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Covariant Loop Quantum Gravity

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Tato práce skýtá široký úvod do teorie smyčkové kvantové gravitace na pozadí všech ostatních přístupů ke kvantování gravitace. Věnuje se jak kanonické, tak kovariantní verzi této teorie. Ve druhém ze zmíněných přístupů posléze zkoumá dynamiku spjatou s množinou vybraných jednoduchých konfigurací. K jejím zjištěním patří, že naivní přístup k definování konzistentní dynamiky, kdy je partiční funkce dráhového integrálu chápána jako suma amplitud odpovídajících všem hraničním a vnitřním stavům, selhává, vzhledem k výskytu divergencí. Tento fakt otevírá prostor pro použití jiných, vice sofistikovaných přístupů.

Klíčová slova: smyčková kvantová gravitace, diskrétní dráhový integrál, Ponzanův-Reggeho model

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Abstract: In this thesis we offer a broad introduction into loop quantum gravity against the backdrop of the quantum gravity research as a whole. We focus on both the canonical and covariant version of the theory. In the latter version we investigate the dynamics of some simple configurations in the simplified setting of Ponzano-Regge model. We ascertain that the naïve approach to define a consistent dynamics, where the path integral's partition function is computed as a sum of amplitudes corresponding to all boundary and bulk states, fails in this case, on account of an appearance of divergences. This opens up space for the utilization of some more sophisticated methods.

Keywords: loop quantum gravity, discrete path integral, Ponzano-Regge model

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Introduction

"Whereof one cannot speak, thereof one must remain silent."*
—Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*

0.1 The why of quantum gravity

In their contemporary form, the two principal physical theories that we have at our disposal, well as they work in their respective application domains, do not fit very coherently together. Quantum theory and quantum field theory, on the one hand, provide a formulation of the dynamics of a system in terms of wave-functions, probabilities, Hermitian operators and Hilbert spaces. The crucial ingredient in this description is an exogenously supplied time and a spacetime structure which is set and does not evolve itself during system's evolution. General relativity, on the other hand, is a prescription for how the spacetime structure evolves (this time in a non-set fairly arbitrary time parameter) in function of its matter content. Finding a common ground between these two frameworks would amount either to finding a prescription for the quantum evolution without any reference to the external temporal structure or being able to formulate the quantum dynamics of space and time along with its matter content with respect to some of the arbitrary time parameters. Unfortunately, none of the tools needed to tackle any of these possibilities came with the theories themselves.

But besides the displeasing fact that both of these theories are, as they stand, written in different mathematical formalisms that cannot be reasonably mapped one onto another, each of them suffers from internal problems of its own, even if we forget for a while about the viewpoint of the other theory. These hint at their fundamental incompleteness. In the case of the quantum field theory, the issue is that one often gets UV divergences - infinite results in places where one would expect finite quantities. These can be avoided by renormalization procedures which, however, do not follow some coherent mathematical prescription and are much closer in spirit to *ad hoc* mix-and-match methods. The fact that predictions made in this way still match the experimental results to an astonishingly high degree makes these shortcomings to a staunchly empiricist eye perhaps of less acute nature.

General relativity, on the other side, while being in much better position mathematically, predicts its own breakdown for large enough densities of matter and energy inside of black holes and in the early Universe. That this is a generic feature of the theory which cannot be avoided was rigorously proved in the well-known theorems by Hawking and Penrose. One is then lead to believe that in these regimes its classical description gets replaced by an as-yet-unknown theory. But besides that, there is another reason for the insufficiency of general relativity

* In its original nonstilted form: *"Wovon man nicht sprechen kann, davon muss man schweigen."*

in its current form. In Einstein's equations the right-hand side couples stress-energy tensor which characterizes matter and energy content to geometrical left-hand side. But we know that the matter fields are at a fundamental level described by quantum theory. One, therefore, needs either to come up with a way how to consistently couple classical and quantum entities, or one needs to reexpress the geometry on the left-hand side in fully quantum mechanical terms, i.e. to construct a framework for a quantum theory of the metric tensor.

Considerations of this kind, along with a natural tendency for unification, have given birth to *quantum gravity* as a new area of physics. Actually, it is not uninteresting to note in this connection that the first person to explicitly accentuate the need for it was Einstein himself when he gave an argument about the instability of the electron orbits[1]. Needless to say, this was, of course, meant with respect to the old quantum theory which missed its stochastic element. For time being, we use the term *quantum gravity* in a purposefully ambiguous sense as any theory that addresses any of the issues mentioned above. Its more concrete definition will be given in a while.

0.2 Contact with experiment

Let us now consider how a purportive theory of quantum gravity could be probed experimentally. A simple dimensional argument goes that as a theory encompassing special relativity, gravitation, and quantum dynamics, its natural units should be given in terms of combinations of fundamental constants connected with each of these different facets, i.e. respectively the speed of light c, gravitational constant G and (reduced) Planck constant $\hbar$. For *Planck units* defined in this way we get in numerical terms $l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}m$ and $t_p = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}s$. These determine the spatiotemporal resolution at which one expects quantum gravitational effects to become significant with the corresponding energy level being $E_p = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{19}J \approx 10^{28}eV$. One of the experimental options how to test directly the predictions of *quantum gravity* would, therefore, be constructing a hadron collider capable of observing collisions with beam energies comparable to E_p.

The fact that the direct option mentioned in the last paragraph lies 15 orders of magnitude beyond our current technological reach would make one adopt at best a cynic attitude, perhaps somewhat inspired in manner by the quote given at the beginning of this introduction. It came therefore as a bit of surprise[2] when it became clear that the direct observation route might not be the only feasible one. The first step in recognizing this was the discovery or relict microwave background radiation in 1960's, the only available imprint from the era when Universe was hot and dense enough for the quantum gravitational effects to play an important role. Even though concrete proposals for how to extract measurable

[1] As cited in 1916: *"Nevertheless, due to the inter-atomic movement of electrons, atoms would have to radiate not only electro-magnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation."*[31]

[2] We fully admit that we may be overindulging in the poetic license here, as there might be some, to which it was no surprise at all. Still, we posit in general the result is not self-evident.

predictions were missing at the time, a potential experimental channel was firmly established. The second important step in this direction was when the works [3] and others in 1990's showed that another way to test for quantum gravity effects would be in measuring high-energy radiation bursts. The idea is this: these sources of radiation emit at various energies, durations, and intensities and when the circumstances are right, the difference in the arrival times of high and low energy photons is directly related to the dispersion factor that the prospective quantum gravity theories predict [43]. The result of both of these developments was a gradual incarnation of a fully fledged subfield of physics - *quantum gravity phenomenology*, despite the supposed technological setbacks.

Without going into too much detail - some recent reviews of the field can be found in [27], [4] - let us cite two newer additions into the possible observation scenarios, each exploring a different aspect of the underlying *quantum gravity* theory. The first one of them is a black hole phenomenology, which gained extra relevance with the recent observation of gravitational waves. Even though the highest curvature region, the singularity, is hidden and shielded under the event horizon, there are various scenarios for how the quantum gravity effects could pile up and leak over to the outer region [24], [46], [8]. Or there might be specific signatures of evaporating black holes formed during high-energy collisions in particle accelerators [18], [5]. The second tantalizing possibility is an observation of quantum gravity effects in condensed matter systems due to a gauge/gravity duality. This concept asserts a fundamental equivalence between gravitational systems in the bulk of a spacetime with quantum field theory systems on its boundary, so once a precise dictionary of terms between both of these systems is known, one can indirectly probe one by means of the another. Actually, due to the extensive amount of experimental knowledge of condensed matter systems, it is not completely unimaginable that we already detected an effect with quantum gravitational origin [16], [17], [15].

0.3 Quantum gravity

In light of the conceptual arguments of the first subsection and a marked experimental appeal of the second subsection, let us consider what forms the sought quantum gravity could take. When considering a theory combining the conceptual insights of quantum mechanics, special relativity, and gravity, one can proceed at three distinct levels.

First, one may consider an implementation of quantum theory on a kinematical and dynamical level, while merging special relativity and gravity into general relativity and restricting its role to a (kinematical) background. This general relativistic quantum mechanics approach would be one concrete instance of *hybrid dynamics* - a consistent coupling of classical and quantum systems mentioned in the very first subsection. It is important to note that this approach is the only one where corroborated experimental results exist, however, studies show that the absence of backreaction of matter content (particles) on geometry forestalls any insight into the deeper questions one is expected to address. The role of this approach, therefore, lies in parametrizing the low energy limit of other more involved theories.

The second option of combining relativistic, quantum and gravity, is to implement a quantum *field* theory on a kinematical and dynamical level while allowing for perturbative backreaction on the general relativistic background. This approach is another example of hybrid dynamics. Interestingly enough, the backreaction of the field is not necessary for getting non-trivial results, so it can actually be without a loss of generality considered again an *a priori* static background. Along with many technical issues that arise when one tries to quantize a field on a general non-symmetric spacetime and lacks standard Poincare invariance as a heuristic principle, most notably a lack of a proper renormalization procedure, *quantum field theory on curved backgrounds*, as this approach came to be known, has yielded important insights into the quantum nature of the gravitational field. In this respect, it stands in contrast to the previous case. One of these insights is a prediction of black hole thermal radiation which causes the black hole to gradually lose mass. This surprising occurrence is an example of a process of purely quantum origin that violates the classical laws of black hole thermodynamics and in the following, we will give its simple heuristic derivation.

Since it turns out that the desired result is quite generic [36] and does not depend on particular choices of the spacetime solution or quantum field, we are free to make the choices so as to make the underlying argument the most transparent. Let us thus consider a $1+1$-dimensional Schwarzschild solution

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}}, \tag{1}$$

and a massless scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}m^2\phi^2, \tag{2}$$

$$S = \int \mathrm{d}^4x \mathcal{L}(\phi, \partial_\mu\phi). \tag{3}$$

Now the 'converting procedure' that transforms Minkowski QFT into curved background QFT is *minimal coupling.* It consists in replacing $\eta_{\mu\nu} \to g_{\mu\nu}$, $\phi_{,\mu} \to \phi_{;\mu}$ and $\mathrm{d}^4x \to \mathrm{d}^4x\sqrt{-g}$, in other words, instead of the flat spacetime metric we use the relevant metric tensor, instead of the partial derivatives we use the covariant derivatives and instead of a trivial Lorentz-invariant volume element we utilize the generally covariant one. The equation of motion then has a simple form

$$D^\mu D_\mu \phi = 0. \tag{4}$$

Intuitively speaking it is now clear that the space of solutions to this equation somehow contains all possible one-particle states that can be used in the construction of the full Fock space. It turns out that this is indeed the case as will become clear in the following. Let us now show explicitly how the non-trivial result of black hole radiation arises and what is the point of difference with the flat space quantization.

Transforming (1) into lightcone tortoise and Kruskal-Szekeres coordinates

$$r^*(r) = r - r_g + r_g \log\left(\frac{r}{r_g} - 1\right), \tag{5}$$

$$\tilde{u} = t - r^*, \tilde{v} = t + r^*, \tag{6}$$

resp.

$$u = 2r_g \exp\left(-\frac{\tilde{u}}{2r_g}\right), v = 2r_g \exp\left(\frac{\tilde{v}}{2r_g}\right), \tag{7}$$

we get

$$ds^2 = \left(1 - \frac{r_g}{r(\tilde{u}, \tilde{v})}\right) d\tilde{u}d\tilde{v}, \tag{8}$$

$$ds^2 = \frac{r_g}{r(u,v)} \exp\left(1 - \frac{r(u,v)}{r_g}\right) dudv. \tag{9}$$

Now, by inspecting equation (4), it is clear that the solutions will have a form of a superposition of planar waves. The respective angular frequency will be in general given with respect to the proper time $d\tau = \sqrt{g_\mu \nu dx^\mu dx^\nu}$, so for the metrics (8), (9) we can write

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega u} \hat{a}_\omega^- + e^{i\omega u} \hat{a}_\omega^+] + \text{(left-moving)} \tag{10}$$

$$= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega\tilde{u}} \hat{b}_\Omega^- + e^{i\Omega\tilde{u}} \hat{b}_\Omega^+] + \text{(left-moving)}. \tag{11}$$

where the left-moving part is given by the same expression except for $u \to v$, resp. $\tilde{u} \to \tilde{v}$ and creation and annihilation operators $\hat{a}_\omega^\pm$,$\hat{b}_\Omega^\pm$ create and annihilate modes as one would expect from standard QFT and give rise to different vacua according to[3]

$$\hat{a}_\omega^- |0_K\rangle = 0, \hat{b}_\Omega^- |0_B\rangle = 0. \tag{12}$$

Now for the general relation between $\hat{a}_\omega^\pm$ and $\hat{b}_\Omega^\pm$ (called Bogolyubov transformation)

$$b_\Omega^- = \int_0^\infty d\omega [\alpha_{\Omega\omega} \hat{a}_\omega^- - \beta_{\Omega\omega} \hat{a}_\omega^+], \tag{13}$$

it can be shown using (5)-(7), (10), (11), (13) that

$$|\alpha_{\Omega\omega}|^2 = e^{2\pi\Omega \cdot 2r_g} |\beta_{\Omega\omega}|^2. \tag{14}$$

If one considers the expectation value of b-particle number operator in $|0_K\rangle$, one eventually gets

$$\langle \hat{N}_\Omega \rangle \equiv \langle 0_K | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0_K \rangle = \int d\omega |\beta_{\omega\Omega}|^2 = [\exp(2\pi\Omega \cdot 2r_g) - 1]^{-1} \delta(0), \tag{15}$$

[3]In accordance with the literature we call $|0_K\rangle$ Kruskal vacuum and $|0_B\rangle$ Boulware vacuum. Physical considerations show that the first one of them corresponds to an observer far from the black hole in asymptotical regions whereas the second one is vacuum of an observer falling through the event horizon.

where the factor $\delta(0)$ corresponds to an infinite volume of space. The density of the b-particles in $|0_K\rangle$, using the definition of the Hawking temperature $T_H = 1/4\pi r_g$, therefore is

$$n_\Omega = [\exp(T_H \Omega) - 1]^{-1}. \tag{16}$$

This is the desired formula for the Hawking black hole radiation for a two-dimensional Schwarzschild solution.

Since the procedure sketched above seems quite generic, one might be lead to believe that the results are not a unique prediction of curved background QFT and might be reproduced under different settings. Let us now show why the standard QFT is insufficient in this respect.

First, for two Lorentz-related reference frames and Minkowski background, (5)-(7) are replaced by

$$x' = \Lambda x, \tag{17}$$

$$u = t - x, v = t + x \tag{18a}$$

$$u' = t' - x', v' = t' + x' \tag{18b}$$

and the connection components are identically zero, so the analog of equation (4) contains only partial derivatives

$$\partial^\mu \partial_\mu \phi = 0. \tag{19}$$

Now the crucial difference consists in the fact that after expanding the general solution into positive and negative frequency modes labeled by creation and annihilation operators along the lines of (10), (11), the Bogolyubov transformation between them yields $\beta_{xx'} = 0$. In other words, there is only one canonical choice of positive frequency modes and Lorentz transformations only act by changing basis in it, not by intermixing positive and negative frequency modes as in the case of general diffeomorphisms (5)-(7). The space of all solutions of (19) is therefore in a sense 'flat' whereas that of (4) is not. This is why a state filled with particles can never be a vacuum state for a different choice of physical observers. It deserves mentioning that similar effect as the one described above occurs also for accelerated observers in Minkowski spacetime (Unruh effect) and therefore it can be seen as a consequence of losing Lorenz symmetry as an aspect of one particular GR solution and replacing it with more general diffeomorphism symmetry.

Another important aspect of this second line of reasoning are the semiclassical Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi \langle \Phi | \hat{T}_{\mu\nu} | \Phi \rangle \tag{20}$$

which are expected to hold in the mean-field approximation regime. The fact that they are, however, not likely to be true at the more fundamental level seems to be in an indirect way suggested by the following arguments. First, assuming that there is a suitable Hilbert space to support the quantum states $|\Phi\rangle$ (which is not trivial), these equations are solved in an iterative manner. Beginning with some non-vacuum state $|\Phi_0\rangle$ and a flat metric, the resulting metric tensor $g_{\mu\nu}$ is fed into both $\hat{T}_{\mu\nu}$ and the state $|\Phi_1\rangle$ which however is built upon different vacuum, as follows from the discussion of Hawking radiation above. In general, this procedure does not converge. Second, the renormalization procedure intended to fix these divergences entails an addition of extra terms quadratic in curvature which change the original setup of the theory as well as its solutions. One of the consequences of this is a possible instability of Minkowski space. Third, the inherent non-linearity of these equations seems to be both at odds with the principles of quantum theory (which is inextricably built around the notion of linearity), as well as with the available empirical evidence.

Finally, the third option[4] of addressing the issue of *quantum gravity* is to implement both quantum field theory and general relativity on both a kinematical and dynamical level, much like in the previous case but this time going beyond the mere perturbative backreaction and considering fully coupled spacetime-matter system evolution. This diffeomorphism-invariant quantum field theory of spacetime and matter amounts to finding a quantum field theory for a GR action

$$S_{EH} = \frac{1}{16}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{8}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K + S_{matter}, \tag{21}$$

where h is a determinant of a 3-metric and K is a trace of the extrinsic curvature on the boundary ∂M. This is the case on which we focus the attention from now on. In the following, we drop the cursive script and use the term quantum gravity to refer only to this third option. Given the shortcomings of the previous options, it does not take a huge leap of faith to state that this approach is in a very good position to address the conceptual issues that were briefly described in the beginning and that initiated and spurred research in these directions in the first place.

0.4 Approaches and options

Despite considerable efforts, a fully satisfactory quantum theory of the action (21) is not yet available, even though that does not mean that no distinct programs and methods associated to them appeared where some degree of progress would be achieved. We will eventually settle on one particularly promising research direction and justify the selection in the next section, and so in this section, we try to paint a more broad picture by giving a short overview of a cornucopia of approaches that one may employ when addressing the issue of quantization of action (21). We are necessarily parsimonious with details and neglect the historical progression of events[5].

The fact that the action (21) does not have a fully consistent quantum theory can be interpreted in (at least) two different ways. On the one hand, one may conclude that no such theory exists and therefore in order to address the problems of singularities and the natural cut-off for QFT one needs to add further terms to (21) which reproduce the Einstein-Hilbert action only in the low energy limit. Or, on the other hand, one might conclude that the set of techniques for constructing

[4]The reason why there are not more options is that general relativity can be seen as a kinematical prescription for valid theories whereas quantum mechanics can be seen as a dynamical prescription. Thus, dynamical GR, kinematical QM combinations do not make sense.

[5]An exhaustive historical account can be found in [44]

QFTs is insufficient and that the peculiar properties that set general relativity apart from other interactions (general covariance and 'background' as a dynamical entity) necessitate the development of new special methods. Related to this first question is the notion of *primary* and *secondary* quantum gravity theories. Primary quantum gravity theories begin with the action (21) and apply heuristic quantization rules to it in order to arrive at a relevant quantum theory. A similar methodology was successfully applied to a quantization of electromagnetism and its merit is a tight control over the assumptions and the logical structure of the theory. Secondary quantum gravity theories, on the other hand, assume that the problem of quantizing gravity is intertwined with the unification of all interactions and therefore posit that the correct framework for finding it is to postulate validity of unified field theory which will reduce to quantum gravity only in suitable limiting cases. A similar approach was successful in finding a renormalizable theory of weak interactions and it is the viewpoint taken by e.g. string theory. A common consequence of this standpoint would be that general relativity is not quantized as it is but contains extra structures that are necessitated by the unification framework. A significant drawback of this set-up connected to its speculative premise is more involved 'accounting' of the results of the theory, i.e. in case a noncoherent result is obtained, it is hard to tell apart effects that are a consequence of general non-viability of the framework from the effect that follows from details of the unification scheme. This leads to a looser logical structure.

An important point of departure which is made explicitly in the primary quantum gravity theories (but in some form is present also in the secondary ones) is a choice of what structures related to the spacetime are to be left classical and which ones to quantize. In general terms the hierarchy is the following:

$$\begin{aligned}&\text{sets of events} \to \text{topological structure} \to \text{differential structure}\\&\to \text{causal structure} \to \text{metric structure}\end{aligned}$$

This in principle offers one way of classifying prospective theories. Most of them, however, make the same choice of subjecting to the quantization only the last two options, leaving set, topological and differential structure intact.

Despite the wide variety of approaches to quantization,[6] only a handful of them have seen a wider application in physics[7]. This situation is also reflected in approaches to quantum gravity. The three main directions that are being actively

[6]This is because from the mathematical point of view, when quantization as a mapping between functions on a phase space and Hermitian operators on the corresponding Hilbert space is defined with respect to a reasonable set of axioms, the construction in general, i.e. the axioms themselves, can be shown to be inconsistent. Somewhat vaguely put, these are *a)* linearity, *b)* unit being mapped to unit, *c)* composition law, *d)* canonical representation for position and momentum operators guaranteed by the Stone-von Neuman theorem and *e)* correspondence between commutator and Poisson brackets. Working around this situation invites a multitude of approaches. A review is available in [1].

[7]So far a theory is either quantizable by some the 'mainstream' methods, or it is not quantizable at all. It is important to note that the relationship between classical and quantum theories is not one-to-one and onto, while not every classical theory needs to have a quantum counterpart, more than one quantum theories may lead in the classical limit to the same classical theory.

pursued are perturbative quantization, canonical quantization, and path integral (covariant) quantization. In the following we briefly describe each one of them.

The starting point of the perturbative quantization is a split of the spacetime metric into a flat background and a small perturbation representing a weak gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{22}$$

This allows us to write Einstein's gravitational law in the form of a wave equation

$$\Box \gamma_{\mu\nu} = 0 \tag{23}$$

for $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. One can now proceed along generic lines (not completely unlike those of the calculation of the Hawking radiation) - the particle excitations can be then shown to be necessarily of spin 2, computing one-loop contributions to transition amplitudes, however, leads to a UV divergence. Renormalization of this divergence requires adding a term quadratic in curvature to the original action. This new term, however, causes another divergence which can be renormalized either by adding another term with a quartic power in the curvature (leading to another divergence of higher order *ad infinitum*) or postulating ghost fields with asymptotical states (for $\Lambda = 0$). In this sense, one can hear said that general relativity is a perturbatively non-renormalizable theory.

There are two ways of coping with this difficulty, the first one of them is abandoning the 'pointiness' and localization of the particles from the very beginning, the second one is generalizing the concept of renormalizability of a quantum theory. The first option is the conceptual foundation of a string theory whereas the second one leads to a scenario called asymptotic safety.

It can be said that string theory is a roundabout answer to the question posed in the beginning of this section. The fact that action (21) does not have a quantum theory is interpreted as meaning that gravity can be only quantized when unified all the other interactions. A similar scenario is realized in the case of the theory of weak interactions - the Fermi four point theory of weak interactions - which is non-renormalizable and can be viewed only as a mean field approximation of the fundamental theory - a $U(1) \times SU(2)$-based electroweak model. Therefore in the terminology introduced above, quantum gravity is here necessarily a secondary theory. This assumption of extra structure implies higher energy corrections to (21). We do not intend to give here a comprehensive overview of the subject which unfortunately is too rich to fit the present confines. We do sketch however in what concrete sense does the string theory satisfy the definition of quantum gravity.

The starting point of the string theory is a quantization of an open or closed string propagating in a D-dimensional Minkowski spacetime. Later considerations show that the closed string case corresponds to the case with bosonic degrees of freedom so we start by considering it first. In analogy with the relativistic particle where the action is proportional to the proper time along the worldline, the action of a string is proportional to the area of the worldsheet

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \sqrt{|\det g_{\alpha\beta}|} \tag{24}$$

where $g_{\alpha\beta}$ is the metric on the worldsheet induced by the embedding $X(\sigma, \tau)$, $d^2\sigma = d\sigma d\tau$ and the term $\frac{1}{2\pi\alpha'}$ represents the string tension. This Nambu-Goto action is not very suitable for quantization because of the non-linear square root. One can, however, use it to derive classically equivalent Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \sqrt{h} h^{\alpha\beta}(\sigma, \tau) \partial_\alpha X^\mu \partial_\beta X_\mu \tag{25}$$

where $h^{\alpha\beta}$ now denotes an intrinsic metric on the worldsheet, i.e. one with respect to which one varies the action. This variation gives a Euler-Lagrange equation a wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2} \right) X^\mu(\sigma, \tau) = 0 \tag{26}$$

whose solution, after taking carefully into account boundary conditions and required commutation relations between postulated creation and annihilation operators, necessarily contains a spin-2 particle in 4 dimensions. But it was shown [31] that Lorentz invariance of a spin-2 massless particle is a sufficient condition for the equivalence principle and thus GR. Thus, as a consequence of our initial assumption (24) resp. (25), string theory automatically (to the lowest order) contains a quantum gravity in four dimensions. Besides this argument, consistency of the string quantization on an arbitrary background requires that the Einstein's equations be satisfied for this background. In this sense one says that string theory provides a consistent theory of quantum gravity. It should be mentioned, however, that this does not come without problems of its own - most notably an avoidance of Weyl anomaly requires the spacetime dimensionality to be $D = 26$ for a bosonic string and $D = 11$ for a fermionic string which contradicts experimental observations as well as everyday experience, the theory also seems to rest on a validity of some form of supersymmetric scenario which has proved so far difficult to establish empirically. One can thus conclude that despite an undeniable elegance of the framework the difficulty of connecting conceptual premises of the theory with falsifiable predictions presents an ongoing serious challenge that the theory has to overcome in order to deliver on its initial goals.

The second approach of dealing the issue of perturbative non-renormalizability of GR is the asymptotic safety scenario. Here one abandons the use of the perturbative expansion and generalizes the notion of renormalizable theory using renormalization group methods instead. Still, it is far more close in spirit to the previous approach than any of the following which is why it is included here in the perturbative approaches. The view taken by the asymptotic safety scenario is this: to a particular set of particles and symmetries one considers a manifold of all possible coupling parameters g_i. On this manifold, changing the energy leads to a renormalization group flow according to the equation

$$\partial_t \Gamma_\mu = \frac{1}{2} \mathrm{Tr} \Big[\left(\frac{\delta^2 \Gamma_\mu}{\delta\phi\delta\phi} + R_\mu \right)^{-1} \partial_t R_\mu \Big] \tag{27}$$

where Γ_μ is an effective action dependent on the energy scale μ, R_μ is an infrared regulator and $t = \log \mu$. The absence of divergences in physical quantities is guaranteed by the existence of a non-Gaussian fixed point[8] of this flow whereas all

[8] A Gaussian fixed point corresponds to the free theory with $g_i = 0 \quad \forall i$.

the trajectories that lead to it form a *critical surface*. Since in general, for a non-renormalizable theory the manifold of parameters will be infinite-dimensional, the finite dimensional critical surface reestablishes the predictive power of the theory. Unfortunately, equation (27) can only be solved by truncating the degrees of freedom of the original theory and considering a finite dimensional problem instead. Thus, a conclusive proof of the existence of fixed point of general relativity is still missing. The truncated systems, however, do possess a non-Gaussian fixed point which is an appealing piece of indirect evidence and a necessary condition for the general viability of this approach.

In contrast with the perturbative quantization, the canonical approach does not take as a starting point the split of the spacetime into a static background and a small perturbation. Rather, the main idea is that different reformulations of the underlying theory lead to differently tractable functional relations between variables which one seeks to represent as operator equations or to differently tractable constraints, if these are present. Once one has chosen the variables, one implements the general canonical quantization procedure:

1. One identifies configuration variables and their conjugate momenta. For these fundamental variables V_i, one demands

$$V_i = \{V_j, V_k\} \to \hat{V}_i = -\frac{i}{\hbar}[\hat{V}_j, \hat{V}_k]. \tag{28}$$

2. One decides which of the non-fundamental variables are to be quantized according to (28) and which are not. Since the left-hand side features commutative classical variables whereas the right-hand side does not, this is the point where factor ordering ambiguities are encountered. In general, it is impossible to implement (28) for all the composite variables without contradicting the irreducibility of the representation.

3. One constructs an appropriate representation space $\mathcal{F}$ for the dynamical variables $\hat{V}_i$, the elements of this space are called wave functionals in anticipation of their physical interpretation. In general, besides the variables $\hat{V}_i$, the system will also feature *constraints* $\hat{C}_i$ which select physical states in $\mathcal{F}$. At this point, neither does the space $\mathcal{F}$ have to have a Hilbert space structure, nor do the variables $\hat{V}_i$ and the constraints have to be self-adjoint, precisely because of a presence of the non-physical states in $\mathcal{F}$. Actually, demanding the self-adjointness of $\hat{C}_i$ might even lead to mathematical inconsistencies.

4. One implements the constraints

$$\hat{C}_i \Psi = 0, \tag{29}$$

this condition singles out candidates for the physical states $\Psi' \in \mathcal{F}_0$. Additional requirements for the physical states might be imposed to obtain the genuine $\mathcal{F}_{phys} \subset \mathcal{F}_0 \subset \mathcal{F}$[9].

[9]Actually, this is not exactly the case of LQG where the diffeomorpism-invariant states are not normalizable in $\mathcal{F}_0$ and they must be constructed as element in the dual space to $\mathcal{F}_0$.

5. One constructs a complete set of observables. In the presence of constraints, a concept of Dirac observable is needed for which the classical requirement $\{O, C_i\} \approx 0$ translates in quantum theory into

$$[\hat{O}, \hat{C}_i]\Phi_{phys} = 0. \tag{30}$$

6. Finally, one ascertains the structure of $\mathcal{F}_{phys}$. For constrained systems with finite-dimensional Lorentz gauge groups, a group averaging procedure yields unique $\mathcal{F}_{phys}$. Additionally, depending on the spectrum of the operators corresponding to fundamental variables $\hat{V}_i$, a rigged Hilbert space construction with Gel'fand triples may be needed.

Following this recipe one gets a fully viable quantum theory for the classical system one started with. Depending on the initial choice of variables, doing this, albeit to a partial extent, gives rise to either *geometrodynamics* or *loop* approach to quantum gravity. As both of these approaches form an important part of our thesis in the latter chapters, we do not elaborate on them here much further and postpone their more detailed discussion until then.

Finally, the third conceptual option of approaching quantum gravity might be called covariant quantization or path-integral quantization. In contrast to both perturbative and canonical quantizations, which one might identify as top-down methods, the path integral approach employs rather a bottom-up methodology. The core issue is that diffeomorphism invariance precludes the existence of a measure on the space of all metrics which one would need in order to advance with the straightforward application of the formula

$$\langle h_0 | h_1 \rangle = \int_{g[\partial \mathcal{M}] = h_0, h_1} D[g_{\mu\nu}] e^{i S_{EH}[g]}, \tag{31}$$

h_0, h_1 being the induced metrics on the boundary ∂M. This can, however, be bypassed if one performs the integral over diffeomorphism-invariant geometries instead of metrics and imposes suitable continuity and smoothness restrictions on the set of admissible geometries. These can be approximated by discrete configurations which lead to an ansatz of a form

$$\langle \Delta_{h_0} | \Delta_{h_1} \rangle = \sum_{conf} \sum_{j_0, j_1, \ldots} \prod_p \mathcal{A}_p[j_0] \prod_l \mathcal{A}_l[j_1] \cdots \prod_{4s} \mathcal{A}_{4s}[j_4] \tag{32}$$

where the first sum is taken with respect to configurations that are consistent with the boundary discretizations Δ_{h_0}, Δ_{h_1}, p, l, $4s$ denote point, line and 4-simplex respectively, j_i are possible degrees of freedom connected with i-dimensional constituents and $\mathcal{A}$'s are elementary contributions to the amplitude. This is how the bottom-up methodology concretely realized.

Taking this line of thought as a starting point leads to two well-developed approaches to quantum gravity - causal dynamical triangulations (CDT) and spinfoams. CDT restricts the set of configurations to be summed over to those formed by triangulations of fixed spatial slices by tetrahedra with a fixed spacelike length l_s and linking these triangulations with links with a fixed timelike length l_t [2]. This gluing together of two subsequent slices so as to ensure local preservation of the causal structure[10] requires four different 4-simplices - a (4,1)-simplex and a (3,2)-simplex[11] and their time-reversed versions. The amplitudes $\mathcal{A}_i$ are the ones reproducing discretized Regge action for a given discretization which can be shown to depend only on the total number of vertices N_0 and numbers of (4,1)- and (3,2)-simplices $N_4^{(4,1)}$, $N_4^{(3,2)}$. Spinfoam approach, on the other hand, does not restrict possible discretizations by enforcing a fixed foliation into spacelike hypersurfaces, its geometrical starting point is rather a general discretization of the bulk with variable line lengths (which can be taken as corresponding to an internal variable j_1) which are subject only to triangular inequalities[12]. The amplitudes are then chosen with respect to group theoretic considerations such as invariance with respect to local gauge transformations with different choices corresponding to different spinfoam models. This second approach will be a topic of interest in latter parts of this thesis which is why we do not expand on it further.

These are then the three most developed approaches to quantum gravity. They can be traced back to their historical roots and have largely retained their distinct character despite various attempts at convergence and mutual cross-pollination. Let us now consider anew the question at the beginning of this section. It was stated in there that the fact that the action of general relativity does not have a fully consistent quantum theory yet can be interpreted either in a way that the set of methods currently used for quantization are inadequate for the case of gravity (point of view that is more in line with the *primary* quantum gravity theories) or that general relativity may require additional structure in order to make it quantizable (point of view more in line with the *secondary* quantum gravity theories). Do these two options fully exhaust all the possibilities? In terms of some speculative orthogonality of ideas, assuming the question is well posed, they do, which however does not mean that some linear combination of them could not be realized in Nature. Such a linear combination could, for example, require general relativity to be reformulated in terms of as yet unknown mathematical structures that would possibly also imply its completion and that would on their own suggest a novel suitable quantization method. Due to the non-orthodox nature of this option, it is clear that research in this direction is necessarily less developed than that of the other options. There have however been concrete results and in the following we present some of the ideas that have arisen in this sector. We also give an argument *against* it as far as adding new structure to general relativity is concerned.

Some of the non-orthodox approaches to quantum gravity that are worth mentioning are non-commutative geometry, causal sets, and twistors. Some closely related ideas concern quantization of Einstein-Cartan theory and either general relativity or quantum mechanics as a non-fundamental but emergent paradigm.

Rather than being an independent approach to quantum gravity, noncommutative geometry is a part of mathematics that has gradually found its way into physical applications. Its basic idea is to extend, roughly speaking, the dual re-

[10] a conceptual assumption around which the theory is built

[11] Numbers in brackets refer to how many vertices of the simplex lie on a t hyperplane and how many on the subsequent one.

[12] This holds true for the Ponzano-Regge model, the other models can be seen as its generalizations.

lation between topological spaces and commutative rings of functions on them to a broader class of structures [52], [30]. This dual relation often allows, for example, to reconstruct the geometrical structure of a given space from the algebraic properties of the functions defined on it. Broadening the analogy so as to include the case of noncommutative algebras and using them to define and explore generalized geometrical structures constitutes the core of non-commutative geometry *per se*. As such, the idea has found applications in quantum field theory, string theory, and quantum spacetime, though it cannot be stated at this stage that the results are as thorough as that of the more mainstream approaches.

Causal sets, on the other hand, build the theory around two notions considered to be fundamental - discreteness and causality [29]. The kinematical setup of the theory is that of a discretized spacetime manifold where each of the chunks of spacetime is represented by an event. These events are connected among themselves with links that embody the relations of causal past/future. The whole set of events is then called a causal set and the causality relation endows it with a partial order structure. The choice of events in the underlying manifold must be subject to Poisson process in order to guarantee Lorentz invariance of the constant density of the selected points which is in turn needed in order to interpret the number of events as a spacetime volume. The dynamical content of the theory is provided by a process called *classical sequential growth* which from the mathematical point of view can be seen as a probability measure on the set of all causal sets with a given number of events. Despite these facts, the theory still faces serious challenges mainly at the kinematical stage where it is in general difficult ascertain a consistent correspondence between causal sets on the one hand and classical solutions to Einstein's equations on the other hand, as well as in finding a coherent *quantum* formulation of the dynamics [29]. From the point of view of the diagram shown above, causal sets are a rare instance of an approach that attempts to apply the quantization procedure already at the level of point set structure.

Twistors are objects in a four-dimensional complex vector space, all of whose mathematical properties can be deduced from the correspondence with Minkowski spacetime through

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}. \tag{33}$$

Such a construction is only tenable in four dimensions and its initial motivation was to provide a unifying mathematical framework for quantum gravity with the complexity of the space naturally accounting for the complex quantum mechanics. Besides that, the twistor formalism naturally arises in the equations of motion of massless particles of arbitrary spin on a flat background. Even though one cannot say that the framework is developed enough to provide a fully functioning quantum theory of gravity - compared with other approaches the research in this direction has received less attention - there has been a recent upsurge of interest in twistors because of their connection with perturbative calculations in string theory.

For the sake of completeness, let us now mention some of the more speculative options besides the more conventional ones mentioned up until now. These options include Einstein-Cartan theory, thermodynamic gravity or non-fundamental quantum mechanics. Out of many possible completions of general relativity we chose Einstein-Cartan theory because of well-known uniqueness theorems that state that any extension of general relativity must reduce to it or to standard GR for low enough energies. One starts here by relaxing the requirement for the connection to be torsion-less and metric compatible. The resulting theory then naturally allows for coupling between spin currents and spacetime torsion and seems to provide a satisfactory resolution of a Big Bang singularity. Some general aspects of its quantization have been recently clarified although from the general point of view the viability of the theory remains in question. The idea of gravity as a force of thermodynamical origin, in turn, finds its basis in considerations of an entanglement entropy and heat fluxes across event horizons. The result is that general relativity can be consistently interpreted as an equation of state for some as yet unknown microscopic degrees of freedom [28]. The consequence is that "it may be no more appropriate to quantize the Einstein equation than it would be to quantize the wave equation for sound in air"[28]. Related to this point is the idea of entropic gravity which does away with the need of quantization completely, since gravity is interpreted as a force of entropic origin caused by a change in the amount of information associated with the positions of bodies of matter. Here the most important consequence is that at cosmological distances, the gravitational force decreases with an inverse linear law instead of the squared inverse law of general relativity or Newtonian dynamics. Surprisingly enough, this is capable of accounting for the rotational curves of galaxies without the need to invoke dark matter[13] Despite these partial successes, however, there is an ongoing argument about whether the theory implies gravity-induced quantum decoherence in a double-slit experiment which would be at odds with the observations and would decisively disprove it [32]. Finally the most speculative and the least developed of the three options is the non-fundamental quantum mechanics scenario. Even though it does not find support in available experimental evidence, it very closely bears on the interpretational issues of quantum mechanics, in particular when considered in the cosmological settings where no reasonable notion of an external observer exists. There have been some ideas linking this scenario with holographical principle and quantum information theory [34]. It is clear that if the framework of quantum mechanics is not fundamental, it may not be the best idea to try to construct a theory of quantum gravity but it might make rather sense to unify or make gravity compatible with the 'true' underlying framework.

Having gone through some of the approaches and more or less exotic ideas with respect to what form the quantum gravity could take[14], let us now try to come up with some hopefully useful criterion that somehow distinguishes between them. With a scarce empirical input and an absence of exact falsifiable predictions, one of the most important criteria is that of mathematical consistency. Fortunately, in simplified circumstances this question can be settled directly. One concrete example of such simplified circumstances on which we focus here is general relativity

[13] Dark energy is explained endogenously in this framework [50].

[14] We purposefully omitted Hawking's Euclidean quantum gravity proposal whose methods have been gradually absorbed into other approaches mentioned above. We also skipped some more speculative proposals, such as the DGP model or the superfluid vacuum theory, that so far lack enough structure to make their quantum gravitational aspects transparent.

in 2+1-dimensions. Without going into too much detail, let us summarize that in this case, gravity can be quantized either by first getting rid of the gauge freedom and postulating commutation relations in the reduced phase space[15] or alternatively by following literally the Dirac recipe for constrained systems outlined in the following and solving the resulting *Wheeler-de Witt* equation[16]. Furthermore, one may successfully employ path integral techniques to achieve again a consistent result bypassing the requirement of no topology change imposed by the two last options. The conclusion then is that no additional structure is needed to obtain a quantum general relativity theory in $2+1$ dimensions. This constitutes an important piece of indirect evidence against secondary quantum gravity theories and some of the non-orthodox approaches in general.

0.5 Loop quantum gravity

In the last section, we have given a discussion of various different approaches to quantum gravity, sketching their respective points of departure and how they relate between themselves. This allowed us to show and compare their respective assets and drawbacks from a more general perspective. In this section we build upon this and justify our selection of one particular approach - loop quantum gravity, as a promising candidate for quantum theory of gravity. Under this designation we mean a union of *loop* approach discussed under canonical approaches and explicitly covariant spinfoams.

The discussion in the last paragraph of the last section represents an argument against addition of extra structure to general relativity solely in order to be able to carry out its quantization. By this it is meant that this extra structure may be realized in Nature but it is by no means necessary for the attainability and consistency of quantum gravity. Formulated in this way, this gives a slight counterevidence[17] against the assertions and conceptual bases of string theory, Einstein-Cartan theory, thermodynamic gravity and non-fundamental quantum mechanics. This, however, cannot be extended to positive evidence for some the remaining approaches. What one can nevertheless do is that one can argue, following the discussion in [49], that what sets loop quantum gravity apart from the other approaches is that the theory is explicitly constructed by closely paying heed to a closely related concept: the concept of minimality. Under its premises, one takes the two empirically validated properties of Nature and tries to explore the consequences of consistently combining them into a single logical unit. If this combination turns out to be untenable then one stops and sees exactly what extra structures are needed in order to do so, rather than guessing them. This methodological clarity can be taken to be the first major argument in favor of LQG.

Second, an up-until-now failure of perturbation expansion-based approaches to provide a working theory of quantum gravity does not imply that the same cannot by achieved by different means. This pertains to string theory only to a limited degree, because as we noted above, the theory cannot be seen as a mere extrapolation of a perturbation quantization of non-interacting gravitons, but it

[15] with several distinct ways how to implement this [14]

[16] the same remark applying here

[17] It is clear that such an assertion cannot be taken too literally unless explicitly proven.

does apply to it because the concept of graviton features prominently and *neccessarily* in it and because non-renormalizable perturbative gravity plus corrections is naturally contained in it as a low-energy limit. The same assertion works also in the other direction: virtually all known interacting theories cannot be quantized exactly (non-perturbatively) but they can be separated into a free field part plus a small correction coupled through interaction constant which is responsible for the nontrivial processes, i.e. a first step of perturbative quantization.

Related to this, one can say that even if the perturbative quantization of a given system does make sense, it may happen that it provides wrong physical insights. An example of such a situation is a standard harmonic oscillator with a Hamiltonian $H = p^2 + \omega^2 q^2$. Treating the potential energy part $\omega^2 q^2$ as a small perturbation in a parameter ω to a free field part p^2, one does not get the discrete energy states as one would expect, what one gets instead, no matter how large ω is, are the continuous eigenstates of the free theory. Because of the peculiar properties of the gravitational field mentioned in the very beginning, one is lead to believe that similar scenario is very likely to be relevant also for quantum gravity.

The third argument in favor of LQG, connected with the terminological remark above, is that it turns out that as a framework LQG admits two independent formulations that yield (very probably) the same physical results. This equivalence between canonical and path-integral formulation has been proven rigorously in 3 dimensions where an exact form of a projector to a physical Hilbert subspace can be given, this projector induces a physical scalar product on H_{phys} and it can be given a rigorous interpretation as a sum of transition amplitudes associated to discrete structures connecting an initial and final triangulation (spinfoams)[39]. Numerical evidence seems to support the claim that this equivalence is also valid in higher dimensions. This result, therefore, reveals internal robustness of loop quantum gravity with respect to what starting point is chosen in its investigation.

Finally, the fourth reason for the well-posedness of LQG as a quantum theory of gravity is its relatively high level of maturity which puts it in a good position with respect to the phenomenological considerations of the previous section. In this respect it differs from most of the other approaches that we enumerated. Generally speaking, the research into applications of LQG has sprung up in two principal directions: the first one of them is concerned with giving a microscopic statistical derivation of black hole entropy (implied by the Hawking radiation derived above), while the second one is the application of symmetry reduced models to problems in cosmology (loop quantum cosmology). In the first case, the breakthrough insight of LQG is that the black hole entropy can be evaluated by considering a number of spin network states with a link piercing the event horizon. This link naturally carries a quantum of area that contributes to event horizon's area which in turn is associated with the black hole's entropy according to the Hawking-Bekenstein formula

$$S_{BH} = \frac{A}{4l_p^2}. \tag{34}$$

To replicate this, two methods can be used. One the one hand, microcanonical considerations, that take into account only geometrical excitations with no degeneracy induced by non-geometric degrees of freedom, lead to a result

$$S = \frac{\gamma_0}{\gamma} \frac{A}{4l_p^2} \tag{35}$$

where γ_0 is a numerical factor of order 1 and γ is a Barbero-Immirzi constant. Since the value of the later parameter is arbitrary, one can interpret this as a condition for its value $\gamma = \gamma_0$. Improved canonical methods on the other hand give

$$S = \frac{A}{4l_p^2} + \sqrt{\frac{\pi A}{6\gamma l_p^2}}, \tag{36}$$

i.e. Barbero-Immirzi constant enters only as a quantum correction to the semi-classical formula. Here one makes use of a qualitative behavior of matter-induced degeneracies suggested by QFT with a cut-off near the black hole horizon [9]. In the case of loop quantum cosmology, the motivation is to probe the framework of loop quantum gravity under simplified circumstances. These circumstances are provided by a symmetry reduction of the full Einstein action, in other words, in the full phase space of general relativity one chooses a subset consistent with some given Killing vectors. This subset is then parameterized in terms of convenient variables and one proceeds with the quantization according to the general recipe. Depending on the degrees of freedom remaining after the symmetry reduction, one distinguishes minisuperspace models (no field-theory degrees of freedom) and midisuperspace models (some field-theory degrees of freedom, but less than in the full theory). One of the most important robust results of this approach is the avoidance of the initial cosmological singularity. On the other hand, a rigorous perturbation theory on top of mini- and midisuperspace models is in a position to provide falsifiable predictions regarding the evolution of the early Universe. That being said, it is important to note that the exact relation between LQC models and cosmological sector of full LQG has not yet been established.

The rest of this thesis is written guided and elucidated by these arguments. It is clear that at a bare minimum they constitute a solid and quite waterproof starting point for studying the deep, mysterious and tantalizing questions of quantum gravity.

0.6 Literature

The previous sections tried to approach the issue of quantum gravity and LQG in particular from a fairly systematic general standpoint. In this section, in accordance with the pedagogical side of this text, which will be clarified in the next section, we offer a review of literature devoted to LQG. The aim is to present the books and articles that deal with LQG and comment on them in a self-contained way. This should constitute, so we hope at least, a useful knowledge for whoever is setting out to study LQG.

We divide our discussion of literature into two parts. In the first one, we consider books and monographs and in the second one review articles. First, let us take into account books devoted solely to LQG [49], [44], [47] and [22]. Beginning with the first one, [49] is both by extent and by depth the magnum opus of the field. This comes as no surprise given author's prominent role in the development of the theory. Its strongest point is the discussion of mathematical underpinnings of canonical LQG where the exposition is impeccable. Other strong points include detailed discussion of the problem of time and prequantization reformulation of general relativity. The parts on applications and spinfoam formalism, however, do not offer an up-to-date treatment as there has been significant development in those areas since the book's publication and these lay somewhat further apart from the book's main emphasis. [44] is another book by one LQG's eminent figures. While covering similar material, compared to the previous publication, it focuses more on the philosophical aspects and conceptual framework. On the other hand, because of this, the exposition of the kinematical and dynamical aspects of LQG is perhaps more crisp and compact than in the previous case. [47] is a book explicitly focused on spinfoam formalism and the only one to be so. As such, some of the cornerstones of canonical LQG such as Ashtekar-Barbero variables are missing entirely in the exposition (not to the detriment of the matter). It presents the minimum amount of knowledge needed to get a grasp of the spinfoam formalism in a self-contained way, along with a clear exposition of the Euclidean 3D gravity and recent applications of spinfoam formalism in cosmology and black hole physics. On the drawback side, vast material covered and small amount of pages come at the price of detailed explanations which, however, would not be very in keeping with the explicit introductory character of the book. The last book [22] supposes substantially lower knowledge of general relativity and quantum field theory and is meant to be an introductory text for undergraduate students. Despite this, the second half of the book raises many pertinent technical points about some important conceptual issues, such as the role of quantization ambiguities or various attempts to access phenomenology.

Another group of books is the one from which one can draw the information about LQG even though the subject matter is broader than that [31], [6], [21], [14], [26]. The advantage in doing so is that one gets a broader perspective on the issue at hand. Thoroughly recommended are especially the books [31], [14], [26]. From the books mentioned, [31] is the most general one applying a holistic and systematic approach to quantum gravity. Besides detailed exposition of path integral and canonical methods, there is also a succinct introduction to supergravity. [14], on the other hand, focuses solely on the quantization programs as they relate to the gravity in $2+1$ dimensions. This simplified setting makes many conceptual issues of quantum gravity more transparent, such as the problem of time or topology change. The approach taken is descriptive and impartial. From the books mentioned, [26] is the one written most from a particle physics perspective and as such it covers much larger amount of material than exclusively quantum gravity. It generally deals with systems with gauge symmetry and mathematical framework for its description. In this sense it is very instructive to see how gauge symmetry is implemented in general relativity compared to other field theories since this has repercussions for the quantum theory. Both books [6] and [21] carry a distinct mathematical tone with the first one of them being somewhat more elementary in its exposition. They both focus on the mathematical underpinnings of the loop quantum gravity ([21] is somewhat more centered on providing a comprehensive overview of how the concept of *loop* features in current physical theories.), but they are written from a markedly less practitioner's point of view compared with [26].

Let us now focus our attention on the second part of literature - LQG reviews and other articles written with the aim to introduce the reader into LQG. It is clear that it is not possible to be entirely exhaustive in giving a review of this literature so let us rather try to pinpoint some salient features of it which could be found useful when reading it.

Publications [12], [13], [7], [48] are all introductory-level texts dealing with the canonical LQG. All of them contain all the important relevant points to be made and can thus be read interchangeably. It is worth mentioning, though, that [12] supposes markedly lower level of initial knowledge and the exposition seems to lose clarity at some points while [13], on the other hand, offers the most up-to-date review and the discussion contains a lot of valuable insights, especially in connection to loop quantum cosmology. [45] is a compact and comprehensive review of covariant LQG and it offers very interesting 'definitory' approach in the exposition while rigorous derivations are exposed only at the end. There is a detailed discussion of all the important points of the theory and a practical demonstration of the introduced concepts in a concrete computation in spinfoam cosmology.

Another wide group of literature is theses and dissertations of researchers in the field. Usually, they contain a review and a research part with the review part being a convenient source of information. [25], [19] are both Masters theses devoted to canonical LQG whereas [41], [42], [35] are all dissertations on covariant LQG. [35] puts more emphasis on providing intuitive understanding with the scope going from foundational issues to the geometrical aspects whereas [42] delves deeper into the mathematical details and discusses also the matter couplings and coherent states. Both [42] and [35] are very well-structured introductions into path-integral LQG discussing all relevant aspects in comprehensible terms with [35] being written in probably somewhat less demanding style and [42] centering more on enumerative exposition of various 4D spinfoam models. [41], on the other hand, besides providing a cogent explanation of the basics of LQG, deals predominantly with the asymptotical analysis. It is important to point out that every one of the cited works offers slightly different perspective and is rich in interesting insights so that it is well worth of reader's attention even though the main points and arguments logically repeat themselves.

Let us conclude this review of literature by mentioning some of the works that do not fit exactly either category brought up so far. These would be the works [37], [38], [33], [40]. [37] and [38] are especially interesting because they are reviews written by researchers working in string theory and so they present LQG from the point of view of this theory. Unfortunately, both are already a bit dated but well worth reading. [33] and [40] in turn present spinfoam formalism from the viewpoint of *group field theory* - a suitable generalization and a mathematical backbone of the spinfoam approach. Let us mention that [40] contains especially interesting discussion of a *continuity* limit of GFT and LQG models rather than the semiclassical one, a topic often not explicitly mentioned in other works. Even though it is unfortunately not possible to comment on every article one could possibly turn to in search of an understanding of LQG, we hope we have at least pointed reader to some of the more convenient places to start.

0.7 Structure of the thesis

Having turned to issues of clear bibliographical streak, this last part finally clarifies the contents and organization of this thesis. It is composed of two parts - a review part and a research part. The review part one takes up the first and the second chapter where the introductions to the canonical and covariant LQG are provided. The organization of the first chapter follows more or less the traditional approach to the canonical LQG. The subsections 1.2, 1.3 sum up the necessary prequantization reformulation of GR so as to be amenable to the chosen quantization method. The subsection points out some salient features of GR turn up time and again during the procedure. The quantization itself is examined in the subsections 1.4, 1.5 and 1.8. The reason for leaving the treatment of the dynamical aspects for the last part is, besides a less corroborated nature of the results, the substantial increase in difficulty compared to that involved in the treatment of the kinematical aspects. The sections 1.6, 1.7 explore some of the consequences and results of the quantization with respect to some physically relevant questions. We also ponder the question of *uniqueness* of the canonical LQG derivation in there.

The second, shorter, chapter introduces the covariant approach to the quantization of GR and follows more or less an idiosyncratic path. This is allowed by the less standardized approach to the subject. The subsection 1.1 presents the core idea of the path integral in the context of a simple mechanical system - a point particle moving in a potential field in one spatial dimension. The main aspects of the construction remain valid in the quantum gravity setting as well. The second subsection introduces the microscopic ansatz that is used to categorize all varieties of the spinfoam models - i.e. the concrete implementations of the path integral quantization idea. The latter part of this subsection is devoted to the Ponzano-Regge model, given its historical importance as well as practical importance within the scope of this thesis. In part 2.3, we finally discuss various of describing realistic general relativity through a spinfoam model, the key vehicle being here so-called simplicity constraints which transform a non-physical theory to GR.

It should be pointed out that, generally speaking, we tried to keep the exposition style somewhat dense in order to avoid any hand-waving arguments.

In the research part, in the third chapter, we summmarize the results of our calculations. For these, we chose the Ponzano-Regge model as it is the simplest option with possibly non-trivial results. The structure of the chapter follows a standard pattern - first we clarify the questions and methodology, then we present the results and these results we then discuss in the final subsection. We also show explicitly the Mathematica code used in our calculations in the appendix.

1. Canonical loop quantum gravity

Our final goal in this chapter is to see the structure of the resulting theory when one applies the canonical quantization algorithm mentioned above to the action (21) and employing the loop rather than the geometrodynamics approach. Each of the following subchapters is devoted to one logically self-contained unit of this procedure.

1.1 Gauge invariance and general covariance

We begin our exposition with a detour into two important aspects of action (21), which we repeat here for convenience

$$S_{EH} = \frac{1}{16}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{8}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K + S_{matter}, \tag{1.1}$$

namely a *gauge symmetry* and a *general covariance*.

Let us begin with the first one mentioned. Generally speaking, gauge symmetry is an aspect of a theory with a redundancy in the description. This redundancy is manifested in using more degrees of freedom than what is strictly necessary with the extra degrees of freedom being *pure gauge*. This can be interpreted as a freedom to choose a reference frame in each moment during the dynamical evolution, the general solution of the equations of motion than invariably contains arbitrary functions of time.

In practice, gauge theories are encountered for two reasons. Often one is given a Lagrangian with given symmetries and does not know beforehand which degrees of freedom are physical and which are a pure gauge. Or, because of mathematical considerations, it might be more expedient to describe the system with more redundancy and symmetry rather than the other way around. Since gauge symmetries and the mathematical description thereof is a vast research area in both mathematics and physics that in some way touches upon many recent results in both of these areas, we are necessarily incomplete in our subsequent treatment and restrict our attention only to the aspects most relevant for the gravity.

One aspect of gauge theories that is important with respect to our subsequent discussion is that of a *constraint* as a means of cutting back on the redundancy introduced by gauge principle. Considering a theory with a Lagrangian $L(q,\dot{q})$, a tell-tale sign that it contains constraints is that its Hessian matrix is not invertible

$$\det \frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}} = 0. \tag{1.2}$$

This results in the phase (q,p)-space not having the same dimensionality as the $(q,\dot{q})$-space of the Lagrangian formulation with the equations of a form

$$\phi_m(q,p) = 0, \quad m = 1,\dots,M \tag{1.3}$$

defining a surface in it. This surface is called the *constraint surface* and the relevant equations are called primary constraints. Constraints of higher order (secondary, tertiary...) are the requirements of time preservation of the constraints of lower order. So, for example, in case there are only M primary constraints that give rise to J secondary ones, the dimensionality of the constraint surface is $\dim P = 2N - M - J$, considering N physical as well as gauge degrees of freedom. Another criterion that differentiates between constraints is that of being first- or second-class[1]. For the *first-class* constraints, the Poisson bracket of it with all the remaining constraints vanishes on the constraint surface[2]

$$[\phi, \phi_i] \approx 0, \tag{1.4}$$

otherwise one has a *second-class* constraint. Unlike the previous case, the distinction between first- and second-class constraints is going to have important implications for the quantization program which we will describe later on. One important point to note is that primary first-class constraints always generate gauge transformations. These transformations in infinitesimal form read

$$\delta x_i = \delta u^a(x,p)[x_i, \phi_a(x,p)] \tag{1.5a}$$
$$\delta p_i = \delta u^a(x,p)[p_i, \phi_a(x,p)], \tag{1.5b}$$

where $u^a(x,p)$ is the Lagrange multiplier corresponding to the constraint $\phi_a(x,p)$. A point of certain subtlety here is that one usually hears that *all* the first class constraints generate these transformations. This is actually the contents of Dirac conjecture. It is not actually true but it holds in all physically relevant situations. The way around this is that postulates the validity of the Dirac conjecture with the point of view being taken that not necessarily does the original Lagrangian formulation reflect the gauge symmetry structure of the system[3].

Some important properties of the first-class constraints are the following:

1. Gauge transformations preserve the constraint surface. This follows directly from the definition.

2. Poisson bracket of two first-class constraints is again a first-class constraint, yielding a structure of algebra of constraint functions. To see this, let us consider two first-class constraints ϕ_I, ϕ_{II}. We can then write

$$[\phi_I, \phi_j] = f_j^{j'} \phi_{j'}, \quad [\phi_{II}, \phi_j] = g_j^{j'} \phi_{j'}. \tag{1.6}$$

For the Poisson bracket of ϕ_I, ϕ_{II} we then get using Jacobi's identity

$$\begin{aligned}
[[\phi_I, \phi_{II}], \phi_j] &= [\phi_I, [\phi_{II}, \phi_j]] - [\phi_{II}, [\phi_I, \phi_j]] \\
&= \left[\phi_I, g_j^{j'} \phi_{j'}\right] - \left[\phi_{II}, f_j^{j'} \phi_{j'}\right] \\
&= \left[\phi_I, g_j^{j'}\right] \phi_{j'} + g_j^{j'} f_{j'}^{j''} \phi_{j''} - \left[\phi_{II}, f_j^{j'}\right] \phi_{j'} + f_j^{j'} g_{j'}^{j''} \phi_{j''} \\
&\approx 0.
\end{aligned}$$

[1]This dichotomy in terminology reflects several distinct research groups working on the issue in the formative years.

[2]For these cases we reserve the notation $\approx$.

[3]For an example of this, see [26], p.19

3. All second-class constraints can be interpreted as a gauge fixing condition in systems with higher symmetry. This can be seen as follows. The gauge fixing is done by imposing gauge fixing conditions of a general form

$$C_b(q, p) \approx 0. \tag{1.7}$$

In order for this to be well-posed one needs to ensure that

- from any point on the constraint surface, there exists a gauge transformation that maps it to the surface spanned by (1.7) (i.e. the surface spanned by (1.7) intersects all the *gauge orbits*[4] on the constraint surface) and
- no more gauge freedom is left after fixing the gauge, i.e. no non-trivial gauge transformation preserves (1.7).

This second condition can be reformulated as requiring that

$$\delta u^a [C_b, \phi_a] \approx 0 \tag{1.8}$$

should imply $\delta u^a = 0$. This is nothing else than stating that a phase space function C_b, when viewed as a constraint, is second-class.

Given our introduction of the notion of a constraint as a means of cutting back on redundant degrees of freedom, it follows that any gauge theory is automatically also a constrained theory. Let us consider the question whether this also remains valid the other way around, if any constrained theory is necessarily also a gauge theory. Given the up-until-now discussion, the answer is easy to find and is in the negative. Since it is the first-class constraints that generate gauge transformations according to our convention, all that is needed to provide a counterexample to the assertion is to consider a system with only second-class constraints. Then after their imposition, there is a one-to-one correspondence between physical states of the system and points on the constraint surface. An elementary example of such a system would be

$$L = \frac{1}{2} m (v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2), \tag{1.9}$$

which describes a particle moving on a circle with a radius r under radial force q_3 in the $q_1 - q_2$-plane. It is straightforward to check that all the relevant properties hold.

In ending this brief summary of some of the important points related to the gauge symmetry, let us give a brief remark on the different treatment of first- and second-class constraints in quantization. The generic canonical quantization algorithm, as succinctly introduced in elementary terms in the previous chapter, was formulated quite generally and did not differentiate between these two types of constraints. One should bear in mind that it is meant to apply *only* to first-class constraints, whereas second-class constraints are dealt with according to the following procedure. At the classical level, the structure of Poisson bracket is changed according to

[4] curves obtained through successive iteration of (1.5a),(1.5b)

$$[A,B]_* = [A,B] - [A,\chi_a]\Lambda^{ab}[\chi_b,B], \tag{1.10}$$

where Λ^{ab} is the matrix inverse of the matrix of Poisson brackets between the second-class constraints χ_a, χ_b. This structure is called the Dirac bracket and one uses it in standard correspondence rules

$$[\hat{A},\hat{B}] = i\hbar\widehat{[A,B]_*} \tag{1.11}$$

to go about building quantum theory structure. Second-class constraints then effectively reduce to on-shell identities on the constraint surface whenever Λ^{ab} is finite everywhere on it.

1.1.1 General covariance

Let us now turn our attention to the second notion brought up at the beginning - general covariance. In the default view of the Hamiltonian dynamics, time is implicitly assumed to be an externally given observable quantity that in an objective manner continuously parametrizes individual solutions of equations of motion. Such an identification then gives direct physical significance to the value of a canonical variable for a given value of t. It turns out however that such a step is not necessary and we can treat time t, on a formal level, as one of the degrees of freedom. The dynamical content of the theory is then alternatively expressed by means of correlations of physical variables among themselves. A system with such a feature is called *generally covariant*.

There are two consequences for the structure of the theory. First, there is an invariance with respect to time reparametrizations $t \to \tau(t)$ for $\tau(t)$ monotonous and second, Hamiltonian vanishes on the constraint surface, i.e. Hamiltonian-induced time-flow on the physical configuration ceases. These two points can be understood as contents of a term one often hears in relation to quantum general relativity - a *problem of time*. Put another way, the problem of time in generally covariant theories is that there is no well-defined useful notion of time. Generally speaking, generally covariant systems can come in two forms - one might perhaps be called *obtained* and the other one *constructed*[5].

We illustrate the meaning of all these points on a concrete example of a system with canonical variables q^n, p_n, a Hamiltonian H_0 and a first-class and second-class constraints γ_a and χ_b respectively. The dynamics of this system can be obtained through variation of the action

$$S[q^n(t),p_n(t),u^a(t),u^b(t)] = \int_{t_1}^{t_2}\left(p_n\frac{dq^n}{dt} - H_0 - u^a\gamma_a - u^b\chi_b\right)dt, \tag{1.12}$$

where extremization with respect to u^a, u^b enforces the constraints as required. Now introducing $q^0 \equiv t$ with its canonically conjugate momentum p_0, the action that yields identical dynamics reads

[5] This is incidentally somewhat reminiscent of the old *nature* vs. *nurture* distinction and debate.

$$S[q^0(\tau), q^n(\tau), p_0(\tau), p_n(\tau), u^0(\tau), u^a(\tau), u^b(\tau)] =$$
$$= \int_{\tau_1}^{\tau_2} \left(p_0\dot{q}^0 + p_n\dot{q}^n - u^0(p_0 + H_0) - u^0u^a\gamma_a - u^0u^b\chi_b\right) d\tau. \quad (1.13)$$

But this action is of a form

$$S = \int_{\tau_1}^{\tau_2} \left(p_\mu\dot{q}^\mu - u^{a'}\gamma_{a'} - u^{b'}\chi_{b'}\right) d\tau, \quad (1.14)$$

with substitutions $u^0u^a = u^{a'}$, $u^0u^b = u^{b'}$, $\gamma_0 = p_0 + H_0$, where it is apparent that the Hamiltonian is but a linear combination of first-class and second-class constraints that vanish on-shell, one of them encoding the full dynamical description of the system. This is a generic procedure of "general-covariantization" of a general dynamical system. It is, therefore, an example of a constructed general covariant theory. In case the physical Hamiltonian H_0 is identically zero, the extended Hamiltonian already *is* a linear combination of constraints and one would denote it as general covariant theory "obtained".

Regarding the point about reparametrization invariance, let us proceed in a roundabout way. Instead of proving the statement directly, let us infer what conditions does the validity of the invariance impose on the transformation properties of particular variables, concluding that they are indeed valid for physically interesting cases. The action (1.14) changes under an infinitesimal reparametrization $\tau \to \bar{\tau} = \tau - \varepsilon(\tau)$ according to

$$\delta S = \int_{\tau_1}^{\tau_2} (\delta p_\mu\dot{q}^\mu + p_\mu\delta\dot{q}^\mu - \delta u^{a'}\gamma_{a'} - \delta u^{b'}\chi_{b'} - u^{b'}\delta\chi_{b'})d\tau. \quad (1.15)$$

Putting this equal to zero requires

$$\delta q^\mu = \dot{q}^\mu\varepsilon \quad (1.16a)$$
$$\delta p_\mu = \dot{p}_\mu\varepsilon \quad (1.16b)$$
$$\delta u^{a'} = (u^{a'}\varepsilon)^\cdot, \quad \delta u^{b'} = (u^{b'}\varepsilon)^\cdot \quad (1.16c)$$
$$\delta\gamma_{a'} = \dot{\gamma}_{a'}\varepsilon, \quad \delta\chi_{b'} = \dot{\chi}_{b'}\varepsilon \quad (1.16d)$$

because taking into consideration, for example, the first two terms in (1.15), one has

$$\delta p_\mu\dot{q}^\mu + p_\mu\delta\dot{q}^\mu = \dot{p}_\mu\varepsilon\dot{q}^\mu + p_\mu(\dot{q}^\mu\varepsilon)^\cdot = \dot{p}_\mu\varepsilon\dot{q}^\mu - \dot{p}_\mu\dot{q}^\mu\varepsilon = 0, \quad \varepsilon(\tau_1) = \varepsilon(\tau_2) = 0, \quad (1.17)$$

with the rest of the terms following a similar logic. Equations (1.16a), (1.16b), (1.16d) are valid for coordinates and momenta q^μ, p_μ that one can measure in their standard units whereas one can always redefine Lagrange multipliers $u^{a'}$, $u^{b'}$ so as to match equation (1.16c). In closing, it also deserves mentioning that general covariance is sometimes equivalently called reparametrization invariance.

1.2 ADM decomposition

Having reviewed some important properties of general relativity - gauge invariance and general covariance, the natural question is what consequences do these have for the quantization. Quite generally, as far as canonical techniques are concerned, there are two ways that constraints can be dealt with on a quantum level. The first one of them is to impose constraints on a classical level and then to carry on with the quantization on the reduced phase space. This amounts to quantizing only the gauge invariant functions that are defined on the space of equivalence classes of gauge orbits. In other words, one must find a complete set of gauge invariant functions and define the quantum space as the irreducible representation space of the commutation relations of this complete set. This procedure does encounter some difficulties on a technical level which is why it is not, in spite of its intuitive appeal, the usual method of choice. These difficulties include the fact that after imposing the gauge fixing conditions, one may spoil a manifest invariance under an important symmetry or destroy locality in the case of field theories. If this is not the case, one may still find the resulting commutation relations too cumbersome to represent irreducibly on a quantum level.

The second approach is to first postulate commutation relations on the extended set of both physical and gauge degrees of freedom and to select the physical states as the ones annihilated by a suitably chosen condition. If we define the physical states as those that are invariant with respect to gauge transformations of a form

$$\hat{U} = e^{i\varepsilon^{\alpha}\hat{G}_{\alpha}} \tag{1.18}$$

then it is not difficult to see that the relevant condition on the physical states reads

$$\hat{G}_{\alpha}|\psi\rangle = 0. \tag{1.19}$$

Imposing the commutation relations on the whole set of variables that one starts with bypasses the technical issue of finding a suitable representation for the possibly highly non-trivially related reduced phase space functions since one can always opt for a reasonably simple one (possibly a standard representation) in the extended space. It needs to be pointed out however that a special care needs to be taken in order to avoid gauge anomalies and wrongly defined scalar product [49]. Fortunately, in the following, we will not encounter any of these difficulties.

Loop quantum gravity in its canonical version is based, on the fundamental level, on this second approach, The Dirac approach to quantization of gauge theories, and the overarching heuristics that guides its development is to make rigorous sense of the operator equations (1.19). One part of this is to provide a sufficient mathematical framework to support these equations. In this and the next section we do this on a prequantization level, i.e. we reformulate general relativity in terms of variables that are appropriate for both the Hamiltonian analysis and the subsequent steps, while in the following section we proceed to the quantization proper and in particular explain in detail how the Dirac algorithm for generic canonical theory is implemented in the case of GR. While a

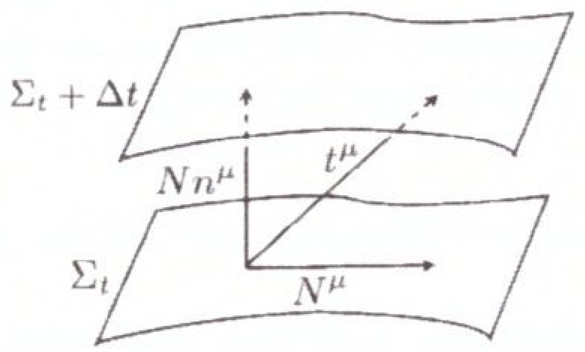

Figure 1.1: ADM decomposition

mere reformulation does not of course in general change the substance, from the practical point of view on a quantum level it might mean a difference between tractable and intractable mathematical formulations. This is the sense in terms of which we gauge "appropriateness". Let us now develop the *ADM decomposition of general relativity*, historically a first important step towards the canonical LQG.

Our aim is to derive the ADM variables and to perform a basic Hamiltonian analysis in them. Beginning with a spacetime $\mathcal{M}$ and the action (1.1), the first assumption that needs to be made is that the spacetime has a topology $\mathbb{R} \times \Sigma$. This is justified by introducing a foliation of $\mathcal{M}$ by a family of hypersurfaces Σ_t parametrized by a time variable t. Picking up a preferred time variable is a necessary step in finding a Hamiltonian formulation of the theory but it is important to point out that this does not destroy the reparametrization invariance of general relativity, which is conserved by the fact that the foliation is *arbitrary*.[6]

As a first step, the selection of the preferred time is done by choosing a vector field t^μ on $\mathcal{M}$ or equivalently a scalar function t which is constant on Σ_t with the relation between the two being $t^\mu \nabla_\mu t = 0$. t^μ is next decomposed into components normal and tangential to Σ_t via

$$t^\mu = N^\mu + Nn^\mu, \tag{1.20}$$

with the normal n^μ normalized as $n^\mu n_\mu = -1$ and N, N^μ being called the lapse function and shift vector respectively (see Fig.1). This decomposition allows one to introduce a three-dimensional metric on Σ_t through

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{1.21}$$

One would have a minus sign in front of the normal term in the case the signature of the spacetime were Riemannian. It is easy to check that the quantity h^μ_ν acts as a projector to the three-dimensional space of the spatial slice Σ_t, it can therefore safely be used to raise and lower indices of any three-dimensional objects.

[6]In fact, choosing a given foliation can be viewed as a partial gauge fixing in the full diffeomorphism group on account of the requirement on the hypersurface to have a spacelike metric.

Choosing coordinates such that $t^\mu = (1,0,0,0)$ and plugging this to (1.20), (1.21) one gets

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab} \end{pmatrix} \tag{1.22}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N_b}{N^2} \\ \frac{N_c}{N^2} & h^{ab} - \frac{N^a N^b}{N^2} \end{pmatrix}. \tag{1.23}$$

This allows one to establish that there is an equivalent information contained in the four-dimensional metric $g_{\mu\nu}$ and in the objects N, N^a, h_{ab} that live on the three-dimensional slice Σ_t. This, in turn, suggests that we can view the four-dimensional spacetime $\mathcal{M}$ as a dynamical evolution of fields defined on a fixed three-dimensional slice. Thus we are lead to choose the variable h_{ab} as the new "position" variable for the Hamiltonian reformulation.

In order to get to the canonically conjugate momentum for h_{ab}, let us consider a quantity

$$K_{\mu\nu} = h_\mu^\sigma \delta_\sigma n_\nu. \tag{1.24}$$

It can be shown that this tensor is a purely spatial quantity that describes how the vectors change during parallel transport as a result of embedding or external curvature - this corresponds to the occurrence of a normal in the definition. Alternatively, one can express it using Lie derivative as

$$K_{ab} = \frac{1}{2}\mathcal{L}_{n^\mu} h_{ab} \tag{1.25}$$

or, using covariant derivative compatible with the three-metric h_{ab}

$$K_{ab} = \frac{1}{2N}(N_{a|b} - N_{b|a} - \partial_t h_{ab}). \tag{1.26}$$

Now, we are ready to express the action (1.1), using the relation between the four-dimensional Riemann tensor of $\mathcal{M}$ and the three-dimensional Riemann tensor of Σ_t, the Gauss-Codazzi equation

$${}^{(3)}R_{\mu\nu\lambda}{}^{\rho} = h_\mu^{\mu'} h_\nu^{\nu'} h_\lambda^{\lambda'} h^\rho_{\rho'}\, {}^{(4)}R_{\mu'\nu'\lambda'}{}^{\rho'} - K_{\mu\lambda}K_n^\rho u + K_{\nu\lambda}K_\mu^\rho, \tag{1.27}$$

in terms of variables h_{ab}, K_{ab}. We get, omitting the boundary term for sake of convenience,

$$\mathcal{L} = \frac{1}{2\kappa}\sqrt{h}N[{}^{(3)}R - 2\Lambda + K_{ab}K^{ab} - K^2]. \tag{1.28}$$

The momentum conjugate to h_{ab} is

$$p^{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \frac{1}{2\kappa}\sqrt{h}(K^{ab} - Kh^{ab}) \tag{1.29}$$

and we easily see that variation of this action with respect to N^a, N imposes the constraints

$$V^b[h,p] = -2D_a(\frac{1}{\sqrt{h}}p^{ab}) = 0, \tag{1.30}$$

$$S[h,p] = -\sqrt{h}({}^{(3)}R - 2\Lambda - \frac{1}{h}p^{cd}p_{cd} + \frac{1}{2h}p^2) = 0, \tag{1.31}$$

called the the diffeomorphism (or vector) constraint and the Hamiltonian (or scalar) constraint respectively, where

$$D_a p^{ab} = \partial_a p^{ab} + \varepsilon^b_{\ jk} A^j_a p^{ak}. \tag{1.32}$$

Applying Legendre transform the action we obtain the Hamiltonian

$$\mathcal{H} = \frac{1}{2\kappa}\sqrt{h}(N(-{}^{(3)}R + \frac{1}{h}p^{cd}p_{cd} - \frac{1}{2h}p^2) - 2N_b D_a(\frac{1}{\sqrt{h}}p^{ab})). \tag{1.33}$$

Finally, we can summarize the whole dynamical content of general relativity as the equations (1.30), (1.31) plus the dynamical equations, obtained by variation of (1.28) with respect to q_{ab}, p^{ab},

$$\dot{h}_{ab} = \frac{\delta H}{\delta p^{ab}}, \tag{1.34}$$

$$\dot{p}_{ab} = -\frac{\delta H}{\delta h_{ab}}. \tag{1.35}$$

This is the ADM reformulation of general relativity. The Poisson brackets between the canonical variables are trivially

$$\{h_{ab}(x), p^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}. \tag{1.36}$$

We do not investigate any deeper the Hamiltonian structure of the theory because we are going to do so in the next section after introducing another set of variables.

1.3 Tetrads, connections and Ashtekar variables

In this section, we give a complete characterization of the phase space of general relativity. This analysis will serve as a point of departure for the latter quantization. We proceed in doing so in the following steps - first, we introduce triad variables on the spatial slice Σ_t which introduce an extra $SO(3)$ symmetry into the theory. Next, taking into account Palatini method we consider the form of connection compatible with the triad reformulation. This along with the exterior curvature of the ADM decomposition allows us to write down the sought-for Ashtekar-Barbero variables. We reformulate the constraints in terms of them, making the case for their use, and show the structure of the constraint algebra.

Since the space-time split made in the preceding section was made so as to make Σ_t spacelike for all t, we can introduce a triad field e^i_a on it via

$$h_{ab} = e^i_a e^j_b \delta_{ij}. \tag{1.37}$$

This quantity is represented by a 3×3 invertible matrix and we can interpret it as a triplet of 1-forms with the index i labeling the 1-form as a whole and the index a giving its coordinate components. As such, it is clear that one introduced a new $SO(3)$ symmetry into the theory because (1.37) is invariant with respect to the rotation in the Euclidean (or SO(3)) index i, $\bar{e}_i^a = S_i^j e_j^a$, on account of $S_i^j S_k^l \delta_{jl} = \delta_{ik}$ for $\forall S \in SO(3)$. This SO(3) invariance can be viewed as a left-over from the original Lorenzian symmetry after the partial gauge fixing was imposed by the foliation Σ. The coordinate index a, on the other hand, is acted on by coordinate transformations. The 1-form character of e_a^i can be made more explicit by contracting it with a basis of one-forms $f^i = e_a^i dx^a$. The equation (1.37) suggests that these quantities can be viewed as a square root of metric h_{ab}. Another way of interpreting them is as an isomorphism between the cotangent bundle at Σ_t and a Euclidean space. Inverting e_a^i we can, of course, get triad fields e_i^a that define an orthonormal basis at each spacetime point.

Let us now consider a densitized version of the fields e_a^i

$$E_a^i = \det e e_a^i. \tag{1.38}$$

The equation (1.37) together with (1.38) readily gives an alternative expression $E_a^i = \sqrt{\det h_{ab}} e_a^i$. Thus defined densitized form of the dual triad field will eventually play the role of a momentum. To derive the canonically conjugate quantity, let us consider

$$K_a^i = \frac{1}{\sqrt{\det E}} K_{ab} E_j^b \delta^{ij} \tag{1.39}$$

where K_{ab} is the external curvature as defined in (1.25), (1.26). A direct substitution of (1.38), (1.39) into (1.33) then yields the canonical Poisson brackets

$$\{E_j^a(x), K_b^i(y)\} = \kappa \delta_b^a \delta_j^i \delta(x-y), \tag{1.40a}$$

$$\{E_j^a(x), E_i^b(y)\} = \{K_a^j(x), K_b^i(y)\} = 0. \tag{1.40b}$$

One also finds that this also creates another term in (1.33) called the Gauss constraint of a form

$$G_i = \varepsilon_{ijk} E^{aj} K_a^k = 0. \tag{1.41}$$

Let us review what has just been achieved. By introducing a new variable, the densitised (dual) triad field e_i^a (resp. e_a^i), on top of the ADM formulation of general relativity, we found a new canonically conjugate pair of variables $(E_j^a(x), K_b^i(y))$. We want to proceed to the quantization which requires appropriately expressed constraints. In the case of a generally covariant theory their form is of utmost importance since, as was shown in the previous section, they hold the whole dynamical content of the theory. In the case of these variables, one finds that the constraints (1.30), (1.31) do not reduce to a convenient form and one is, therefore, motivated to inspect yet another reformulation of general relativity. This reformulation is provided by the Palatini method in the tetrad-connection formalism. We develop it in the following.

Beginning with the concept of a tetrad one finds that it is a straightforward generalization of the notion of triad introduced above. Forgetting for a while about the foliation Σ, the definitory relation for a dual tetrad reads $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$. Both indices now run through the values $\{0, 1, 2, 3\}$ and there is now a Lorentz symmetry connected with the I-index instead of $SO(3)$, with the respective transformations being $\bar{e}^I = \Lambda^I{}_J e^I$ acting on $e^I = e^I_\mu dx^\mu$ and the orthogonality condition ensuring invariance of the definition with respect to these transformations $\Lambda^I_K \Lambda^J_L \eta^{KL} = \eta_{IJ}$.

The Palatini formalism takes off the requirement of a torsion-free metric compatible connection. The connection thus becomes one of the dynamical degrees of freedom and in presence of tetrads, one usually denotes it as a *spin connection* ω_μ^{IJ}. One may contract it with dx^μ to get an object with purely Lorentzian indices $\omega^{IJ} = \omega_\mu^{IJ} dx^\mu$. The covariant derivative is then defined according to $\nabla_\mu v^I = \partial_\mu v^I + \omega_\mu{}^I{}_J v^J$. In the case of tetrads, the requirement of metricity is expressed by

$$\nabla_\mu e^I_\nu = 0 \tag{1.42}$$

which is equivalent to an antisymmetry in the Lorentzian indices of ω^{IJ}. Torsion freeness, one the other hand, is present whenever

$$de^I + \omega^I{}_J \wedge e^J = 0. \tag{1.43}$$

This second equation is the first Cartan equation of structure. The solution to these two equations is unique, it is usually denoted $\omega^I{}_J[e]$ to emphasize the dependence on a tetrad, and can be shown to read

$$\omega_\mu{}^I{}_J[e] = e^I_\rho \Gamma^\rho_{\mu\nu}[g] e^\nu{}_J + e^I_\rho \partial_\mu e^\rho{}_J \tag{1.44}$$

where $\Gamma^\rho_{\mu\nu}[g]$ are the Christoffel symbols. In order now to use these objects to express the Einstein-Hilbert action as a function of both tetrad and connection let us give a *second Cartan equation of structure* which relates the curvature expressed in the tetrad basis to the connection

$$R^I{}_J \equiv \frac{1}{2} R_{\mu\nu}{}^I{}_J dx^\mu \wedge dx^\nu = \mathrm{d}\omega^I{}_J + \omega^I{}_K \wedge \omega^K_J \tag{1.45}$$

where $R_{\mu\nu}{}^I{}_J = e_\rho{}^I e^\nu{}_J R_{\mu\nu}{}^\rho{}_\sigma$. As an important sidenote, one can show that under the Lorentz transformations $\bar{e}^I = \Lambda^I{}_J e^J$ this connection transforms as $\bar{\omega}_\mu{}^I{}_J = \Lambda^I{}_K \omega_\mu{}^K{}_L \Lambda^L_J + \Lambda^I{}_K \partial_\mu \Lambda^K_J$.

We are now ready to give the action in the Palatini tetrad-connection formulation

$$S[e,\omega] = \frac{1}{2} \int dt dx^3 N e e^\mu_I e^\nu_J R_{\mu\nu}{}^{IJ}[\omega] \tag{1.46}$$

where the only difference with the "standard" (second-order) tetrad-connection is in the functional dependence ($S = S[e]$ for second-order action, $S = S[e,\omega]$ for the first-order action. The condition of zero torsion now follows from the variation of (1.46) with respect to ω.

Let us now finally combine the 3+1 formalism with the developments of the previous paragraphs. In doing so we derive the Ashtekar variables that are the basis of the subsequent quantization. First, the action of (1.46) is extended so as to include a topological term

$$S[e,\omega] = \frac{1}{2} \int dt dx^3 N e e^\mu_I e^\nu_J (R_{\mu\nu}{}^{IJ}[\omega] - \frac{1}{2\beta} \varepsilon^{IJ}{}_{KL} R_{\mu\nu}{}^{KL}[\omega]). \tag{1.47}$$

This new term does not influence the equations of motion in any way but its importance will be shown later during the quantization. β is called the Barbero-Immirzi parameter. Next, we decompose the connection $\omega_\mu^{\ IJ}$ to account for the imposed foliation. We denote

$$\Gamma_a^{\ ij} = \omega_a^{\ ij} \quad K_a^{\ i} = \omega_a^{\ i0} \tag{1.48a}$$

$$B^{ij} = \omega_0^{\ ij} \quad C^i = \omega_0^{i0}. \tag{1.48b}$$

Applying the Hodge dual operator to $\Gamma_a^{\ jk}$ one obtains an object with two indices $\Gamma_a^{\ i} = -\frac{1}{2}\varepsilon^i_{\ jk}\Gamma_a^{\ jk}$. The Ashtekar variables are then the Ashtekar connection

$$A_a^i = \Gamma_a^i(E) + \gamma K_a^i \tag{1.49}$$

and E_j^b as defined in (1.38). It is important to note that $\Gamma_a^i(E)$ corresponds to the spatial part of the Levi-Civita connection obtained as a solution to the equations of motion of action (1.47) as well as the tetrad-compatible connection corresponding to the triads e_a^i on the spatial slice Σ_t. Also, on-shell, the quantity $K_a^{\ i}$ coincides with the external curvature from (1.39). The Poisson brackets of the canonical pair read

$$\{E_j^a(x), A_b^i(y)\} = \kappa\gamma\delta_b^a\delta_j^i\delta(x,y). \tag{1.50}$$

This implies that the transformation

$$E_i^a, K_a^i \to \frac{1}{\gamma}E_i^a, A_a^i \tag{1.51}$$

is a canonical one. Now we finally substitute Ashtekar variables into (1.47). This yields a Lagrangian density

$$\mathcal{L} = E_i^a\partial_t A_a^{\ i} - \Lambda^i G_i - N^b V_b - NS \tag{1.52}$$

with the constraints being

$$V_b = E_i^a F_{ab}^i - (1+\gamma^2)K_b^i G_i = 0 \tag{1.53}$$

$$S = \frac{E_i^a E_j^b}{\sqrt{\det E}}(\varepsilon^{ij}_{\quad k}F_{ab}^k - 2(1+\gamma^2)K_{[a}^i K_{b]}^j) \tag{1.54}$$

$$G_i = D_a E_i^a = 0 \tag{1.55}$$

where

$$D_a v_i = \partial_a v_i - \varepsilon_{ijk}A_a^j v^k, \tag{1.56}$$

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \varepsilon^i_{jk}A_a^j A_b^k. \tag{1.57}$$

The same form of the constraints would have been obtained if we plugged E, A into (1.30), (1.31), (1.41). The last step of the Hamiltonian analysis now consists in working out the Poisson structure of the constraint algebra. Doing so necessitates smearing the constraints with test functions of a corresponding structure. We compute thus

$$G(\alpha) = \int_\Sigma d^3x \alpha^i G_i, \tag{1.58}$$

$$V(f) = \int_\Sigma d^3x f^a V_a, \tag{1.59}$$

$$S(N) = \int_\Sigma d^3x N S \tag{1.60}$$

where the functions α^i, f^a, N are a Lorentz vector, a tangent (coordinate) vector and a scalar. Computing the Poisson brackets between all the constraints yields[7]

$$\{G(\alpha), G(\beta)\} = G([\alpha, \beta]), \tag{1.61a}$$

$$\{G(\alpha), V(f)\} = G(\mathcal{L}_f \alpha), \tag{1.61b}$$

$$\{G(\alpha), S(N)\} = 0, \tag{1.61c}$$

$$\{V(f), V(g)\} = V([f, g]) \tag{1.61d}$$

$$\{S(N), V(f)\} = -S(\mathcal{L}_f N) \tag{1.61e}$$

$$\{S(N), S(M)\} = V(h^{ab}(N\partial_b M - M\partial_b N)) + G(.). \tag{1.61f}$$

The appearance of the inverse metric h^{ab} in the last equation prevents the coefficients from being constant in the phase space. Technically speaking, the constraint functions then do not form a Lie algebra, one of the structure coefficients being rather a *structure function*. This will have its consequences for the quantization in the following section. The equations (1.50), (1.52)-(1.55) together with the constraint algebra (1.61a)-(1.61f) form the complete characterization of the canonical structure of general relativity that will form the basis for quantization in the next section.

1.4 Regularization, quantization, kinematical Hilbert space

Knowing the results of the last section, one would like to proceed to the canonical quantization. At an intuitive level, the quantum states of the quantized general relativity should be wave functionals of the classical configuration variable, in our case the Ashtekar connection, yielding thus an object with a generic form $\Psi[A]$. The operators corresponding to the canonically conjugate pair (A, E) should then act on these functionals as a multiplication with $A(x)$ or as a functional derivative with respect to $A(x)$ (modulo numerical factor). This form of the canonical operators should then be substituted into the constraint equations to give restrictions on $\Psi(A)$, i.e. to limit the set of permissible functional dependencies. Moreover, these restrictions, when viewed as operators, should reproduce the algebra structure (1.61).

The implementation of this intuitive idea and giving it precise contents is the program of refined algebraic quantum theory which constitutes rigorous mathematical underpinnings of the canonical quantization approach. It turns out that

[7] $G(.)$ denotes a term proportional to the Gauss constraint.

its realization requires considering a holonomy-flux algebra rather than working directly with the original $A-E$ algebra. The reason for this is mainly a mathematical convenience. First, the constraints (1.53)-(1.55) are much better defined when the *regularized* versions of the operators are used in them. This procedure of regularization consists in taking into account versions of the Ashtekar-Barbero connection and densitised triads integrated along or over suitable chosen geometrical objects. Second, implementation of the Gauss constraint requires taking carefully into account the transformation properties of the constituent object with respect to the local $SO(3)$ rotations. These are greatly simplified and easier to implement for the holonomy-flux algebra. Let us now consider the case of connection first.

The first point to notice is that the dualized connection acts as a one-form in its coordinate index. This index can naturally be contracted with a coordinate vector to give an antisymmetric Lorentz (internal) matrix. Let us derive the character of this matrix. The basic kinematical setup of general relativity, on the classical level, suggested by the previous section implies that the configuration of spacetime at one instant is determined either by specifying the triad field in each spacetime point or by specifying the three-dimensional connection. Upon specifying the coordinates, any two triads at two arbitrary spacetime points are related by an $SO(3)$ rotation. The physical interpretation of the connection with three (two) indices, when one of them is contracted with a direction, is an (antisymmetric) matrix specifying infinitesimal rotation of the tetrad in that direction. It can be thus viewed as an element of an $so(3)$ algebra. Later considerations in LQG actually make use of the $su(2)$ algebra with which $so(3)$ is isomorphic. One can therefore write

$$A_a = A^i_a \tau_i \in su(2) \tag{1.62}$$

where τ_i are the $su(2)$ generators.

The concept of holonomy gives a precise meaning to how the triads at different spacetime points are connected, as mentioned before. Let us consider a path $\gamma : [0,1] \to \Sigma$, the holonomy is then defined as

$$H[A,\gamma] = P \exp \int_0^1 ds A^i(s)\tau_i = P \exp \int_\gamma A \tag{1.63}$$

where

$$P \exp \int_0^s d\tilde{s} A^i(\tilde{s})\tau_i \equiv h(s) \tag{1.64}$$

is a solution to the differential equation

$$\frac{d}{ds}h(s) + \dot{x}^\mu(s) A_\mu(\gamma(s)) h(s) = 0 \tag{1.65}$$

and the *path-ordered exponential* is defined by the series

$$P \exp \int_0^s d\tilde{s} A(\gamma(\tilde{s})) \equiv \sum_{n=0}^{\infty} \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_n - 1} A(\gamma(s_n)) \ldots A(\gamma(s_1)). \tag{1.66}$$

Considering a path γ with end points x_0 and x_1, the holonomy transforms under local $SU(2)$ transformation as

$$\bar{H}[A,\gamma] = g(x_1)H[A,\gamma]g^{-1}(x_0) \tag{1.67}$$

The transition from one degree of freedom per spacetime point to holonomies inside of the argument of the functional Ψ is an effective step towards tractability inside of the functional dependence. That such a step in fact does not discard some information is the contents of the *Giles theorem* which is valid for a subset of all paths - loops. These are the paths for which $x_0 = x_1$ and $x_i \neq x_j, i \neq j$. Its more precise statement is that the traces of holonomy for *all* possible loops on a manifold hold all gauge-invariant information in a connection [22]. For the case of paths that are not necessarily loops, one can give the following intuitive argument. In the same way that a connection unambiguously determines arbitrary holonomy, the holonomies along short paths give arbitrarily precise information about the value of the connection in an arbitrary point. Thus, at a kinematical level, the set of all functions of a finite set of holonomies $\{H_1[\omega,\gamma_1],\ldots,H_n[\omega,\gamma_n]\}$ is dense in the set of all functions of the connection $\omega(x)$. This key insight will guide the construction of the kinematical Hilbert space for LQG.

The function of a finite number of holonomies along particular paths is called a *cylindrical function*[8]. It is indexed by a collection of oriented paths $\gamma_l (l = 1,\ldots,L)$ also denoted a *graph* and a smooth complex valued function $f(g_1,\ldots,g_L)$ of L group elements. We can thus write

$$\psi_{\Gamma,f}[A] = f(H[A,\gamma_1],...,H[A,\gamma_L]). \tag{1.68}$$

Proceeding in a mathematically rigorous way requires that we make some restrictions on the set of connections A that we consider. We make the choice of requiring that the connections A be defined everywhere on Σ, except perhaps for a countable number of points, and smooth thereon, writing $A \in \mathcal{A}$. Let us consider a vector space of all linear combinations of functions (1.68), denoting is $Cyl(\mathcal{A})$. One can then define a scalar product on this space according to

$$\langle \psi_{\Gamma,f}, \psi_{\Gamma,g} \rangle = \int dg_1 \ldots dg_L \overline{f(g_1,...,g_L)} g(g_1,\ldots,g_L) \tag{1.69}$$

for two functionals that are supported on the same graph Γ and through extension of the functionals and the same formula in case they are not defined on the same graph. This extension is defined on the union of the two graphs that the functional are defined on by

$$\tilde{f}(g_1,\ldots,g_l,g_{l+1},\ldots,g_{l+l'}) \equiv f(g_1,\ldots,g_l) \tag{1.70}$$

where the indices $l+1,\ldots,l+l'$ correspond to the paths disjoint with the ones the functional was originally defined on. The functional labeled by g is extended analogously. The measure dg_i is the Haar measure over $SU(2)$[9]. This fact that

[8] The origin of this denomination is the most transparent in one-dimensional case where, if the function depends on the value of "connection" in a finite number of particular points, the connection may oscillate on the rest of the domain without affecting the cylindrical function, filling up in this way imagined rectangles (2D cylinders).

[9] This, in particular, ensures that the measure is invariant with respect to both right and left translations in $SU(2)$.

the set of all graphs is partially ordered by the relation of inclusion is taken account of in that the cylindrical function is, mathematically speaking, defined on the *projective limit* of all possible graphs.

The Hilbert kinematical space $\mathcal{H}_{kin}$ is defined as a completion of $Cyl(\mathcal{A})$ with respect to the scalar product (1.69), symbolically $\mathcal{H}_{kin} = \bar{Cyl}(\mathcal{A})$. Taking into account also distributional states, one has to consider the Gelfand triple $Cyl(\mathcal{A}) \subset \bar{Cyl}(\mathcal{A}) \subset Cyl(\mathcal{A})'$ where the last term is the space of all linear functionals on the set of all cylindrical functions $Cyl(\mathcal{A})$. One useful way of looking at the kinematical Hilbert space is as an L^2 space

$$\mathcal{H}_{kin} = L^2(\bar{\mathcal{A}}, d\mu_{AL}) \tag{1.71}$$

where $\mathcal{A}$ is a suitable distributional extension of $\mathcal{A}$ and $d\mu_{AL}$ is the Ashtekar-Lewandowski measure which is constructed in such a way that two diffeomorphically-related states have a unitary scalar product, otherwise it is zero.

This concludes our construction of the kinematical Hilbert space for the canonical LQG. It is important to point out that this construction can be put on solid mathematical footing which we largely skimmed over for sake of clarity. The construction of the basis in the Hilbert space and discussion of physical interpretation will be made in the next section together with the implementation of the Gauss and diffeomorphism constraint. The regularization of the other half of the canonical variables will be given in section 1.6. In this respect let us just note that $\mathcal{H}_{kin}$ as defined now cannot support the canonical operator $\hat{A}^i_a$ that would act as

$$\hat{A}^i_a \psi[A] = A^i_a(x)\psi[A], \tag{1.72}$$

but it can support the holonomy operator corresponding to (1.63). This operator maps the cylindrical functions to cylindrical functions dependent on one additional holonomy. As for the operator $\hat{E}^a_i$, one has for its action on a state $\psi[A]$

$$\hat{E}^a_i \psi[A] = -i\hbar\kappa\gamma \frac{\delta}{\delta A^i_a(x)} \psi[A], \tag{1.73}$$

which is a distribution yielding a well-defined operator on $\mathcal{H}_{kin}$ only upon integration over a 2-dimensional surface.

1.5 Implementation of constraints

In the last section, the points 1 and 3 of the canonical quantization procedure were addressed. The configuration variables and their momenta were chosen to be the connection A and densitised triad E and the basic representation space for the canonical pair of operators was constructed. The second point will be dealt with closer in section 1.6, this is made possible by the fact that even though the procedure fixes the steps needed to be taken, it does not necessarily fix their order. In this section, we approach the point 4 of the canonical quantization recipe, i.e. the implementation of the gauge and diffeomorphism constraints. We leave the Hamiltonian constraint for the last part.

Given the interpretation of the kinematical Hilbert space $\mathcal{H}_{kin}$ as the space of square-integrable functions of distributionally-extended connections[10], one can use Peter-Weyl theorem to construct an orthonormal basis in it. The theorem states that given a compact group G, the orthonormal basis in the Hilbert space $L^2(G, dg)$ is formed by the matrix elements of its unitary irreducible representations $D^{(j)}_{mn}(U)$. In case of $SU(2)$, j attains non-negative half-integer values $j \in \mathbb{N}/2$. We can then write

$$\int dU D^{\bar{j'}}_{m'n'}(U) D^{j}_{mn}(U) = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'}. \tag{1.74}$$

where $d_j = 2j + 1$ is the dimension of the representation. These states are eigenstates of the area operator defined later in (1.113) so that we can write[11]

$$\hat{A} D^{j}_{mn} = \sqrt{j(j+1)} D^{j}_{mn}. \tag{1.75}$$

Orthonormal bases for an arbitrary graph with l links are then obtained by tensoring l $L^2(G, dg)$ spaces among themselves and exterior multiplying matrices $D^{(j)}_{mn}(U)$. Adopting a Dirac-like convention $D^{(j)}_{mn}(U) \equiv \langle U|j, m, n\rangle$, this would read

$$|\Gamma, j_1, m_1, n_1, \ldots, j_l, m_l, n_l\rangle = |j_1, m_1, n_1\rangle \ldots |j_l, m_l, n_l\rangle. \tag{1.76}$$

The problem with this construction is that in absence of suitably-imposed constraints the basis is not countable and the Hilbert space is therefore unseparable. The reason is that there is *a priori* no reason to identify states indexed by different graphs - an infinitesimal change in one the paths of a graph leads to an orthogonal state by definition. But the individual group elements can of course, on the other hand, be isomorphic. This case, as we will see later will be taken care of by implementing the diffeomorphism constraint.

1.5.1 Gauss constraint

Let us now first consider the Gauss constraint (1.55) given the kinematical setup of the previous paragraphs. We proceed in two steps. First, we show that the transformation properties of canonical variables imply that any function of them transforms under local $SO(3)$ rotation through Poisson bracket. This purely classical result is then used to derive the consequences of the imposition of Gauss constraint on a general connection functional $\Psi[A]$.

Local $SO(3)$ transformations act on the canonical variables as

$$\bar{E}^{a}_{\ j} = S^{a}_{\ b} E^{b}_{\ j}, \tag{1.77}$$

$$\bar{A}^{\ ij}_{a} = S^{i}_{\ k} S^{j}_{\ l} A^{\ kl}_{a} + S^{i}_{\ k} \partial_a S^{jk}. \tag{1.78}$$

In an infinitesimal form, the transformation matrices S become

$$S^{i}_{\ j} = \delta^{i}_{\ j} + \lambda^{i}_{\ j} + \ldots \tag{1.79}$$

[10] i.e. functions giving a number for every field configuration

[11] In 3-dimensional GR, it has an interpretation of a *length*.

Plugging this into (1.77), (1.78) one gets for the infinitesimal change in A and E

$$\delta A_a^{\ i} = -\mathcal{D}_a\lambda^i \equiv \partial_a\lambda^i + A_{a\ \ j}^{\ i}\lambda^j = \partial_a\lambda^i + \varepsilon^i_{\ jk}A_a^{\ j}\lambda^k, \tag{1.80}$$

$$\delta E_a^{\ i} = \lambda_{ij}E^{aj} = \varepsilon_{ijk}\lambda^j E^{ak} \tag{1.81}$$

where we can alternate between $\lambda^i = -\frac{1}{2}\varepsilon^i_{\ jk}\lambda^{jk}$, $\lambda^{ij} = -\varepsilon^{ij}_{\ \ k}\lambda^k$ on account of the antisymmetry in the ij index pair.

Now taking into account the Gauss constraint (1.55) in its smeared-out version (1.58), one finds that

$$\delta A_a^{\ i}(x) = \{A_a^{\ i}(x), G[\lambda]\}, \tag{1.82}$$

$$\delta E_a^{\ i}(x) = \{E_a^{\ i}(x), G[\lambda]\}, \tag{1.83}$$

This, in particular, implies that any functional of these variables $F[A,E]$ transforms in the same way

$$\delta F = \{F, G[\lambda]\}. \tag{1.84}$$

Working out explicitly the derivatives in this expression and integrating per partes gives

$$\{F, G[\lambda]\} = \int_\Sigma d^3x\lambda^i(x)\mathcal{D}_a\frac{\delta F}{\delta A_a^i(x)} = -\int_\Sigma d^3x(\mathcal{D}_a\lambda^i(x))\frac{\delta F}{\delta A_a^i(x)}. \tag{1.85}$$

Note that the expression contains functional derivatives with respect to A already on a classical level, and without invoking the canonical representation.

Let us now prove, considering a general element of the Hilbert space $\Psi[A]$ expandable into the basis (1.76), that invariance with respect to transformations (1.80) is a necessary and sufficient condition for satisfying the quantum Gauss constraint

$$\hat{\mathcal{G}}^i = \frac{1}{\beta}(\partial_a\hat{E}^{ai} + \hat{A}_{a\ \ k}^{\ i}\hat{E}^{ak}) = i\hbar\left(\partial_a\frac{\delta}{\delta A_{ai}} + \varepsilon^i_{\ jk}A_a^{\ j}\frac{\delta}{\delta A_{ai}}\right). \tag{1.86}$$

Making use of (1.85) for $\Psi[A] \equiv |\Gamma, j_1, m_1, n_1, \ldots, j_l, m_l, n_l\rangle$

$$\delta\Psi[A] = \int d^3x\delta A_a^i(x)\frac{\delta}{\delta A_a^i(x)} = -\int_\Sigma d^3x(\mathcal{D}_a\lambda^i)\frac{\delta}{\delta A_a^i(x)}\Psi[A] \tag{1.87}$$

and integrating by parts yields

$$\delta\Psi[A] = \int d^3x\lambda^i\left(\partial_a\frac{\delta}{\delta A_a^{\ i}} + \varepsilon_{ij}^{\ \ k}A_a^{\ j}\frac{\delta}{\delta A_a^{\ k}}\right) = i\hbar\int d^3x\lambda^i\hat{\mathcal{G}}_i\Psi[A] \tag{1.88}$$

which proves the point. In order to select states annihilated by the Gauss constraint, one has to look for the states satisfying $\Psi[\bar{A}] = \Psi[A]$. One way to look at this equation is that it selects a certain subset of all possible states of a form

$$|\Psi\rangle \equiv \Psi_{\Gamma,f}[A] = \sum_{\substack{j_1...j_L \\ m_1...m_L \\ n_1...n_L}} \mathcal{C}_{j_1...j_l m_1...m_L n_1...n_L} D^{j_1}_{m_1 n_1} \dots D^{j_L}_{m_L n_L}. \tag{1.89}$$

Taking into account how an $SO(3)$ transformation acts on the holonomies, given its action on the connection (1.78),

$$U[A_g, \gamma] = g(x_f) U[A, \gamma] g(x_i)^{-1}, \tag{1.90}$$

it is possible to show that the action on $\psi_{\Gamma,f}[A]$ reads

$$U(g)\psi_{\Gamma,f}[A] \equiv \psi_{\Gamma,f}[A_g^{-1}] = f(g(x_f^{\gamma_1})g_1 g(x_i^{\gamma_1})^{-1}, ..., g(x_f^{\gamma_L})g_L g(x_i^{\gamma_L})^{-1}). \tag{1.91}$$

In the equation (1.89) this manifests itself as

$$\begin{aligned} U|\Psi\rangle &= \sum_{\substack{j_1...j_L \\ m_1...m_L \\ n_1...n_L}} \mathcal{C}^{m_1...m_L n_1...n_L}_{j_1...j_L} D^{j_1}_{m_1 n_1}(g(x_{1f})g_1 g(x_{1i})^{-1}) \dots D^{j_L}_{m_L n_L}(g(x_{Lf})g_L g(x_{Li})^{-1}) \\ &= \sum_{\substack{j_1...j_L \\ m_1...m_L \\ n_1...n_L}} \mathcal{C}^{m_1...m_L n_1...n_L}_{j_1...j_L} U^{m_1'}_{m_1} D^{j_1}_{m_1' n_1'} U^{n_1'}_{n_1} \dots U^{m_L'}_{m_L} D^{j_L}_{m_L' n_L'} U^{n_L'}_{n_L}. \end{aligned} \tag{1.92}$$

One can, therefore, give the condition on the invariance of the state as a whole as

$$\mathcal{C}^{m_1...m_L n_1...n_L}_{j_1...j_L} U^{m_1'}_{m_1} U^{n_1'}_{n_1} \dots U^{m_L'}_{m_L} U^{n_L'}_{n_L} = \mathcal{C}^{m_1'...m_L' n_1'...n_L'}_{j_1...j_L} \tag{1.93}$$

with $2L$ transformation matrices acting directly on the coefficients C. This equation is a definition of an *intertwiner*. More precisely, (1.89) contained a product of functions D per link of the graph, but the requirement of gauge invariance singles out only certain combinations of the coefficients C which can be interpreted as components of a tensor object, i.e. the intertwiner. This tensor object is connected with a particular *node* where the links associated to the holonomies that the intertwiner contracts with meet. Taking this into account we can write for the gauge-invariant state

$$\Psi_{\Gamma,j_l,i_n}[A] = \sum_{\alpha_l beta_l} v_{i_1}{}^{\beta_1...\beta_{n_1}}{}_{\alpha_1...\alpha_{n_1}} ... v_{i_N}{}^{\beta_{n_{N-1}+1}...\beta_L}{}_{\alpha_{n_{N-1}+1}...\alpha_L} \psi_{\Gamma,j_l,\alpha_l,\beta_l}[A] \tag{1.94}$$

where i_n indexes the nodes and $\psi_{\Gamma,j_l,\alpha_l,\beta_l}[A] \equiv D^{j_l}_{\alpha_l\beta_l}$. It is customary to call (1.94) a *spin network state*. It is also customary to label given link with the quantum number j to which a given eigenstate D^j_{mn} belongs.

The requirement of invariance imposes some important conditions on the adjacent representations j. It is in particular impossible, if we consider only three-valent nodes[12], to have invariant combinations of the Wigner matrices if

[12]the case of interest in 3-dimensional GR

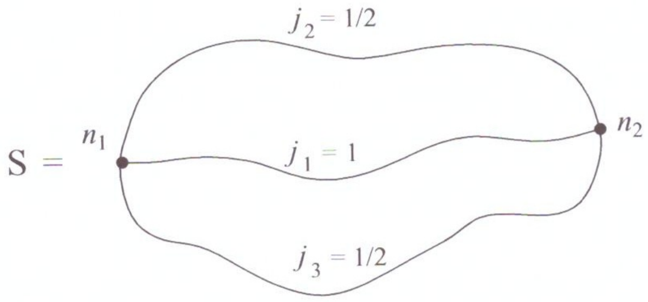

Figure 1.2: A simple spin network state

- the sum of the three adjacent spins j_i is not an integer and if
- the sum of any two spins is not larger or equal to the third spin (i.e. if *triangular inequalities* are not satisfied).

As a practical demonstration of these considerations, let us illustrate them on a simple example. Let us consider a spin network with two nodes and three links as depicted in the fig. 2.

The links carry eigenfunctions corresponding to $j_1 = 1$, $j_2 = 1/2$, $j_3 = 1/2$. The only gauge invariant state is then given by

$$\Psi_S[A] = \frac{1}{3}\sigma_{i,AB} D^{1i}_{\ j} D^{\frac{1}{2}A}_{\ C} D^{\frac{1}{2}B}_{\ D} \sigma^{j,CD} \tag{1.95}$$

where the indices i, j belong to the representation $j = 1$ ($\equiv$ *adjoint representation*) and take on values $i, j = \{1, 2, 3\}$ and the indices A, B, C, D belong to the representation $j = \frac{1}{2}$ ($\equiv$ *fundamental representation*) and take on values $A, \ldots, D = \{0, 1\}$. The intertwiner in this case are suitably normed Pauli matrices $v_{i,AB} = \frac{1}{\sqrt{3}}\sigma_{i,AB}$.

1.5.2 Diffeomorphism constraint

Having constructed the gauge invariant part $\mathcal{H}_0$ of the kinematical Hilbert space $\mathcal{H}_{kin}$, the orthonormal basis still suffers from the issue mentioned in the beginning of this section, i.e. of being uncountable. In order to remedy this, one is lead to try to construct a diffeomorphism invariant part of $\mathcal{H}_{kin}$. Mathematical considerations show that this is actually not feasible and that the corresponding space must be constructed as a subset of $\mathcal{S}'$. Moreover, there are difficulties that arise as a consequence of the set of all diffeomorphisms having no measure. But apart from that, this step changes the character of the theory in a profound way. The basic intuitive idea up until now in deriving the theory was that of an embedded graphs on a spatial slice Σ_t evolving the parameter t. Imposing the diffeomorphism constraint effectively erases all information that such a graph might have contained and replaces it with the concept of a *knot*, a gauge- and a diffeomorphism-invariant state that constitutes a basic atom of space. The physical contents of general relativity are preserved in a roundabout form only in the operator algebra of the physical observables whose commutation relation preserve the original structure present in the general relativistic Lagrangian. This is a subtle point that deserves being kept in mind. In a way, it is a manifestation of an idea of "kicking the ladder down once one's climbed it". That such a procedure may, in fact, be necessary is suggested by a need to effectively describe topology change in the context of quantum gravity.

Let us now first make precise what we mean by a diffeomorphism - under this designation we understand an invertible function $\phi : \Sigma \to \Sigma$ which is smooth everywhere on Σ. In the following, we will work with a *extended diffeomorphism* which satisfies this definition except for a finite number of points. Let us consider how such an extended diffeomorphism[13] acts on a state $\Psi \in \mathcal{S}'$. For $\Phi \in \mathcal{S}'$, $\Psi \in \mathcal{S}$ and ϕ a diffeomorphism, we have

$$(U_\phi \Phi)(\Psi) \equiv \Phi(U_{\phi^{-1}}\Psi) \tag{1.96}$$

where $U_\phi \Psi_{\Gamma,f} = \Psi_{\phi\Gamma,f}$ acts by shifting the graph Γ around on the slice Σ_t. A diffeomorphism-invariant state Φ in $\mathcal{S}'$ then satisfies

$$\Phi(U_\phi \Psi) = \Phi(\Psi). \tag{1.97}$$

Formally, such a state is obtained through a projection $P_{Diff*} : \mathcal{S} \to \mathcal{S}'$ given by

$$(P_{Diff^*})(\psi') = \sum_{\psi''=U_\phi\psi} \langle \psi'', \psi \rangle \tag{1.98}$$

where $\langle .,. \rangle$ is the "kinematical" scalar product (1.69) and the sum is over all states Ψ'' in $\mathcal{S}'$ for which there exists a diffeomorphism $\psi \in Diff^*$ such that $\Psi'' = U_\psi \Psi$. One postulates this sum to be finite and well defined - considering pure spin network states for simplicity, the action of any diffeomorphism that changes the underlying graph is that it projects it to a state orthogonal to the original one. Considering all the diffeomorphisms $\psi \in Diff^*$ that do not change the graph, their contribution is infinitesimally small and equally distributed[14]. In this way, one obviates the need for a measure on space of all diffeomorphisms.

Let us show however that the definition (1.98) is not *ad hoc* and that if the measure were available, one would recover it in the same form. Granting an existence of a measure on $Diff^*$, the diffeomorphism invariant state would be obtained by an integration on a gauge orbit

$$P_{Diff}\Psi = \int_{Diff^*} [d\phi] U_\phi \Psi. \tag{1.99}$$

Its action on a state $\Psi' \in \mathcal{S}$ would then be

$$(P_{Diff}\Psi)(\Psi') = \int_{Diff^*} [d\phi] \langle \Psi' U_\phi \Psi \rangle \tag{1.100}$$

[13]In the following we drop the word *extended* and mean an extended diffeomorphism whenever a diffeomorphism is used.

[14]There are also diffeomorphisms that do no change the underlying graph Γ, but reverse orientation of some of its links. The contribution of these is weighted by an appropriate sign factor.

Now, considering the scalar product (1.69) and pure states for simplicity, only such ϕ's that take Γ to Γ' do have a non-zero contribution to the integral. In other words

$$(P_{Diff}\Psi)(\Psi') = \sum_{\Psi''=U_\phi\Psi} \int_{D\Psi''} [d\phi]\langle\Psi'\Psi''\rangle \tag{1.101}$$

where $D\Psi''$ is a subgroup of $Diff^*$ the leaves Ψ'' invariant. We can factor the scalar product out of the integral

$$(P_{Diff}\Psi)(\Psi') = \sum_{\Psi''=U_\phi\Psi} \langle\Psi'\Psi''\rangle \int_{D\Psi''} [d\phi] \tag{1.102}$$

and making the assumption that the volume of $D\Psi''$ is constant for all Ψ'', one gets, modulo a possible multiplicative factor, the equation (1.98).

One great advantage of this definition is that it readily provides us with a scalar product on the space of diffeomorphism-invariant functionals $\mathcal{H}_{diff}$

$$\langle P_{diff}\Psi_S, P_{diff}\Psi_{S'}\rangle_{\mathcal{H}_{diff}} \equiv (P_{diff}\Psi_S)(\Psi_{S'}) = \langle\Psi|P_{diff}|\Psi'\rangle. \tag{1.103}$$

A general element of $\mathcal{H}_{diff}$ is called a *knot* to emphasize that it is determined by its invariant topological properties and not by the particular way it was embedded into Σ_t.

There are some points to be observed in this argument- first, one does not need to take into account the term proportional the Gauss constraint in (1.53) because the Gauss constraint has already been imposed by constructing the spin network states. Second, it turns out that one cannot make a rigorous sense of the equation

$$E^a_i F^i_{ab}|\Psi\rangle = 0 \tag{1.104}$$

because the operator F^i_{ab} is ill-defined. This corresponds to the absence of an infinitesimal generator of diffeomorphism transformations. However, one can make the following heuristic argument: by formally substituting the operator expressions (1.72), (1.73) into (1.104) one gets a term proportional to $\delta/\delta A$. By virtue of the definition (1.98), the diffeomorphism-invariant state is maximally "spread-out" on the gauge orbit in $\mathcal{H}_{diff}$ and therefore any derivative of it with respect to A, were it to be defined rigorously, is bound to be zero.

We have succeeded in building up the space $\mathcal{H}_{diff}$ annihilated by both the Gauss and the diffeomorphism constraint (within the respective Gelfand triple). The last and most non-trivial step consists in implementing the Hamiltonian constraint (1.54). But since doing so constitutes a highly non-trivial technical task and it is not necessary in order to derive the most immediate physical consequences of the framework, we postpone it to a later section and focus on properties of spin network states and knots instead.

1.6 Electric flux, area and volume operators

In the previous sections, we have built the basic kinematical[15] structure of the canonical loop quantum gravity. One important step in doing so was the regularization of the connection operator A and introduction of a holonomy. But it is clear that in the sense of the Poisson algebra, holonomy on its own is not complete and one needs to regularize the canonically conjugate triad operators as well. This turns out to be a first preliminary step in giving preceding construction a physical interpretation. Building on the regularized triad operators, we proceed to introduce operators of more direct geometrical relevance - the area and volume operators.

Let us consider a two-dimensional surface S embedded in a spatial slice Σ_t, let $\sigma = (\sigma^1, \sigma^2)$ be the coordinates on S. Then a regularization of the triad E_i^a over S is classically given by

$$E_i(S) \equiv -i\hbar\kappa\gamma \int_S d\sigma^1 d\sigma^2 n_a(\sigma) E_i^a(\sigma) \tag{1.105}$$

where the normal $n_a(\sigma)$ is

$$n_a(\sigma) = \varepsilon_{abc} \frac{\partial x^b(\sigma)}{\partial \sigma^1} \frac{\partial x^c(\sigma)}{\partial \sigma^2}. \tag{1.106}$$

We quantize this expression by substituting $E_i^a \to \hat{E}_i^a$ in (1.118). One can then show that the action of this operator on a holonomy is

$$\hat{E}_i(S) H[A, \gamma] = -i\hbar\kappa\gamma \int_S \int_\gamma d\sigma^1 d\sigma^2 ds \varepsilon_{abc} \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^3} \delta^3(x(\sigma), x(s)) H(A, \gamma_1) \tau_i H(A, \gamma_2). \tag{1.107}$$

The result of this integral depends on the configuration details of the surface S and the path γ, in case there is only no intersection between them, the integral vanishes. For one intersection, assuming H in the fundamental $j = \frac{1}{2}$ representation, one has

$$\hat{E}_i(S) H[A, \gamma] = \pm i\hbar\kappa\gamma H(A, \gamma_1) \tau_i H(A, \gamma_2) \tag{1.108}$$

and for multiple intersections

$$\hat{E}_i(S) H[A, \gamma] = \sum_p \pm i\hbar\kappa\gamma H(A, \gamma_1^p) \tau_i H(A, \gamma_2^p) \tag{1.109}$$

where the sign is given by the relative orientation of the path with respect to the surface. For arbitrary representations, one has an analogous expression

$$\hat{E}_i(S) H^j[A, \gamma] = \sum_p \pm i\hbar\kappa\gamma H^j(A, \gamma_1^p) \tau_i^{(j)} H^j(A, \gamma_2^p). \tag{1.110}$$

Defined in this way, this operator has an straightforward interpretation of a flux of densitised triad vector field E through the surface S. Expressed in terms of holonomies and fluxes, the canonical commutation relations become

[15] kinematical in the sense of up until the implementation of the Hamiltonian constraint

$$\{E^i(S), H[\gamma, A]\} = \gamma\kappa\tau^i H[\gamma, A]. \tag{1.111}$$

It is clear that the flux operator cannot represent a physically measurable quantity on account of its $SU(2)$ gauge index. This, however, suggests also a possible remedy - summing over this index with the same quantity should yield an invariant entity. While it is not actually true, because the integral over the surface S complicates the transformation properties, it is a step in the right direction. Let us, therefore, consider an entity $\hat{E}^2(S) = \sum_{i=1}^{3} \hat{E}^i(S)\hat{E}_i(S)$. Its action on the spin network state $\Psi_{\Gamma,f}$ is given by

$$\hat{E}^2(S)\Psi_{\Gamma,f} = (\hbar\kappa\gamma)^2 j(j+1)\Psi_{\Gamma,f}, \tag{1.112}$$

but only if the spin j link punctures the surface S exactly once. Taking this into account we define

$$A(S) = \int_S d^2\sigma \sqrt{n_a E_i^a n_b E_i^b} = \lim_{N\to\infty} \sum_n \sqrt{E^2(S_n)}. \tag{1.113}$$

Then in the limit $N \to \infty$ every infinitesimal surface S_n is punctured exactly once and the action on the spin network state is

$$\hat{A}(S)\Psi_{\Gamma,f} = \hbar\kappa\gamma \sum_p \sqrt{j_p(j_p+1)}\Psi_{\Gamma,f}. \tag{1.114}$$

This result is valid for the simplified case of no node being on the surface. The general result reads

$$\hat{A}(S)\Psi_{\Gamma,f} = \hbar\kappa\gamma \sum_{u,d,t} \sqrt{\frac{1}{2}j_u(j_u+1) + \frac{1}{2}j_d(j_d+1) + \frac{1}{2}j_t(j_t+1)}\Psi_{\Gamma,f} \tag{1.115}$$

where u and d differentiates between two possible orientations of the surface and the path and t labels links tangent to the surface S. In this way one obtains a quantum operator corresponding to the area giving a set of *discrete values* for this observable. It is important to note that the area defined in this way is, of course, gauge invariant but *not* diffeomorphism invariant.

Let us now construct an operator representing a volume of a three-dimensional subset of spacetime. One proceeds in broad lines in a manner analogous to the previous case. The classical expression for volume is

$$V(\mathcal{R}) = \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!}\left|\varepsilon_{abc}\varepsilon_{ijk}E^{ai}E^{bj}E^{ck}\right|}. \tag{1.116}$$

Substituting operators for the classical quantities E and partitioning the region $\mathcal{R}$ into infinitesimal cubes where the triad E is roughly constant one has that

$$V(\mathcal{R}) = \lim_{N\to\infty} \sum_n \sqrt{\varepsilon_{abc}E_i(S^a)E_j(S^b)E_k(S^c)\varepsilon^{ijk}}. \tag{1.117}$$

Letting the resulting infinitesimal operator act on an intertwiner, one can prove that

$$\hat{\mathcal{V}}_{\mathcal{R}} i_v = \hbar\kappa\gamma \sum_{(e,f,g)} \sqrt{\frac{1}{8}\frac{\varepsilon^{ijk}}{48} E_i^e E_j^f E_k^g i_v} \tag{1.118}$$

where the fact that one expects non-zero contributions only from nodes of the graph is justified by the appearance of three distinct derivative operators under the square root[16]. One noteworthy aspect of this calculation with respect to our subsequent discussion is that the expression (1.118) vanishes unless there are four or more links connected to the node.

There are three important points to be made about the constructed operators in general. First, we got discrete eigenvalues of the area and volume operators as a generic result of the quantization procedure. This can be considered a first empirical prediction of canonical loop quantum gravity. In effect, the fundamental reason for it is the finiteness of the $SO(3)$ group. Second, the computation of the eigenvalues of the area and volume operators clarifies the role that the Barbero-Immirzi parameter γ plays in the theory. Even though the term that is coupled through it to the general relativity action (1.46) does not influence the dynamics, the equations of motion, it does have kinematical consequences in the geometrical setup of the theory. Third, it needs to be pointed out that canonical loop quantum gravity is not spared the generic problem ordering ambiguities. These turn up in the process of regularization of the expression (1.117). Even though a reasonable choice has been made, it can be shown that there exists an alternative ordering which leads to *equidistant* spectrum of the area operator, instead of the *main sequence* proportional to $j(j+1)$. It deserves to be said that further investigation of the properties of operators needs to be carried out in order to put the general geometrical consideration of canonical LQG on even more solid footing.

1.7 Uniqueness results, physical interpretation

The following section is motivated by the question of to what extent is the construction of canonical LQG given so far unique. Besides putting many arguments of the introductory chapter on a solid ground, establishing concrete results for this question has also important consequences for the physical interpretation of the geometrical operators just introduced.

In full generality the question still cannot be confronted in a fully exhaustive manner. Doing so would require a lot more concrete knowledge of functional space underlying the formulation of diffeomorphism invariant theories, some of them hinging crucially on the delicacies of set theory. What can be tackled however is a closely related question of how unique the construction of canonical LQG is conditional on the form of holonomy-flux algebra we have given in (1.111). There the answer turns out to be available and giving it involves a particular mathematical construction called the *algebraic quantization*. In terms of the taxonomy of approaches in section 0.3, this approach stands firmly in the canonical camp as it provides a rigorous mathematical framework to implement the Dirac algorithm. Let us sketch its basic tools and structure.

[16]This expectation is of course vindicated by explicit calculation, unless there is a node present in a general region of spacetime $\mathcal{R}$, the operator $\mathcal{V}$ returns zero.

In broad terms, one can say that the algebraic quantization is a recipe for how to construct a consistent quantum theory given a classical algebra of phase space functions. Going one step further, *refined algebraic quantization* is a recipe pertaining particularly to constrained systems, a case of highest relevance for general relativity. But we focus concretely on the first option which directly bears on the question motivating this section. A starting point of the construction is an algebra $\mathcal{P}$ of classical elementary observables on a phase space $\mathcal{M}$. These are functions $f(m), m \in \mathcal{M}$ that *i)* separate points in $\mathcal{M}$, for any $m \neq m'$ there exists $f(m) \in \mathcal{P}$ such that $f(m) \neq f(m')$, *ii)* are closed under Poisson bracket, $f(m), f'(m) \in \mathcal{P} \implies \{f(m), f'(m)\} \in \mathcal{P}$ and that are *iii)* closed under complex conjugation, $f(m) \in \mathcal{P} \implies \bar{f}(m) \in \mathcal{P}$. This algebra is a subalgebra of $C^\infty(\mathcal{M})$.

Given the algebra of classical elementary observables, one constructs the quantum of elementary observables by *i)* considering finite products $(f_1 \dots f_n)$ of classical observables $f_k \in \mathcal{P}$, *ii)* giving the set of these products a structure of a free *-algebra $F(\mathcal{P})$ of $\mathcal{P}$ by defining the operations of multiplication and involution as

$$(f_1 \dots f_n) \cdot (f'_1 \dots f'_m) \equiv (f_1 \dots f_n f'_1 \dots f'_m)$$
$$(f_1 \dots f_n)^* \equiv (\bar{f}_1 \dots \bar{f}_n)$$

and allowing general sums of elements of variable length $\sum_{k=1}^{N} (f_1^{(k)} \dots f_{n_k}^{(k)})$ where $f_{n_i}^{(i)} \in \mathcal{P}$ and *iii)* defining a quotient *-algebra $\mathcal{U} \equiv F(\mathcal{P}/\mathcal{I})$ with respect to a 2-sided ideal $\mathcal{I}$ formed by the elements of a form

$$(f + f') - (f) - (f'), \quad (zf) - z(f), \quad [(f), (f')] - i\hbar(\{f, f'\})$$

where $z \in \mathbb{C}$ and $[(f), (f')] \equiv (f) \cdot (f') - (f') \cdot (f)$. It is the representation of $\mathcal{U}$ on a suitable Hilbert space $\mathcal{H}$ that defines the quantum theory.

A representation of an algebra $\mathcal{U}$ is, precisely speaking, a morphism $\pi : \mathcal{U} \to \mathcal{L}(\mathcal{H})$ from $\mathcal{U}$ into the algebra of linear operators on $\mathcal{H}$ with a common dense domain. A way how to construct a representation for a given algebra $\mathcal{U}$ is supplied by a *GNS construction* which we now explain. But before doing so, several other notions need to be introduced.

- **Cyclic representation** A representation is is cyclic if there exists a normed vector $\Omega \in \mathcal{H}$ in the common domain of all $a \in \mathcal{U}$ such that $\pi(\mathcal{U})\Omega$ is dense in $\mathcal{H}$. Ω is then called a *cyclic vector*.

- **Equivalent representations** Two representations are said to be equivalent if there exists a Hilbert space isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for all $a \in \mathcal{U}$, i.e.,for $\mathcal{H}_1 = \mathcal{H}_2$, one can view them as related by a mere change of coordinates.

- **State** Importantly, a state on a *-algebra is a linear functional $\omega : \mathcal{U} \to \mathbb{C}$ which is positive, $\omega(a^*a) \geq 0$ and for unital algebras also satisfying $\omega(1) = 1$.

- **Automorphism** An automorphism of a *-algebra is an isomorphism of $\mathcal{U}$ that is compatible with the algebraic structure. Considering a group G, it is represented on a group of automorphisms through $\alpha : G \to \mathrm{Aut}(\mathcal{U}); g \to \alpha_g$.

Importantly, a state ω on $\mathcal{U}$ is invariant with respect to α_g, respectively G, if $\omega \circ \alpha_g = \omega$, where in the latter case the equality holds for all $g \in G$.

With these definitions in mind, the GNS construction can be summarized as follows. For a state ω on $\mathcal{U}$, the null space $\mathcal{N}_\omega$ with respect to ω is defined as $\mathcal{N}_\omega \equiv \{a \in \mathcal{U} | \omega(a^* \cdot a) = 0\}$. This is a left ideal in $\mathcal{U}$. Subsequently, a quotient projection is defined as $[.] : \mathcal{U} \to \mathcal{U}/\mathcal{N}_\omega; a \to [a] \equiv \{a + b | b \in \mathcal{N}_\omega\}$. Now the GNS representation consists of a triplet $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ where $\mathcal{H}_\omega \equiv \langle \mathcal{U}/\mathcal{N}_\omega \rangle$, i.e. a completion of $\mathcal{U}/\mathcal{N}_\omega$ with respect to the inner product

$$\langle [a] | [b] \rangle_{\mathcal{H}_\omega} \equiv \omega(a^* \cdot b),$$

π_ω is a representation map defined by

$$\pi_\omega(a)[b] \equiv [a \cdot b], \forall a \in \mathcal{U}, b \in \mathcal{H}_\omega$$

and Ω_ω is a cyclic vector satisfying

$$\omega(a) = \langle \Omega_\omega | \pi_\omega(a) \Omega_\omega \rangle_{\mathcal{H}_\omega}. \tag{1.120}$$

In the case there are gauge symmetries, they are represented on $\mathcal{H}_\omega$ via

$$U(g)\pi_\omega(a)\Omega_\omega = \pi_\omega(\alpha_g(a))\Omega_\omega,$$

i.e. the group of gauge automorphisms acts as a group of unitary operators on the Hilbert space and one chooses, of course, ω satisfying $\omega \circ \alpha_g = \omega$.

The GNS construction applied to the holonomy-flux algebra (1.111) results in the following celebrated theorem by Lewandowski, Okolow, Sahlmann and Thiemann:

LOST theorem

1. There exists a unique $SU(2)$ gauge and diffeomorphism invariant state on the holonomy-flux algebra (1.111).

2. The GNS representation associated to this state is given by the Hilbert space $L^2(\bar{A}, d\mu_{AL})$ and the holonomy and flux operators act on it.

This result generalizes the Stone-von Neumann theory of ordinary quantum mechanics which states that there exists only one continuous, irreducible, unitary representation of the Weyl algebra[17], up to unitary equivalence. The invariant state stipulated in the first part of the theorem is analogous to a vacuum state of perturbative quantum field theory where this state is selected so that it is annihilated by all symmetry generators of the system. Another rephrasing of this uniqueness theorem can be made [41], which reinforces robustness of the uniqueness statement. One point to note is that in the present discussion we contented ourselves with the discussion of classical part of the algebraic quantization program, the latter part, the refined algebraic quantization is concerned with giving a precise meaning to Dirac constraint equations (29), resp. (1.19). The main technical obstacle there is that, strictly speaking, there is no $|\Psi\rangle \in \mathcal{H}_{kin}$ satisfying them and one has to look for the solutions of this equation in the algebraic

[17] Algebra generated by $e^{i\sum_{j=1}^{n} s_j \cdot x_j}$ and $e^{i\sum_{j=1}^{n} r_j \cdot p_j}$ where s_j, r_j are real numbers and $2n$ is the dimensionality of the phase space.

dual space of a dense subset of $\mathcal{H}_{kin}$, as, after all, we did it with the diffeomorphism constraint. This subspace is then nothing else than the space of functional on the cylindrical functions $Cyl(\mathcal{A})$ introduced in section 1.4.

The consequences of this for the geometrical interpretation of the construction are intuitively easy to develop. Modulo some minor quantization ambiguities discussed in the previous section, the expressions (1.113) and (1.117) constitute the unique way of representing quantities with the dimensions of area and volume respectively, given the form of the algebra (1.111). Each link of a spin network $\Psi_{\Gamma,f}$ carries quanta of area with the eigenvalues determined by (1.114), whereas each node a spin network corresponds to a certain amount of quanta of volume determined via (1.118). Two volumes of space are contiguous if they are connected with some link l. The spin of this link determines the quantum of area that it carries, whereas the volume associated with the nodes is determined by an intertwiner number v. This volume can be nonzero only if the there are more than 3 links connected with the node. Importantly, the graph Γ, which can be seen as a kind of a continuous index, can be also interpreted as a dual graph of a cellular decomposition of a physical space.

But it is important to point out that the intuitive direct geometrical interpretation is only tenable up until the implementation of the diffeomorphism constraint as already mentioned in subsection 1.5.2. If one understands the diffeomorphisms that one averages over in the sum (1.98) as active ones, i.e. the ones that move the graph around on the surface Σ_t, then the original geometrical information is clearly lost and one only hopes to retain it in a suitable limit[18]. In case one interprets the diffeomorphisms as passive ones, i.e. mere relabeling of the points on the surface, then the point is more subtle but one at the end will still need some kind of a limiting procedure to extract physically relevant information. So even though the title of this section alludes to the physical interpretation of the framework developed until now, it turns out that as it is it still needs to be taken with a grain of salt in certain aspects and in its current form, it necessitates further mathematical procedures to fully establish the equivalence with the geometrically well-interpretable general relativity. But one important aspect is apparent even at this stage - it is the *relationalistic* description inherent in the spin network language. This means that the quanta of volume and area are not localized in some outer manifold in which they are embedded, they themselves constitute the space and the only important information is contained in the quantum numbers and contiguity relations between themselves. This desirable result can be seen as a fruit of paying full heed to the philosophical contents of general relativity of which the relationalistic description is an indispensable component.[19].

1.8 Dynamics, the road to spinfoams

In this last part of the construction of canonical LQG, we approach the quantization of the scalar constraint S. This turns out to be a difficult task for two reasons - first, the constraint is highly non-linear and not even polynomial in the

[18]In this case, large-spin or continuity.

[19]More on this in [44].

fundamental fields A, E. This leads to various quantization ambiguities. Second, the action of the constraint does not have a clear physical interpretation without which it is difficult to assess whether one particular quantization is the physically relevant one.

Let us describe the most popular approach to the quantization of S. The main trick is to express the constraint in terms of Poisson brackets between variables that are well represented on $\mathcal{H}_{kin}$ and then replace them by canonical commutators. Writing for the smeared scalar constraint

$$S(N) = \frac{\gamma^2}{2}\int_\Sigma d^3xN\frac{E_i^aE_j^b}{\sqrt{|\det E|}}(\varepsilon^{ij}{}_kF^k_{ab}-2(1+\gamma^2)K^i_{[a}K^j_{b]}) \equiv S_E(N)-2(1+\gamma^2)T(N) \tag{1.121}$$

and making use of the classical identities

$$e^i_a(x) = \frac{\gamma^2}{2}\eta_{abc}\varepsilon^{ijk}\frac{E^b_jE^c_k}{\sqrt{\det E}}(x) = \frac{2}{\gamma}\{A^i_a(x), V_R\}, \tag{1.122}$$

where

$$\varepsilon_{ijk} \equiv h^I_ih^J_jh^K_kn^L\varepsilon_{LIJK},$$
$$\eta_{abc} \equiv h^\alpha_ah^\beta_bh^\gamma_ct^\mu\varepsilon_{\mu\alpha\beta\gamma}$$

and V_R is a volume functional for the neighborhood R containing x and

$$K^i_a(x) = \frac{1}{\gamma}\{A^i_a(x), K\}, \tag{1.124}$$

$$K = \frac{1}{\gamma^2}\{\mathcal{S}_E(1), V_\Sigma\}, \tag{1.125}$$

we obtain for the scalar constraint an equivalent classical expressions

$$S_E(N) = -\frac{2}{\gamma}\int_\Sigma d^3xN(x)\eta^{abc}\delta_{ij}F^j_{ab}\{A^i_c, V\}$$
$$T(N) = -\frac{2}{\gamma^3}\int_\Sigma d^3xN(x)\eta^{abc}\mathrm{Tr}(\{\mathbf{A}_a(x), K\}\{\mathbf{A}_b(x), K\}\{\mathbf{A}_c(x), V_{R_x}\})$$

In the next step, one introduces a triangulation $T(\varepsilon)$, parametrized by a parameter ε such that for $\varepsilon \to 0$ the triangulation fills up the whole Σ. This serves as a regulator in a way similar to that of the cut-off parameter in the perturbative quantum field theory. Using this, the regularized version of the constraint (1.21) can be written as

$$\hat{S}^\varepsilon_{E,\gamma} = \frac{16}{i\hbar\gamma}\sum_{\nu\in V(\gamma)}\frac{N(\nu)}{E(\nu)}\sum_{\nu(\Delta)=\nu}\varepsilon^{ijk}$$
$$\mathrm{Tr}(\hat{A}(\alpha_{ij}(\Delta))^{-1}\hat{A}(s_k(\Delta))^{-1}[\hat{A}(s_k(\Delta)), \hat{V}_{U^\varepsilon_\nu}]), \tag{1.127}$$

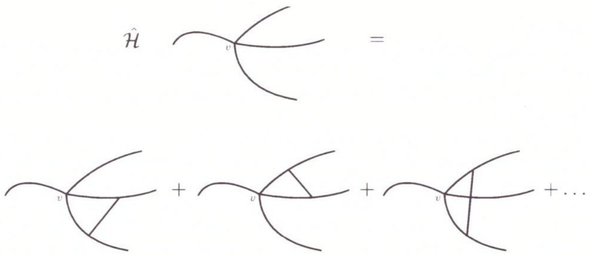

Figure 1.3: Action of the Hamiltonian constraint on a vertex

$$\hat{T}^{\varepsilon}_{\gamma}(N) = -\frac{4\sqrt{2}}{3i\hbar^3\gamma^3} \sum_{\nu\in V(\gamma)} \frac{N(\nu)}{E(\nu)} \sum_{\nu(\Delta)=\nu} \varepsilon^{ijk}$$
$$\mathrm{Tr}(\hat{A}(s_i(\Delta))^{-1}[\hat{A}(s_i(\Delta)), \hat{K}^{\varepsilon}]\hat{A}(s_j(\Delta))^{-1}[\hat{A}(s_j(\Delta)), \hat{K}^{\varepsilon}]$$
$$\hat{A}(s_k(\Delta))^{-1}[\hat{A}(s_k(\Delta)), \hat{V}_{U^{\varepsilon}_{\nu}}]), \quad (1.128)$$

where γ is a graph embedded in $T(\varepsilon)$, ν denotes a vertex, $N(\nu)$ is the value of the smearing function N in ν, $E(\nu) \equiv \binom{n(\nu)}{3}$ where $n(\nu)$ is the valence of ν, $\{s_i(\Delta)\}_{i=1,2,3}$ denotes the three outgoing segments with a common beginning point in $\nu(\Delta)$ and $\alpha_{ij}(\Delta)$ is the loop $s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}$ where $a_{ij}(\Delta)$ is the segment connecting the end points of $s_i(\Delta)$ and $s_j(\Delta)$. The action of this operator on the spin network state is then by definition

$$\hat{S}^{\varepsilon}(N)\psi_{\gamma} = \sum_{\nu\in V(\gamma)} N(\nu)\hat{S}^{\varepsilon}_{\nu}\psi_{\gamma} \quad (1.129)$$

where $\hat{S}^{\varepsilon}_{\nu}$ is the part of $\hat{S}^{\varepsilon}(N)$ regularized on ν. Further analysis shows that the action of the operator $\hat{S}^{\varepsilon}_{\nu}$ is that it creates new links around ν, with the precise form of the action depending on the spins of the relevant segments s_i. Importantly, the limit $\varepsilon \to 0$ is well-defined and does not lead to any divergences.

One can further check that the operator constructed above satisfies the necessary consistency conditions. One of these is a reproduction of the form of the classical constraint algebra. Leaving aside the terms not containing the scalar constraint one can show that the (dualized) Hamiltonian constraint $S'(N)$ and a finite diffeomorphism transformation $\hat{U}_{\phi}$ reproduce, on the diffeomorphism-invariant states, the Poisson algebra relations between $S(M)$ and $V(N)$

$$(-([\hat{S}(N), \hat{U}_{\phi}])'\Psi_{diff})[\phi_{\gamma}] = (\hat{S}'(N - \phi^* N)\Psi_{diff})[\phi_{\gamma}] \quad (1.130)$$

and that the commutator between two Hamiltonian constraints is

$$[\hat{S}(N), \hat{S}(M)]\phi_{\gamma} = \frac{1}{2} \sum_{\nu,\nu'\in V(\gamma),\nu\neq\nu'} [M(\nu)N(\nu')-N(\nu)M(\nu')][(\hat{U}_{\phi_{\nu',\nu}}-\hat{U}_{\phi_{\nu,\nu'}})\hat{S}^{\varepsilon}_{\nu'}\hat{S}^{\varepsilon}_{\nu}]\phi_{\gamma} \quad (1.131)$$

where $\phi_{\nu,\nu'}$ is an extended diffeomorphism such that $\hat{S}^{\varepsilon}_{\nu}\hat{S}^{\varepsilon}_{\nu'} = \hat{U}_{\phi_{\nu,\nu'}}\hat{S}^{\varepsilon}_{\nu'}\hat{S}^{\varepsilon}_{\nu}$. The fact that the commutator (1.131) vanishes on a diffeomorphism-invariant state

$$([\hat{S}(N), \hat{S}(M)])'\Psi_{diff} = 0$$

whereas it is non-zero on a general kinematical state is important as it ensures, in a certain sense, anomaly[20] freedom. However, it is still an open issue whether (1.131) reproduces the classical Poisson brackets outside of the $\mathcal{H}_{diff}$.[21]

Besides the procedure explained above, there are two more approaches to implementing the scalar constraint. First, one can define an explicitly *graph-preserving* regularization. The advantages of doing so are easier access to the symmetry-reduced sector of the theory[22]. Second, one may instead try to impose all the constraints $C(x)$ at once, instead of proceeding in steps. This corresponds to a single *Master constraint*

$$\mathbb{M} = \int_{\Sigma} d^3x C_{\alpha}(x) K^{\alpha\beta}(x) C_{\beta}(x) \tag{1.132}$$

with $K^{\alpha\beta}$ being some invertible positive-definite matrix of phase space functions. The main advantages of this approach are the fact that the constraint algebra trivializes, hence there are no structure functions appearing in it, and that one can give a rigorous proof of existence of the physical Hilbert space.

In summary, it can be said that obtaining a rigorously defined Hamiltonian constraint is an immense non-trivial consistency check for the canonical LQG culminating the program laid out in the present chapter. Its construction, however, is not without issues. Besides the points mentioned in the beginning, i.e. that a high non-linearity in the canonical variables and a lack of a clear geometrical interpretation of their action hinders the insight into the semiclassical regime of the theory, these issues can be traced back to the classical level in the form of the constraint algebra. First, the (dualized) Hamiltonian constraint operator does not leave the Hilbert space $\mathcal{H}_{diff}$ invariant. This can be seen in that the Poisson bracket relation (1.61e) is non-zero and, in particular, it implies that one cannot use the diffeomorphism-invariant inner product in the construction of an inner product in the physical Hilbert space H_{phys}. Second, the collection of Hamiltonian constraints does not form a Lie subalgebra in the Poisson constraint algebra, unlike the Gauss and the diffeomorphism constraints taken together. This, in particular, implies that one cannot employ the group averaging strategy used in the construction of $\mathcal{H}_{diff}$. This, in turn, invites a multitude of approaches that are *a priori* in absence of the access to the semiclassical regime of the theory all equally viable. If we were to summarize it, these are the important consequences of (1.61) not being a Lie algebra that were explicitly mentioned earlier.

Given this situation, one is motivated to search for alternative approaches of expressing the dynamics which would shed light on the issues given above. Such an alternative approach is provided by the *spinfoam formalism* reviewed in the next chapter.

[20] Anomaly is a situation in quantization when the quantum theory has less symmetry than the original classical system.

[21] Establishing this would ensure the correct classical limit of the theory [25].

[22] More in [13]

2. Covariant loop quantum gravity

In analogy to the case of classical mechanics which allows two fundamental formulations, a lagrangian one and a hamiltonian one, the quantum mechanics can also be built up in two complementary versions - a hamiltonian-based one and a lagrangian-based one. These are often called, in somewhat loose parlance, canonical and covariant approaches, techniques, or methods. The first one of these was used in the context of loop quantum gravity in the previous chapter, it is more general and far more rigorous when making the transition to the quantum theory. The second one is more intuitive, more simple and notably, keeps all the symmetries of the system manifest. Even though each of these two formulations possesses different virtues and uses somewhat different techniques, at the fundamental level they should be of course fully equivalent.

The lagrangian-based approach, when applied to the loop quantum gravity, bears a lot of different names. The one used in the introduction is the spinfoam approach. This probably comes closest to how the theory is perceived and regarded at present (spinfoam is considered a central dynamical object there) but there are also some other designations bearing evidence of the field's wide ideological and methodological grounds. Thus, besides a spinfoam approach and covariant LQG one can also encounter names such as path integral, state sum, sum-over-surfaces or sum-over-paths approach, all meaning, to a high degree of accuracy, the same thing[1]. But it is important to bear in mind in this respect that this equivalence is only valid within the field of loop quantum gravity. For some authors, the term path-integral quantization of GR, for example, necessarily involves perturbative effects and therefore diverges to a large extent from the contents and tenets of LQG.

It must be said at the very outset that, in line with the general quality of the lagrangian formulation, the spinfoam approach to LQG is mathematically somewhat less rigorous than canonical LQG and it lacks some of the mathematical underpinnings that provide a such a strong evidence in favor of the canonical LQG. However, from a purely operationalistic viewpoint the framework is not affected by this so neither is the ability to make falsifiable predictions, which rests on the notion of an operationalistic consistency rather than mathematical rigor, lessened in any way.

Expanding upon the preliminary discussion in the introductory chapter, one can say that the major aim of the covariant quantization is to provide an explicit tool for computing transition amplitudes between different states of geometry on the boundary of a region that is considered (implicit in equations (31), (32)). These amplitudes consist of individual contributions - *paths*, or equivalently *spinfoams*. These spinfoams can be interpreted as worldsheet histories swept out by spin network states. They also form a notion of what a general spacetime solution to the Einstein's equations looks like at the microscopic level, geomet-

[1] An exception to a this and a possible source of confusion is the way the word *covariant* is used, for some authors it means, very roughly speaking, a set of Hamiltonian inspired methods without the preliminary $3+1$ split. See e.g. [35].

rically and conceptually. Each of these spinfoams is weighted by a factor which is a function of the quantum numbers of nodes and links. Its form is dictated by group-theoretic considerations - simply speaking it has to be invariant with respect to right and left group multiplication. The node quantum number manifests itself as a quantum of volume concentrated at the node while the links quantum number corresponds to the area carried by a given link. In summing over everything, i.e. over all possible histories in between, spinfoams, we make therefore two kinds of expansions - first a vertex expansion, where we sum over all possible geometrical configurations of vertices and links that are compatible with the discretized boundary geometry. The limitations for these configurations will mostly come from the particular model considered and will be made on physical grounds. Second, there is a large spin expansion where the sum is made for a given configuration of vertices and links over all possible values of relevant quantum numbers. Let us now build the intuition of how this works and why it is equivalent to the Hamiltonian picture by inspecting the simplest case imaginable - a path integral for a quantum mechanical system with a finite number of degrees of freedom. This intuition and all its pertinent points carry over to all of the more advanced applications.

2.1 Path integral in quantum mechanics

In this section, we build up the basic intuition behind the sum over histories (Feynman path integral). Let's consider a Hamiltonian for a particle in d dimensions

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(\hat{\mathbf{x}}). \tag{2.1}$$

Then the *evolution operator* is defined as

$$U(T) = \exp(-\frac{i}{\hbar}HT). \tag{2.2}$$

Given the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \tag{2.3}$$

it is not difficult to see that the evolution operator represents an amplitude of probability for a system to transition from a position eigenstate $|\mathbf{x}'\rangle$ to a different position eigenstate $|\mathbf{x}''\rangle$ during a period of time T. Let us now write for the operator U

$$U(T) = \left(\exp(-\frac{i}{\hbar}H\frac{T}{n})\right)^n. \tag{2.4}$$

This can be done on account of the composition property of the evolution operator. Substituting for H one gets

$$U(T) = \left(\exp(\frac{i}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}\frac{T}{n})\exp(-\frac{i}{\hbar}V(\hat{\mathbf{x}}\frac{T}{n}))\right)^n + \mathcal{O}(\frac{1}{n}). \tag{2.5}$$

Contracting this expression with the initial and the final state in the x-representation, one obtains

$$\begin{aligned}\langle \mathbf{x}''|U(t)|\mathbf{x}'\rangle &= \langle \mathbf{x}''|\left(\exp(\frac{i}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}\frac{T}{n})\exp(-\frac{i}{\hbar}V(\hat{\mathbf{x}}\frac{T}{n}))\right)^n|\mathbf{x}'\rangle + \mathcal{O}(\frac{1}{n})\\ &= \int d^dx_1\dots d^dx_{n-1}\langle \mathbf{x}''|e^{-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}\frac{T}{n}}e^{-\frac{i}{\hbar}V(\hat{\mathbf{x}})\frac{T}{n}}|\mathbf{x}_{n-1}\rangle\dots\langle \mathbf{x}_1|e^{-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}\frac{T}{n}}e^{-\frac{i}{\hbar}V(\hat{\mathbf{x}})\frac{T}{n}}|\mathbf{x}'\rangle\\ &\quad+\mathcal{O}(\frac{1}{n})\end{aligned} \tag{2.6}$$

where we have made use of the fact that

$$\int d^d\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}| = 1, \quad \langle\mathbf{x}|\mathbf{y}\rangle = \delta^{(d)}(\mathbf{x}-\mathbf{y}). \tag{2.7}$$

Inserting

$$\int d^d\mathbf{p}|\mathbf{p}\rangle\langle\mathbf{p}| = 1, \quad \langle\mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{d/2}}\exp(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}), \tag{2.8}$$

one can write

$$\begin{aligned}\langle \mathbf{x}_j|e^{-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}\frac{T}{n}}e^{-\frac{i}{\hbar}V(\hat{\mathbf{x}})\frac{T}{n}}|\mathbf{x}_{j-1}\rangle &= \int d^d\mathbf{p}_j\exp(-\frac{i}{\hbar}(\frac{\mathbf{p}_j^2}{2m}+V(\mathbf{x}_{j-1}))\frac{T}{n})\langle\mathbf{x}_j|\mathbf{p}_j\rangle\langle\mathbf{p}_j|\mathbf{x}_{j-1}\rangle\\ &= \int\frac{d^d\mathbf{p}_j}{(2\pi\hbar)^d}\exp(\frac{i}{\hbar}\mathbf{p}_j\cdot(\mathbf{x}_j-\mathbf{x}_{j-1})-\frac{i}{\hbar}(\frac{\mathbf{p}_j^2}{2m}+V(\mathbf{x}_{j-1}))\frac{T}{n}).\end{aligned} \tag{2.9}$$

Taking the limit of this expression one has finally

$$\langle \mathbf{x}''|U(t)|\mathbf{x}'\rangle = \lim_{n\to\infty}\int\frac{d^d\mathbf{p}_n}{(2\pi\hbar)^d}\left(\prod_{j=1}^{n-1}\frac{d^d\mathbf{p}_jd^d\mathbf{x}_j}{(2\pi\hbar)^d}\right)\exp(\frac{i}{\hbar}\sum_{j=1}^{n}(\mathbf{p}_j\cdot(\mathbf{x}_j-\mathbf{x}_{j-1})-\frac{i}{\hbar}(\frac{\mathbf{p}_j^2}{2m}+V(\mathbf{x}_{j-1}))\frac{T}{n})) \tag{2.10}$$

which with substitutions $\prod_{j=1}^{n}\frac{d^d\mathbf{p}_j}{(2\pi\hbar)^d}\equiv\mathcal{D}p(t)$, $\prod_{j=1}^{n-1}d^d\mathbf{x}_j\equiv\mathcal{D}x(t)$ can be rewritten as

$$\langle \mathbf{x}''|U(t)|\mathbf{x}'\rangle = \int_{\mathbf{X}(0)=\mathbf{x}'}^{\mathbf{X}(T)=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\mathcal{D}\mathbf{p}(t)\exp\left(\frac{i}{\hbar}\int_0^T dt(\mathbf{p}\dot{\mathbf{x}}-H)\right). \tag{2.11}$$

This has exactly the structure of (31), with the term $\mathcal{D}\mathbf{x}(t)\mathcal{D}\mathbf{p}(t)$ representing an infinitesimal contribution of a form $D[g_{\mu\nu}]$ and the term $\exp\left(\frac{i}{\hbar}\int_0^T dt(\mathbf{p}\dot{\mathbf{x}}-H)\right)$ being a classically equivalent reformulation of the weighting factor $e^{iS_{EH}[g]}$.

Let us show explicitly that this analogy has a mathematically rigorous base. Let us consider a phase space with the canonical coordinates $(\mathbf{x},\mathbf{p})$. Then for a path piecewise constant in $\mathbf{p}$

$$\mathbf{p}(t) = \mathbf{p}_j \quad \text{for} \quad \frac{j-1}{n}T < t < \frac{j}{n}T \tag{2.12}$$

and piecewise linear in $\mathbf{x}$

$$\mathbf{x}(t) = \mathbf{x}_{j-1} + \frac{n}{T}(\mathbf{x}_j - \mathbf{x}_{j-1})(t - \frac{j-1}{n}T) \quad \text{for} \quad \frac{j-1}{n}T < t < \frac{j}{n}T \tag{2.13}$$

with n being the number of individual "chunks", $j = 1, \ldots, n-1$ and $\mathbf{x}_0 = \mathbf{x}'$, $\mathbf{x}_n = \mathbf{x}''$, the action functional reads

$$S(\mathbf{x}', \mathbf{x}'', T) = \sum_{j=1}^{n} (\mathbf{p}_j \cdot (\mathbf{x}_j - \mathbf{x}_{j-1}) - \frac{i}{\hbar}(\frac{\mathbf{p}_j^2}{2m} + V(\mathbf{x}_{j-1}))\frac{T}{n} + \mathcal{O}((\frac{T}{n})^2)), \tag{2.14}$$

which, up to a prefactor $\frac{i}{\hbar}$ and terms of the order $\mathcal{O}((\frac{T}{n})^2)$ is identical with the term inside of the exponential in (2.10).

Using the Gaussian integral identity

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} \exp\left(-\frac{1}{2}ax^2 + bx\right) = \left(\frac{1}{2\pi a}\right)^{1/2} \exp\left(\frac{b^2}{2a}\right), \tag{2.15}$$

we can partially solve the integral (2.10) to obtain

$$\int \frac{d^d p}{(2\pi\hbar)^d} \exp(\frac{i}{\hbar} - \frac{\mathbf{p}^2}{2m}\frac{T}{n} + \mathbf{p}\cdot\Delta\mathbf{x})) = \left(\frac{nm}{2\pi i\hbar T}\right)^{d/2} \exp\left(\frac{i}{\hbar} m \Delta\mathbf{x}^2 \frac{n}{2T}\right), \tag{2.16}$$

$$\langle \mathbf{x}''|U(t)|\mathbf{x}'\rangle =$$
$$\lim_{n\to\infty} \left(\frac{nm}{2\pi i\hbar T}\right)^{nd/2} \int \prod_{j=1}^{n-1} d^d\mathbf{x}_j \exp\left(\frac{i}{\hbar}\sum_{j=1}^{n}\left(m(\mathbf{x}_j - \mathbf{x}_{j-1})^2\frac{n}{2T} - V(\mathbf{x}_{j-1})\frac{T}{n}\right)\right). \tag{2.17}$$

Absorbing the prefactor in this expression into the measure one then gets an alternative expression for (2.11)

$$\langle \mathbf{x}''|U(t)|\mathbf{x}'\rangle = \int_{\mathbf{X}(0)=\mathbf{x}'}^{\mathbf{X}(T)=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left(\frac{i}{\hbar}\int_0^T dt(m\frac{\dot{\mathbf{x}(t)}^2}{2} - V(\mathbf{x}(t)))\right). \tag{2.18}$$

Both expressions (2.11), (2.18) form the basis of the numerical computation of the integral. One important point to make is that one makes these integrations on a quantum mechanical level, position, therefore, does not determine momentum, as in the classical mechanics, because doing so would mean breaking Heisenberg's uncertainty relations. Hence there is nothing incorrect about integrating with respect to both $\mathbf{x}$ and $\mathbf{p}$ here.

spin network	spinfoam
$(d-1)$-dimensional space	d-dimensional space
graph Γ	2-complex $\mathcal{C}$
spin representations j_l on links l	spin representations j_f on faces f
intertwiners i_n on nodes n	intertwiners i_e on edges e

Table 2.1: The relationship between a spin network state and a spinfoam

4-dim. triangulation	spinfoam 2-complex
4-simplex	spinfoam vertex σ
tetrahedron T	spinfoam edge e
triangle t	face f

Table 2.2: Duality relations in 4-dimensional space

2.2 Regge calculus

The last section has shown a derivation of a path integral formalism in quantum mechanics for a very simple and general type of Hamiltonian. Subsequently, a very gentle sketch of how path integrals can be solved and enumerated was outlined.

In this part, the attention is focused on the discretized path integral, we categorize the resulting spinfoam models and then focus on the simplest one. Its discussion is done both from the point of view of the general ansatz as well as of the original idea leading to its discovery - Reggae calculus.

An intuitively appealing way of looking at spinfoams is as histories of evolving spin networks. This means that assuming a spin network state on a graph Γ with a gauge group $\mathcal{G}$ its nodes n sweep out spinfoam edges e during time evolution and its links l sweep out spinfoam faces f. The intertwiners naturally associated to the spin network nodes become attached to the edges while the spin representations j_l become attached to the spinfoam faces. The relationship between a spin network state and a corresponding spinfoam is schematically summarized in Table 2.1. Of course, reciprocally, taking a slice on a 2-complex $\mathcal{C}$ produces a graph Γ', which is possibly homeomorphic to the original Γ.

One important point to note in this construction is that the 2-complex is dual to a 2-skeleton created through a triangulation of the underlying space. This duality relation works through assigning a $d-n$-dimensional (2-complex) object to an n-dimensional (2-skeleton) object - in case of 4-dimensional triangulation one thus assigns a spinfoam vertex σ to every 4-simplex S, a spinfoam edge e to each tetrahedron T and a spinfoam face f to each triangle t and in case of 3-dimensional triangulation one has a vertex σ for each tetrahedron T, an edge e for every triangle t and a face f for every segment s of the 2-skeleton. These relations are summarized in tables 2.2, 2.3.

Whichever point of view is taken, the spinfoam model is defined by a local ansatz where only the objects with the three respective dimensionalities have non-zero amplitudes. Taking the spinfoam point of view, this is

$$\mathcal{A}_\Delta = \sum_{\Delta':\partial\ Delta'=\Delta} \sum_{j_f, i_e} \prod_f A_f[j_f] \prod_e A_e[i_e] \prod_\sigma A_\sigma[j_f, i_e]. \tag{2.19}$$

3-dim. triangulation	spinfoam 2-complex
tetrahedron T	spinfoam vertex σ
triangle t	spinfoam edge e
segment s	face f

Table 2.3: Duality relations in 3-dimensional space

model	theory	2-complexes	representation	vertex
Ponzano-Regge	Euc. 3D GR	3-valent σ	SO(2)	$\{6j\}$
Turaev-Viro	Euc. 3D GR with Λ	3-valent σ	$SU(2)_q$	$\{6j\}_q$
Ooguri	Euc. 4D BF	4-valent σ	SO(4)	$\{15j\}$
Crane-Yetter	Euc. 4D BF with Λ	4-valent σ	$SO(4)_q$	$\{15j\}_q$
Barrett-Crane	Euc. 4D GR	4-valent σ	SO(4), simple	$\{15j\}$ or $\{10j\}_{BC}$
EPRL	Lor. 4D GR (with Λ)	4/valent σ	SU(2) or $SU(2)_q$	$\{10j\}$ or $\{10j\}_q$

Table 2.4: Spinfoam models

In the formula (32) this corresponds to a sum where the configurations Δ' summed over are chosen according to $\partial\Delta' = \Delta$ and $A_p[j_0] = A_\sigma[j_f, i_e]$, $A_l[j_1] = A_e[i_e]$ and $A_t[j_2] = A_f[j_f]$. Notice that in passing from (32) to (2.19) we implicitly got rid of the assumption that the discretization under consideration is formed by disjunct initial and final states of geometries and a bulk in between, but allowed also for the possibility of a compact boundary state where the "bulk" is only its refinement. This is a necessary step in the generally covariant path integral formulation where one has to allow also for more general boundary conditions. The symbols j_f, i_e stand for quantum numbers indexing group representations and intertwiners respectively.

The spinfoam model is then defined by three choices:

1. a choice of allowed 2-complexes
2. a choice of representations j and intertwiners i
3. a choice of a vertex amplitude $A_\sigma[j_f, i_e]$, an edge amplitude $A_e[i_e]$ and a face amplitude A_f.

In the third point, it is customary to absorb the edge amplitude $A_e[i_e]$ into the vertex amplitude $A_\sigma[j_f, i_e]$ while the face amplitude is usually the dimension of the representation j.

Keeping in mind these simplifications, let us now look at what the resulting options are. They are summarized in the Table 2.4.

The first column of this matrix specifies the name of the model and the second one its physical meaning, the classical theory that forms its basis. The next three columns give the information needed to specify a model - the set of 2-complexes that define it, the group and its representations that the model is based on, and in the last column the vertex amplitude A_v.

There are several important points to note in the table. First, the theory column and the representation column are of course not independent. Spinfoam models based on Euclidean general relativity are necessarily based on the representation of groups of ordinary rotations matrices $SO(n)$ whereas Lorentzian models usually feature the group $SL(2,\mathbb{C})$ (or possibly $SO(1,2)$).

Second, because of the duality relation between a spinfoam and a triangulation of the space, the set of admissible 2-complexes of the theory is effectively characterized by the number of edges incoming or outgoing from a given vertex and this number is constant for a given space dimensionality.

Third, there is a very effective way of quantizing general relativity with the cosmological term Λ through the use of quantum group deformations to the underlying symmetry group. The theory is parametrized by a parameter $q = e^{i\pi/r}$, while the non-deformed version is retained for $q = 1$. This choice of deformation group has its consequences for both face and vertex amplitude, thus one needs to use $d_q(j_f)$ and $\{6j\}_q$ instead of $d(j_f)$ and $\{6j\}$[2]

The fourth remark is that it is only the Barrett-Crane model that makes a nontrivial choice of representations of a symmetry group. Here one sums only over *simple* unitary irreducible representations of $SO(4)$, the motivation for this is to implement the simplicity constraint which will be clarified later. This constraint transforms a topological BF theory into general relativity. Simple representation means that its highest weight[3] has a structure $\Lambda = (N, 0, \ldots, 0)$[4]. The condition for simpleness can be alternatively expressed as

$$X_{[ij}X_{ij]} \cdot V_N = 0 \tag{2.20}$$

where X_{ij} is a basis in the Lie algebra and V_N is the representation space. BF theory, on the other hand, is a suitable generalization of the topological Ponzano-Regge model to four dimensions that will be dealt with in more detail in the following section. The object $\{10j\}_{BC}$ is built up only from intertwiners of a form

$$i_{BC}^{(aa')(bb')(cc')(dd')} = \sum_j (2j+1) v^{abf} v^{fcd} v^{a'b'f'} v^{f'c'd'} \tag{2.21}$$

where v^{abf} is a generic trivalent intertwiner.

Finally, it is worth noting that in terms of the exposition made so far, it is only the EPRL model (named after Engle, Pereira, Rovelli and Livine) which is dimensionally consistent with the 3+1-split since upon performing it one gets a space with $N - 1$ dimensions. The rest of the models make use of the basic variables corresponding to the dimension of the unfoliated spacetime. This fact can be explained in heuristic terms insofar as the group representations of $SU(2)$ are concerned. First, only the case of $SU(2)$ satisfies the basic kinematical requirement of the same number of position and momentum variables in the canonical commutation relations (1.50). This explains, for example, why the Ponzano-Regge model is not based on $SO(2)$ plane rotations instead of the full

[2]Quantum group-based quantization is discussed more in detail e.g. in [47], d stands for the dimension of the representation j_f.

[3]For a Lie group, a weight is a generalization of a notion of eigenvalue defined with respect to the action of a maximal subalgebra with has vanishing Lie brackets of its Lie algebra on the whole Lie algebra. The corresponding generalization of eigenspace is called weight space and is, of course, a subset of the whole representation space.

[4]More in detail in [44], appendix A3.

group $SU(2)$. Second, it happens to be the case that for the groups $SO(4)$ as well as $SL(2.\mathbb{C})$ the Lie algebras are isomorphic to the direct sum of two $SU(2)$ Lie algebras $so(4) \sim su(2) \oplus su(2)$. Hence any formulation of a model based on $SO(4)$ or $SL(2, \mathbb{C})$ can be given an alternative reformulation in terms of $SU(2)$ which, for reasons related to a historical happenstance rather than a rigorous argument, is more convenient and common. This explains why the EPRL model makes use of the $3 + 1$ split. Note that the structure of the operator algebra and the constraint algebra in the canonical formalism differs in the cases, with and without the split, but that is not a major concern in the construction since it does not enter the theory as an initial assumption, but rather its structure is to be replicated from the bottom up during the development of the formalism.

Out of all the models introduced, in the following, we concentrate on the first one, the Ponzano-Regge model, and the most developed one, the EPRL model. In doing so one exhausts the range of ideas that are crucial in the construction of the models while restraining oneself to the physically most relevant results.

2.2.1 Ponzano-Regge model

In this section, we proceed to the exposition of the Ponzano-Regge model. This turns out to be the simplest setting to understand in what sense is the physics of continuous general relativity encoded in a combinatorial discrete language and so we introduce it as the first model. The statement that both of these theories are fundamentally equivalent rests on the form that the action of Ponzano-Regge model takes - one writes

$$S_{Regge} = \sum_{v} S_v, \tag{2.22}$$

with the sum going over the tetrahedrons v of the triangulation (or alternatively vertices of the spinfoam) and with

$$S_v = \sum_{f} l_f \theta_f(l_f). \tag{2.23}$$

Here f denotes segments (or spinfoam faces), l_f is its length and θ_f is the *dihedral angle* associated to the segment f (i.e. the angle between the outward normals of the triangles incident to the segment). This action converges in the limit of large quantum to the Einstein-Hilbert action (1.1). This statement is what is meant by *Regge calculus* on the classical level.

The tetrahedra filling up the spacetime are then assumed to be flat in a discretized metric g_Δ which approximates the "true" metric g. The curvature is consequently nonzero only along the segments of the triangulation. The resulting geometrical picture is, therefore, that of a base manifold being approximated by a set of tetrahedra which, extruding into the "fourth dimension", capture the effects of curvature. This corresponds in (2.23) to the sum of adjacent θ_f's being different from 2π. The important point is that one can achieve in this manner an arbitrarily exact approximation by adjusting accordingly the length parameter of the discretization. Generalizing this construction to an arbitrary dimension D

one can easily deduce that the curvature is, in general, concentrated on $D-2$ -dimensional objects[5].

The discrete construction just introduced opens up some interesting substance overlap with another part of quantum field theory where discrete structures also play an important role - namely with lattice Yang-Mills theories. It is instructive to bring some of these points up and to touch upon briefly some of the similarities and differences of both theoretical frameworks.

First, both of these theories possess an infinite amount of degrees of freedom (one field variable per spacetime point, there is an uncountable infinity of them) which one tries to alter by choosing some subset of them (again uncountable, but of measure zero in the whole spacetime). Now in the case of a Yang-Mills theory, one integrates the field variables along these subsets to obtain a holonomy-type variable (much like in the canonical LQG) and constructing a Hilbert space corresponding to the excitation of respective Hamiltonian taking into account relevant symmetries. In the case of Ponzano-Regge model one follows a different route. In the model's simple setting the discrete scaffold that one constructed in order to simplify the theory plays, in a way, the role of the dynamical variable. Even though one might interpret the segment lengths l_f as integrals of densitised triads (diads) along the segments of the triangulation, one usually does not need to go that far and defines a discretized path integral fully in terms of these lengths. One might conclude, allegorically speaking, that whereas in the case of Yang-Mills theory, during this alteration, one reduced the "theater stage" so that less "actors" fit on it, in the case of Ponzano-Regge model, one reduces the very actors because they form the stage themselves. As a side note, it deserves mentioning that *Wilson loops*, the variables that one gets in this way in the Yang-Mills case, served as an important impetus in the development analogous techniques in LQG.

The second point of contact between these two theories that yields different insights in the case of both is a presence of an ultraviolet/infrared cut-off. What is meant by this is an appropriate device in the theory that does away with too high or too low degrees of freedom in the theory which would manifest themselves in a form of unphysical singular results. In lattice Yang-Mills theory, the number of elementary cells N serves as an infrared cut-off: arbitrarily long wavelength excitations can be defined only when measured with discrete chunks with a fixed extensive dimension. This fixed number, on the other hand, provides an ultraviolet cut-off - the minimal wavelength that an excitation can have is given by it. In Ponzano-Regge model, on the other hand, the discretization serves as a cut-off that is neither infrared nor ultraviolet. This is because a fixed triangulation Δ can represent arbitrarily small or large geometry - according to the metric that is assigned to it, which is a fully independent choice[6]. The cut-off of Ponzano-Regge model therefore rather limits the *ratio* between the smallest allowed wavelength and the total size of the spacetime region under consideration.

Let us now consider how one proceeds in translating Ponzano-Regge model to a quantum level. The main idea is to try to make sense of a path integral expression

[5] Corresponding in e.g. two dimensions to points and surrounding "hilltops".

[6] Note that this is in stark contrast to the canonical LQG where the geometry is encoded in discrete quantum numbers which *do* have a minimum value.

$$Z = \int dl_1 \dots dl_N e^{iS_{Regge}}. \tag{2.24}$$

The lengths of segments are *a priori* continuous quantities. A breakthrough insight is achieved if we makes use of an asymptotic formula of a form

$$\{6j\} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(S_v(j_n) + \frac{\pi}{4}\right) \tag{2.25}$$

where the $6j$-symbol is associated to a tetrahedron and j_n are the $SU(2)$ representations along its six edges. The relation $\sim$ denotes an approximate equality in the large j limit. Since, as previously noted, j is a discrete quantity labeling individual representations of $SU(2)$, one should take j to take values in $\mathbb{Z}/2$ and restrict it to a discrete set of possible values. Assuming this then, one can show that the state sum for the path integral (2.24) can be expressed as

$$Z_{PR} = \sum_{j_1 \cdots j_N} \prod_f \dim(j_f) \prod_v \{6j\}_v. \tag{2.26}$$

But this is nothing else than a special case of the general formula (2.19). This is a Regge calculus derivation of a Ponzano-Regge model. One could of course also proceed directly by imposing respective choices in (2.19) (which would correspond to a state sum derivation).

Let us now show how this formula can be brought to a computational use. Let us define a holonomy of an $SU(2)$ spin connection of 3-dimensional euclidean GR along an edge e of the 2-complex as $h_e = \mathcal{P}\exp(\int_e \omega^i \tau_i)$ and let l_f^a denote the line integral of the triad e^i along a segment f of the discretization which intersects a face f of the 2-complex. Then the discretized Einstein-Hilbert action in terms of these variables reads

$$S[l_f, h_e] = \sum_f l_f^i \mathrm{Tr}[h_f \tau_i] = \sum_f \mathrm{Tr}[h_f l_f] \tag{2.27}$$

where

$$h_f = h_{e_1^f} \dots h_{e_n^f}, \tag{2.28}$$

f denoting the face intersected by the segment. Varying the action (2.27) with respect to l_f^i and h_e one gets the equations of motion

$$h_f = 1, \tag{2.29}$$

resp.

$$l_{f_1}^i + l_{f_2}^i + l_{f_3}^i = 0, \tag{2.30}$$

the first one of which states that the holonomy of a connection around a face is trivial, i.e. the manifold is flat, and the second one of which is nothing else than a discretized form of a Cartan structure equation for the three sides of each triangle. The reason for the three summands in (2.30) is, of course, the fact that the discretization is composed only of triangles.

Given that the holonomy h_e is an element of $SU(2)$, one can write for a partition sum corresponding to (2.27)

$$Z = \int dl^i_f dh_e \exp iS[l_f, h_e] \tag{2.31}$$

where dh_e is the Haar measure on $SU(2)$. Substituting (2.27) into the last equation and using the Plancherel expansion

$$\delta(h) = \sum_j \dim(j) \mathrm{Tr} M^{(j)}(h), \tag{2.32}$$

where $M^{(j)}$ is the full representation matrix of the spin j representation, one can compute

$$Z = \int dh_e \prod_f \delta(h_{e^f_1} \dots h_{e^f_n}) = \sum_{j_1 \cdots j_N} \prod_f \dim(j_f) \int dh_e \prod_f \mathrm{Tr} M^{(j_f)}(h_{e^f_1} \dots h_{e^f_n}). \tag{2.33}$$

Each of the integrals after the last equal sign, before taking trace, is of a form

$$\int_{SU(2)} dg M^{(j_1)}(g)^a_{\ a'} M^{(j_2)}(g)^b_{\ b'} M^{(j_3)}(g)^c_{\ c'} = v^{abc} v_{a'b'c'} \tag{2.34}$$

because there are exactly three segments forming each triangle and so in the product Π, f takes on exactly three distinct values. $M^{(j_f)}$ denotes the full representation matrix labeled by j_f which is being summed over: $f \in \{1, \dots, N\}$. The symbol v^{abc} stands for a normalized intertwiner among the three representations which upon contraction yields the $\{6j\}$-symbol. This vindicates the step taken in (2.27)[7] and proves that (2.33) is indeed equivalent to the state sum (2.26). This last expression forms the basis of numerical computations in the Ponzano-Regge model.

As a closing remark, let us point out that the model just introduced has many unphysical properties which manifest itself in flatness condition (2.29) and subsequent absence of any true dynamical degrees of freedom. The type of theory represented is denoted a *topological* quantum field theory to refer to the possibility that the theory may still possess nontrivial *topological* degrees of freedom. Despite these shortcomings, the value of this model consists in showing the consistency of spinfoam quantum gravity model under simplified circumstances.

2.3 BF theory

Let us now consider the case of four spacetime dimensions and a Lorentzian signature. The heuristical path to the relevant model, the EPRL model, is to first extend the topological setting of the Ponzano-Regge model to four spacetime dimensions obtaining thus a BF theory, and, second, to impose *simplicity constraints* which reinstate physical degrees of freedom so as to obtain standard general relativity. It turns out that this step allows various realizations all of which in some sense reproduce the general relativity, the EPRL model, however, only corresponds to their *weak* imposition whereas *strong* imposition of the

[7] i.e. defining the holonomies and the connections and considering the Einstein-Hilbert action

simplicity constraint gives rise to models of Barret-Crane type. Let us illustrate these statements in the following in more detail.

A starting point for a definition of a BF theory is an action of a form

$$S[B,\omega] = \int B_{IJ} \wedge F^{IJ}[\omega] \tag{2.35}$$

which can be seen as a direct generalization of action (2.27). Here B_{IJ} stands for an *arbitrary* two-form in the Lorentz indices. Taking into account the connection curvature

$$F^{IJ} = d\omega^{IJ} + \omega^{I}_{\ K} \wedge \omega^{KJ}, \tag{2.36}$$

the Lie-algebra-valued 2-form B^{IJ} associated to the faces of the dual 2-complex corresponds to a Lie algebra element l_f whereas F^{IJ} corresponds to the holonomy h_f (the generalization thus consists in relaxation of the definition of l_f as an integral of triad e along a segment f). This action leads to equations of motion that posit that the connection ω is flat, $F[\omega] = 0$, and that the parallel transport of B along an arbitrary segment is trivial, $d_\omega B = 0$. This flatness of the solutions, of course, corresponds to the absence of any propagating physical degrees of freedom, as it was the case in the Ponzano-Regge model.

Let us now proceed to the quantum level. Progressing straight to the final result and considering a gauge group $SU(2)$, the path integral partition function can be derived by a sequence of steps analogous to that of the previous section and it has a form

$$Z = \sum_{j_f, i_e} \prod_f \dim(j_f) \prod_v \{15\}_v = \int dh_e \prod_f \delta(h_{e_1^f} \dots h_{e_n^f}) \tag{2.37}$$

The Wigner 15j-symbol which is taking place of the Wigner 6j-symbol in the formula (2.26) is accounted for by the occurrence of integrals of products of *four* representation matrices in the partition sum instead of just three, as in (2.34). It has 10 indices associated to the faces and 5 indices associated with the choice of the intertwiner which represents a non-zero volume concentrated on a node[8]. This is in accordance with the physical requirement that the spin network states should carry quanta of 3-dimensional volume. The case of the $SO(4)$ group is very similar to this one just given because of the decomposition $SO(4) \simeq SU(2) \times SU(2)$ - the group $SO(4)$ is isomorphic to a product of two $SU(2)$ groups as mentioned earlier. As a consequence, in the expressions for partition function one just gets contributions from the two copies of $SU(2)$ representation labeled by a $j^\pm$

$$Z = \sum_{j_f^\pm, i_e^\pm} \prod_f \dim(j_f^+) \dim(j_f^-) \prod_v \{15j^+\}_v \{15j^-\}_v. \tag{2.38}$$

The case of $SL(2,\mathbb{C})$ is covered in the following.

As a closing remark before moving on to the implementation of the simplicity constraints, let us point out that the path integral formulation of a spin foam models (the second term in (2.37)) is dual to the state sum formulation (the first

[8]It is the fact that there are four matrices instead of three that ensures the volume being non-zero for the group $SU(2)$.

term therein) in the sense that in the first case one integrates over the eigenstates of h_e whereas in the second case one sums over the eigenstates l_f and these two operators are canonically conjugated. Both formulations carry the same extent of information about the spinfoam model, even though the path integral formulation is more suitable for numerical calculations.

Let us now proceed to the implementing simplicity constraints. Classically these constraints fix the form of B to be

$$B^{IJ} = \varepsilon^{IJ}_{\ \ KL} e^K \wedge e^L \tag{2.39}$$

The quantum theory formalism allows a significant freedom in their imposition, so let us consider two important cases which lead to well-defined spinfoam models.

One possible way of approaching the imposition of equation (2.39) in the context of a spinfoam model is to discretize the B-field and to impose its discretized version in every 4-simplex v. Making use of an expression equivalent to (2.39),

$$B^{IJ} \wedge B^{KL} = V\varepsilon^{IJKL}, \tag{2.40}$$

these discretized equations read

$$\star B_f \cdot B_f = 0, \tag{2.41a}$$

$$\star B_f \cdot B_{f'} = 0, \tag{2.41b}$$

$$\star B_f \cdot B_{f'} = \pm 2V(v), \tag{2.41c}$$

where the case (2.40b) applies if f, f' share an edge and (2.40c) applies if f, f' are the opposite faces of v and $\star$ denotes Hodge dual. It can be shown then that the equation (2.41a) restricts the representations summed over in the definition of the partition function. Since these are labeled by two spins $j^{\pm}$, it can be shown that they require vanishing of a group's Casimir operator of a form

$$\star \hat{B}_f^{IJ} \hat{B}_f^{KL} = \hat{B}_+^{IJ} \hat{B}_{+IJ} - \hat{B}_-^{IJ} \hat{B}_{-IJ} = [j_+(j_+ + 1) - j_-(j_- + 1)]1 = 0. \tag{2.42}$$

This leads to the restriction of the representations $j^{\pm}$ summed over in equation (2.38) - only the ones that satisfy $j_+ = j_-$ - the simple representations we introduced in (2.20), do get a non-zero contribution. Similarly, the equation (2.41b) restricts the intertwiners being summed over in (2.38) (they are hidden inside of the $15j$ symbols) to those of a form

$$i_{BC}^{(aa')(bb')(cc')(dd')} = \sum_i (2i+1) v^{abg} v^{gcd} v^{a'b'g'} v^{g'c'd'} \tag{2.43}$$

corresponding to simple representations as well.

The model obtained by this procedure, the Barret-Crane model, represents an earnest attempt at approximating euclidean GR. However, its deeper analysis uncovers some crucial difficulties. The first one of them is that one does not get an expected form of a graviton propagator in the low energy limit - some components of it are suppressed too much. The second one is that there does not seem to be a consistent way how to connect the framework with the $SU(2)$ spin networks which hinders its desirable physical interpretation.

One is, therefore, motivated to look for alternative approaches to the imposition of the simplicity constraints. The way to proceed, which leads to the EPRL model, is to enforce them *only* in the limit of large quantum numbers. Let us explain in the rest of this chapter how this comes about.

As a first step, let us recall some facts about $SL(2,\mathbb{C})$, the group underlying the "true" Lorentzian formulation of GR. Any function on this group can be expanded into its irreducible unitary representations. These representations are labeled by a positive real number p and a non-negative half-integer k and the relevant representation space $V^{(p,k)}$ can be decomposed into irreducible representation space of the subgroup $SU(2)$ according to[9]

$$V^{(p,k)} = \bigoplus_{j=k}^{\infty} H_j. \tag{2.44}$$

The basis in this space can be therefore denoted as $|p,k;j,m\rangle$ with $j = k, k+1, \ldots$ and $m = -j, -j+1, \ldots, j$. $SL(2,\mathbb{C})$ has two Casimir operators, in terms of a boost $\vec{K} \in SL(2,\mathbb{C})$ and a rotation $\vec{L} \in SL(2,\mathbb{C})$, they are $C_1 = |\vec{K}|^2 - |\vec{L}|^2$ and $C_2 = \vec{K} \cdot \vec{L}$. What is important is their action on $V^{(p,k)}$ written in terms of the quantum numbers p, k - it reads

$$|\vec{K}|^2 - |\vec{L}|^2 = p^2 - k^2 + 1, \tag{2.45a}$$

$$|\vec{K}| \cdot |\vec{L}| = pk. \tag{2.45b}$$

The key insight here is that the very same group generators $\vec{K}$, $\vec{L}$ can be seen as components of the B field discretized on every face f which allows us to reformulate advantageously the simplicity constraints. Considering a definition of B adjusted to extended action (1.47) $B = \star e \wedge e + \frac{1}{\gamma} e \wedge e$, one can identify, upon choosing a normal n to a spatial slice Σ K^i, L^i as

$$K^i = B^{i0}, \quad L^i = \frac{1}{2}\varepsilon^i{}_{jk} B^{jk}. \tag{2.46}$$

Then the definition of B given above is shown to imply simplicity constraints in a form

$$\vec{K} = \gamma \vec{L} \tag{2.47}$$

because one has

$$n_I B^{IJ} = n_I (\star e \wedge e + \frac{1}{\gamma} e \wedge e)^{IJ} = n_I (\varepsilon^{IJ}{}_{KL} e^K \wedge e^L + \frac{1}{\gamma} e^I \wedge e^J) = n_I (\star e \wedge e)^{IJ} \tag{2.48}$$

and

$$n_I (\star B)^{IJ} = n_I (\star (\frac{1}{\gamma} e^I \wedge e^J))^{IJ} = \frac{1}{\gamma} n_I B^{IJ}. \tag{2.49}$$

The key requirement now is to demand the validity of (2.47) in the limit $p, k \to \infty$. The equations (2.45) then give

$$|\vec{K}|^2 - |\vec{L}|^2 = (\gamma^2 - 1)|\vec{L}|^2, \tag{2.50a}$$

[9]Note that the spaces corresponding to different p's are isomorphic.

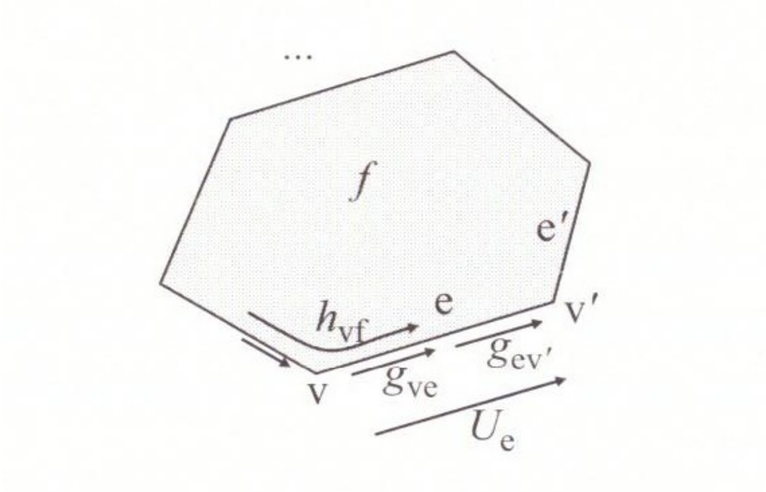

Figure 2.1: Group variables U_e notation

$$\vec{K} \cdot \vec{L} = \gamma |\vec{L}|^2 \tag{2.50b}$$

which in terms of the quantum numbers translates to

$$p^2 - k^2 + 1 = (\gamma^2 - 1) j(j+1) \tag{2.51a}$$

$$pk = \gamma j(j+1). \tag{2.51b}$$

The large numbers limit of this is $p^2 - k^2 = (\gamma^2 - 1) j^2$, $pk = \gamma j^2$ which is solved by $p = \gamma k$, $k = j$. The element of $V^{(p,k)}$ satisfying the simplicity constraint can thus be written as $|\gamma j, j; j, m\rangle$. We thus see how in the case of realistic general relativity a "weak" imposition of the simplicity constraint effectively transforms the system from a 4-dimensional spinfoam language to the language of $SU(2)$ spinfoams explained earlier. The map $|j, m\rangle \to |\gamma j, j; j, m\rangle$ is in the literature usually denoted as the Y_γ map.

By virtue of this isomorphism of structures, one would expect that the transition amplitudes will to a large extent copy the structure of those for the Ponzano-Regge model. It turns out that this is indeed the case - the expressions (2.33), (2.34) are equivalent to a transition amplitude $W(h_l)$ of a form

$$W(h_l) = \int_{SU(2)} dh_{vf} \prod_f \delta(h_f) \prod_v A_v(h_{vf}), \tag{2.52}$$

$$A_v(h_{vf}) = \sum_{j_f} \int_{SU(2)} dg'_{ve} \prod_f d_{f_f} (Tr)_{j_f} [g_{e'v} g_{ve} h_{vf}] \tag{2.53}$$

where the group variable is split according to $U_e = g_{ve} g_{ev'}$ with $g_{ev} = g_{ve}^{-1}$ and $h_{vf} = g_{ev} g_{ve'}$ as depicted in figure 2.1.

A detailed computation then shows that the form of the transition amplitude for the EPRL model is formally the same as the second equation (2.53) being replaced by an expression

$$A_v(h_{vf}) = \sum_{j_f} \int_{sl(2,\mathbb{C})} dg'_{ve} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f} [Y_\gamma^+ g_{e'v} g_{ve} Y_\gamma h_{vf}]. \tag{2.54}$$

Alternatively, this can be expressed as

$$A(j_{ab}, i_a) = \sum_{n_a} \int dp_a (k_a^2 + p_a^2) \left(\prod_a f^{i_a}_{k_a p_a}(j_{ab}) \right) \{15j\}_{SL(2,\mathbb{C})}((2j_{ab}, 2j_{ab}\gamma); (k_a, p_a)) \tag{2.55}$$

where indices a, b denote the edges of the 2-complex, j_{ab} is therefore naturally associated to a face as expected, f^i_{kp} being defined as $f^i_{kp} \equiv i^{abcd} v^{(k,p)}_{abcd}$, i^{abcd} and $v^{(k,p)}_{abcd}$ are $SU(2)$ and $SL(2,\mathbb{C})$ intertwiners respectively and $\{15j\}_{SL(2,\mathbb{C})}$ is, of course, an $SL(2,\mathbb{C})$ $15j$-symbol.

Both (2.52), (2.54) and (2.52), (2.55) are the culmination of the review part of this thesis. They form the base for the understanding of the problem of quantum gravity. Before listing the arguments to support this point, let us review some general characteristics of the amplitude (2.54), resp. (2.55).

First, the amplitude satisfies the *superposition principle* as an integral element of quantum mechanics. This means that (2.52) is a sum over independent histories in between the boundary state. One can thus schematically write

$$\langle W | \psi_{boun} \rangle = \sum_\sigma W(\sigma) \tag{2.56}$$

with σ labeling individual paths. This fact is in perfect agreement with the elementary computation at the beginning of this chapter.

Second, the transition amplitude is *local*. This is an important and often used requirement in quantum field theory - the inside of the transition amplitude should depend only on the values of fields and mathematical structures that represent it in individual spacetime points and no product of fields evaluated at different spacetime points should be allowed. Schematically, one can express it as

$$W(\sigma) \sim \prod_v W_v. \tag{2.57}$$

Third, although somewhat less direct to ascertain, the amplitude is Lorentz invariant, as one would expect from the construction.

It should be mentioned that the exposition up until now neglected one important moment in the construction of spinfoam models. As specified in this chapter, the spinfoam model is defined by a choice of set of allowed 2-complexes, by a choice of representations and intertwiners corresponding to a given group and by a choice vertex, edge and face amplitudes. It has been commented that the set of allowed 2-complexes is dictated by the fact that it is dual to the discretization of the spacetime. Nowhere, however, has the sum over all these 2-complexes been taken into account - according to the path integral formalism, each allowed field configuration should contribute to the transition amplitude and the exposition so far only assumed a fixed arbitrary 2-complex. This sum over 2-complexes is carried out through the so-called *group field formalism.* In it, remarkably, a spinfoam amplitude of the spinfoam formalism corresponds to a Feynman diagram of an auxiliary quantum field theory so summing over the contributions of different Feynman graphs corresponds to summing over different geometrical configurations. As a simple example, let us mention a model defined by an action

$$
\begin{aligned}
S[\phi] =& \frac{1}{2} \int \prod_{i=1}^{4} dg_i \phi^2(g_1, g_2, g_3, g_4) \\
&+ \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7)\phi(g_7, g_3, g_8, g_9) \\
&\times \phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1)
\end{aligned}
\tag{2.58}
$$

which generates the sum correspond to the four-dimensional BF theory [35], i.e. the Ooguri model (see Table 2.1). Notice that the potential term in (2.58) reproduces the structure of a 4-simplex.

Let us now list some of the arguments that point in favor of its viability. The following treatment is necessarily comprehensive rather than detailed so for each point we give references to original literature where the reader can learn about the topic in detail.

First, one can show that the model encapsulated by the equations (2.52), (2.54) is related, in the large quantum numbers limit, to general relativity. This result is far more difficult to establish here then it is for the Ponzano-Regge model as the construction involves building up reasonable semiclassical states for the model. The end result is that the WKB-type approximation of the transition amplitudes for the *physical* configurations is composed of two terms

$$
A(j_{ab}) \sim c e^{iS_{Regge}(j_{ab})} + c' e^{-iS_{Regge}(j_{ab})} \tag{2.59}
$$

where the constants c, c' are not equal because of the nontrivial geometrical effects[10] and the Regge action is in this case enumerated on a *four*-dimensional discretization. For the unphysical configurations, on the other hand, the amplitude is suppressed. Surprisingly, thus, one sees that an amplitude analogous to the Ponzano-Regge model is recovered in the classical limit in the EPRL model as well.

Second, as with the other models, EPRL model allows an inclusion of a non-zero and positive cosmological constant through a quantum deformation of $SL(2,\mathbb{C})$. Without going into the full details of the construction, let us state that the transition amplitudes of the deformed are finite and that their classical limit is related again to Regge action with a non-zero cosmological constant.

The first two points thus emphasized the mathematical consistency inherent in the EPRL model. The next two points clarify its position with respect to the empirical considerations listed in the introduction. The first area of these considerations is the black hole entropy derivation. The covariant path integral approach does provide in this respect a corroboration of the Bekenstein-Hawking black hole entropy formula, confirming in this way the results of the canonical-theory-based computations given in the introduction. The key idea in the covariant derivation is that the black hole entropy is identified with an *entanglement entropy*[11] across the horizon. On the other hand, the spinfoam formalism based

[10] More in [47].

[11] For a quantum system composed of two parts with a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, starting with a general state of the full system ρ, the entanglement entropy is defined as a von Neumann entropy of the reduced density matrix $S_A \equiv -\mathrm{Tr}\rho_A \log \rho_A$ where $\rho_A = \mathrm{Tr}_B \rho$.

on the EPRL model has been successfully applied to cosmological setting as well [10], [11], [51]. Choosing an initial state of geometry and a final one, provided that one can find spin network configurations that approximate these state to a required degree of precision, one can with the help of covariant formalism proceed to the computation of individual contributions to the transition amplitude. This is what is meant in this context by a *vertex expansion*. A subsequent step which turns out to be necessary in order to extract dynamics in the semiclassical regime is the *large spin expansion* which consists, as the name suggests, in passing in limit to a large-scale geometry. One of the non-trivial results of this approach is a prediction of a maximal acceleration during a gravitational collapse [47].

Let us close this chapter by pointing out some of the issues that are pending further development. This serves in a way also as a "bridge" to the research last chapter.

Let us again proceed sequentially. The first area of great interest that has not been conclusively settled so far is the relation between both canonical and covariant frameworks. This is related to the methodological remark made in the introduction - the covariant approach is from the beginning conceived as a bottom-up approach that is aimed to provide an alternative formulation to the dynamics of the theory and thus perhaps to circumvent some of the difficulties that plague the canonical theory. If this has indeed been achieved is, of course, not clear in advance and requires a meticulous mathematical, rather than physical, analysis. So far the issue has been conclusively settled - positively - only in the case of three-dimensional Ponzano-Regge model in [39]. However, no similar proof of equivalence exists so far for the realistic EPRL model. There are strong indications in its favor consisting in the fact that the EPRL model framework reduces through the Y_γ map to the Ponzano-Regge spin networks as well as the fact that one can directly cross-validate the discreteness of the area operator in purely EPRL setting [41].

Second, given that one's goal in the spinfoam program is to reproduce the dynamics of general relativity, one needs to have sufficient instruments at one's disposal to substantiate that this has indeed been achieved. Given the peculiar properties of quantum general relativity theory stemming from the fact that the field that is subject to quantization represent in a well-defined sense *space* and *time*, these instruments in this concrete case fall into two-fold distinction - namely a classical (i.e. large quantum numbers) and a continuum limit[12]. However, it is only the classical that has been investigated extensively so far. As a matter of fact, one has little to no insight into the behavior in the continuum regime.

Third, similar to the case of canonical LQG there are some quantization ambiguities that have important consequences for the structure of the theory and whose implications have not been studied so far. One of these includes the definition of the Y_γ map where the option $p = \gamma(j + 1)$ is equally viable in addition to $p = \gamma j$ considered.

[12]Unlike standard QM or QFT where there is only a classical limit.

3. Dynamics of simple configurations in Ponzano-Regge model

In this chapter we get to the computational part of this thesis. It is composed of two parts, in the first one, we clarify the research questions and methodology, in the second one, we proceed to the exposition of the results.

3.1 Questions and methodology

The last part of the previous chapter provides a natural point of departure for considerations made in this computational part of this thesis. One of the points that one could observe in it was a relative lack of understanding in the sector of the theory which is characterized by many small spins, i.e. the continuum limit, as opposed to the classical limit of large quantum numbers (actually both in the canonical and covariant version of the theory). In order to be able to gain some insight into this issue, one has to have as a prerequisite a sufficiently good understanding of the theory's dynamics. One way of gaining this is by investigating the transition amplitude - because we will be working in the covariant formulation of the theory - on some simple elementary configurations. This is a description in the simplest terms of what we have set out to investigate.

To facilitate the insight into this question, we focused on the simplest model available - the model of Ponzano and Regge introduced in section 2.2. We chose four different configurations, depicted in Figure 3.1. Our previous review of the basic structure of covariant LQG facilitates to a great extent the task at hand. We know that the objects depicted can be interpreted in two distinct ways - as a discretization of the spacetime or as its dual 2-complex. We choose the former interpretaton and thus one has a spin number j per each segment of the configuration. Our starting point for the dynamics is an amplitude similar to (2.26) given in [47]

$$W_{\Delta}(j_l) = N_{\Delta} \sum_{j_s} \prod_{s} (-1)^{2j_s} \dim(j_s) \prod_{T} (-1)^{J_T} \{6j\} \tag{3.1}$$

where the subscript Δ suggests a dependence on the discretization Δ, N_{Δ} is a normalization factor, j_s is the spin associated to a given segment, a link l is a segment on the boundary and J_T is the sum of spins belonging to the segments forming a given tetrahedron T.

The first question one has to find a satisfactory answer to is how to define the normalization factor N_{Δ}. In our understanding, the logically cleanest way to do so is in the standard thermodynamical way, i.e. as $N_{\Delta} = 1/Z_{\Delta}$ where Z_{Δ} is a partition sum for a given fixed configuration. This partition sum is composed of contributions enumerated as $\prod_s (-1)^{2j_s} \dim(j_s) \prod_T (-1)^{J_T} \{6j\}$ on this configuration corresponding to all possible boundary conditions, whereas the rest of the formula (3.1), the unnormed amplitude, is of course composed only of the contributions consistent with the boundary conditions j_l.

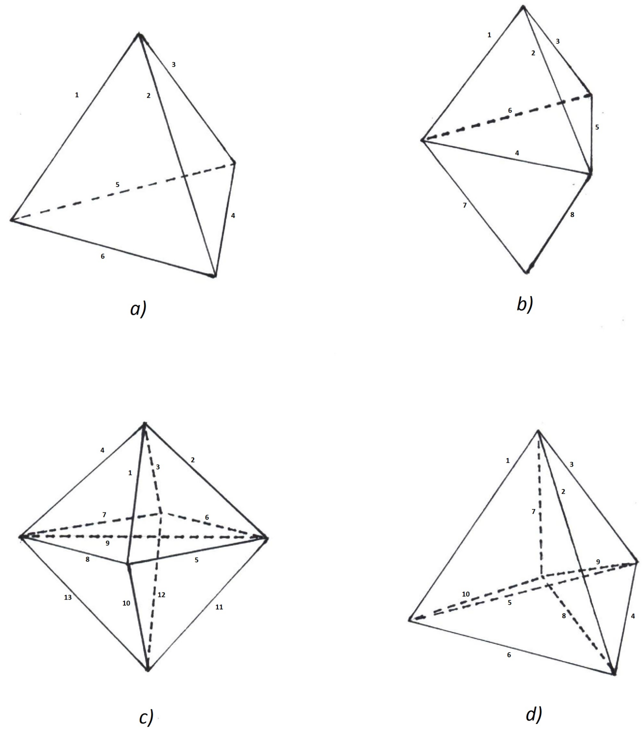

Figure 3.1: The four configurations considered: a) tetrahedron, b) double tetrahedron, c) octahedron, d) bubble

Given this preliminary setup, one can define the dynamics of the theory only for systems with a finite number of degrees of freedom, i.e. one must choose the "universe" of configurations that one takes into account and proceed in two steps - first, to define an amplitude of a given state j_l on a given discretization Δ *relative to other states* j'_l *on the same* Δ according to (3.1), and, second, to define an amplitude of one state with respect to another one on a *different* configuration as

$$w_{12} = \frac{W_{\Delta_1}}{W_{\Delta_2}} = \frac{W_1^{unnormed}(j_{l_1})Z_2}{W_2^{unnormed}(j_{l_2})Z_1} \tag{3.2}$$

where the Z_1, Z_2 are the partition sums of the two configurations. The set of all amplitudes for all allowed states can then be normed so as to yield a consistent dynamics.

The set of objects we chose is depicted in Fig. 3.1 - namely we consider a tetrahedron, a double tetrahedron, an octahedron and a tetrahedron with a point in its interior, also called a *bubble*[1]. According to the literature [47], the first three ones should yield a consistent dynamics whereas the bubble should diverge. Similar dynamical questions in the context of a so-called melon graph have already been investigated in the context of EPRL model [?].

Let us note that the first two configuration have only boudary segments j_l whereas the latter two do posses inner "bulk" segments j_s. All four of them

[1] This is because in the dual 2-complex, this point manifests itself as an extra tetrahedron inside of a larger one.

can be obtained from an elementary building block - a tetrahedron, through deforming and identification: the double tetrahedron is, of course, composed of two tetrahedra identified along a common face, the octahedron is composed of four tetrahedra identified along a common segment and the "bubble" is composed of four suitably identified tetrahedra sharing a common vertex.

One can even give these four configurations a tentative physical interpretation. A tetrahedron represents a zeroth order amplitude for a point-to-triangle process, of which a "bubble" is a first order approximation, whereas a double tetrahedron can be seen as a third-order contribution to a point-to-point process. Octahedron, on the other hand, represents either a fourth order contribution to a point-to-point process or a zeroth order contribution to a tringle-to-triangle process. This nonuniqueness in interpretation is given by the covariant character of the theory[2]

3.2 Results

As a first step, we proceeded to the computation of the partition functions. The results are in the Tables 3.1, 3.2, 3.3, 3.4. Contrary to the expectations, using a code in Mathematica enclosed in the appendix, **we encountered** (probable) **divergences in all four cases**. Furthermore, the computations proved so time- and computer power-consuming, that we were able to get only a modest restricted number of data points, depending on the triangulation. This is an unexpected result that requires a careful interpretation as precludes any further steps in the construction of the dynamics.

Given this unexpected outcome, in the next step we therefore focused on a "phenomenological" description of these divergences. We tried to ascertain their character - whether it is polynomial or exponential and what is the relevant parameter value. For this, we calibrated two linear models,

$$|Z| = \beta_0 + \beta_1 j + \cdots + \beta_n j^n \tag{3.3}$$

and

$$\log |Z| = \beta'_0 + \beta'_1 j. \tag{3.4}$$

The calibration of the former one takes place in two steps - first, one considers a high degree linear model (3.3) (i.e. high n) and runs a regression whereupon one chooses statistically significant coefficients, the highest of which determines the "degree of divergence". In the first step, of course, one cannot choose n higher than the $\#_{datapoints} - 1$ as it yields by definition an ideal fit[3] and the results would therefore be invalid. For the tetrahedron, double tetrahedron, octahedron and the bubble we have therefore chosen respectively $n = 15$, $n = 5$, $\nexists n$ and $n = 5$. In case of the octahedron, we limited ourselves to a descriptive analysis since the number of data points is too low. The calibration of the later model, on the other hand, is straightforward, except for the fact that one must consistently disregard the first data point which is zero.

[2]The options mentioned do not exhaust all the possibilities, only in some sense the more "apparent", basic ones.

[3]Any two points uniquely determine a line - a first degree polynomial with two parameters $ax + b$, three points uniquely determine a second degree polynomial with three parameters etc.

j max	Re Z	Im Z	$\lvert Z\rvert$
1	0	0	0
1.5	121.5	$2.68 \cdot 10^{-13}$	121.5
2	-1145.7	-728.59	1357.74
2.5	7343.93	7417.28	10437.88
3	-31751.1	47429.5	57076.18
3.5	77531	199895	214403.98
4	-136543	611912	626961.15
4.5	191608	$1.46 \cdot 10^{6}$	$1.46 \cdot 10^{6}$
5	-442724	$3.25 \cdot 10^{6}$	$3.28 \cdot 10^{6}$
5.5	808467	$6.73 \cdot 10^{6}$	$6.8 \cdot 10^{6}$
6	$-1.45 \cdot 10^{6}$	$1.33 \cdot 10^{7}$	$1.34 \cdot 10^{7}$
6.5	$2.22 \cdot 10^{6}$	$2.44 \cdot 10^{7}$	$2.45 \cdot 10^{7}$
7	$-3.84 \cdot 10^{6}$	$4.34 \cdot 10^{7}$	$4.36 \cdot 10^{7}$
7.5	$6.14 \cdot 10^{6}$	$7.44 \cdot 10^{7}$	$7.47 \cdot 10^{7}$
8	$-8.36 \cdot 10^{6}$	$1.25 \cdot 10^{8}$	$1.25 \cdot 10^{8}$
8.5	$1.18 \cdot 10^{7}$	$1.99 \cdot 10^{8}$	$1.99 \cdot 10^{8}$
9	$-1.57 \cdot 10^{7}$	$3.13 \cdot 10^{8}$	$3.13 \cdot 10^{8}$
9.5	$2.38 \cdot 10^{7}$	$4.79 \cdot 10^{8}$	$4.80 \cdot 10^{8}$
10	$-2.93 \cdot 10^{7}$	$7.22 \cdot 10^{8}$	$7.23 \cdot 10^{8}$

Table 3.1: The tetrahedron partition sum

j max	Re Z	Im Z	$\lvert Z\rvert$
1	0	0	0
1.5	546.75	$2.01 \cdot 10^{-12}$	546.75
2	8395.71	7978.05	11581.76
2.5	84296.4	129973	154915.67
3	420568	$1.38 \cdot 10^{6}$	$1.44 \cdot 10^{6}$
3.5	$1.59 \cdot 10^{6}$	$7.46 \cdot 10^{6}$	$7.63 \cdot 10^{6}$
4	$-4.14 \cdot 10^{6}$	$3.73 \cdot 10^{7}$	$3.76 \cdot 10^{7}$
4.5	$-6.83 \cdot 10^{6}$	$9.57 \cdot 10^{7}$	$9.59 \cdot 10^{7}$

Table 3.2: The double tetrahedron partition sum

j max	Re Z	Im Z	$\lvert Z\rvert$
1	0	0	0
1.5	1230.19	$-7.53 \cdot 10^{-12}$	1230.19
2	9199.98	$-2.10 \cdot 10^{-10}$	9199.98
2.5	804230	$1.06 \cdot 10^{-10}$	804230
3	$1.77 \cdot 10^{7}$	$-2.55 \cdot 10^{-7}$	$1.77 \cdot 10^{7}$

Table 3.3: The octahedron partition sum

j max	Re Z	Im Z	|Z|
1	0	0	0
1.5	45.56	$-2.46 \cdot 10^{-13}$	45.56
2	-674.82	-1007.72	1212.80
2.5	3391.85	11658.90	12142.26
3	8054.16	-120166	120435.61
3.5	50011.80	834920	836416.52
4	713364	$-5.34 \cdot 10^{6}$	$5.39 \cdot 10^{6}$

Table 3.4: The bubble partition sum

	Estimate	Std. Error	t value	$\Pr(>\|t\|)$
β_0	-9.185e+08	7.416e+08	-1.239	0.270
β_1	3.599e+09	2.837e+09	1.269	0.260
β_2	-6.097e+09	4.683e+09	-1.302	0.250
β_3	5.930e+09	4.435e+09	1.337	0.239
β_4	-3.705e+09	2.698e+09	-1.373	0.228
β_5	1.574e+09	1.117e+09	1.409	0.218
β_6	-4.676e+08	3.238e+08	-1.444	0.208
β_7	9.827e+07	6.650e+07	1.478	0.200
β_8	-1.452e+07	9.620e+06	-1.509	0.192
β_9	1.465e+06	9.528e+05	1.538	0.185
β_{10}	-9.410e+04	6.017e+04	-1.564	0.179
β_{11}	3.119e+03	1.965e+03	1.587	0.173
β_{12}	NA	NA	NA	NA
β_{13}	-2.777e+00	1.708e+00	-1.626	0.165
β_{14}	NA	NA	NA	NA
β_{15}	2.492e-03	1.506e-03	1.655	0.159
Multiple R-squared:1, Adjusted R-squared:1				

Table 3.5: Polynomial model calibration - tetrahedron

The question of which fit is better is solved by comparing the goodnesses of fit of both models and by visual inspection of the graphs. Whichever one it is, it determines the character of the divergence. The results of the regressions are shown in tables 3.5 - 3.8 and the graphs of both the polynomial fit and the exponential fit are given in fig. 3.2 - 3.9.

We can take the following comment on the procedure: regarding the tetrahedron configuration, we can see in the tables 3.5, 3.6 both the coefficients β_{13}, β_{15} are statistically significant, however the value of β_{15} is of the order 10^{-3}, whereas β_{13} is of the order one. From this we conclude that the partition sum diverges rather as j^{13}[4]. Second, regarding the octahedron and "bubble", the estimation of the polynomial model yielded all coefficients significant already in the first step so there was no need to do a second estimation.

Let us now proceed to the interpretation of the results. The graphs in figures 3.2 - 3.8, with the exception of 3.6, consistently show that the growth of the partition function has a polynomial, rather than exponential, character. This is

[4]Even though the second regression yields a statistically insignificant β_{13}.

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β_0	1.997e+08	3.465e+08	0.576	0.585
β_1	-7.285e+08	1.248e+09	-0.584	0.581
β_2	1.131e+09	1.916e+09	0.590	0.577
β_3	-9.932e+08	1.668e+09	-0.595	0.573
β_4	5.517e+08	9.222e+08	0.598	0.572
β_5	-2.051e+08	3.425e+08	-0.599	0.571
β_6	5.249e+07	8.796e+07	0.597	0.572
β_7	-9.332e+06	1.576e+07	-0.592	0.575
β_8	1.143e+06	1.954e+06	0.585	0.580
β_9	-9.339e+04	1.621e+05	-0.576	0.586
β_{10}	4.707e+03	8.331e+03	0.565	0.593
β_{11}	-1.173e+02	2.123e+02	-0.552	0.601
β_{12}	NA	NA	NA	NA
β_{13}	4.592e-02	8.762e-02	0.524	0.619
Multiple R-squared:1, Adjusted R-squared:1				

Table 3.6: Polynomial model calibration - tetrahedron

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β'_0	5.5441	0.7423	7.469	1.34e-06
β'_1	1.6409	0.1177	13.944	2.27e-10
Multiple R-squared:0.924, Adjusted R-squared:0.9192				

Table 3.7: Exponential model calibration - tetrahedron

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β_0	6.397E+07	6.282E+07	1.018	0.416
β_1	-1.546E+08	1.473E+08	-1.049	0.404
β_2	1.348E+08	1.276E+08	1.057	0.401
β_3	-5.244E+07	5.160E+07	-1.016	0.416
β_4	8.774E+06	9.844E+06	0.891	0.467
β_5	-4.455E+05	7.148E+05	-0.623	0.597
Multiple R-squared:0.9995, Adjusted R-squared:0.9981				

Table 3.8: Polynomial model calibration - double tetrahedron

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β'_0	1.2894	0.9773	1.319	0.244
β'_1	4.0208	0.3091	13.010	4.78e-05
Multiple R-squared:0.9713, Adjusted R-squared:0.9656				

Table 3.9: Exponential model calibration - double tetrahedron

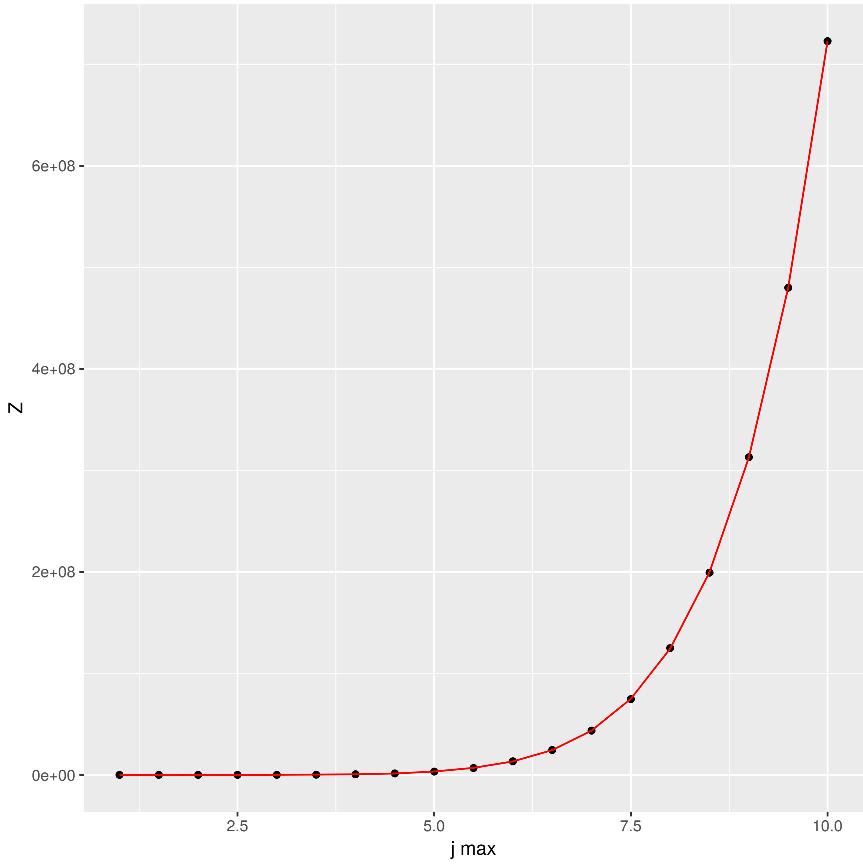

Figure 3.2: Polynomial fit corresponding to the table 3.6

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β_0	-1.069e+07	4.569e+06	-2.339	0.257
β_1	2.956e+07	1.143e+07	2.586	0.235
β_2	-3.104e+07	1.068e+07	-2.908	0.211
β_3	1.558e+07	4.697e+06	3.317	0.186
β_4	-3.760e+06	9.813e+05	-3.832	0.163
β_5	3.517e+05	78411	4.486	0.140
Multiple R-squared:0.9998, Adjusted R-squared:0.999				

Table 3.10: Polynomial model calibration - “bubble”

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
β'_0	-2.4262	0.6782	-3.578	0.0232
β'_1	4.5889	0.2355	19.485	4.09e-05
Multiple R-squared:0.9896, Adjusted R-squared:0.987				

Table 3.11: Exponential model calibration - “bubble”

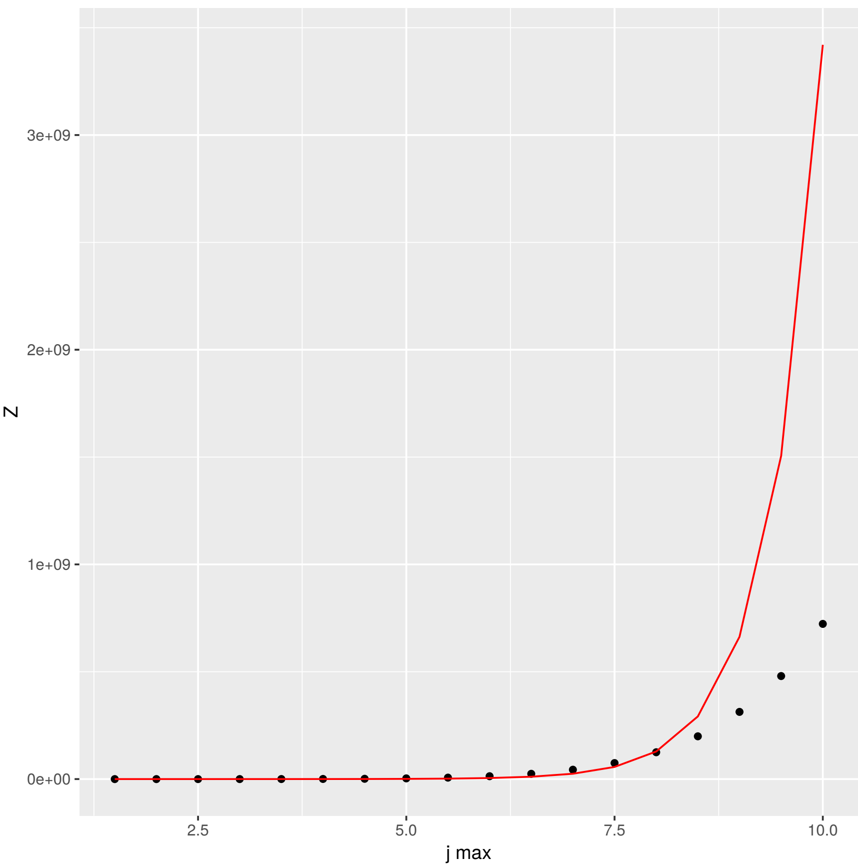

Figure 3.3: Exponential fit corresponding to the table 3.7

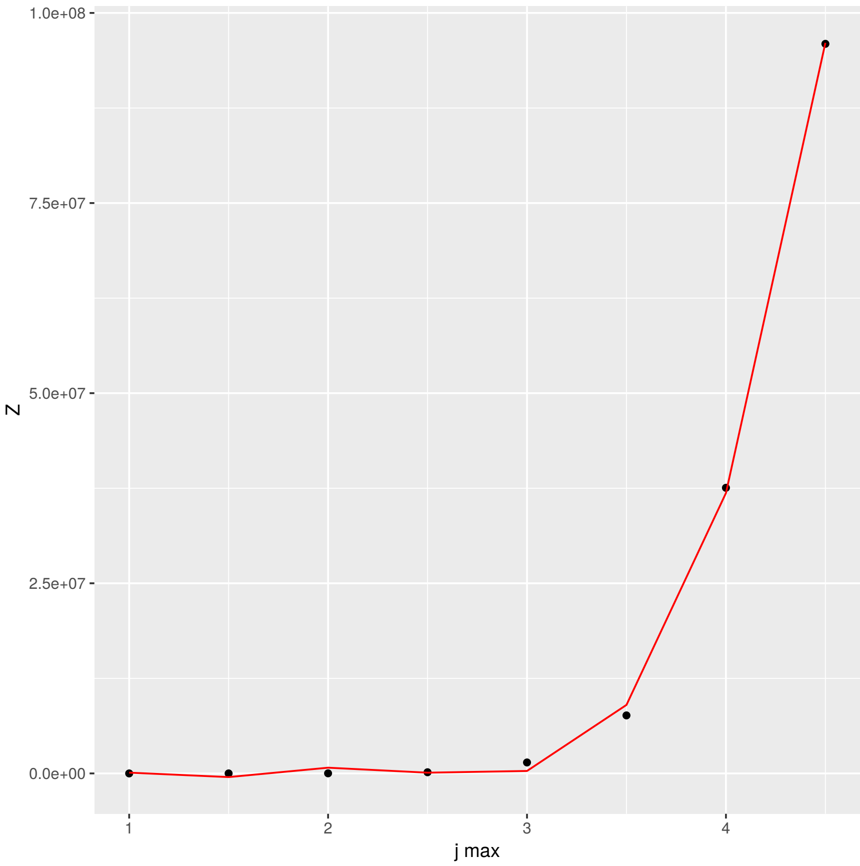

Figure 3.4: Polynomial fit corresponding to the table 3.8

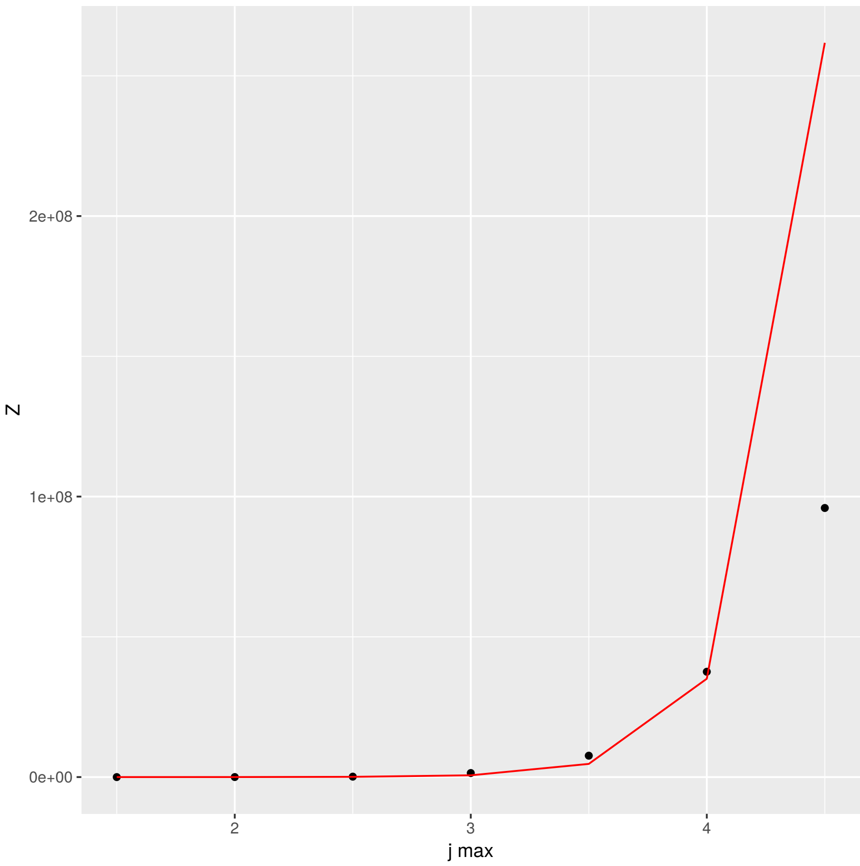

Figure 3.5: Exponential fit corresponding to the table 3.9

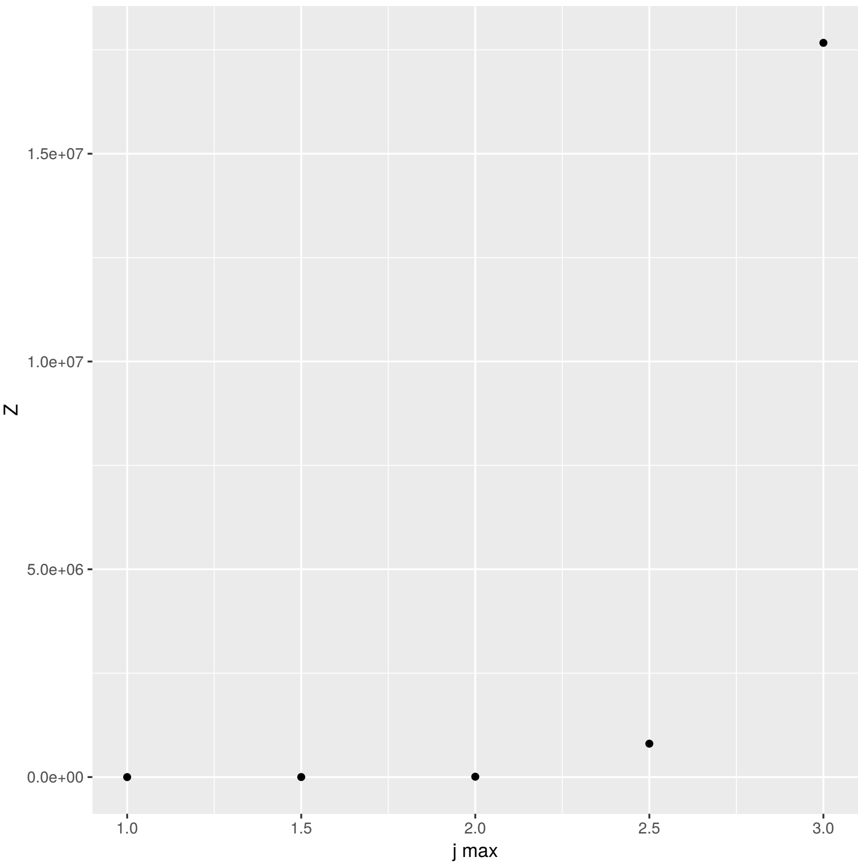

Figure 3.6: Octahedron partition sum values

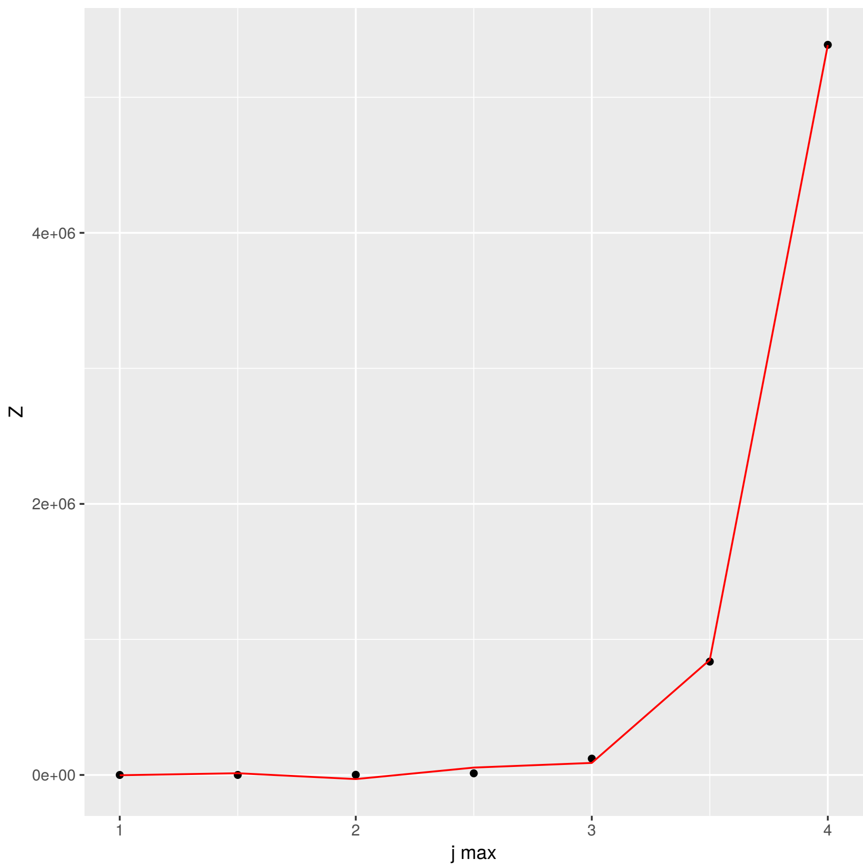

Figure 3.7: Polynomial fit corresponding to the table 3.10

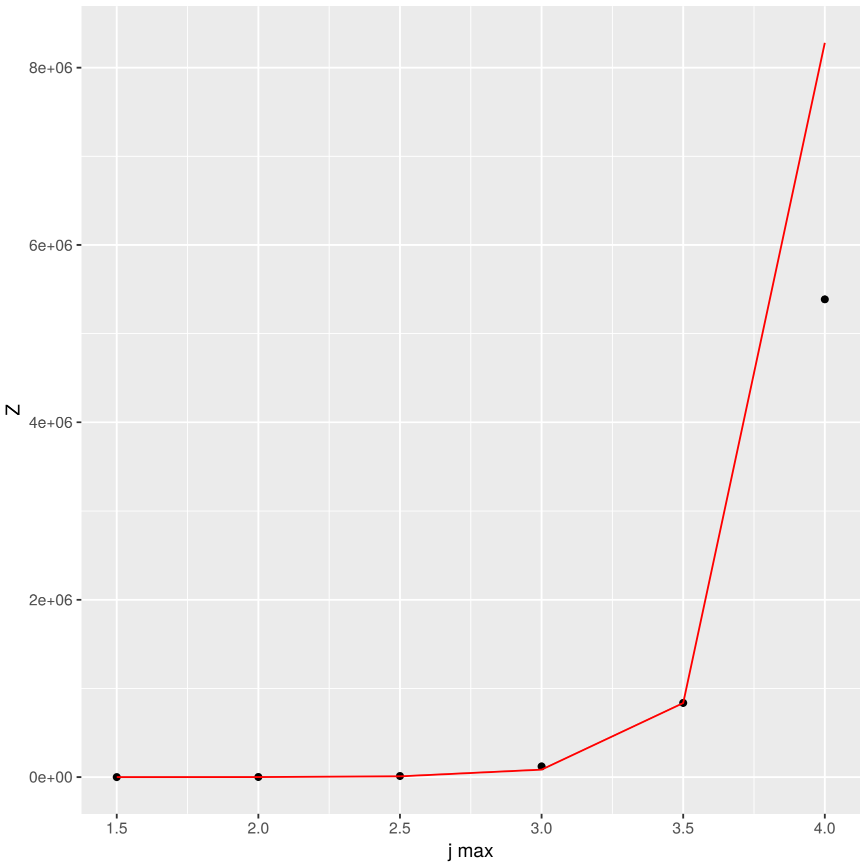

Figure 3.8: Exponential fit corresponding to the table 3.11

	Estimate	Std. Error	t value	Pr(> $|t|$)
β_0	-1.5404	0.3538	-4.354	0.0489
β_1	13.6317	2.6084	5.226	0.0347
β_2	-39.1829	6.0529	-6.473	0.0230
β_3	36.6737	4.4249	8.288	0.0142
Multiple R-squared:0.9993, Adjusted R-squared:0.9982				

Table 3.12: Normalized amplitude inverse-powers fitting - "bubble"

also reflected in the goodness of fit measures being higher for the model (3.3), even after adjustment for the number of regressands (adjusted R-squared). In the case of the tetrahedron, one thus gets a divergence of the order at least 13 and in the cases of the double tetrahedron and the bubble of the order at least 5. The qualitative behavior of the octahedron partition sum is consistent with any of these two options.

As the next step in the analysis of the divergences we considered an arbitrary fixed boudary state j_l. We put $j_l = 1$ for $\forall l$, for all the configurations. It is clear that this cannot change the convergence properties of the transition amplitude of the first two considered configurations, as there are no inner segments in them, but for the octahedron and "bubble" configuration, it could diverge in such a way so as to cancel the divergence in the partition function.

It turns out that, at least for the case of the boundary state considered, this is not true. The sum corresponding to the unnormed amplitude for the octahedron containts a single contribution when $j_s = 1$, so the normed amplitude (3.1) converges to zero. In the case of the bubble, on the other hand, one gets an infinite amount of contributions, the sum of which apparently diverges, as depicted in figure 3.9. It is apparent that the character of this sum is n-th root or logarithmic, even without further statistical analysis, one can therefore conclude that it will not save the amplitude (3.1) from converging to zero. To confirm this, we ran a regression of a form

$$W = \beta_0 + \beta_1 j^{-1} + \beta_2 j^{-2} + \beta_3 j^{-3}, \tag{3.5}$$

whose results are depicted in the Table 3.12.

It is clear that the fact that all of the coefficients are statistically significant means, at the very least, that asymptiotically the normed amplitude is indeed zero - a result one was trying to avoid. One can therefore conclude that the divergent behavior of the partition sum precludes a proper definition of the dynamics for all the configurations considered, in the sense of both (3.1) and (3.2).

3.3 Discussion

The fact that our results are at odds with the literature [**?**] can be interpreted in several ways. One possibility is that the formula used (3.1) is not equivalent to the simpler (2.26). Even though it is possible that both are related to the Einstein-Hilbert action in the large-spin limit in a required way, they may on the microscopic level define different dynamical systems. Another, more mundane, possibility is that there is a programming error in our code. Due to the absence of reproducible results in our area of interest, we were not able to calibrate our code to see if it gives the right results. However, to the naked eye we dare to say that there is not an error of an egregious nature. The third possibility is that Mathematica computes the $6j$-symbols in an inefficient and error-prone way so that small rounding errors get compounded over many computation cycles and give a wrong mathematical result in the end.

There are further options to ponder. Maybe the sums do actually converge, but we have not seen it because enumerating too little data points. Such a scenario is hard to believe but it cannot be dismissed out of hand. Another possibility is that the results are right and one needs to get rid of them by some sort of regularization. Such a scheme could be provided by the GFT theory duality. It is also possible that the way of defining the normalization factor as we did through a partition sum may not be correct in these circumstances. Logical clearness might not imply mathematical correctness in this instance, one possible example of a different definition of it could be through a partial fixing some of the spins on the boundary, perhaps within some sort of a causal past-future distinction, and computing the partition sum with respect to these fixed conditions.

It should be mentioned that there have been some studies of the asymptotic properties of the $6j$-formula [23], [20], these, however, do not concern exactly our case of interest - [23] offers a proof of the asymptotic formula (2.25), whereas [20] does study the properties of the $6j$-symbol but only within the context of an isosceles tetrahedron. There, however, no signs of divergent behavior seem to be found.

Conclusion

In this thesis, we have offered a review of the issue of quantum gravity in general and of loop quantum gravity in particular. A wish for novel insight into the continuity limit of the LQG theory, in the simplified setting of the Ponzano-Regge model, led us to consider a simple dynamical system composed of four configurations. There, it did not prove consistent to define a dynamics through weighing particular histories with an inverse of the sum of all amplitudes consistent with a given configuration. As a possible explanation of this we discussed various causes, all of which serve as a pointer for a subsequent research.

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Attachments

```
Import["labels.jpeg"]
```

Import::nffil : File not found during Import. >>

```
$Failed
```

```
Import["labels.jpg"]
```

Import::nffil : File not found during Import. >>

```
$Failed
```

```
Directory[]
```

```
C:\Users\Pavel\Documents
```

```
SetDirectory["C:\Users\Pavel\Desktop\Mathematica Directory"]
```

Syntax::stresc : Unknown string escape \U.

Syntax::stresc : Unknown string escape \P.

Syntax::stresc : Unknown string escape \D.

Syntax::stresc : Unknown string escape \M.

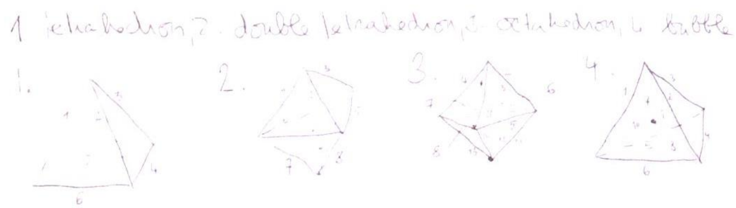
```
C:\Users\Pavel\Desktop\Mathematica Directory
```

```
"C:Users\\Pavel\\Desktop\\Mathematica Directory"
```

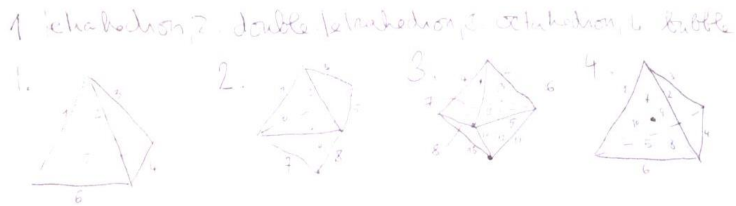
```
SetDirectory["C:/Users/Pavel/Desktop/Mathematica Directory"]
```

```
C:\Users\Pavel\Desktop\Mathematica Directory
```

```
Import["labels.jpg"]
```



```
Import["labels.jpg"]
```



Common functions

```
fun[x_] := (2 * x + 1) * (-1) ^ (2 * x)
```

Conditions

```
(*for the configurations 1,2,3,4, the conditions are c,
d,e,l and the spin representations j,k,l,m respectively*)
```

1.

```
c126 = j6 > Abs[j2 - j1] && j6 < j1 + j2 && j2 > Abs[j6 - j1] && j2 < j1 + j6 &&
    j1 > Abs[j2 - j6] && j1 < j6 + j2 && FractionalPart[j1 + j2 + j6] == 0;
c234 = j4 > Abs[j3 - j2] && j4 < j2 + j3 && j3 > Abs[j4 - j2] && j3 < j2 + j4 &&
    j2 > Abs[j3 - j4] && j2 < j4 + j3 && FractionalPart[j2 + j3 + j4] == 0;
c135 = j5 > Abs[j3 - j1] && j5 < j1 + j3 && j3 > Abs[j5 - j1] && j3 < j1 + j5 &&
    j1 > Abs[j3 - j5] && j1 < j5 + j3 && FractionalPart[j1 + j3 + j5] == 0;
c456 = j6 > Abs[j4 - j5] && j6 < j4 + j5 && j4 > Abs[j6 - j5] && j4 < j5 + j6 &&
    j5 > Abs[j4 - j6] && j5 < j6 + j4 && FractionalPart[j4 + j5 + j6] == 0;

c = c126 && c234 && c135 && c456;
```

2.

```
d124 = k4 > Abs[k2 - k1] && k4 < k1 + k2 && k2 > Abs[k4 - k1] && k2 < k1 + k4 &&
    k1 > Abs[k2 - k4] && k1 < k4 + k2 && FractionalPart[k1 + k2 + k4] == 0;
d235 = k5 > Abs[k3 - k2] && k5 < k2 + k3 && k3 > Abs[k5 - k2] && k3 < k2 + k5 &&
    k2 > Abs[k3 - k5] && k2 < k5 + k3 && FractionalPart[k2 + k3 + k5] == 0;
d136 = k6 > Abs[k3 - k1] && k6 < k1 + k3 && k3 > Abs[k6 - k1] && k3 < k1 + k6 &&
    k1 > Abs[k3 - k6] && k1 < k6 + k3 && FractionalPart[k1 + k3 + k6] == 0;
d456 = k6 > Abs[k4 - k5] && k6 < k4 + k5 && k4 > Abs[k6 - k5] && k4 < k5 + k6 &&
    k5 > Abs[k4 - k6] && k5 < k6 + k4 && FractionalPart[k4 + k5 + k6] == 0;
d478 = k8 > Abs[k7 - k4] && k8 < k4 + k7 && k7 > Abs[k8 - k4] && k7 < k4 + k8 &&
    k4 > Abs[k7 - k8] && k4 < k8 + k7 && FractionalPart[k4 + k7 + k8] == 0;
d589 = k9 > Abs[k8 - k5] && k9 < k5 + k8 && k8 > Abs[k9 - k5] && k8 < k5 + k9 &&
    k5 > Abs[k8 - k9] && k5 < k9 + k8 && FractionalPart[k5 + k8 + k9] == 0;
d679 = k9 > Abs[k7 - k6] && k9 < k6 + k7 && k7 > Abs[k9 - k6] && k7 < k6 + k9 &&
    k6 > Abs[k7 - k9] && k6 < k9 + k7 && FractionalPart[k7 + k6 + k9] == 0;

d = d124 && d235 && d136 && d456 && d478 && d589 && d679;
```

3.

```
e125 = l5 > Abs[l2 - l1] && l5 < l1 + l2 && l2 > Abs[l5 - l1] && l2 < l1 + l5 &&
    l1 > Abs[l2 - l5] && l1 < l5 + l2 && FractionalPart[l1 + l2 + l5] == 0;
e236 = l6 > Abs[l3 - l2] && l6 < l2 + l3 && l3 > Abs[l6 - l2] && l3 < l2 + l6 &&
    l2 > Abs[l3 - l6] && l2 < l6 + l3 && FractionalPart[l2 + l3 + l6] == 0;
e347 = l7 > Abs[l4 - l3] && l7 < l3 + l4 && l4 > Abs[l7 - l3] && l4 < l3 + l7 &&
    l3 > Abs[l4 - l7] && l3 < l7 + l4 && FractionalPart[l3 + l4 + l7] == 0;
e148 = l8 > Abs[l4 - l1] && l8 < l1 + l4 && l4 > Abs[l8 - l1] && l4 < l1 + l8 &&
    l1 > Abs[l4 - l8] && l1 < l8 + l4 && FractionalPart[l1 + l4 + l8] == 0;
e589 = l9 > Abs[l8 - l5] && l9 < l5 + l8 && l8 > Abs[l9 - l5] && l8 < l5 + l9 &&
    l5 > Abs[l8 - l9] && l5 < l9 + l8 && FractionalPart[l5 + l8 + l9] == 0;
e679 = l9 > Abs[l7 - l6] && l9 < l6 + l7 && l7 > Abs[l9 - l6] && l7 < l6 + l9 &&
    l6 > Abs[l7 - l9] && l6 < l9 + l7 && FractionalPart[l6 + l7 + l9] == 0;
e81013 = l13 > Abs[l10 - l8] && l13 < l8 + l10 && l10 > Abs[l13 - l8] && l10 < l8 + l13 &&
    l8 > Abs[l10 - l13] && l8 < l13 + l10 && FractionalPart[l8 + l10 + l13] == 0;
e51011 = l11 > Abs[l10 - l5] && l11 < l5 + l10 && l10 > Abs[l11 - l5] && l10 < l5 + l11 &&
    l5 > Abs[l10 - l11] && l5 < l11 + l10 && FractionalPart[l5 + l10 + l11] == 0;
e61112 = l12 > Abs[l11 - l6] && l12 < l6 + l11 && l11 > Abs[l12 - l6] && l11 < l6 + l12 &&
    l6 > Abs[l11 - l12] && l6 < l12 + l11 && FractionalPart[l6 + l11 + l12] == 0;
e71213 = l13 > Abs[l12 - l7] && l13 < l7 + l12 && l12 > Abs[l13 - l7] && l12 < l7 + l13 &&
    l7 > Abs[l12 - l13] && l7 < l13 + l12 && FractionalPart[l7 + l12 + l13] == 0;
e249 = l9 > Abs[l4 - l2] && l9 < l2 + l4 && l4 > Abs[l9 - l2] && l4 < l2 + l9 &&
    l2 > Abs[l4 - l9] && l2 < l9 + l4 && FractionalPart[l4 + l2 + l9] == 0;
e91113 = l13 > Abs[l11 - l9] && l13 < l9 + l11 && l11 > Abs[l13 - l9] && l11 < l9 + l13 &&
    l9 > Abs[l11 - l13] && l9 < l13 + l11 && FractionalPart[l11 + l9 + l13] == 0;

e = e125 && e236 && e347 && e148 && e589 &&
    e679 && e81013 && e51011 && e61112 && e71213 && e249 && e91113;
```

4.

```
f126 = m6 > Abs[m2 - m1] && m6 < m1 + m2 && m2 > Abs[m6 - m1] && m2 < m1 + m6 &&
    m1 > Abs[m2 - m6] && m1 < m6 + m2 && FractionalPart[m1 + m2 + m6] == 0;
f234 = m4 > Abs[m3 - m2] && m4 < m2 + m3 && m3 > Abs[m4 - m2] && m3 < m2 + m4 &&
    m2 > Abs[m3 - m4] && m2 < m4 + m3 && FractionalPart[m2 + m3 + m4] == 0;
f135 = m5 > Abs[m3 - m1] && m5 < m1 + m3 && m3 > Abs[m5 - m1] && m3 < m1 + m5 &&
    m1 > Abs[m3 - m5] && m1 < m5 + m3 && FractionalPart[m1 + m3 + m5] == 0;
f456 = m6 > Abs[m4 - m5] && m6 < m4 + m5 && m4 > Abs[m6 - m5] && m4 < m5 + m6 &&
    m5 > Abs[m4 - m6] && m5 < m6 + m4 && FractionalPart[m4 + m5 + m6] == 0;
f1710 = m10 > Abs[m7 - m1] && m10 < m1 + m7 && m7 > Abs[m10 - m1] && m7 < m1 + m10 &&
    m1 > Abs[m7 - m10] && m1 < m10 + m7 && FractionalPart[m1 + m7 + m10] == 0;
f278 = m8 > Abs[m7 - m2] && m8 < m2 + m7 && m7 > Abs[m8 - m2] && m7 < m2 + m8 &&
    m2 > Abs[m7 - m8] && m2 < m8 + m7 && FractionalPart[m2 + m7 + m8] == 0;
f379 = m9 > Abs[m7 - m3] && m9 < m3 + m7 && m7 > Abs[m9 - m3] && m7 < m3 + m9 &&
    m3 > Abs[m7 - m9] && m3 < m9 + m7 && FractionalPart[m3 + m7 + m9] == 0;
f6810 = m10 > Abs[m8 - m6] && m10 < m6 + m8 && m8 > Abs[m10 - m6] && m8 < m6 + m10 &&
    m6 > Abs[m8 - m10] && m6 < m10 + m8 && FractionalPart[m6 + m8 + m10] == 0;
f489 = m9 > Abs[m8 - m4] && m9 < m4 + m8 && m8 > Abs[m9 - m4] && m8 < m4 + m9 &&
    m4 > Abs[m8 - m9] && m4 < m9 + m8 && FractionalPart[m4 + m8 + m9] == 0;
f5910 = m10 > Abs[m9 - m5] && m10 < m5 + m9 && m9 > Abs[m10 - m5] && m9 < m5 + m10 &&
    m5 > Abs[m9 - m10] && m5 < m10 + m9 && FractionalPart[m5 + m9 + m10] == 0;
```

```
f = f126 && f234 && f135 && f456 && f1710 && f278 && f379 && f6810 && f489 && f5910;
```

Setting boundary spin representations

1.

```
j1 = 1;
j2 = 1;
j3 = 1;
j4 = 1;
j5 = 1;
j6 = 1;
```

2.

```
k1 = 1;
k2 = 1;
k3 = 1;
k4 = 1;
k5 = 1;
k6 = 1;
k7 = 1;
k8 = 1;
k9 = 1;
```

3.

```
l1 = 1;
l2 = 1;
l3 = 1;
l4 = 1;
l5 = 1;
l6 = 1;
l7 = 1;
l8 = 1;
l10 = 1;
l11 = 1;
l12 = 1;
l13 = 1;
```

4.

```
m1 = 1;
m2 = 1;
m3 = 1;
m4 = 1;
m5 = 1;
m6 = 1;
```

Amplitudes

```
Import["vzorec.jpg"]
```

$$W_\Delta(j_e) = \sum_{j_f} \prod_f (-1)^{2j_f} d_{j_f} \prod_v (-1)^{J_v} \{6j\}$$
$$J_v = \sum^{6} j_e$$

```
ImageResize[%3, 590.]
```

$$W_\Delta(j_e) = \sum_{j_f} \prod_f (-1)^{2j_f} d_{j_f} \prod_v (-1)^{J_v} \{6j\}$$
$$J_v = \sum^{6} j_e$$

1.

```
a1 = 0;

If[c, a1 += Apply[Times, Map[fun, {j1, j2, j3, j4, j5, j6}]] *
    (-1) ^ (j1 + j2 + j3 + j4 + j5 + j6) * SixJSymbol[{j4, j5, j6}, {j1, j2, j3}]]
```

$\frac{243}{2}$

```
a1
```

$\frac{243}{2}$

2.

```
a2 = 0;

If[d, a2 += Apply[Times, Map[fun, {k1, k2, k3, k4, k5, k6, k7, k8, k9}]] *
    (-1) ^ (k1 + k2 + k3 + k4 + k5 + k6) * SixJSymbol[{k4, k5, k6}, {k3, k1, k2}] *
    (-1) ^ (k4 + k5 + k6 + k7 + k8 + k9) * SixJSymbol[{k4, k5, k6}, {k9, k7, k8}]]
```

$\frac{2187}{4}$

```
a2
```

$\frac{2187}{4}$

3.

```
a3 = 0;

For[l9 = 0.5, l9 < 2, l9 += 0.5, If[e, a3 +=
    Apply[Times, Map[fun, {l1, l2, l3, l4, l5, l6, l7, l8, l9, l10, l11, l12, l13}]] *
     (-1) ^ (l1 + l2 + l4 + l5 + l8 + l9) * SixJSymbol[{l5, l8, l9}, {l4, l2, l1}] *
     (-1) ^ (l2 + l3 + l4 + l6 + l7 + l9) * SixJSymbol[{l6, l7, l9}, {l4, l2, l3}] *
     (-1) ^ (l5 + l8 + l9 + l10 + l11 + l13) * SixJSymbol[{l5, l8, l9}, {l13, l11, l10}] *
     (-1) ^ (l6 + l7 + l9 + l11 + l12 + l13) * SixJSymbol[{l6, l7, l9}, {l13, l11, l12}]]]

a3
```

$1230.19 - 3.91702 \times 10^{-12}\ i$

4.

```
a4 = 0;

For[m7 = 0.5, m7 < 2.5, m7 += 0.5,
 For[m8 = 0.5, m8 < 2.5, m8 += 0.5,
  For[m9 = 0.5, m9 < 2.5, m9 += 0.5,
   For[m10 = 0.5, m10 < 2.5, m10 += 0.5,
    If[f, a4 += Apply[Times, Map[fun, {m1, m2, m3, m4, m5, m6, m7, m8, m9, m10}]] *
        (-1) ^ (m1 + m3 + m5 + m9 + m10 + m7) *
        SixJSymbol[{m1, m3, m5}, {m9, m10, m7}] * (-1) ^ (m2 + m3 + m4 + m9 + m8 + m7) *
        SixJSymbol[{m2, m3, m4}, {m9, m8, m7}] * (-1) ^ (m1 + m2 + m6 + m8 + m10 + m7) *
        SixJSymbol[{m1, m2, m6}, {m8, m10, m7}] * (-1) ^ (m4 + m5 + m6 + m10 + m8 + m9) *
        SixJSymbol[{m4, m5, m6}, {m10, m8, m9}]]]]]]

a4
```

$82.665 - 4.54465 \times 10^{-13}\ i$

Partition functions

1.

```
z1 = 0;

For[j1 = 0.5, j1 < 7.5, j1 += 0.5,
 For[j2 = 0.5, j2 < 7.5, j2 += 0.5,
  For[j3 = 0.5, j3 < 7.5, j3 += 0.5,
   For[j4 = 0.5, j4 < 7.5, j4 += 0.5,
    For[j5 = 0.5, j5 < 7.5, j5 += 0.5,
     For[j6 = 0.5, j6 < 7.5, j6 += 0.5,
      If[c, z1 += Apply[Times, Map[fun, {j1, j2, j3, j4, j5, j6}]] * (-1) ^
          (j1 + j2 + j3 + j4 + j5 + j6) * SixJSymbol[{j4, j5, j6}, {j1, j2, j3}]]]]]]]]

z1
```

$6.13813 \times 10^{6} + 7.44096 \times 10^{7}\ i$

2.

```
z2 = 0;

For[k1 = 0.5, k1 < 3, k1 += 0.5,
 For[k2 = 0.5, k2 < 3, k2 += 0.5, For[k3 = 0.5, k3 < 3, k3 += 0.5,
   For[k4 = 0.5, k4 < 3, k4 += 0.5, For[k5 = 0.5, k5 < 3, k5 += 0.5,
     For[k6 = 0.5, k6 < 3, k6 += 0.5, For[k7 = 0.5, k7 < 3, k7 += 0.5,
       For[k8 = 0.5, k8 < 3, k8 += 0.5, For[k9 = 0.5, k9 < 3, k9 += 0.5,
         If[d, z2 += Apply[Times, Map[fun, {k1, k2, k3, k4, k5, k6, k7, k8, k9}]] *
            (-1) ^ (k1 + k2 + k3 + k4 + k5 + k6) * SixJSymbol[{k4, k5, k6},
             {k3, k1, k2}] * (-1) ^ (k4 + k5 + k6 + k7 + k8 + k9) *
            SixJSymbol[{k4, k5, k6}, {k9, k7, k8}]]]]]]]]]]]

z2
```

420 568. + 1.38044 × 10^6 i

3.

```
z3 = 0;

For[l1 = 0.5, l1 < 2, l1 += 0.5,
 For[l2 = 0.5, l2 < 2, l2 += 0.5, For[l3 = 0.5, l3 < 2, l3 += 0.5,
   For[l4 = 0.5, l4 < 2, l4 += 0.5, For[l5 = 0.5, l5 < 2, l5 += 0.5,
     For[l6 = 0.5, l6 < 2, l6 += 0.5, For[l7 = 0.5, l7 < 2, l7 += 0.5,
       For[l8 = 0.5, l8 < 2, l8 += 0.5, For[l9 = 0.5, l9 < 2, l9 += 0.5,
         For[l10 = 0.5, l10 < 2, l10 += 0.5, For[l11 = 0.5, l11 < 2, l11 += 0.5,
           For[l12 = 0.5, l12 < 2, l12 += 0.5, For[l13 = 0.5, l13 < 2, l13 += 0.5,
             If[e, z3 += Apply[Times, Map[fun, {l1, l2, l3, l4, l5, l6, l7, l8,
                    l9, l10, l11, l12, l13}]] * (-1) ^ (l1 + l2 + l4 + l5 + l8 + l9) *
                SixJSymbol[{l5, l8, l9}, {l4, l2, l1}] * (-1) ^ (l2 + l3 +
                    l4 + l6 + l7 + l9) * SixJSymbol[{l6, l7, l9}, {l4, l2, l3}] *
                (-1) ^ (l5 + l8 + l9 + l10 + l11 + l13) * SixJSymbol[{l5, l8, l9},
                 {l13, l11, l10}] * (-1) ^ (l6 + l7 + l9 + l11 + l12 + l13) *
                SixJSymbol[{l6, l7, l9}, {l13, l11, l12}]]]]]]]]]]]]]]]

z3
```

9199.98 − 2.10091 × 10^{-10} i

4.

```
z4 = 0;
```

```
For[m1 = 0.5, m1 < 1.5, m1 += 0.5,
  For[m2 = 0.5, m2 < 1.5, m2 += 0.5, For[m3 = 0.5, m3 < 1.5, m3 += 0.5, For[m4 = 0.5,
     m4 < 1.5, m4 += 0.5, For[m5 = 0.5, m5 < 1.5, m5 += 0.5, For[m6 = 0.5, m6 < 1.5,
       m6 += 0.5, For[m7 = 0.5, m7 < 1.5, m7 += 0.5, For[m8 = 0.5, m8 < 1.5, m8 += 0.5,
         For[m9 = 0.5, m9 < 1.5, m9 += 0.5, For[m10 = 0.5, m10 < 1.5, m10 += 0.5, If[f,
            z4 += Apply[Times, Map[fun, {m1, m2, m3, m4, m5, m6, m7, m8, m9, m10}]] *
              (-1) ^ (m1 + m3 + m5 + m9 + m10 + m7) * SixJSymbol[{m1, m3, m5},
               {m9, m10, m7}] * (-1) ^ (m2 + m3 + m4 + m9 + m8 + m7) * SixJSymbol[
               {m2, m3, m4}, {m9, m8, m7}] * (-1) ^ (m1 + m2 + m6 + m8 + m10 + m7) *
              SixJSymbol[{m1, m2, m6}, {m8, m10, m7}] * (-1) ^ (m4 + m5 + m6 +
                 m10 + m8 + m9) * SixJSymbol[{m4, m5, m6}, {m10, m8, m9}]]]]]]]]]]]]

z4
```

$45.5625 - 2.45511 \times 10^{-13}\, i$

Conditions II

1.

```
C126 = J6 >= Abs[J2 - J1] && J6 <= J1 + J2 && J2 >= Abs[J6 - J1] && J2 <= J1 + J6 &&
    J1 >= Abs[J2 - J6] && J1 <= J6 + J2 && FractionalPart[J1 + J2 + J6] == 0;
C234 = J4 >= Abs[J3 - J2] && J4 <= J2 + J3 && J3 >= Abs[J4 - J2] && J3 <= J2 + J4 &&
    J2 >= Abs[J3 - J4] && J2 <= J4 + J3 && FractionalPart[J2 + J3 + J4] == 0;
C135 = J5 >= Abs[J3 - J1] && J5 <= J1 + J3 && J3 >= Abs[J5 - J1] && J3 <= J1 + J5 &&
    J1 >= Abs[J3 - J5] && J1 <= J5 + J3 && FractionalPart[J1 + J3 + J5] == 0;
C456 = J6 >= Abs[J4 - J5] && J6 <= J4 + J5 && J4 >= Abs[J6 - J5] && J4 <= J5 + J6 &&
    J5 >= Abs[J4 - J6] && J5 <= J6 + J4 && FractionalPart[J4 + J5 + J6] == 0;

CII = C126 && C234 && C135 && C456;
```

2.

```
D124 = K4 > Abs[K2 - K1] && K4 < K1 + K2 && 12 > Abs[K4 - K1] && K2 < K1 + K4 &&
    K1 > Abs[K2 - K4] && K1 < K4 + K2 && FractionalPart[K1 + K2 + K4] == 0;
D235 = K5 > Abs[K3 - K2] && K5 < K2 + K3 && K3 > Abs[K5 - K2] && K3 < K2 + K5 &&
    K2 > Abs[K3 - K5] && K2 < K5 + K3 && FractionalPart[K2 + K3 + K5] == 0;
D136 = K6 > Abs[K3 - K1] && K6 < K1 + K3 && K3 > Abs[K6 - K1] && K3 < K1 + K6 &&
    K1 > Abs[K3 - K6] && K1 < K6 + K3 && FractionalPart[K1 + K3 + K6] == 0;
D456 = K6 > Abs[K4 - K5] && K6 < K4 + K5 && K4 > Abs[K6 - K5] && K4 < K5 + K6 &&
    K5 > Abs[K4 - K6] && K5 < K6 + K4 && FractionalPart[K4 + K5 + K6] == 0;
D478 = K8 > Abs[K7 - K4] && K8 < K4 + K7 && K7 > Abs[K8 - K4] && K7 < K4 + K8 &&
    K4 > Abs[K7 - K8] && K4 < K8 + K7 && FractionalPart[K4 + K7 + K8] == 0;
D589 = K9 > Abs[K8 - K5] && K9 < K5 + K8 && K8 > Abs[K9 - K5] && K8 < K5 + K9 &&
    K5 > Abs[K8 - K9] && K5 < K9 + K8 && FractionalPart[K5 + K8 + K9] == 0;
D679 = K9 > Abs[K7 - K6] && K9 < K6 + K7 && K7 > Abs[K9 - K6] && K7 < K6 + K9 &&
    K6 > Abs[K7 - K9] && K6 < K9 + K7 && FractionalPart[K7 + K6 + K9] == 0;

DII = D124 && D235 && D136 && D456 && D478 && D589 && D679;
```