FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

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Characterization of functions vanishing at the boundary

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Study programme:	Mathematics
Study branch:	General Mathematics

Prague 2017

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I thank my supervisor Aleš Nekvinda for suggesting a perfect topic of the thesis, for his thorough and comprehensive guidance and plenty of consultation. I thank as well the consultant Luboš Pick for the constant supply of good advice. I would also like to thank my parents for their enormous support and patience.

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Abstract: Let $\Omega \subset \mathbb{R}^n$ be a domain with a moderate boundary regularity, $p \in (1, \infty)$ and let d be the distance function defined by $d(t) = \operatorname{dist}(t, \partial\Omega)$, $t \in \mathbb{R}^n$. Assume that u belongs to the Sobolev space $W^{1,p}(\Omega)$. A classical result states that $u \in W_0^{1,p}(\Omega)$ if and only if $\frac{u}{d} \in L^p(\Omega)$ and $\nabla u \in L^p(\Omega)$. This fact has been several times consecutively refined, and each time the required condition $\frac{u}{d} \in L^p(\Omega)$ was relaxed to a weaker one. The first such improvement shows that the condition $\frac{u}{d} \in L^{p,\infty}(\Omega)$ is sufficient. In the next such result the condition $\frac{u}{d} \in L^1(\Omega)$ was considered. Moreover, this result was extended to Sobolev spaces of higher order. In this thesis we improve the previous results in the case when $n = 1$ and Ω is an open interval I. In our principal result we prove that $u \in W_0^{1,p}(I)$ if and only if $\frac{u}{d} \in L^{1,p}(I)$ and $u' \in L^p(I)$.

Keywords: Banach function spaces, Lorentz spaces, Sobolev spaces

Contents

Introduction

The theory of Sobolev spaces is widely used in the theory of partial differential equations. For a solution to the Dirichlet problem, especially important spaces are $W^{1,p}$ and $W_0^{1,p}$. The classical definition describes the space $W_0^{1,p}$ as the closure of smooth functions with a compact support in $W^{1,p}$.

It is proved in [1, Theorem V.3.4] that for certain regular domains $\Omega \subset \mathbb{R}^n$ the following equivalence holds: $u \in W_0^{1,p}(\Omega)$ if and only if $\frac{u}{d} \in L^p(\Omega)$ and $\nabla u \in L^p(\Omega)$, where the function $d(t)$ is defined as the distance of t from the boundary of Ω. This result was later improved. In [2] the same conclusion was shown under a weaker condition, namely: $u \in W_0^{1,p}(\Omega)$ if and only if $\frac{u}{d} \in L^{p,\infty}(\Omega)$ and $\nabla u \in L^p(\Omega)$. In [3], the assumption was further relaxed. It was shown that $u \in W_0^{1,p}(\Omega)$ if and only if $\frac{u}{d} \in L^1(\Omega)$ and $\nabla u \in L^p(\Omega)$. This result was extended in [4] to Sobolev spaces of higher order. More precisely, it is proved there that $u \in W_0^{k,p}(\Omega)$ if and only if $\frac{u}{d^k} \in L^p(\Omega)$ and $|D^k u| \in L^p(\Omega)$, where $D^k u$ denotes the vector of all weak derivatives of the order k.

The main aim of this thesis is to prove that in the case when $n = 1$ and Ω is an interval $I \subset \mathbb{R}$, a function u belongs to $W_0^{1,p}(I)$ if and only if $\frac{u}{d} \in L^{1,p}(I)$ and $u' \in L^p(I)$. We believe the method of the proof developed in this thesis will enable us to obtain an analogous result for $\Omega \in \mathbb{R}^n$. This will be done in our future research.

1. Preliminaries

In this chapter we give a survey of concepts and results from functional analysis and notation that will be used in this thesis. All this background material can be found in various books and articles that will be cited when appropriate.

Let μ be the one-dimensional Lebesgue measure. For $\Omega \subset \mathbb{R}$ let us denote by $\mathscr{M}(\Omega)$ the set of all μ-measurable real functions on Ω and by $\mathscr{M}_+(\Omega)$ the set of all μ-measurable functions on Ω whose values are nonnegative and finite μ-a.e. in Ω. We denote by χ_E the characteristic function of a set E.

If $m, n \colon \mathscr{M}(\Omega) \to [0, \infty]$, we write $m(u) \lesssim n(u)$ if there exists a positive constant c independent of u such that $m(u) \le c \cdot n(u)$.

1.1 Definition and background results on function spaces

Our goal is to prove some relations between function spaces. However, first we need to define several fundamental function spaces that will be used in thesis and specify certain relations between them.

Lemma 1.1. ([5, Section 23.4]) *Let I be an interval, $g \in L^1(I)$ and let f be an indefinite Lebesgue integral of g on I. Then f is absolutely continuous (a fact which we denote by $f \in AC(I)$) and $f' = g$ almost everywhere.*

In what follows we shall denote by u' the weak derivative of a given function u.

Theorem 1.2. *Let I be an interval and let u' be the weak derivative of u. Let $u' \in L^1(I)$. Then there exists a function u_0 such that $u = u_0$ μ-a.e. and u_0 is absolutely continuous on I.*

Proof. Let $I = (a, b)$, $a, b \in \mathbb{R}$. Following the definition of the weak derivative, we obtain

$$\int_a^b u(t)\varphi'(t)\,dt = -\int_a^b u'(t)\varphi(t)\,dt$$

for each test function φ from $C_0^\infty(I)$. Using the Fubini theorem, we have

$$\begin{aligned}\int_a^b \int_a^t u'(s)\,ds\varphi'(t)\,dt &= \int_a^b u'(s) \int_s^b \varphi'(t)\,dt\,ds \\ &= \int_a^b u'(s)(-\varphi(s))\,ds = \int_a^b u(s)\varphi'(s)\,ds\end{aligned}$$

for each test function φ with support in I. Therefore by Theorem 5.49 from [6] there exists $c \in \mathbb{R}$ such that $\int_a^t u'(s)\,ds + c = u(t)$ μ-a.e. Denote $u_0 = \int_a^t u'(s)\,ds + c$. As a consequence of Lemma 1.1, u_0 is absolutely continuous on I. □

Remark 1.3. As a consequence of Theorem 1.2, every function u satisfying $u' \in L^1(I)$ has a continuous representant on I.

Definition 1.4. ([7, Definition 1.1]) Let (Ω, μ) be a subspace of $(\mathbb{R}, \mu)$. We say that a function $\rho : \mathscr{M}_+(\Omega) \to [0, \infty]$ is a *Banach function norm* if, for all f, g

and $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{M}_+(\Omega)$, for every $\lambda \geq 0$ and for all μ-measurable subsets E of Ω, the following five properties are satisfied:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(\lambda f) = \lambda\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ μ-a.e. in $\Omega \Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \nearrow f$ μ-a.e. in $\Omega \Rightarrow \rho(f_n) \nearrow \rho(f)$;

(P4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$

(P5) $\mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$ for some constant $C_E \in (0, \infty)$ possibly depending on E and ρ but independent of f.

Definition 1.5 (BFS)**.** ([7, Definition 1.3]) Let ρ be a Banach function norm. We then say that the set $X = X(\rho)$ of those functions in $\mathscr{M}(\Omega)$ for which $\rho(|f|) < \infty$ is a *Banach function space*. For each $f \in X$ we then define $\|f\|_X := \rho(|f|)$.

We will work even with more general spaces, where conditions (P4) and (P5) for Banach function norm are omitted.

Definition 1.6 (GBFS)**.** Let (Ω, μ) is a subspace of $(\mathbb{R}, \mu)$. Let ρ be a function on $\mathscr{M}_+(\Omega)$ satisfying conditions (P1), (P2) and (P3) in Definition 1.4. We then say that the set $X = X(\rho)$ of those functions in $\mathscr{M}(\Omega)$ for which $\rho(|f|) < \infty$, is a *generalized Banach function space.* For each $f \in X$ we then define $\|f\|_X := \rho(|f|)$.

Theorem 1.7. ([8, Lemma 2.5]) *Let $(X; \|\cdot\|_X)$ be a generalized Banach function space. Assume that $f_n \in X$ and $\sum_{k=1}^{\infty} \|f_k\|_X < \infty$. Then $\sum_{k=1}^{\infty} f_k$ converges to a function f in X and $\|f\|_X \leq \sum_{k=1}^{\infty} \|f_k\|_X$. Consequently, X is complete and, as such, a Banach space.*

Definition 1.8 (A continuous embedding)**.** ([9, Definition 1.15.5]) Let X, Y be two quasinormed linear spaces and let $X \subset Y$. We define the identity operator Id from X into Y as the operator which maps every element $u \in X$ onto itself: $\mathrm{Id}(u) = u$, regarded as an element of Y. We say that the space X is *continuously embedded* into the space Y if the identity operator is continuous, that is, if there exists a constant $c > 0$ such that

$$\|u\|_Y \leq c\,\|u\|_X \text{ for every } u \in X.$$

We shall call the operator Id the *embedding operator* from X to Y.

Definition 1.9 (Lebesgue space)**.** ([9, Notation 3.2.2 and Definition 3.10.2]) Let $p \in [1, \infty)$. Let Ω be a Lebesgue measurable subset of $\mathbb{R}$. We denote by $L^p(\Omega)$ the set of all real-valued measurable functions f defined almost everywhere on Ω and such that

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

is finite.

We denote by $L^\infty(\Omega)$ the set of all real-valued measurable functions f defined almost everywhere on Ω and such that

$$\|f\|_{L^\infty(\Omega)} := \inf_{\mu(E)=0} \{ \sup_{x \in \Omega \setminus E} |f(x)| \}$$

is finite.

Lemma 1.10. ([9, Lemma 3.2.3 and Exercise 6.1.18]) *Let* $p \in [1, \infty]$. *Then* $L^p(\Omega)$ *endowed with* $\|f\|_{L^p(\Omega)}$ *is a Banach function space.*

Definition 1.11. ([9, Definition 7.1.6]) Let f be a measurable function on $\Omega \subset \mathbb{R}$. Then the function $f^* : [0, \infty) \to [0, \infty)$ defined by

$$f^*(t) := \inf\{\lambda : \mu(\{x \in \Omega : |f(x)| > \lambda\}) \leq t\}, \quad t \in [0, \infty),$$

is called a *nonincreasing rearrangement* of f.

Definition 1.12 (Lorentz space)**.** ([9, Definition 8.1.1]) Assume that $0 < p \leq \infty$ and $0 < q \leq \infty$, $\Omega \subset \mathbb{R}$. The *Lorentz space* $L^{p,q}(\Omega)$ is the collection of all measurable f such that $\|f\|_{L^{p,q}(\Omega)} \leq \infty$, where

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty \\ \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

Remark 1.13. The functional $\|\cdot\|_{L^{1,p}(\Omega)}$ is not a norm, but it is a quasinorm, and there is no norm equivalent to it. (We recall that a quasinorm constitutes a weaker concept than that of a norm in the sense that the triangle inequality is replaced with

$$\|u + v\|_{L^{1,p}(\Omega)} \leq c\left(\|u\|_{L^{1,p}(\Omega)} + \|v\|_{L^{1,p}(\Omega)}\right)$$

for each $u, v \in L^{1,p}(\Omega)$ and some $c \geq 1$.)

Definition 1.14 (Sobolev spaces)**.** ([10, Section 3.1]) Let $\Omega \subset \mathbb{R}$. We define the functional $\|\cdot\|_{W^{m,p}(\Omega)}$, where m is a nonnegative integer, $1 \leq p \leq \infty$ and $D^\alpha u$ is the weak derivative of order α of a function u, as follows:

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{0 \leq \alpha \leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty,$$
$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{0 \leq \alpha \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}$$

for every function u for which the right side is defined. We define the set

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq \alpha \leq m\}$$

and the set $W_0^{m,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Lemma 1.15. [10, Section 3.1] *The sets* $W^{m,p}(\Omega)$ *and* $W_0^{m,p}(\Omega)$ *equipped with the functional* $\|\cdot\|_{W^{m,p}(\Omega)}$ *are Banach spaces.*

The spaces from Lemma 1.15 are called *Sobolev spaces.*

1.2 Mean continuity and mollifiers

In this section we will point out certain useful properties of the space L^p. The facts listed in this section can be found e.g. in [9, parts 3.3. and 3.4].

Convention 1.16. We will, when convenient, consider a function f defined almost everywhere on $\Omega \subset \mathbb{R}$ to be extended outside of Ω by zero. We thus obtain a function F defined for almost all $x \in \mathbb{R}$ by

$$F(x) := \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Instead of $F(x)$ we shall often simply write $f(x)$ also for $x \notin \Omega$.

Definition 1.17 (Mean continuity)**.** Let $p \in [1, \infty)$, $\Omega \subset \mathbb{R}$ and $f \in L^p(\Omega)$. The function f is said to be *p-mean continuous* if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $\delta \in \mathbb{R}$, $\delta > 0$ such that

$$\left(\int_\Omega |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}} < \varepsilon$$

provided $h \in \mathbb{R}$, $|h| < \delta$.

Theorem 1.18. *Let $p \in [1, \infty)$ and let Ω be a nonempty open subset of $\mathbb{R}$ having finite measure. Then any function $f \in L^p(\Omega)$ is p-mean continuous.*

Definition 1.19. We will denote by S the nonempty set of all functions φ_0 satisfying
(i) $\varphi_0 \in C_0^\infty(\mathbb{R})$,
(ii) $\varphi_0(x) \geq 0$ for all $x \in \mathbb{R}$,
(iii) $\int_\mathbb{R} \varphi_0(x) \, dx = 1$,
(iv) $\operatorname{supp} \varphi_0 = \{x \in \mathbb{R}; |x| \leq 1\}$.

Definition 1.20. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, $\Omega \subset \mathbb{R}$ and let $\varphi_0 \in S$. For $u \in L^1(\Omega)$, set

$$(R_\varepsilon u)(x) := \frac{1}{\varepsilon} \int_\Omega \varphi_0 \left(\frac{x-y}{\varepsilon} \right) u(y) \, dy.$$

The mapping R_ε is called a *mollifier*.

Theorem 1.21. *Let $p \in [1, \infty)$ and let Ω be a nonempty bounded open subset of $\mathbb{R}$. Let $u \in L^p(\Omega)$. Then:*
(i) $R_\varepsilon u \in C^\infty(\mathbb{R})$;
(ii) $\lim_{\varepsilon \to 0+} \|R_\varepsilon u - u\|_{L^p(\Omega)} = 0$.

1.3 Inequalities

In this section we state some known inequalities which we will use in the thesis. We include proofs only for those which are slightly modified.

Theorem 1.22 (Hölder inequality)**.** [9, Theorem 3.1.6] *Let $p \in (1, \infty)$, $\Omega \subset \mathbb{R}$ and let $p' = p/(p-1)$ be the conjugate Lebesgue index of p, let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$\left| \int_\Omega f(x)g(x)dx \right| \leq \int_\Omega |f(x)g(x)| \, dx \leq \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^{p'} \, dx \right)^{\frac{1}{p'}}.$$

Theorem 1.23 (Hardy inequality)**.** ([11, Theorem 6.8.7]) *Let $a, b \in (-\infty, \infty)$, $a < b$, $u \in L^p(a,b)$ and let $p \in (1,\infty)$. Then*

$$\int_a^b \left(\frac{1}{t-a}\int_a^t |u(s)|\,ds\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_a^b |u(x)|^p\,dx,$$
$$\int_a^b \left(\frac{1}{b-t}\int_t^b |u(s)|\,ds\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_a^b |u(x)|^p\,dx.$$

Theorem 1.24 (Jensen inequality)**.** ([9, Theorem 4.2.11]) *Let Φ be a convex function on $\mathbb{R}$. Let $t_1, ..., t_n \in \mathbb{R}$ and let $\alpha_1, ..., \alpha_n$ be positive numbers. Then*

$$\Phi\left(\frac{\alpha_1 t_1 + ... + \alpha_n t_n}{\alpha_1 + ... + \alpha_n}\right) \le \frac{\alpha_1 \Phi(t_1) + ... + \alpha_n \Phi(t_n)}{\alpha_1 + ... + \alpha_n}.$$

Theorem 1.25 (Jensen inequality for infinitely many terms)**.** *Let Φ be a convex function on $\mathbb{R}$. Let $t_n \in \mathbb{R}$, $n \in \mathbb{N}$, and let α_n be a positive number for each $n \in \mathbb{N}$. Then*

$$\Phi\left(\frac{\sum_{n=1}^\infty \alpha_n t_n}{\sum_{n=1}^\infty \alpha_n}\right) \le \frac{\sum_{n=1}^\infty \alpha_n \Phi(t_n)}{\sum_{n=1}^\infty \alpha_n}.$$

Proof. By the Jensen inequality (Theorem 1.24) we have

$$\Phi\left(\frac{\alpha_1 t_1 + ... + \alpha_n t_n}{\alpha_1 + ... + \alpha_n}\right) \le \frac{\alpha_1 \Phi(t_1) + ... + \alpha_n \Phi(t_n)}{\alpha_1 + ... + \alpha_n}.$$

Now, Φ is a convex function and therefore also continuous. Letting $n \to \infty$ we have easily

$$\Phi\left(\frac{\sum_{n=1}^\infty \alpha_n t_n}{\sum_{n=1}^\infty \alpha_n}\right) \le \frac{\sum_{n=1}^\infty \alpha_n \Phi(t_n)}{\sum_{n=1}^\infty \alpha_n}.$$

□

Lemma 1.26. *Let $a, b \in \mathbb{R}$. Then for each $p \in [1,\infty)$ we have*

$$|a+b|^p \le 2^{p-1}(|a|^p + |b|^p).$$

Proof. Using the Jensen inequality (Theorem 1.24) for a convex function $|\cdot|^p$, we get

$$\frac{|a+b|^p}{2^p} \le \frac{|a|^p + |b|^p}{2},$$

and in turn

$$|a+b|^p \le 2^{p-1}(|a|^p + |b|^p).$$

□

2. Basic definitions and assertions

2.1 Definitions of used function spaces

Definition 2.1 ($C^\infty_{\{a\}}$)**.** Let (a,b), $a,b\in\mathbb{R}$, $a<b$, be an open interval. By the expression $C^\infty_{\{a\}}(a,b)$ we denote the set of all functions defined on [a,b] where each of functions has derivatives of any order on $[a,b]$ and its support is a subset of $[a+\varepsilon,b]$ for some $\varepsilon>0$. $C^\infty_{\{b\}}(a,b)$ is defined analogously.

Definition 2.2 ($W^{1,p}_{\{a\}}$)**.** Let (a,b), $a,b\in\mathbb{R}$, $a<b$, be an open interval. By the expression $W^{1,p}_{\{a\}}(a,b)$ we denote the set of all functions u defined on (a,b), such that for each u there exists a sequence $\{u_n\}_{n=1}^\infty$, $u_n\in C^\infty_{\{a\}}(a,b)$, satisfying

$$\lim_{n\to\infty}\|u_n-u\|_{W^{1,p}(a,b)}=0.$$

$W^{1,p}_{\{b\}}(a,b)$ is defined analogously.

Definition 2.3 ((w)-$W^{1,p}$)**.** Let $p\in(1,\infty)$. Let us define the function space (w)-$W^{1,p}$ as follows: a function $u\in\mathscr{M}(0,1)$ is an element of (w)-$W^{1,p}$ if and only if it satisfies

- $\frac{u(t)}{t}\in L^{1,p}(0,1)$,
- $u'(t)\in L^p(0,1)$,

and the quasinorm is defined by $\|u\|_{(w)\text{-}W^{1,p}}:=\left\|\frac{u(t)}{t}\right\|_{L^{1,p}(0,1)}+\|u'\|_{L^p(0,1)}$.

Definition 2.4 (T_p)**.** Let $p\in(1,\infty)$ and $\{I_n\}$ be a sequence of intervals, $I_n:=(\frac{1}{2^n},\frac{1}{2^{n-1}})$. Let us define the function space T_p as follows: a function $u\in\mathscr{M}(0,1)$ is an element of T_p if $\|u\|_{T_p}<\infty$, where

$$\|u\|_{T_p}:=\left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p+\int_0^1|u'(t)|^p\,dt\right)^{\frac{1}{p}}$$

is the norm of the space T_p.

Both the above definitions are corect, (w)-$W^{1,p}$ and T_p satisfy the properties of a vector space, and $\|u\|_{(w)\text{-}W^{1,p}}$ satisfies the properties of a quasinorm. In the next theorem, we shall prove that $\|u\|_{T_p}$ is a norm.

Theorem 2.5. *If $p\in(1,\infty)$, then the functional $\|\cdot\|_{T_p}$ is a norm.*

Proof. Obviously, $\|\cdot\|_{T_p}$ is positively homogeneous and

$$\|u\|_{T_p}=0\Leftrightarrow(u=0\text{ almost everywhere}).$$

Let us focus on the triangle inequality.

Let us define the functional $\|\cdot\|_{A_p}$ at an appropriate function u by

$$\|u\|_{A_p} = \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}. \tag{2.1}$$

Then for functions u, v we have

$$\begin{aligned}\|u+v\|_{A_p} &= \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)+v(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt + \int_{I_n}\frac{|v(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

By the fact that $\|\cdot\|_{\ell^p}$ is a norm we have

$$\begin{aligned}\|u+v\|_{A_p} &\le \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|v(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}\\ &= \|u\|_{A_p} + \|v\|_{A_p}.\end{aligned}$$

Now, following the fact that $\|\cdot\|_{L^p}$ is a norm we compute

$$\begin{aligned}\|u+v\|_{T_p} &= \left(\|u+v\|_{A_p}^p + \|u'+v'\|_{L^p}^p\right)^{\frac{1}{p}}\\ &\le \left((\|u\|_{A_p} + \|v\|_{A_p})^p + (\|u'\|_{L^p} + \|v'\|_{L^p})^p\right)^{\frac{1}{p}}\\ &= \left\|\left(\|u\|_{A_p} + \|v\|_{A_p}, \|u'\|_{L^p} + \|v'\|_{L^p}, 0, \ldots\right)\right\|_{\ell^p}\\ &\le \left(\|u\|_{A_p}^p + \|u'\|_{L^p}^p\right)^{\frac{1}{p}} + \left(\|v\|_{A_p}^p + \|v'\|_{L^p}^p\right)^{\frac{1}{p}} = \|u\|_{T_p} + \|v\|_{T_p},\end{aligned}$$

which completes the proof. □

Remark 2.6. By Remark 1.3, elements of (w)-$W^{1,p}$, T^p and $W^{1,p}_{\{0\}}(0,1)$ have continuous representants on $(0,1)$. Throughout the following text we will always work with the continuous representants.

2.2 Basic properties of T_p

Lemma 2.7. *Let us denote by A_p the space of functions u such that $\|u\|_{A_p} < \infty$, where $\|\cdot\|_{A_p}$ is the norm defined in* (2.1)*. Then A_p is a generalized Banach function space.*

Proof. We shall prove that there exists a function ρ satisfying conditions (P1), (P2) and (P3) of the Definition 1.4 such that

$$\rho(|u|) = \|u\|_{A_p} = \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}.$$

Take

$$\rho(u) = \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p\right)^{\frac{1}{p}}. \tag{2.2}$$

Then ρ satisfies (P1), as was proved above. The condition (P2) is an obvious consequence of (P3). Let us show that (P3) holds.

Let $\{u_N\}$ be a sequence of A_p functions such that $0 \leq u_N \nearrow u$. Then also $0 \leq \frac{u_N(t)}{t} \nearrow \frac{u(t)}{t}$. Since L^1 is a Banach function space,

$$\int_{I_n} \frac{|u_N(t)|}{t} dt \nearrow \int_{I_n} \frac{|u(t)|}{t} dt \qquad \text{for each } n \in \mathbb{N}.$$

Therefore and since ℓ^p is a Banach function space,

$$\left(\sum_{n=1}^{\infty} \left(\int_{I_n} \frac{|u_N(t)|}{t} dt \right)^p \right)^{\frac{1}{p}} \nearrow \left(\sum_{n=1}^{\infty} \left(\int_{I_n} \frac{|u(t)|}{t} dt \right)^p \right)^{\frac{1}{p}}.$$

Consequently,

$$\rho(u_N) = \|u_N\|_{A_p} \nearrow \|u\|_{A_p} = \rho(u).$$

This proves that A_p is a generalized Banach function space and, by the Theorem 1.7, it is complete. □

Remark 2.8. The function ρ defined in (2.2) does not satisfy the condition (P4) for $E = (0,1)$, the space A_p is not a Banach function space.

Theorem 2.9. *The space T_p defined in 2.4 is a Banach space.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space T_p, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \exists n_0 \in \mathbb{N}\ \forall n, m \in \mathbb{N}, n, m > n_0 :$$

$$\|u_n - u_m\|_{T_p} = \left(\sum_{N=1}^{\infty} \left(\int_{I_N} \frac{|u_n(t) - u_m(t)|}{t} dt \right)^p + \int_0^1 |u_n'(t) - u_m'(t)|^p \, dt \right)^{\frac{1}{p}} < \varepsilon.$$

Hence

$$\|u_n - u_m\|_{A_p} = \left(\sum_{N=1}^{\infty} \left(\int_{I_N} \frac{|u_n(t) - u_m(t)|}{t} dt \right)^p \right)^{\frac{1}{p}} < \varepsilon$$

and

$$\|u_n' - u_m'\|_{L^p} = \left(\int_0^1 |u_n'(t) - u_m'(t)|^p \, dt \right)^{\frac{1}{p}} < \varepsilon.$$

By Lemma 2.7, A_p is a Banach space. Since $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A_p, it converges in A_p. Denote $u := \lim_{n\to\infty} u_n$. The sequence $\{u_n'\}_{n=1}^{\infty}$ is a Cauchy sequence in a Banach space L^p, and so it is convergent in L^p. Denote $v := \lim_{n\to\infty} u_n'$.

The function u_n' is a weak derivative of u_n so for each $n \in \mathbb{N}$ and each test function φ we have $\int u_n(t)\varphi'(t)\, dt = -\int u_n'(t)\varphi(t)\, dt$. Passing to a limit on both sides we get $\int u(t)\varphi'(t)\, dt = -\int v(t)\varphi(t)\, dt$. Therefore $v = u'$ and the function u is an element of T_p. The sequence $\{u_n\}_{n=1}^{\infty}$ is convergent in T_p, hence T_p is a Banach space. □

3. An embedding of weak Sobolev space into space T_p

3.1 A useful inequality

Lemma 3.1. *Let $a, b \in \mathbb{R}$, $p \in (1, \infty)$. There exists a positive constant c such that, for each $u \in AC(a, b)$, there holds*

$$\int_a^b |u(x)|^p \, dx \le c \cdot \left(\inf_{x \in (a,b)} |u(x)|^p + \int_a^b |u'(x)|^p \, dx \right)$$

and $c = 2^{p-1}(b-a) \cdot \max\{(b-a)^{p-1}, 1\}$.

Proof. Let $x, y \in (a, b)$. Then, using the Newton-Leibniz formula and applying the triangle inequality, we get

$$|u(x)| - |u(y)| \le |u(x) - u(y)| = \left| \int_{\min\{x,y\}}^{\max\{x,y\}} u'(t)dt \right| \le \int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)| \, dt.$$

Thus, using Lemma 1.26

$$|u(x)|^p \le 2^{p-1} \left(\left(\int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)| \, dt \right)^p + |u(y)|^p \right)$$

and integrating the previous inequality over (a, b) we have

$$\int_a^b |u(x)|^p \, dx \le 2^{p-1} \left(\int_a^b \left(\int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)| \, dt \right)^p dx + (b-a) \, |u(y)|^p \right). \tag{3.1}$$

Using the Hölder inequality we obtain

$$\begin{aligned}
&\int_a^b \left(\int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)| \, dt \right)^p dx \\
&\le \int_a^b \left(\int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)|^p \, dt \right) \left(\int_{\min\{x,y\}}^{\max\{x,y\}} 1^{p'} dt \right)^{\frac{p}{p'}} dx \\
&\le \int_a^b \left(\int_{\min\{x,y\}}^{\max\{x,y\}} |u'(t)|^p \, dt \right) |y-x|^{\frac{p}{p'}} \, dx \\
&\le \left(\int_a^b |u'(t)|^p \, dt \right) \left(\int_a^b |y-x|^{\frac{p}{p'}} \, dx \right) \le \left(\int_a^b |u'(t)|^p \, dt \right) (b-a)^{\frac{p}{p'}+1}.
\end{aligned}$$

Together with (3.1), this gives

$$\int_a^b |u(x)|^p \, dx \le 2^{p-1} \left((b-a)^p \left(\int_a^b |u'(t)|^p \, dt \right) + (b-a) \, |u(y)|^p \right).$$

Since the last inequality holds for each $y \in (a, b)$, then

$$\int_a^b |u(x)|^p \, dx \le c \cdot \left(\inf_{y \in (a,b)} |u(y)|^p + \int_a^b |u'(t)|^p \, dt \right),$$

where $c = 2^{p-1}(b-a) \cdot \max\{(b-a)^{p-1}, 1\}$.

□

3.2 The embedding

Lemma 3.2. *Let $\{I_n\}$ be a sequence of intervals, $I_n := (\frac{1}{2^n}, \frac{1}{2^{n-1}})$. Then there exists a positive constant c such that for each $u \in (w)$-$W^{1,p}$ and $n \in \mathbb{N}, n > 1$, it holds that*

$$\left(\int_{I_n} \frac{|u(t)|}{t} dt\right)^p \leq c \left(\int_{I_n} t^{p-1} \left[\left(\frac{u(\cdot)}{\cdot}\right)^* (t)\right]^p dt + \|u'\|^p_{L^p(I_n \cup I_{n-1})}\right).$$

Proof. Fix $u \in (w)$-$W^{1,p}$ and $n \in \mathbb{N}$. For brevity, let us denote $g(t) := \frac{u(t)}{t}$. By the assumption $u \in (w)$-$W^{1,p}$,

$$\int_0^1 t^{p-1} (g^*(t))^p \, dt < \infty, \qquad \int_0^1 |u'(t)|^p \, dt < \infty.$$

Let us denote

$$a_n^p := \int_{I_n} t^{p-1} (g^*(t))^p \, dt, \tag{3.2}$$

then $\sum_{n=1}^{\infty} a_n^p < \infty$.

We claim that there exists a constant $c_1 \in [1, 2^{p-1}]$ such that

$$a_n^p = c_1 \cdot |I_n|^{p-1} \int_{I_n} (g^*(t))^p \, dt.$$

Indeed, this follows from

$$\left(\frac{1}{2^n}\right)^{p-1} \int_{I_n} (g^*(t))^p \, dt \leq a_n^p \leq \left(\frac{1}{2^{n-1}}\right)^{p-1} \int_{I_n} (g^*(t))^p \, dt$$

and $|I_n| = \frac{1}{2^n}$.

As a consequence of the first mean value theorem for definite integral there exists $t_n \in I_n$ that

$$a_n^p = c_1 \cdot |I_n|^{p-1} \int_{I_n} (g^*(t))^p \, dt = c_1 \cdot |I_n|^p \, g^*(t_n)^p$$

and

$$g^*(t_n) = \frac{a_n}{\sqrt[p]{c_1} \, |I_n|}. \tag{3.3}$$

Now, let us show that there exists $s_n \in I_{n-1}$ such that

$$g(s_n) \leq g^*(t_n). \tag{3.4}$$

Assume the contrary. It means that $g(t) > g^*(t_n)$ holds for all $t \in I_{n-1}$. Then for some $\varepsilon > 0$

$$\mu\{t \in (0,1) : |g(t)| > g^*(t_n) + \varepsilon\} \geq |I_{n-1}| = \sum_{m=n}^{\infty} |I_m| > t_n,$$

which contradicts the definition of $g^*(t_n) := \inf\{\lambda : \mu\{t \in (0,1) : |g(t)| > \lambda\} \leq t_n\}$. This proves (3.4). Using (3.4) we obtain an inequality

$$u(s_n) \leq s_n \cdot g^*(t_n). \tag{3.5}$$

Now let us regard $t \in I_n \cup I_{n-1}$. Easily we have

$$\frac{1}{4} < \frac{t}{s_n} < 2. \tag{3.6}$$

Using the Newton-Leibniz formula and applying (3.5) we get

$$|u(t)| \leq \left| \int_{\min\{t,s_n\}}^{\max\{t,s_n\}} u'(x)dx \right| + |u(s_n)| \leq \int_{\min\{t,s_n\}}^{\max\{t,s_n\}} |u'(x)|\, dx + s_n \cdot g^*(t_n).$$

Thus

$$\frac{|u(t)|}{s_n} \leq \int_{\min\{t,s_n\}}^{\max\{t,s_n\}} \frac{|u'(x)|}{s_n} dx + g^*(t_n) \leq \int_{I_n \cup In-1} \frac{|u'(x)|}{s_n} dx + g^*(t_n),$$

and then by (3.6)

$$\frac{|u(t)|}{t} \leq 8 \left(\int_{I_n \cup In-1} \frac{|u'(x)|}{t} dx + g^*(t_n) \right).$$

Using (3.3) we have

$$\frac{|u(t)|}{t} \leq 8 \left(\int_{I_n \cup In-1} \frac{|u'(x)|}{t} dx + \frac{a_n}{\sqrt[p]{c_1}\, |I_n|} \right) \leq 8 \left(\frac{1}{|I_n|} \int_{I_n \cup I_{n-1}} |u'(x)|\, dx + \frac{a_n}{|I_n|} \right).$$

Integrating over I_n we obtain

$$\int_{I_n} \frac{|u(t)|}{t} dt \leq 8 \left(\int_{I_n \cup I_{n-1}} |u'(x)|\, dx + a_n \right)$$

and then raising to the p and using Lemma 1.26

$$\left(\int_{I_n} \frac{|u(t)|}{t} dt \right)^p \leq 2^{p-1} \cdot 2^{3p} \left(\left(\int_{I_n \cup I_{n-1}} |u'(x)|\, dx \right)^p + a_n^p \right).$$

Applying the Hölder inequality and (3.2) we have

$$\begin{aligned} \left(\int_{I_n} \frac{|u(t)|}{t} dt \right)^p &\leq 2^{4p-1} \left((3 \cdot |I_n|)^{\frac{1}{p'}} \int_{I_n \cup I_{n-1}} |u'(x)|^p\, dx + \int_{I_n} t^{p-1} (g^*(t))^p dt \right) \\ &\leq 2^{4p-1} \left(\int_{I_n} t^{p-1} \left[\left(\frac{u(\cdot)}{\cdot} \right)^* (t) \right]^p dt + \int_{I_n \cup I_{n-1}} |u'(x)|^p\, dx \right), \end{aligned}$$

which proves the lemma. □

Lemma 3.3. *There exists a positive constant c such that for each function $u \in (w)$-$W^{1,p}$ it holds that*

$$\left(\int_{\frac{1}{2}}^1 \frac{|u(t)|}{t} dt \right)^p \leq c \left(\inf_{t \in (0,1)} |u(t)|^p + \int_0^1 |u'(t)|^p\, dt \right). \tag{3.7}$$

Proof. Using the Hölder inequality we have

$$\left(\int_{\frac{1}{2}}^{1} \frac{|u(t)|}{t} dt\right)^{p} \leq \left(\int_{\frac{1}{2}}^{1} |u(t)|^{p}\, dt\right)\left(\int_{\frac{1}{2}}^{1} \frac{1}{t^{p'}} dt\right)^{\frac{p}{p'}} \leq \left(\int_{0}^{1} |u(t)|^{p}\, dt\right)\left(\int_{\frac{1}{2}}^{1} \frac{1}{t^{p'}} dt\right)^{\frac{p}{p'}}.$$

Clearly $\left(\int_{\frac{1}{2}}^{1} \frac{1}{t^{p'}} dt\right)^{\frac{p}{p'}} = ((p-1)(2^{\frac{1}{p-1}}-1))^{p-1}$.

Under the assertion of Lemma 3.1,

$$\left(\int_{\frac{1}{2}}^{1} \frac{|u(t)|}{t} dt\right)^{p} \leq (2p(1-2^{-p}))^{p-1}\left(\inf_{t\in(0,1)} |u(t)|^{p} + \int_{0}^{1} |u'(t)|^{p}\, dt\right),$$

which is our claim. □

Corollary 3.4. *The space* (w)-$W^{1,p}$ *is continuously embedded into the space* T_p.

Proof. By Lemma 3.2 we have

$$\sum_{n=2}^{\infty}\left(\int_{I_n} \frac{|u(t)|}{t} dt\right)^{p} \lesssim \sum_{n=2}^{\infty}\int_{I_n} t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)\right]^{p} dt + \sum_{n=2}^{\infty}\int_{I_n\cup I_{n-1}} |u'(x)|^{p}\, dx. \tag{3.8}$$

Adding inequality (3.7) to (3.8) and adding a non-negative number on the right side gives

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\int_{I_n} \frac{|u(t)|}{t} dt\right)^{p} &\lesssim \sum_{n=1}^{\infty}\int_{I_n} t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)\right]^{p} dt + \sum_{n=2}^{\infty}\int_{I_n\cup I_{n-1}} |u'(x)|^{p}\, dx \\ &+ \int_{0}^{1} |u'(t)|^{p}\, dt + \inf_{t\in(0,1)} |u(t)|^{p} \\ &\lesssim \int_{0}^{1} t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)\right]^{p} dt + 3\int_{0}^{1} |u'(t)|^{p}\, dt + \inf_{t\in(0,1)} |u(t)|^{p}.\end{aligned} \tag{3.9}$$

Let us denote $a := \inf_{t\in(0,1)} |u(t)|$. Then we have $a \leq |u(t)|$ and $a \leq \frac{a}{t} \leq \frac{|u(t)|}{t}$ for each $t \in (0,1)$. It means also that

$$a \leq \left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)$$

for each $t \in (0,1)$. It implies that

$$\frac{a^p}{p} = \int_{0}^{1} t^{p-1} a^p dt \leq \int_{0}^{1} t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)\right]^{p} dt$$

and, consequently,

$$\inf_{t\in(0,1)} |u(t)|^{p} \leq p\int_{0}^{1} t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^{*}(t)\right]^{p} dt. \tag{3.10}$$

We denote by c a positive constant. The latter inequality together with (3.9) gives

$$\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p \leq c\left((p+1)\int_0^1 t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^*(t)\right]^p dt + 3\int_0^1 |u'(t)|^p\, dt\right).$$

Finally,

$$\begin{aligned}
\|u\|_{T_p} &= \left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p + \int_0^1 |u'(t)|^p\, dt\right)^{\frac{1}{p}} \\
&\leq c^{\frac{1}{p}}\left((p+1)\int_0^1 t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^*(t)\right]^p dt + (3+\frac{1}{c})\int_0^1 |u'(t)|^p\, dt\right)^{\frac{1}{p}} \\
&\leq \left(c\max\left\{p+1, 3+\frac{1}{c}\right\}\right)^{\frac{1}{p}} \|u\|_{(w)\text{-}W^{1,p}} < \infty,
\end{aligned}$$

proving our claim. □

Remark 3.5. The space (w)-$W^{1,p}$ is continuously embedded into $L^p(0,1)$.

Proof. Let u be a function from the space (w)-$W^{1,p}$. Using Lemma 3.1 we have

$$\begin{aligned}
\int_0^1 |u(t)|^p\, dt &\leq 2^{p-1}\left(\inf_{t\in(0,1)}|u(t)|^p + \int_0^1 |u'(t)|^p\ dt\right) \\
&\leq 2^{p-1}\left(\inf_{t\in(0,1)}\left|\frac{u(t)}{t}\right|^p + \int_0^1 |u'(t)|^p\ dt\right) \\
&= 2^{p-1}\left(p\int_0^1 t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^*(1)\right]^p dt + \int_0^1 |u'(t)|^p\ dt\right) \\
&\leq 2^{p-1}\left(p\int_0^1 t^{p-1}\left[\left(\frac{u(\cdot)}{\cdot}\right)^*(t)\right]^p dt + \int_0^1 |u'(t)|^p\ dt\right).
\end{aligned}$$

Therefore,

$$\|u\|_{L^p(0,1)} \leq 2p^{\frac{1}{p}}\left(\left\|\frac{u(t)}{t}\right\|_{L^{1,p}(0,1)} + \|u'\|_{L^p(0,1)}\right) \leq 2p^{\frac{1}{p}}\|u\|_{(w)\text{-}W^{1,p}},$$

which proves our claim. □

4. An embedding of T_p into a Sobolev space with zero boundary value

4.1 Approximation functions vanishing at zero

Lemma 4.1. *Let $u \in T_p$. Then there exists a sequence of functions $\{v_N\}_{N=1}^{\infty}$, $v_N \in W^{1,p}(0,1)$, such that each of v_N is continuous on $[0,1)$, linear on some right neighborhood of 0, $v_N(0) = 0$ and, moreover,*

$$\|v_N - u\|_{W^{1,p}(0,1)} \xrightarrow{N\to\infty} 0.$$

Proof. By the first mean-value theorem for definite integral, because u is continuous, for each $n \in \mathbb{N}$ we choose some $t_n \in I_n$ such that

$$\int_{I_n} \frac{|u(t)|}{t} dt = \frac{|I_n|}{t_n} \cdot |u(t_n)|\,.$$

Thus

$$\frac{1}{2}|u(t_n)| \leq \int_{I_n} \frac{|u(t)|}{t} dt \leq |u(t_n)|\,.$$

Consequently, following the assumption, we obtain

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}|u(t_n)|\right)^p \leq \sum_{n=1}^{\infty} \left(\int_{I_n} \frac{|u(t)|}{t} dt\right)^p < \infty,$$

and therefore the sum on the left is convergent. By the necessary condition for convergence of series,

$$\lim_{n\to\infty} |u(t_n)| = 0. \tag{4.1}$$

Now, let us construct the sequence of functions $\{v_N\}_{N=1}^{\infty}$ as follows: for $N \in \mathbb{N}$, let

$$v_N(t) = \begin{cases} \frac{u(t_N)}{t_N} t, & t \in (0, t_N], \\ u(t), & t \in (t_N, 1). \end{cases}$$

It follows from the construction that

$$\lim_{t\to 0} v_N(t) = 0.$$

We first prove that

$$\int_0^1 |v_N' - u'(t)|^p \, dt \xrightarrow{N\to\infty} 0.$$

Let us compute

$$\begin{aligned}\int_0^1 |v_N' - u'(t)|^p \, dt &= \sum_{n=N}^{\infty} \int_{t_{n+1}}^{t_n} \left|\frac{u(t_N)}{t_N} - u'(t)\right|^p dt \\ &\leq 2^{p-1} \sum_{n=N}^{\infty} \int_{t_{n+1}}^{t_n} \left(\left|\frac{u(t_N)}{t_N}\right|^p + |u'(t)|^p\right) dt \qquad \text{(by Lemma 1.26)} \\ &= 2^{p-1} \left(t_N \left|\frac{\sum_{n=N}^{\infty}(t_n - t_{n+1})\frac{u(t_n)-u(t_{n+1})}{t_n - t_{n+1}}}{\sum_{n=N}^{\infty}(t_n - t_{n+1})}\right|^p + \int_0^{t_N} |u'(t)|^p \, dt\right),\end{aligned}$$

where $\sum_{n=N}^{\infty}(u(t_n)-u(t_{n+1}))$ converges since

$$\sum_{n=N}^{\infty}|u(t_n)-u(t_{n+1})| \le \sum_{n=N}^{\infty}\int_{t_{n+1}}^{t_n}|u'(t)|\;dt = \int_0^{t_N}|u'(t)|\;dt \lesssim \|u'\|_{L^p(0,1)}^p < \infty.$$

Using the Jensen inequality (Threorem 1.25) for the convex function $t \mapsto |t|^p$, we have

$$\left|\frac{\sum_{n=N}^{\infty}(t_n-t_{n+1})\frac{u(t_n)-u(t_{n+1})}{t_n-t_{n+1}}}{\sum_{n=N}^{\infty}(t_n-t_{n+1})}\right|^p \le \frac{\sum_{n=N}^{\infty}(t_n-t_{n+1})\left|\frac{u(t_n)-u(t_{n+1})}{t_n-t_{n+1}}\right|^p}{\sum_{n=N}^{\infty}(t_n-t_{n+1})},$$

which yields

$$\begin{aligned}
&\int_0^1 |v_N'-u'(t)|^p\,dt \\
&\quad\le 2^{p-1}\left(t_N\frac{\sum_{n=N}^{\infty}(t_n-t_{n+1})\left|\frac{u(t_n)-u(t_{n+1})}{t_n-t_{n+1}}\right|^p}{\sum_{n=N}^{\infty}(t_n-t_{n+1})}+\int_0^{t_N}|u'(t)|^p\,dt\right)\\
&\quad= 2^{p-1}\left(\sum_{n=N}^{\infty}\frac{|u(t_n)-u(t_{n+1})|^p}{(t_n-t_{n+1})^{p-1}}+\int_0^{t_N}|u'(t)|^p\,dt\right).
\end{aligned}$$

Let us focus on

$$\begin{aligned}
\sum_{n=N}^{\infty}\frac{|u(t_n)-u(t_{n+1})|^p}{(t_n-t_{n+1})^{p-1}} &= \sum_{n=N}^{\infty}\left((t_n-t_{n+1})^{1-p}\left|\int_{t_{n+1}}^{t_n}u'(t)dt\right|^p\right)\\
&\le \sum_{n=N}^{\infty}\left((t_n-t_{n+1})^{1-p}\left(\int_{t_{n+1}}^{t_n}|u'(t)|\,dt\right)^p\right).
\end{aligned}$$

Applying the Hölder inequality, we have

$$\begin{aligned}
\sum_{n=N}^{\infty}\frac{|u(t_n)-u(t_{n+1})|^p}{(t_n-t_{n+1})^{p-1}} &\le \sum_{n=N}^{\infty}\left((t_n-t_{n+1})^{1-p}(t_n-t_{n+1})^{\frac{p}{p'}}\int_{t_{n+1}}^{t_n}|u'(t)|^p\,dt\right)\\
&= \sum_{n=N}^{\infty}\int_{t_{n+1}}^{t_n}|u'(t)|^p\,dt = \int_0^{t_N}|u'(t)|^p\,dt.
\end{aligned}$$

This gives us the estimate

$$\int_0^1 |v_N'(t)-u'(t)|^p\,dt \le 2^p\int_0^{t_N}|u'(t)|^p\,dt.$$

Since $\int_0^1|u'(t)|^p\,dt < \infty$, we have by the absolute continuity of the Lebesgue integral that for each $\varepsilon > 0$ we can choose $N_0 \in \mathbb{N}$ such that for each $N \in \mathbb{N}, N > N_0$, we have $\int_0^{t_N}|u'(t)|^p < \varepsilon$, which proves

$$\lim_{N\to\infty}\|v_N'-u'\|_{L^p(0,1)} = 0.$$

We now turn to $\lim_{N\to\infty}\|v_N-u\|_{L^p(0,1)}$. Since $\left|\frac{t}{t_N}\right|^p \le 1$ for $t \in [0,t_N]$, we have by Lemma 1.26

$$\begin{aligned}\int_0^1 |v_N(t)-u(t)|^p\,dt &= \int_0^{t_N} \left|\frac{u(t_N)}{t_N}t - u(t)\right|^p dt\\ &\le 2^{p-1}\left(\int_0^{t_N} |u(t_N)|^p \left|\frac{t}{t_N}\right|^p dt + \int_0^{t_N} |u(t)|^p\,dt\right)\\ &\le 2^{p-1}\left(\int_0^{t_N} |u(t_N)|^p\,dt + \int_0^{t_N} |u(t)|^p\,dt\right)\\ &= 2^{p-1}\left(t_N\,|u(t_N)|^p + \int_0^{t_N} |u(t)|^p\,dt\right).\end{aligned}$$

As a consequence of Remark 3.5, we arrive at $\int_0^{t_N} |u(t)|^p\,dt < \infty$. Hence we can obtain $\int_0^{t_N} |u(t)|^p\,dt \xrightarrow{N\to\infty} 0$ in the same manner as above and following (4.1) we have $t_N\,|u(t_N)|^p\,dt \xrightarrow{N\to\infty} 0$. This proves

$$\lim_{N\to\infty} \|v_N - u\|_{L^p(0,1)} = 0,$$

which completes the proof. □

Lemma 4.2. *Let $u \in T_p$. Then there exists a sequence of functions $\{w_N\}_{N=1}^{\infty}$, $w_N \in W^{1,p}(0,1)$, such that each of w_N is continuous on $(0,1)$, piecewise linear on some reduced right neighborhood of 0, vanishing on small reduced right neighborhood of 0 and, moreover,*

$$\|w_N - u\|_{W^{1,p}(0,1)} \xrightarrow{N\to\infty} 0.$$

Proof. Let the symbols t_N, v_N have the same meaning as in Lemma 4.1.

For each $N \in \mathbb{N}$ we construct functions $w_{N,\eta}$, $\eta \in (0, t_N)$, as follows:

$$w_{N,\eta} := \begin{cases} 0, & t \in (0,\eta],\\ \frac{t-\eta}{t_N-\eta}u(t_N), & t\in(\eta,t_N],\\ u(t), & t\in(t_N,1).\end{cases}$$

Then

$$\begin{aligned}\|w_{N,\eta} - v_N\|_{L^p(0,1)}^p &= \int_0^{\eta} |v_N(t)|^p\,dt + \int_{\eta}^{t_N} \left|\frac{t-\eta}{t_N-\eta}u(t_N) - \frac{u(t_N)}{t_N}t\right|^p dt\\ &= \int_0^{\eta} |v_N(t)|^p\,dt + \frac{\eta^p\,|u(t_N)|^p}{t_N^p(t_N-\eta)^p}\int_{\eta}^{t_N}(t_N-t)^p dt\\ &= \int_0^{\eta} |v_N(t)|^p\,dt + \frac{\eta^p\,|u(t_N)|^p}{t_N^p(t_N-\eta)^p}\frac{(t_N-\eta)^{p+1}}{p+1}\\ &= \int_0^{\eta} |v_N(t)|^p\,dt + \frac{\eta^p\,|u(t_N)|^p}{(p+1)t_N^p}(t_N-\eta),\end{aligned}$$

and, following the construction of v_N, the last expression tends to zero if η tends to zero. Similary

$$\left\|w'_{N,\eta} - v'_N\right\|_{L^p(0,1)}^p \xrightarrow{\eta\to 0} 0.$$

Consequently, for each $N \in \mathbb{N}$, one has:

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \exists \eta \in \mathbb{R}, \eta > 0 : \\ (\|w_{N,\eta} - v_N\|_{L^p(0,1)} < \varepsilon) \wedge (\|w'_{N,\eta} - v'_N\|_{L^p(0,1)} < \varepsilon).$$

Then for each $N \in \mathbb{N}$ we choose $\eta_N > 0$ such that

$$(\|w_{N,\eta_N} - v_N\|_{L^p(0,1)} < \frac{1}{N}) \wedge (\|w'_{N,\eta_N} - v'_N\|_{L^p(0,1)} < \frac{1}{N}).$$

We denote $w_N := w_{N,\eta_N}$.

Choose $\varepsilon_0 \in \mathbb{R}$, $\varepsilon_0 > 0$ arbitrarily. We find $N_0 \in \mathbb{N}$ such that for each $N \in \mathbb{N}$, $N > N_0$, we have $\frac{1}{N} < \frac{\varepsilon_0}{4}$ and, as a consequence of Lemma 4.1,

$$\|v_N - u\|_{L^p(0,1)} < \frac{\varepsilon_0}{4}, \qquad \|v'_N - u'\|_{L^p(0,1)} < \frac{\varepsilon_0}{4}.$$

Then also for each $N \in \mathbb{N}$, $N > N_0$, it holds that

$$\|w_N - v_N\|_{L^p(0,1)} < \frac{\varepsilon_0}{4}, \qquad \|w'_N - v'_N\|_{L^p(0,1)} < \frac{\varepsilon_0}{4}.$$

Therefore for each $N \in \mathbb{N}$, $N > N_0$, we have

$$\|w_N - u\|_{L^p(0,1)} + \|w'_N - u'\|_{L^p(0,1)} \\ \leq \|w_N - v_N\|_{L^p(0,1)} + \|v_N - u\|_{L^p(0,1)} + \|w'_N - v'_N\|_{L^p(0,1)} + \|v'_N - u'\|_{L^p(0,1)} < \varepsilon_0.$$

By the definition of a limit, this is the desired conclusion. □

Lemma 4.3. *Let $u \in T_p$. Then there exists a sequence of functions $\{z_N\}_{n=1}^{\infty}$ such that each of z_N is an element of $C^{\infty}_{\{0\}}(0,1)$ and*

$$\|z_N - u\|_{W^{1,p}(0,1)} \xrightarrow{N\to\infty} 0.$$

Proof. The proof will be divided into two steps.

1. Shift. Fix $N \in \mathbb{N}$ and consider $\varepsilon_0 \in \mathbb{R}$, $\varepsilon_0 > 0$.

   We have $u \in L^p(0,1)$ by Remark 3.5 and, by the assumption, $u' \in L^p(0,1)$. Therefore functions w_N defined in Lemma 4.2 and w'_N are also elements of $L^p(0,1)$. By Theorem 1.18, the functions w_N and w'_N are p-mean continuous. Hence we can find δ such that for each $h \in \mathbb{R}$, $|h| < \delta$, it holds (in conformity with Convention 1.16)

$$\left(\int_0^1 |w_N(t+h) - w_N(t)|^p \, dt\right)^{\frac{1}{p}} < \varepsilon_0$$

   and

$$\left(\int_0^1 |w'_N(t+h) - w'_N(t)|^p \, dt\right)^{\frac{1}{p}} < \varepsilon_0.$$

   Fix $h \in \mathbb{R}$, $0 < h < \delta$, and define

$$g_N(t) := \begin{cases} 0, & t \in (-h, h], \\ w_N(t-h), & t \in (h, 1+h). \end{cases}$$

2. Mollification. Let S be the set of functions defined in Definition 1.19
   Let $\varphi_0 \in S$ and let $\eta \in \mathbb{R}$, $0 < \eta < h$. Then we define the functions

$$(R_\eta g_N)(t) := \frac{1}{\eta}\int_{-h}^{1+h} \varphi_0\left(\frac{t-x}{\eta}\right) g_N(x)dx$$

   and

$$(R_\eta g_N')(t) := \frac{1}{\eta}\int_{-h}^{1+h} \varphi_0\left(\frac{t-x}{\eta}\right) g_N'(x)dx.$$

   By Theorem 1.21,

   (a) $R_\eta g_N, R_\eta g_N' \in C^\infty(\mathbb{R})$,

   (b) $\lim_{\eta\to 0+} \|R_\eta g_N - g_N\|_{L^p(0,1)}$ and $\lim_{\eta\to 0+} \|R_\eta g_N' - g_N'\|_{L^p(0,1)}$.

   Moreover, following the construction of g_N there exists η_0 such that for every η, $0 < \eta < \eta_0$, we have $R_\eta g_N, R_\eta g_N' \in C^\infty_{\{0\}}(0,1)$.
   It follows from (b) above that there exists η, $0 < \eta < \eta_0$, such that

$$\|R_\eta g_N - g_N\|^p_{L^p(0,1)} < \varepsilon_0^p \tag{4.2}$$

   and, simultaneously,

$$\|R_\eta g_N' - g_N'\|^p_{L^p(0,1)} < \varepsilon_0^p. \tag{4.3}$$

   Further, using the Leibniz integral rule and then integration by parts we compute

$$\begin{aligned}
&\|(R_\eta g_N)' - g_N'\|^p_{L^p(0,1)} \\
&= \int_0^1 \left| \left(\frac{1}{\eta}\int_{-h}^{1+h} \varphi_0\left(\frac{t-x}{\eta}\right) g_N(x)dx\right)' - g_N'(t)\right|^p dt \\
&= \int_0^1 \left| \frac{1}{\eta^2}\int_{-h}^{1+h} \varphi_0'\left(\frac{t-x}{\eta}\right) g_N(x)dx - g_N'(t)\right|^p dt \\
&= \int_0^1 \left| \frac{1}{\eta^2}\int_{t-\eta}^{t+\eta} \varphi_0'\left(\frac{t-x}{\eta}\right) g_N(x)dx - g_N'(t)\right|^p dt \\
&= \int_0^1 \left| \frac{1}{\eta}\left[-\varphi_0\left(\frac{t-x}{\eta}\right) g_N(x)\right]_{t-\eta}^{t+\eta} + \frac{1}{\eta}\int_{t-\eta}^{t+\eta} \varphi_0\left(\frac{t-x}{\eta}\right) g_N'(x)dx - g_N'(t)\right|^p dt \\
&= \int_0^1 \left| \frac{1}{\eta}\int_{t-\eta}^{t+\eta} \varphi_0\left(\frac{t-x}{\eta}\right) g_N'(x)dx - g_N'(t)\right|^p dt < \varepsilon_0^p,
\end{aligned}$$

   where the last inequality follows from (4.3).
   Thus, $\lim_{\eta\to 0+} \|(R_\eta g_N)' - g_N'\|_{L^p(0,1)}$ and also

$$\|w_N - R_\eta g_N\|_{L^p(0,1)} \le \|w_N - g_N\|_{L^p(0,1)} + \|g_N - R_\eta g_N\|_{L^p(0,1)} < 2\varepsilon_0.$$

   Furthermore,

$$\|w_N' - (R_\eta g_N)'\|_{L^p(0,1)} \le \|w_N' - g_N'\|_{L^p(0,1)} + \|g_N' - (R_\eta g_N)'\|_{L^p(0,1)} < 2\varepsilon_0.$$

In other words, for each $N \in \mathbb{N}$ it holds that

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \exists \eta : \\ \Big(\|w_N - R_\eta g_N\|_{L^p(0,1)} < \varepsilon \Big) \wedge \Big(\|w_N' - (R_\eta g_N)'\|_{L^p(0,1)} < \varepsilon \Big)$$

and $R_\eta g_N \in C^\infty_{\{0\}}$.

Then for each $N \in \mathbb{N}$ we find η_N, $\eta_N > 0$, small enough to guarantee that $R_{\eta_N} g_N \in C^\infty_{\{0\}}$ and

$$\left(\|w_N - R_{\eta_N} g_N\|_{L^p(0,1)} < \frac{1}{N} \right) \wedge \left(\|w_N' - (R_{\eta_N} g_N)'\|_{L^p(0,1)} < \frac{1}{N} \right).$$

Let us denote $z_N := R_{\eta_N} g_N$.

Then $z_N \in C^\infty_{\{0\}}(0,1)$ and, using Lemma 4.2, for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for each $N \in \mathbb{N}$, $N > N_0$, one has $\|z_N - w_N\|_{L^p(0,1)} < \frac{1}{N} < \varepsilon$, $\|z_N' - w_N'\|_{L^p(0,1)} < \frac{1}{N} < \varepsilon$, $\|w_N - u\|_{L^p(0,1)} < \varepsilon$, $\|w_N' - u'\|_{L^p(0,1)} < \varepsilon$.

Hence

$$\|z_N - u\|_{L^p(0,1)} \le \|z_N - w_N\|_{L^p(0,1)} + \|w_N - u\|_{L^p(0,1)} < 2\varepsilon$$

and

$$\|z_N' - u'\|_{L^p(0,1)} \le \|z_N' - w_N'\|_{L^p(0,1)} + \|w_N' - u'\|_{L^p(0,1)} < 2\varepsilon,$$

which establishes our claim.

□

Lemma 4.4. *The space T_p is continuously embedded into $W^{1,p}_{\{0\}}(0,1)$.*

Proof. Following Lemma 4.3, for each function $u \in T^p$ there exists a sequence $\{z_N\}_{N=1}^\infty$ of $C^\infty_{\{0\}}(0,1)$ functions such that u is the limit of $\{z_N\}_{N=1}^\infty$ in the space $L^p(0,1)$ and u' is limit of $\{z_N'\}_{N=1}^\infty$ in the space $L^p(0,1)$. Then, by the definition of $W^{1,p}_{\{0\}}(0,1)$, the function u is an element of $W^{1,p}_{\{0\}}(0,1)$.

Now let us prove that there exists a constant $c \ge 1$ such that for every $u \in T_p$ we have

$$\|u\|_{W^{1,p}_{\{0\}}(0,1)} \le c \|u\|_{T_p}.$$

Using Lemma 3.1 and Lemma 1.26 we compute

$$\begin{aligned} \|u\|^p_{W^{1,p}_{\{0\}}(0,1)} &= \Big(\|u\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)} \Big)^p \\ &\le 2^{p-1} \Big(\|u\|^p_{L^p(0,1)} + \|u'\|^p_{L^p(0,1)} \Big) \\ &\le 2^{p-1} \left(2^{p-1} \inf_{t \in (0,1)} |u(t)|^p + (2^{p-1} + 1) \int_0^1 |u'(t)|^p \, dt \right). \end{aligned}$$

Denote $a := \inf_{t \in (0,1)} |u(t)|$. Then we have $a \le |u(t)|$ and $a \le \frac{a}{t} \le \frac{|u(t)|}{t}$ for each $t \in (0,1)$. Consequently,

$$\frac{1}{2^p - 1} a^p = \sum_{n=1}^\infty \left(\int_{I_n} a \cdot dt \right)^p \le \sum_{n=1}^\infty \left(\int_{I_n} \frac{|u(t)|}{t} dt \right)^p.$$

Therefore

$$\|u\|^p_{W^{1,p}_{\{0\}}(0,1)} \leq 2^{p-1}\left(2^{p-1}(2^p-1)\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p + (2^{p-1}+1)\int_0^1 |u'(t)|^p\, dt\right)$$
$$\leq 2^{p-1}(2^{2p-1}+1)\left(\sum_{n=1}^{\infty}\left(\int_{I_n}\frac{|u(t)|}{t}dt\right)^p + \int_0^1 |u'(t)|^p\, dt\right) \leq 2^{3p}\,\|u\|^p_{T_p}$$

Hence, $\|u\|_{W^{1,p}_{\{0\}}(0,1)} \leq 8\,\|u\|_{T_p}$, which is our claim. □

4.2 An embedding of a weak Sobolev space into a Sobolev space with zero boundary trace

Theorem 4.5. *The space* (w)-$W^{1,p}$ *is continuously embedded into* $W^{1,p}_{\{0\}}(0,1)$.

Proof. Let us focus on an existence of a positive constant c such that for every $u \in (w)\text{-}W^{1,p}$ one has

$$\|u\|_{W^{1,p}_{\{0\}}(0,1)} \leq c\,\|u\|_{(w)\text{-}W^{1,p}}\,.$$

It follows from assertions of Corollary 3.4 and Lemma 4.4 that

$$\|u\|_{W^{1,p}_{\{0\}}(0,1)} \leq c_1\,\|u\|_{T^p} \leq c_1\cdot c_2\,\|u\|_{(w)\text{-}W^{1,p}} < \infty,$$

which completes the proof of theorem. □

5. An embedding of a Sobolev space with zero boundary trace into a weak Sobolev space

Theorem 5.1. *The space $W^{1,p}_{\{0\}}(0,1)$ is continuously embedded into (w)-$W^{1,p}$.*

Proof. The Hardy inequality (Theorem 1.23) applied to the function u', where $u \in W^{1,p}_{\{0\}}(0,1)$, gives

$$\int_0^1 \left|\frac{u(t)}{t}\right|^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^1 |u'(t)|^p \, dt.$$

Then for $u \in W^{1,p}_{\{0\}}(0,1)$

$$\begin{aligned} \left\|\frac{u(t)}{t}\right\|^p_{L^{1,p}(0,1)} &= \int_0^1 t^{p-1} \left[\left(\frac{u(\cdot)}{\cdot}\right)^*(t)\right]^p dt \le \int_0^1 \left[\left(\frac{u(\cdot)}{\cdot}\right)^*(t)\right]^p dt \\ &= \int_0^1 \left|\frac{u(t)}{t}\right|^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^1 |u'(t)|^p \, dt \\ &= \left(\frac{p}{p-1}\right)^p \|u'\|^p_{L^p(0,1)} . \end{aligned} \tag{5.1}$$

Since $u \in W^{1,p}_{\{0\}}(0,1)$, we get $\|u\|_{W_0^{1,p}(0,1)} := \|u\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)} < \infty$. Therefore $\|u\|_{(w)\text{-}W^{1,p}} := \left\|\frac{u(t)}{t}\right\|_{L^{1,p}(0,1)} + \|u'\|_{L^p(0,1)} < \infty$. Thus, $u \in (w)$-$W^{1,p}$. As a consequence of (5.1), one also obtains

$$\begin{aligned} \|u\|_{(w)\text{-}W^{1,p}} &= \left\|\frac{u(t)}{t}\right\|_{L^{1,p}(0,1)} + \|u'\|_{L^p(0,1)} \\ &\le \left(\left(\frac{p}{p-1}\right) + 1\right) \left(\|u\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)}\right) \\ &= \left(\frac{2p-1}{p-1}\right) \|u\|_{W^{1,p}_{\{0\}}(0,1)} , \end{aligned}$$

which completes the proof. □

6. The principal result

In this chapter we shall establish the main goal of this theses. It will be formulated in form of a fairly general theorem whose particular cases we have seen in previous chapters. We denote by $d(t)$ the distance function from $t \in (a,b)$, $a,b \in \mathbb{R}$, to the boundary of (a,b), i.e. $d(t) = \min\{|t-a|, |t-b|\}$.

Lemma 6.1. *Let $a,b \in \mathbb{R}$, $p \in (1,\infty)$. Then the following conditions are equivalent:*

(i) $\frac{u}{d} \in L^{1,p}(a,b)$,

(ii) $\frac{u(t)}{t-a} \in L^{1,p}(a,b)$ *and* $\frac{u(t)}{b-t} \in L^{1,p}(a,b)$.

Proof. (i)⇒(ii). Clearly, $\frac{1}{t-a} \le \frac{1}{d(t)}$ and $\frac{1}{b-t} \le \frac{1}{d(t)}$, which gives

$$\frac{|u(t)|}{t-a} \le \frac{|u(t)|}{d(t)}, \qquad \frac{|u(t)|}{b-t} \le \frac{|u(t)|}{d(t)} \qquad \text{for each } t \in (a,b).$$

Thus

$$\left\|\frac{u(t)}{t-a}\right\|_{L^{1,p}(a,b)} \le \left\|\frac{u(t)}{d(t)}\right\|_{L^{1,p}(a,b)}, \qquad \left\|\frac{u(t)}{b-t}\right\|_{L^{1,p}(a,b)} \le \left\|\frac{u(t)}{d(t)}\right\|_{L^{1,p}(a,b)}.$$

(ii)⇒(i). Let $c = \frac{a+b}{2}$. Then

$$d(t) = \begin{cases} t-a, & t \in (a,c], \\ b-t, & t \in [d,c). \end{cases}$$

Consequently

$$\frac{|u(t)|}{d(t)} = \frac{|u(t)|}{t-a}\chi_{(a,c]}(t) + \frac{|u(t)|}{b-t}\chi_{(c,b)}(t)$$

and, following the fact that $\|.\|_{L^{1,p}}$ is a quasinorm with a constant C, we arrive at

$$\begin{aligned}\left\|\frac{u(t)}{d(t)}\right\|_{L^{1,p}(a,b)} &\le C\left(\left\|\frac{u(t)}{t-a}\chi_{(a,c]}(t)\right\|_{L^{1,p}(a,b)} + \left\|\frac{u(t)}{b-t}\chi_{(c,b)}(t)\right\|_{L^{1,p}(a,b)}\right)\\ &\le C\left(\left\|\frac{u(t)}{t-a}\right\|_{L^{1,p}(a,b)} + \left\|\frac{u(t)}{b-t}\right\|_{L^{1,p}(a,b)}\right).\end{aligned}$$

□

Lemma 6.2. *Let $a,b \in \mathbb{R}$, $p \in (1,\infty)$. Then the following statements hold:*

(i) $u \in W^{1,p}_{\{a\}}(a,b) \;\Leftrightarrow\; \frac{u(t)}{t-a} \in L^{1,p}(a,b) \;\wedge\; u' \in L^p(a,b)$ *and the norms are equivalent,*

(ii) $u \in W^{1,p}_{\{b\}}(a,b) \;\Leftrightarrow\; \frac{u(t)}{b-t} \in L^{1,p}(a,b) \;\wedge\; u' \in L^p(a,b)$ *and the norms are equivalent.*

Proof. It suffices to prove (i). The statement (ii) can then be proved analogously.

To prove the "if" part, set $v(t) = u(a + (b - a)t)$. Denote $s = a + (b - a)t$. Then

$$\frac{u(s)}{s-a} = \frac{u(a+(b-a)t)}{a+(b-a)t-a} = \frac{v(t)}{(b-a)t}.$$

This gives

$$\left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(a,b)} = \frac{1}{b-a} \left\| \frac{v(t)}{t} \right\|_{L^{1,p}(0,1)}.$$

Clearly $v' \in L^p(0,1)$ and

$$(b-a)^{\frac{p-1}{p}} \|u'\|_{L^p(a,b)} = \|v'\|_{L^p(0,1)}, \qquad (b-a)^{-\frac{1}{p}} \|u\|_{L^p(a,b)} = \|v\|_{L^p(0,1)}.$$

By Lemma 4.4 there exists a sequence $v_n \in C^\infty_{\{0\}}(0,1)$ with

$$v_n \to v \text{ in } W^{1,p}(0,1)$$

and for some positive constant c_1

$$\|v\|_{W^{1,p}_{\{0\}}(0,1)} \le c_1 \left(\left\| \frac{v(t)}{t} \right\|_{L^{1,p}(0,1)} + \|v'\|_{L^p(0,1)} \right).$$

Set $u_n(t) = v_n(\frac{t-a}{b-a})$. Then $u_n \in C^\infty_{\{a\}}(a,b)$, and by an easy calculation one gets

$$u_n \to u \text{ in } W^{1,p}(a,b),$$

and so $u \in W^{1,p}_{\{a\}}(a,b)$. Denote $c_2 := (b-a)^{\frac{1}{p}} \max\{1, \frac{1}{b-a}\}$. Then

$$\begin{aligned}
\|u\|_{W^{1,p}_{\{a\}}(a,b)} &= \|u\|_{L^p(a,b)} + \|u'\|_{L^p(a,b)} \le c_2 \|v\|_{W^{1,p}_{\{0\}}(0,1)} \\
&\le c_2 c_1 \left(\left\| \frac{v(t)}{t} \right\|_{L^{1,p}(0,1)} + \|v'\|_{L^p(0,1)} \right) \\
&= c_2 c_1 \left((b-a) \left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(0,1)} + (b-a)^{\frac{p-1}{p}} \|u'\|_{L^p(a,b)} \right) \\
&\le C_1 \left(\left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)} \right)
\end{aligned}$$

for an appropriate value of C_1.

Conversely, in order to prove the "only if" part, note first that the proof of the implication $u \in W^{1,p}_{\{a\}}(a,b) \;\Rightarrow\; \frac{u(t)}{t-a} \in L^{1,p}(a,b) \;\wedge\; u' \in L^p(a,b)$ is similar to that of Theorem 5.1. The Hardy inequality (Theorem 1.23) gives

$$\begin{aligned}
\left\| \frac{u(t)}{t-a} \right\|^p_{L^{1,p}(a,b)} &= \int_0^{b-a} t^{p-1} \left[\left(\frac{u(\cdot)}{\cdot - a} \right)^* (t) \right]^p dt \\
&\le (b-a)^{p-1} \int_0^{b-a} \left[\left(\frac{u(\cdot)}{\cdot - a} \right)^* (t) \right]^p dt = (b-a)^{p-1} \int_a^b \left| \frac{u(t)}{t-a} \right|^p dt \\
&\le (b-a)^{p-1} \left(\frac{p}{p-1} \right)^p \int_a^b |u'(t)|^p \, dt \\
&= (b-a)^{p-1} \left(\frac{p}{p-1} \right)^p \|u'\|^p_{L^p(a,b)}.
\end{aligned}$$

Thus we have

$$\left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(0,1)} + \|u'\|_{L^p(a,b)} \le C_2 \|u\|_{W^{1,p}_{\{a\}}(a,b)}$$

for an appropriate constant C_2. □

Lemma 6.3. *Let $a, b \in \mathbb{R}$, $p \in (1, \infty)$. Then $W^{1,p}_0(a,b) = W^{1,p}_{\{a\}}(a,b) \cap W^{1,p}_{\{b\}}(a,b)$.*

Proof. (Inclusion $\subseteq$). Let $u \in W^{1,p}_0(a,b)$. Then $u \in W^{1,p}_{\{a\}}(a,b)$ and $u \in W^{1,p}_{\{b\}}(a,b)$, hence $u \in W^{1,p}_{\{a\}}(a,b) \cap W^{1,p}_{\{b\}}(a,b)$.

(Inclusion $\supseteq$). Take a function $\varphi \in C^\infty(a,b)$, $0 \le \varphi(t) \le 1$, $\varphi(t) = 1$ for $t \in (a, a + \frac{1}{3}(b-a))$ and $\varphi(t) = 0$ for $t \in (a + \frac{2}{3}(b-a), b)$. Now assume that $u \in W^{1,p}_{\{a\}}(a,b) \cap W^{1,p}_{\{b\}}(a,b)$. We find sequences $\{u_n\} \subset C^\infty_{\{a\}}(a,b)$ and $\{v_n\} \subset C^\infty_{\{b\}}(a,b)$ such that

$$\|u - u_n\|_{W^{1,p}(a,b)} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|u - v_n\|_{W^{1,p}(a,b)} \xrightarrow{n\to\infty} 0.$$

Choose $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Find u_{n_0}, v_{n_0} such that for each $n \ge n_0$, $n \in \mathbb{N}$:

$$\|u - u_n\|_{W^{1,p}_{\{a\}}(a,b)} < \varepsilon, \qquad \|u - v_n\|_{W^{1,p}_{\{b\}}(a,b)} < \varepsilon$$

For each $n \in \mathbb{N}$, set $\overline{u}_n(t) = u_n(t)\varphi(t)$, $\overline{v}_n(t) = v_n(t)(1 - \varphi(t))$. Clearly, $\overline{u}_n, \overline{v}_n \in C^\infty_0(a,b)$, and so $w_n := \overline{u}_n + \overline{v}_n \in C^\infty_0(a,b)$. We now calculate

$$\begin{aligned}
\|u - w_n\|_{W^{1,p}(a,b)} &= \|(1 - \varphi + \varphi)u - u_n\varphi - v_n(1-\varphi)\|_{W^{1,p}(a,b)} \\
&\le \|(u - u_n)\varphi\|_{W^{1,p}(a,b)} + \|(u - v_n)(1-\varphi)\|_{W^{1,p}(a,b)} \\
&\le (1 + \max_{t\in(a,b)} |\varphi'(t)|)\|u - u_n\|_{W^{1,p}(a,b)} + (1 + \max_{t\in(a,b)} |\varphi'(t)|)\|u - v_n\|_{W^{1,p}(a,b)} \\
&\le 2(1 + \max_{t\in(a,b)} |\varphi'(t)|)\varepsilon.
\end{aligned}$$

Thus $\|u - w_n\|_{W^{1,p}(a,b)} \xrightarrow{n\to\infty} 0$, whence $u \in W^{1,p}_0(a,b)$. □

Theorem 6.4. *Let $a, b \in \mathbb{R}$, $p \in (1, \infty)$. Let us denote by (w)-$W^{1,p}(a,b)$, where $a, b \in \mathbb{R}$, the space of functions satisfying conditions*

- $\frac{u(t)}{d(t)} \in L^{1,p}(a,b)$,
- $u'(t) \in L^p(a,b)$

with the quasinorm $\|u\|_{(w)\text{-}W^{1,p}(a,b)} := \left\|\frac{u}{d}\right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)}$. Then

$$(w)\text{-}W^{1,p}(a,b) = W^{1,p}_0(a,b)$$

and the quasinorms are equivalent for each $p \in (1, \infty)$.

Proof. Let $u \in (w)$-$W^{1,p}(a,b)$. By Lemma 6.1, this is equivalent to saying that both conditions

- $\frac{u(t)}{t-a} \in L^{1,p}(a,b) \ \wedge \ u' \in L^p(a,b)$,
- $\frac{u(t)}{b-t} \in L^{1,p}(a,b) \ \wedge \ u' \in L^p(a,b)$

hold. Following Lemma 6.2, this is equivalent to

$$u \in W^{1,p}_{\{a\}}(a,b) \wedge u \in W^{1,p}_{\{b\}}(a,b)$$

which is satisfied, by the Lemma 6.3, if and only if $u \in W_0^{1,p}(a,b)$. Consequently, (w)-$W^{1,p}(a,b) = W_0^{1,p}(a,b)$.

Also, by Lemma 6.1, Lemma 6.2 and Lemma 6.3 we have

$$\begin{aligned}
\|u\|_{W_0^{1,p}(a,b)} &\le \|u\|_{W^{1,p}_{\{a\}}(a,b)} + \|u\|_{W^{1,p}_{\{b\}}(a,b)} \\
&\le C_1 \left(\left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)} \right) + C_1 \left(\left\| \frac{u(t)}{b-t} \right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)} \right) \\
&\le 2C_1 \left(\left\| \frac{u(t)}{d(t)} \right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)} \right) = 2C_1 \|u\|_{(w)\text{-}W^{1,p}(a,b)}
\end{aligned}$$

and

$$\begin{aligned}
\|u\|_{(w)\text{-}W^{1,p}(a,b)} &= \left\| \frac{u(t)}{d(t)} \right\|_{L^{1,p}(a,b)} + \|u'\|_{L^p(a,b)} \\
&\le C \left\| \frac{u(t)}{t-a} \right\|_{L^{1,p}(0,1)} + C \left\| \frac{u(t)}{b-t} \right\|_{L^{1,p}(0,1)} + \|u'\|_{L^p(a,b)} \\
&\le C \cdot C_2 \|u\|_{W^{1,p}_{\{a\}}(a,b)} + C \cdot C_2 \|u\|_{W^{1,p}_{\{b\}}(a,b)} \\
&\le 2C \cdot C_2 \|u\|_{W_0^{1,p}(a,b)},
\end{aligned}$$

which completes the proof. □

Remark 6.5 (a counterexample for $p = \infty$)**.** Let $u(t) = 1$, $t \in (0,1)$. Then $\frac{u(t)}{d(t)} = \frac{1}{d(t)} \in L^{1,\infty}(0,1)$ and $1' = 0 \in L^\infty(0,1)$, however, for each $v_n \in C_0^\infty(0,1)$, one has

$$\|u - v_n\|_{W_0^{1,\infty}(0,1)} = \sup_{t\in(0,1)} |u(t) - v_n(t)| + \sup_{t\in(0,1)} |u'(t) - v_n'(t)| \ge 1$$

and $u \notin W_0^{1,\infty}(0,1)$.

Remark 6.6 (optimality)**.** Let $p \in [1,\infty)$, $q \in [1,\infty]$ and $a,b \in \mathbb{R}$. It is natural to pose the question whether $u \in W_0^{1,p}(a,b)$ if and only if $\frac{u(t)}{d(t)} \in L^{1,q}(a,b)$ and $u'(t) \in L^p(a,b)$ within the following ranges of parameters:

(i) $q \le p$,

(ii) $q > p$,

(iii) $q = \infty$.

In case (i) the positive answer follows from Theorem 6.4 and the fact $L^{1,1} \subset L^{1,q} \subset L^{1,p}$. In case (iii) the answer is negative as we shall demonstrate by a counterexample below. In case (ii) is question remains open.

Counterexample for the case (iii). Set $(a,b) = (0,1)$ and $u(t) = 1$ for each $t \in (0,1)$. Then u satisfies $\frac{u}{d} \in L^{1,\infty}(0,1)$ and $u' \in L^p(0,1)$. Assume for a contradiction that $u \in W_0^{1,p}(0,1)$. Thus there exists a sequence $\{u_n\}$ of functions from

$C_0^\infty(0,1)$ such that $|u - u_n|_{W^{1,p}} \xrightarrow{n\to\infty} 0$. Choose ε, $0 < \varepsilon < \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\int_0^1 |u(t) - u_n(t)|^p + \int_0^1 |u'(t) - u_n'(t)|^p < \varepsilon^{2p} \tag{6.1}$$

holds for each $n > n_0$. Let us denote $M := \{t \in (0,1); |u(t) - u_n(t)| > \varepsilon\}$ for some $n > n_0$. Hence

$$\varepsilon^{2p} > \int_0^1 |u(t) - u_n(t)|^p \geq \int_M |u(t) - u_n(t)|^p > |M|\, \varepsilon^p,$$

and thus $|M| < \varepsilon^p$. There exists t_0, $0 < t_0 < \varepsilon^p$, such that $u_n(t_0) > 1 - \varepsilon$. Therefore, using the Hölder inequality, we obtain

$$\begin{aligned}\int_0^1 |u'(t) - u_n'(t)|^p &= \int_0^1 |u_n'(t)|^p \geq \left(\int_0^1 |u_n'(t)|\right)^p \geq \left(\int_0^{t_0} |u_n'(t)|\right)^p \\ &\geq \left|\int_0^{t_0} u_n'(t)\right|^p = |u_n(t_0)|^p \geq (1-\varepsilon)^p > \varepsilon^{2p},\end{aligned}$$

which contradicts the inequality (6.1). □

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