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BACHELOR THESIS

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Typical continuous and integrable functions

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In Prague on May 27, 2016 David Hruška

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Abstract: In this thesis we use the Baire categories to define the concept of "typical functions". Then we prove several theorems generally asserting that a typical function from a space of functions having some nice property does not have a stronger property. In particular we prove that a typical continuous or $\alpha-$Hölder continuous function is nowhere differentiable, a typical continuous monotone function does not satisfy the Luzin (N) condition and a typical integrable function is nowhere continuous.

Keywords: typical function, Baire category, Baire category theorem, complete metric space

I dedicate this thesis studying the properties of function spaces to Kateřina Nová in the hope of improving her view on functions. I would like to thank my supervisor doc. RNDr. Stanislav Hencl, Ph.D. very much for his guidance, patience and pleasant cooperation.

Contents

1. Introduction

In 1872, the German mathematician Karl Weierstrass[1] surprised the mathematical world with an example of a continuous but nowhere differentiable function. Until then, it was believed that a continuous function on real line is differentiable everywhere except for an isolated set of points. Weierstrass defined his function as a sum of suitable function series[2].

Since then, more general and elegant approach has been developed, although it is not a constructive one. We are talking about the Baire categories and the Baire category theorem. Let us introduce them here. We begin with several elementary definitions and propositions. More about the topic can be found in [4, Chap. 9.6.].

Definition 1. *Let (X, ρ) be a metric space. Then*

- *we say that the sequence $x_n \in X$ is* Cauchy *if*

  for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ for every $m, n \geq n_0$,

- *we say that (X, ρ) is* complete *if every Cauchy sequence in X converges to some $x \in X$,*

- *for $x \in X$ and $\varepsilon > 0$ we denote $B_X(x, \varepsilon) := \{y \in X : \rho(x, y) < \varepsilon\}$,*

- *a normed linear space $(X, \|\cdot\|)$ is called* Banach space *if (X, ρ) is complete metric space, where $\rho(x, y) = \|x - y\|$.*

Proposition 2 (Cantor)**.** *Let X be a complete metric space and $F_n \subset X$ be closed sets such that* $\operatorname{diam}(F_n) \xrightarrow{n\to\infty} 0$*. Then there is $x \in X$ such that*

$$\bigcap_{n=1}^{\infty} F_n = x.$$

The proof can be found in [4, Thm 9.6.19.].

Definition 3. *Let X be a metric space.*

- *A set $A \subset X$ is called* dense *in X if $\overline{A} = X$ and* nowhere dense *in X if $X \setminus \overline{A}$ is dense.*

- *A set $A \subset X$ is* of first (Baire) category *(or* meagre*) in X if $A = \bigcup_{i=1}^{\infty} B_n$ where $B_n \subset X$ are nowhere dense.*

- *A set $D \subset X$ is called G_δ* set *if $D = \bigcap_{i=1}^{\infty} G_i$ where $G_i \subset X$ are open.*

Proposition 4. *Let X be a metric space, $A \subset X$ be of first category in X and let $B \subset A$. Then B is also of first category in X. Moreover, if $C = \bigcup_{i=1}^{\infty} C_i$ where C_i are of first category then C is of first category.*

Proof. If $A = \bigcup_{i=1}^{\infty} A_i$ where A_i are nowhere dense sets we observe that $B = \bigcup_{i=1}^{\infty} (B \cap A_i)$ is a countable union of nowhere dense set and thus of first category in X. The second assertion follows easily from the fact that a countable union of countable unions is again a countable union. □

[1] https://en.wikipedia.org/wiki/Karl_Weierstrass

[2] https://math.berkeley.edu/~jcalder/104F14/weierstrass-function.pdf

Theorem 5 (Baire)**.** *Let X be a complete metric space. Then*

- *X is not of first category in itself,*
- *$A \subset X$ is of first category if and only if $X \setminus A$ contains a dense G_δ set.*

Proof. Obviously the first assertion follows from the second one so let us prove just the second one.

At first suppose that A is of first category in X. It follows from the Definition 3 that $A = \bigcup_{i=1}^{\infty} B_n$ where $B_n \subset X$ are nowhere dense, that is $C_n := X \setminus \overline{B_n}$ is open and dense in X for every $n \in \mathbb{N}$. Thus we obtain

$$X \setminus A = \bigcap_{i=1}^{\infty} (X \setminus B_n) \supset \bigcap_{i=1}^{\infty} C_n.$$

We denote $C := \bigcap_{i=1}^{\infty} C_n$ and observe that C is a G_δ set. The density of C is equivalent to the claim

$$\bigcap_{i=1}^{\infty} C_n \cap G \neq \emptyset \text{ for every open } G \subset X. \tag{1.1}$$

Since C_1 is dense and open we have $B_X(x_1, \varepsilon_1) \subset C_1 \cap G$ for some $x_1 \in X$ and $\varepsilon_1 < 1$. Let us set $G_1 := B_X(x_1, \frac{\varepsilon_1}{2})$. Now we proceed by induction. Suppose we already have an open set G_n such that

$$\bigcap_{i=1}^{n} C_i \cap G_n \neq \emptyset.$$

Then since C_{n+1} is dense and open we have $B_X(x_n, \varepsilon_n) \subset C_{n+1} \cap G_n$ for some $x_{n+1} \in X$ and $\varepsilon_{n+1} < \frac{1}{n+1}$. We set $G_{n+1} := B_X(x_{n+1}, \frac{\varepsilon_{n+1}}{2})$.

Now for any $n \in \mathbb{N}$ we have

$$B_X(x_{n+1}, \varepsilon_{n+1}) \subset \overline{B_X(x_{n+1}, \varepsilon_{n+1})} \subset \overline{B_X(x_n, \varepsilon_n/2)} \subset B_X(x_n, \varepsilon_n).$$

Since $\varepsilon_n \xrightarrow{n\to\infty} 0$ we can use Proposition 2 to obtain

$$x \in \bigcap_{n=1}^{\infty} \overline{B_X(x_n, \varepsilon_n)} = \bigcap_{n=1}^{\infty} B_X(x_n, \varepsilon_n).$$

In view of

$$B_X(x_n, \varepsilon_n) \subset C_n$$

we have $x \in \bigcap_{i=1}^{\infty} C_n \cap G$ and the conclusion follows.

For the opposite implication consider $A \subset X$ such that there is a dense G_δ set D such that $D \subset X \setminus A$. Then $D = \bigcap_{i=1}^{\infty} D_i$ where D_i are open and dense since even their intersection is dense. Thus $X \setminus D_i = \overline{X \setminus D_i}$ is nowhere dense for every $i \in \mathbb{N}$ and we observe that $A \subset X \setminus D = \bigcup_{i=1}^{\infty} (X \setminus D_i)$ is of first category in X.

□

Now we are ready to introduce typical functions.

Definition 6. *Let X be a complete metric space. We say that* a typical function from X has the property $\mathcal{P}$ *if and only if the set*

$$P := \{f \in X : f \text{ has the property } \mathcal{P}\}$$

is comeagre*, that is $X \setminus P$ is of first category.*

According to Theorem 5, typical functions have two significant intuitive properties justifying their name. They are everywhere in their space X (which rigorously means that they are dense in X) and they are topologically nice, in particular "nearly open" (rigorously speaking they contain a G_δ set). For instance one can use Theorem 5 to prove that the set of all rational numbers is dense but not G_δ in $\mathbb{R}$. On contrary, they obviously form a set of first category and thus we can demonstrate our notation by saying that a typical real number is irrational.

In this thesis we prove four main theorems about properties of typical functions in (complete metric) function spaces. In the second chapter we give a nonconstructive proof of density of nowhere differentiable functions in space of continuous functions and then we generalize the result for $\alpha-$Hölder continuous functions (Theorems 7 and 11). Next we introduce the Luzin (N) condition and prove that a typical monotone continuous function does not satisfy this condition (Theorem 21) and we end by proving Theorem 27 stating that a typical integrable function is nowhere bounded and thus nowhere continuous. We express the conclusions of these theorems by claiming that some set is of first category. Considering the previous paragraph we obtain the desired properties of the complements of these sets and consequently of the typical functions in respective spaces.

Except for the Theorem 7 and Theorem 5, which are standard, all the proofs are original. However, it is possible that they can be found somewhere in literature.

2. Nowhere differentiable functions

In this chapter we introduce two "nice" function spaces and show that their typical element is nowhere differentiable.

2.1 Continuous functions

Here we prove that a typical continuous function is nowhere differentiable. It is well known that the set of all continuous functions on $[0,1]$ equipped with the supremum norm is a Banach space and it is denoted by $\mathcal{C}\big([0,1]\big)$.

Theorem 7. *Let us denote*

$$D := \big\{ f \in \mathcal{C}\big([0,1]\big) : \textit{there is } x_f \in (0,1) \textit{ such that } f'(x_f) \in \mathbb{R} \big\}.$$

Then D is of first category in $\mathcal{C}\big([0,1]\big)$.

Proof. For $n \in \mathbb{N}$ we define

$$A_n = \Big\{ f \in \mathcal{C}\big([0,1]\big) : \text{there is } x_0 \in [0,1] \quad \text{such that } \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq n \text{ for all } x \in [0,1] \Big\}. \tag{2.1}$$

Now we proceed in three steps:

Step 1: $D \subset \bigcup_{n=1}^{\infty} A_n$

Let us take $g \in D$. Since $g'(x_g)$ exists and is finite we get $\delta > 0$ such that for every $x \in [0,1]$ satisfying $0 < |x - x_g| < \delta$ we have

$$\frac{|g(x) - g(x_g)|}{|x - x_g|} < |g'(x_g)| + 1.$$

Function

$$h(x) := \frac{|g(x) - g(x_g)|}{|x - x_g|}$$

is continuous on the compact set $[0,1] \setminus (x_g - \delta, x_g + \delta)$ and thus it is bounded by some $b \in \mathbb{R}$. Considering $n \in \mathbb{N}$, $n > \max\{|g'(x_g)| + 1, b\}$ we see that $g \in A_n$ with $x_0 = x_g$. Hence $D \subset \displaystyle\bigcup_{n=1}^{\infty} A_n$.

Step 2: A_n are closed

Let us fix $n \in \mathbb{N}$ and consider functions $f_k \in A_n$ such that $f_k \xrightarrow{\mathcal{C}([0,1])} f$, that is $f_k \rightrightarrows f$ on $[0,1]$. By definition of A_n, we obtain $t_k \in [0,1]$ such that

$$\text{for all } x \in [0,1] \text{ and } k \in \mathbb{N} \text{ we have } |f_k(x) - f_k(t_k)| \leq n\,|x - t_k|\,. \tag{2.2}$$

Since $[0,1]$ is compact there is a subsequence (t_{k_l}) such that $t_{k_l} \to t \in [0,1]$. We can relabel this sequence to obtain $t_k \to t \in [0,1]$. Then for arbitrary $x \in [0,1]$ and $k \in \mathbb{N}$ we have using (2.2)

$$\begin{aligned} |f(x) - f(t)| \leq & \, |f(x) - f_k(x)| + |f_k(x) - f_k(t_k)| + \\ & + |f_k(t_k) - f_k(t)| + |f_k(t) - f(t)| \leq \\ & \leq |f(x) - f_k(x)| + n\,|x - t_k| + n\,|t_k - t| + |f_k(t) - f(t)|\,. \end{aligned} \tag{2.3}$$

Since $f_k \rightrightarrows f$ we obtain

$$\lim_{k\to\infty} |f(x) - f_k(x)| = 0 \text{ and } \lim_{k\to\infty} |f(t) - f_k(t)| = 0.$$

Moreover, as $t_k \to t$ we have

$$\lim_{k\to\infty} n\,|x - t_k| = n\,|x - t| \text{ and } \lim_{k\to\infty} n\,|t_k - t| = 0.$$

Passing to limit in (2.3) therefore yields

$$|f(x) - f(t)| \leq n\,|x - t|$$

and hence $f \in A_n$ as x is arbitrary and we set $t = x_0$ in (2.1).

Step 3: A_n are nowhere dense

Let us take $f \in \mathcal{C}\big([0,1]\big)$ and $\varepsilon > 0$. Since f is continuous on compact set $[0,1]$ it is also uniformly continuous and there exists $\delta > 0$ such that

$$\text{for every } x, y \in [0,1] \text{ satisfying } |x - y| < \delta \text{ we have } |f(x) - f(y)| < \frac{\varepsilon}{3}\,. \tag{2.4}$$

Let us fix $k \in \mathbb{N}$ such that

$$k > \max\left\{\frac{3n}{\varepsilon}, \frac{1}{2\delta}\right\}$$

and define the "saw" function $s : [0,1] \to [0,\varepsilon]$ as $\frac{2}{k}$−periodic extension of function $\widetilde{s}$ defined as

$$\widetilde{s}(x) = \begin{cases} k\varepsilon x & \text{on } [0, \frac{1}{k}] \\ 2\varepsilon - k\varepsilon x & \text{on } [\frac{1}{k}, \frac{2}{k}] \end{cases}$$

(See Figure 2.1). We claim that

$$f + s \in B_{\mathcal{C}([0,1])}(f, 2\varepsilon) \setminus A_n.$$

Since $\|s\|_\infty \leq \varepsilon$ we have $f + s \in B_{\mathcal{C}([0,1])}(f, 2\varepsilon)$. Now for any $x_0 \in [0,1]$ we find $x \in [0,1]$ such that

$$|x_0 - x| = \frac{1}{2k} \text{ and } |s(x_0) - s(x)| = \frac{\varepsilon}{2} \tag{2.5}$$

(see Figure 2.1).

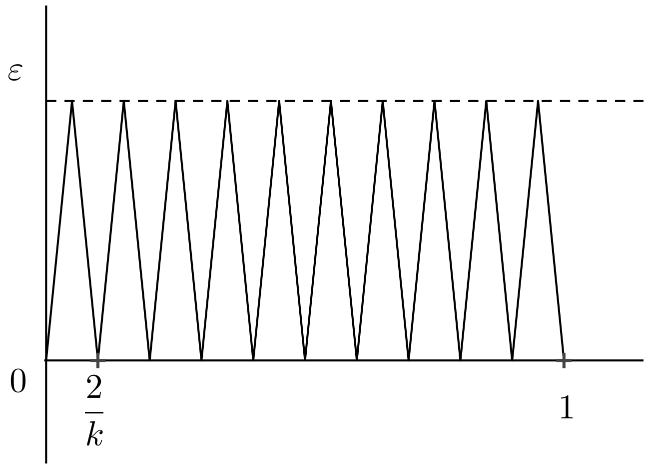

Figure 2.1: "saw" function $s(x)$

Using (2.4) and (2.5) we obtain

$$|(f+s)(x_0)-(f+s)(x)| \geq |s(x_0)-s(x)| - |f(x_0)-f(x)| \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{3} = \frac{\varepsilon}{6}$$

since $|x_0 - x| = \frac{1}{2k} < \delta$. On the other hand

$$k\varepsilon > 3n \Rightarrow n\,|x_0 - x| = \frac{n}{2k} < \frac{\varepsilon}{6}.$$

For arbitrary $x_0 \in [0,1]$ we have found x such that the condition in (2.1) fails and therefore $f+s \notin A_n$.

We have proved that D is a subset of countable union of nowhere dense sets and hence D is of first category in $\mathcal{C}\big([0,1]\big)$.

□

Corollary. *The set $D' \subset \mathcal{C}(\mathbb{R})$ consisting of functions f for which there is $x_f \in \mathbb{R}$ such that $f'(x_f) \in \mathbb{R}$ is of first category in $\mathcal{C}(\mathbb{R})$.*

Proof. We observe that $D' \subset \bigcup_{n=1}^{\infty} D_n$ where D_n denote set of continuous functions on $\mathbb{R}$ differentiable at some point belonging to $[-n,n]$. Now we claim that the proof of Theorem 7 works for any D_n as for D. In all steps we work just with the restrictions of functions from $\mathcal{C}(\mathbb{R})$ on the set $[-n,n]$ which clearly belongs to $\mathcal{C}\big([-n,n]\big)$. Since $\|\cdot\|_{\mathcal{C}(\mathbb{R})} \geq \|\cdot\|_{\mathcal{C}\left([-n,n]\right)}$, one can check that all conclusions remain true and D_n are of first category in $\mathcal{C}(\mathbb{R})$. Using Proposition 4 we obtain that D' is of first category.

□

2.2 Hölder continuous functions

One can ask if an analogue of Theorem 7 holds for even nicer space than continuous functions on real line or a compact interval. In this section we present $\alpha-$*Hölder continuity* and show that a typical $\alpha-$Hölder continuous function is nowhere differentiable.

Definition 8. *Let $\alpha \in (0,1)$. We say that $f : [0,1] \to \mathbb{R}$ is $\alpha-$Hölder continuous if (see Figure 2.2)*

$$\sup_{x,y\in[0,1],\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} < \infty. \tag{2.6}$$

Remark. Hölder condition for $\alpha = 0$ gives boundedness of f and for $\alpha = 1$ it gives the Lipschitz condition. If $f : [0,1] \to \mathbb{R}$ satisfies $\sup_{x,y\in[0,1],\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\beta}} < \infty$ for some $\beta > 1$ it is easy to see that $f' = 0$ everywhere on $[0,1]$ and hence f is constant on $[0,1]$. This is why α is set to be in $(0,1)$.

Proposition 9. *The set of all functions $f : [0,1] \to \mathbb{R}$ which are $\alpha-$Hölder continuous equipped with the norm*

$$\|f\|_{\alpha} = \|f\|_{\infty} + \sup_{x,y\in[0,1],\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$$

is Banach space. We denote it by $C^{0,\alpha}\big([0,1]\big)$.

For the proof 9 see [1, Thm 2.3.3.].

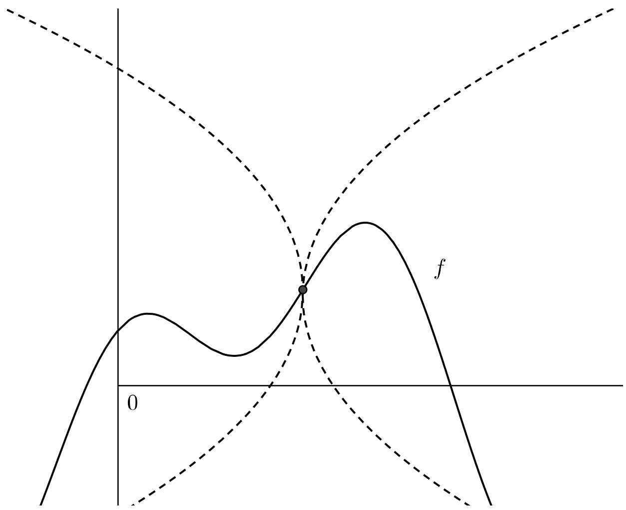

Figure 2.2: Graph of an $\alpha-$Hölder continuous function lies in two suitable symmetrical parabolic areas centered at any of its points.

Proposition 10. *For $0 < \alpha \le \alpha' \le 1$ we have $C^{0,\alpha'}([0,1]) \subset C^{0,\alpha}\big([0,1]\big)$. Moreover, every $f \in C^{0,\alpha}\big([0,1]\big)$ is uniformly continuous on $[0,1]$.*

Proof. Suppose that $f \in C^{0,\alpha'}([0,1])$. Then

$$\sup_{x,y\in[0,1],\,x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \le \sup_{x,y\in[0,1],\,x\neq y} |x-y|^{\alpha'-\alpha} \cdot \sup_{x,y\in[0,1],\,x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha'}} < \infty.$$

The second assertion is trivial.

□

Theorem 11. *Let $\alpha \in (0,1)$ and denote*

$$E := \big\{ f \in C^{0,\alpha}\big([0,1]\big) : \textit{there exists } x_f \in (0,1) \textit{ such that } f'(x_f) \in \mathbb{R} \big\}.$$

Then E is of first category in $C^{0,\alpha}\big([0,1]\big)$.

Proof. We will follow the ideas from the proof of Theorem 7 and use its parts. Let us define

$$B_n = \Big\{ f \in C^{0,\alpha}\big([0,1]\big) : \text{there is } x_0 \in [0,1] \\ \text{such that } \frac{|f(x)-f(x_0)|}{|x-x_0|} \le n \text{ for all } x \in [0,1] \Big\}. \tag{2.7}$$

Step 1: $E \subset \bigcup_{n=1}^{\infty} B_n$

Since f is continuous, the analogy of first step from proof of Theorem 7 works.

Step 2: B_n are closed

Clearly $C^{0,\alpha}\big([0,1]\big) \subset \mathcal{C}\big([0,1]\big)$, considered as sets, and for any $n \in \mathbb{N}$ we have $B_n = A_n \bigcap C^{0,\alpha}\big([0,1]\big)$. By definition from Proposition 9 we have

$$\|\cdot\|_{C^{0,\alpha}([0,1])} \ge \|\cdot\|_{\mathcal{C}([0,1])}$$

and thus if $f_k \in B_n$, $f \in C^{0,\alpha}\big([0,1]\big)$ and $f_k \xrightarrow{C^{0,\alpha}([0,1])} f$, then also $f_k \xrightarrow{\mathcal{C}([0,1])} f$ and hence $f \in C^{0,\alpha}\big([0,1]\big) \bigcap A_n$ since A_n is closed in $\mathcal{C}\big([0,1]\big)$. It follows that $f \in B_n$ and thus B_n is closed in $C^{0,\alpha}\big([0,1]\big)$.

Step 3: B_n are nowhere dense

Let us take $f \in C^{0,\alpha}\big([0,1]\big)$, $\varepsilon > 0$ and $n \in \mathbb{N}$. We fix $k \in \mathbb{N}$ such that

$$k > \max\left\{ \frac{5n}{\varepsilon}, \left(\frac{5n}{\varepsilon}\right)^{\frac{1}{1-\alpha}} \right\} \tag{2.8}$$

and define functions "upper jump" $u : [0, \frac{1}{k}] \to \mathbb{R}$ such that $u(x) = 5nx$ and "lower jump" $l : [0, \frac{1}{k}] \to \mathbb{R}$ such that $l(x) = -5nx$. Now we define "broken saw" function $b : [0,1] \to \mathbb{R}$ inductively.

For start we set $b(0) := 0$. If $\big|f(\frac{1}{k}) - f(0)\big| > \frac{2n}{k}$, then we set $b \equiv 0$ on $[0, \frac{1}{k}]$ and otherwise we set $b \equiv u$ on $[0, \frac{1}{k}]$. Let us suppose we have already defined b as piecewise linear continuous function on $[0, \frac{a}{k}]$ for some $a \in \{1, 2, \ldots, k-1\}$.

If $\left|f(\frac{a+1}{k}) - f(\frac{a}{k})\right| > \frac{2n}{k}$ we extend b continuously and constantly on $[\frac{a}{k}, \frac{a+1}{k}]$. Otherwise we continuously extend b on $[\frac{a}{k}, \frac{a+1}{k}]$ by opposite type of jump function we used last time, i.e.

$$b(x) = u\left(x - \frac{a}{k}\right) \quad \text{or} \quad b(x) = \frac{5n}{k} - l\left(x - \frac{a}{k}\right) \quad \text{on} \quad \left[\frac{a}{k}, \frac{a+1}{k}\right] \tag{2.9}$$

(see Figure 2.3).

It follows from the construction that for every $a \in \{0, 1, \dots, k-1\}$ we have

$$\left|(f+b)\left(\frac{a}{k}\right) - (f+b)\left(\frac{a+1}{k}\right)\right| > \frac{2n}{k} \tag{2.10}$$

since either (2.10) holds just for f and then $b \equiv 0$ on $[\frac{a}{k}, \frac{a+1}{k}]$ and otherwise (2.9) gives

$$\left|(f+b)\left(\frac{a}{k}\right) - (f+b)\left(\frac{a+1}{k}\right)\right| \geq \left|b\left(\frac{a}{k}\right) - b\left(\frac{a+1}{k}\right)\right| - \left|f\left(\frac{a}{k}\right) - f\left(\frac{a+1}{k}\right)\right| \geq$$
$$\geq \frac{5n}{k} - \frac{2n}{k} = \frac{3n}{k}.$$

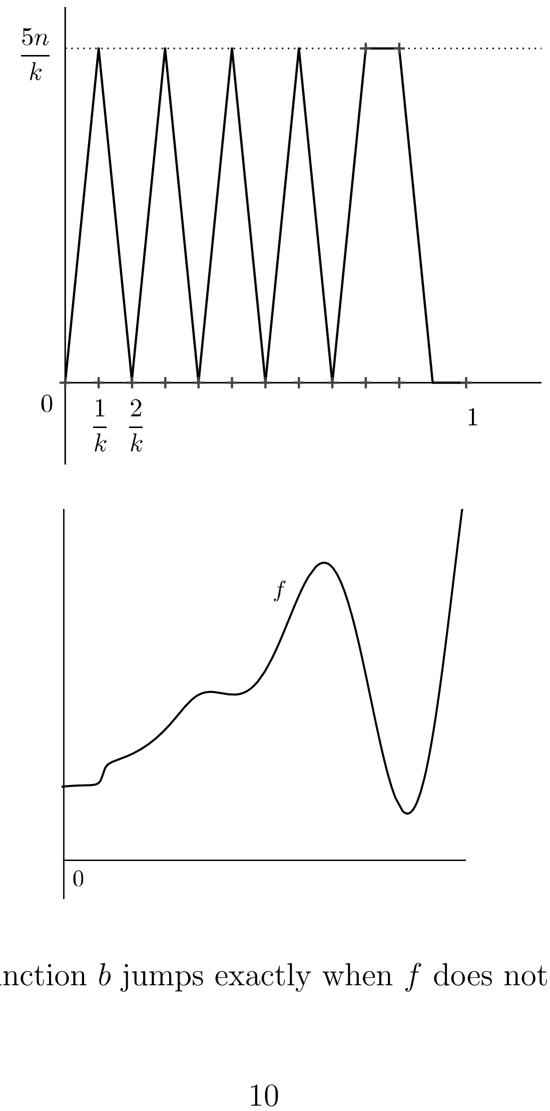

Figure 2.3: Function b jumps exactly when f does not jump much.

We claim that

$$f + b \in B_{C^{0,\alpha}([0,1])}(f, 2\varepsilon) \setminus B_n.$$

It is clear from the construction of b that $\|b\|_\infty \le \frac{5n}{k} < \varepsilon$. Since b is piecewise linear and its graph is made only of shifted graphs of u and l we have

$$\sup_{x,y\in[0,1],\,x\neq y} \frac{|b(x)-b(y)|}{|x-y|^\alpha} = \sup_{x,y\in\left[0,\frac{1}{k}\right],\,x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} = \sup_{t\in\left[0,\frac{1}{k}\right]} \frac{5nt}{t^\alpha} =$$
$$= \sup_{t\in\left[0,\frac{1}{k}\right]} 5nt^{1-\alpha} \le \frac{5n}{k^{1-\alpha}} < \varepsilon$$

where last the two inequalities follow from (2.8) and from the fact that $x \mapsto x^{1-\alpha}$ is increasing function on $(0,\infty)$. Combining the last two estimates yields

$$\|(f+b) - f\|_{C^{0,\alpha}([0,1])} = \|b\|_{C^{0,\alpha}([0,1])} < 2\varepsilon.$$

For any $x_0 \in [0,1]$ we find $a \in \{0,1,\dots,k\}$ such that $\frac{a}{k} \le x_0 < \frac{a+1}{k}$. We recall (2.10) and obtain

$$\frac{2n}{k} < \left|(f+b)\left(\frac{a}{k}\right) - (f+b)\left(\frac{a+1}{k}\right)\right| \le$$
$$\le \left|(f+b)\left(\frac{a}{k}\right) - (f+b)(x_0)\right| + \left|(f+b)(x_0) - (f+b)\left(\frac{a+1}{k}\right)\right|$$

and hence either

$$\left|(f+b)\left(\frac{a}{k}\right) - (f+b)(x_0)\right| > \frac{n}{k} \ge n\left|x_0 - \frac{a}{k}\right|$$

or

$$\left|(f+b)(x_0) - (f+b)\left(\frac{a+1}{k}\right)\right| > \frac{n}{k} \ge n\left|\frac{a+1}{k} - x_0\right|.$$

For arbitrary $x_0 \in [0,1]$ we have found $x\,(=\frac{a}{k}$ or $\frac{a+1}{k})$ such that the condition in (2.7) fails and therefore $f + s \notin B_n$.

We have proved that E is a subset of a countable union of nowhere dense sets and hence E is of first category in $C^{0,\alpha}\big([0,1]\big)$.

□

2.3 Notes

As was claimed in the first chapter, applying Theorem 7 together with Theorem 5 for complete metric spaces $\mathcal{C}\big([0,1]\big)$ and $C^{0,\alpha}\big([0,1]\big)$ gives us density of nowhere differentiable functions in both mentioned spaces. This fact is interesting as such and furthermore it gives us some other properties of typical continuous functions. For instance using Lebesgue theorem about differentiability of monotone functions (see [2, Thm 22.5.]) we easily obtain that typical continuous (resp. $\alpha-$Hölder) function is not monotone on any interval.

One could ask whether Theorem 11 can be further generalized and whether a typical function from some even nicer space is nowhere differentiable. After Definition 8 of $\alpha-$Hölder continuous functions for $\alpha \in (0,1)$ we mentioned that $1-$Hölder continuous are Lipschitz. Since any Lipschitz function is differentiable almost everywhere (see [2, Thm 30.3.]), our result is optimal at least in this scale.

3. Luzin ($\mathcal{N}$) condition

We start this chapter by defining the Luzin (N) condition and some related propositions we will need to prove that this condition is not fulfilled by a typical continuous monotone function.

3.1 Preliminaries

Definition 12. *Let $f : [0,1] \to \mathbb{R}$ be a measurable function with respect to the Lebesgue measure. We say that f satisfies the* Luzin (N) condition *if $|f(M)| = 0$ for every $M \subset [0,1]$ such that $|M| = 0$, where $|A|$ denotes the Lebesgue measure of measurable set A.*

Definition 13. *We say that $f : [0,1] \to \mathbb{R}$ is* absolutely continuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

whenever $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \le 1$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

The validity of the Luzin (N) condition is closely related to the validity of the change of variables formula (see [3]). Let us also recall that each absolutely continuous function satisfies the Luzin (N) condition (see [2, Ex. 21.6.]).

Following proposition is trivial but useful and we state it here without proof.

Proposition 14. *Let us consider a function $f : X \to Y$ between the sets X and Y. Then we have*

$$f\Big(\bigcup_{\alpha\in\mathcal{A}} M_\alpha\Big) = \bigcup_{\alpha\in\mathcal{A}} f\Big(M_\alpha\Big)$$

for every system $\{M_\alpha \subset X : \alpha \in \mathcal{A}\}$. Moreover, if f is injective then also

$$f\Big(\bigcap_{\alpha\in\mathcal{A}} M_\alpha\Big) = \bigcap_{\alpha\in\mathcal{A}} f\Big(M_\alpha\Big).$$

Proposition 15. *A set $G \subset \mathbb{R}$ is open if and only if $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where (a_i, b_i) are pairwise disjoint.*

For the proof see [4, Thm 9.2.12.].

Next we recall continuity of measure and regularity of the Lebesgue measure.

Proposition 16. *Let X be a space and μ be a measure on X. Then*

- *if $A_1 \subset A_2 \subset \dots$ are measurable subsets of X, then*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n),$$

- *if $A_1 \supset A_2 \supset \dots$ and $\mu(A_1) < \infty$, then*

$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n).$$

For proof see [2, Thm 2.6.].

Proposition 17. *For every Lebesgue measurable set $M \subset \mathbb{R}$ and $\varepsilon > 0$ there exists an open set $A \subset \mathbb{R}$ such that $M \subset A$ and $|A \setminus M| < \varepsilon$.*

For the proof see [2, Thm 1.21.].

3.2 Monotone continuous functions

In this section we concentrate on nondecreasing continuous functions on $[0,1]$ and their behavior concerning the Luzin (N) condition. We show that they form a complete metric space and prove that its subset consisting of functions satisfying the Luzin (N) condition is of first category.

Proposition 18. *The set*

$$X := \Big\{ f \in \mathcal{C}([0,1]) : f \text{ is nondecreasing on } [0,1] \Big\}$$

is closed subset of $\mathcal{C}([0,1])$. In particular, $(X, \|\cdot\|_\infty)$ is a complete metric space.

Proof. If $f_n \xrightarrow{\mathcal{C}([0,1])} f \in \mathcal{C}([0,1])$, $f_n \in X$, then for any $x, y \in [0,1]$, $x < y$ we have

$$f(y) - f(x) = \lim_{n\to\infty} f_n(y) - f_n(x) \geq 0$$

since for all $n \in \mathbb{N}$ we have $f_n(y) - f_n(x) \geq 0$ and thus $f \in X$.

□

We will work with subspaces of X consisting of functions with additional lower bound of the size of their images.

Proposition 19. *Let us take $k \in \mathbb{N}$. Then the set*

$$X_k := \left\{ f \in \mathcal{C}([0,1]) : f \text{ is nondecreasing on } [0,1] \text{ and } f(1) - f(0) \geq \frac{1}{k} \right\}$$

is closed subset of X. Moreover,

$$X = \left\{ f \in X : f \text{ is constant on } [0,1] \right\} \cup \bigcup_{k=1}^{\infty} X_k.$$

Proof. An analogic argument as in the previous proof shows that since

$$f(1) - f(0) = \lim_{n\to\infty} f_n(1) - f_n(0) \geq \frac{1}{k},$$

the sets X_k are closed in X. The second assertion is trivial.

□

Now we prove a lemma covering the essential part of the proof of Theorem 21.

Lemma 20. *For $k \in \mathbb{N}$ we denote*

$$A_k := \left\{ f \in X_k : \text{for every } M \subset [0,1] \text{ such that } |M| = 0 \text{ we have } |f(M)| < \frac{1}{2k} \right\}.$$

Then A_k is of first category in X_k.

Proof. Let us define

$$B_{n,k} := \left\{ f \in X_k : \text{for every } M \subset [0,1],\ |M| < \frac{1}{n^2} \text{ we have } |f(M)| \leq \frac{1}{2k} \right\}. \tag{3.1}$$

<u>Step 1: $A_k \subset \bigcup_{n=1}^{\infty} B_{n,k}$</u>

Let us take some

$$f \in X_k \setminus \bigcup_{n=1}^{\infty} B_{n,k} = \bigcap_{n=1}^{\infty} (X_k \setminus B_{n,k}).$$

We know that for every $n \in \mathbb{N}$ there is $M_n \subset [0,1]$ satisfying

$$|M_n| < \frac{1}{n^2} \text{ and } |f(M_n)| > \frac{1}{2k}. \tag{3.2}$$

We define

$$M := \limsup_{n\to\infty} M_n = \bigcap_{j=1}^{\infty} \left(\bigcup_{i=j}^{\infty} M_i \right).$$

For any $j \in \mathbb{N}$ we have $M \subset \bigcup_{i=j}^{\infty} M_i$ and hence

$$|M| \leq \left| \bigcup_{i=j}^{\infty} M_i \right| \leq \sum_{i=j}^{\infty} \frac{1}{i^2} \xrightarrow{j\to\infty} 0$$

and we obtain $|M| = 0$. Since f is nondecreasing on $[0,1]$ we observe that the preimages of points are (possibly degenerate or empty) intervals. Every nondegenerate interval from the set $\{f^{-1}(x) : x \in \mathbb{R}\}$ in $[0,1]$ contains a unique rational number. Thus the set

$$N^* := \big\{x \in \mathbb{R} : f^{-1}(x) \text{ is nondegenerate interval}\big\}$$

is countable and thus $|N^*| = 0$. Since f is injective on $[0,1] \setminus f^{-1}(N^*)$ it follows from Proposition 14 that

$$\begin{aligned} f(M) = f\Big(\bigcap_{j=1}^{\infty}\Big(\bigcup_{i=j}^{\infty} M_i\Big)\Big) \supset f\Big(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} M_i\Big) \setminus N^* = \\ = \Big(\bigcap_{j=1}^{\infty} f\Big(\bigcup_{i=j}^{\infty} M_i\Big)\Big) \setminus N^* = \bigcap_{j=1}^{\infty}\Big(\bigcup_{i=j}^{\infty} f(M_i)\Big) \setminus N^* \end{aligned} \tag{3.3}$$

and from Proposition 16 using $\big|f\big([0,1]\big)\big| < \infty$, $|N^*| = 0$ and (3.2) we get

$$|f(M)| = \lim_{j\to\infty} \left| f\left(\bigcup_{i=j}^{\infty} M_i\right)\right| \geq \frac{1}{2k}.$$

We found

$$M \subset [0,1] \text{ such that } |M| = 0 \text{ and } |f(M)| \geq \frac{1}{2k}$$

and hence $f \notin A_k$.

<u>Step 2: $B_{n,k}$ are closed</u>

Let us fix $n \in \mathbb{N}$ and consider functions $f_j \in B_{n,k}$ such that $f_j \xrightarrow{X_k} f$, that is $f_j \rightrightarrows f$ on $[0,1]$. Now let us take $\varepsilon > 0$ and $M \in [0,1]$ such that $|M| < \frac{1}{n^2}$ arbitrarily. We would like to show that $|f(M)| \leq \frac{1}{2k}$. Using Proposition 17 and Proposition 15 we obtain an open set $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where (a_i, b_i) are disjoint, such that $M \subset G$ and $|G| < \frac{1}{n^2}$.

Since f is continuous and nondecreasing $f\big((a_i, b_i)\big)$ are intervals (possibly degenerate) with pairwise disjoint interiors. Thus

$$\sum_{i=1}^{\infty} f(b_i) - f(a_i) = \left|\bigcup_{i=1}^{\infty} f\big((a_i, b_i)\big)\right| < \infty$$

and hence there is $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0}^{\infty} f(b_i) - f(a_i) < \varepsilon.$$

For any $i \in \mathbb{N}$ we have

$$f(b_i) - f(a_i) = \lim_{j\to\infty} f_j(b_i) - f_j(a_i)$$

and thus there exists $j_i \in \mathbb{N}$ such that for any $j \geq j_i$ we have

$$f(b_i) - f(a_i) < f_j(b_i) - f_j(a_i) + \frac{\varepsilon}{2^i}. \tag{3.4}$$

Let us set

$$j := \max_{1 \le m \le i_0} j_m.$$

Since $f_j \in B_{n,k}$ and $|G| < \frac{1}{n^2}$ we have

$$\begin{aligned} |f(M)| \le |f(G)| &= \sum_{i=1}^{i_0} f(b_i) - f(a_i) + \sum_{i=i_0+1}^{\infty} f(b_i) - f(a_i) \\ &\le \sum_{i=1}^{i_0} \left(f_j(b_i) - f_j(a_i) + \frac{\varepsilon}{2^i} \right) + \varepsilon \le |f_j(G)| + \sum_{i=1}^{i_0} \frac{\varepsilon}{2^i} + \varepsilon < \frac{1}{2k} + 2\varepsilon \end{aligned} \tag{3.5}$$

and by $\varepsilon \to 0+$ we obtain $|f(M)| \le \frac{1}{2k}$. Since M was arbitrary we get $f \in B_{n,k}$ and thus $B_{n,k}$ are closed.

Step 3: $B_{n,k}$ are nowhere dense

Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $f \in X_k$. We need to find $g \in B_{X_k}(f, \varepsilon) \setminus B_{n,k}$. Thanks to the previous step we are about to find $g \in B_{X_k}(f, \varepsilon)$ such that there exists $M \subset [0,1]$ with $|M| < \frac{1}{n^2}$ and $|g(M)| > \frac{1}{2k}$.

We fix some $N \in \mathbb{N}$ such that $\frac{f(1)-f(0)}{N} < \varepsilon$. For $i \in \{1, 2, \ldots, N+1\}$ we find some

$$a_i \in f^{-1}\left(f(0) + \frac{i-1}{N}\big(f(1) - f(0)\big) \right).$$

For $i \in \{1, 2, \ldots, N\}$ we set

$$b_i := \min\left\{ \frac{a_{i+1} - a_i}{2}, \frac{1}{2n^2 N} \right\}$$

and define

$$g(x) = \begin{cases} f(0) + \left(\frac{i-1}{N} + \frac{x - a_i}{N(b_i - a_i)} \right) \big(f(1) - f(0)\big) & \text{on } [a_i, b_i] \\ f(0) + \frac{i}{N}\big(f(1) - f(0)\big) & \text{on } [b_i, a_{i+1}] \end{cases} \tag{3.6}$$

(See Figure 3.2).

One can easily check that g is continuous piecewise linear function and

$$\|f - g\|_\infty \le \frac{f(1) - f(0)}{N} < \varepsilon$$

and hence

$$g \in B_{X_k}(f, \varepsilon).$$

On the other hand for

$$M := \bigcup_{i=1}^{N} (a_i, b_i)$$

we have

$$|M| \le N \frac{1}{2n^2 N} < \frac{1}{n^2}$$

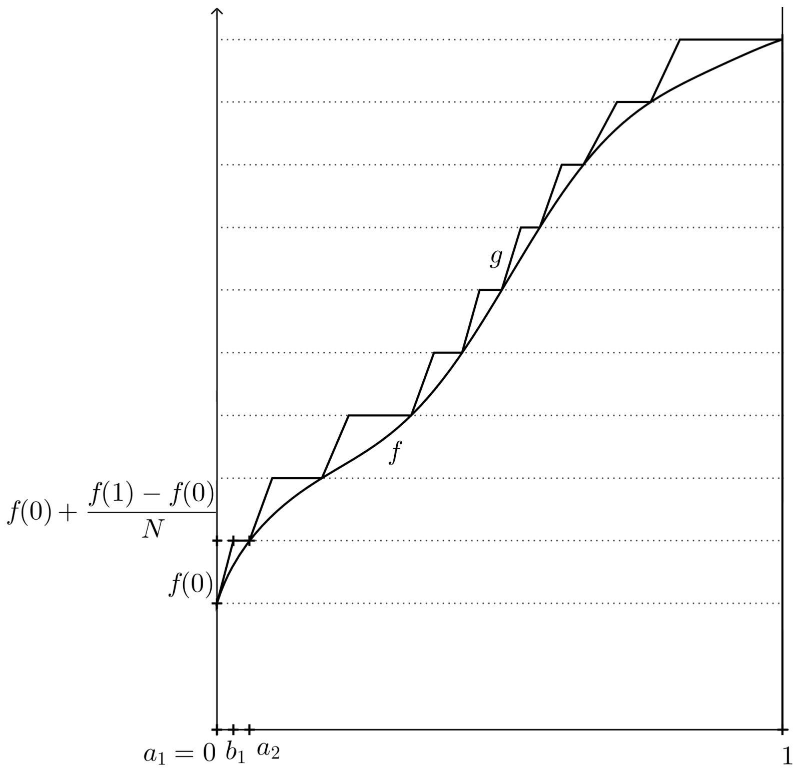

Figure 3.1: The function g approximates the function f but it maps a small set to a set of measure $f(1) - f(0)$.

and

$$|g(M)| = \left|\bigcup_{i=1}^{N} \left(f(0) + \frac{(i-1)(f(1)-f(0))}{N}, f(0) + \frac{i(f(1)-f(0))}{N} \right)\right| =$$
$$= N\frac{f(1)-f(0)}{N} \geq \frac{1}{k} > \frac{1}{2k}. \tag{3.7}$$

We found M such that the condition in (3.1) fails and hence

$$g \notin B_{n,k}$$

which proves that $B_{n,k}$ are nowhere dense in X_k.

We have proved that $A_k \subset \bigcup_{n=1}^{\infty} B_{n,k}$ is a subset of countable union of nowhere dense sets and thus a set of first category in X_k.

□

Now we can easily prove that a typical monotone continuous function does not satisfy the Luzin (N) condition.

Theorem 21. *The set*

$$L := \{f \in X : f \textit{ satisfies the (N) condition}\}$$

is of first category in X.

Proof. The set

$$C := \{f \in X : f \text{ is constant on } [0,1]\}$$

is obviously closed and nowhere dense in X since an uniform limit of constant function is constant and for any $\varepsilon > 0$ and $c \in C$ there is a nonconstant function $h \in B_X(c, \varepsilon)$. Using Proposition 19 and Lemma 20 we obtain

$$L = \bigcup_{k=1}^{\infty} \{f \in X_k : f \text{ satisfies the (N) condition}\} \cup C \subset \bigcup_{k=1}^{\infty} A_k \cup C \tag{3.8}$$

and using Proposition 4 we conclude that L is countable union of sets of first category and thus of first category in X.

□

Remark. The continuity assumption wasn't necessary and we could state previous definitions and theorems just for the metric space of nondecreasing functions on $[0,1]$ with the supremum norm. The only nontrivial obstacle is the lower bound of the size (measure) of the images of such functions. However, one could prove the density of monotone functions breaking the Luzin (N) condition by suitable approximation of general nondecreasing function on $[0,1]$ by nonconstant continuous nondecreasing function and then using Theorem 21.

Using [2, Ex. 21.6.] we immediately get the following corollary.

Corollary. *A typical monotone continuous function is not absolutely continuous.*

4. Discontinuity of L^1 functions

In this chapter we prove that a typical integrable function on bounded interval is nowhere bounded and thus nowhere continuous. We start with some essential definitions and propositions.

4.1 Preliminaries

Convention. Measure and integral are always meant to be the Lebesgue measure on $\mathbb{R}$ and the Lebesgue integral with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 22. *Let $\widetilde{L}^1(0,1)$ be the set of all measurable functions $f : (0,1) \to \mathbb{R}$ such that*

$$\|f\|_1 := \int_0^1 |f(x)| \, \mathrm{d}x < \infty.$$

We call such functions integrable.

We would like the set of all integrable functions equipped with the norm $\|\cdot\|_1$ to form a normed linear space. However, there exists a nonzero integrable function $g \in \widetilde{L}^1(0,1)$ such that $\|g\|_1 = 0$ (for instance the characteristic function of any set of zero measure). Thus the classes of equivalence are used instead.

Proposition 23. *For $f \in \widetilde{L}^1(0,1)$ we denote*

$$[f] := \left\{ g \in \widetilde{L}^1(0,1) : f = g \text{ almost everywhere in } (0,1) \right\}.$$

Then the set $\left\{ [f] : f \in \widetilde{L}^1(0,1) \right\}$ equipped with the norm $\|[f]\|_1 = \int_0^1 |g(x)| \, \mathrm{d}x$, where $g \in [f]$, is a well defined Banach space. We denote it by $L^1(0,1)$ and write just $f \in L^1(0,1)$ instead of $[f] \in L^1(0,1)$.

For the proof see [2, Theorem 10.6].

Our purpose here is to prove a theorem claiming that a typical integrable function is nowhere continuous (or even bounded) and such statements make no sense for the mentioned classes of functions. We will therefore define such properties for these classes. The following theorem will help us to do it:

Theorem 24 (Lebesgue differentiation theorem)**.** *Let $f \in \widetilde{L}^1(0,1)$. Then for almost every $x \in (0,1)$ we have*

$$\lim_{h \to 0_+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \, \mathrm{d}t = 0.$$

For the proof see [2, Theorem 23.9.].

Lemma 25. *For every $[f] \in L^1(0,1)$ the function*

$$\widetilde{f}(x) := \limsup_{h \to 0_+} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\,\mathrm{d}t$$

is well defined, $\widetilde{f} \in [f]$ and if f is continuous at $x_0 \in (0,1)$ (resp. bounded on some $(a,b) \subset (0,1)$), then so is $\widetilde{f}$.

Proof. For any $x \in (0,1)$ there is $h_0 > 0$ such that $(x - h_0, x + h_0) \subset (0,1)$ and since $f \in \widetilde{L}^1(0,1)$, we observe that $\widetilde{f}$ is indeed well defined. By Theorem 24 we have $\widetilde{f} = f$ almost everywhere in $(0,1$ and hence $\widetilde{f} \in [f]$.

Suppose that $f(x_0) = \lim_{x \to x_0} f(x)$. In other words for any $\varepsilon > 0$ we have $\delta > 0$ such that

$$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \text{ for every } x \in (x_0 - \delta, x_0 + \delta).$$

For any such x and $h < \delta$ we obtain

$$\begin{aligned} f(x_0) - \varepsilon = \frac{1}{2h} \int_{x-h}^{x+h} (f(x_0) - \varepsilon)\,\mathrm{d}t \leq \frac{1}{2h} \int_{x-h}^{x+h} f(t)\,\mathrm{d}t \leq \\ \leq \frac{1}{2h} \int_{x-h}^{x+h} (f(x_0) + \varepsilon)\,\mathrm{d}t = f(x_0) + \varepsilon. \end{aligned} \tag{4.1}$$

Using this for $x = x_0$ and passing to a limsup in h and limit in ε give

$$\widetilde{f}(x_0) = \limsup_{h \to 0_+} \frac{1}{2h} \int_{x_0-h}^{x_0+h} f(t)\,\mathrm{d}t = f(x_0).$$

We can prove analogously that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0 + \delta)$ we have $\widetilde{f}(x) \in \big(\widetilde{f}(x_0) - \varepsilon, \widetilde{f}(x_0) + \varepsilon\big)$ and thus $\widetilde{f}$ is continuous at x_0. The proof of boundedness of $\widetilde{f}$ assuming that f is bounded can be also done analogously.

□

Definition 26. *We say that $[f] \in L^1(0,1)$ is continuous at $x_0 \in (0,1)$ (resp. bounded on $(a,b) \subset (0,1)$) if $\widetilde{f}$ is continuous at x_0 (resp. bounded on $(a,b) \subset (0,1)$).*

Lemma 25 guarantees that we allow the most elements of $L^1(0,1)$ to be continuous at some point (resp. bounded on some interval) as possible. From now on, we identify $[f] \in L^1(0,1)$ with $\widetilde{f}$.

4.2 Nowhere continuous functions in $L^1(0,1)$

Now we prove that a typical integrable function is nowhere bounded. Again we express this result in the language of Baire categories.

Theorem 27. *Let us denote*

$$\mathcal{B} := \left\{ f \in L^1(0,1) : f \text{ is bounded on some open interval } (a,b) \subset (0,1) \right\}.$$

Then $\mathcal{B}$ is of first category in $L^1(0,1)$.

Proof. We just recall that every $f \in L^1(0,1)$ is represented by the function $\widetilde{f}$. For $n \in \mathbb{N}$ we define

$$A_n := \Big\{ f \in L^1(0,1) : \text{there is } (a,b) \subset (0,1) \text{ such that } b - a \geq \frac{1}{n} \\ \text{and } \widetilde{f}\big((a,b)\big) \subset [-n,n] \Big\}. \tag{4.2}$$

Let us proceed in three standard steps.

Step 1: $\mathcal{B} \subset \bigcup_{n=1}^{\infty} A_n$

Let $f \in L^1(0,1)$ be bounded on some interval of length $\omega > 0$ by some $K > 0$. Then taking an integer $n > \max\left\{K, \frac{1}{\omega}\right\}$ we observe that $f \in A_n$.

Step 2: A_n are closed

Suppose that $f_k \in A_n$, $f_k \xrightarrow{L^1(0,1)} f \in L^1(0,1)$. Since $f_k \in A_n$ we have a_k, b_k, $k = 1, 2, \dots$ such that

$$b_k - a_k \geq \frac{1}{n} \text{ and } f_k\big((a_k, b_k)\big) \subset [-n, n].$$

Since $[0,1]$ is compact, there are a_{k_l} such that $a_{k_l} \xrightarrow{l\to\infty} a$ and $b_{k_{l_m}}$ such that $b_{k_{l_m}} \xrightarrow{m\to\infty} b$. Without loss of generality we assume that $a_k \xrightarrow{k\to\infty} a$ and $b_k \xrightarrow{k\to\infty} b$. Moreover,

$$b - a = \lim_{k\to\infty} b_k - a_k \geq \frac{1}{n}.$$

We claim that $f\big((a,b)\big) \subset [-n,n]$. Let $x \in (a,b)$. Since (a,b) is open we find $\delta > 0$ such that $(x - 2\delta, x + 2\delta) \subset (a,b)$. Since $a_k \xrightarrow{k\to\infty} a$ and $b_k \xrightarrow{k\to\infty} b$ there is $k_0 \in \mathbb{N}$ such that

$$(a_k, b_k) \supset (x - \delta, x + \delta) \text{ for all } k \geq k_0. \tag{4.3}$$

From $\|f_k - f\|_1 \xrightarrow{k\to\infty} 0$ we obtain

$$\int_I f_k(t)\,\mathrm{d}t \xrightarrow{k\to\infty} \int_I f(t)\,\mathrm{d}t$$

for any interval $I \subset (0,1)$. Thus

$$\widetilde{f}(x) = \limsup_{h\to 0_+} \left(\frac{1}{2h} \int_{x-h}^{x+h} f(t)\,\mathrm{d}t \right) = \limsup_{h\to 0_+} \left(\frac{1}{2h} \lim_{k\to\infty} \int_{x-h}^{x+h} f_k(t)\,\mathrm{d}t \right). \tag{4.4}$$

Now (4.3) gives $-n \leq f_k(t) \leq n$ for every $t \in (x-\delta, x+\delta)$ and $k \geq k_0$ and hence for $h < \delta$ we have

$$-n \leq \frac{1}{2h} \lim_{k\to\infty} \int_{x-h}^{x+h} f_k(t)\,\mathrm{d}t \leq n$$

which together with (4.4) implies $\left|\widetilde{f}(x)\right| \leq n$. Since $x \in (a,b)$ was arbitrary, we proved that $f \in A_n$ and thus A_n are closed in $L^1(0,1)$.

Step 3: A_n are nowhere dense

Since A_n are closed, it is sufficient to prove that for any $f \in L^1(0,1)$ and $\varepsilon > 0$ there is $h \in L^1(0,1)$ such that

$$\widetilde{h} \in B_{L^1(0,1)}(f, \varepsilon) \setminus A_n.$$

Let us take $\varepsilon > 0$ and $f \in L^1(0,1)$ arbitrarily. We find some $\delta > 0$ satisfying

$$\delta < \min\left\{\frac{\varepsilon}{6n^2}, \frac{1}{2n}\right\}$$

and define a jump function $j : (0, \frac{1}{2n}) \to \mathbb{R}$

$$j(x) = \begin{cases} 3n & \text{if } x \in (0, \delta) \\ 0 & \text{otherwise} \end{cases}. \tag{4.5}$$

For $k = 1, \ldots, 2n$ we define g for $x \in \left(\frac{k-1}{2n}, \frac{k}{2n}\right)$:

$$g(x) = \begin{cases} j(x - \frac{k-1}{2n}) & \text{if } \widetilde{f}\left((\frac{k-1}{2n}, \frac{k}{2n})\right) \subset [-n, n] \\ 0 & \text{otherwise} \end{cases}. \tag{4.6}$$

We defined g on $(0,1)$ except finitely many points. Therefore $\widetilde{g}$ is well defined and since g is piecewise constant we observe that

$$g = \widetilde{g} \text{ up to finitely many points in } (0,1). \tag{4.7}$$

We claim that for $h := f + g$ we have

$$\widetilde{h} \in B_{L^1(0,1)}(f, \varepsilon) \setminus A_n.$$

Using Theorem 24 and the linearity of limit and integral we obtain

$$\widetilde{h} = \widetilde{f} + \widetilde{g} \text{ almost everywhere in } (0,1). \tag{4.8}$$

Every interval $(a,b) \subset (0,1)$ of length at least $\frac{1}{n}$ contains an interval $I = (\frac{k-1}{2n}, \frac{k}{2n})$ for some $k = 1, \ldots, 2n$. It follows from (4.5) and (4.6) that either

$$\widetilde{f}(I) \not\subset [-n, n] \text{ and then } g \equiv 0 \text{ on } I$$

or

$$\left|\widetilde{f}(x) + \widetilde{g}(x)\right| \geq 3n - n \text{ for } x \in \left(\frac{k-1}{2n}, \frac{k-1}{2n} + \delta\right).$$

Thanks to 4.7 and 4.8
we obtain

$$\widetilde{h}\,(I) \not\subset [-n, n] \text{ and thus } \widetilde{h} \notin A_n.$$

On the other hand

$$\|h - f\|_1 = \|g\|_1 \leq 2n \int_0^\delta j(x)\,\mathrm{d}x = 2n \cdot 3n \cdot \delta < \varepsilon.$$

Combining the last two results we obtain

$$\widetilde{h} \in B_{L^1(0,1)}(f, \varepsilon) \setminus A_n$$

which we wanted to prove.

We proved that $\mathcal{B}$ is a subset of countable union of nowhere dense sets and thus a set of first category in $L^1(0,1)$.

$\square$

If $f \in L^1(0,1)$ is continuous at some point x_0 it is clearly bounded on some interval centered at x_0 and hence we can summarize our results in the following corollary.

Corollary. *A typical function $f \in L^1(0,1)$ is nowhere bounded and consequently nowhere continuous.*

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