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Matching covers of cubic graphs

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Title: Matching covers of cubic graphs

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Abstract: Berge and Fulkerson conjectured that for each cubic bridgeless graph there are six perfect matchings such that each edge is contained in exactly two of them. Another conjecture due to Berge says that we can cover cubic bridgeless graphs by five perfect matchings. Both conjectures are studied for over forty years. Abreu et al. [2016] introduce a new class of graphs (called treelike snarks) which cannot be covered by less then five perfect matchings. We show that their lower bound on number of perfect matchings is tight. Moreover we prove that a bigger class of cubic bridgeless graphs admits Berge conjecture. Finally, we also show that Berge–Fulkerson conjecture holds for treelike snarks.

Keywords: cubic graphs, perfect matching, Berge–Fulkerson conjecture, treelike snarks

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Contents

Introduction

We are using basic graph theory terminology. For an introduction to graph theory see Diestel [2000]. Through this thesis we consider only unoriented graphs without multiple edges and without loops.

We study the number of perfect matchings which is needed to cover a cubic graph. We know that if a cubic graph contains a bridge then it cannot be covered by any number of perfect matchings because each perfect matching has to contain a bridge edge (we prove this fact in Lemma 1). On the other hand each edge of a bridgeless cubic graph is contained in some perfect matching (see Lovász and Plummer [2009]). Thus we can cover any such graph by $\mathcal{O}\Big(m\Big)$ perfect matchings. Let us remind Vizing theorem which says that the edge chromatic number of cubic graphs is either three or four. If we can color edges of G by three colors then we can cover G by three perfect matchings (each matching corresponds to one color class). The first conjecture states that something similar holds for all bridgeless cubic graphs: more precisely that any bridgeless cubic graph can be covered by a constant number of perfect matchings.

Conjecture 1 (Berge)**.** *Let G be a bridgeless cubic graph then there are five perfect matchings $M_1, \ldots, M_5$, which cover the graph G.*

The second conjecture is based on coloring of cubic graphs, namely on fractional edge-coloring of cubic bridgeless graphs. An $a\colon b$ edge-coloring of a graph G is a proper coloring of edges of G by a colors such that each edge is colored by exactly b colors. We say that G has a *fractional c edge-coloring* if G has a proper $a\colon b$ coloring, where $\frac{a}{b} = c$. It is known that there exists a fractional three edge-coloring for every bridgeless cubic graph. The result follows from the Edmonds perfect matching polytope (Edmonds [1965]). This three edge-coloring of a cubic graph G can be viewed as a covering of G by perfect matchings, as each color class forms one perfect matching. But this covering of G may require more than a constant number of perfect matchings. The open question is whether we can use only a constant number of perfect matchings, especially whether there exists a 6:2 coloring of each bridgeless cubic graph.

Conjecture 2 (Berge–Fulkerson)**.** *Let G be a bridgeless cubic graph then there exist six perfect matchings $M_1, \ldots, M_6$ such that each edge is contained in exactly two of them.*

The conjecture was stated by Fulkerson [1971]. It is easy to see that Berge–Fulkerson conjecture implies Berge conjecture.

Both conjectures trivially hold for a graph G which has a proper coloring of its edges by three colors. Bridgeless cubic graphs which are not three edge-colorable are called *snarks*. The class of snarks is very important as many conjectures about properties of cubic graphs hold if and only if we can prove them for all snarks. Thus this class is well known as a class of counterexamples for various conjectures. Let us remark that the first known snark and also the smallest snark is the Petersen graph which is famous for being a counterexample to many conjectures.

Previous work

As the conjecture was stated in 1971, there has already been a lot of work done. First of all we mention two books containing an overview of conjectures about properties of bridgeless cubic graphs and known results, namely books Zhang [1997] and Zhang [2012]. Note that both of them are more focused on the related problem of circuit covering.

Another overview was written by Fiol et al. [2017]. They mention known results for perfect matching covering as well as known results for related circuit covering. They also present various relations between these two problems. An interested reader could also find a historical excursion of searching for different snarks and infinite families of snark in their paper.

We give a brief overview of results about perfect matching coverings of cubic bridgeless graphs. There are papers focusing on the relations between different conjectures about properties of bridgeless cubic graphs. Mazzuoccolo [2011a] shows that both conjectures are equivalent. Note that the fact that Conjecture 1 implies Conjecture 2 does not hold for a particular graph but for the whole class of bridgeless cubic graphs. The proof uses the fact that if G is a counterexample for Conjecture 2 then there exists another graph G' which is a counterexample for Conjecture 1. Thus showing that Conjecture 1 holds for some subclass of cubic graphs does not imply that Conjecture 2 holds for the same class of cubic graphs.

Hao et al. [2009] show that a bridgeless cubic graph G satisfies Conjecture 2 if and only if G has two edge-disjoint matchings M_1, M_2 such that their union is a two-regular subgraph and both $\overline{G \setminus M_1}$ and $\overline{G \setminus M_2}$ are three edge-colorable. Here $\overline{H}$ is the graph obtained from H by removing all cycle components and contracting each subdivided edge.

Note that Conjecture 2 is closely related to the circuit double cover conjecture. In the terminology of circuit covering the word circuit stands for an even subgraph. Thus it may be the case that one circuit consists of many cycles.

Conjecture 3 (Circuit double cover)**.** *Each cubic bridgeless graph has a collection of circuits such that each edge is contained in exactly two of them.*

Conjecture 2 holds for a bridgeless cubic graph G if and only if it can be covered by six circuits containing each edge precisely four times. Bermond et al. [1983] prove that there exists a covering by circuits such that each edge is contained in exactly four of them. On the other hand Fan and Raspaud [1994] show that Conjecture 2 implies that every bridgeless cubic graph G has a three circuit cover of length at most $\frac{22}{15}|E(G)|$ and Jamshy and Tarsi [1992] prove that covering graph by circuits of length at most $\frac{21}{15}|E(G)|$ implies Conjecture 3.

Another important result was obtained by Kaiser et al. [2005], who studied the number of edges covered by constant number of perfect matchings. Let m_i denote the fraction of graph covered by i perfect matchings. Using a technique based on the Edmonds perfect matching polytope (Edmonds [1965]) they show a tight upper bound on $m_2(G) \leq \frac{3}{5}$ for all G. Their upper bound corresponds to a fraction of edges of the Petersen graph covered by two perfect matchings. Using the same technique they also show upper bounds on the fraction of edges that can be covered by three or more perfect matchings. But these upper bounds differ from the lower bounds obtained from the Petersen graph. Especially they showed that $m_3(G) \leq \frac{27}{35}$ for all G and the best known lower bound on $m_3(G)$ is $\frac{4}{5}$ obtained from the Petersen graph.

Based on this result Mazzuoccolo [2011b] proves that each bridgeless cubic graph can be covered by $\mathcal{O}\Big(\log n\Big)$ perfect matchings. This result is still the best known upper bound on the number of perfect matchings in the covering. Patel [2006] shows that Conjecture 2 implies the tightness of upper bounds on m_i obtained from the Petersen graph.

As we have already mentioned it suffices to show that the above conjectures hold for snarks. There are some partial results for different classes of snarks. Sun [2017] show that Conjecture 1 holds for all bridgeless cubic graphs that are hypohamiltonian or that has a spanning subgraph which consists of two cycles. Where *hypohamiltonian graph* is a graph G such that for each vertex $v \in V(G)$ there exists a cycle which contains all vertices of G except of v.

Another direction of research concerns cubic bridgeless graphs that cannot be covered by less than five perfect matchings. Fouquet and Vanherpe [2009] asked for an existence of three edge connected and cyclically four edge connected cubic graphs that cannot be covered by four perfect matchings and that are different from the Petersen graph. Where a graph is a *cyclically k edge connected* if removing any subset of at most $k-1$ edges the remaining graph does not contain two components containing a cycle. Hägglund [2016] finds an example of such a graph on 34 vertices. Esperet and Mazzuoccolo [2014] improve this result by showing that there exists an infinite subclass of snarks which cannot be covered by four perfect matchings.

Finally, Abreu et al. [2016] introduce another class of snarks which they call treelike snarks. Roughly speaking, a treelike snark is a Halin graph with cycle vertices substituted by a modified Petersen graph, called Petersen fragment. Abreu et al. [2016] show that for any treelike snark at least five perfect matchings are needed to cover it. We complement their result by showing an upper bound on the number of matchings needed to cover any treelike snark.

Our work

We show that there is a larger class of snarks based on Petersen fragments cycle. Namely we allow to substitute the tree in treelike snarks by an arbitrary graph $G_{\mathcal{I}}$ in such a way that the resulting graph is bridgeless and cubic. We prove that even those graphs cannot be colored by three colors.

The second chapter is focused on Conjecture 1. We show that if edges of $G_{\mathcal{I}}$ can be colored by three colors, then the whole graph can be covered by five perfect matchings. Together with the lower bound due to Abreu et al. [2016] we get that the number of perfect matchings needed to cover treelike snarks is exactly five.

In the third chapter we restrict our class of graphs even more, specifically to have a coloring of edges of $G_{\mathcal{I}}$ by three colors with one extra restriction on colors. We prove that this restricted class admits Conjecture 2. Furthermore we show that treelike snarks belong to this class of snarks and thus we get that also Conjecture 2 holds for them.

1. Petersen fragment based snarks

In this chapter we define the studied class of graphs. We prove some auxiliary lemmas about properties of perfect matchings on these graphs. First of all we define the Petersen fragment. As the name suggests it is a graph, which can be obtained from the Petersen graph. The following figures show how we modify the Petersen graph to get Petersen fragment.

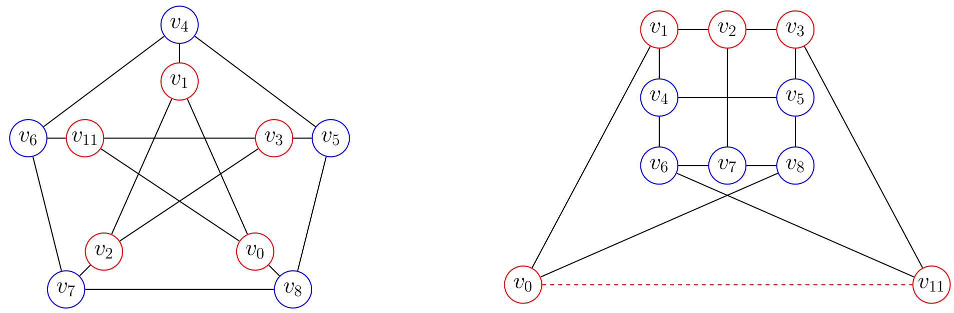

Figure 1.1: Transforming the Petersen graph into a Petersen fragment – step 1

First of all we redraw the Petersen graph in a way we use later for drawing Petersen fragment. Then we delete the red dashed edge between vertices v_0, v_{11}.

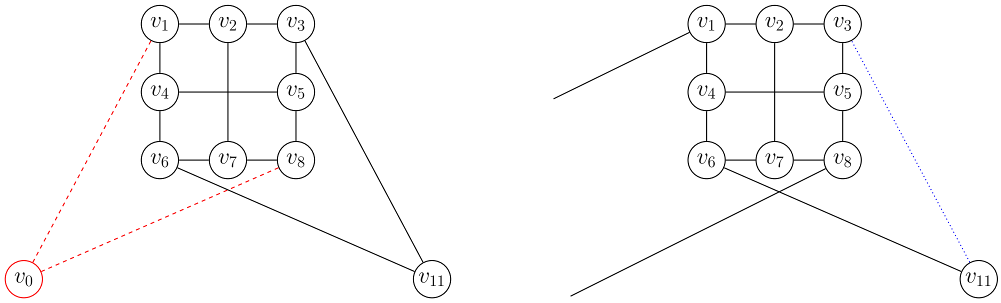

Figure 1.2: Transforming the Petersen graph into a Petersen fragment – step 2

We delete the red vertex v_0 but we preserve edges incident with this vertex (red dashed edges). These edges are used to connect the Petersen fragment with the rest of the graph. The next step is to subdivide the blue dotted edge by vertices v_9, v_{10}.

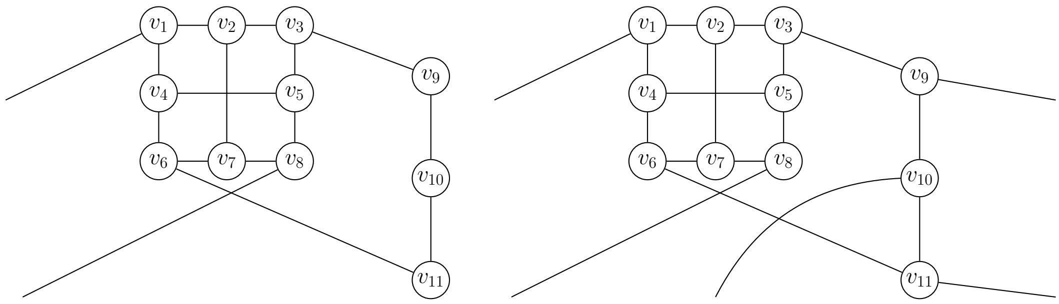

Figure 1.3: Transforming the Petersen graph into a Petersen fragment – step 3

We add new edges to vertices of degree two. Thus we get another three edges used to connect Petersen fragment with the rest of the graph. Last figure represents the subgraph we call *Petersen fragment*. As there is no possibility of confusion we often use only a word *fragment* instead of Petersen fragment.

Note that letters f, g usually denote Petersen fragments in this thesis. Edges are usually denoted by e. We often need to refer to the edges connecting a Petersen fragment with the rest of the graph. Thus we introduce the following notation.

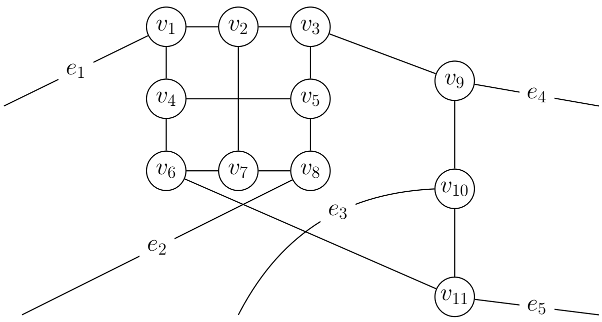

Figure 1.4: A *Petersen fragment*

Edges e_1, e_4 are edges of the *outer cycle*, similarly e_2, e_5 are edges of the *inner cycle*. The edge e_3 is called *inside edge*. We call the edges e_1, e_2 *left edges* and e_4, e_5 *right edges*. The symbol e_i^f denotes the e_i edge of fragment f. Edges $e_1, \ldots, e_5$ are called *outside edges* and the set containing all outside edges of a fragment f is denoted by $\mathcal{O}_f$.

Now we can formally define treelike snarks and our generalization of this class of graphs.

Definition 1 (Petersen fragment based snarks). *Let $G_{\mathcal{I}}$ be any graph, which has k vertices of degree one and all other vertices are of degree three. Let ℓ be an enumeration of vertices of degree one and ℓ_i be the $(i+1)$-th vertex in this enumeration. G is the graph consisting of Petersen fragments $f_0, \ldots, f_{k-1}$ such that*

1. *We connect all Petersen fragments to the cycle by unifying edges $e_4^{f_i} = e_1^{f_{(i+1 \bmod k)}}$ and $e_5^{f_i} = e_2^{f_{(i+1 \bmod k)}}$ (see Figure 1.5).*
2. *For all $i \in \{0, 1, \ldots, k-1\}$ we merge vertex ℓ_i with vertex $v_{10}^{f_i}$ in such a way that the inside edge e_3^f is the edge connecting ℓ_i to the rest of $G_{\mathcal{I}}$.*

We say that $G_{\mathcal{I}}$ is an inside graph *of G and $\mathcal{G}$ denotes the class of all graphs described above that are also bridgeless (for examples of such graphs see Figure 1.6).*

Definition 2 (Neighboring fragments). *We say that fragments f, g are* neighboring *(f on the left, g on the right) if and only if $e_1^f = e_4^g$ and $e_2^f = e_5^g$ (see Figure 1.5).*

Definition 3 (Left-nearest fragment). *Let f be any fragment, we can order all fragments of $\mathcal{F} \setminus \{f\}$ according to their position on the cycle such that left-neighbor of f is the first in this ordering and for each fragment g its successor in this ordering is the fragment h such that h is the left-neighbor of g.*

We say that g is the left-nearest fragment *of f with a property $\mathcal{P}$ if g is the first fragment with the property $\mathcal{P}$ in the ordering described above.*

We define a right-nearest fragment *with a property $\mathcal{P}$ in a similar manner. The only difference is that g is the last fragment with the property $\mathcal{P}$ in the ordering.*

Note that the left-neighboring fragment of f is exactly the counterclockwise nearest fragment of f and its right-neighboring fragment is the clockwise nearest fragment of f.

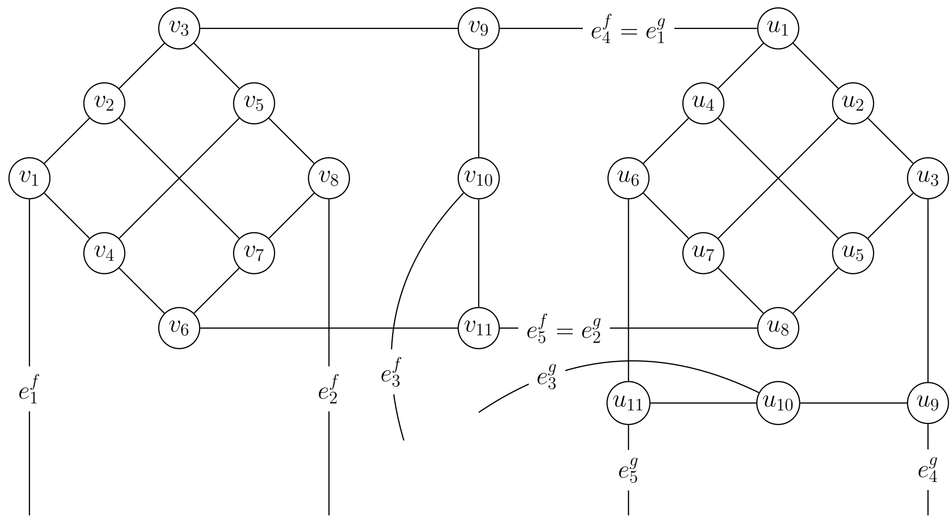

Figure 1.5: Connecting neighboring fragments

We connect edges e_1, e_2 of a fragment g with edges e_4, e_5 of a fragment f. We say that f and g are neighboring fragments f on the left and g on the right.

Definition 4 (Halin graph)**.** *Let T be a tree on at least four vertices which does not contain vertices of degree two. A* Halin graph *is a planar graph which is constructed by connecting leaves of T to a cycle.*

Definition 5 (Treelike snarks, Abreu et al. [2016])**.** *Let H be a Halin graph consisting of a tree T with inner vertices of degree three and a cycle C. A* treelike snark *is a graph G constructed as follows:*

1. *For each leaf ℓ we add a copy of the Petersen fragment and connect it to the tree by its inside edge e_3.*

2. *We fix a direction of a cycle C and for each Petersen fragment f and its successor g we unify edges $e_1^f = e_4^g$ and $e_2^f = e_5^g$.*

For examples of treelike snarks see Figure 1.6.

Observation 1. *Let G be a treelike snark, then G belongs to $\mathcal{G}$.*

We define a class of graphs called snarks. Note that the definition of this class differs through the literature. All definitions require that a snark is a graph which is not three edge-colorable but sometimes more requirements on the connectivity of these graphs or on the length of a minimal cycle appear in the literature.

Definition 6 (Snark)**.** *We say that a graph G is a* snark *if G is a cubic bridgeless graph such that its edges cannot be colored by three colors.*

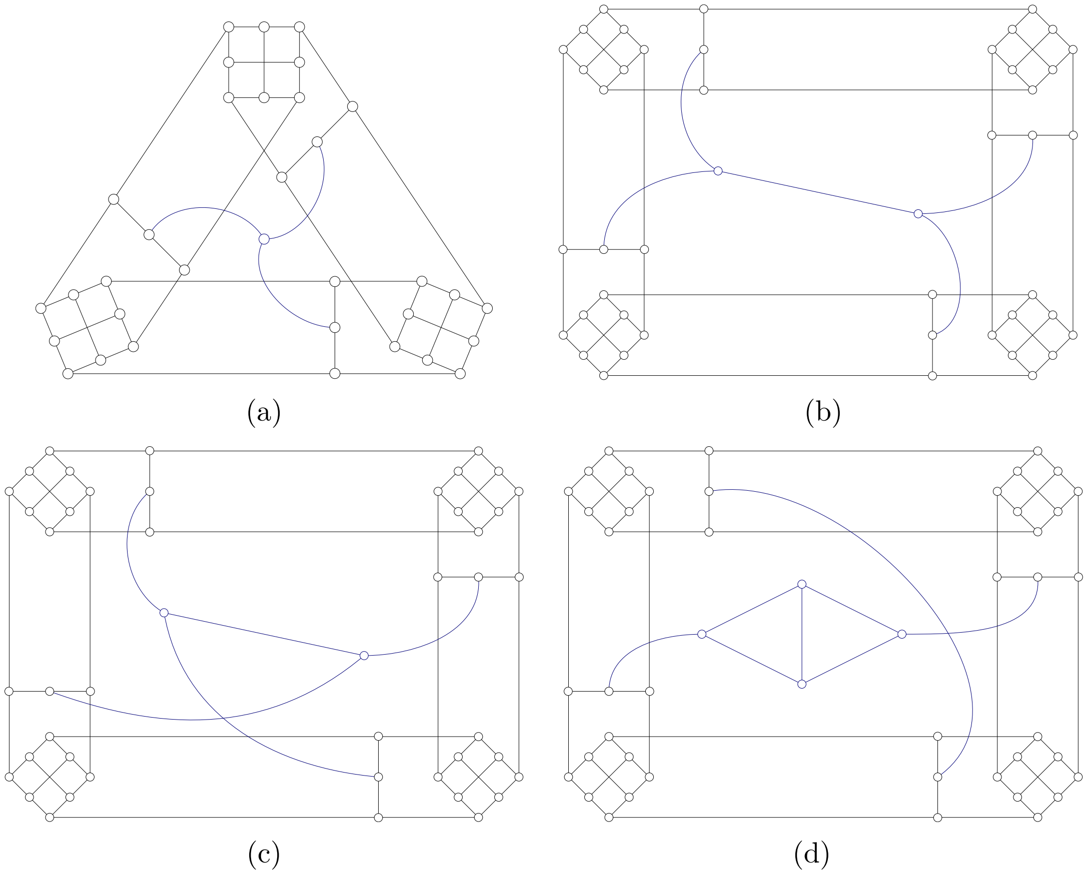

Figure 1.6: Examples of graphs from class $\mathcal{G}$
Inside graphs are colored blue. The graph (a) is the Hägglund [2016] example of a graph which cannot be cover by four perfect matchings. Graphs (a),(b) are treelike snarks. Our class $\mathcal{G}$ contains also graphs (c),(d) as we allow the inside graph to be any graph. Note that the inside graph does not have to be connected as it is shown on example (d).

In the following theorems we use the folklore parity lemma.

Lemma 1 (Parity lemma)**.** *Let G be a cubic graph, F be an inclusion minimal edge-cut in G and M be any perfect matching, then*

$$|M \cap F| \bmod 2 = |F| \bmod 2.$$

Proof. Let A be the set of vertices of a component of $G \setminus F$. Summing degrees of vertices in A we get:

$$3 \cdot |A| = \sum_{a \in A} \deg(a) = 2 \cdot |\{e \mid e = (u, v),\ u, v \in A\}| + |F|.$$

Let G' be a graph induced by M. Then $V(G') = V(G)$ and each vertex of G' has degree one. And summing degrees of vertices from the set A in the graph G' we get:

$$|A| = \sum_{a \in A} \deg_{G'}(a) = 2 \cdot |\{e \mid e = (u, v),\ u, v \in A\}| + |M \cap F|.$$

Finally combining those equalities and counting modulo two give us:

$$|F| \bmod 2 = |A| \bmod 2 = |M \cap F| \bmod 2.$$

□

We show that all graphs in $\mathcal{G}$ are snarks. As we define class $\mathcal{G}$ in such a way that it contains only cubic bridgeless graphs it suffices to show that edges of any graph from $\mathcal{G}$ cannot be colored by three colors.

Theorem 2. *Let G be a graph from $\mathcal{G}$ then G is not three edge-colorable.*

The proof is inspired by the proof of Proposition 3 from Abreu et al. [2016], but we slightly modify it to show that all graphs in class $\mathcal{G}$ has edge chromatic number bigger than three.

Proof. We prove the theorem by contradiction. Let us suppose that we have a three edge-coloring $c\colon E(G) \to [3]$. We show that for each fragment f both left outside edges are of the same color.

We assume that $c(e_1^f) = i$ and $c(e_2^f) = j$, where $i \neq j$. Note that each color class forms a perfect matching on G as c is the three edge-coloring and G is cubic. Thus from Lemma 1 we know that one of the edges (v_3, v_9) and (v_8, v_{11}) has to be colored by color i and one of them has to be colored by color j. Which is a contradiction as this coloring can be modified to the coloring of the Petersen graph (see construction of the Petersen fragment on Figures 1.1, 1.2 and 1.3).

We show that left outside edges of a fragment f are of the same color i. From Lemma 1 we know that also edges (v_3^f, v_9^f) and (v_8^f, v_{11}^f) are of the same color j (which may or may not be the same as i). Notice that also right outside edges of the fragment f are colored by the same color k as these edges are left outside edges of the right-neighboring fragment of f.

But this is a contradiction as color j has to be different from color k and we have to color both edges (v_9, v_{10}) and (v_{10}, v_{11}) by the same color $\{1, 2, 3\} \setminus \{j, k\}$ (see Figure 1.7). □

Notation. The set of all fragments of $G \in \mathcal{G}$ is denoted by $\mathcal{F}$ and the set of all outside edges is denoted by $\mathcal{O}_\mathcal{F} = \bigcup_{f \in \mathcal{F}} \mathcal{O}_f$.

Our construction of a graph covering by perfect matchings uses a three edge-coloring of the inside graph. Based on this coloring we define a color of a fragment to be inherited from the appropriate inside edge.

Definition 7 (Fragment color)**.** *For any graph $G \in \mathcal{G}$ and any three coloring of edges of its inside graph we define a coloring of fragments $c\colon \mathcal{F} \to [3]$ such that for each fragment f: $c(f) = i$ if and only if f is connected to the inside graph $G_\mathcal{I}$ by an edge of color i.*

We define $\mathcal{F}_i$ to be the set of fragments colored by i. Similarly $\mathcal{F}_{\neq i}$ is a set of fragments which are not colored by i.

Definition 8 (Matching based on color)**.** *Let $G \in \mathcal{G}$ be such that there exists a three edge-coloring c of its inside graph. We say that a matching M is based on a color $i \in \{1, 2, 3\}$ iff $M \cap G_\mathcal{I} = \{e \mid c(e) = i\}$.*

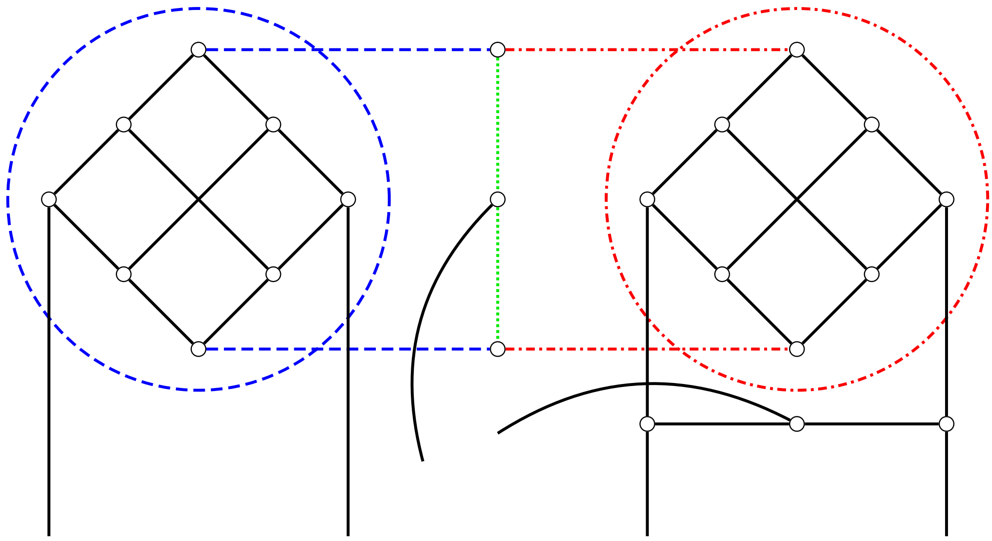

Figure 1.7: Proof of not three edge-colorability

We get that blue dashed edges are of the same color due to the edge-cut determined by blue dashed circle. Similarly red dash-dotted edges are of the same color due to the even edge-cut determined by red dash-dotted circle. We cannot color both green dotted edges by the same color.

As our construction is based on expanding a matching on a subgraph to a bigger subgraph we give conditions for a matching under which we can expand it to a perfect matching on the whole graph. We also show that each perfect matching satisfies these conditions.

Theorem 3. *Let $G \in \mathcal{G}$ be a graph and M be a matching such that $M \subseteq \mathcal{O}_{\mathcal{F}}$. Then M can be extended to a perfect matching M' in such a way that $M' \cap \mathcal{O}_{\mathcal{F}} = M$ if and only if the following conditions hold:*

1. *M can be extended to the inside graph $G_{\mathcal{I}}$ in such a way that for each inner vertex v of $G_{\mathcal{I}}$ there exists an incident edge, which is contained in the extension of M.*

2. *For each fragment f, M contains an odd number of outside edges $\mathcal{O}_f$.*

3. *For each fragment f, if M contains both its right edges e_4^f, e_5^f then it also contains the inside edge e_3^f.*

Proof. First of all we prove that for any perfect matching M' and $M = M' \cap \mathcal{O}_{\mathcal{F}}$ the above conditions hold. The first condition holds because we can extend M into the inside graph $G_{\mathcal{I}}$ by edges $M' \cap G_{\mathcal{I}}$. G is a cubic graph thus by Lemma 1 each perfect matching has an odd intersection with each inclusion-minimal odd edge-cut in G. The outside edges of any fragment f form an edge-cut of size five. Thus the matching M has to contain odd number of edges $\mathcal{O}_f$. If the last condition is broken then the vertex v_{10} is not incident with any edge contained in the matching M', which is a contradiction.

It remains to show that we can define a perfect matching M' if all the conditions hold for M. From condition 1 we know that M can be extended into the inside graph $G_{\mathcal{I}}$. Now we show that we can extend M into each fragment in such a way that M' is a perfect matching. From the last condition we know that we can extend M into vertices v_9, v_{10}, v_{11} and edges incident to them. Thus we extend M to v_9, v_{10}, v_{11} arbitrarily and count how many of edges $e_1^f, e_2^f, (v_3, v_9), (v_6, v_{11})$ are contained in this extension of M (denote it by M^{ext}). Notice that

$$|\{((v_3, v_9), (v_6, v_{11})\} \cap M^{ext}| \mod 2 \neq |\{e_3^f, e_4^f, e_5^f\} \cap M| \mod 2.$$

Thus from the second condition we know that there is even number of edges $e_1^f, e_2^f, (v_3, v_9), (v_6, v_{11})$ contained in M^{ext}. Figure 1.8 shows that if we choose any even subset $S \subseteq \{v_1, v_3, v_6, v_8\}$ then there exists a perfect matching on the graph induced by the vertices $\{v_1, \ldots, v_8\} \setminus S$. $\square$

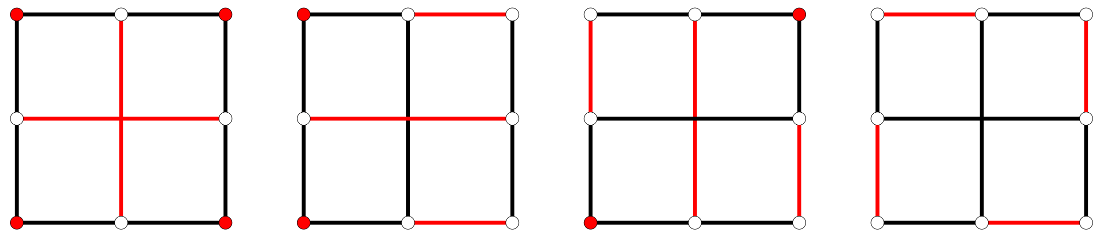

Figure 1.8: Extensible matchings – proof

Red vertices are vertices of the set S. Red edges are edges we add into M to obtain a matching containing all inner vertices of the fragment.

Previous theorem says that the fragments which inside edge is in M have the same parity of right edges in M and left edges in M. On the other hand if an inside edge of a fragment is not in M, then the parity of right edges in M differs from the parity of right edges in M.

We define

$$P = (M \cap \mathcal{O}_{\mathcal{F}}) \setminus \{e_1^f, e_2^f \mid \forall f\ (e_1^f \in M \ \&\ e_2^f \in M)\}.$$

Notice that P pairs fragments which tree edge is not in M. Suppose that P pairs fragments f and g together in such a way that f has odd number of edges from the right and g has odd number of edges from the left. Let I be the set of fragments between f, g in the part of cycle which is on the right from f and on the left from g (see Figure 1.9, where $f = f_6, g = f_2$ and $I = \{f_1\}$). Then I contains only fragments with the corresponding inside edge in M. Thus we distinguish following 4 types of fragments.

Definition 9 (Types of fragments)**.** *For a graph $G \in \mathcal{G}$ and a perfect matching M we define possible types of a fragment as follows. We say that a fragment f is*

1. *a* left fragment *iff $e_3^f \notin M$ and $|M \cap \{e_4^f, e_5^f\}| = 1$ (fragments f_4, f_6 in Figure 1.9),*

2. *a* right fragment *iff* $e_3^f \notin M$ *and* $|M \cap \{e_1^f, e_2^f\}| = 1$ *(fragments* f_2, f_5 *in Figure 1.9),*

3. *an* inner fragment *iff* $e_3^f \in M$ *and its left-neighbor is an inner fragment or a left fragment (the fragment* f_1 *in Figure 1.9),*

4. *an* outer fragment *iff* $e_3^f \in M$ *and its left-neighbor is an outer fragment or a right fragment (the fragment* f_3 *in Figure 1.9),*

Do not confuse the previous definition with the definition of a left-neighboring fragment (see Definition 2).

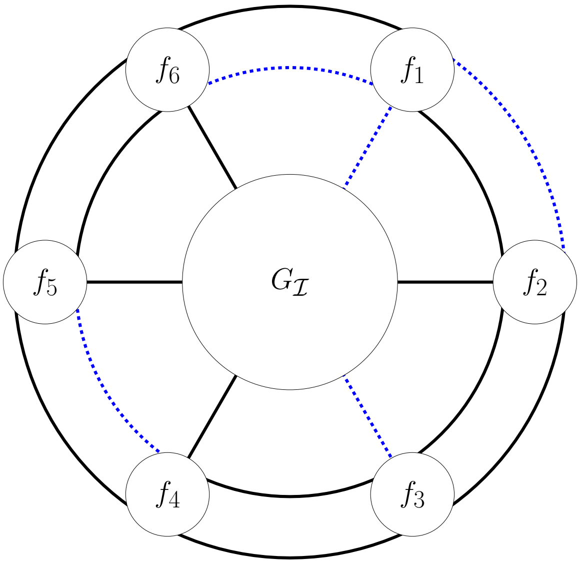

Figure 1.9: Types of fragments

This example illustrates how we pair fragments with the inside edge which is not contained in the matching M. Blue dashed edges are exactly edges which are contained in the matching. In this case we set P to be the set of blue dashed cycle edges. We can see two pairs of fragments, namely (f_4, f_5) and (f_6, f_2).

Lemma 4. *Let* $G \in \mathcal{G}$ *be a graph and* M *be a perfect matching on* G *such that there exists a fragment* f *such that its inside edge* e_3^f *is not contained in* M*. If* f *is an inner fragment then an odd number of left outside edges of* f *is contained in* M *and an odd number of right outside edges of* f *is contained in* M*.*

Similarly if f *is an outer fragment then an even number of its left outside edges is in* M *and an even number of its right outside edges in contained in* M*.*

Proof. We prove the lemma by induction on the distance of f from the left-nearest left fragment, respectively left-nearest right fragment. Note that it suffices to show that f has an odd, respectively even, number of left outside edges contained in M as Lemma 1 gives us that it has also an odd, respectively even, number of right outside edges contained in M.

If f is a right-neighbor of a left fragment g then from the definition f is an inner fragment and $|M \cap \{e_1^f, e_2^f\}| = |M \cap \{e_4^g, e_5^g\}| = 1$, where the last equality is given by the definition of a left fragment. On the other hand if f is a right-neighbor of a right fragment g then from the definition f is an outer fragment. We know that

$$|M \cap \{e_1^f, e_2^f\}| = |M \cap \{e_4^g, e_5^g\}| = |M \cap \mathcal{O}_g| - |M \cap \{e_1^g, e_2^g, e_3^g\}|.$$

From the parity lemma (Lemma 1) we know that $|M \cap \mathcal{O}_g|$ is odd and from the definition of a right fragment $|M \cap \{e_1^g, e_2^g, e_3^g\}| = |M \cap \{e_1^g, e_2^g\}|$ is also odd. Thus f has an even number of left outside edges in M.

If the distance of an inner fragment f from the left-nearest left fragment is bigger, then f is a right-neighbor of another inner fragment g. By induction we get that g has an odd number of right outside edges contained in M. Thus also f has an odd number of left outside edges contained in M, as left outside edges of f are exactly right outside edges of g. Similarly, if the distance of an outer fragment f from the left-nearest right fragment is bigger then f is a right-neighbor of another outer fragment g. By induction we get that f has an an even number of left outside edges contained in M. $\square$

We defined types of fragments dependently on a perfect matching M. In the following theorems we use it in the opposite way. We start by defining the type of each fragment, then we define a perfect matching which corresponds to given types. We show that given a matching on the inside graph there exist only two possible choices of types of all fragments.

Lemma 5. *Let $G \in \mathcal{G}$ be a graph and M be a matching on its inside graph $G_{\mathcal{I}}$ such that each inner vertex of $G_{\mathcal{I}}$ is incident to an edge contained in M. Then we can determine types of all fragments by choosing a type of one fragment f.*

Proof. We gradually fix a type of each fragment according to a clockwise direction. At the beginning we have already fixed the type of f. Let $g_1, g_2, \ldots g_m$ be a sequence of fragments in clockwise direction such that:

1. $g_1 = f$,
2. g_{i+1} is a right-neighbor of g_i and
3. g_m is a left-neighbor of f

Suppose that the last fixed fragment is a fragment g_i then we fix the type of g_{i+1} to be:

1. an inner fragment iff $e_3^{g_{i+1}} \in M$ and g_i is inner or left, or
2. an outer fragment iff $e_3^{g_{i+1}} \in M$ and g_i is outer or right, or
3. a left fragment iff $e_3^{g_{i+1}} \notin M$ and g_i is outer or right, or
4. a right fragment iff $e_3^{g_{i+1}} \notin M$ and g_i is inner or left.

Note that we defined types correctly. The only case we need to consider is that g_m forces f to be a different type then the chosen one (otherwise the correctness follows from Definition 9 and Lemma 4). Note that by the proof of Lemma 1 and by our assumptions on the matching M we know that there exists an even number of fragments such that their inside edge is not contained in M. Due to our choice of types we alternate left and right fragments. In the sense that it is never the case that the nearest fragments such that their inside edges are not in M are both left or both right. Let $g_{\min}$ be a fragment with the minimal index such that its inside edge is not contained in M and $g_{\max}$ be a fragment with the maximal index such that its inside edge is not contained in M. We distinguish two cases to show that g_m never forces f to be a different type.

1. If f is an inner or right fragment then $g_{\min}$ is a right fragment, thus $g_{\max}$ is a left fragment and g_m is a left fragment or an inner fragment.

2. If f is an outer or left fragment then $g_{\min}$ is a left fragment, thus $g_{\max}$ is a right fragment and g_m is a right fragment or an outer fragment.

Thus it suffices to show that we cannot choose a type of g_i differently for any $i \in \{2, \dots, m\}$. This follows from Lemma 4 and the definition of types of fragments. □

2. Covering graphs by perfect matchings

In this chapter we show that $G \in \mathcal{G}$ can be covered by five perfect matchings if the inside graph is three edge-colorable. Let us start by showing that this assumption holds for treelike snark.

Lemma 6. *Each tree T with inner vertices of degree three is three edge-colorable.*

Proof. We choose any vertex as a root r and color edges greedily depending on the distance from r (starting with coloring edges incident to r). Suppose that we are coloring the edge $e = (u, v)$, where u is closer to r than v. Inner vertices of tree T are of degree three and at most two edges incident with u are already colored. Also any edge incident to v is not colored yet because its distance from r is higher than the distance of the edge e from r. Thus we can color the edge e by some color i. $\square$

Theorem 7. *Let G be a graph from $\mathcal{G}$ such that its inside graph $G_{\mathcal{I}}$ has the edge-chromatic number at most three. Then G can be covered by five perfect matchings.*

First of all we define the intersection of perfect matchings and edges of the inside graph. Then we extend these matchings into edges $\mathcal{O}_{\mathcal{F}}$. Finally we define the intersection of matchings with each fragment and show that we have five perfect matchings covering the graph.

Proof. Let $c \colon E(G_{\mathcal{I}}) \to [3]$ be a three coloring of edges of $G_{\mathcal{I}}$, which exists due to assumptions. We define intersections of $M_1, M_2, \ldots, M_5$ with edges of $G_{\mathcal{I}}$ as follows:

1. An edge $e \in G_{\mathcal{I}}$ is contained in matchings M_1 and M_2 iff $c(e) = 1$,
2. Similarly e is in matchings M_3 and M_4 iff $c(e) = 2$ and
3. $e \in M_5$ iff $c(e) = 3$.

As we have already mentioned we choose a type of each fragment and then we define a matching such that its fragment types correspond to the chosen ones. For the matching M_1 we choose one of two possible choices of types of Petersen fragments (see Lemma 5). Then M_1 does not contain edges of the inner cycle. It means that

$$(M_1 \cap \mathcal{O}_{\mathcal{F}}) \setminus E(T) = \left\{ e_1^f \mid f \text{ is inner or right} \right\} \cup \left\{ e_4^f \mid f \text{ is inner or left} \right\}$$

We define the matching M_2 similarly but we change the choice of types of Petersen fragments to the another one. It means that each right fragment in the matching M_1 is the left fragment in the matching M_2 and vice versa. Similarly, each fragment which is inner in M_1 is outer in M_2 and vice versa. M_2 again does not contain edges of the inner cycle.

Our definition of matchings M_3, M_4 is analogous. The only difference is that they contain edges of the inner cycle instead of edges of the outer cycle. For the last matching M_5 we can choose the types of fragments arbitrarily and define the matching in such a way that it contains only the edges of the outer cycle.

We show that except of edges inside fragments we covered the graph by five perfect matchings. If e is an edge of the inside graph then it is contained in the perfect matching $M_{2 \cdot c(e)-1}$. Observe that all edges of the outer cycle are contained in exactly one of matchings M_1, M_2. Because any $f \in \mathcal{F}_1$ is an inner fragment in the matching M_1 iff it is an outer fragment in M_2. Similarly, each fragment $f \in \mathcal{F}_{\neq 1}$ is left in M_1 if it is right in M_2. The same holds for the inner cycle and matchings M_3, M_4.

Matchings $M_1, M_2, \ldots, M_5$ satisfy assumptions of Theorem 3, thus we can extend them into fragments in such a way that we get five perfect matchings. It remains to show that we can extend them into each fragment in such a way that they are not only perfect matchings but they also cover all edges of the fragment.

We consider three different cases dependency on the fragment color and type.

1. The fragment is not colored by color three.
2. The fragment is colored by color three and it is an outer fragment in matching M_5.
3. The fragment is colored by color three and it is an inner fragment in matching M_5.

These are the only cases we need to distinguish as fragments of color different from three are covered already by matchings $M_1, \ldots, M_4$. From our definition of matchings $M_1, \ldots, M_4$ we also do not have to distinguish whether the fragment is left or right (inner or outer) in matchings M_1, M_2 as we can change this property by renumbering these matchings. The same holds for matchings M_3, M_4. Due to symmetry of Petersen fragment we also do not have to consider whether the fragment is colored by color one or two. As we can swap edges of the inner cycle with edges of the outer cycle due to a symmetry of the Petersen fragment. Thus by renumbering of matchings and possible swapping of the edges of inner and outer cycles we get one of the three cases showed in Figures 2.1, 2.2 and 2.3. These figures also show how to extend the matchings into these fragments and thus complete the proof.

□

Corollary. Treelike snarks satisfy Conjecture 1.

Proof. Follows directly from Lemma 6 and Theorem 7. □

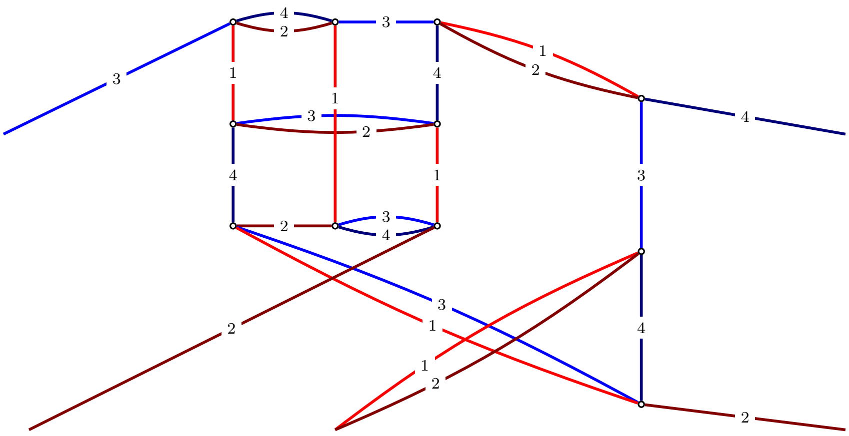

Figure 2.1: A fragment of color one or two

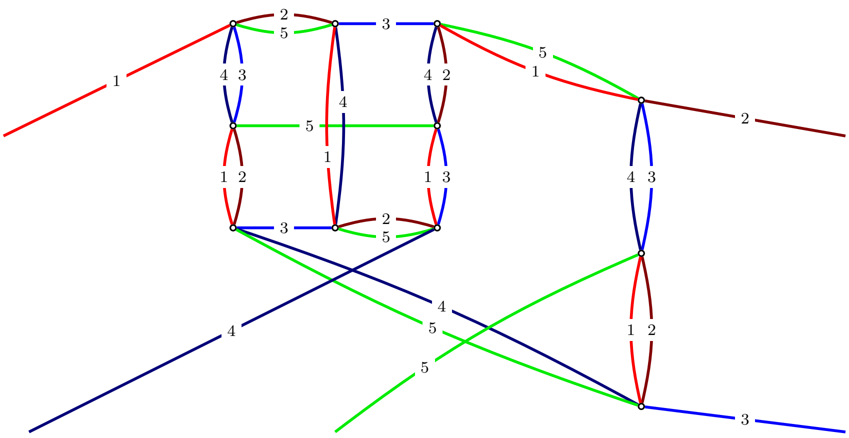

Figure 2.2: An outer fragment of color three

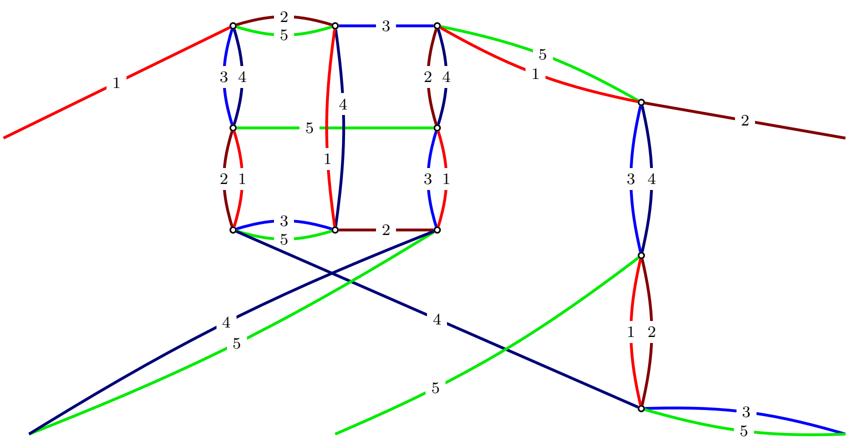

Figure 2.3: An inner fragment of color three

Note that the proof of the Theorem 7 works also for some graphs $G \in \mathcal{G}$ such that their inside graph cannot be colored by three colors. It suffices to find matchings $M_1, \ldots, M_5$ such that:

3. For each $i \in \{1, 2\}$, edges which are incident with leaves are contained in M_{2i} iff they are contained in M_{2i-1}.

1. Their union covers the inside graph $G_{\mathcal{I}}$.
2. For each $i \in [5]$ and each inner vertex v of $G_{\mathcal{I}}$, M_i contains an edge incident to v.
3. For each $i \in \{1, 2\}$, edges which are incident with leaves are contained in M_{2i} iff they are contained in M_{2i-1}.

We can satisfy these conditions also in the case when the inside graph is a copy of the cycle of Petersen fragment. Although due to Theorem 2 the edge chromatic number of this graph is bigger then three.

Note that it is not sufficient to suppose that the inside graph has a covering by five perfect matchings. For example if the graph contain a fragment which is a right fragment in four matchings and an inner fragment in one matching we cannot cover one of its right edges by any matching (see Figure 2.4). The same holds in the case when a fragment is four times left and once inner. Then one edge from $(v_3, v_9), (v_8, v_{11})$ cannot be covered by these matchings (see again Figure 2.4). Let us call these fragments bad.

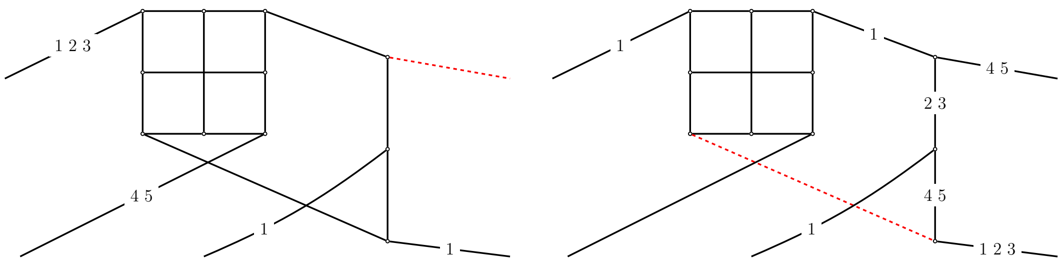

Figure 2.4: Uncoverable types of fragments – example

On the left picture we can see an example of a fragment which is for times a right fragment (matchings 2,3,4,5) and once an inner fragment (the matching 1). On the right picture there is an example of a fragment which is for times a left fragment (matchings 2,3,4,5) and once an inner fragment (the matching 1). We cannot cover red dashed edges.

Unfortunately, there could be a covering of an inside graph such that even if we choose types of fragments arbitrarily for each matching, there exists a bad fragment (see Figure 2.5).

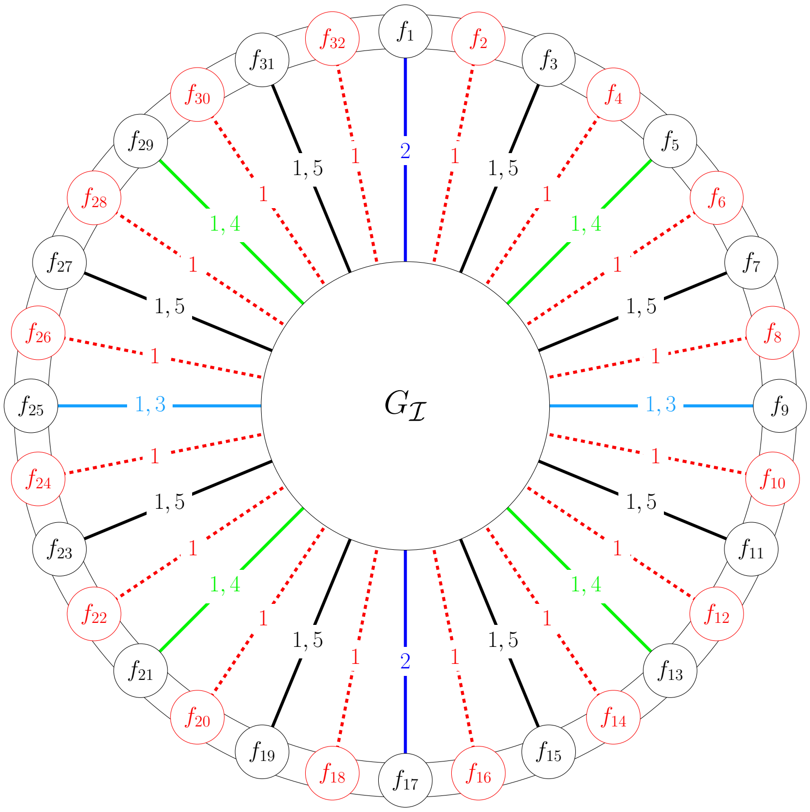

Figure 2.5: A bad covering of the inside graph

This is an example of a possible covering of the inside graph which cannot be extended into the rest of the graph. No matter how we choose types of fragments one of the red fragments is bad.

Remind that the best known upper bound on the number of perfect matchings which cover each cubic graph is $\mathcal{O}(\log n)$. In case of graphs from class $\mathcal{G}$ we can bound the number of perfect matchings by the number of matchings needed to cover the inside graph.

Theorem 8. *Let $G \in \mathcal{G}$ be such that $G_{\mathcal{I}}$ can be covered by $k \geq 2$ matchings $M_1, \dots M_k$. Then we can cover G by $k+2$ perfect matchings.*

Proof. We define matchings $N_1, \dots, N_{k+2}$ such that:

$$\forall i \in \{1, \dots, k\} \;\; N_i \cap G_{\mathcal{I}} = M_i \text{ and}$$

$$\forall i \in \{k+1, k+2\} \;\; N_i \cap G_{\mathcal{I}} = M_{i \bmod k}.$$

We choose types of fragments in matchings $N_1, N_2, N_{k+1}, N_{k+2}$ in the same manner as in the proof of Theorem 7. It means that N_1, N_{k+1} contain edges of the outer cycle and N_2, N_{k+2} contain edges of the inner cycle and we choose types of fragments differently in matchings N_i, N_{i+k}. We can choose types of fragments in matchings $N_3, \dots, N_k$ arbitrarily.

From the proof of Theorem 7 we know that we can cover the edges of fragments such that their inside edge is not contained in both matchings M_1, M_2 (see Figures 2.1, 2.2 and 2.3). Thus the only case we have to consider is a fragment f such that their inside edge is contained in both matchings M_1, M_2. We have to distinguish following two cases:

1. There exists a fragment $g \neq f$ such that $e_3^f = e_3^g$. Then we can simply remove this edge from matching M_2 and we can cover f, g as we have already shown.

2. The edge is incident with an inner vertex of $G_{\mathcal{I}}$. Because $\bigcup_{i \in 1,\dots,k} M_i = G_{\mathcal{I}}$ and inner vertices of $G_{\mathcal{I}}$ are of degree three there exist two matchings M_i, M_j which do not contain the edge e_3^f. Thus the fragment is two times left or right. We also know that the fragment is twice inner and twice outer. Figures 2.6, 2.7 and 2.8 show how we can extend matchings $N_1, \dots, N_{k+2}$ to cover the whole fragment.

Note that in Figures 2.6, 2.7 we can suppose without loss of generality that one of the matchings N_i, N_j does not contain edges of the outer cycle and one of the matchings N_i, N_j does not contain edges of the inner cycle. In contrary to the previous approach we do not choose which edges are contained in matchings $N_3, \dots, N_k$ the same way for the whole graph, but we choose it for each pair of neighboring fragments separately.

Let $\mathcal{L} \subseteq \mathcal{F}$ be the set of fragments such that each $f \in \mathcal{L}$ is an inner or outer fragment in matchings $N_1, N_2, N_{k+1}, N_{k+2}$ and it is twice left in matchings $N_3, \dots N_k$. Similarly let $\mathcal{R} \subseteq \mathcal{F}$ denote the set of fragments such that each $f \in \mathcal{R}$ is an inner or outer fragment in matchings $N_1, N_2, N_{k+1}, N_{k+2}$ and it is twice right in matchings $N_3, \dots, N_k$. And let i_f, j_f be indices of two matchings from $N_3, \dots, N_k$ in which the fragment f is twice left (respectively right).

For neighboring fragments f, g (f on the left, g on the right) we choose intersections of their common edges with matchings $\{e_4^f, e_5^f\} \cap N_i = \{e_1^g, e_2^g\} \cap N_i$ for all $i \in \{3, \dots, k\}$ in the following way:

1. If $f \in \mathcal{L}$ and $g \notin \mathcal{R}$ then matching N_{i_f} contains the edge e_5^f of the inner cycle and N_{j_f} contains the edge e_4^f of the outer cycle.

2. If $f \notin \mathcal{L}$ and $g \in \mathcal{R}$ then matching N_{i_g} contains the edge e_2^g of the inner cycle and N_{j_g} contains the edge e_1^g of the outer cycle.

3. If $f \in \mathcal{L}$ and $g \in \mathcal{R}$ and $i_f = i_g, j_f = j_g$ then matching $N_{i_f} = N_{i_g}$ contains the edge e_2^f of the inner cycle and $N_{j_f} = N_{j_g}$ contains the edge e_1^f of the outer cycle.

4. If $f \in \mathcal{L}$ and $g \in \mathcal{R}$ and $i_f = i_g, j_f \neq j_g$ then matching $N_{i_f} = N_{i_g}$ contains the edge e_2^f of the inner cycle and matchings N_{j_f}, N_{j_g} contain the edge e_1^f of the outer cycle.

5. If $f \in \mathcal{L}$ and $g \in \mathcal{R}$ and $i_f \neq i_g, j_f \neq j_g$ then matchings N_{i_f}, N_{i_g} contain the edge e_2^f of the inner cycle and N_{j_f}, N_{j_g} contain the edge e_1^f of the outer cycle.

Other intersections of matchings $N_3, \dots, N_k$ with outside edges of fragments are chosen arbitrarily. We only preserve conditions given by types of fragments f, g in these matchings (see Lemma 4 and Definition 9).

Note that this way we could get a matching N_i and a fragment f such that f is an inner fragment in N_i and N_i contains the left inner and the right outer edge of f. This case is not described in Figure 2.3 as in this figure we assume that matchings contain either edges of the outer cycle or edges of the inner cycle. From Theorem 3 we know that we can extend N_i into a perfect matching even if we use the outer edge on one side of a fragment and the inner edge on the another one. Figure 2.9 shows that such fragments can be also covered by five perfect matchings. □

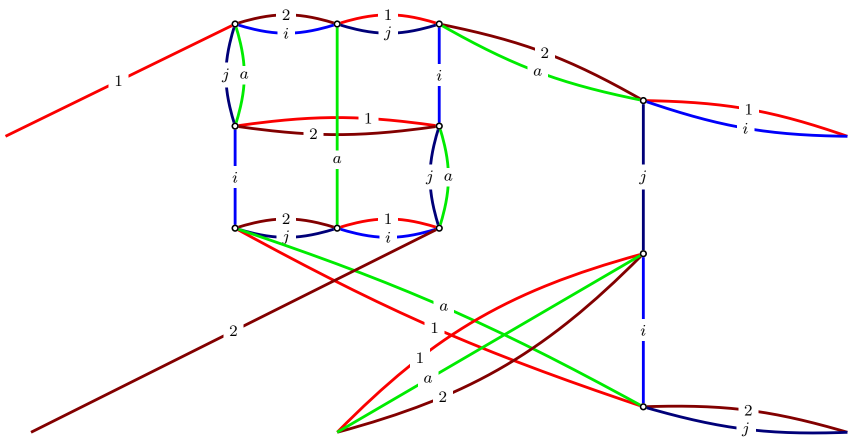

Figure 2.6: A fragment which is two times left

Edges of the matching N_{k+1} are labeled by a. Without loss of generality we suppose that the matching N_i contains the edge e_5 and N_j contains the edge e_4 as we can always choose them such that they do not contain the same right edge.

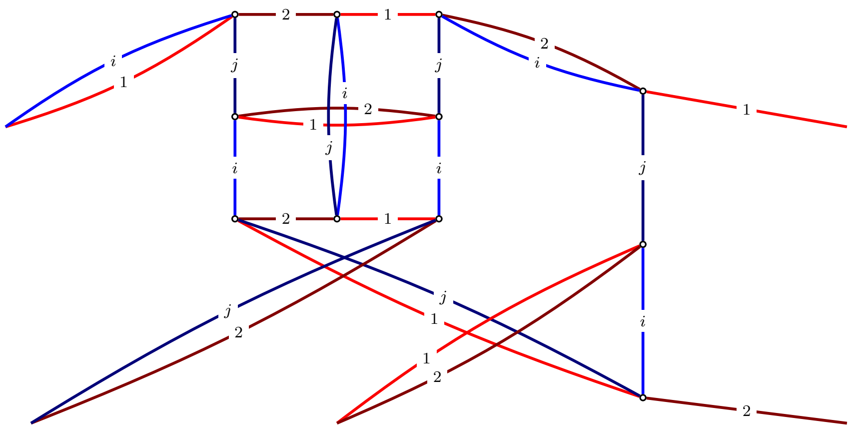

Figure 2.7: A fragment which is two times right

Without loss of generality we suppose that matching N_i contains the edge e_1 and N_j contains the edge e_2 as we can always choose them such that they do not contain the same left edge.

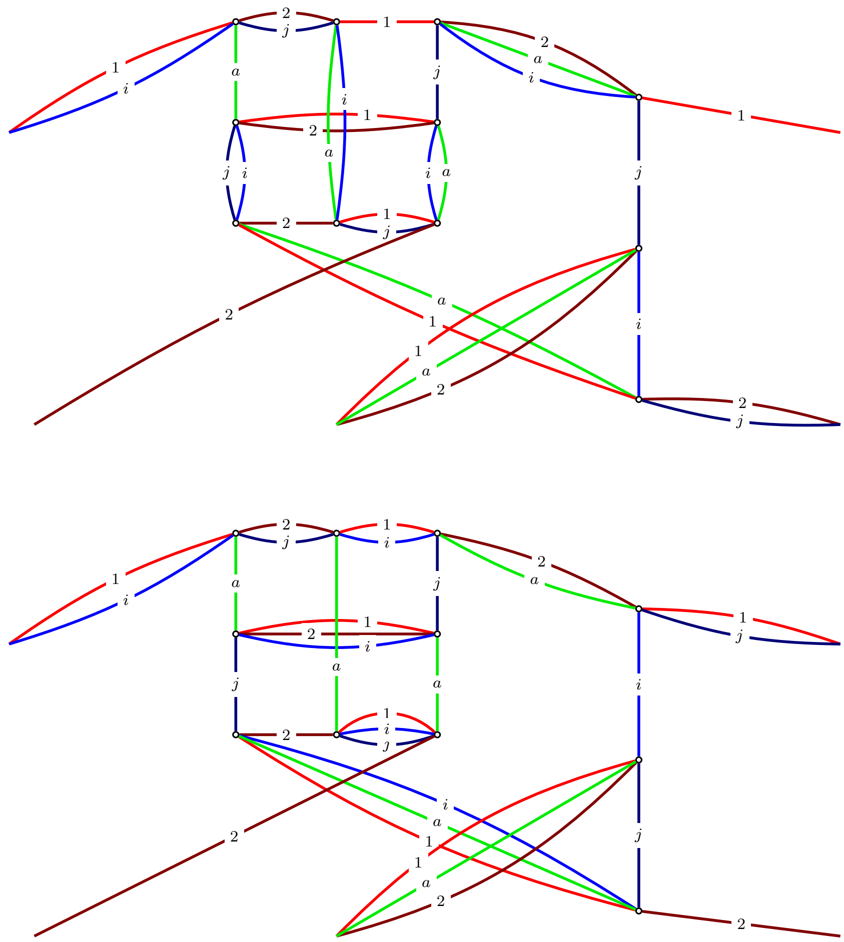

Figure 2.8: A fragment which is right in matching N_i and left in N_j

Edges of the matching N_{k+1} are labeled by a. If the intersection of matchings N_i, N_j with edges $\mathcal{O}_f$ differs from the given examples we can change the intersection into one of these two cases by swapping edges of the outer cycle with edges of the inner cycle due to a symmetry of Petersen fragment.

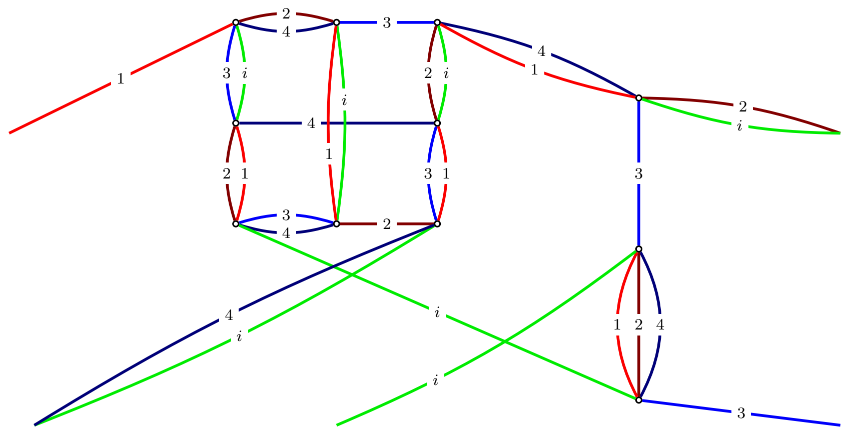

Figure 2.9: An inner fragment – swapping cycles

Matching N_i contain an edge of the inner cycle on one side of f and an edge of the outer cycle on the another side of f.

3. Berge–Fulkerson conjecture and treelike snarks

We have already proved that any treelike snark can be covered by five perfect matchings. In this chapter we improve this result. Namely we show that treelike snarks admit Berge–Fulkerson conjecture (Conjecture 2).

Suppose that we define matchings $M_1, \dots, M_6$ in the same manner as before in the proof of Theorem 7. We can compute that for each fragment f:

$$\sum_{i=1}^{6} |M_i \cap \mathcal{O}_f| = 8.$$

But this means that not all edges $\mathcal{O}_f$ are covered twice, otherwise

$$\sum_{i=1}^{6} |M_i \cap \mathcal{O}_f| = 10.$$

Thus we need to modify our matchings. We know that all edges of the tree are contained in exactly two perfect matchings M_{2i} and M_{2i+1} according to their color i. Moreover for each fragment f exactly one of its left edges and one of its right edges are covered only once.

The first attempt to fix it is to add those edges into some matching. If we do not add all these edges into one matching we change the parity of $|\mathcal{O}_f \cap M|$ for some f. On the other hand if we add all those edges into one matching M then we might violate the condition that both right edges can be in M only if M contains the inner edge of a fragment f.

We overcame these problems by defining matchings M_1 and M_2 slightly differently. More precisely the types of fragments in matchings M_1, M_2 are the same and thus edges covered only once are on the same side of a fragment. Unfortunately the graph could contain a combination of fragments we are not able to cover by our construction.

We avoid having these bad fragments by using a special edge-coloring of the inside graph $G_{\mathcal{I}}$. More precisely we find a coloring which does not use the same color for neighboring fragments except for one color class. The following lemma shows that for treelike snarks we can find this special coloring.

Lemma 9. *Let G be a treelike snark. Then there exists a three edge-coloring c of $G_{\mathcal{I}} = T$ such that for all f, g neighboring fragments $c(f) \neq 1$ implies $c(f) \neq c(g)$.*

Proof. We prove the lemma by induction. Due to Abreu et al. [2016] we know that treelike snarks can be created inductively. In each step we have two parts of the resulting graph each having five half-edges (edges we have not specified both of its end vertices yet). Two of those half edges are left edges of some fragment, let us denote them by left edges. Two of them are right edges of some fragment and we denote them by right edges. The last edge is called *tree edge*.

Given two parts of the resulting graph we unify the left edges of one part with the right edges of the another one and connect both tree edges by a new vertex and add a new tree edge. This operation is denoted by + in the following text. In the last step of the inductive construction we connect two parts of the resulting graph by unifying all of their outside edges (see Figure 3.1).

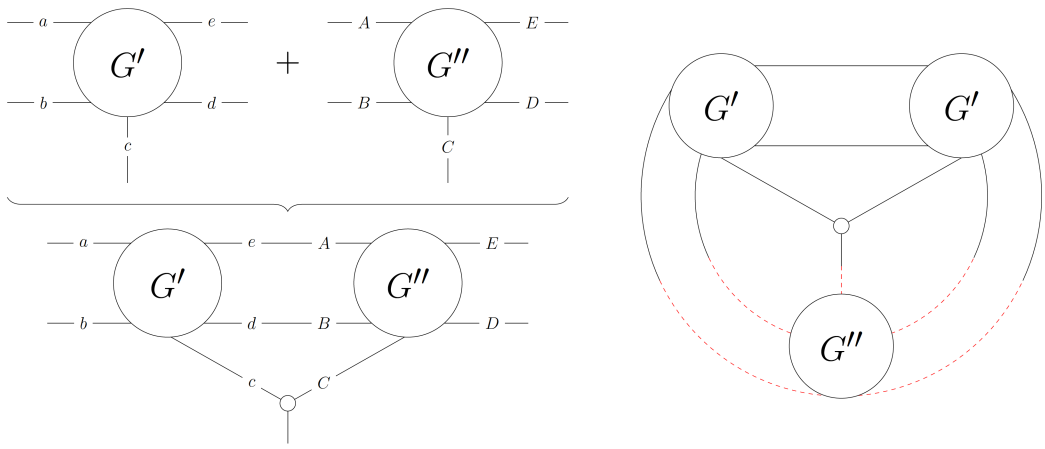

Figure 3.1: Inductive construction of treelike snarks

Figure on the left shows the operation $+$. Edges a, b are left edges of the left most fragment of G', similarly d, e are right edges of the right most fragment G'. Both these fragments are called *end fragments* of G'.

We follow the inductive construction of G and color the tree such that except of the last step of the construction we never repeat colors on neighboring fragments. If our inside graph consists only of one inner edge we can use any color and we get a proper three coloring without repeating colors on neighboring fragments.

Let G be a graph created by $G' + G''$, we fix a coloring of G' given by induction. Let $c'' \colon E(T(G'')) \to [3]$ be a coloring of tree edges of G'' by three colors which is again given by induction. We have to satisfy two restrictions given by the coloring of G'. The first restriction is on the color of the tree edge of G'' and it is given by the coloring of the tree edge of G'. Another restriction follows from the color of the right most fragment of G' which has to be different from the color of the left most fragment of G''.

Let e denote the tree edge of G'' and f denote the left most fragment of G''. There are two cases $c''(e) \neq c''(f)$ and $c''(e) = c''(f) = i$. In the first case we choose a color of edge e to be different from the color of the tree edge of G'. We have two possible choices for the color of f to be different from $c''(e)$, thus one of them differs from the color of the right most fragment of G'. In the second case we have two restrictions on the color i. We choose i to be the last not used color. Thus we can permute colors of c'' to have a proper edge-coloring of G without repeating colors on neighboring fragments.

If G is created by the last step of the inductive construction from graphs G', G'', then we have already prescribed one color class. Namely we have determined the color a of the tree edge of G'' (see Figure 3.1). Let us suppose that at most one end fragment of G'' has its corresponding end fragment of G' colored by the same color or both end fragments of G'' are colored by the same color. Then there is at most one color class containing neighboring fragments.

On the other hand if end fragments of G'' are colored by different colors. We know that at least one of them is different from a. Thus if both have a neighbor of their color in G' we can swap colors different from a. We again get at most one color class containing neighboring fragments. In both cases we can assume without loss of generality that these fragments are of color one. $\square$

Now we can prove that Conjecture 2 holds for treelike snarks.

Theorem 10. *Let $G \in \mathcal{G}$ be a graph, $c\colon E(G_{\mathcal{I}}) \to [3]$ be a three edge-coloring of its inside graph $G_{\mathcal{I}}$ and M be a matching such that $M = \{e \mid c(e) = 1\}$. Let us suppose that we can choose types of fragments in the matching M in such a way that for all neighboring fragments f, g, where f is the left-neighbor of g, one of the following holds:*

1. *$c(f) = c(g) = 1$ or*
2. *$c(f) \neq c(g)$ or*
3. *f is a left fragment.*

Then there exist six perfect matchings, which cover each edge of G exactly twice.

The proof is similar to the proof of Theorem 7. We again start with matchings on the inside graph and extend them into the rest of the graph G.

Proof. First of all we define intersections of matchings $M_1, \dots, M_6$ with the inside graph $G_{\mathcal{I}}$ as follows:

$$M_1 \cap G_{\mathcal{I}} = M_2 \cap G_{\mathcal{I}} = \{e \mid c(e) = 1\},$$

$$M_3 \cap G_{\mathcal{I}} = M_4 \cap G_{\mathcal{I}} = \{e \mid c(e) = 2\},$$

$$M_5 \cap G_{\mathcal{I}} = M_6 \cap G_{\mathcal{I}} = \{e \mid c(e) = 3\}.$$

In contrast with the proof of Theorem 7 we fix same types of fragments for both matchings M_1, M_2. Types of fragments in matchings $M_3, \dots, M_6$ are chosen similarly as before. Which means that the left fragment in the matching M_3 is the right fragment in the matching M_4. The same holds for matchings M_5, M_6.

Matchings M_1, M_3, M_4 use edges of the outer cycle to pair the fragments with inside edge which is not contained in the given matching. Other matchings use the edges of the inner cycle. More precisely if $M \in \{M_1, M_3, M_4\}$, $N \in \{M_2, M_5, M_6\}$ and f is any fragment then

1. $e_1^f \in M$ if f is inner or right in the matching M,
2. $e_4^f \in M$ if f is inner or left in the matching M,
3. $e_2^f \in N$ if f is inner or right in the matching N,
4. $e_5^f \in N$ if f is inner or left in the matching N.

We can observe that matchings $M_3, \dots, M_6$ cover all edges of the outer cycle and all edges of the inner cycle precisely once. Both right edges of a fragment f are covered twice if f is inner or left in the matching M_1. We have to add edges covered only once into some matching.

Let f be any fragment colored by color $k \in \{1, 2, 3\}$ such that its right edges are covered once then there is a matching M in which f is an outer fragment.

We add both right edges of the fragment f into M. For all fragments f and for all $i \in \{1, \ldots, 6\}$ we do not change the parity of outside edges of f contained in M_i, as we always add an even number of those edges into M_i. Matchings created by our construction contain both right edges of some fragment f only if f is an outer fragment. Thus if M_i contains both right edges it contains the inside edge, too. Due to Lemma 3 we can extend matchings $M_1, \ldots, M_6$ to perfect matchings.

The rest of the proof is a case analysis. We use the fact that f changes its type in matchings M_3, M_4 (M_5, M_6), thus the type of f in those matchings depends only on the color of f. Notice that intersections of matchings and outside edges of the fragment f depends on:

1. $c(f)$,
2. the type of f in matchings M_1, M_2 and
3. the type of its left-neighboring fragment g in M_1, M_2.

The type of g changes intersections of f with matchings only if g is right or outer. In this case we add both left edges of f into a matching in which g is an outer fragment. From Lemma 5 we know that the type of matching g is determined by the color of g and the type of f.

We reduce the number of cases even more. We observe that fragments of color two and three have similar intersections with matchings $M_1, \ldots, M_6$. The only difference is that a fragment of color three has swapped matchings M_3, M_4 for matchings M_5, M_6 relative to a fragment of color two. Figures 3.2, 3.3 and 3.4 show how to extend perfect matchings into different fragments of color one. Figures 3.5, 3.6 and 3.7 show extensions of perfect matchings into fragments of colors two and three and complete the proof. Note that from assumptions we know that it is never the case that a fragment is left in M_1 and its right-neighbor is of the same color. □

Corollary. Let G be a treelike snark then G satisfy Conjecture 2.

Proof. The corollary directly follows from Theorem 10 and Lemma 9. □

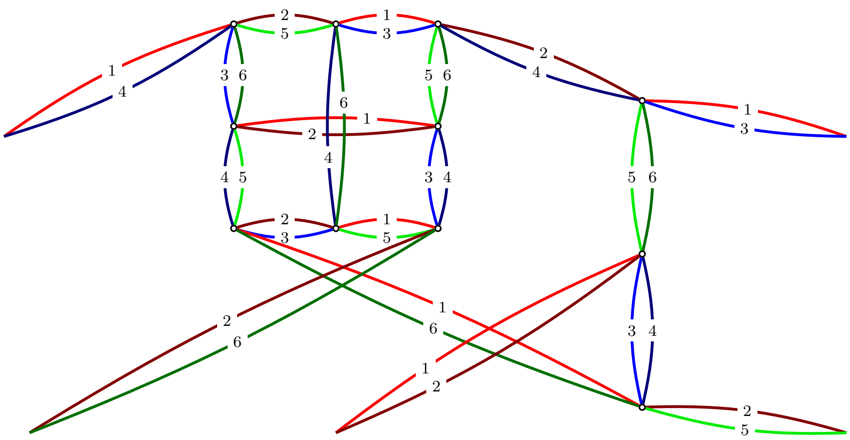

Figure 3.2: Conjecture 2 – an inner fragment

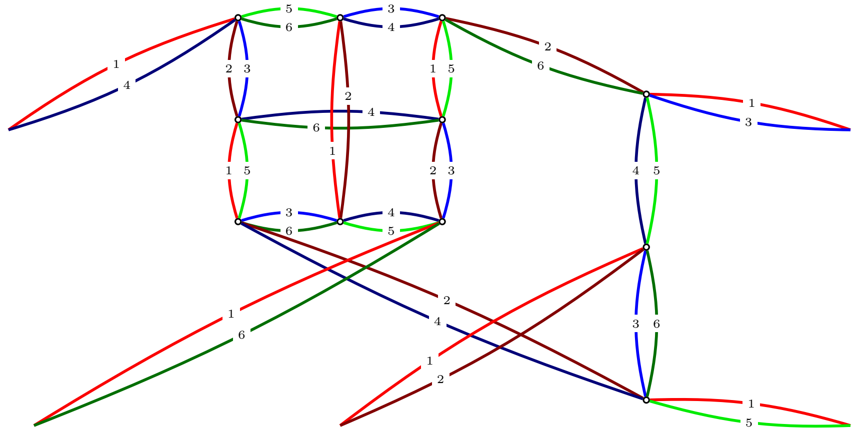

Figure 3.3: Conjecture 2 – an outer fragment with the left-neighbor of color one

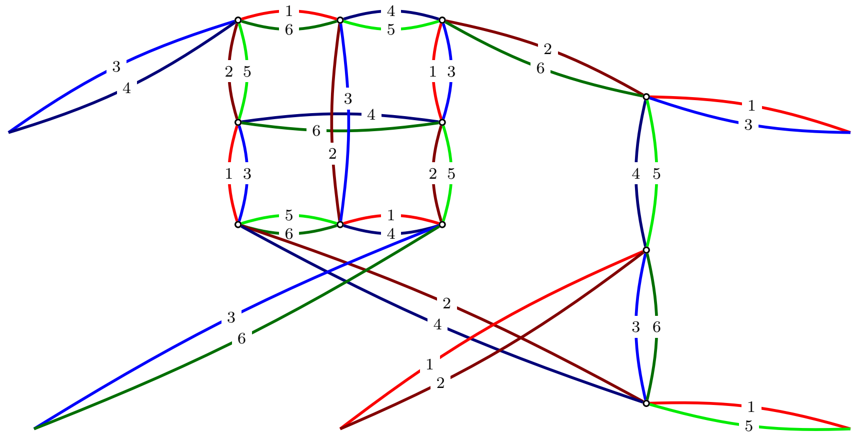

Figure 3.4: Conjecture 2 – an outer fragment with the left-neighbor of color two or three

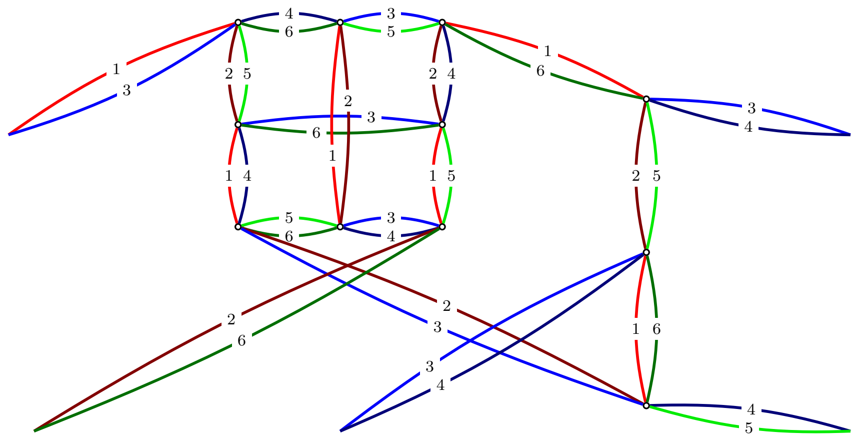

Figure 3.5: Conjecture 2 – a right fragment

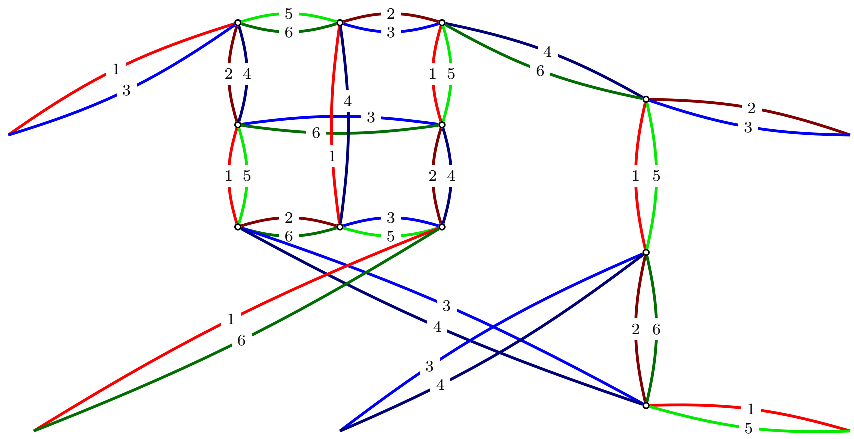

Figure 3.6: Conjecture 2 – a left fragment with the left-neighbor of color one

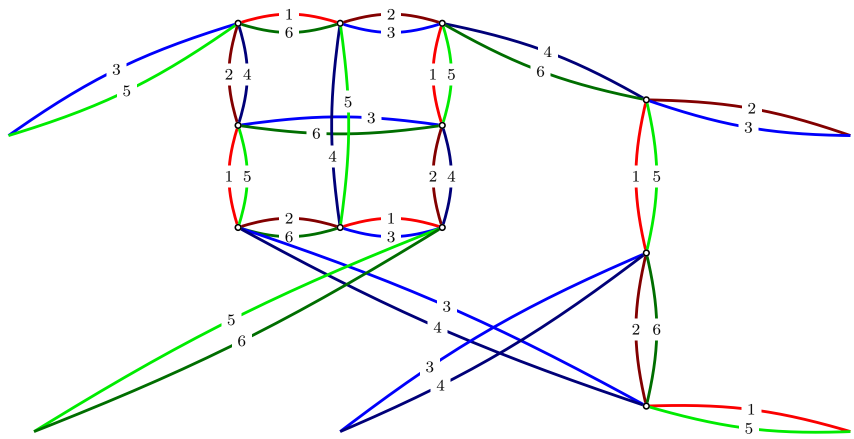

Figure 3.7: Conjecture 2 – a left fragment with the left-neighbor of color two or three
The color of the left-neighbor is different from the color of fragment, due to assumptions.

Conclusion

We have shown that both Conjectures 1 and 2 hold for treelike snarks. Conjecture 1 holds even if we substitute the inside tree by any three edge-colorable graph. A natural question is whether we can improve our result to hold for remaining graphs in $\mathcal{G}$. Another natural question is whether we can weaken the restrictions on the inside graph from Theorem 10 to proper three coloring of its edges.

As we already mention Berge–Fulkerson conjecture (Conjecture 2) is closely related to the well known circuit double cover conjecture (Conjecture 3). In particular Berge–Fulkerson conjecture easily implies that there exists covering of a bridgeless cubic graph by six even subgraphs such that each edge is in exactly four of them. Furthermore Abreu et al. [2016] showed that treelike snarks have cycle double cover consisting of five circuits. We can ask what can be proven about these circuit coverings for other graphs in $\mathcal{G}$.

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