a,b]$ is symmetric around 0, that is, $a = -b$. Otherwise we work with the function g, $g(t) = f\left(t + \frac{a+b}{2}\right) - \frac{a+b}{2}$, which is conjugate to f.

Let $K \in \mathcal{K}(\mathbb{R}, f)$. We will show that $K \subset \tilde{I}_f$ for some compact interval $\tilde{I}_f$ independent of K. Let us discuss several cases (See Figure 3.2 for illustration).

<u>Case 1</u>: $s_+ \geq 0, s_- \geq 0$. Here both intervals I_f^+, I_f^- are invariant. We will show that $K \subset \tilde{I}_f^-$, where $\tilde{I}_f^- = (-\infty, \tilde{b}]$ for some $\tilde{b} \geq b$. We will split this case into subcases.

<u>Case 1.1</u>: $s_+ = 1$, $f(b) = b$. In this special case we have $f|_{(b,\infty)} = \text{id}$. By using Lemma 3.3 we get $h_{\text{top}}(f|_K) = h_{\text{top}}(f|_{K \cap I_f^-})$, so without loss of generality we can assume that $K \subset I_f^-$.

<u>Case 1.2</u>: $s_+ \geq 1$, $f(b) \geq b$. This is either Case 1.1 or $f(t) > t$ for every $t > b$. In particular, $K \subset I_f^-$, because otherwise $\max K < f(\max K) \notin K$ and K would not be invariant.

<u>Case 1.3</u>: $1 \geq s_+ > 0$, $f(b) \leq b$. This is either Case 1.1 or $f(t) < t$ for every $t > b$. In particular, $K \subset I_f^-$, because otherwise $\max f(K) = f(\max K) < \max K \notin f(K)$ and K would not be strictly invariant. (The equality $\max f(K) = f(\max K)$ holds because f is increasing on (b, ∞).)

<u>Case 1.4</u>: $s_+ > 1$, $f(b) < b$. The map f has a fixed point $\tilde{b} = \frac{f(b) - s_+ b}{1 - s_+} > b$ and $f(t) > t$ for any $t > \tilde{b}$. Taking $\tilde{b}$ instead of b, $\tilde{I}_f^- = (-\infty, \tilde{b}]$ instead of I_f^-, this is the same as Case 1.2.

<u>Case 1.5</u>: $1 > s_+ > 0$, $f(b) \geq b$. The map f has a fixed point $\tilde{b} = \frac{f(b) - s_+ b}{1 - s_+} > b$ and $f(t) < t$ for any $t > \tilde{b}$. Taking $\tilde{b}$ instead of b, $\tilde{I}_f^+ = (-\infty, \tilde{b}]$ instead of I_f^+, this is the same as Case 1.3.

<u>Case 1.6</u>: $s_+ = 0$. Here $f(\mathbb{R}) \subset I_f^-$, so $K \subset I_f^-$.

Applying Cases 1.1–1.6 to the function $g(t) = -f(-t)$, we get that $K \subset \tilde{I}_f^+ = [\tilde{a}, \infty)$ for some $\tilde{a} \leq a$. Therefore $K \subset \tilde{I}_f = \tilde{I}_f^- \cap \tilde{I}_f^+ = [\tilde{a}, \tilde{b}]$.

<u>Case 2.1</u>: $s_+ \geq 0$, $s_- \leq 0$. Here $f(\mathbb{R}) \subset I_f^+$ and we can apply one of the cases 1.1–1.6, so we have $K \subset [a, \tilde{b}]$.

<u>Case 2.2</u>: $s_+ \leq 0$, $s_- \geq 0$. This is Case 2.2 for the function $g(t) = -f(-t)$.

<u>Case 3</u>: $s_+ \leq 0$, $s_- \leq 0$. Consider the piecewise affine map $f^2 = f \circ f$. Now for any t large enough (so that $t \notin I_{f^2}$, $f(t) \notin I_{f^2}$) we have $(f^2)'(t) = f'(f(t))f'(t) = s_+ s_-$. Therefore the slopes of f^2 on both the unbounded intervals are equal to $s_+ s_- \geq 0$. Now we can apply Case 1 (with the exception that in Case 1.1 we have to use the remark after Lemma 3.3 about 2-periodic points) to f^2 and get $K \subset I_{f^2} \cup I_f = \tilde{I}_f$.

We have thus proved that

$$\text{ent}(f) = \sup\left\{h_{\text{top}}(f|_K) : K \in \mathcal{K}(\mathbb{R}, f), K \subset \tilde{I}_f\right\},$$

which gives $\text{ent}(f) = h_{\text{top}}(f|_{(\tilde{I}_f)_\infty})$. So we denote $K = (\tilde{I}_f)_\infty$.

This will enable us to calculate the entropy of f by calculating the entropy of "f restricted to $\tilde{I}_f$". However, we cannot directly say $\text{ent}(f) = h_{\text{top}}(f|_{\tilde{I}_f})$, because $\tilde{I}_f$ may not be forward invariant for f. This happens in Case 1.2 when the inequality $f(\tilde{b}) \geq \tilde{b}$ is strict and so there is $c \in (\tilde{a}, \tilde{b})$ such that $f(t) > \tilde{b}$ for every $t \in (c, \tilde{b}]$. We can fix this by redefining f on $(c, \tilde{b}]$ to be constantly $\tilde{b}$. Symmetrically this can also happen at the left end of $\tilde{I}_f$. That is, there is $d \in (\tilde{a}, \tilde{b})$ such that $f(t) < \tilde{a}$ for $t \in [\tilde{a}, d)$. Similarly for Case 3. Thus we define $\tilde{f} : \tilde{I}_f \to \tilde{I}_f$ by

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } f(t) \in \tilde{I}_f, \\ \tilde{b} & \text{if } f(t) > \tilde{b}, \\ \tilde{a} & \text{if } f(t) < \tilde{a}. \end{cases}$$
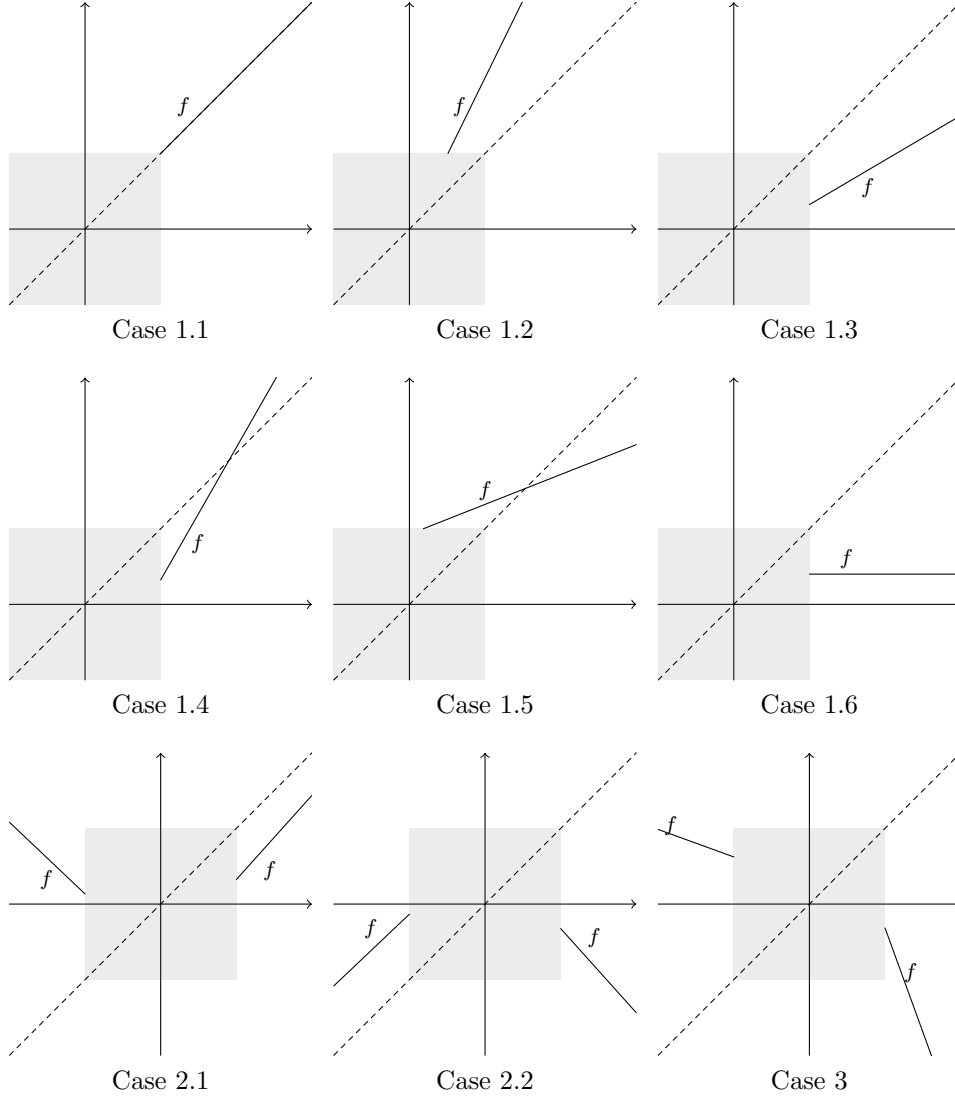
Figure 3.2: Cases in Claim 3.5. (Dashed line is the identity.)

Now the inequality $h_{\text{top}}(f|_K) \leq h_{\text{top}}(\tilde{f})$ is clear because $\tilde{f}$ is an extension of $f|_K$. Conversely, we have $\tilde{I}_f = K \cup \bigcup_{n=1}^{\infty} f^{-n}((c, \tilde{b}]) \cup \bigcup_{n=1}^{\infty} f^{-n}([\tilde{a}, d))$. Then $\bigcup_{n=1}^{\infty} \tilde{f}^n(\tilde{I}_f) = K \cup \{\tilde{a}, \tilde{b}\}$ and since entropy does not change after adding two points, this proves the converse inequality. We have proved the following result.

Claim 3.5. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be piecewise affine, let $\tilde{I}_f$ and $\tilde{f} \colon \tilde{I}_f \to \tilde{I}_f$ be as above. Then $\text{ent}(f) = h_{\text{top}}(\tilde{f})$.*

The following is a result for piecewise monotone interval maps given in Brucks and Bruin [2004], Theorem 9.5.1. A map $f \colon [a, b] \to [a, b]$ is *piecewise monotone* if there exists a finite sequence $x_1, \dots, x_n \in [a, b]$, $a = x_1 < \cdots < x_n = b$, such that each $f|_{[x_i, x_{i+1}]}$, $i = 1, \dots, n-1$ is monotone. In particular, every piecewise affine map is piecewise monotone.

Theorem 3.6. *Let $f \colon [a, b] \to [a, b]$ be a continuous piecewise monotone map such that $h_{\text{top}}(f) > 0$. Then there exists a semiconjugacy to a piecewise affine map $g \colon [a, b] \to [a, b]$ with a constant slope, where the slope of g is $e^{h_{\text{top}}(f)}$. Moreover, if f is topologically transitive, then the semicojugacy is a conjugacy.*

Similar result cannot be true for piecewise monotone maps on the real line, as the next example shows.

Example 3.7. Define a piecewise affine map $f\colon \mathbb{R} \to \mathbb{R}$ by setting $x_1 = 0$, $x_2 = 1/3$, $x_3 = 2/3$, $x_4 = 1$, $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, $y_4 = 1$, $s_+ = s_- = 1$. Then $I_f = [0, 1]$ and f is identity outside I_f. We know that $\mathrm{ent}(f) = h_{\mathrm{top}}(f|_{[0,1]})$, which by Theorem 3.6 is equal to the logarithm of its slope, $h_{\mathrm{top}}(f|_{[0,1]}) = \log 3$.

Suppose that there exists a semiconjugacy $\varphi\colon \mathbb{R} \to \mathbb{R}$ of f to a piecewise affine map $g\colon \mathbb{R} \to \mathbb{R}$. Since φ is continuous, the sets $\varphi((-\infty, 0))$, $\varphi((1, \infty))$ are connected and $\varphi([0, 1])$ is compact and in particular, it is bounded. Since $\mathbb{R} \setminus \{f(0), f(1)\}$ has two unbounded components, it follows that $\varphi((-\infty, 0))$, $\varphi((1, \infty))$ are the two unbounded intervals. In particular g has to be identity on these intervals, so g has slope 1 there. If g has a constant slope, then necessarily g is identity on the entire line and g has constant slope $1 \neq e^{\mathrm{ent}(f)}$.
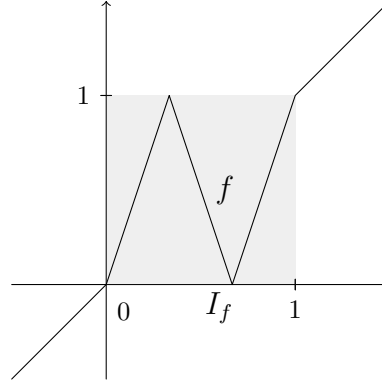

Figure 3.3: Counterexample for semiconjugacy to a map of a constant slope on the real line.

There remains the question whether for every piecewise affine map $f\colon \mathbb{R} \to \mathbb{R}$ there exists a semiconjugacy to some piecewise affine map g on a closed interval such that the slope of g is equal to $e^{\mathrm{ent}(f)}$.

The answer is positive in the case when $s_+, s_- > 0$. In this case the semiconjugacy to $\tilde{f}\colon \tilde{I}_f \to \tilde{I}_f$ can be chosen as $\varphi\colon \mathbb{R} \to \tilde{I}_f$, where

$$\varphi(x) = \begin{cases} \tilde{a} & \text{if } x < \tilde{a}, \\ \tilde{f}(x) & \text{if } x \in \tilde{I}_f, \\ \tilde{b} & \text{if } x > \tilde{b}. \end{cases}$$

The map φ is a semiconjugacy and it preserves entropy by Claim 3.5.

For other cases, the question remains unanswered.

Conclusion

Our discussion started by defining topological dynamical systems. We defined the notion of conjugacy, which says when two systems are the same from dynamical point of view. Then some examples of topological dynamical systems were given, with emphasis to those properties which are interesting in relation to topological entropy. In particular, we studied rotations and expanding maps on the circle, shifts on symbolic spaces, tent and quadratic maps. We have seen that some of these maps are conjugate.

We continued by defining topological entropy for compact metric spaces. This was done by defining metrics measuring the distance of the first n iterations of two points. We showed that the definition of topological entropy is independent of the particular choice of metric, given that the metric generates the same topology. An important result was that topological entropy is invariant under conjugacy. Then we calculated topological entropy for the examples shown in the beginning.

Next, we extended the previous definition to the realm of noncompact metric spaces. This was done by looking at entropies of strictly invariant compact subsets. In more detail we looked at piecewise affine maps and showed that entropy of piecewise affine map on the real line can be calculated by restricting it to a bounded interval. We concluded by remarking the known result about semiconjugacies to a map of a constant slope on the interval and discussed its version on the entire real line.

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