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Stationary distribution of time series

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Contents

1 Introduction

Suppose that $\{X_t\}$ is a strictly stationary time series. We wish to evaluate its stationary distribution, say Q. This is quite complicated problem if an analytic solution in closed form is desired. If $\{X_t\}$ is an ergodic Markov chain, then Q satisfies an integral equation

$$p(A) = \int_{\mathbb{R}} \mathsf{P}(A|x)Q(\mathrm{d}x) \tag{1.1}$$

for arbitrary Borel set A.

Explicit solution of equation (1.1) is known only in few cases. The most trivial of them is a linear AR(1) model

$$X_t = \rho X_{t-1} + \eta_t,$$

$\rho \in (-1, 1)$, driven by Gaussian white noise $\eta_t \overset{\text{iid}}{\sim} \mathsf{N}(0, \sigma^2)$ since we can write

$$X_t = \eta_t + \rho\eta_{t-1} + \rho^2\eta_{t-2} + \cdots$$

and the distribution of the expression on the right hand side is again normal with zero mean and variance $\sigma^2/(1-\rho^2)$. Similar arguments can be used for AR models of higher order.

If the exact solution of this problem is not known, we construct at least some approximation of the stationary distribution. In Section 2, we describe and study two procedures which can be used for class of linear models. In Section 3, one of them is generalized to multidimensional time series.

In Section 4, we deal with two nonlinear models where an exact stationary distribution has been found – the absolute autoregression AAR(1) given by

$$X_t = a|X_{t-1}| + \eta_t,$$

$a \in (-1, 1)$, with innovations η_t having Gaussian, Cauchy, Laplace and discrete rectangular distributions, and the threshold autoregression TAR(1)

$$X_t = \begin{cases} \alpha X_{t-1} + \eta_t & \text{if } X_{t-1} \geq 0 \\ \beta X_{t-1} + \eta_t & \text{if } X_{t-1} < 0 \end{cases}$$

where the noise process η_t has Laplace distribution.

New results in this thesis include

- assertions on the speed of convergence of algorithm of Anděl and Hrach for innovations with uniform and general distribution – Proposition 2.6 and Theorem 2.10

- relaxing the assumptions of Haiman's procedure, in particular finitness of support of density of innovations and its differentiability – Theorems 2.13, 2.14 and 2.16
- extension of Haiman's procedure to general causal linear process and ARMA processes – Theorem 2.20
- assertions on the properties of convolution – Theorems 2.25, 2.27, 2.30
- extension of algorithm of Anděl and Hrach to multidimensional AR(1) model – Section 3
- derivation of explicit closed form of stationary density of AAR(1) process for several types of distribution (normal, Cauchy, discrete uniform and Laplace) – Section 4.2
- approximation of stationary density of AAR(1) process driven by symmetric distribution – Thorem 4.6
- illustrative examples

2 One-dimensional linear processes

In this section we study the class of linear models. Even in this relatively simple case, it is not easy to find a closed form of stationary density given the distribution of the innovations.

First we focus on the AR(1) model. We describe two algoritms (proposed by Anděl and Hrach [4] and Haiman [12]) which yield a sequence of densities converging to the desired stationary density. We also prove assertions on the speed of convergence and illustrate the described methods on several examples. In the second part of the section we extend these algorithms to models AR(2) (Anděl and Hrach) and a general causal linear process (Haiman), respectivelly.

2.1 Model AR(1)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary AR(1) process defined by

$$X_t = \rho X_{t-1} + \eta_t, \tag{2.1}$$

where $\rho \in (-1, 1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ are i.i.d. random variables with finite second moment. We can rewrite (2.1) into the form

$$X_t = \eta_t + \rho\eta_{t-1} + \rho^2\eta_{t-2} + \cdots . \tag{2.2}$$

The series on the right hand side obviously converges in the quadratic mean and the proces $\{X_t\}$ is strictly stationary. In [4], it was proved that the stationary distribution of such process is continuous (see also [10]). Under an additional assumption of absolute continuity of the white noise η_t, the proof is very simple.

Using (2.2) we can write $X_t = \eta_t + Z_t$ where $Z_t = \rho\eta_{t-1} + \rho^2\eta_{t-2} + \cdots$. If η_t has a density, then since η_t and Z_t are independent, their sum X_t has an absolutely continuous distribution (see [20], p. 196).

One of the first attempts to find a connection between the distributions of X_t and η_t in a non-normal case was published in [8]. For some special distributions of η_t (continuous and discrete rectangular distributions, Laplace distribution) the stationary distribution of X_t is calculated in [2]. It is also known that if η_t has a stable distribution of exponent θ ($0 < \theta \le 2$), then X_t also has a stable distribution of the same exponent (see e.g. [22], p. 208, Ex. 11).

2.1.1 Algorithm of Anděl and Hrach

First we recall well known equations for the stationary density of X_t and a formula for the corresponding characteristic function.

Theorem 2.1. *Let η_t have a density f. Then the stationary density h of X_t satisfies the integral equation*

$$h(x) = \int_{\mathbb{R}} f(x - \rho u) h(u) \, \mathrm{d}u. \tag{2.3}$$

Proof. See [4], Theorem 1.5. □

Theorem 2.2. *Let $\psi(t)$ be the characteristic function of η_t and let $\lambda(t)$ be the characteristic function of X_t. Then*

$$\lambda(t) = \prod_{n=0}^{\infty} \psi(\rho^n t).$$

Proof. See [4], Theorem 1.6. □

Now we review an algorithm proposed by Anděl and Hrach [4] which approximates the solution of equation (2.3). Let h_0 be an arbitrary density. For $n \geq 1$ define

$$h_n(x) = \int_{\mathbb{R}} f(x - \rho u) h_{n-1}(u) \, \mathrm{d}u. \tag{2.4}$$

It is obvious that every function h_n is a density.

Theorem 2.3. *Let h_0 be a density. Define h_n by* (2.4)*. Assume that there exists an integer $m \geq 0$ such that*

$$\int_{\mathbb{R}} |\psi(t)\psi(\rho t) \cdots \psi(\rho^m t)| \, \mathrm{d}t < \infty. \tag{2.5}$$

Then $h_n(x) \to h(x)$ for every x as $n \to \infty$.

Proof. See [4], Theorem 2.1. □

In some special cases it was proved that the procedure converges geometrically fast (see [4] and [18]).

Proposition 2.4. *Let $\{X_t\}$ be the AR(1) process* (2.1)*, let η_t have exponential distribution with expectation equal to 1, $\rho \in (0, 1)$. Choose $h_0(x) = f(x) = \exp(-x)$. Then, for $h_n(x)$ defined by* (2.4)*, we have $h_n(x) \to h(x)$ and*

$$|h_{n+1}(x) - h_n(x)| \leq \frac{1}{4} \rho^{n-3}$$

for every $n \geq 3$.

Proof. See [4], p. 314–315. □

Remark 2.5. Let the assumptions of the previous Proposition hold. Define

$$\Delta_{n+1} = \sup_x |h_{n+1}(x) - h_n(x)|.$$

It was also proved (see [4]) that

$$\Delta_2 \leq \frac{\pi - 2\,\mathrm{arctg}(\rho^2/\sqrt{1-\rho^4})}{2\pi\sqrt{1-\rho^4}}$$

and

$$\Delta_3 \leq -\frac{2\rho \ln \rho}{\pi(1-\rho^4)}.$$

Proposition 2.6. *Let $\{X_t\}$ be the AR(1) process* (2.1) *with η_t uniformly distributed on interval $[0,1]$, $\rho \in (-1,1)$, $\rho \neq 0$. Choose $h_0(x) = f(x) = \chi_{[0,1]}(x)$. Then, for $h_n(x)$ defined by* (2.4)*, we have $h_n(x) \to h(x)$,*

$$|h_{n+1}(x) - h_n(x)| \leq \frac{3}{\pi}|\rho|^{n-2}$$

and

$$|h_n(x) - h(x)| \leq \frac{3}{\pi(1-|\rho|)}|\rho|^{n-2}$$

for every $n \geq 2$.

Proof. Clearly,

$$\psi(t) = \mathsf{E}\mathrm{e}^{\mathrm{i}t\eta_1} = \frac{\mathrm{e}^{\mathrm{i}t} - 1}{\mathrm{i}t} = \frac{\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t})}{t}, \qquad t \neq 0.$$

Since for $t \neq 0$

$$\begin{aligned} |\psi(t)| &= \sqrt{\psi(t)\overline{\psi(t)}} = \sqrt{\frac{\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t})}{t} \cdot \frac{-\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t})}{t}} \\ &= \frac{\sqrt{2(1-\cos t)}}{|t|} \leq \begin{cases} \frac{2}{|t|} & \text{for } |t| \geq 2 \\ 1 & \text{for } |t| < 2, \end{cases} \end{aligned} \tag{2.6}$$

we have

$$\int_{\mathbb{R}} |\psi(t)\psi(\rho t)|\,\mathrm{d}t \leq \int_{-2}^{2} \mathrm{d}t + 2\int_2^\infty \frac{4}{|\rho| t^2}\,\mathrm{d}t < \infty,$$

the assumption (2.5) is fulfilled for $m = 1$ and Theorem 2.3 implies $h_n(x) \to h(x)$.

Let λ_n be the characteristic function corresponding to h_n. Using (2.4) we obtain

$$\begin{aligned}\lambda_n(t) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} h_n(x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\left[\int_{\mathbb{R}} f(x-\rho u)h_{n-1}(u)\,\mathrm{d}u\right]\mathrm{d}x\\ &= \int_{\mathbb{R}} h_{n-1}(u)\left[\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} f(x-\rho u)\,\mathrm{d}x\right]\mathrm{d}u\\ &= \int_{\mathbb{R}} h_{n-1}(u)\left[\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\rho u+\mathrm{i}ty} f(y)\,\mathrm{d}y\right]\mathrm{d}u\\ &= \psi(t)\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\rho u} h_{n-1}(u)\,\mathrm{d}u = \psi(t)\lambda_{n-1}(\rho t).\end{aligned}$$

Thus

$$\lambda_n(t) = \psi(t)\psi(\rho t)\psi(\rho^2 t)\cdots\psi(\rho^{n-1}t)\lambda_0(\rho^n t) = \prod_{j=0}^{n}\psi(\rho^j t). \tag{2.7}$$

Let $n \geq 2$. Then $|\lambda_n(t)| \leq \psi(t)\psi(\rho t) \in L^1(\mathbb{R})$ and we can use inverse formula

$$h_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx}\lambda_n(t)\,\mathrm{d}t. \tag{2.8}$$

Hence

$$\begin{aligned}|h_{n+1}(x)-h_n(x)| &= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}\lambda_{n+1}(t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}\lambda_n(t)\,\mathrm{d}t\right|\\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}[\lambda_{n+1}(t)-\lambda_n(t)]\,\mathrm{d}t\right|\\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}}|\lambda_{n+1}(t)-\lambda_n(t)|\,\mathrm{d}t \end{aligned}\tag{2.9}$$

and from (2.7) and (2.9) we get

$$\begin{aligned}|h_{n+1}(x)-h_n(x)| &\leq \frac{1}{2\pi}\int_{\mathbb{R}}\left|\left[\prod_{j=0}^{n}\psi(\rho^j t)\right][\psi(\rho^{n+1}t)-1]\right|\mathrm{d}t\\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}}|\psi(t)|\cdot|\psi(\rho t)|\cdot|\psi(\rho^2 t)|\cdot|\psi(\rho^{n+1}t)-1|\,\mathrm{d}t.\end{aligned}\tag{2.10}$$

for any $x \in \mathbb{R}$. Further, for $t \neq 0$,

$$\begin{aligned}|\psi(\rho^{n+1}t)-1| &= \sqrt{\frac{\mathrm{i}(1-\mathrm{e}^{\mathrm{i}\rho^{n+1}t})-\rho^{n+1}t}{\rho^{n+1}t}\cdot\frac{-\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}\rho^{n+1}t})-\rho^{n+1}t}{\rho^{n+1}t}}\\ &= \frac{\sqrt{[\cos(\rho^{n+1}t)-1]^2+[\sin(\rho^{n+1}t)-\rho^{n+1}t]^2}}{|\rho^{n+1}t|}.\end{aligned}\tag{2.11}$$

We will show that $g(x) = x^{-4}\left[(\cos x - 1)^2 + (\sin x - x)^2\right] \leq 1/4$ for any $x \neq 0$. It suffices to prove that $1/4 - g(x) > 0$ for any $x > 0$ since $g(x)$ is an even function. First, define $l(x) = x^2 + 2(\cos x - 1)$. We can see that $l(0) = 0$ and $l'(x) = 2(x - \sin x) > 0$ for any $x > 0$. Thus, $l(x)$ is increasing and therefore positive on $(0, \infty)$. Obviously

$$\frac{1}{4} - g(x) = \frac{x^4 - 8 + 8\cos x + 8x\sin x - 4x^2}{4x^4} =: \frac{k(x)}{4x^4}.$$

Then, $k(0) = 0$ and $k'(x) = x[4x^2 + 8(\cos x - 1)] = 4x \cdot l(x) > 0$ for x positive, which implies $k(x) > 0$ for $x > 0$. Hence, $g(x) < 1/4$ for any $x \neq 0$. Using this result in (2.11), we get

$$|\psi(\rho^{n+1}t) - 1| \leq |\rho^{n+1}t|/2. \tag{2.12}$$

Substitution of (2.6) and (2.12) to (2.10) yields

$$\begin{aligned} |h_{n+1}(x) - h_n(x)| &\leq \frac{1}{\pi}\int_0^2 \frac{|\rho^{n+1}|t}{2}\,\mathrm{d}t + \frac{1}{\pi}\int_2^\infty \frac{8}{|\rho^3|t^3} \cdot \frac{|\rho^{n+1}|t}{2}\,\mathrm{d}t \\ &= \frac{|\rho|^{n+1}}{\pi} + \frac{2|\rho|^{n-2}}{\pi} \leq \frac{3}{\pi}|\rho|^{n-2}. \end{aligned}$$

Moreover,

$$|h_n(x) - h(x)| \leq \sum_{k=n}^{\infty} |h_{k+1}(x) - h_k(x)| \leq \frac{3}{\pi}\sum_{k=n}^{\infty} |\rho|^{k-2} = \frac{3}{\pi(1 - |\rho|)}|\rho|^{n-2}.$$

□

Now we will show that under some rather mild conditions on the density f, the sequence h_n converges to the stationary density h uniformly w.r.t. x and geometrically fast. First, we state some preliminary results.

Definition 2.7. Let $I \subset \mathbb{R}$ be an interval. We say that function $f : I \to \mathbb{R}$ is *absolutely continuous* on I, if f is absolutely continuous on every closed subinterval $J \subset I$.

Definition 2.8. Let $\mathcal{S}$ be a system of functions $f \in L^1(\mathbb{R})$ for which there exist numbers $-\infty = a_{-1} < a_0 < a_1 < \cdots < a_r < a_{r+1} = \infty$, $r \in \mathbb{N}$, such that

(i) function f is absolutely continuous on (a_{j-1}, a_j), $j = 0, \ldots, r+1$;

(ii) $f' \in L^1(\mathbb{R})$.

Then $\mathcal{S}$ is called a system of piecewise smooth functions.

Theorem 2.9. *Let $f \in \mathcal{S}$. Let $\hat{f}(t)$ denote the characteristic function corresponding to f. Then there exists number $b > 0$ such that*

$$|\hat{f}(t)| \leq \frac{b}{1+|t|}, \quad t \in \mathbb{R}.$$

Proof. See Appendix, Theorem 2.29. □

Now we are ready to prove the main result.

Theorem 2.10. *Let $\{X_t\}$ be the AR(1) process given by (2.1) with $\rho \neq 0$ and η_1 such that $\mathsf{E}\,|\eta_1| < \infty$. Further, let the density f of η_1 be piecewise smooth. Then $h_n(x) \to h(x)$ uniformly as $n \to \infty$ and for $n \geq 2$*

$$|h_n(x) - h(x)| \leq K \frac{\ln|\rho|}{(\rho^2-1)(1-|\rho|)} |\rho|^n$$

for every $x \in \mathbb{R}$ and some $K \in \mathbb{R}$, independent of ρ.

Proof. Since f is piecewise smooth, Theorem 2.9 yields

$$|\psi(\rho^k t)| \leq \frac{b}{1+|\rho^k t|} \tag{2.13}$$

for every $k = 0, 1, 2, \ldots$ and some $b \in \mathbb{R}$. The condition (2.5) is fulfilled for $m = 1$ since

$$\int_{\mathbb{R}} |\psi(t)\psi(\rho t)|\,\mathrm{d}t \leq \int_{\mathbb{R}} \frac{b^2}{(1+|t|)(1+|\rho t|)}\,\mathrm{d}t < \infty$$

and therefore $h_n(x) \to h(x)$ pointwise as $n \to \infty$. Similarly to (2.10), for $n \geq 2$ we have

$$|h_{n+1}(x) - h_n(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\psi(t)| \cdot |\psi(\rho t)| \cdot |\psi(\rho^2 t)| \cdot |\psi(\rho^{n+1} t) - 1|\,\mathrm{d}t. \tag{2.14}$$

Further,

$$\left| \frac{\partial}{\partial t} [\mathrm{e}^{\mathrm{i}tx} f(x)] \right| = |x| f(x) \in L^1(\mathbb{R})$$

(because of finitness of the first absolute moment of η_1). Therefore $\psi(t) = \int \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x$ is differentiable and thus Lipschitz with some constant, say L. This implies that $|\psi(t) - 1|$ is Lipschitz with the same constant since $\big||x| - |y|\big| \leq |x - y|$ for any x, y. Using this fact at point 0, we get

$$|\psi(\rho^{n+1} t) - 1| \leq L |\rho^{n+1} t|. \tag{2.15}$$

Combination of (2.14), (2.13) and (2.15) gives

$$
\begin{aligned}
|h_{n+1}(x) - h_n(x)| &\le \frac{1}{2\pi}\int_{\mathbb{R}} \frac{b}{1+|t|}\cdot\frac{b}{1+|\rho t|}\cdot\frac{b}{1+|\rho^2 t|}\cdot L|\rho^{n+1}t|\,\mathrm{d}t \\
&= \frac{b^3L|\rho|^{n+1}}{2\pi}\int_{\mathbb{R}} \frac{|t|}{(1+|t|)(1+|\rho t|)(1+|\rho^2 t|)}\,\mathrm{d}t \\
&= \frac{b^3L|\rho|^{n+1}}{2\pi}\cdot 2\frac{\ln\rho^2(1-|\rho|)+\ln|\rho|(\rho^2-1)}{2(\rho^2-\rho^4)+|\rho|(\rho^4-1)} \\
&= \frac{b^3L|\rho|^{n+1}}{\pi}\cdot\frac{\ln|\rho|}{|\rho|(\rho^2-1)}.
\end{aligned}
$$

Thus, the sequence $\{h_n\}$ has a limit h and

$$
|h_n(x) - h(x)| \le \sum_{k=n}^{\infty} |h_k(x) - h_{k+1}(x)| \le \frac{b^3L}{\pi}\cdot\frac{\ln|\rho|}{(\rho^2-1)(1-|\rho|)}\cdot|\rho|^n.
$$

Choosing $K = b^3L/\pi$ completes the proof. □

2.1.2 Haiman's procedure

Consider again an AR(1) process (2.1) with $\rho \in (0,1)$. Suppose that η_t have a density $f(x)$ with respect to Lebesgue measure and its support $\operatorname{spt} f = \operatorname{cl}\{x : f(x) > 0\}$ is a compact subset of the interval $[0,1]$. Further, assume that the derivative $f'(x)$ exists for every $x \in [0,1]$ and $\sup_{x\in(0,1)} |f'(x)| \le D$ for some constant $D \in \mathbb{R}$.

It is obvious that the stationary distribution of $\{X_t\}$ defined by (2.1) is the same as that of the random variable $\sum_{i=1}^{\infty} \eta_i \rho^{i-1}$. Consider the sequence of partial sums $Y_n = \eta_1 + \rho\eta_2 + \cdots \rho^n \eta_{n+1}$, $n \ge 1$. Let $h_n(u)$ denote the density of Y_n and define $h_0(u) = f(u)$. Haiman [12] proved the following theorem.

Theorem 2.11. *Under the above assumptions, the stationary distribution of $\{X_t, t \in \mathbb{Z}\}$ has a density $h(x)$ with $\operatorname{spt} h \subseteq [0, 1/(1-\rho)]$ such that $h \in C^\infty$. Furthermore, we have, for $n \ge 1$,*

$$
\sup_{0\le x\le 1/(1-\rho)} |h_n(x) - h(x)| \le \frac{D\rho^{n+1}}{1-\rho}. \tag{2.16}
$$

Remark 2.12. The author introduced factor 2 on the right hand side of the last inequality, which is not necessary.

This theorem gives not only the convergence of sequence $h_n(x)$ to the stationary density $h(x)$, but the speed of convergence, which is geometrical, as well. On the other hand, its assumptions are quite strong and most of widely used distributions do not satisfy them.

However, in the following we relax some assumptions mentioned above, in particular the finitness of the support of density f and its differentiability. Moreover, we improve the boundary on the right hand side of (2.16). First, we deal with the support and positivity of parameter ρ.

Theorem 2.13. *Let $\{X_t\}$ be the AR(1) process* (2.1)*, $\rho \in (-1, 1)$. Let η_1 have a density $f(x)$ such that $\mathsf{E}\,|\eta_1| < \infty$. Assume that the derivative $f'(x)$ exists for every $x \in \mathbb{R}$ and $\sup_x |f'(x)| \le D$ for some constant $D \in \mathbb{R}$. Then the stationary distribution of $\{X_t\}$ has a density $h(x) \in C^\infty$ and for $n \ge 1$ we have*

$$\sup_{x\in\mathbb{R}} |h_n(x) - h(x)| \le \frac{D|\rho|^{n+1}\,\mathsf{E}\,|\eta_1|}{1-|\rho|}.$$

Proof. Stronger assertion will be proved later, see Theorem 2.20. We only need to mention that the first important step of the proof is the change of integration and derivative in

$$\frac{\mathrm{d}}{\mathrm{d}u} h_1(u) = \frac{\mathrm{d}}{\mathrm{d}u} \int f(u-x) \frac{1}{|\rho|} f\left(\frac{x}{\rho}\right) \mathrm{d}x \tag{2.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u} h_n(u) = \frac{\mathrm{d}}{\mathrm{d}u} \int h_{n-1}(u-x) \frac{1}{|\rho|} f\left(\frac{x}{\rho}\right) \mathrm{d}x, \qquad n \ge 2. \tag{2.18}$$

□

In the next step we will deal with the assumption of differentiability of the density f. It allows to change the order of integration and derivative in (2.17) and (2.18). However, to be allowed to do that, it suffices to assume that $h_m(x)$ is differentiable on $\mathbb{R}$ for some $m \ge 0$ (and thus for every $k > m$) and the assertions of Theorem 2.13 will hold for every $n \ge m$ (the rest of the proof remains the same).

Theorem 2.14. *Let $\{X_t\}$ be the AR(1) process* (2.1)*, $\rho \in (-1, 1)$. Let η_1 have a density $f(x)$ such that $\mathsf{E}\,|\eta_1| < \infty$. Assume that there exists an integer $m \ge 0$ such that $\eta_1 + \rho\eta_2 + \cdots + \rho^m \eta_{m+1}$ has a density $h_m(x)$ differentiable on $\mathbb{R}$, $\sup_x |h'_m(x)| \le D$ for some constant $D \in \mathbb{R}$. Then the stationary distribution of $\{X_t\}$ has a density $h(x) \in C^\infty$ and for $n \ge m$ we have*

$$\sup_{x\in\mathbb{R}} |h_n(x) - h(x)| \le \frac{D|\rho|^{n+1}\,\mathsf{E}\,|\eta_1|}{1-|\rho|}.$$

Consider the case of uniformly distributed innovations, i.e. $\eta_t \overset{\text{iid}}{\sim} \mathsf{U}([0,1])$. Their density $f(x) = \chi_{[0,1]}(x)$ is not differentiable at points 0 and 1 and thus the assumptions of Theorem 2.13 are not fulfilled. However, after some algebra, it is possible to show that

$$h_1(x;\rho) = \begin{cases} x/\rho & x \in [0,\rho], \\ 1 & x \in [\rho,1], \\ (1+\rho-x)/\rho & x \in [1,1+\rho], \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_2(x;\rho) = \begin{cases} x^2/(2\rho^3), & x \in [0,\rho^2], \\ (2x-\rho^2)/(2\rho), & x \in [\rho^2,\rho], \\ (2\rho^2 x + 2\rho x - x^2 - \rho^2 - \rho^4)/(2\rho^3), & x \in [\rho,\rho+\rho^2], \\ 1, & x \in [\rho+\rho^2,1], \\ (2x - x^2 - 1 + 2\rho^3)/(2\rho^3), & x \in [1,1+\rho^2], \\ (2\rho + 2 + \rho^2 - 2x)/(2\rho), & x \in [1+\rho^2,1+\rho], \\ (2\rho + 3\rho^2 + 2\rho^3 - 2\rho x + 1 & \\ \quad - 2x + \rho^4 - 2\rho^2 x + x^2)/(2\rho^3), & x \in [1+\rho,1+\rho+\rho^2], \\ 0 & \text{otherwise} \end{cases}$$

for ρ such that $\rho + \rho^2 \leq 1$ and

$$h_2(x;\rho) = \begin{cases} (x^2)/(2\rho^3), & x \in [0,\rho^2], \\ (2x-\rho^2)/(2\rho), & x \in [\rho^2,\rho], \\ (2\rho^2 x + 2\rho x - x^2 - \rho^2 - \rho^4)/(2\rho^3), & x \in [\rho,1], \\ (2\rho^2 x + 2\rho x + 2x - 2x^2 & \\ \quad - \rho^4 - \rho^2 - 1)/(2\rho^3), & x \in [1,\rho+\rho^2], \\ (2x - x^2 - 1 + 2\rho^3)/(2\rho^3), & x \in [\rho+\rho^2,1+\rho^2], \\ (2\rho + 2 + \rho^2 - 2x)/(2\rho), & x \in [1+\rho^2,1+\rho], \\ (2\rho + 3\rho^2 + 2\rho^3 - 2\rho x + 1 & \\ \quad - 2x + \rho^4 - 2\rho^2 x + x^2)/(2\rho^3), & x \in [1+\rho,1+\rho+\rho^2], \\ 0 & \text{otherwise} \end{cases}$$

for ρ such that $\rho + \rho^2 > 1$.

Thus, $h_2(x)$ is a (continuously) differentiable function for every $x \in \mathbb{R}$, $\sup_{x\in\mathbb{R}} |h_2'(x)| \leq \rho^{-1}$ and the assumptions of Theorem 2.14 hold for $m = 2$. Figure 1 shows densities h_1 and h_2 for $\rho = 0.7$.

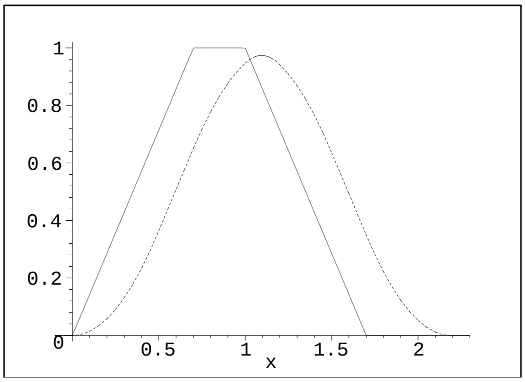

Figure 1: Densities $h_1(x;\rho)$ (solid) and $h_2(x;\rho)$ (dashed) for $\rho = 0.7$

Notice that h_1 and h_2 are densities of $\eta_1 + \rho\eta_2$ and $\eta_1 + \rho\eta_2 + \rho^2\eta_3$, respectively, and thus, they are convolutions of two and three densities, respectively. Density h_0 is discontinuous, h_1 is continuous, but not differentiable, h_2 is continuously differentiable. This suggests that the convolution has a "smoothing effect". In the Appendix we study the behaviour of convolution and derive conditions under which convolution of n densities is $(n-2)$-times continuously differentiable and the $(n-2)$th derivative is bounded (see also [5]). In such case, assumptions of Theorem 2.14 are satisfied for m equal to (at most) two, since h_2 is the convolution of three densities.

Theorem 2.15. *Let $n \geq 3$ and let $f_1, \ldots, f_n$ be piecewise smooth densities (see Definition 2.8). Then $f_1 * \cdots * f_n \in C^{n-2}(\mathbb{R})$ and*

$$\left| \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} (f_1 * \cdots * f_n)(x) \right| \leq D$$

for some $D \in \mathbb{R}$ and every $x \in \mathbb{R}$.

Proof. See Appendix, Theorem 2.30. □

Obviously, if a density f of random variable η_1 is piecewise smooth, then the density of $\rho^k\eta_1$, $k = 1, 2, \ldots$, is piecewise smooth as well. According to Theorem 2.15, the density h_2 of $\eta_1 + \rho\eta_2 + \rho^2\eta_3$ is a continuously differentiable function with bounded derivative.

Theorem 2.16. *Let $\{X_t\}$ be the AR(1) process* (2.1), $\rho \in (-1, 1)$. *Let η_1 have a piecewise smooth density $f(x)$ such that $\mathsf{E}\,|\eta_1| < \infty$. Let h_n denote the* *density of the random variable* $\eta_1 + \rho\eta_2 + \cdots + \rho^n \eta_{n+1}$. *Then the stationary distribution of* $\{X_t\}$ *has a density* $h(x) \in C^\infty$ *and for* $n \geq 2$ *we have*

$$\sup_{x \in \mathbb{R}} |h_n(x) - h(x)| \leq \frac{D|\rho|^{n+1} \, \mathsf{E}\,|\eta_1|}{1 - |\rho|},$$

where $D = \sup_x |h_2'(x)|$.

Proof. The assertion follows from Theorems 2.14 and 2.15. □

2.2 Model AR(2)

The iterative method for calculating stationary density (Theorem 2.3) can be generalized to autoregressive process of higher order (see [4]). Here we present a derivation for AR(2) model.

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary AR(2) process defined by

$$X_t = \rho_1 X_{t-1} + \rho_2 X_{t-2} + \eta_t, \tag{2.19}$$

where $\{\eta_t\}$ are i.i.d. random variables with denisty f and finite second moment. Let ψ be the characteristic function of η_t and

$$\boldsymbol{F} = \begin{pmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{pmatrix}.$$

It is known that

$$X_t = \sum_{j=0}^{\infty} a_j \eta_{t-j},$$

where a_j denotes the (1,1)-element of the matrix $\boldsymbol{F}^j$ (see [13], p. 57). It follows from the assumption of stationarity that all eigenvalues of $\boldsymbol{F}$ lie inside the unit circle and thus the series (2.19) converges in the quadratic mean. If we define $\boldsymbol{c} = (1,0)^\top$ then

$$a_j = \boldsymbol{c}^\top \boldsymbol{F}^j \boldsymbol{c} = \boldsymbol{c}^\top \left(\boldsymbol{F}^\top\right)^j \boldsymbol{c}$$

and the characteristic function λ of X_t is given by

$$\lambda(t) = \prod_{j=0}^{\infty} \psi(ta_j) = \prod_{j=0}^{\infty} \psi\left(t\boldsymbol{c}^\top \left(\boldsymbol{F}^\top\right)^j \boldsymbol{c}\right). \tag{2.20}$$

Since we assumed that η_t have a density, it follows from (2.19) that the random vector $(X_t, X_{t-1})^\top$ has a joint denisty, say $q(x,y)$. The stationary density of $\{X_t\}$ is $h(x) = \int q(x, y)\,\mathrm{d}y$. Since $\{X_t\}$ is stationary, the vector $(X_{t-1}, X_{t-2})^\top$ has also density q. The joint denisty of $(X_t, X_{t-1}, X_{t-2})^\top$ is the product of the conditional density of X_t given $(X_{t-1}, X_{t-2})^\top$ and the joint density of $(X_{t-1}, X_{t-2})^\top$, i.e. $q(x_{t-1}, x_{t-2})f(x_t - \rho_1 x_{t-1} - \rho_2 x_{t-2})$. This leads to the integral equation

$$q(x_t, x_{t-1}) = \int q(x_{t-1}, x_{t-2}) f(x_t - \rho_1 x_{t-1} - \rho_2 x_{t-2})\,\mathrm{d}x_{t-2}. \tag{2.21}$$

Let $q_0(y, z)$ be an arbitrary joint density. Formula (2.21) suggests that a method for calculating q can be based on the recurrent relation

$$q_n(x, y) = \int q_{n-1}(y, z) f(x - \rho_1 y - \rho_2 z)\,\mathrm{d}z, \qquad n = 1, 2, \ldots \tag{2.22}$$

It was proved that under some conditions concerning ψ and $\boldsymbol{F}$ the functions q_n converge to q pointwise.

Theorem 2.17. *Let λ_n be the characteristic function corresponding to q_n. Then for arbitrary $\boldsymbol{t} = (t_1, t_2)^\top$ we have $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$.*

Proof. Using (2.22) we get

$$\begin{aligned}
\lambda_n(t_1, t_2) &= \iint \mathrm{e}^{\mathrm{i}t_1 x + \mathrm{i}t_2 x} q_n(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iiint \mathrm{e}^{\mathrm{i}t_1 x + \mathrm{i}t_2 y} q_{n-1}(y, z) f(x - \rho_1 y - \rho_2 z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&= \iiint \mathrm{e}^{\mathrm{i}t_1(w + \rho_1 y + \rho_2 z) + \mathrm{i}t_2 y} q_{n-1}(y, z) f(w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}y \\
&= \iint \mathrm{e}^{\mathrm{i}(t_1\rho_1 + t_2)y + \mathrm{i}t_1\rho_2 z} q_{n-1}(y, z)\,\mathrm{d}y\,\mathrm{d}z \int \mathrm{e}^{\mathrm{i}t_1 w} f(w)\,\mathrm{d}w \\
&= \lambda_{n-1}(t_1\rho_1 + t_2, t_1\rho_2)\psi(t_1) \\
&= \lambda_{n-1}(\boldsymbol{F}^\top \boldsymbol{t})\psi(\boldsymbol{c}^\top \boldsymbol{t}).
\end{aligned}$$

This gives

$$\lambda_n(\boldsymbol{t}) = \psi(\boldsymbol{c}^\top \boldsymbol{t})\psi(\boldsymbol{c}^\top \boldsymbol{F}^\top \boldsymbol{t}) \cdots \psi\left(\boldsymbol{c}^\top \left(\boldsymbol{F}^\top\right)^{n-1} \boldsymbol{t}\right) \lambda_0\left(\left(\boldsymbol{F}^\top\right)^n \boldsymbol{t}\right).$$

Since $\boldsymbol{F}^n \to \boldsymbol{0}$ as $n \to \infty$, we have $\lambda_0\left((\boldsymbol{F}^n)^\top \boldsymbol{t}\right) \to 1$ and from (2.20) it follows that $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$. □

Theorem 2.18. *Let q_0 be a density. Assume that there exists an integer $m \geq 0$ such that*

$$\iint \left|\psi(\boldsymbol{c}^\top \boldsymbol{t})\psi(\boldsymbol{c}^\top \boldsymbol{F}^\top \boldsymbol{t}) \cdots \psi\left(\boldsymbol{c}^\top \left(\boldsymbol{F}^\top\right)^m \boldsymbol{t}\right)\right|\,\mathrm{d}t_1\,\mathrm{d}t_2 < \infty.$$

Then $q_n(x, y) \to q(x, y)$ for all (x, y) as $n \to \infty$.

Proof. For $n \geq m$ we have

$$|\lambda_n(t_1,t_2)| \leq \left|\psi(\boldsymbol{c}^\top \boldsymbol{t})\psi(\boldsymbol{c}^\top \boldsymbol{F}^\top \boldsymbol{t})\cdots\psi\left(\boldsymbol{c}^\top \left(\boldsymbol{F}^\top\right)^m \boldsymbol{t}\right)\right|$$

and thus

$$\iint |\lambda_n(t_1,t_2)|\,\mathrm{d}t_1\,\mathrm{d}t_2 < \infty.$$

Then $q_n(x,y)$ is continuous and

$$q_n(x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(xt_1+yt_2)}\lambda_n(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

(c.f. [11], formula 7.12). Theorem 2.17 and Lebesgue theorem imply

$$\lim_{n\to\infty}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(xt_1+yt_2)}\lambda_n(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 = q(x,y).$$

□

Example 2.19. Consider the AR(2) model (2.19) with $\rho_1 = 0.7$, $\rho_2 = -0.1$ and $\eta_t \sim \mathsf{Exp}(1)$. We choose

$$q_0(x,y) = \begin{cases} \exp\{-x-y\} & \text{if } x \geq 0, y \geq 0,\\ 0 & \text{otherwise,}\end{cases}$$

and compute joint densities $q_i(x,y)$, $i = 1,2,3$, according to (2.22). Graphs of these functions can be found in Figures 2, 3 and 4.

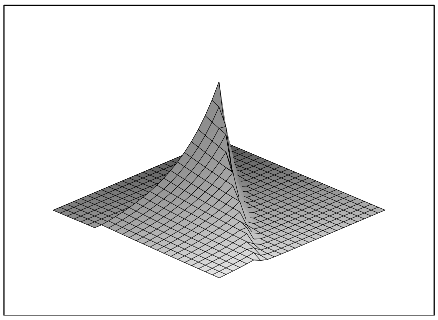

Fig. 2
Function $q_1(x,y)$

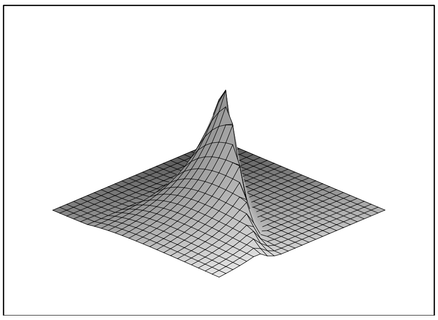

Fig. 3
Function $q_2(x,y)$

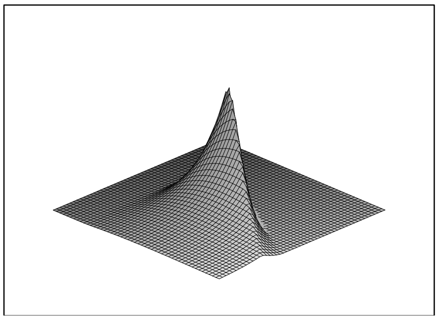

Fig. 4
Function $q_3(x, y)$

2.3 General linear process

Now we extend Haiman's approach to general causal linear processes. Let $\{X_t\}_{t\in\mathbb{Z}}$ be stationary process defined by

$$X_t = \sum_{j=0}^{\infty} c_j \eta_{t-j}, \qquad t \in \mathbb{Z}, \tag{2.23}$$

where $\{\eta_t\}_{t\in\mathbb{Z}}$ are i.i.d. random variables with denisty f and finite second moment. Constants $c_j \in \mathbb{R}$ are assumed to satisfy conditions $\sum_{j=0}^{\infty} |c_j| < \infty$, $c_0 = 1$ and infinitely many of them are nonzero. Then we know that the sum on the right hand side of (2.23) converges almost surely.

Obviously, the stationary distribution of (2.23) is the same as that of the random variable $\sum_{j=0}^{\infty} c_j \eta_j$. Consider the sequence of partial sums

$$Y_n = c_0\eta_0 + \cdots + c_n\eta_n.$$

Let $h_n(u)$ denote the density of Y_n and define $h_0 = f$.

Theorem 2.20. *Let $\{X_t\}$ be a linear process defined by (2.23). Let η_1 have a piecewise smooth density $f(x)$. Let ν be the smallest integer such that among the coefficients $c_0, c_1, \ldots, c_\nu$ there are three nonzero. Then the stationary distribution of $\{X_t\}$ has a density $h \in C^\infty$ and for $n \geq \nu$ we have*

$$\sup_{x\in\mathbb{R}} |h_n(x) - h(x)| \leq D\,\mathsf{E}\,|\eta_1| \sum_{k=n}^{\infty} |c_{k+1}|,$$

where $D = \sup_x |h'_\nu(x)| < \infty$.

Proof. Density h_ν is the convolution of three densities which are all piecewise smooth. Therefore it is continuously differentiable and $|h'_\nu(x)| \le D$ for some $D \in \mathbb{R}$ and every $x \in \mathbb{R}$ according to Theorem 2.15.

If $c_{\nu+1} \neq 0$, we get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}u}h_{\nu+1}(u)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}u}\int h_\nu(u-x)\frac{1}{|c_{\nu+1}|}f\left(\frac{x}{c_{\nu+1}}\right)\mathrm{d}x\right| \\ &\le \int \left|\frac{\mathrm{d}}{\mathrm{d}u}h_\nu(u-x)\right|\frac{1}{|c_{\nu+1}|}f\left(\frac{x}{c_{\nu+1}}\right)\mathrm{d}x \le D. \qquad (2.24)\end{aligned}$$

If $c_{\nu+1} = 0$, then $h_{\nu+1} = h_\nu$ and $|h'_{\nu+1}(x)| = |h'_\nu(x)| \le D$. Thus, $|h'_{\nu+1}(x)| \le D$ for every $c_{\nu+1}$. Similarly, by complete induction, $|h'_k(x)| \le D$ for every $k \ge \nu + 1$.

Choose $n \ge \nu$. Assume that $c_{n+1} \neq 0$. We have

$$h_{n+1}(u) = \int h_n(y)\cdot\frac{1}{|c_{n+1}|}f\left(\frac{u-y}{c_{n+1}}\right)\mathrm{d}y. \qquad (2.25)$$

Mean value theorem gives

$$h_n(y) = h_n(u) + (y-u)\frac{\mathrm{d}}{\mathrm{d}u}h_n[u+\theta_u(y-u)] \qquad (2.26)$$

for some $\theta_u \in [0,1]$. Substituting (2.26) into (2.25), we get

$$\begin{aligned}h_{n+1}(u) = h_n(u)&\int \frac{1}{|c_{n+1}|}f\left(\frac{u-y}{c_{n+1}}\right)\mathrm{d}y \\ &+ \int \frac{y-u}{|c_{n+1}|}\cdot f\left(\frac{u-y}{c_{n+1}}\right)\frac{\mathrm{d}}{\mathrm{d}u}h_n[u+\theta_u(y-u)]\,\mathrm{d}y \\ = h_n(u) &+ \int \frac{y-u}{|c_{n+1}|}\cdot f\left(\frac{u-y}{c_{n+1}}\right)\frac{\mathrm{d}}{\mathrm{d}u}h_n[u+\theta_u(y-u)]\,\mathrm{d}y. \qquad (2.27)\end{aligned}$$

Thus, we have

$$\begin{aligned}|h_{n+1}(u) - h_n(u)| &= \left|\int \frac{y-u}{|c_{n+1}|}f\left(\frac{u-y}{c_{n+1}}\right)\frac{\mathrm{d}}{\mathrm{d}u}h_n[u+\theta_u(y-u)]\,\mathrm{d}y\right| \\ &\le D\int\left|\frac{y-u}{c_{n+1}}\right|\cdot f\left(\frac{u-y}{c_{n+1}}\right)\mathrm{d}y \\ &= D|c_{n+1}|\int |z|f(z)\,\mathrm{d}z = D|c_{n+1}|\,\mathsf{E}\,|\eta_1|.\end{aligned}$$

If $c_{n+1} = 0$ then $h_{n+1} = h_n$ and therefore

$$|h_{n+1}(u) - h_n(u)| \le D|c_{n+1}|\,\mathsf{E}\,|\eta_1|$$

for every c_{n+1}. This implies that the sequence $\{h_n\}$ is Cauchy with respect to supremum norm and thus, it has a limit h. Finally,

$$|h_n(x) - h(x)| \le \sum_{k=n}^{\infty} |h_k(x) - h_{k+1}(x)| \le D \,\mathsf{E}\, |\eta_1| \sum_{k=n}^{\infty} |c_{k+1}|.$$

□

Special case of linear process is an ARMA(p, q) process. It is defined by equation

$$X_t + \varphi_1 X_{t-1} + \cdots + \varphi_p X_{t-p} = \eta_t + \vartheta_1 \eta_{t-1} + \cdots + \vartheta_q \eta_{t-q},$$

where the polynomials $A(\lambda) = \lambda^p + \varphi_1 \lambda^{p-1} + \cdots + \varphi_p$ and $B(\lambda) = \lambda^q + \vartheta_1 \lambda^{q-1} + \cdots + \vartheta_q$ have no common roots. Assume that that the roots of $A(\lambda)$ lie inside the unit circle. Then the process $\{X_t\}$ is stationary and expressible as a linear causal process (2.23). We also know that the coefficients c_j decay exponentially, say $|c_j| \le K|\gamma|^j$, where $K > 0$ and $\gamma \in (-1, 1)$ are some real constants.

If the density of η_1 is piecewise smooth, we can apply Theorem 2.20 and we get

$$\sup_{x \in \mathbb{R}} |h_n(x) - h(x)| \le \frac{DK \,\mathsf{E}\, |\eta_1|}{1 - |\gamma|} |\gamma|^{n+1}.$$

Again, we showed that the densities h_n converge to the stationary density h uniformly and geometrically fast.

2.4 Appendix – Properties of convolution

2.4.1 Criterion for differentiability of characteristic functions

Let f be the density of a distribution with respect to Lebesgue measure. Then the characteristic function $\hat{f}$ corresponding to this density is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} f(x) \,\mathrm{d}x, \quad t \in \mathbb{R}.$$

Let $h = f * g$ be the convolution of densities f and g. It is well known that

$$\hat{h}(t) = \hat{f}(t)\hat{g}(t), \qquad t \in \mathbb{R}. \tag{2.28}$$

If density f is continuous and its characteristic function $\hat{f}$ absolutely integrable ($\hat{f} \in L^1(\mathbb{R})$), then according to characteristic function's inverse theorem

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} \hat{f}(t) \,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{2.29}$$

Let $C^k(\mathbb{R})$ denote the set of functions with continuous kth derivative.

Lemma 2.21. *Let the density f be continuous, $k \in \mathbb{N}$, $a \in \mathbb{R}$ and*

$$|\hat{f}(t)| \le \frac{a}{(1+|t|)^{k+2}}, \quad t \in \mathbb{R}.$$

Then $f \in C^k(\mathbb{R})$ and $|f^{(k)}(x)| \le D$ for some $D \in \mathbb{R}$ and every $x \in \mathbb{R}$.

Proof. It is obvious that $\hat{f} \in L^1(\mathbb{R})$. Define

$$h(t) = \frac{a|t|^k}{(1+|t|)^{k+2}}, \quad t \in \mathbb{R}.$$

Then $h \in L^1(\mathbb{R})$ and

$$\left| \frac{\partial^k}{\partial x^k} \left[\mathrm{e}^{-\mathrm{i}tx} \hat{f}(t) \right] \right| = |(-\mathrm{i}t)^k \mathrm{e}^{-\mathrm{i}tx} \hat{f}(t)| = |t|^k \cdot |\hat{f}(t)| \le h(t)$$

for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$. From (2.29), we get $f \in C^k(\mathbb{R})$. Moreover,

$$|f^{(k)}(x)| \le \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\partial^k}{\partial x^k} \left[\mathrm{e}^{-\mathrm{i}tx} \hat{f}(t) \right] \right| \mathrm{d}t \le \frac{1}{2\pi} \int_{\mathbb{R}} h(t)\, \mathrm{d}t.$$

Choosing $D = \|h\|_1/(2\pi)$ completes the proof. □

2.4.2 Continuity of convolution

Lemma 2.22. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function such that*

(i) φ is continuous on $\mathbb{R}$,

(ii) there exists $a > 1$ such that $\varphi(x) = 0$ for $x \notin (-a+1, a-1)$.

Then

$$\lim_{s \to 0} \int_{-\infty}^{\infty} |\varphi(s+t) - \varphi(t)|\, \mathrm{d}t = 0. \tag{2.30}$$

Proof. Define $\omega_s(t) = |\varphi(s+t) - \varphi(t)|$, $s, t \in \mathbb{R}$. Let $\varepsilon > 0$. Since function φ is uniformly continuous on $\mathbb{R}$, there exists $\delta \in (0,1)$ such that for every $s \in (-\delta, \delta)$ and every $t \in \mathbb{R}$ we have $\omega_s(t) \le \varepsilon$. Since $\delta \in (0,1)$, $\omega_s(x) = 0$ for $|x| \ge a$. Therefore

$$\int_{-\infty}^{\infty} \omega_s(t) \mathrm{d}t = \int_{-a}^{a} \omega_s(t) \mathrm{d}t \le 2a\varepsilon.$$

□

Lemma 2.23. *Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function and $\varepsilon > 0$. Then there exists a function φ satisfying conditions (i) and (ii) in Lemma 2.22 such that*

$$\int_{-\infty}^{\infty} |f(t) - \varphi(t)| \,\mathrm{d}t \le \varepsilon. \tag{2.31}$$

Proof. See [21], Theorem 3.14. □

Lemma 2.24. *Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function. Then*

$$\lim_{s\to 0} \int_{-\infty}^{\infty} |f(s+t) - f(t)| \,\mathrm{d}t = 0. \tag{2.32}$$

Proof. Fix $\varepsilon > 0$ and choose φ by Lemma 2.23. Then

$$\int_{-\infty}^{\infty} |f(s+t) - \varphi(s+t)| \,\mathrm{d}t \le \varepsilon, \quad s \in \mathbb{R}, \tag{2.33}$$

holds. It follows from (2.30) that there exists $\delta > 0$ such that

$$\int_{-\infty}^{\infty} |\varphi(s+t) - \varphi(t)| \,\mathrm{d}t \le \varepsilon \tag{2.34}$$

whenever $|s| < \delta$. From the triangle inequality, (2.31), (2.33) and (2.34), we get the inequality

$$\begin{aligned}\int_{-\infty}^{\infty} |f(s+t) - f(t)| \,\mathrm{d}t \le& \int_{-\infty}^{\infty} |f(s+t) - \varphi(s+t)| \,\mathrm{d}t \\ &+ \int_{-\infty}^{\infty} |\varphi(s+t) - \varphi(t)| \,\mathrm{d}t \\ &+ \int_{-\infty}^{\infty} |f(t) - \varphi(t)| \,\mathrm{d}t \le 3\varepsilon\end{aligned}$$

for every $s \in (-\delta, \delta)$. □

Theorem 2.25. *Let f and g be densities such that for some $c > 0$, $|g(x)| \le c$ almost everywhere. Then density $h = f * g$ is uniformly continuous on $\mathbb{R}$.*

Proof. Let $x, x_1 \in \mathbb{R}$. Then

$$\begin{aligned}|h(x_1) - h(x)| &\le \left| \int_{-\infty}^{\infty} [f(x_1 - y) - f(x - y)] g(y) \,\mathrm{d}y \right| \\ &\le c \int_{-\infty}^{\infty} |f(x_1 - y) - f(x - y)| \,\mathrm{d}y \\ &= c \int_{-\infty}^{\infty} |f(x_1 - x + t) - f(t)| \,\mathrm{d}t.\end{aligned}$$

The assertion follows from Lemma 2.24. □

Now we prove that previous theorem is optimal in some sense. Let $\mathbb{R}^*$ denote the extended real line. Let $g : \mathbb{R} \to \mathbb{R}^*$ be a measurable function. Define

$$\|g\|_\infty = \inf\{c \in [0, \infty); |g| \leq c \ \text{ a.e.}\}$$

(cf. 3.7 in [21]), so that $\|g\|_\infty$ is the essential supremum of function $|g|$.

For a measurable function g satisfying the condition $\|g\|_\infty < \infty$, the mapping

$$\Phi : h \mapsto \int_{-\infty}^{\infty} hg, \quad h \in L^1(\mathbb{R})$$

is a linear functional on $L^1(\mathbb{R})$ with $\|\Phi\| = \|g\|_\infty$ (see 6.16 in [21]).

Further, we remember a version of Banach-Steinhaus Theorem.

Theorem 2.26 (Banach-Steinhaus). *Let X be a Banach space. Let Φ_n, $n \in \mathbb{N}$, be continuous linear functionals on X and let*

$$\sup\{|\Phi_n(x)|; n \in \mathbb{N}\} < \infty$$

for every $x \in X$. Then

$$\sup\{\|\Phi_n\|; n \in \mathbb{N}\} < \infty.$$

So a *pointwise bounded* sequence of functionals is *uniformly bounded* as well, cf. [21], Theorem 5.8.

For $f : \mathbb{R} \to \mathbb{R}^*$ and $x \in \mathbb{R}$ define the function f_x by

$$f_x(y) = f(x - y), \quad y \in \mathbb{R}.$$

Let f and g be measurable functions. Denote

$$M(f, g) = \{x \in \mathbb{R}; f_x \cdot g \in L^1(\mathbb{R})\}.$$

For $x \in M(f, g)$, define function $k(x)$ by

$$k(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\, \mathrm{d}y.$$

It is known that for f, $g \in L^1(\mathbb{R})$, the set $\mathbb{R} \setminus M(f, g)$ has zero measure and k, as an element of $L^1(\mathbb{R})$, is equal to convolution $f * g$ (see [21], Theorem 8.14).

Theorem 2.27. *Let g be a measurable function on $\mathbb{R}$. Then the following conditions are equivalent.*

(i) For every function $f \in L^1(\mathbb{R})$, we have $M(f, g) = \mathbb{R}$.

(ii) There exists $x \in \mathbb{R}$ such that $x \in M(f,g)$ whenever $f \in L^1(\mathbb{R})$.

(iii) $\|g\|_\infty < \infty$.

(iv) Function k is uniformly continuous for every $f \in L^1(\mathbb{R})$.

Proof. Implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are obvious. Implication (iii) $\Rightarrow$ (iv) follows from Theorem 2.25. Therefore, it suffices to prove (ii) $\Rightarrow$ (iii). Let (ii) hold. Then $f_x \cdot g \in L^1(\mathbb{R})$ for every function $f \in L^1(\mathbb{R})$, and thus $h \cdot g \in L^1(\mathbb{R})$ for every $h \in L^1(\mathbb{R})$. Define $g_n = \max\{-n, \min\{g, n\}\}$, $n \in \mathbb{N}$. Then $\|g_n\|_\infty \le n$ and the norm of the linear functional

$$\Phi_n : h \mapsto \int_{-\infty}^{\infty} h g_n, \quad h \in L^1(\mathbb{R}),$$

is equal to $\|g_n\|_\infty$. Note that $|g_n| \nearrow |g|$ for $n \to \infty$ and thus, according to Lebesgue's monotone convergence theorem,

$$\int_{-\infty}^{\infty} |hg_n| \to \int_{-\infty}^{\infty} |hg| \in \mathbb{R}$$

for every $h \in L^1(\mathbb{R})$. Hence, the sequence $\left\{\int_{-\infty}^{\infty} |hg_n|\right\}$ is bounded for every $L^1(\mathbb{R})$ and by Banach-Steinhaus Theorem there exists $c \in \mathbb{R}$ such that $\|g_n\|_\infty \le c$ for every n. Thus, $\|g\|_\infty \le c < \infty$, which completes the proof. $\square$

2.4.3 Boundedness of convolution

Now we deal with boundedness of convolution of two unbounded densities. First, define

$$f_n(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} \mathrm{e}^{-\frac{x}{2}}, \qquad x > 0, \tag{2.35}$$

and $f_n(x) = 0$ otherwise. It is known that $f_n(x)$ is a density of χ^2_n distribution. Density $f_1(x)$ is not bounded, but $f_2 = f_1 * f_1$ is a non-continuous bounded density. Thus, it is an example of the situation when the convolution of two unbounded densities is bounded.

Now we show that the convolution of two unbounded densities can be an unbounded function. First we derive an auxiliary result. For $t \in (0,1)$ define $u(s) = \ln\left(s + \sqrt{s^2+t}\right)$. Then $u'(s) = 1\big/\sqrt{s^2+t}$ and for $t \in (0,1)$ we have

$$\int_0^{\sqrt{1-t}} \frac{\mathrm{d}s}{\sqrt{s^2+t}} = \ln\left(1 + \sqrt{1-t}\right) - \ln\sqrt{t}.$$

Further, define function

$$f(x) = \frac{1}{4\sqrt{|x|}} \qquad \text{for } x \in (-1,0) \cup (0,1) \tag{2.36}$$

and $f(x) = 0$ otherwise. Function f is a density. Choose $x \in (0,1)$. Then

$$\begin{aligned}
(f * f)(x) &= \frac{1}{16} \int_{x-1}^{1} \frac{\mathrm{d}y}{\sqrt{|x-y|}\sqrt{|y|}} \geq \frac{1}{16} \int_{x-1}^{0} \frac{\mathrm{d}y}{\sqrt{|x-y|}\sqrt{|y|}} \\
&= \frac{1}{16} \int_{0}^{1-x} \frac{2}{\sqrt{|x+y|}} \frac{\mathrm{d}y}{2\sqrt{|y|}} = \frac{1}{8} \int_{0}^{\sqrt{1-x}} \frac{\mathrm{d}s}{\sqrt{s^2+x}} \\
&= \frac{1}{8} \left[\ln\left(1+\sqrt{1-x}\right) - \ln\sqrt{x}\right] \geq \frac{1}{16} \ln\frac{1}{x}.
\end{aligned}$$

Hence it follows that $f * f$ is not a bounded function.

It seems that the main difference between density f_1 from (2.35) and density f from (2.36) is the fact that f_1 has "one-sided singularity" while f has "two-sided singularity". Instead of density f, consider density $g(x) = \frac{1}{2\sqrt{x}}$ for $x \in (0,1)$, $g(x) = 0$ otherwise. We get

$$(g * g)(x) = \begin{cases} \frac{\pi}{4} & \text{for } x \in (0,1), \\ \frac{1}{2} \arcsin \frac{1}{\sqrt{x}} - \frac{1}{2} \arcsin \sqrt{\frac{x-1}{x}} & \text{for } x \in (1,2), \\ 0 & \text{otherwise.} \end{cases}$$

Function $g * g$ is bounded. Its graph is introduced in Figure 5.

2.4.4 Smoothness of convolution

Lemma 2.28. *Let $f : (\alpha,\beta) \to \mathbb{R}$ be absolutely continuous function on (α,β), $-\infty \leq \alpha < \beta \leq \infty$ and $f' \in L^1\big((\alpha,\beta)\big)$. Then f is bounded on (α,β). If $\beta < \infty$, then $f(\beta-)$ exists and the function $\tilde{f}$ defined by $\tilde{f} = f$ on (α,β) and $\tilde{f}(\beta) = f(\beta-)$ is absolutely continuous on $(\alpha,\beta]$. Similarly for $\alpha > -\infty$.*

Proof. It is known that $f'(x)$ exists for almost every $x \in (\alpha,\beta)$. Let $\gamma \in (\alpha,\beta)$. Then for every $x \in (\alpha,\beta)$ we have

$$|f(x)| - |f(\gamma)| \leq |f(x) - f(\gamma)| = \left|\int_\gamma^x f'\right| \leq \int_\gamma^x |f'| \leq \int_\alpha^\beta |f'| < \infty.$$

(Equality $|f(x) - f(\gamma)| = \left|\int_\gamma^x f'\right|$ follows from [21], Theorem 7.18.) We can see that f is bounded on (α,β). Let $\alpha < x < y < \beta$. Then

$$|f(y) - f(x)| \leq \int_x^y |f'|.$$

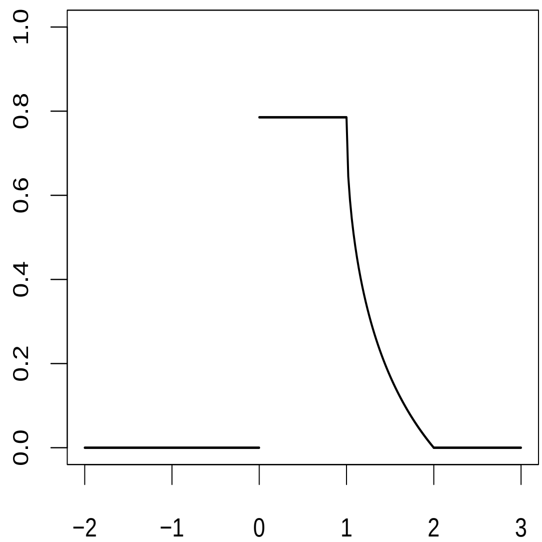

Figure 5: Graph of function $g * g$

Since measure μ defined by $\mu(A) = \int_A |f'|$ for a Borel set $A \subset (\alpha, \beta)$ is absolutely continuous with respect to Lebesgue measure, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A |f'| \le \varepsilon$ whenever A is a set with Lebesgue measure less than δ. In particular, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| \le \varepsilon$ for any $x, y \in (\alpha, \beta)$, $|x - y| < \delta$. Thus, function f is uniformly continuous on interval (α, β). Hence $f(\beta-)$ exists if $\beta < \infty$.

Let $\beta < \infty$, $\gamma \in (\alpha, \beta)$ and $x \in [\gamma, \beta)$. Theorem 7.18 in [21] yields

$$f(x) - f(\gamma) = \int_\gamma^x f',$$

because f is absolutely continuous on $[\gamma, x]$. Since $f(\beta-)$ exists and

$$\lim_{x \to \beta-} \int_\gamma^x f' = \int_\gamma^\beta f',$$

(recall that $f' \in L^1(\mathbb{R})$), we have $\tilde{f}(x) - \tilde{f}(\gamma) = \int_\gamma^x f'$ for any $x \in [\gamma, \beta]$. According to [21], Theorem 7.18, $\tilde{f}$ is absolutely continuous on $[\gamma, \beta]$ and therefore on $(\alpha, \beta]$. Similarly for $\alpha > -\infty$. □

Theorem 2.29. *Let f be a piecewise smooth function. Let $\hat{f}(t)$ denote the characteristic function corresponding to f. Then there exists number $b > 0$ such that*

$$|\hat{f}(t)| \le \frac{b}{1 + |t|}, \quad t \in \mathbb{R}.$$

Proof. Let $a_{-1}, a_0, \dots, a_{r+1}$ have the same meaning as in Definition 2.8. Lemma 2.28 implies the existence of a number M such that $|f| \le M$ on $\mathbb{R}$. Let $j \in \{1, \dots, r\}$ and $t \neq 0$. For absolutely continuous functions we can use integration by parts. We get

$$\int_{a_{j-1}}^{a_j} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x = -\frac{\mathrm{i}}{t}\mathrm{e}^{\mathrm{i}ta_j} f(a_j-) + \frac{\mathrm{i}}{t}\mathrm{e}^{\mathrm{i}ta_{j-1}} f(a_{j-1}+) + \frac{\mathrm{i}}{t}\int_{a_{j-1}}^{a_j} \mathrm{e}^{\mathrm{i}tx} f'(x)\,\mathrm{d}x.$$

If we define $d = 2M + \|f'\|_1$, we get

$$\left|\int_{a_{j-1}}^{a_j} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \le \frac{d}{|t|}. \tag{2.37}$$

Similarly, for $\alpha < a_0$, we get

$$\left|\int_{\alpha}^{a_0} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \le \frac{d}{|t|}$$

and therefore

$$\left|\int_{-\infty}^{a_0} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \le \frac{d}{|t|} \tag{2.38}$$

since the integrand belongs to $L^1(\mathbb{R})$. Similarly,

$$\left|\int_{a_r}^{\infty} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \le \frac{d}{|t|}. \tag{2.39}$$

It follows from (2.37), (2.38) and (2.39) that for $t \neq 0$

$$\begin{aligned}
|\hat{f}(t)| &= \left|\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \\
&\le \left|\int_{-\infty}^{a_0} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| + \sum_{j=1}^{r}\left|\int_{a_{j-1}}^{a_j} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| + \left|\int_{a_r}^{\infty} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x\right| \\
&\le \frac{(r+2)d}{|t|}.
\end{aligned}$$

Thus, we have

$$|\hat{f}(t)| \le \frac{2d(r+2)}{1+|t|}$$

for $|t| \ge 1$ and

$$|\hat{f}(t)| \le \|f\|_1 \le \frac{2}{1+|t|}\|f\|_1$$

for $|t| < 1$. Now it suffices to choose $b = 2[(r+2)d + \|f\|_1]$. □

Theorem 2.30. *Let $n \geq 3$ and let $f_1, \ldots, f_n$ be piecewise smooth densities (see Definition 2.8). Then $f_1 * \cdots * f_n \in C^{n-2}(\mathbb{R})$ and*

$$\left| \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} (f_1 * \cdots * f_n)(x) \right| \leq D$$

for some $D \in \mathbb{R}$ and every $x \in \mathbb{R}$.

Proof. According to Theorem 2.9 there exist $b_j \in \mathbb{R}$ such that

$$|\hat{f}_j(t)| \leq \frac{b_j}{1 + |t|}, \quad t \in \mathbb{R}, \quad j = 1, \ldots, n.$$

If we take

$$a = \prod_{j=1}^{n} b_j, \quad f = f_1 * \cdots * f_n,$$

then density f is continuous according to Theorem 2.25 and from (2.28) we get

$$|\hat{f}(t)| = \left| \prod_{j=1}^{n} \hat{f}_j(t) \right| \leq \frac{a}{(1 + |t|)^n}, \quad t \in \mathbb{R}.$$

The assertion now follows from Lemma 2.21. □

3 Multidimensional AR(1) model

In this section we extend the algorithm of Anděl and Hrach to multidimensional case. Let $\{\boldsymbol{\eta}_t\}$ be p-dimensional i.i.d. random vectors with a density f. Consider a p-dimensional stationary AR(1) process $\{\boldsymbol{X}_t\}$ defined by

$$\boldsymbol{X}_t = \boldsymbol{B}\boldsymbol{X}_{t-1} + \boldsymbol{\eta}_t$$

where $\boldsymbol{B}$ is a regular $p \times p$ matrix such that the equation $\mathsf{Det}(\boldsymbol{I}x - \boldsymbol{B}) = 0$ has all roots inside the unit circle. (Symbol $\boldsymbol{I}$ denotes the eigenmatrix.)

Theorem 3.1. *The process $\{\boldsymbol{X}_t\}$ has a stationary density h which satisfies equation*

$$h(\boldsymbol{z}) = \int h(\boldsymbol{w}) f(\boldsymbol{z} - \boldsymbol{B}\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}. \tag{3.1}$$

Proof. Existence of the density h can be proved analogously as in the one-dimensional case (see p. 5). If $\boldsymbol{X}_{t-1}$ has a density h then the density of $\boldsymbol{B}\boldsymbol{X}_{t-1}$ is

$$g(\boldsymbol{y}) = \left|\mathsf{Det}\,\boldsymbol{B}^{-1}\right| h(\boldsymbol{B}^{-1}\boldsymbol{y}).$$

For sake of completeness remember that if $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are two independent random vectors with densities p_1 and p_2, respectivelly, then $\boldsymbol{Z} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$ has the density

$$q(\boldsymbol{z}) = \int p_1(\boldsymbol{z} - \boldsymbol{u}) p_2(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}.$$

Since $\boldsymbol{X}_t$ has also the density h, combining the results above we get

$$h(\boldsymbol{z}) = \left|\mathsf{Det}\,\boldsymbol{B}^{-1}\right| \int h[\boldsymbol{B}^{-1}(\boldsymbol{z} - \boldsymbol{u})] f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}.$$

Substitution $\boldsymbol{w} = \boldsymbol{B}^{-1}(\boldsymbol{z} - \boldsymbol{u})$ yields formula (3.1). □

Theorem 3.2. *Let $\psi(\boldsymbol{t}) = \mathsf{E}\,\mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top \boldsymbol{\eta}_t}$ be the characteristic function of the random vector $\boldsymbol{\eta}_t$. Then the characteristic function $\lambda(\boldsymbol{t})$ of the vector $\boldsymbol{X}_t$ is*

$$\lambda(\boldsymbol{t}) = \prod_{k=0}^{\infty} \psi\left(\left(\boldsymbol{B}^\top\right)^k \boldsymbol{t}\right). \tag{3.2}$$

Proof. Define

$$\boldsymbol{X}_{t,n} = \boldsymbol{\eta}_t + \boldsymbol{B}\boldsymbol{\eta}_{t-1} + \cdots + \boldsymbol{B}^n \boldsymbol{\eta}_{t-n}.$$

It is known that $\boldsymbol{X}_{t,n} \to \boldsymbol{X}_t$ in the quadratic mean as $n \to \infty$. Thus $\boldsymbol{X}_{t,n} \to \boldsymbol{X}_t$ in distribution and the characteristic functions of vectors $\boldsymbol{X}_{t,n}$ converge pointwise to the characteristic function of the vector $\boldsymbol{X}_t$. It is easy to check that the characteristic function of $\boldsymbol{X}_{t,n}$ is $\psi(\boldsymbol{t})\psi\left(\boldsymbol{B}^\top\boldsymbol{t}\right)\cdots\psi\left(\left(\boldsymbol{B}^\top\right)^n\boldsymbol{t}\right)$ and thus the characteristic function $\lambda(\boldsymbol{t})$ of the vector $\boldsymbol{X}_t$ is

$$\lambda(\boldsymbol{t}) = \prod_{k=0}^{\infty}\psi\left(\left(\boldsymbol{B}^\top\right)^k\boldsymbol{t}\right).$$

□

Let $h_0(\boldsymbol{z})$ be a density. Define a sequence $\{h_n(\boldsymbol{z})\}$ by

$$h_n(\boldsymbol{z}) = \int h_{n-1}(\boldsymbol{w})f(\boldsymbol{z}-\boldsymbol{B}\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}, \quad n \geq 1. \tag{3.3}$$

It is clear that each function h_n is a density.

Theorem 3.3. *Let λ_n be the characteristic function corresponding to h_n. Then $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$ for all $\boldsymbol{t}$.*

Proof. For $n \geq 1$, we have

$$\begin{aligned}
\lambda_n(\boldsymbol{t}) &= \int \mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{z}}h_n(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \\
&= \int \mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{z}}\left[\int h_{n-1}(\boldsymbol{w})f(\boldsymbol{z}-\boldsymbol{B}\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}\right]\mathrm{d}\boldsymbol{z} \\
&= \int h_{n-1}(\boldsymbol{w})\int\left[\mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{z}}f(\boldsymbol{z}-\boldsymbol{B}\boldsymbol{w})\,\mathrm{d}\boldsymbol{z}\right]\mathrm{d}\boldsymbol{w} \\
&= \int \mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{B}\boldsymbol{w}}h_{n-1}(\boldsymbol{w})\left[\int \mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{u}}f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\right]\mathrm{d}\boldsymbol{w} \\
&= \psi(\boldsymbol{t})\int \mathrm{e}^{\mathrm{i}\boldsymbol{t}^\top\boldsymbol{B}\boldsymbol{w}}h_{n-1}(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w} \\
&= \psi(\boldsymbol{t})\lambda_{n-1}(\boldsymbol{B}^\top\boldsymbol{t}).
\end{aligned}$$

Thus,

$$\lambda_n(\boldsymbol{t}) = \psi(\boldsymbol{t})\psi(\boldsymbol{B}^\top\boldsymbol{t})\cdots\psi\left(\left(\boldsymbol{B}^\top\right)^{n-1}\boldsymbol{t}\right)\lambda_0\left(\left(\boldsymbol{B}^\top\right)^n\boldsymbol{t}\right).$$

Since $\left(\boldsymbol{B}^\top\right)^n \to \boldsymbol{0}$ as $n \to \infty$, we obtain $\lambda_0\left(\left(\boldsymbol{B}^\top\right)^n\boldsymbol{t}\right) \to 1$ and $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$ for all $\boldsymbol{t}$. □

Theorem 3.4. *Let $h_0(\boldsymbol{z})$ be a density. Define $h_n(\boldsymbol{z})$ by formula* (3.3). *If there exists an integer $m \geq 0$ such that*

$$\int \left|\psi(\boldsymbol{t})\psi\left(\boldsymbol{B}^\top\boldsymbol{t}\right)\cdots\psi\left(\left(\boldsymbol{B}^\top\right)^m\boldsymbol{t}\right)\right|\,\mathrm{d}\boldsymbol{t} < \infty$$

then $h_n(\boldsymbol{z}) \to h(\boldsymbol{z})$ as $n \to \infty$ for all $\boldsymbol{z}$.

Proof. If $n > m$ then

$$|\lambda_n(\boldsymbol{t})| \leq \left|\psi(\boldsymbol{t})\psi\left(\boldsymbol{B}^\top\boldsymbol{t}\right)\cdots\psi\left(\left(\boldsymbol{B}^\top\right)^m\boldsymbol{t}\right)\right|$$

and thus $\int |\lambda_n(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t} < \infty$. Then $h_n(\boldsymbol{z})$ is bounded, continuous, and

$$h_n(\boldsymbol{z}) = \frac{1}{(2\pi)^p}\int \mathrm{e}^{-\mathrm{i}\boldsymbol{t}^\top\boldsymbol{z}}\lambda_n(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

(see [11], formula 7.12). Lebesgue's dominated convergence theorem yields $\lim_{n\to\infty} h_n(\boldsymbol{z}) = h(\boldsymbol{z})$ for arbitrary $\boldsymbol{z}$. □

4 Some nonlinear models

In this chapter we first describe a procedure which can be used to approximate stationary density of a nonlinear autoregressive process of first order. Then we study two models for which an explicit form of stationary distribution was found.

4.1 Approximation of stationary density in nonlinear autoregressive processes of first order

Let the process $\{X_t\}$ follow general autoregression of first order

$$X_t = \lambda(X_{t-1}) + \eta_t, \qquad t \in \mathbb{N},$$

where $\{\eta_t\}$ are i.i.d. random variables with known density f with respect to Lebesgue measure, zero mean and finite positive variance σ^2. The model is assumed to be stationary. We do not have any further restrictions on function λ.

If an analytic solution of this problem is not known, we try to find a numerical approximation. Several methods have been proposed, see e.g. [14] or [16]. We describe an algorithm based on Chapman-Kolmogorov relation (for details see [22], p. 152). Its basic idea is to evaluate the sequence of conditional densities which converge to the desired stationary density.

Let $h(x_{t+m}|x_t)$ denote the conditional density, which we assume to exist, of X_{t+m} given $X_t = x_t$. Chapman-Kolmogorov relation states that

$$h(x_{t+m}|x_t) = \int_{\mathbb{R}} h(x_{t+m}|x_{t+1})h(x_{t+1}|x_t)\,\mathrm{d}x_{t+1}.$$

Let F be the distribution function of η_1. In the first step we calculate the initial density $h(x_{t+1}|x_t)$. We compute the corresponding conditional distribution function

$$\begin{aligned} H(x_{t+1}|x_t) &= \mathsf{P}(X_{t+1} \le x_{t+1}|X_t = x_t) = \mathsf{P}(\lambda(x_t) + \eta_{t+1} \le x_{t+1}) \\ &= F(x_{t+1} - \lambda(x_t)). \end{aligned}$$

Further,

$$h(x_{t+1}|x_t) = \frac{\partial}{\partial x_{t+1}} H(x_{t+1}|x_t) = f(x_{t+1} - \lambda(x_t)) \tag{4.1}$$

and

$$h(x_{t+m}|x_t) = \int_{\mathbb{R}} h(x_{t+m}|x_{t+1}) f(x_{t+1} - \lambda(x_t))\,\mathrm{d}x_{t+1}. \tag{4.2}$$

Iterating the last equality, we obtain

$$
\begin{aligned}
h(x_{t+2}|x_t) &= \int_{\mathbb{R}} f(x_{t+2} - \lambda(x_{t+1}))f(x_{t+1} - \lambda(x_t))\,\mathrm{d}x_{t+1} \\
h(x_{t+3}|x_t) &= \iint_{\mathbb{R}^2} f(x_{t+3} - \lambda(x_{t+2}))f(x_{t+2} - \lambda(x_{t+1}))\times \\
&\qquad\qquad \times f(x_{t+1} - \lambda(x_t))\,\mathrm{d}x_{t+2}\,\mathrm{d}x_{t+1} \\
&\ \vdots \\
h(x_{t+m}|x_t) &= \int\cdots\int_{\mathbb{R}^{m-1}} f(x_{t+m} - \lambda(x_{t+m-1}))\times\cdots\times \\
&\qquad\qquad \times f(x_{t+1} - \lambda(x_t))\,\mathrm{d}x_{t+m-1}\cdots\mathrm{d}x_{t+1}. \qquad (4.3)
\end{aligned}
$$

Starting with (4.1), we evaluate $h(x_{t+m}|x_t)$, $m = 2, 3, \ldots$, from (4.2) or (4.3) using numerical integration. By stationarity, the sequence $h(x_{t+m}|x_t)$ converges to the stationary density as $m \to \infty$ for every x_0 (see [22], p. 153).

It is possible to reduce the computation time as follows. Note that (4.2) still holds if we replace its left hand side by $h(x_{t+2m}|x_t)$, the integrand by $h(x_{t+2m}|x_{t+m})h(x_{t+m}|x_t)$ and $\mathrm{d}x_{t+1}$ by $\mathrm{d}x_{t+m}$, i.e.

$$h(x_{t+2m}|x_t) = \int_{\mathbb{R}} h(x_{t+2m}|x_{t+m})h(x_{t+m}|x_t)\,\mathrm{d}x_{t+m}.$$

Thus, instead of obtaining iterates $h(x_{t+m}|x_t)$, $m = 1, 2, \ldots$, we get a sequence $h(x_{t+2^m}|x_t)$, $m = 0, 1, 2, \ldots$, which increases the speed of convergence significantly.

Remark 4.1. The method may also be extended to cope with higher-order dependence, i.e.

$$X_t = \lambda^*(X_{t-1}, \ldots, X_{t-m}) + \eta_t.$$

Example 4.2 (Absolute autoregression). As an illustration of the method described above, we compare an approximate stationary density obtained by this algorithm with the exact density h of the AAR(1) process $\{X_t\}$ given by

$$X_t = \frac{1}{2}|X_{t-1}| + \eta_t,$$

where $\eta_t \overset{\text{iid}}{\sim} \mathsf{N}(0, 1)$. In (4.5) we show that

$$h(x) = \sqrt{\frac{3}{2\pi}}\exp\{-3x^2/8\}\Phi(x/2).$$

Chapman-Kolmogorov method yields the following results. Conditional densities $h_j = h(x_j|x_0 = 1)$, $j = 1, 2, 3$, and the exact stationary density are shown in Figure 6. We can see that the convergence is quite fast since the curves illustrating h_2, h_3 and h almost coincide. In Table 1 we find maximum differences between densities h_j and the exact density h. Table 2 summarizes approximate and exact moments of $\{X_t\}$.

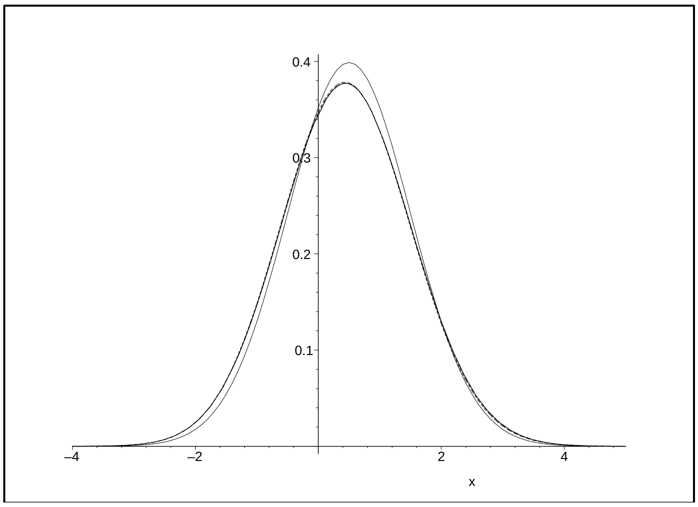

Figure 6: Conditional densities h_1 (solid), h_2 (dashed), h_3 (dotted) and stationary density h (solid thick)

Table 1: Maximum differences between h_j and h.

j	1	2	3
$\max_x \lvert h_j(x) - h(x)\rvert$	0.02513	0.00272	0.00075

In the next example we study the case when the exact form of stationary density is unknown.

Example 4.3 (Threshold autoregression). Let $\{X_t\}_{t\geq 0}$ be generated by a TAR(1) model

$$X_t = \begin{cases} 1 + 0.6X_{t-1} + \eta_t & \text{if } X_{t-1} \leq 0 \\ -1 + 0.4X_{t-1} + \eta_t & \text{if } X_{t-1} > 0 \end{cases}$$

Table 2: Conditional and exact moments of $\{X_t\}$.

k	1	2	3	4
$\mathsf{E}(X_1^k \mid X_0 = 1)$	0.500	1.250	1.625	4.560
$\mathsf{E}(X_2^k \mid X_0 = 1)$	0.448	1.312	1.618	5.150
$\mathsf{E}(X_3^k \mid X_0 = 1)$	0.457	1.328	1.670	5.280
$\mathsf{E}\, X_t^k$	0.461	1.333	1.689	5.333

where the noise process η_t has standard normal distribution (see [22], p. 100). Approximate densities $h_1, \ldots, h_4$ and corresponding moments are shown in Figure 2 and Table 3, respectively. We can see that the convergence is much slower than in the previous example.

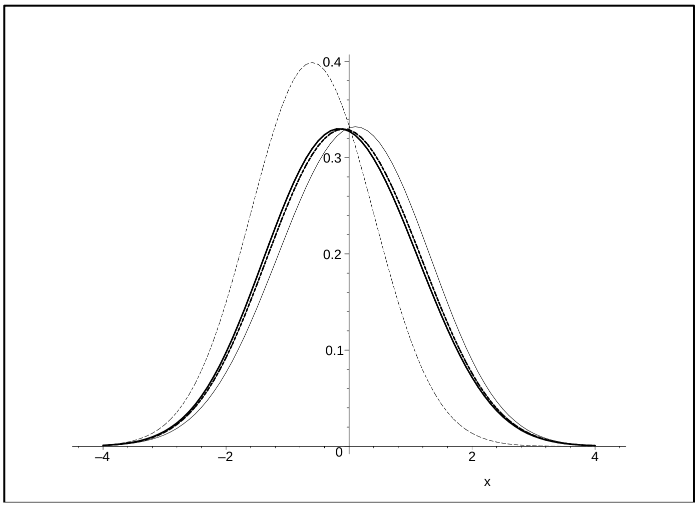

Figure 7: Densities h_1 (dashed), h_2 (solid), h_3 (dashed thick) and h_4 (solid thick)

4.2 Absolute autoregression

Consider the model of absolute autoregression of first order AAR(1) given by

$$X_t = a|X_{t-1}| + \eta_t \tag{4.4}$$

Table 3: Conditional moments of $\{X_t\}$.

k	1	2	3	4
$\mathsf{E}(X_1^k \mid X_0 = 1)$	-0.600	1.360	-2.016	5.290
$\mathsf{E}(X_2^k \mid X_0 = 1)$	0.058	1.390	0.171	5.507
$\mathsf{E}(X_3^k \mid X_0 = 1)$	-0.112	1.410	-0.399	5.630
$\mathsf{E}(X_4^k \mid X_0 = 1)$	-0.070	1.402	-0.260	5.582

where $a \in (-1, 1)$ and η_t is a strict white noise. Anděl et al. [6] proved that for $a \in (-1, 0)$ and for $\eta_t \sim \mathsf{N}(0, 1)$ the stationary density of (4.4) is

$$h(x) = \sqrt{\frac{2(1-a^2)}{\pi}} \exp\{-(1-a^2)x^2/2\}\Phi(ax) \tag{4.5}$$

where Φ is the distribution function of $\mathsf{N}(0, 1)$. The expectation is given by

$$\begin{aligned} \mathsf{E}\, X_t &= \int_{\mathbb{R}} x \sqrt{\frac{2(1-a^2)}{\pi}} \exp\left\{-\frac{(1-a^2)x^2}{2}\right\} \int_{-\infty}^{ax} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \mathrm{d}y\, \mathrm{d}x \\ &= \frac{\sqrt{1-a^2}}{\pi} \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2}\right\} \frac{-1}{1-a^2} \int_{-\infty}^{-\frac{1}{2}(1-a^2)(y/a)^2} \mathrm{e}^z\, \mathrm{d}z\, \mathrm{d}y \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{\sqrt{1-a^2}}. \end{aligned} \tag{4.6}$$

Before computing the variance $\mathsf{var}\, X_t$ we prove an auxiliary lemma.

Lemma 4.4. *Let f be an arbitrary density symmetric around zero. Let F be a distribution function of an arbitrary symmetric distribution. Then*

$$\int_{\mathbb{R}} f(y)F(y)\, \mathrm{d}y = \frac{1}{2}.$$

Proof. Since $f(-x) = f(x)$ and $F(-x) = 1 - F(x)$ for every x, we have

$$\begin{aligned} \int_{\mathbb{R}} f(y)F(y)\, \mathrm{d}y &= \int_{-\infty}^{0} f(y)F(y)\, \mathrm{d}y + \int_{0}^{\infty} f(y)F(y)\, \mathrm{d}y \\ &= \int_{0}^{\infty} f(z)[1 - F(z)]\, \mathrm{d}z + \int_{0}^{\infty} f(z)F(z)\, \mathrm{d}z \\ &= \int_{0}^{\infty} f(z)\, \mathrm{d}z = \frac{1}{2}. \end{aligned}$$

□

Now, we can calculate the second moment

$$
\begin{aligned}
\mathsf{E}\, X_t^2 &= \int_{\mathbb{R}} x^2 h(x)\,\mathrm{d}x \\
&= \frac{\sqrt{1-a^2}}{\pi} \int_{\mathbb{R}} \mathrm{e}^{-\frac{y^2}{2}} \int_{-\infty}^{\frac{y}{a}} x^2 \exp\left\{-\frac{1}{2}(1-a^2)x^2\right\} \mathrm{d}x\,\mathrm{d}y. \qquad (4.7)
\end{aligned}
$$

First, we evaluate the inner integral

$$
\begin{aligned}
\int_{-\infty}^{\frac{y}{a}} x^2 \exp\left\{-\frac{1}{2}(1-a^2)x^2\right\} \mathrm{d}x &= \left[\frac{-x\exp\left\{-\frac{1}{2}(1-a^2)x^2\right\}}{1-a^2}\right]_{-\infty}^{\frac{y}{a}} + \\
&\quad + \int_{-\infty}^{\frac{y}{a}} \frac{\exp\left\{-\frac{1}{2}(1-a^2)x^2\right\}}{1-a^2}\,\mathrm{d}x \\
&= \frac{-y\exp\left\{-\frac{1}{2}(1-a^2)(y/a)^2\right\}}{a(1-a^2)} + \\
&\quad + \frac{\sqrt{2\pi}}{(1-a^2)^{3/2}} \Phi\left(\frac{y}{a}\sqrt{1-a^2}\right).
\end{aligned}
$$

Substitution back to (4.7) yields

$$
\mathsf{E}\, X_t^2 = \frac{-\sqrt{1-a^2}}{\pi a} \int_{\mathbb{R}} y\mathrm{e}^{-\frac{y^2}{2a^2}}\,\mathrm{d}y + \frac{\sqrt{2}}{\sqrt{\pi}(1-a^2)} \int_{\mathbb{R}} \mathrm{e}^{-\frac{y^2}{2}}\,\Phi\left(\frac{y}{a}\sqrt{1-a^2}\right) \mathrm{d}y.
$$

The first integral is zero since the integrand is an odd function. The second integral is equal to $\sqrt{\pi/2}$ according to Lemma 4.4. Thus $\mathsf{E}\, X_t^2 = (1-a^2)^{-1}$ and

$$
\mathsf{var}\, X_t = \mathsf{E}\, X_t^2 - (\mathsf{E}\, X_t)^2 = \frac{\pi - 2a^2}{\pi(1-a^2)}. \qquad (4.8)
$$

By similar means, it was derived that the correlation coefficient between X_t and X_{t-1} is

$$
\rho(a) = \frac{|a|\pi + 2a^2\sqrt{1-a^2} - 2a^2 - 2|a| \operatorname{arctg} \sqrt{a^{-2}-1}}{\pi - 2a^2} \qquad (4.9)
$$

(see [5], Theorem 4.3).

Let $\mathsf{C}(\alpha, \beta)$ be the Cauchy distribution with the density

$$
f(x) = \frac{1}{\pi} \frac{\beta}{\beta^2 + (x-\alpha)^2}.
$$

Consider the model (4.4) with $a \in (-1, 0)$ and $\eta_t \sim \mathsf{C}(0, 1)$. Define $A = |a|/(1-|a|)$. Anděl and Bartoň [3] proved that X_t in (4.4) has the stationary density

$$h(x) = \frac{2A}{\pi^2}\left\{\frac{(1+A)\pi}{2A[(1+A)^2+x^2]} - \right. \\ \left. - \frac{x\ln[A^{-2}(1+x^2)] + (A^2-1+x^2)\operatorname{arctg} x}{4A^2x^2 + (1-A^2+x^2)^2}\right\}. \quad (4.10)$$

Note that the distribution with density h does not have its first moment finite.

Chan and Tong [9] and Tong [22] (p. 141) simplified the methods used for derivation of (4.5) and (4.10). Their procedure can be summarized as follows. Let η_t in (4.4) have a symmetric density f. Let g be the stationary density of the AR(1) process ξ_t given by

$$\xi_t = a\xi_{t-1} + \eta_t. \quad (4.11)$$

The stationary density h of X_t in (4.4) clearly satisfies

$$\begin{aligned} h(y) &= \int_{\mathbb{R}} h(x) f(y - a|x|)\,\mathrm{d}x \\ &= \int_0^\infty h(x) f(y-ax)\,\mathrm{d}x + \int_{-\infty}^0 h(x) f(y+ax)\,\mathrm{d}x. \end{aligned} \quad (4.12)$$

By symmetry of f, we also have

$$h(-y) = \int_0^\infty h(x) f(y+ax)\,\mathrm{d}x + \int_{-\infty}^0 h(x) f(y-ax)\,\mathrm{d}x. \quad (4.13)$$

Let $h'(y) = \frac{1}{2}[h(y) + h(-y)]$. Then from (4.12) and (4.13), we get

$$\begin{aligned} h'(y) &= \frac{1}{2}\int_{\mathbb{R}} h(x) f(y-ax)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} h(x) f(y+ax)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}} h(x) f(y-ax)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} h(-x) f(y-ax)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} h'(x) f(y-ax)\,\mathrm{d}x \end{aligned} \quad (4.14)$$

which is the integral equation for the stationary density of ξ_t in (4.11) and thus $g = h'$. Now, from (4.12),

$$\begin{aligned} h(y) &= \int_0^\infty h(x) f(y-ax)\,\mathrm{d}x + \int_0^\infty h(-x) f(y-ax)\,\mathrm{d}x \\ &= 2\int_0^\infty g(x) f(y-ax)\,\mathrm{d}x. \end{aligned} \quad (4.15)$$

Remark 4.5. The authors overlooked that the factor 2 must be introduced in the last formula.

Note that if we have a guess that a function h could be a stationary density of X_t then it is easy to verify it from (4.4).

4.2.1 Normal distribution

We mentioned above that formulae (4.5), (4.6), (4.8) and (4.9) were derived under the assumptions that $\eta_t \sim \mathsf{N}(0,1)$ and $a \in (-1,0)$. We generalize the results to $a \in (-1,1)$.

If $\eta_t \sim \mathsf{N}(0,1)$ then ξ_t in (4.11) has the distribution $\mathsf{N}(0, \frac{1}{1-a^2})$. From (4.15) we get that the stationary density h of X_t is

$$h(y) = 2\int_0^\infty \sqrt{\frac{1-a^2}{2\pi}}\,\exp\{-(1-a^2)x^2/2\}\,\frac{1}{\sqrt{2\pi}}\exp\{-(y-ax)^2/2\}\,\mathrm{d}x.$$

Direct integration leads to

$$\begin{aligned} h(y) &= \frac{\sqrt{1-a^2}}{\pi}\exp\left\{-\frac{1}{2}y^2(1-a^2)\right\}\int_0^\infty \exp\left\{-\frac{1}{2}(x-ay)^2\right\}\mathrm{d}x \\ &= \sqrt{\frac{2(1-a^2)}{\pi}}\exp\left\{-\frac{1}{2}y^2(1-a^2)\right\}\Phi(ay). \end{aligned}$$

We can see that (4.5), (4.6), (4.8) and (4.9) are valid for $a \in (-1,1)$.

The density h is plotted in Fig. 8 (for $a = -0.8$) and in Fig. 9 (for $a = 0.8$). Expectation $\mathsf{E}\,X_t$ and variance $\mathsf{var}\,X_t$ as functions of a given by (4.6) and (4.8) are introduced in Fig. 10 and Fig. 11, respectively. In Fig. 12 we can see $\rho(a)$, which is defined by (4.9).

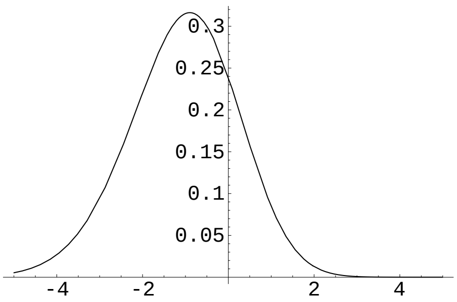

Fig. 8
Function h for $a = -0.8$

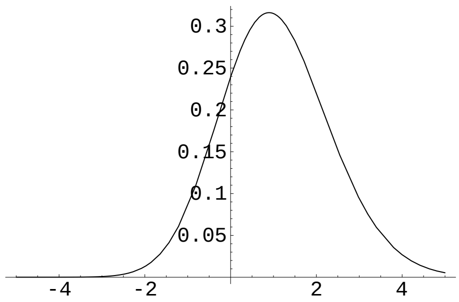

Fig. 9
Function h for $a = 0.8$

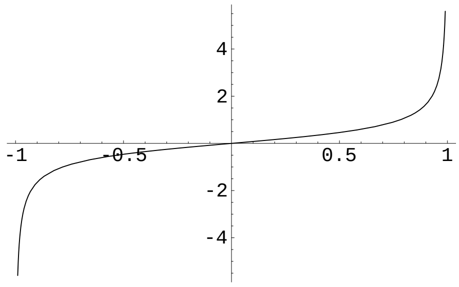

Fig. 10
Expectation $\mathsf{E}\, X_t$

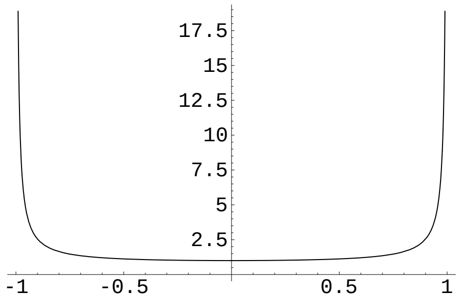

Fig. 11
Variance $\mathsf{var}\, X_t$

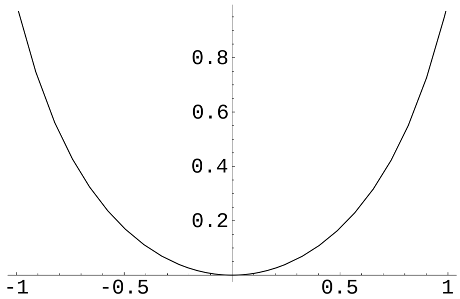

Fig. 12
Correlation coefficient $\rho(a)$

Let $p_1(x_s, x_{s-1})$ denote the joint stationary density of (X_s, X_{s-1}) and let $q_1(x_s|x_{s-1})$ denote the conditional density of X_s given $X_{s-1} = x_{s-1}$. Then

$$\begin{aligned} q_1(x_s|x_{s-1}) &= \frac{\partial}{\partial x_s} \mathsf{P}(X_s \le x_s | X_{s-1} = x_{s-1}) \\ &= \frac{\partial}{\partial x_s} \mathsf{P}(a|X_{s-1}| + \eta_s \le x_s | X_{s-1} = x_{s-1}) \\ &= \frac{\partial}{\partial x_s} \mathsf{P}(\eta_s \le x_s - a|x_{s-1}|) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(x_s - a|x_{s-1}|)^2 \right\} \end{aligned}$$

and

$$\begin{aligned} p_1(x_s, x_{s-1}) &= h(x_{s-1}) \cdot q_1(x_s|x_{s-1}) \\ &= \frac{\sqrt{1-a^2}}{\pi} \exp\left\{ -\frac{1}{2}[(1-a^2)x_{s-1}^2 + (x_s - a|x_{s-1}|)^2] \right\} \Phi(a x_{s-1}). \end{aligned}$$

The joint stationary density of (X_s, X_{s-2}) is

$$\begin{aligned}
p_2(x_s, x_{s-2}) &= \int_{\mathbb{R}} q_1(x_s|x_{s-1})q_1(x_{s-1}|x_{s-2})h(x_{s-2})\,\mathrm{d}x_{s-1} \\
&= \sqrt{\frac{1-a^2}{2\pi^3}} \exp\left\{-\frac{1-a^2}{2}x_{s-2}^2\right\} \Phi(ax_{s-2}) \exp\left\{-\frac{x_s^2 + x_{s-2}^2}{2}\right\} \\
&\times \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\big[(1+a^2)x_{s-1}^2 - \right. \\
&\qquad\left. - 2a(x_s|x_{s-1}| + x_{s-1}|x_{s-2}|)\big]\right\} \mathrm{d}x_{s-1}. \qquad (4.16)
\end{aligned}$$

We split the last integral into two parts, $\int_{\mathbb{R}} = \int_{-\infty}^{0} + \int_{0}^{\infty} = I_1 + I_2$. Then

$$\begin{aligned}
I_1 &= \int_{-\infty}^{0} \exp\left\{-\frac{1}{2}[(1+a^2)x_{s-1}^2 + 2ax_{s-1}(x_s - |x_{s-2}|)]\right\} \mathrm{d}x_{s-1} \\
&= \exp\left\{\frac{a^2}{2(1+a^2)}(x_s - |x_{s-2}|)^2\right\} \times \\
&\times \int_{-\infty}^{0} \exp\left\{-\frac{1}{2}\left[\sqrt{1+a^2}x_{s-1} + \frac{a}{\sqrt{1+a^2}}(x_s - |x_{s-2}|)\right]^2\right\} \mathrm{d}x_{s-1} \\
&= \exp\left\{\frac{a^2}{2(1+a^2)}(x_s - |x_{s-2}|)^2\right\} \times \\
&\times \int_{-\infty}^{\frac{a}{\sqrt{1+a^2}}(x_s - |x_{s-2}|)} \frac{1}{\sqrt{1+a^2}} \exp\left\{-\frac{1}{2}y^2\right\} \mathrm{d}y \\
&= \sqrt{\frac{2\pi}{1+a^2}} \exp\left\{\frac{a^2}{2(1+a^2)}(x_s - |x_{s-2}|)^2\right\} \Phi\left(\frac{a}{\sqrt{1+a^2}}(x_s - |x_{s-2}|)\right) \qquad (4.17)
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \int_{0}^{\infty} \exp\left\{-\frac{1}{2}[(1+a^2)x_{s-1}^2 - 2ax_{s-1}(x_s + |x_{s-2}|)]\right\} \mathrm{d}x_{s-1} \\
&= \sqrt{\frac{2\pi}{1+a^2}} \exp\left\{\frac{a^2}{2(1+a^2)}(x_s - |x_{s-2}|)^2\right\} \Phi\left(\frac{a}{\sqrt{1+a^2}}(x_s - |x_{s-2}|)\right). \qquad (4.18)
\end{aligned}$$

Substituting (4.17) and (4.18) into (4.16), we get

$$p_2(x_s, x_{s-2}) = \frac{1}{\pi}\sqrt{\frac{1-a^2}{1+a^2}} \exp\left\{-\frac{1-a^2}{2}x_{s-2}^2\right\} \Phi(ax_{s-2}) \\ \times \left(\exp\left\{-\frac{x_s^2 - 2a^2 x_s|x_{s-2}| + x_{s-2}^2}{2(1+a^2)}\right\} \Phi\left[\frac{a(x_s + |x_{s-2}|)}{\sqrt{1+a^2}}\right] \right. \\ \left. + \exp\left\{-\frac{x_s^2 + 2a^2 x_s|x_{s-2}| + x_{s-2}^2}{2(1+a^2)}\right\} \Phi\left[\frac{a(x_s - |x_{s-2}|)}{\sqrt{1+a^2}}\right] \right).$$

The functions p_1 and p_2 for $a = 0.8$ are introduced in Fig. 13 and Fig. 14, respectively.

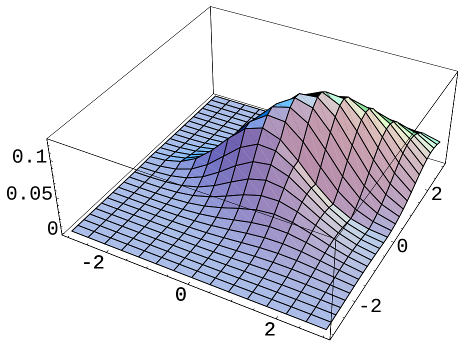

Fig. 13
Function $p_1(x_s, x_{s-1})$

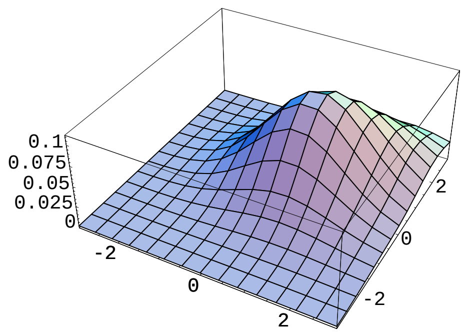

Fig. 14
Function $p_2(x_s, x_{s-2})$

4.2.2 Cauchy distribution

If $\eta_t \sim \mathsf{C}(0,1)$ then the stationary distribution of the process $\xi_t = a\xi_{t-1} + \eta_t$ with $|a| < 1$ is $\mathsf{C}(0, Q)$ where $Q = 1/(1-|a|)$. The corresponding density is

$$g(x) = \frac{1}{\pi}\frac{Q}{Q^2 + x^2}.$$

The stationary density of X_t can be calculated from (4.15). We obtain

$$h(y) = \frac{2Q}{\pi^2}\int_0^\infty \frac{1}{Q^2 + x^2}\frac{1}{1 + (y - ax)^2}\,\mathrm{d}x.$$

Again define $A = |a|/(1-|a|)$. Let $a \in (-1, 1)$. After some computations we get

$$h(y) = \frac{2A}{\pi^2}\left\{\frac{(1+A)\pi}{2A[(1+A)^2 + y^2]} + \right. \\ \left. + \frac{y\ln[A^{-2}(1+y^2)] + (A^2 - 1 + y^2)\operatorname{arctg} y}{4A^2y^2 + (1 - A^2 + y^2)^2}\right\}. \quad (4.19)$$

The density h for $a = -0.8$ defined by (4.10) is plotted in Fig. 15. If $a = 0.8$ then h is defined by (4.19) and its graph can be found in Fig. 16.

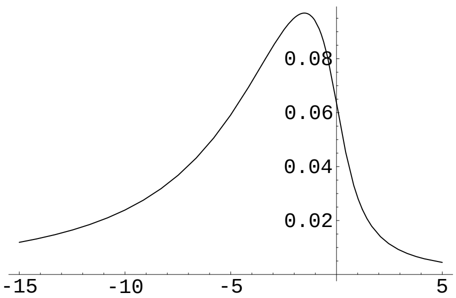

Fig. 15
Function h for $a = -0.8$

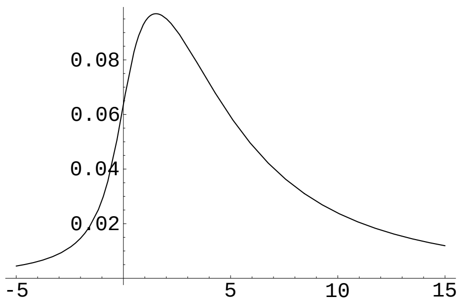

Fig. 16
Function h for $a = 0.8$

4.2.3 Discrete uniform distribution

Assume that $a = \frac{1}{2n}$ and

$$\eta_t = \frac{2i-1}{2n} b \;\text{ with probability }\; \frac{1}{2n}$$

for $i = -n+1, \ldots, n$ where $b > 0$ and $n = 1, 2, \ldots$. Then the uniform distribution $\mathsf{U}(-b, b)$ is the stationary distribution of the process ξ_t in (4.11) (see [1] and [2]). Let χ_B be the characteristic function of the set B and let $\delta_c(x)$ be the Dirac δ-function, i.e.

$$\delta_c(x) = \begin{cases} \infty & \text{for } x = c \\ 0 & \text{otherwise} \end{cases}$$

and $\int_{\mathbb{R}} \delta_c(x)\,\mathrm{d}x = 1$. The distribution of η_t can be described by the generalized density

$$f(x) = \frac{1}{2n} \sum_{i=-n+1}^{n} \delta_{\frac{2i-1}{2n} b}(x),$$

A random variable with the density f is discrete and it reaches values $\frac{2i-1}{2n} b$, $i = -n+1, \ldots, n$, each with probability $\frac{1}{2n}$. A straightforward calculation gives

$$
\begin{aligned}
h(y) &= \frac{1}{b}\int_0^\infty \chi_{[-b,b]}(x) \sum_{i=-n+1}^{n} \frac{1}{2n}\delta_{\frac{2i-1}{2n}b}\left(y - \frac{1}{2n}x\right) \mathrm{d}x \\
&= \frac{1}{b}\sum_{i=-n+1}^{n} \int_{y-\frac{b}{2n}}^{y} \delta_{\frac{2i-1}{2n}b}(z)\,\mathrm{d}z \\
&= \frac{1}{b}\sum_{i=-n+1}^{n} \chi_{\left[\frac{2i-1}{2n}b,\frac{i}{n}b\right]}(y)
\end{aligned}
$$

It is easy to verify that h is really the stationary density of the AAR process (4.4). Further we obtain

$$
\mathsf{E}\,X_t = \frac{1}{b}\sum_{i=-n+1}^{n}\int_{\frac{2i-1}{2n}b}^{\frac{i}{n}b} x\,\mathrm{d}x = \frac{b}{8n^2}\sum_{i=-n+1}^{n}(4i-1) = \frac{b}{4n},
$$

$$
\mathsf{E}\,X_t^2 = \frac{1}{b}\sum_{i=-n+1}^{n}\int_{\frac{2i-1}{2n}b}^{\frac{i}{n}b} x^2\,\mathrm{d}x = \frac{b^2}{24n^3}\sum_{i=-n+1}^{n}(12i^2-6i+1) = \frac{b^2}{3}
$$

and

$$
\mathsf{var}\,X_t = \mathsf{E}\,X_t^2 - (\mathsf{E}\,X_t)^2 = \frac{b^2(16n^2-3)}{48n^2}.
$$

Since

$$
X_t = \frac{1}{2n}|X_{t-1}| + \eta_t \quad \text{and} \quad \mathsf{E}\,\eta_t = 0,
$$

we have

$$
\begin{aligned}
\mathsf{E}\,X_tX_{t-1} &= \frac{1}{2n}\,\mathsf{E}\,|X_{t-1}|X_{t-1} \\
&= -\frac{1}{2n}\int_{-\infty}^{0} x^2 h(x)\,\mathrm{d}x + \frac{1}{2n}\int_0^\infty x^2 h(x)\,\mathrm{d}x \\
&= -\frac{1}{2bn}\sum_{i=-n+1}^{0}\int_{\frac{2i-1}{2n}b}^{\frac{i}{n}b} x^2\,\mathrm{d}x + \frac{1}{2bn}\sum_{i=1}^{n}\int_{\frac{2i-1}{2n}b}^{\frac{i}{n}b} x^2\,\mathrm{d}x \\
&= -\frac{b^2}{48n^4}\sum_{i=-n+1}^{0}(12i^2-6i+1) + \frac{b^2}{48n^4}\sum_{i=1}^{n}(12i^2-6i+1) \\
&= \frac{b^2}{8n^2}
\end{aligned}
$$

and thus

$$
\rho = \mathsf{corr}(X_t, X_{t-1}) = \frac{\mathsf{E}\,X_tX_{t-1} - (\mathsf{E}\,X_t)^2}{\mathsf{var}\,X_t} = \frac{3}{16n^2-3}.
$$

A simple case arises for $n = 1$ when we have the process $X_t = \frac{1}{2}|X_{t-1}| + \eta_t$ where

$$\eta_t = \begin{cases} -\frac{b}{2} & \text{with probability } \frac{1}{2}, \\ \frac{b}{2} & \text{with probability } \frac{1}{2}. \end{cases}$$

The stationary density of X_t is

$$h(y) = \frac{1}{b} \left\{ \chi_{[-\frac{b}{2},0]}(y) + \chi_{[\frac{b}{2},b]}(y) \right\}$$

and $\rho = 3/13 = 0.231$.

Now, we consider the case $a = \frac{1}{2n+1}$ and

$$\eta_t = \frac{2i}{2n+1} b \ \text{ with probability } \ \frac{1}{2n+1}$$

for $i = -n, \ldots, n$ where $b > 0$ and $n = 1, 2, \ldots$. The stationary distribution of the process ξ_t in (4.11) is again the uniform distribution $\mathsf{U}(-b, b)$ (see [1] and [2]). The generalized density of η_t is

$$f(x) = \frac{1}{2n+1} \sum_{i=-n}^{n} \delta_{\frac{2i}{2n+1}b}(x)$$

and direct integration gives

$$\begin{aligned} h(y) &= \frac{1}{b} \int_0^\infty \chi_{[-b,b]}(x) \frac{1}{2n+1} \sum_{i=-n}^{n} \delta_{\frac{2i}{2n+1}b} \left(y - \frac{1}{2n+1} x \right) \mathrm{d}x \\ &= \frac{1}{b} \sum_{i=-n}^{n} \int_{y-\frac{b}{2n+1}}^{y} \delta_{\frac{2i}{2n+1}b}(z) \, \mathrm{d}z \\ &= \frac{1}{b} \sum_{i=-n}^{n} \chi_{\left[\frac{2i}{2n+1}b, \frac{2i+1}{2n+1}b\right]}(y). \end{aligned}$$

It can be verified that h is the stationary density of the process $\{X_t\}$ in (4.4). Further we obtain

$$\mathsf{E}\, X_t = \frac{1}{b} \sum_{i=-n}^{n} \int_{\frac{2i}{2n+1}b}^{\frac{2i+1}{2n+1}b} x \, \mathrm{d}x = \frac{b}{2(2n+1)^2} \sum_{i=-n}^{n} (4i+1) = \frac{b}{2(2n+1)},$$

$$\mathsf{E}\, X_t^2 = \frac{1}{b} \sum_{i=-n}^{n} \int_{\frac{2i}{2n+1}b}^{\frac{2i+1}{2n+1}b} x^2 \, \mathrm{d}x = \frac{b^2}{3(2n+1)^3} \sum_{i=-n}^{n} (12i^2 + 6i + 1) = \frac{b^2}{3}$$

and

$$\operatorname{\mathsf{var}} X_t = b^2 \frac{16n^2 + 16n + 1}{12(2n+1)^2}.$$

Finally,

$$\begin{aligned}
\mathsf{E}\, X_t X_{t-1} &= \mathsf{E}\, |X_{t-1}| X_{t-1} \\
&= -\frac{1}{b(2n+1)} \sum_{i=-n}^{-1} \int_{\frac{2i}{2n+1}b}^{\frac{2i+1}{2n+1}b} x^2 \, \mathrm{d}x + \frac{1}{b(2n+1)} \sum_{i=0}^{n} \int_{\frac{2i}{2n+1}b}^{\frac{2i+1}{2n+1}b} x^2 \, \mathrm{d}x \\
&= -\frac{b^2}{3(2n+1)^4} \sum_{i=-n}^{-1} (12i^2 + 6i + 1) + \frac{b^2}{3(2n+1)^4} \sum_{i=0}^{n} (12i^2 + 6i + 1) \\
&= b^2 \frac{6n^2 + 6n + 1}{3(2n+1)^4}
\end{aligned}$$

and

$$\operatorname{\mathsf{corr}}(X_t, X_{t-1}) = \frac{\mathsf{E}\, X_t X_{t-1} - (\mathsf{E}\, X_t)^2}{\operatorname{\mathsf{var}} X_t} = \frac{12n^2 + 12n + 1}{(2n+1)^2 (16n^2 + 16n + 1)}.$$

4.2.4 Laplace distribution

Laplace distribution $\mathsf{La}(b)$ has the density

$$p(x) = \frac{1}{2b} \exp\left\{ -\frac{|x|}{b} \right\}$$

where $b > 0$ is a parameter. Assume that $a \in (-1, 1)$ and that $\{Z_t\}$ are i.i.d. $\mathsf{La}(b)$ random variables. Let the strict white noise be defined by

$$\eta_t = \begin{cases} 0 & \text{with probability } \; a^2, \\ Z_t & \text{with probability } \; 1 - a^2. \end{cases}$$

Then $\xi_t = a\xi_{t-1} + \eta_t$ has the stationary density $p(x)$ (see [1] and [2]) and

$$h(y) = 2 \int_0^\infty p(x) [a^2 \delta_0(y - ax) + (1 - a^2) p(y - ax)] \, \mathrm{d}x.$$

Assume that $a \in (0, 1)$. If $y < 0$ then

$$\int_0^\infty p(x) \delta_0(y - ax) \, \mathrm{d}x = 0$$

and

$$h(y) = \frac{2(1-a^2)}{4b^2} \int_0^\infty \exp\left\{-\frac{1}{b}[x(1+a)-y]\right\} \mathrm{d}x = \frac{1-a}{2b} \exp\left\{\frac{y}{b}\right\}.$$

For $y > 0$ we have

$$\begin{aligned} 2a^2 \int_0^\infty p(x)\delta_0(y-ax)\,\mathrm{d}x &= 2a \int_{-\infty}^{y} p\left(\frac{y-z}{a}\right)\delta_0(z)\,\mathrm{d}z = 2a \cdot p\left(\frac{y}{a}\right) \\ &= \frac{a}{b} \exp\left\{-\frac{y}{ab}\right\} \qquad (4.20) \end{aligned}$$

and

$$\begin{aligned} 2(1-a^2) \int_0^\infty p(x)p(y-ax)\,\mathrm{d}x &= \frac{1-a^2}{2b^2} \Bigg(\mathrm{e}^{-y/b} \int_0^{\frac{y}{a}} \exp\left\{-\frac{1-a}{b}x\right\} \mathrm{d}x + \\ &\qquad + \mathrm{e}^{y/b} \int_{\frac{y}{a}}^\infty \exp\left\{-\frac{1+a}{b}x\right\} \mathrm{d}x \Bigg) \\ &= \frac{1-a^2}{2b} \left[\frac{1}{1-a} \left(\mathrm{e}^{-y/b} - \mathrm{e}^{-y/(ab)}\right) + \right. \\ &\qquad \left. + \frac{1}{1+a}\mathrm{e}^{-y/(ab)} \right]. \qquad (4.21) \end{aligned}$$

Summing (4.20) and (4.21) we get

$$h(y) = \frac{1+a}{2b}\mathrm{e}^{-y/b} + \frac{1}{2b}\mathrm{e}^{-y/(ab)}[-(1+a)+(1-a)+2a] = \frac{1+a}{2b}\mathrm{e}^{-y/b}.$$

Similar calculations can be done if $a \in (-1, 0)$. Finally, we obtain that for $a \in (-1, 1)$ the stationary density of the process $\{X_t\}$ is given by

$$h(y) = \begin{cases} \dfrac{1+a}{2b} \exp\left\{-\dfrac{y}{b}\right\} & \text{for } y > 0, \\ \dfrac{1-a}{2b} \exp\left\{\dfrac{y}{b}\right\} & \text{for } y < 0. \end{cases}$$

It can be easily shown that the moments of the stationary distribution are

$$\mathsf{E}\,X_t = ab, \qquad \mathsf{E}\,X_t^2 = 2b^2, \qquad \mathsf{var}\,X_t = b^2(2-a^2).$$

Further,

$$\mathsf{E}\,X_t X_{t-1} = a\,\mathsf{E}\,|X_{t-1}|X_{t-1} = 2a^2b^2$$

and the correlation ρ between X_t and X_{t-1} is

$$\rho = \frac{\mathsf{E}\,X_tX_{t-1} - (\mathsf{E}\,X_t)^2}{\mathsf{var}\,X_t} = \frac{a^2}{2-a^2}.$$

4.2.5 Approximation of stationary density

Consider again the AAR(1) process (4.4) with innovations η_t having symmetric density. Above we studied several cases where we were able to derive the exact form of its stationary distribution. Using formula (4.15) simplified the problem – we searched for the stationary density in a linear AR(1) model instead of the original nonlinear AAR(1).

If we are not able to compute the density g of the AR(1) process (4.11), we construct at least some numerical approximation. In section 3, we described two approaches to such problem. In the following theorem we show that if we replace the unknown density g by a sequence g_n obtained by either algorithm of Anděl and Hrach or Haiman's procedure, we get a sequence of densities which converges to the stationary density of the AAR(1) process uniformly and geometrically fast.

Theorem 4.6. *Let $\{X_t\}$ be the AAR(1) process* (4.4) *and $\{\xi_t\}$ the AR(1) process* (4.11)*. Let the innovations η_t have density f which is symmetric around zero, piecewise smooth and $\mathsf{E}\,|\eta_t| < \infty$. Let the sequence of densitites g_n be defined by* (2.4)*, i.e. $g_0 = f$ and*

$$g_n(x) = \int_{\mathbb{R}} f(x - au) g_{n-1}(u) \,\mathrm{d}u, \qquad n \geq 1. \tag{4.22}$$

Further, define

$$h_n(y) = 2 \int_0^\infty g_n(x) f(y - ax) \,\mathrm{d}x, \qquad n \geq 0. \tag{4.23}$$

Then $\{h_n\}$ converges to the stationary density h of $\{X_t\}$ uniformly and there exists $C \in \mathbb{R}$ such that

$$|h_n(y) - h(y)| \leq C|a|^n$$

for $n \geq 2$.

Proof. According to Theorem 2.10 there exists $M \in \mathbb{R}$ such that

$$|g_n(x) - g(x)| \leq M|a|^n, \qquad n \geq 2.$$

Then (4.23) and (4.15) imply

$$\begin{aligned} |h_n(y) - h(y)| &= 2 \left| \int_0^\infty [g_n(x) - g(x)] f(y - ax) \,\mathrm{d}x \right| \\ &\leq 2 \int_0^\infty |g_n(x) - g(x)| f(y - ax) \,\mathrm{d}x \\ &\leq 2M|a|^n \int_0^\infty f(y - ax) \,\mathrm{d}x \\ &\leq 2M|a|^{n-1}. \end{aligned}$$

Choosing $C = 2M/|a|$ completes the proof. □

Remark 4.7. We can achieve very similar result by applying Haiman's procedure.

Example 4.8 (Laplace distribution). Let $\{X_t\}$ follow the AAR(1) model (4.4) with $a = 1/2$ and innovations η_t having Laplace distribution $\mathsf{La}(1)$, i.e. $f(x) = \frac{1}{2}\mathrm{e}^{-|x|}$, $x \in \mathbb{R}$. We compute first four members of sequences g_n and h_n using (4.22) and (4.23), respectively. We get

$$\begin{aligned} g_0(x) &= \frac{1}{2}\mathrm{e}^{-|x|}, \\ g_1(x) &= \frac{1}{3}(2\mathrm{e}^{-|x|} - \mathrm{e}^{-2|x|}), \\ g_2(x) &= \frac{2}{45}(16\mathrm{e}^{-|x|} - 10\mathrm{e}^{-2|x|} + \mathrm{e}^{-4|x|}), \\ g_3(x) &= \frac{4}{2835}(512\mathrm{e}^{-|x|} - 336\mathrm{e}^{-2|x|} + 42\mathrm{e}^{-4|x|} - \mathrm{e}^{-8|x|}) \end{aligned}$$

and

$$h_0(x) = \begin{cases} \frac{1}{3}\mathrm{e}^{x} & x < 0 \\ \mathrm{e}^{-x} - \frac{2}{3}\mathrm{e}^{-2x} & x \geq 0, \end{cases}$$

$$h_1(x) = \begin{cases} \frac{14}{45}\mathrm{e}^{x} & x < 0 \\ \frac{2}{45}(25\mathrm{e}^{-x} - 20\mathrm{e}^{-2x} + 2\mathrm{e}^{-4x}) & x \geq 0, \end{cases}$$

$$h_2(x) = \begin{cases} \frac{124}{405}\mathrm{e}^{x} & x < 0 \\ \frac{4}{2835}(807\mathrm{e}^{-x} - 672\mathrm{e}^{-2x} + 84\mathrm{e}^{-4x} - 2\mathrm{e}^{-8x}) & x \geq 0, \end{cases}$$

$$h_3(x) = \begin{cases} \frac{31496}{103275}\mathrm{e}^{x} & x < 0 \\ \frac{8}{722925}(103513\mathrm{e}^{-x} - 87040\mathrm{e}^{-2x} + 11424\mathrm{e}^{-4x} - \\ \qquad -340\mathrm{e}^{-8x} + 2\mathrm{e}^{-16x}) & x \geq 0. \end{cases}$$

Densities g_n and h_n, $n = 0, \dots, 3$, are shown in Figures 17 and 18, respectively. We can see that the actual speed of convergence is very high since the densities h_2 and h_3 almost coincide (the maximum difference between them is approximately 0.0012).

4.3 Threshold autoregression

In [15], Loges studied the model of threshold autoregression of first order driven by innovations with Laplace distribution. He derived explicit (but not closed) form of stationary marginal density.

First, we state some preliminary results concerning linear AR(1) model driven by Laplace noise.

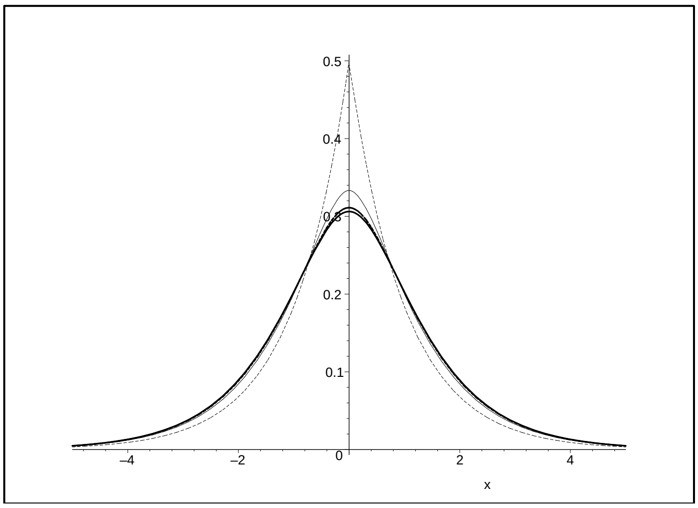

Figure 17: Densities g_0 (dashed), g_1 (solid), g_2 (dashed thick) and g_3 (solid thick)

4.3.1 Auxiliary results for AR(1) model with Laplace innovations

Consider model

$$X_t = \alpha X_{t-1} + \eta_t, \tag{4.24}$$

where $|\alpha| < 1$ and η_t are i.i.d. random variables with Laplace distribution, i.e. the density of η_t is given by $(1/2\lambda)\exp\{-|x|/\lambda\}$, $\lambda > 0$. Without loss of generality we can assume that $\lambda = 1$ [since we can rescale equation (4.24)]. All moment of X_t can be given in closed form.

Proposition 4.9. *Let $\varphi_\alpha(t)$ be the characteristic function corresponding to the stationary marginal density $f_\alpha(x)$ of X_t. Then*

$$\varphi_\alpha(t) = \left(\prod_{j=0}^{\infty}(1 + \alpha^{2j}t^2)\right)^{-1}, \quad t \in \mathbb{R}, \tag{4.25}$$

and

$$\mathsf{E}\, X_{2p+1} = 0 \quad \text{and} \quad \mathsf{E}\, X_{2p} = (2p)! \cdot \left(\prod_{j=1}^{p}(1 - \alpha^{2j})\right)^{-1}, \quad p \in \mathbb{N}. \tag{4.26}$$

The density f_α is an even function and $f_\alpha \in C^\infty(\mathbb{R})$.

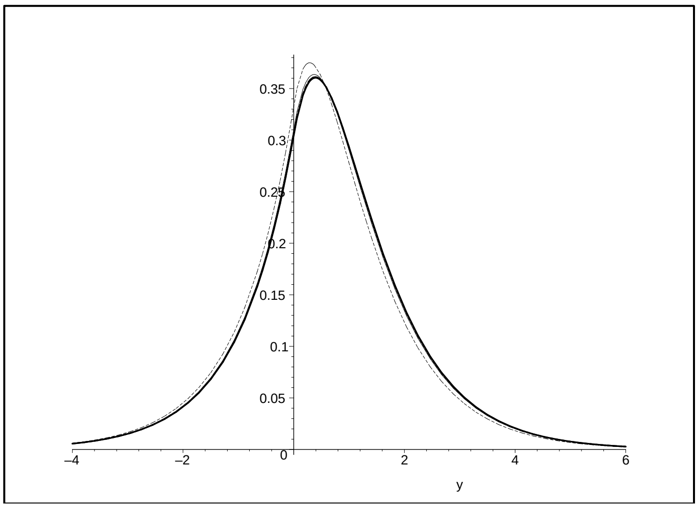

Figure 18: Densities h_0 (dashed), h_1 (solid), h_2 (dashed thick) and h_3 (solid thick)

Proof. Let ψ denote the characteristic function of η_t. Then according to Theorem 2.2

$$\varphi_\alpha(t) = \prod_{j=0}^{\infty} \psi(\alpha^j t) = \left(\prod_{j=0}^{\infty} (1 + \alpha^{2j} t^2) \right)^{-1}.$$

Since X_{t-1} and η_t are independent, the function φ_α satisfies equality

$$\varphi_\alpha(t) = \varphi_\alpha(\alpha t)\psi(t) = \frac{\varphi_\alpha(\alpha t)}{1 + t^2}. \qquad (4.27)$$

To prove (4.26), we derive power series expansion of φ_α. Let

$$\varphi_\alpha(t) = \sum_{k=0}^{\infty} \gamma_k(\alpha) \cdot t^k. \qquad (4.28)$$

With (4.27) rewritten as

$$(1 + t^2)\varphi_\alpha(t) = \varphi_\alpha(\alpha t)$$

the r.h.s. equals to

$$\sum_{k=0}^{\infty} \alpha^k \gamma_k(\alpha) t^k$$

whereas the l.h.s. is

$$\gamma_0(\alpha) + \gamma_1(\alpha)t + \sum_{k=2}^{\infty}[\gamma_k(\alpha) + \gamma_{k-2}(\alpha)]t^k.$$

Comparing the coefficients gives $\gamma_1(\alpha) = 0$ and $\gamma_k(\alpha) = (\alpha^k - 1)^{-1}\gamma_{k-2}(\alpha)$, $k \geq 2$. The condition $\varphi_\alpha(0) = 1$ implies $\gamma_0(\alpha) = 1$. Thus

$$\gamma_{2k+1}(\alpha) = 0 \quad \text{and} \quad \gamma_{2k}(\alpha) = \left(\prod_{j=1}^{k}(\alpha^{2j} - 1)\right)^{-1}, \quad k \in \mathbb{N}.$$

It is known that $\varphi_\alpha^{(k)}(0) = \mathrm{i}^k \mathsf{E}\, X_k$ and obviously $\varphi_\alpha^{(k)}(0) = k!\gamma_k(\alpha)$. Hence

$$\mathsf{E}\, X^k = \frac{k!}{\mathrm{i}^k}\gamma_k(\alpha), \quad k \in \mathbb{N},$$

and (4.26) is proved.

The symmetry of f_α follows from the fact that the convolution of two symmetric densities is symmetric. Lemma 2.21 and (4.25) yields $f_\alpha \in C^\infty(\mathbb{R})$. □

4.3.2 TAR(1) model with positive parameters

Now, consider the threshold TAR(1) process defined by

$$X_t = \tau(X_{t-1}) + \eta_t \tag{4.29}$$

where

$$\tau(x) = \begin{cases} \alpha x & \text{if } x \geq 0 \\ \beta x & \text{if } x < 0. \end{cases}$$

In this section we discuss the case $0 < \alpha < 1$ and $0 < \beta < 1$. It is known (see e.g. [17]) that under these conditions the stationary marginal distribution of X_t exists and is unique.

Let P_τ denote so called Perron-Frobenius operator associated with the map τ. It describes the transformation of a probability density under the action of τ, i.e. if a r.v. X has a density f, then $\tau(X)$ has a density $P_\tau f$.

Proposition 4.10. *Let f be an arbitrary density, $\alpha, \beta > 0$. Then*

$$P_\tau f(x) = \frac{1}{\beta} f\left(\frac{x}{\beta}\right)\chi_{(-\infty,0)} + \frac{1}{\alpha} f\left(\frac{x}{\alpha}\right)\chi_{[0,\infty)}. \tag{4.30}$$

Proof. Since $\alpha, \beta > 0$, we have

$$\begin{aligned}\mathsf{P}(\tau(X) \le x) &= \mathsf{P}(\tau(X) \le x, X \ge 0) + \mathsf{P}(\tau(X) \le x, X < 0) \\ &= \mathsf{P}\left(X \le \frac{x}{\alpha}, X \ge 0\right) + \mathsf{P}\left(X \le \frac{x}{\beta}, X < 0\right) \\ &= \begin{cases} \mathsf{P}\left(X \le \frac{x}{\alpha}\right) & \text{if } x \ge 0 \\ \mathsf{P}\left(X \le \frac{x}{\beta}\right) & \text{if } x < 0. \end{cases}\end{aligned}$$

Differentiating with respect to x yields (4.30). □

Instead of working with densities, it is more convenient to deal with characteristic functions. Let $\mathcal{F}$ denote the Fourier transform, $\mathcal{F}^{-1}$ the inverse Fourier transform and $\mathcal{P} = P_\tau$ the Perron-Frobenius operator.

Define the operator $\mathcal{H}$ on the space of characteristic functions by $\mathcal{H} = \mathcal{F} \circ \mathcal{P} \circ \mathcal{F}^{-1}$. It describes the transformation of characteristic function under the action of τ.

In the following we deal with four types of functions. For any $\gamma > 0$, define

$$k_1[\gamma](t) = \frac{1}{1+\gamma^2 t^2}, \qquad k_2[\gamma](t) = \frac{t}{1+\gamma^2 t^2},$$
$$j^+[\gamma](t) = \frac{1+\mathrm{i}\gamma t}{1+\gamma^2 t^2}, \qquad j^-[\gamma](t) = \frac{1-\mathrm{i}\gamma t}{1+\gamma^2 t^2}.$$

Proposition 4.11. *For all* $\gamma > 0$,

$$\mathcal{H}(k_1[\gamma]) = \frac{1}{2}\left(k_1[\alpha\gamma] + k_1[\beta\gamma] + \mathrm{i}\alpha\gamma k_2[\alpha\gamma] - \mathrm{i}\beta\gamma k_2[\beta\gamma]\right) \tag{4.31}$$

and

$$\mathcal{H}(k_2[\gamma]) = \frac{1}{2}\left(\alpha k_2[\alpha\gamma] + \beta k_2[\beta\gamma] + \frac{\mathrm{i}}{\gamma} k_1[\beta\gamma] - \frac{\mathrm{i}}{\gamma} k_1[\alpha\gamma]\right). \tag{4.32}$$

Proof. It is easy to show that

$$\mathcal{F}^{-1}(k_1[\gamma]) = \frac{1}{2\gamma}\mathrm{e}^{-|x|/\gamma}.$$

According to Proposition 4.10,

$$\mathcal{P}\left(\frac{1}{2\gamma}\mathrm{e}^{-|x|/\gamma}\right) = \frac{1}{2\beta\gamma}\mathrm{e}^{-|x|/(\beta\gamma)}\chi_{(-\infty,0)} + \frac{1}{2\alpha\gamma}\mathrm{e}^{-|x|/(\alpha\gamma)}\chi_{[0,\infty)}.$$

Finally,

$$\begin{aligned}\mathcal{H}(k_1[\gamma]) &= \frac{1}{2\beta\gamma}\int_{-\infty}^{0}\exp\left\{\mathrm{i}tx+\frac{x}{\beta\gamma}\right\}\mathrm{d}x+\frac{1}{2\alpha\gamma}\int_{0}^{\infty}\exp\left\{\mathrm{i}tx-\frac{x}{\alpha\gamma}\right\}\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{1-\mathrm{i}t\beta\gamma}{1+t^2\beta^2\gamma^2}+\frac{1+\mathrm{i}t\alpha\gamma}{1+t^2\alpha^2\gamma^2}\right).\end{aligned}$$

Formula (4.32) may be proved in the same manner. □

Proposition 4.12. *For all* $\gamma > 0$,

$$\mathcal{H}(j^+[\gamma]) = j^+[\alpha\gamma] \qquad \textit{and} \qquad \mathcal{H}(j^-[\gamma]) = j^-[\beta\gamma].$$

Proof. Obviously

$$j^+[\gamma] = k_1[\gamma] + \mathrm{i}\gamma k_2[\gamma], \qquad j^-[\gamma] = k_1[\gamma] - \mathrm{i}\gamma k_2[\gamma]$$

and

$$k_1[\gamma] = \frac{1}{2}(j^+[\gamma] + j^-[\gamma]), \qquad k_2[\gamma] = \frac{1}{2\mathrm{i}\gamma}(j^+[\gamma] - j^-[\gamma]).$$

The assertion now follows from linearity of the operator $\mathcal{H}$ and Proposition 4.11. □

Now we define operator $\mathcal{G}$ which transforms the characteristic function of X_{t-1} to the characteristic function of X_t. Let

$$\mathcal{G}(\varphi)(t) = \mathcal{H}(\varphi)(t)\cdot\frac{1}{1+t^2}.$$

Proposition 4.13. *For all* $\gamma > 0$ *such that* $\alpha\gamma \neq 1$ *and* $\beta\gamma \neq 1$,

$$\mathcal{G}(j^+[\gamma]) = \frac{(\alpha\gamma)^2}{(\alpha\gamma)^2-1}j^+[\alpha\gamma] + \frac{1}{2(1-\alpha\gamma)}j^+[1] + \frac{1}{2(1+\alpha\gamma)}j^-[1] \tag{4.33}$$

and

$$\mathcal{G}(j^-[\gamma]) = \frac{(\beta\gamma)^2}{(\beta\gamma)^2-1}j^-[\beta\gamma] + \frac{1}{2(1+\beta\gamma)}j^+[1] + \frac{1}{2(1-\beta\gamma)}j^-[1]. \tag{4.34}$$

Proof. By definition of $\mathcal{G}$ and Proposition 4.12,

$$\mathcal{G}(j^+[\gamma])(t) = \frac{1+\mathrm{i}\alpha\gamma t}{1+\alpha^2\gamma^2t^2}\cdot\frac{1}{1+t^2}.$$

Partial fraction expansion yields

$$\begin{aligned}\mathcal{G}(j^+[\gamma])(t) &= \frac{(\alpha\gamma)^2}{(\alpha\gamma^2)-1}\frac{1+\mathrm{i}\alpha\gamma t}{1+(\alpha\gamma)^2t^2} + \frac{1}{1-(\alpha\gamma)^2}\frac{1+\mathrm{i}\alpha\gamma t}{1+t^2} \\ &= \frac{(\alpha\gamma)^2}{(\alpha\gamma^2)-1}j^+[\alpha\gamma] + \frac{1}{1-(\alpha\gamma)^2}(k_1[1]+\mathrm{i}\alpha\gamma k_2[1]).\end{aligned}$$

Substitute

$$k_1[1] = \frac{1}{2}(j^+[1]+j^-[1]) \qquad \text{and} \qquad k_2[1] = \frac{1}{2\mathrm{i}}(j^+[1]-j^-[1])$$

to get (4.33). Formula (4.34) results from similar calculations. □

Proposition 4.14. *Let* $\mu_q^+ = j^+[\alpha^q]$ *and* $\mu_q^- = j^-[\beta^q]$, $q \in \mathbb{N}_0$. *Then for every* $0 \le \alpha, \beta < 1$ *and* $q \in \mathbb{N}_0$ *we have*

$$\mathcal{G}(\mu_q^+) = c_1(\alpha,q)\mu_{q+1}^+ + c_2(\alpha,q)\mu_0^+ + c_3(\alpha,q)\mu_0^- \tag{4.35}$$

and

$$\mathcal{G}(\mu_q^-) = c_1(\beta,q)\mu_{q+1}^- + c_3(\beta,q)\mu_0^+ + c_2(\beta,q)\mu_0^- \tag{4.36}$$

where

$$c_1(\lambda,q) = \frac{\lambda^{2q+2}}{\lambda^{2q+2}-1}, \quad c_2(\lambda,q) = \frac{1}{2(1-\lambda^{q+1})}, \quad c_3(\lambda,q) = \frac{1}{2(1+\lambda^{q+1})}.$$

Proof. A direct consequence of Proposition 4.13. □

Obviously, any fixed point φ of the operator $\mathcal{G}$ satisfying $\varphi(0) = 1$ is the characteristic function of the stationary distribution of the TAR(1) process (4.29). By uniqueness of this distribution, it suffices to solve equation $\mathcal{G}\varphi = \varphi$. Observe that the operator $\mathcal{G}$ maps the linear hull of $\{\mu_q^+, \mu_q^-, q \in \mathbb{N}_0\}$ onto itself. Therefore we search for φ in the form $\sum_{q=0}^{\infty}(h_q^+\mu_q^+ + h_q^-\mu_q^-)$, where $h_q^\pm$, $q \in \mathbb{N}_0$, are unknown real constants (depending on parameters α and β).

Theorem 4.15. *The characteristic function* $\varphi = \varphi_{\alpha,\beta}$ *of the stationary distribution of the TAR(1) proces* (4.29) *is given by*

$$\varphi = h_0^+(\alpha,\beta)\left(\sum_{q=0}^{\infty} d_1(\alpha,q)\mu_q^+ + d_2(\alpha,\beta)\sum_{q=0}^{\infty} d_1(\beta,q)\mu_q^-\right) \tag{4.37}$$

where

$$d_1(\lambda,0) = 1, \qquad d_1(\lambda,q) = \prod_{j=0}^{q-1} c_1(\lambda,j) \qquad \text{for } q \ge 1, 0 \le \lambda < 1,$$

$$d_2(\alpha,\beta) = \frac{1 - \sum_{q=0}^{\infty} c_2(\alpha,q) d_1(\alpha,q)}{\sum_{q=0}^{\infty} c_3(\beta,q) d_1(\beta,q)}$$

and

$$h_0^+ := h_0^+(\alpha,\beta) = \left\{ \sum_{q=0}^{\infty} d_1(\alpha,q) + d_2(\alpha,\beta) \sum_{q=0}^{\infty} d_1(\beta,q) \right\}^{-1}. \tag{4.38}$$

Proof. Set

$$\sum_{q=0}^{\infty} (h_q^+ \mu_q^+ + h_q^- \mu_q^-)$$

and substitute it into $\mathcal{G}\varphi = \varphi$. Linearity of $\mathcal{G}$ and Proposition 4.14 give

$$\begin{aligned}\sum_{q=0}^{\infty} (h_q^+ \mu_q^+ + h_q^- \mu_q^-) = & \sum_{q=0}^{\infty} h_q^+ (c_1(\alpha,q)\mu_{q+1}^+ + c_2(\alpha,q)\mu_0^+ + c_3(\alpha,q)\mu_0^-) \\ & + \sum_{q=0}^{\infty} h_q^- (c_1(\beta,q)\mu_{q+1}^- + c_3(\beta,q)\mu_0^+ + c_2(\beta,q)\mu_0^-)\end{aligned}$$

Comparing the coefficients at $\mu_q^{\pm}$, $q \in \mathbb{N}_0$, yields

$$h_q^+ = c_1(\alpha, q-1) h_{q-1}^+, \quad q \geq 1, \tag{4.39}$$

$$h_q^- = c_1(\beta, q-1) h_{q-1}^-, \quad q \geq 1, \tag{4.40}$$

$$h_0^+ = \sum_{q=0}^{\infty} [c_2(\alpha,q) h_q^+ + c_3(\beta,q) h_q^-] \tag{4.41}$$

and

$$h_0^- = \sum_{q=0}^{\infty} [c_3(\alpha,q) h_q^+ + c_2(\beta,q) h_q^-]. \tag{4.42}$$

From the systems (4.39) and (4.40) we obtain

$$h_q^+ = d_1(\alpha,q) h_0^+, \qquad h_q^- = d_1(\beta,q) h_0^-, \qquad q \geq 1. \tag{4.43}$$

Inserting (4.43) into (4.41) gives the relation

$$h_0^- = d_2(\alpha,\beta) h_0^+$$

and thus, (4.37) is proved. The condition $\varphi(0) = 1$ implies (4.38). The convergence of all infinite series is assured by the fact that

$$d_1(\lambda,q) = (-1)^q \lambda^{q(q+1)} \left\{ \prod_{j=1}^{q} (1 - \lambda^{2j}) \right\}^{-1} = \mathcal{O}(\lambda^{q(q-1)}) \qquad \text{as } q \to \infty.$$

□

The next goal is to express all constants from Theorem 4.15 in simpler form. First, we need an auxiliary assertion.

Lemma 4.16. *For every* $0 \le \alpha < 1$, $d_2(\alpha, \alpha) = 1$.

Proof. By definition, $d_2(\alpha, \alpha) = 1$ iff

$$\sum_{q=0}^{\infty} (1 - \alpha^{2q+2})^{-1} d_1(\alpha, q) = 1.$$

Further,

$$\frac{d_1(\alpha, q)}{d_1(\alpha, q+1)} = \frac{1}{c_1(\alpha, q)} = \frac{\alpha^{2q+2} - 1}{\alpha^{2q+2}},$$

i.e.

$$(1 - \alpha^{2q+2})^{-1} d_1(\alpha, q) = -\alpha^{-2q-2} d_1(\alpha, q+1).$$

Therefore it suffices to prove

$$-\sum_{q=0}^{\infty} \alpha^{-2q-2} d_1(\alpha, q+1) = 1.$$

Since $d_1(\alpha, q) = (1 - \alpha^{-2q-2}) d_1(\alpha, q+1)$, we have

$$\sum_{q=0}^{\infty} d_1(\alpha, q) = \sum_{q=0}^{\infty} d_1(\alpha, q+1) - \sum_{q=0}^{\infty} \alpha^{-2q-2} d_1(\alpha, q+1).$$

Hence

$$-\sum_{q=0}^{\infty} \alpha^{-2q-2} d_1(\alpha, q+1) = \sum_{q=0}^{\infty} d_1(\alpha, q) - \sum_{q=0}^{\infty} d_1(\alpha, q+1) = d_1(\alpha, 0) = 1.$$

□

Now we define functions g and g_i, $i = 1, 2, 3$, which will be used to reformulate Theorem 4.15. Let

$$g_1(\lambda) = \sum_{q=0}^{\infty} d_1(\lambda, q), \quad g_2(\lambda) = 1 - \sum_{q=0}^{\infty} c_2(\lambda, q) d_1(\lambda, q) = \sum_{q=0}^{\infty} c_3(\lambda, q) d_1(\lambda, q),$$

(the last equality follows from Lemma 4.16),

$$g_3(\lambda) = g_1(\lambda)/g_2(\lambda), \qquad g(\alpha, \beta) = \frac{1}{g_3(\alpha) + g_3(\beta)}.$$

It is easy to show that

$$h_0^+(\alpha,\beta) = g(\alpha,\beta)\frac{g_3(\alpha)}{g_1(\alpha)}$$

and

$$h_0^+(\alpha,\beta)d_2(\alpha,\beta) = g(\alpha,\beta)\frac{g_3(\beta)}{g_1(\beta)}.$$

Remark 4.17. It was proved that functions $g_i, i = 1, 2, 3$, can be rewritten in the form

$$g_1(\lambda) = \prod_{j=1}^{\infty}(1-\lambda^{2j}) = \sum_{j=-\infty}^{\infty}(-1)^j\lambda^{(3j-1)j},$$

$$g_2(\lambda) = \frac{1}{2}\prod_{j=1}^{\infty}(1-\lambda^{2j-1})$$

and

$$g_3(\lambda) = 2\prod_{j=1}^{\infty}\frac{1-\lambda^{2j}}{1-\lambda^{2j-1}} = 2\sum_{j=0}^{\infty}\lambda^{j(j+1)/2},$$

see [15], Proposition 9 and Corollary 6.

Now we are ready to state simpler version of Theorem 4.15.

Theorem 4.18. *The stationary characteristic function $\varphi_{\alpha,\beta}$ can be written in the form*

$$\varphi_{\alpha,\beta}(t) = g(\alpha,\beta)\left[\frac{g_3(\alpha)}{g_1(\alpha)}\sum_{q=0}^{\infty}d_1(\alpha,q)\mu_q^+(t) + \frac{g_3(\beta)}{g_1(\beta)}\sum_{q=0}^{\infty}d_1(\beta,q)\mu_q^-(t)\right]. \tag{4.44}$$

Proof. Direct consequence of Theorem 4.15 and calculations above. □

Theorem 4.18 also allows to derive another form of the characteristic function $\varphi_\alpha(t)$ of the linear AR(1) process (4.24) and corresponding stationary density f_α.

Theorem 4.19. *Characteristic funtion φ_α is given by*

$$\varphi_\alpha(t) = \frac{1}{g_1(\alpha)}\sum_{q=0}^{\infty}\frac{d_1(\alpha,q)}{1+\alpha^{2q}t^2}.$$

Proof. The AR(1) process (4.24) is the special case of the TAR(1) process (4.29) when $\alpha = \beta$. Hence (4.44) implies

$$\begin{aligned}\varphi_\alpha = \varphi_{\alpha,\alpha} &= g(\alpha,\alpha)\frac{g_3(\alpha)}{g_1(\alpha)}\left[\sum_{q=0}^{\infty} d_1(\alpha,q)(\mu_q^+ + \mu_q^-)\right] \\ &= \frac{1}{g_1(\alpha)}\sum_{q=0}^{\infty}\frac{d_1(\alpha,q)}{1+\alpha^{2q}t^2}.\end{aligned}$$

□

Theorem 4.20. *The stationary marginal density f_α of the AR(1) process (4.24) is given by*

$$f_\alpha(x) = \frac{1}{2g_1(\alpha)}\sum_{q=0}^{\infty} d_1(\alpha,q)\alpha^{-q}\mathrm{e}^{-\alpha^{-q}|x|}.$$

Proof. Follows immediately from Theorem 4.19 since

$$\mathcal{F}^{-1}\left(\frac{1}{1+\alpha^{2q}t^2}\right) = \frac{1}{2}\alpha^{-q}\mathrm{e}^{-\alpha^{-q}|x|}.$$

□

Remark 4.21. Some properties of functions f_α and $f_{\alpha,\beta}$ can be found in [15], Lemma 4 and Corollary 5.

Theorem 4.22. *The stationary marginal density $f_{\alpha,\beta}$ of the TAR(1) process (4.29) is given by*

$$f_{\alpha,\beta}(x) = 2g(\alpha,\beta)[g_3(\alpha)f_\alpha(x)\chi_{[0,\infty)} + g_3(\beta)f_\beta(x)\chi_{(-\infty,0)}].$$

Proof. Follows from Theorems 4.18 and 4.20 since

$$\mathcal{F}^{-1}(\mu_q^+)(x) = \alpha^{-q}\mathrm{e}^{-\alpha^{-q}|x|}\chi_{[0,\infty)}$$

and

$$\mathcal{F}^{-1}(\mu_q^-)(x) = \beta^{-q}\mathrm{e}^{-\beta^{-q}|x|}\chi_{(-\infty,0)}.$$

□

Theorem 4.23. *The moments of the stationary TAR(1) process (4.29) are given by*

$$\mathsf{E}\,X^p = p!\cdot g(\alpha,\beta)\left(\frac{g_3(\alpha)}{g_1(\alpha)}\sum_{q=0}^{\infty} d_1(\alpha,q)\alpha^{pq} + (-1)^p\frac{g_3(\beta)}{g_1(\beta)}\sum_{q=0}^{\infty} d_1(\beta,q)\beta^{pq}\right). \tag{4.45}$$

for every $p \in \mathbb{N}$.

Proof. According to Theorem 4.22,

$$\mathsf{E}\,X^p = 2g(\alpha,\beta)\left(g_3(\alpha)\int_0^\infty x^p f_\alpha(x)\,\mathrm{d}x + g_3(\beta)\int_{-\infty}^0 x^p f_\beta(x)\,\mathrm{d}x\right),$$

where f_α and f_β are defined in Theorem 4.20. Since

$$\int_0^\infty x^p \mathrm{e}^{-\alpha^{-q}x}\,\mathrm{d}x = \alpha^{pq+q}\int_0^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y = \alpha^{pq+q}p!$$

we have

$$\begin{aligned}\int_0^\infty x^p f_\alpha(x)\,\mathrm{d}x &= \frac{1}{2g_1(\alpha)}\sum_{q=0}^\infty d_1(\alpha,q)\alpha^{-q}\int_0^\infty x^p \mathrm{e}^{-\alpha^{-q}|x|}\,\mathrm{d}x \\ &= \frac{p!}{2g_1(\alpha)}\sum_{q=0}^\infty d_1(\alpha,q)\alpha^{pq}\end{aligned}$$

and similarly

$$\int_{-\infty}^0 x^p f_\beta(x)\,\mathrm{d}x = \frac{p!(-1)^p}{2g_1(\beta)}\sum_{q=0}^\infty d_1(\beta,q)\beta^{pq}.$$

□

Remark 4.24. The moments of the TAR(1) process can be also expressed in the form

$$\mathsf{E}\,X^{2p-1} = 2g(\alpha,\beta)(2p-1)!\left[\left\{\prod_{j=1}^p(1-\alpha^{2j-1})\right\}^{-1} - \left\{\prod_{j=1}^p(1-\beta^{2j-1})\right\}^{-1}\right],$$

$$\mathsf{E}\,X^{2p} = g(\alpha,\beta)(2p)!\left[g_3(\alpha)\left\{\prod_{j=1}^p(1-\alpha^{2j})\right\}^{-1} + g_3(\beta)\left\{\prod_{j=1}^p(1-\beta^{2j})\right\}^{-1}\right]$$

for every $p \in \mathbb{N}$, see [15], Theorem 4.

The stationary densities f_α and $f_{\alpha,\beta}$ for various values of parameters are shown in Figures 19 and 20. Corresponding moments are summarized in Tables 4 and 5.

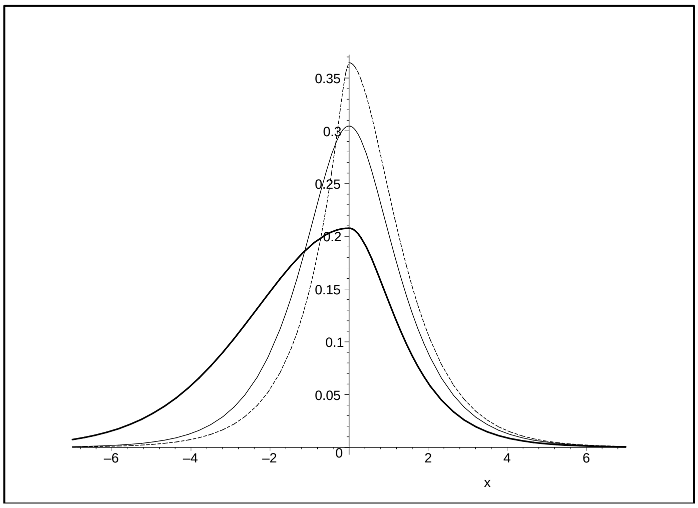

Figure 19: Density $f_{\alpha,\beta}(x)$ for $\alpha = 0.5$ and $\beta = 0.1$ (dashed), 0.5 (solid) and 0.85 (solid thick)

Table 4: Moments of the TAR process with $\alpha = 0.5$

	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.85$
$\mathsf{E}\, X_t$	0.324	0.000	−0.969
$\mathsf{E}\, X_t^2$	2.407	2.667	5.659
$\mathsf{E}\, X_t^3$	2.567	0.000	−18.683
$\mathsf{E}\, X_t^4$	30.164	34.133	130.878

4.3.3 Model with parameters with opposite signs

Now we very briefly state the results for the case $0 < \alpha < 1$, $\beta < 0$. The methods of calculation are very similar to those desrcibed above.

To keep the notation simple, we introduce the following constants and functions (in addition to those already defined). Let

$$\mu_{\alpha,0} = j^+[1], \qquad \mu_{\alpha,q} = j^+[\alpha^q], \quad q \in \mathbb{N},$$

$$\nu_{\alpha,\beta,0} = j^-[1], \qquad \nu_{\alpha,\beta,q} = j^+[|\beta|\alpha^{q-1}], \quad q \in \mathbb{N},$$

$$c_4(\alpha, \beta, q) = \frac{\alpha^{2q}\beta^2}{\alpha^{2q}\beta^2 - 1}, \quad q \in \mathbb{N}_0,$$

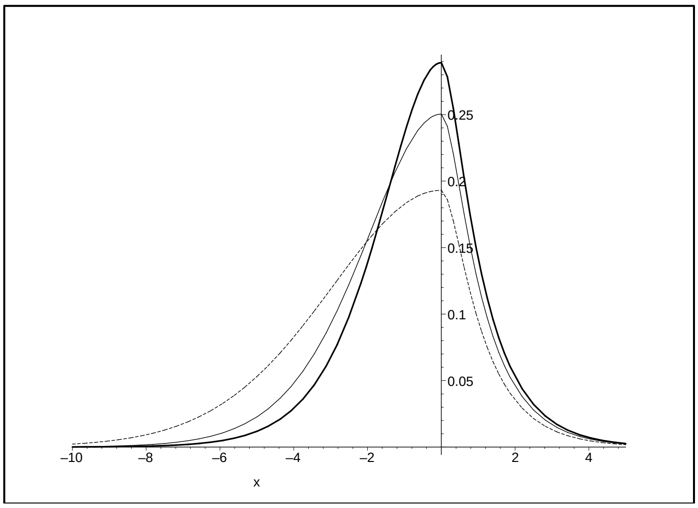

Figure 20: Density $f_{\alpha,\beta}(x)$ for $\alpha = 0.25$ and $\beta = 0.9$ (dashed), 0.8 (solid) and 0.7 (solid thick)

Table 5: Moments of the TAR process with $\alpha = 0.25$

	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
$\mathsf{E}\, X_t$	-1.673	-0.918	-0.578
$\mathsf{E}\, X_t^2$	8.475	4.471	3.267
$\mathsf{E}\, X_t^3$	-41.187	-13.357	-6.449
$\mathsf{E}\, X_t^4$	283.796	85.276	48.674

$$c_5(\alpha, \beta, q) = \frac{1}{2(1 - |\beta|\alpha^q)}, \quad q \in \mathbb{N}_0,$$

$$d_3(\alpha, \beta, 0) = 1, \qquad d_3(\alpha, \beta, q) = \prod_{j=0}^{q-1} c_4(\alpha, \beta, j), \quad q \in \mathbb{N}.$$

Further, let

$$g_5(\alpha, \beta) = \sum_{q=0}^{\infty} d_3(\alpha, \beta, q),$$

$$g_6(\alpha, \beta) = \sum_{q=0}^{\infty} c_5(\alpha, \beta, q) d_3(\alpha, \beta, q),$$

$$g_7(\alpha,\beta,p) = |\beta|^p \sum_{q=1}^{\infty} d_3(\alpha,\beta,q)\alpha^{(q-1)p}, \quad p \in \mathbb{N},$$

$$g_8(\alpha,\beta) = \frac{g_2(\alpha)}{g_6(\alpha,\beta)},$$

$$g_9(\alpha,\beta) = [g_3(\alpha)g_6(\alpha,\beta) + g_5(\alpha,\beta)]^{-1}$$

and finally

$$h(\alpha,\beta) = [g_1(\alpha) + g_5(\alpha,\beta)g_8(\alpha,\beta)]^{-1}.$$

Theorem 4.25. *The characteristic function $\varphi = \varphi_{\alpha,\beta}$ of the stationary distribution of the TAR(1) process* (4.29) *with $0 < \alpha < 1$ and $\beta < 0$ is given by*

$$\varphi = h(\alpha,\beta)\left(\sum_{q=0}^{\infty} d_1(\alpha,q)\mu_{\alpha,q} + g_8(\alpha,\beta)\sum_{q=0}^{\infty} d_3(\alpha,\beta,q)\nu_{\alpha,\beta,q}\right).$$

Theorem 4.26. *The stationary marginal density $f_{\alpha,\beta}$ of the TAR(1) proces* (4.29) *with $0 < \alpha < 1$ and $\beta < 0$ has the form*

$$f_{\alpha,\beta}(x) = h(\alpha,\beta)g_8(\alpha,\beta)\mathrm{e}^{x}\chi_{(-\infty,0)}(x) + h(\alpha,\beta)\left[r_1(x,\alpha) + g_8(\alpha,\beta)r_2(x,\alpha,\beta)\right]\chi_{[0,\infty)}(x)$$

where

$$r_1(x,\alpha) = \sum_{q=0}^{\infty} d_1(\alpha,q)\alpha^{-q}\mathrm{e}^{-\alpha^{-q}x}$$

and

$$r_2(x,\alpha,\beta) = |\beta|^{-1}\sum_{q=1}^{\infty} d_3(\alpha,\beta,q)\alpha^{1-q}\mathrm{e}^{-\alpha^{1-q}|\beta|^{-1}x}.$$

Theorem 4.27. *The moments of the TAR(1) process* (4.29) *with $0 < \alpha < 1$ and $\beta < 0$ are given by*

$$\mathsf{E}\,X^{2p-1} = (2p-1)!g_9(\alpha,\beta)\left[\frac{2g_6(\alpha,\beta)}{\prod_{j=1}^{p}(1-\alpha^{2j-1})} - 1 + g_7(\alpha,\beta,2p-1)\right]$$

$$\mathsf{E}\,X^{2p} = (2p)!g_9(\alpha,\beta)\left[\frac{g_3(\alpha)g_6(\alpha,\beta)}{\prod_{j=1}^{p}(1-\alpha^{2j})} + 1 + g_7(\alpha,\beta,2p)\right]$$

for every $p \in \mathbb{N}$.

Densities $f_{\alpha,\beta}$ for various values of parameters are plotted in Figure 21. Corresponding moments are shown in Table 6.

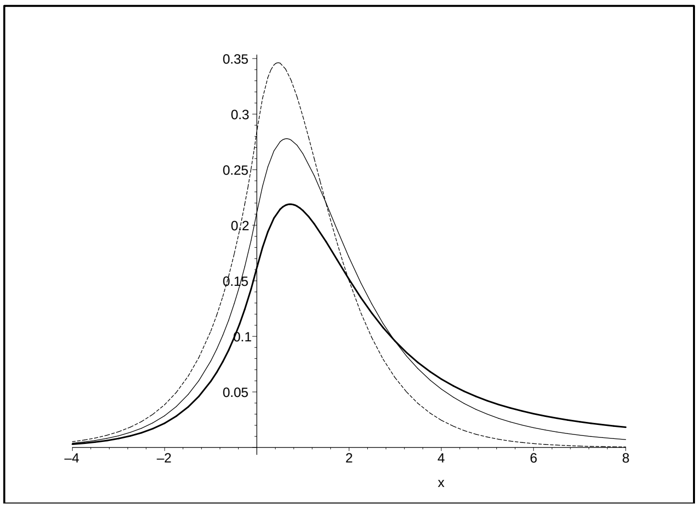

Figure 21: Density $f_{\alpha,\beta}(x)$ for $\alpha = 0.5$ and $\beta = -0.75$ (dashed), -3 (solid) and -10 (solid thick)

Table 6: Moments of the TAR process with $\alpha = 0.5$

	$\beta = -0.75$	$\beta = -3$	$\beta = -10$
$\mathsf{E}\, X_t$	0.711	1.479	3.392
$\mathsf{E}\, X_t^2$	2.904	7.595	45.635
$\mathsf{E}\, X_t^3$	5.939	49.428	1131.065
$\mathsf{E}\, X_t^4$	39.013	534.891	41936.726

4.3.4 Model with parameters with opposite signs and the same absolute value

Consider now TAR(1) model (4.29) with $0 < \alpha < 1$ and $\beta = -\alpha$ which is in fact the AAR(1) model (4.4) with positive parameter a. In such case the results from the previous section simplify significantly.

Theorem 4.28. *The characteristic function φ of the stationary distribution of the TAR(1) process* (4.29) *with* $0 < \alpha < 1$ *and* $\beta = -\alpha$ *is given by*

$$\varphi(t) = -\frac{2\mathrm{i}}{g_3(\alpha)}\frac{t}{1+t^2} + \frac{1}{g_1(\alpha)}\sum_{q=0}^{\infty} d_1(\alpha, q)\mu_{\alpha,q}(t).$$

Theorem 4.29. *The stationary marginal density f of the TAR(1) proces* (4.29) *with* $0 < \alpha < 1$ *and* $\beta = -\alpha$ *has the form*

$$f(x) = \frac{\mathrm{e}^x}{g_3(\alpha)}\chi_{(-\infty,0)}(x) + \left(\frac{\sum_{q=0}^{\infty} d_1(\alpha, q)\alpha^{-q}\mathrm{e}^{-\alpha^{-q}x}}{g_1(\alpha)} - \frac{\mathrm{e}^{-x}}{g_3(\alpha)} \right) \chi_{[0,\infty)}(x).$$

Theorem 4.30. *The moments of the TAR(1) process* (4.29) *with* $0 < \alpha < 1$ *and* $\beta = -\alpha$ *are given by*

$$\mathsf{E}\,X^{2p-1} = \frac{2(2p-1)!}{g_3(\alpha)} \left[\left\{ \prod_{j=1}^{p} (1 - \alpha^{2j-1}) \right\}^{-1} - 1 \right]$$

$$\mathsf{E}\,X^{2p} = (2p)! \left\{ \prod_{j=1}^{p} (1 - \alpha^{2j}) \right\}^{-1}$$

for every $p \in \mathbb{N}$.

Densities f for various values of parameter α are plotted in Figure 22. Corresponding moments are shown in Table 7.

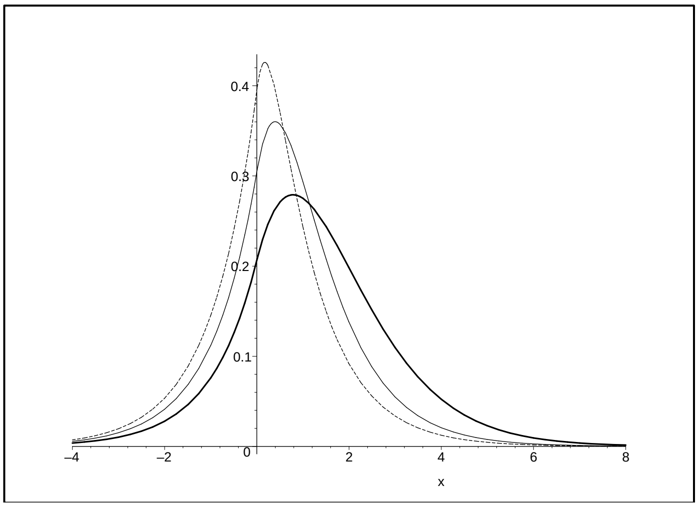

Figure 22: Density $f(x)$ for $\alpha = 0.25$ (dashed), 0.5 (solid) and 0.75 (solid thick), $\beta = -\alpha$.

Table 7: Moments of the TAR process with $\beta = -\alpha$

	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
$\mathsf{E}\, X_t$	0.263	0.609	1.239
$\mathsf{E}\, X_t^2$	2.133	2.667	4.571
$\mathsf{E}\, X_t^3$	1.680	4.699	14.661
$\mathsf{E}\, X_t^4$	25.700	34.133	80.248

Example 4.31 (Absolute autoregression with Laplace innovations, continued). In Example 4.8 (p. 49) we constructed an approximation of the stationary density of the AR(1) and AAR(1) processes driven by Laplace innovations. We compared densities g_3 and h_3 with the explicit results obtained by the methods described above. We received almost exact match, the maximum difference is about 0.0016 and 0.0004, respectively. In Figures 23 and 24, exact stationary densities of AR(1) and AAR(1) processes are plotted. Curves illustrating approximate densities g_3 and h_3, respectively, would be virtually identical.

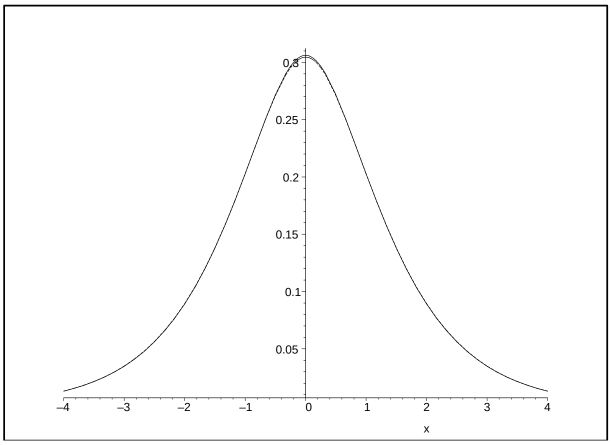

Figure 23: Sstationary density of AR(1) process for $\alpha = 0.5$

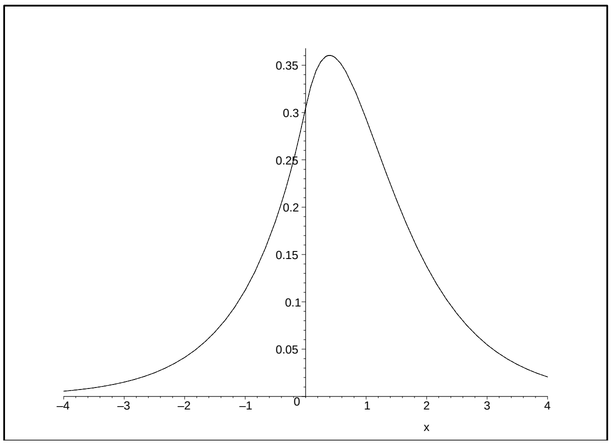

Figure 24: Stationary density of AAR(1) process for $\alpha = 0.5$

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