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Lorentz group and its application in the theory of quantum gravity

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Abstract: In this thesis we are dealing with basic methods of theoretical physics focusing on quantum theory of gravity, that are: Hamilton-Dirac formalism for singular systems, Dirac‘s method of quantizing systems with constraints and its mathematical formulation - refined algebraic quantization, representation of compact groups and representation of Lorentz group. We apply these methods to find eigenstates of Lorentz group and General linear group generators. We construct a physical Hilbert space on temporal part of 3+1 decomposition of Einstein-Cartan theory.

Keywords: Lorentz group, Dirac Formalism, Refined Algebraic Quantization, Quantum Gravity

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Introduction

Einstein-Cartan theory (ECT) is a modification of Einstein's *general theory of relativity* (GR). Basic variables of the theories are either metric tensor, or coframe field and connection. We adopt the latter approach concerning coframe and connection as basic variables. Metric can be reconstructed via coframe field.

GR a priori sets the connection to be RLC[1] connection, that is both *metric compatible* (annihilated by covariant derivative) and *symmetric* (vanishing torsion). In this case the connection is uniquely given by metric which is constructed via coframe. That means the independent variables in GR is just the coframe field. In ECT, on the other hand, we do not set connection to be RLC. We treat the connection as another independent variable, with respect to which we must also perform a variation procedure of a given action functional.

The action functional of GR[2] is

$$S_{EH}[e] = \frac{1}{16\pi G}\int R_g\,\omega_g = \frac{1}{32\pi G}\int \epsilon_{abcd}\,g^{b\bar{b}}\,\Omega^a_{\bar{b}}(\mathrm{g}) \wedge \mathrm{e}^c \wedge \mathrm{e}^d \tag{1}$$

where e^a is the coframe field, $\mathrm{g} = \eta_{ab}\,\mathrm{e}^a \otimes \mathrm{e}^b$ is metric, $\eta_{ab} = diag(-,+,+,+)$ is Minkowski metric, R_g is a Ricci scalar curvature of the RLC connection of metric g, $\omega_g = \sqrt{-\det g}\,\mathrm{d}^4x$ its metric volume form and $\Omega^a_b(g)$ is a curvature 2-form of RLC connection. G is a gravitational constant. For simplicity we have set the speed of light $c = 1$.

If we define a general connection 1-form as

$$\omega^a_b = \Gamma^a_{bc}\mathrm{e}^c$$

with Γ^a_{bc} being coefficients of linear connection[3], then the curvature 2-form Ω is given by one of the two *Cartan structure equations*, namely by

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

where d denotes exterior derivative. This curvature 2-form may be also expressed via Riemann tensor R^a_{bcd} as follows

$$\Omega^a_b = \frac{1}{2}R^a_{bcd}\,\mathrm{e}^c \wedge \mathrm{e}^d.$$

In GR the connection 1-forms are uniquely given by metric g expressed via coframe, therefore the dependence of Ω on g in (1).

The action for ECT looks the same

$$S_{ECT}[e,\omega] = \frac{1}{32\pi G}\int \epsilon_{abcd}\,g^{b\bar{b}}\,\Omega^a_{\bar{b}}(\omega) \wedge \mathrm{e}^c \wedge \mathrm{e}^d \tag{2}$$

only now we treat the connection 1-form ω as independent variable and we do not fix it to be RLC connection. It is important to note that dependance on ω is only in Ω.

[1] *Riemann* or *Levi-Civita* connection

[2] So called *Einstein-Hilbert* action

[3] Christoffel symbols of second kind in case of coordinate frame $\mathrm{e}^i = \mathrm{d}x^i$

ECT and GR are equivalent theories, unless we also take into account a *spinor* fields as possible matter fields. This comes from the fact that spinor field interacts with the connection, which now would explicitly enter the action (2). Without this interaction, connection is implicitly "hidden" in a curvature 2-form and nowhere else. Everything else depends on the coframe (as in case of GR). Contribution in variation of curvature 2-form with respect to the connection 1-form yields a part of the resulting varied form[4] which happens to be an exact form (thus not contributing into the dynamics). Another part of the varied form is just a condition on vanishing torsion. Thus, in this case, ECT and GR are equivalent. If we introduce spinor fields (so action depends *explicitly* on connection), then variation with restpect to connection results in part of the varied form, which in contrast to the "exact" part, contributes into non-vanishing torsion. Now ECT and GR are not equivalent theories. Also the *stress-energy tensor* is no longer traceless, nor symmetric and one needs to introduce improvement terms in order to "repair" this cannonical stress-energy tensor. For more details on difference in ECT and GR see for example book of Rovelli and Vidotto [Rovelli and Vidotto, 2014].

In this thesis we do not consider any matter fields. We are interested in case of pure gravity. Therefore ECT and GR are, in our case, equivalent, so we need not distinguish them.

In this thesis we focus on quantizing ECT in the temporal part of its 3+1 decomposition. In the following chapter we briefly summarize result of its 3+1 decomposition given in [Pilc, 2013b]. It was necessary to use *Dirac approach* to Hamilton dynamics (Appendix A), becouse it turns out that we are dealing with a singular system containing first class constraints. In another chapters we partially quantize the system by introducing a suitable Hilbert space on which we represent basic variables. In constructing physical states of the theory, we use methods of *Refined algebraic quantization* (RAQ), namely *group averaging procedure*.

[4] $\Omega(\omega + \epsilon\sigma)$, ϵ being an infinitesimal parameter

1. Summary of the theory

Our thesis focuses on Eistein-Cartan theory of gravity (see for example [Nikolič, 1995, Kibble, 1961, Hehl et al., 1976] and [Hehl et al., 1995]) and partial quantization of its phase space. This system is impossible to be quantized directly since it is a system with first class constraints. In appendix A we briefly summarize a general approach to systems with singular Lagrangian, according to Dirac. For convenience we set the speed of light $c = 1$ and the Planck‘s constant $\hbar = 1$ throughout the entire thesis.

This chapter briefly summarizes results of 3+1 decomposition of ECT. Following results are a short compilation of two papers [Pilc, 2013b,a].

The basic variables are the coframe components e^a. We assume that we have an orientable space-time manifold M, equipped with metric g (g as in introduction) and that the manifold can be written as product $\mathrm{M} = \mathbb{R} \times \Sigma$. Here we also assume Σ is compact orientable manifold and that M admits existence of *global* orthonormal frame e_a over M. Our frame e_a (or its dual e^a) is chosen and oriented in such a way that Σ is spatial.

A 3+1 decomposition of coframe can be performed

$$\mathbf{e}^a = \lambda^a \mathrm{d}t + \mathbf{E}^a = \lambda^a \mathrm{d}t + E^a_\alpha \mathrm{d}x^\alpha,$$

where $\alpha = 1, 2, 3$ and a configuration space is defined as

$$\mathfrak{Conf} = \{(\lambda^a, \mathbf{E}^a);\ e > 0,\ \eta_{ab}\lambda^a\lambda^b < 0,\ \lambda^0 > 0, \mathbf{q} > 0\},$$

with $e = \frac{1}{3!}\,\epsilon_{abcd}\,\epsilon^{\alpha\beta\gamma}\,\lambda^a E^b_\alpha E^c_\beta E^d_\gamma$ being determinant of matrix (λ^a, E^a_α) and $\mathbf{q} = \eta_{ab}\mathbf{E}^a \otimes \mathbf{E}^b$ is a spatial metric tensor on Σ. We can write $\mathfrak{Conf}$ as $\mathfrak{Conf} = \Lambda \times \mathcal{E}$, where

$$\Lambda := \{\lambda^a;\ \eta_{ab}\lambda^a\lambda^b < 0,\ \lambda^0 > 0\}$$

and

$$\mathcal{E} := \{\mathbf{E}^a;\ \mathbf{q} > 0,\ \forall v^a \in \Lambda :\ e(v) = \frac{1}{3!}\,\epsilon_{abcd}\,\epsilon^{\alpha\beta\gamma}\,v^a E^b_\alpha E^c_\beta E^d_\gamma > 0\}.$$

Reduced phase space was described by coordinates (λ^a, $\mathbf{E}^a$, π_a, $\mathbf{F}_a$) where π_a, $\mathbf{F}_a$ are conjugate momenta of λ^a, $\mathbf{E}^a$ respectively. System possesses second class constraints, therefore its symplectic structure is not given by Poisson brackets but rather by Dirac brackets (see Appendix A). Dirac brackets for this system are discussed in [Pilc, 2013b, equations (97) and (98)].

Kinematical Hilbert space at space-time point x can be constructed as

$$\mathrm{H}_x = \mathbf{L}^2(\Lambda, \mathrm{d}\Lambda) \otimes \mathbf{L}^2(\mathcal{E}, \mathrm{d}\mathcal{E}), \tag{1.1}$$

where $\mathrm{d}\Lambda$ denotes measure on Λ (it will be specified in the next chapter) and $\mathrm{d}\mathcal{E}$ denotes measure on $\mathcal{E}$. Subspaces spanned by (λ^a, π^a) or $(\mathbf{E}^a, \mathbf{F}^a)$ can be treated independently. We focus on the temporal part of coframe decomposition in this thesis.

We cannot quantize the subspace given by (λ^a, π^a) directly, as we would in standard courses of quantum mechanics. We cannot for example represent π^a as $-i\partial_{\pi^a}$ on $\mathcal{H} = \mathbf{L}^2(\Lambda, \mathrm{d}\Lambda)$. The reason is, that its action

$$e^{i\tau\hat{\pi}}\psi(\lambda) = e^{\tau^a\partial_{\lambda^a}}\psi(\lambda^b) = \psi(\lambda^b + \tau^a) \tag{1.2}$$

does not leave the space Λ invariant. Vectors $\psi(\lambda)$ on $\mathcal{H}$ have support on a future-oriented lightcone[1] and for some τ the support is shifted outside the cone.

Fortunatelly, we can use the Stone's theorem [Blank et al., 2008]:

Theorem 1 (Stone)**.** *For any strongly continuous one-parameter unitary group $\{U(s) : s \in \mathbb{R}\}$ there is just one self-adjoint operator A such that $U(s) = e^{isA}$ holds for all $s \in \mathbb{R}$.*

We need to find some group, acting on Λ such that its action preserves this space. A Lorentz group is a good candidate. The transformation is

$$\lambda^a \to \left(e^{-\Lambda\eta}\right)_b^a \lambda^b, \tag{1.3}$$

where $(\Lambda\eta)_b^a = \Lambda^{ac}\eta_{cb}$ and Λ^{ab} is antisymetric Lorentz matrix. It holds

- unitary representation of the group on $\mathcal{H}$: $U(\Lambda^{ab})\psi(\lambda^a) = \psi(\left(e^{-\Lambda\eta}\right)_b^a \lambda^b)$,
- $U_\Lambda(t) = U(t\,\Lambda^{ab})$ is strongly continuous.

Continuity and Stone's theorem yield existance of a self-adjoint generator and we can write

$$U(\Lambda^{ab}) = e^{i\frac{1}{2}\Lambda^{ab}L_{ab}} \tag{1.4}$$

the generator is

$$L(\Lambda) = \frac{1}{2}\Lambda^{ab}L_{ab} = i\Lambda^{ab}\eta_{bc}\lambda^c\partial_{\lambda^a}. \tag{1.5}$$

There is another group, acting on our space and preserving it. It is the $\mathrm{GL}^+(\mathbb{R})$ group that represents the fact that we can arbitrarily stretch the vectors ∂_t. So we define transformation and representation of the group on $\mathcal{H}$ as

- transformation: $\lambda^a \to e^{-\tau}\lambda^a$,
- unitary representation: $U(\tau)\psi(\lambda^a) = \psi(e^{-\tau}\lambda^a)$.

with $U(\tau) = e^{i\tau\pi}$. The self-adjoint generator is

$$\pi = -i\lambda^a\partial_{\lambda^a}. \tag{1.6}$$

More details regarding representations of groups can be found many publlications. For compact groups see for exampla [Bröcker and Dieck, 1998] and for the Lorentz group see [I.M. et al., 1963, Naimark and Farahat, 1964].

[1] Space Λ is a futire-oriented lightcone

2. Representation of generators of gauge group on Hilbert space

We have introduced self-adjoint generators in the previous chapter :

- $\pi = -i\lambda^a \partial_{\lambda^a}$ for the general linear group,
- $L(\Lambda) = i\Lambda^{ab}\eta_{bc}\lambda^c\partial_{\lambda^a}$ for the Lorentz group.

We introduce self-adjoint operators (using a shorthand notation $\partial_a := \partial_{\lambda^a}$)

- $\hat{\pi} := -i\lambda^a\partial_a$
- $\hat{L}_{ab} := -i\eta_{c[a}\lambda^c\partial_{b]}$

on a Hilbert space

$$\mathcal{H} = \mathrm{L}^2\left(\Lambda, \frac{\mathrm{d}^4\lambda}{\lambda^4}\right) \tag{2.1}$$

where $\Lambda := \left\{\lambda^a, \lambda^2 := -\eta_{ab}\lambda^a\lambda^b, \lambda^0 > 0\right\}$ and $\mathrm{d}^4\lambda := \mathrm{d}\lambda^0\mathrm{d}\lambda^1\mathrm{d}\lambda^2\mathrm{d}\lambda^3$. Λ is a future oriented light cone. The measure $\frac{\mathrm{d}^4\lambda}{\lambda^4}$ is both lorentz group $\mathrm{SO}^+(1,3)$ and general linear group $\mathrm{GL}^+(\mathbb{R})$ invariant, as we now prove.

Proof. Lorentz group leaves the space-time interval λ^2 invariant, becouse the Lorentz matrix Λ is orthogonal, therefore the denominator of the measure is Lorentz invariant. Transformation of the numerator reads $(\mathrm{d}\lambda)' = \det\Lambda\,\mathrm{d}\lambda$, however the determinant of Λ is 1 for $\mathrm{SO}^+(1,3)$ part of $\mathrm{SO}(1,3)$.

As for the group GL^+ we have $(\lambda)' = e^{-\tau}\lambda$ and $(\mathrm{d}^4\lambda)' = (e^{-\tau})^4\mathrm{d}^4\lambda$. So indeed, the measure is $\mathrm{SO}^+(1,3)$ and $\mathrm{GL}^+(\mathbb{R})$ invariant.

□

The scalar product on $\mathcal{H}$ is given by

$$(\phi, \psi) = \int \frac{\mathrm{d}^4\lambda}{\lambda^4}\,\phi(\lambda)\,\psi^*(\lambda). \tag{2.2}$$

Let us see what the metric, $\mathrm{g} = g_{ab}\mathrm{d}\lambda^a\mathrm{d}\lambda^b$ on $\mathcal{H}$, looks like. First of all, we have chosen coordinates λ^a to form an orthogonal coordinate system, thus the metric is diagonal:

$$\mathrm{g} = k(\lambda)\eta_{ab}\mathrm{d}\lambda^a\mathrm{d}\lambda^b.$$

Further, we have already introduced an invariant measure $\frac{\mathrm{d}^4\lambda}{\lambda^4} = \sqrt{-\det \mathrm{g}}\,\mathrm{d}^4\lambda$. This condition on the determinant of the metric gives us the final form

$$\mathrm{g} = \frac{1}{\lambda^2}\eta_{ab}\mathrm{d}\lambda^a\mathrm{d}\lambda^b. \tag{2.3}$$

We easily see that this metric is both Lorentz group and $\mathrm{GL}^+(\mathbb{R})$ invariant.

The algebra of operators $\hat{\pi}$ and $\hat{L}_{ab}$ is

$$[\hat{\pi}, \hat{\pi}] = 0, \tag{2.4}$$

$$[\hat{L}_{ab}, \hat{\pi}] = 0, \tag{2.5}$$

$$[\hat{L}_{ab}, \hat{L}_{cd}] = i\left(\eta_{ac}\hat{L}_{bd} + \eta_{bd}\hat{L}_{ac} - \eta_{ad}\hat{L}_{bc} - \eta_{bc}\hat{L}_{ad}\right). \tag{2.6}$$

We prove this in Appendix B.

Operators $\hat{\pi}$ and $\hat{\mathrm{L}}_{ab}$ form a representation of $\mathrm{GL}^+(\mathbb{R})$ and $\mathrm{SO}^+(1,3)$, respectively, on $\mathcal{H}$. These representations are unitary (simply becouse the groups act as a complex exponential[1]) and infinite-dimensional (the representation space $\mathcal{H}$ is infinite-dimensional space of functions). In case of $\hat{\pi}$ the representation is irreducible, since its only invariant subspaces are those trivial ones (the whole $\mathcal{H}$ and that transforming under identity element of $\mathrm{GL}^+(\mathbb{R})$). Otherwise, action of this group transports points on $\mathcal{H}$ along $\lambda = \sqrt{-\eta_{ab}\,\lambda^a\lambda^b}$ by distance λ.

Representation of Lorentz group is also irreducible. Lorentz group acts on hyperboloids (points in λ distance from origin[2].). There exist non trivial invariant subspaces transforming under a subgroup SO(3). That is, if we choose λ^0 to serve as time direction, SO(3) elements preserve this axis. The generators of $\mathrm{SO}^+(1,3)$ can be decomposed into generators of rotations and generators of boosts [Naimark and Farahat, 1964]. We have chosen the coordinates in such a way, that λ^0 is preserved uder rotations. So the representation of Lorenzt group is also a *reducible* representation of SO(3). This representation can be decomposed into SO(3) invariant blocks[3] within which there acts an irreducible representation of rotation group of a given spin. However the representation of the whole Lorentz group itself is irreducible, due to the existence of boost generators that moves from one SO(3) block to another, thus the only invariant subspaces are, again, only the trivial ones.

Boosts[4] act on hyperboloids and each element of boosts can be decomposed as an element of SO(2) and one-parametric boost (Cartan decomposition [Naimark and Farahat, 1964]). This can be pictorially imagined as painting a circle, which we then expand in 'radial' direction, in such way that its 2-dimensional cut, perpendicular to the plane of the circle, is a hyperbola. Hyperboloid is thus a homogeneous[5] space under boosts. It is also homogeneous under whole Lorentz group, that can be writen[6] as $\mathrm{SO}^+(1,3) = \mathrm{SO}(3) \times \mathrm{SO}^+(1,3)/\mathrm{SO}(3)$. The space of elements of boosts, $\mathrm{SO}^+(1,3)/\mathrm{SO}(3)$, can be identified with upper sheet of hyperboloid. If we remind ourselves of the fact that group $\mathrm{GL}^+(\mathbb{R})$ essentially distributes a hyperboloid over whole space Λ (it stratches the hyperboloid's radius λ, thus moving it "upwards" in the light cone), we can say that the whole light cone Λ is homogeneous space under group $\mathrm{GL}^+(\mathbb{R}) \times \mathrm{SO}^+(1,3)$.

[1]Then we have $||\hat{U}(g)\phi|| = ||\phi||$.

[2]That means all points λ^a satisfying a constraint $\lambda = \sqrt{-\eta_{ab}\,\lambda^a\lambda^b}$

[3]In terms of representation by Wigner matricies, which is well known from the theory of angular momentum.

[4]Boosts do not form a subgroup of SO(1,3).

[5]Space S is homogeneous under a group G, if any to points s, s' can be connected by some element $g \in G$, that is $s' = g\, s$.

[6]This must not be understood as a product in sense of groups - Lorentz group is not a direct product of goup of rotations and boost, boosts do not form a subgoup of Lorentz - but we understand it in sense of product of vector spaces!

3. Eigenstates for the representation of generators of the gauge group

We have already introduced representation of the Lorentz group in terms of operators $\hat{L}_{ab} = -i\,\eta_{a[c}\lambda^c\partial_{b]}$ on $\mathcal{H}$. $SO^+(1,3)$ is group of rank 2 (two independent commuting operators are $\hat{L}_{01}$ and $\hat{L}_{23}$). There exist two Casimir operators

$$\begin{aligned} \hat{C}_1 &= \frac{1}{2}\hat{L}_{ab}\hat{L}^{ab}, \\ \hat{C}_2 &= \frac{1}{8}\,\epsilon^{abcd}\hat{L}_{ab}\hat{L}_{cd}. \end{aligned} \tag{3.1}$$

It is proved in appendix B that operators (3.1) are commuting with every group generator. Now we show that the Casimir $\hat{C}_2$ is identically zero in our particular representation.

Proof. We have

$$\begin{aligned} \hat{C}_2 &= \frac{1}{8}\,\epsilon^{abcd}\,i\,\left(\eta_{al}\lambda^l\partial_b - \eta_{bl}\lambda^l\partial_a\right)\,i\,\left(\eta_{ck}\lambda^k\partial_d - \eta_{dk}\lambda^k\partial_c\right) \\ &= -\frac{1}{8}\,\epsilon^{abcd}\left[\eta_{al}\lambda^l\partial_b\left(\eta_{ck}\lambda^k\partial_d - \eta_{dk}\lambda^k\partial_c\right) - \eta_{bl}\lambda^l\partial_a\left(\eta_{ck}\lambda^k\partial_d - \eta_{dk}\lambda^k\partial_c\right)\right] \\ &= 0, \end{aligned} \tag{3.2}$$

since $\hat{C}_2$ contains contractions of symmetric with antisymmetric indeces, only.

□

If we define generators of rotations $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{L}_{jk}$ and boosts $\hat{K}^i = \hat{L}^{0i}$ then Casimirs (3.1) can be rewritten as

$$\begin{aligned} \hat{C}_1 &= \frac{1}{2}\hat{L}_{ab}\hat{L}^{ab} = -\left(|\vec{K}|^2 - |\vec{J}|^2\right), \\ \hat{C}_2 &= \frac{1}{8}\,\epsilon^{abcd}\hat{L}_{ab}\hat{L}_{cd} = \vec{K}\vec{J}. \end{aligned}$$

The minus sign in the first line is due to our choice of signature of Minkowski metric. The representations are labeled by two numbers (p,k) [Rovelli and Vidotto, 2014, I.M. et al., 1963] with p being positive real number and k is non negative half-integer. The numbers (p,k) are related to the values of Casimirs in the following way

$$\begin{aligned} &|\vec{K}|^2 - |\vec{J}|^2 = p^2 - k^2 + 1, \\ &\vec{K}\vec{J} = pk. \end{aligned}$$

In our case, we have $k = 0$[1] (we show below that the spectrum of $\hat{C}_1$ is continuous, therefore $p \neq 0$), so the representation is only labeled by real values of p.

From the set of operators $\hat{\pi}$, $\hat{L}_{ab}$, we choose a complete set of commuting operators. We know that $\hat{\pi}$ commutes with everything, so it is the first member

[1] In general quantum numbers j of Casimir operator $\vec{J}^2$ are related to k by $j = k, k+1, \ldots$.

of the set. Naturally $\hat{C}_1$ also belongs to the set. In the set of operators $\hat{L}_{ab}$ we conveniently choose $\hat{J}^3 = \frac{1}{2}(\hat{L}^{12} - \hat{L}^{21})$ and $\hat{\vec{J}}^2 = (\hat{J}^1)^2 + (\hat{J}^2)^2 + (\hat{J}^3)^2$ to fill the last two remainig places in the set. The complete set of commuting operators is then $\hat{\pi}$, $\hat{C}_1$, $\hat{\vec{J}}^2$ and $\hat{J}^3$. The basis in $\mathcal{H}$ is labeled by (quantum numbers are written in the same order as the operators) $|q\,p\,j\,m\rangle$.

The goal is to find eigenstates of complete set of commuting operators. In order to do so, we introduce new coordinates $\lambda = \sqrt{-\eta_{ab}\lambda^a\lambda^b}$, rapidity u and angular variables θ, ϕ. The transformation reads

$$\begin{aligned}
\lambda^0 &= \lambda \cosh u, \\
\lambda^1 &= \lambda \sinh u \sin\theta \sin\phi, \\
\lambda^2 &= \lambda \sinh u \sin\theta \cos\phi, \\
\lambda^3 &= \lambda \sinh u \cos\theta,
\end{aligned} \tag{3.3}$$

with $\lambda \in [0, \infty)$, $u \in [0, \infty)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

Let us find the spectra and eigenstates of the operators. We already know the result for the two of them from courses of quantum mechanics. Namely the third component of angular momentum $\hat{J}^3$ and Casimir of SO(3), $\hat{\vec{J}}^2$. The basis is written in terms of spherical harmonics

$$\langle\theta\,\phi|j\,m\rangle = Y_{jm}(\theta,\phi) = N_{jm}\,P_{jm}(\cos\theta)\,e^{i\,m\,\phi},$$

where we adopted the normalization $N_{jm} = (-1)^m\sqrt{\frac{2j+1}{4\pi}\frac{(l-m)!}{(l+m!)}}$, so the harmonics are normalized as follows

$$\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, Y_{jm}(\theta\,\phi)\, Y^*_{j'm'}(\theta\,\phi) = \delta_{j\,j'}\delta_{m\,m'}.$$

The functions $P_{jm}(\cos\theta)$ are asociated Legendre polynomials. The spectrum of $\hat{J}^3$ and $\hat{\vec{J}}^2$ is then

$$\begin{aligned}
&\hat{\vec{J}}^2\,|j\,m\rangle = j\,(j+1)\,|j\,m\rangle, \\
&\hat{J}^3\,|j\,m\rangle = m\,|j\,m\rangle,
\end{aligned}$$

with $j = 0, \frac{1}{2}, 1, \cdots$ and $m = -j, \ldots, j$.

To transform $\hat{\pi}$ into new coordinates (3.3), we use the transformation formula $\partial_{\bar{x}^a} = \frac{\partial x^b}{\partial \bar{x}^a}\partial_{x^b}$, yielding

$$\begin{aligned}
\partial_\lambda &= \cosh u\,\partial_0 + \sinh u\,\sin\theta\,\sin\phi\,\partial_1 + \sinh u\,\sin\theta\,\cos\phi\,\partial_2 + \sinh u\,\cos\theta\,\partial_3 \\
&= \frac{\lambda^0}{\lambda}\,\partial_0 + \frac{\lambda^1}{\lambda}\,\partial_1 + \frac{\lambda^2}{\lambda}\,\partial_2 + \frac{\lambda^3}{\lambda}\,\partial_3,
\end{aligned}$$

that is $\lambda^a\partial_a = \lambda\,\partial_\lambda$. So we have $\hat{\pi} = -i\,\lambda\,\partial_\lambda$. The dependence on λ and no other coordinates enables us to factorize the basis states

$$\langle\lambda\,u\,\theta\,\phi|q\,p\,j\,m\rangle = \Psi(\lambda\,u\,\theta\,\phi) = \psi_q(\lambda)\psi_p(u)\psi_{jm}(\theta\,\phi),$$

where we also factorized the part depending on angles (we will see later, that also the form of $\hat{C}_1$ allows for factorization in u).

To find the eigenstates of $\hat{\pi}$, we solve the eigenequation

$$\hat{\pi}\,\psi_q(\lambda) = q\,\psi_q(\lambda) \;\Rightarrow\; \lambda\,\partial_\lambda\psi_q(\lambda) = i\,q\,\psi_q(\lambda).$$

The solution is $\psi_q(\lambda) = N\,\lambda^{i\,q}$, where N stands for normalization.

It is not possible to normalize ψ_q as $||\psi_q||_{\mathcal{H}} = 1$[2], thus $\psi_q \notin \mathcal{H}$. However, as in the case of a plane wave we can normalize it to delta function, that is, $\psi_q \in \mathcal{H}^*$, where $*$ means the dual space to $\mathcal{H}$. To find the norm of ψ_q, we first transform the measure into new coordinates. Using $\omega_g = \sqrt{\det g}\,\mathrm{d}\lambda\,\mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\phi$ yields the transformed measure

$$\frac{\mathrm{d}^4\lambda}{\lambda^4} \to \frac{\sinh^2 u\,\sin\theta}{\lambda}\mathrm{d}\lambda\,\mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The norm of ψ_q is then (using a definition of scalar product on $\mathcal{H}$ (2.2))

$$\begin{aligned}
||\psi_q(\lambda)||_{\mathcal{H}} &= (\psi_q(\lambda), \psi_q(\lambda)) = \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\lambda^{i\,(q-q')} = \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\,e^{i\,(q-q')\,\ln\lambda} \\
&= \ldots[\ln\lambda = x]\cdots = \int_{-\infty}^\infty \mathrm{d}x\,e^{i\,(q-q')\,x} = 2\pi\,\delta(q-q'),
\end{aligned}$$

thus normalized eigenstate ψ_q reads

$$\psi_q(\lambda) = \frac{1}{\sqrt{2\pi}}\,\lambda^{i\,q}, \qquad q \in \mathbb{R}.$$

It remains to solve for the Casimir $\hat{C}_1$. It looks

$$\begin{aligned}
\hat{C}_1 &= \frac{1}{2}\eta^{ac}\,\eta^{bd}\,\hat{L}_{ab}\hat{L}_{cd} = -\frac{1}{2}\eta^{ac}\,\eta^{bd}\left(\eta_{ak}\lambda^k\partial_b - \eta_{bk}\lambda^k\partial_a\right)\left(\eta_{cl}\lambda^l\partial_d - \eta_{dl}\lambda^l\partial_c\right) \\
&= 3\lambda^a\partial_a + \lambda^a\lambda^b\partial_a\partial_b + \lambda^2\eta^{ab}\partial_a\partial_b. \qquad (3.4)
\end{aligned}$$

We rewrite the result into new coordinates. First, we use the relation

$$(\lambda^a\partial_a)(\lambda^b\partial_b) = \lambda^a\partial_a + \lambda^a\lambda^b\partial_a\partial_b$$

for the second term in (3.4) and we obtain

$$\hat{C}_1 = 2\lambda\,\partial_\lambda + (\lambda\,\partial_\lambda)^2 + \lambda^2\Box_{\bar{\eta}}$$

where we have denoted $\Box_{\bar{\eta}}$ the coordinate transformation of the d'Alembert operator in flat minkowski metric $\Box_\eta = \eta^{ab}\partial_a\partial_b$.

Next, we transform the flat matric η_{ab} into the new coordinates. We use the fact that we know how the metric looks like in spherical coordinates. We only substitute for the radial coordinate $r = \lambda\,\sinh u$ and the time coordinate $t = \lambda\,\cosh u$. We then obtain

$$\begin{aligned}
\bar{\eta} &= -\mathrm{d}^2(\lambda\,\cosh u) + \mathrm{d}^2(\lambda\,\sinh u) + \lambda^2\sinh^2 u\,(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\phi) \\
&= -\mathrm{d}^2\lambda + \lambda^2\mathrm{d}^2 u + \lambda^2\sinh^2 u\,(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\phi).
\end{aligned}$$

[2]For the same reason as plane wave cannot be normalized.

Transformation of metric (2.3) on $\mathcal{H}$ is simply accomplished using the relation $\bar{g}_{ab} = \frac{1}{\lambda^2}\bar{\eta}_{ab}$:

$$\bar{\mathrm{g}} = -\frac{1}{\lambda^2}\,\mathrm{d}^2\lambda + \mathrm{d}^2 u + \sinh^2 u\,(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\phi).$$

Computing d'Alembert operator $\Box_g = \frac{1}{\sqrt{-\det g}}\,\partial_a\left(\sqrt{-\det g}\,g^{ab}\partial_b\right)$ for the metric $g_{ab} = \frac{1}{\lambda^2}\,\eta_{ab}$ and following the coordinates transformation, we arrive at

$$\Box_{\bar{g}} = 2\lambda\,\partial_\lambda + \lambda^2\Box_{\bar{\eta}}. \tag{3.5}$$

The Casimir can then be cast into a form

$$\hat{C}_1 = (\lambda\,\partial_\lambda)^2 + \Box_{\bar{g}}$$

We can now either compute $\Box_{\bar{\eta}}$ and substitute into (3.5) or compute $\Box_{\bar{g}}$ explicitly, to arrive at the same result. We perform the latter:

$$\begin{aligned}\Box_{\bar{g}} &= -\lambda\,\partial_\lambda - \lambda^2\partial_{\lambda\lambda} + \frac{2\cosh u}{\sinh u}\,\partial_u + \partial_{uu} + \frac{\cos\theta}{\sinh^2 u\,\sin\theta}\,\partial_\theta + \frac{1}{\sinh^2 u}\,\partial_{\theta\theta}\\ &\quad + \frac{1}{\sinh^2 u\,\sin^2\theta}\partial_{\phi\phi},\end{aligned}$$

where ∂_{xx} denotes second derivative $\partial_x\partial_x$. Realizing that $(\lambda\,\partial_\lambda)^2 = \lambda\,\partial_\lambda + \lambda^2\partial_{\lambda\lambda}$ and adjusting the formula, we get

$$\begin{aligned}\hat{C}_1 &= \frac{1}{\sinh^2 u}\,\partial_u\left(\sinh^2 u\,\partial_u\right) + \frac{1}{\sinh^2 u}\left[\frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\,\partial_{\phi\phi}\right]\\ &=: \Box_{\mathrm{hyp}}.\end{aligned}$$

We used subscript hyp suggesting it is expressed via coordinates on hyperboloid. In $\Box_{\mathrm{hyp}}$ we recongnize, in the square brackets, minus the square of angular momentum $\vec{\hat{J}}^2$ in spherical coordinates. Final form of $\hat{C}_1$ then reads

$$\hat{C}_1 = \Box_{\mathrm{hyp}} = \frac{1}{\sinh^2 u}\,\partial_u\left(\sinh^2 u\,\partial_u\right) - \frac{1}{\sinh^2 u}\vec{\hat{J}}^2.$$

The eigenequation is

$$\hat{C}_1\,\Psi(\lambda\,u\,\theta\,\phi) = -(p^2+1)\Psi(\lambda\,u\,\theta\,\phi) \tag{3.6}$$

$$\left[\frac{1}{\sinh^2 u}\,\partial_u\left(\sinh^2 u\,\partial_u\right) - \frac{1}{\sinh^2 u}\,\vec{\hat{J}}^2\right]\psi_q(\lambda)\psi_p(u)\psi_{jm}(\theta\,\phi)$$

$$= -(p^2+1)\psi_q(\lambda)\psi_p(u)\psi_{jm}(\theta\,\phi)$$

$$\left[\frac{1}{\sinh^2 u}\,\partial_u\left(\sinh^2 u\,\partial_u\right) - \frac{1}{\sinh^2 u}\,j(j+1) + (p^2+1)\right]\psi_{pj}(u) = 0.$$

In the second step we have used the factorization of wave function and in the last step we have used the eigenequation for $\vec{\hat{J}}^2$.

At first we need to find the domain of the operator $\hat{C}_1$ such, that it is self-adjoint. Let $\psi \in \mathrm{D}(\hat{C}_1)$ and let it be the solution to (3.6). To normalize $\psi_p(u)$ we need to compute an integral

$$\int_0^\infty \mathrm{d}u\, \sinh^2 u\, \psi(u)\psi^*(u) = \int_0^\infty \mathrm{d}u\, \tilde{\psi}(u)\tilde{\psi}^*(u),$$

where we have denoted $\tilde{\psi}(u) = \sinh u\, \psi(u)$.

Let us compute the scalar product $\langle\phi|\hat{C}_1\psi\rangle$:

$$\langle\phi|\hat{C}_1\psi\rangle = \int_0^\infty \mathrm{d}u\, \sinh^2 u\, \phi(u)\, \left(\hat{C}_1\psi(u)\right)^* =$$

$$= \left[\phi(u)\sinh^2 u\, \psi^{*\prime}(u) - \phi'(u)\sinh^2 u\, \psi^*(u)\right]_0^\infty + \langle\phi|\hat{C}_1\psi\rangle.$$

In terms of $\tilde{\phi}$ and $\tilde{\psi}$ this looks like

$$\begin{aligned}\langle\phi|\hat{C}_j\psi\rangle &= \left[\left(\frac{\tilde{\phi}(u)}{\sinh u}\right)\sinh^2 u\, \left(\frac{\tilde{\psi}^*(u)}{\sinh u}\right)' - \left(\frac{\tilde{\phi}(u)}{\sinh u}\right)' \sinh^2 u\, \left(\frac{\tilde{\psi}^*(u)}{\sinh u}\right)\right]_0^\infty \\ &\quad + \langle\hat{C}_1\phi|\psi\rangle.\end{aligned}$$

After performing the derivatives we get

$$\langle\phi|\hat{C}_j\psi\rangle = \left[\tilde{\phi}(u)\tilde{\psi}^{*\prime}(u) - \tilde{\phi}'(u)\tilde{\psi}^*(u)\right] + \langle\hat{C}_j\phi|\psi\rangle. \tag{3.7}$$

Let us recall the definition of adjoint operator in unbounded case, e.g. in [Blank et al., 2008]:

Let $\mathrm{D}(\hat{A})$ be a domain of linear unbounded operator $\hat{A}$,such that $\mathrm{D}(\hat{A})$ is dense in some hilbert space $\mathcal{H}$. Then to any $\phi \in \mathcal{H}$ there exists *at most* one ϕ_0 such, that $\langle\phi|\hat{A}\psi\rangle = \langle\phi_0|\psi\rangle\ \forall\psi \in \mathrm{D}(\hat{A})$. Then we define the adjoint operator $\hat{A}^*$ to $\hat{A}$ by $\hat{A}^*\phi = \phi_0$ with domain $\mathrm{D}(\hat{A}^*) := \{\phi;\ \exists\phi_0 :\ \hat{A}^*\phi = \phi_0\}$.

$\hat{A}$ is symmetric if $\hat{A} \subset \hat{A}^*$, or equivalently $\langle\phi|\hat{A}\psi\rangle = \langle\hat{A}\phi|\psi\rangle$. $\hat{A}$ is self-adjoint if $\hat{A} = \hat{A}^*$.

We see from (3.7) that $\hat{C}_1$ is not symmetric for

$$\mathrm{D}(\hat{C}_1) = C_0^\infty(\Lambda_u) \subset \mathcal{H} = \mathrm{L}^2(\Lambda_u, \mathrm{d}\Lambda_u),$$

where $\overline{C_0^\infty(\Lambda_u)} = \mathcal{H}$ is a space of smooth functions of compact support in Λ_u, $\Lambda_u\colon = \{u;\ u \in \Lambda\}$ and

$$\mathrm{d}\Lambda_u = \sinh^2 u\, \mathrm{d}u.$$

The reason is that while in infinity the boundary term in (3.7) vanishes (functions belong to L^2 space), it is generally not guaranteed to vanish in zero.

We now prove some lemmas which we will use for constructing a symmetric extension of $\hat{C}_1$, that will also be self-adjoint.

Let us introduce Sobolev-like space

$$W_2^2 := \{\phi(u); ||\phi(u)||^2_{W_2^2} = \int\limits_0^\infty du \sinh^2 u \left(\phi\phi^* + \hat{C}_1\phi(\hat{C}_1\phi)^*\right) < \infty\}. \tag{3.8}$$

Lemma 1. *Space W_2^2 is a linear space and is a dense subspace of $\mathcal{H}$.*

Proof. Using the linearity of the integral (3.8) in definition of the norm on W_2^2, we immediately get for $\phi \in W_2^2$ that

$$||\phi||^2_{W_2^2} = ||\phi||^2_{\mathrm{L}^2} + a^2\, ||\hat{C}_1\phi||^2_{\mathrm{L}^2}.$$

with $a > 0$ being a finite constant. We require $||\hat{C}_1\phi||^2_{\mathrm{L}^2} < \infty$, so the states $\hat{C}_1\phi \in \mathcal{H}$. Then $||\phi||^2_{\mathrm{L}^2}$ must also be finite. Therefore any function in W_2^2 is also in $\mathrm{L}^2(\Lambda)$ That gives the inclusion $W_2^2 \subset \mathcal{H}$.

Now we use the fact that space $C_0^\infty(\Lambda_u)$ of smooth functions of compact support is dense in $\mathcal{H}$. Any function in $C_0^\infty(\Lambda_u)$ also belongs in W_2^2. This is trivial fact, since all derivatives exist for functions in $C_0^\infty(\Lambda_u)$ and the integral in (3.8) becomes an integral over a compact domain, therefore the norm (3.8) is finite for all functions in $C_0^\infty(\Lambda_u)$. From the triple of inclusions $C_0^\infty(\Lambda_u) \subset W_2^2 \subset \mathcal{H}$ and the fact that $C_0^\infty(\Lambda_u)$ is dense in $\mathcal{H}$, we immediately get that W_2^2 is dense in $\mathcal{H}$ as well.

It remains to prove the linearity. Let $\phi, \psi \in W_2^2$ and $a, b \in \mathbb{C}$. Then for $\Psi = a\,\phi + b\,\psi$ it holds

$$\begin{aligned} ||\Psi||_{W_2^2} &= \int\limits_{\mathbb{R}_0^+} du \sinh^2 u \Big[(a\,\phi + b\,\psi)(a\,\phi + b\,\psi)^* + \hat{C}_1(a\,\phi + b\,\psi)(\hat{C}_1(a\,\phi + b\,\psi))^*\Big] \\ &= \int du \sinh^2 u \Big[|a|^2(\phi\phi^* + \hat{C}_1\phi(\hat{C}_1\phi)^*) + |b|^2(\psi\psi^* + \hat{C}_1\psi(\hat{C}_1\psi)^*) \\ &\quad + a\,b^*(\phi\psi^* + \hat{C}_1\phi(\hat{C}_1\psi)^*) + a^*\,b(\psi\phi^* + \hat{C}_1\psi(\hat{C}_1\phi)^*)\Big]. \end{aligned}$$

We have used linearity of $\hat{C}_1$. Now using $z + z^* = 2\Re(z)$ for $z \in \mathbb{C}$ we get

$$||\Psi||_{W_2^2} = |a|^2||\phi||^2_{W_2^2} + |b|^2||\psi||_{W_2^2} + 2\Re(a^*b\,\langle\phi|\psi\rangle_W) \tag{3.9}$$

where we denoted $\langle\phi|\psi\rangle_W$ scalar product on W_2^2 by

$$\langle\phi|\psi\rangle_W := \int\limits_{\mathbb{R}_0^+} du \sinh^2 u(\phi\psi^* + \hat{C}_1\phi(\hat{C}_1\psi)^*). \tag{3.10}$$

The integral (3.10) is well defined, using the triangle inequality we get

$$\begin{aligned} |\phi\psi^* + \hat{C}_1\phi(\hat{C}_1\psi)^*| &\leq |\phi\psi^*| + |\hat{C}_1\phi(\hat{C}_1\psi)^*| \\ &\leq \frac{1}{2}\left(|\phi|^2 + |(\hat{C}_1\phi)^*|^2 + |\psi|^2 + |(\hat{C}_1\psi)^*|^2\right). \end{aligned}$$

Since $\phi, \psi \in W_2^2$ then $\phi, \psi, \in \mathrm{L}^2$ and $\phi\psi^* + \hat{C}_1\phi(\hat{C}_1\psi)^* \in \mathrm{L}^1$. Right-hand side of (3.9) is finite. The first two terms are finite becouse $\phi, \psi \in W_2^2$, the last term is finite becouse we can estimate $\langle\phi|\psi\rangle_W \leq \max(||\phi||^2_{W^2_2}, ||\psi||^2_{W^2_2})$. This result yields that $\Psi \in W^2_2$ whenever $\phi, \psi \in W^2_2$.

□

We denote the boundary term in (3.7) as B.T. and, as before, $\tilde{\phi} = \sinh u\, \phi$. We now prove another lemmas.

Lemma 2. $D_{\alpha\beta} := \{\phi \in W^2_2;\, \alpha\, \tilde{\phi}(0) + \beta\, \tilde{\phi}'(0) = 0\}$ *is linear subspace of* W^2_2 *dense in* $\mathcal{H}$.

Proof. The linearity is trivial. $D_{\alpha\beta}$ is subspace of linear space W^2_2. The boundary condition for $\Psi = a\,\phi + b\,\psi$ looks

$$\begin{aligned}&\alpha\,[a\,\tilde{\phi}(0) + b\,\tilde{\psi}(0)) + \beta\,(a\,\tilde{\phi}'(0) + b\,\tilde{\psi}'(0)]\\&= a\,[\alpha\,\tilde{\phi}(0) + \beta\,\tilde{\phi}'(0)] + b\,[\alpha\,\tilde{\psi}(0) + \beta\,\tilde{\psi}'(0)] = 0\\&\Rightarrow \quad \alpha\,\tilde{\Psi}(0) + \beta\,\tilde{\Psi}'(0) = 0,\end{aligned}$$

becouse ϕ, $\psi \in D_{\alpha\beta}$

W^2_2 is dense in $\mathcal{H}$, that is, to any element $h \in \mathcal{H}$ there exist a sequence $\psi_n \in W^2_2$ such, that it converges to h in the L^2 norm on $\mathcal{H}$:

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^+_0} \mathrm{d}u\, \sinh^2 u\, |h(u) - \psi_n(u)| = 0.$$

Since $D_{\alpha\beta} \subset W^2_2$, some of those sequences (or its subsequences) lie in $D_{\alpha\beta}$. The boundary condition posted on functions on $D_{\alpha\beta}$ does not spoil the convergence (it is not necessary to mention that this condition holds on a set of measure zero). Therefore, for any $h \in \mathcal{H}$ we can find a sequence $\psi^D_n \in D_{\alpha\beta}$ converging to h, thus $D_{\alpha\beta}$ is dense in $\mathcal{H}$.

□

Lemma 3. *The operator* $\hat{C}_1$ *with* $\mathrm{D}(\hat{C}_1) = D_{\alpha\beta}$ *is self-adjoint for* α, $\beta \in \mathbb{R}$.

Proof. We have already found that

$$\langle\phi|\hat{C}_1\psi\rangle = \text{B.T.} + \langle\hat{C}_1\phi|\psi\rangle.$$

We also know that B.T. = 0 in ∞, since $\phi, \psi \in \mathcal{H}$. For $\hat{C}_1$ to be symmetric, we also require that $\text{B.T.}|_0 = 0$, that is

$$\phi(0)\psi^{*\prime}(0) = \phi'(0)\psi^*(0). \tag{3.11}$$

If $\psi \in D_{\alpha\beta}$, we have from boundary condition

$$\alpha\tilde{\psi}^*(0) + \beta\tilde{\psi}^{*\prime}(0) = 0^3 \tag{3.12}$$

that

$$\psi^*(0) = 0 \quad \Rightarrow \quad \psi^{*\prime} = 0.$$

[3] We complex conjugated the condition and used the fact that α and β are real.

Equation (3.11) then holds identically. If $\psi^*, \psi^{*\prime} \neq 0$, then (3.11) holds either for $\phi(0) = 0$, which implies $\phi'(0) = 0$ (and vice versa) or if

$$\alpha\tilde{\phi}(0) + \beta\tilde{\phi}'(0) = 0.$$

To see that, we multiply (3.12) by $\tilde{\phi}'(0)$[4] to get

$$\begin{aligned}
&\alpha\tilde{\phi}'(0)\tilde{\psi}^*(0) + \beta\tilde{\phi}'(0)\tilde{\psi}^{*\prime}(0) = 0 \quad \Rightarrow \\
&\alpha\tilde{\phi}(0)\tilde{\psi}^{*\prime}(0) + \beta\tilde{\phi}'(0)\tilde{\psi}^{*\prime}(0) = 0 \quad \Rightarrow \\
&\alpha\tilde{\phi}(0) + \beta\tilde{\phi}'(0) = 0.
\end{aligned}$$

In second line we used equation (3.11) and in the last line we used the assumption that $\psi^{*\prime}(0) \neq 0$, otherwise its trivial. Therefore $\hat{C}_1$ is symmetric iff $\phi \in D_{\alpha\beta}$.

We have $D_{\alpha\beta} = \mathrm{D}(\hat{C}_1) \subset \mathrm{D}(\hat{C}_1^*)$. To prove that the operator domains equal, let us assume the contrary. If $D_{\alpha\beta} \subset \mathrm{D}(\hat{C}_1^*)$, then there exists $h \in \mathcal{H}$, $h \notin D_{\alpha\beta}$ such that $\hat{C}_1^* h \in \mathcal{H}$. Then $h \in W_2^2$ since $||h||_{\mathrm{L}^2} < \infty$ and $||\hat{C}_1^* h||_{\mathrm{L}^2} < \infty$. However, if $\psi \in D_{\alpha\beta}$ for given α, β and $h \notin D_{\alpha\beta}$[5], $\hat{C}_1$ can be symmetric only if $\psi^*(0) = 0$[6] for all $\psi^* \in D_{\alpha\beta}$ or $\phi(0) = 0 \wedge \phi' = 0$[7] for all $\phi \in \mathrm{D}(\hat{C}_1^*)$. Neither is true for *all* functions in $D_{\alpha\beta}$ or in $\mathrm{D}(\hat{C}_1^*)$. It follows from it that for $\mathrm{D}(\hat{C}_1^*) \neq D_{\alpha\beta}$ operator $\hat{C}_1$ is not symmetric. Therefore $\mathrm{D}(\hat{C}_1^*) = \mathrm{D}(\hat{C}_1)$ and $\hat{C}_1$ is self-adjoint.

□

Functions $\psi \in \mathrm{D}_{\alpha\beta}$ satisfy

$$\lim_{u \to 0} u^{\frac{3}{2}} \psi(u) = 0.$$

This comes from requirement $||\psi||^2_{\mathrm{L}^2} < \infty$, that is

$$\int \mathrm{d}u \, \sinh^2 u \, |\psi|^2 \sim \int \mathrm{d}u \, u^{2(\kappa+1)} = \frac{u^{2\kappa+3}}{2\kappa+3},$$

therefore $\kappa > -\frac{3}{2}$ for $u \to 0$.

If we solve (3.6) for small u, the solutions can be obtained in the form $\psi(u)_1 \sim u$ and $\psi(u)_2 \sim u^{-(l+1)}$. We immediately see that $\psi_2 \notin \mathrm{D}_{\alpha\beta}$ for $l = 1, 2, \ldots$ from the above requirement. Solution behaving as $\frac{1}{u}$ can also be exluded. The reason for that is desccribed in [Fermi, 1995]. In short, solutions behaving like $\frac{1}{u}$ in zero cause that the Schrödinger equation, in spherical coordinates, has δ - function type singularity in radial coordinate u, while no singularity in other two coordinates. That leads to contradiction. The probability of measuring nonzero energy at points different from $u = 0$ does not vanish, while from the Schrödinger equation it follows that the energy is concentrated only at point zero as δ - function.

General soution of (3.6) can be found in the form

$$\psi_{pj}(u) = C_1 \frac{1}{\sqrt{\sinh u}} P^{j+\frac{1}{2}}_{-\frac{1}{2}+ip}(\cosh u) + C_2 \frac{1}{\sqrt{\sinh u}} Q^{j+\frac{1}{2}}_{-\frac{1}{2}+ip}(\cosh u), \tag{3.13}$$

[4] Which is nonzero by assumption, otherwise (3.11) holds identically and all is trivial.
[5] Or if $h \in D_{\alpha'\beta'}$ for $\alpha' \neq \alpha$ and $\beta' \neq \beta$
[6] Recall that $\psi^{*\prime}(0) = 0$ also implies $\psi^*(0) = 0$.
[7] For $\psi^*(0) \neq 0$

With P, Q being the associated Legendre functions of the first and second kind, respectivelly, and C_1, C_2 are complex constatnts.

We are not able to normalize functions (3.13) for general quantum number j. This task is complicated, therefore we do not work with general j. However, we can accomplish this task for $j = 0$. When $j = 0$ the solution of (3.6) is

$$\psi_p(u) = A\,\frac{\sin pu}{\sinh u} + B\,\frac{\cos pu}{\sinh u} \tag{3.14}$$

with A, B being complex constants. In (3.14) the second term is excluded as we have discussed above. We therefore set $B = 0$. That leads to $\psi_p(u) = 0$. From the boundary condition in definition of $\mathrm{D}_{\alpha\beta}$ we have

$$\alpha\tilde{\psi}(0) + \beta\tilde{\psi}'(0) = 0 \quad \Rightarrow \quad \beta = 0.$$

The normalization integral then reads

$$\int_0^\infty \mathrm{d}u\,\tilde{\psi}_p(u)\,\tilde{\psi}^*_{p'}(u) = \int_0^\infty \mathrm{d}u\,|A|^2 \sin pu \sin p'u.$$

We need to know what $\int_{\mathbb{R}_0^+} \mathrm{d}x\,\sin(kx)\sin(lx)$ is. But that is easy. We know that

$$\int_{-\infty}^\infty \mathrm{d}x\,e^{i(k-l)x} = 2\pi\,\delta(k-l).$$

We also know that $\sin x = e^{ix} - e^{-ix}$. Therefore, after substitition in $\int_{\mathbb{R}_0^+} \mathrm{d}x\,\sin(kx)\sin(lx)$, we get

$$\int_0^\infty \mathrm{d}x\,e^{i(k+l)x} + e^{-i(k+l)x} + e^{i(k-l)x} + e^{-i(k-l)x}$$

Next we use the fact that

$$\int_0^\infty \mathrm{d}x\,e^{-iKx} = \int_{-\infty}^0 \mathrm{d}x\,e^{iKx}$$

and get

$$\int_0^\infty \mathrm{d}x\,\sin(kx)\sin(lx) = \int_{-\infty}^\infty \mathrm{d}x\,e^{i(k+l)x} + \int_{-\infty}^\infty \mathrm{d}x\,e^{i(k-l)x} = 2\pi\,\delta(k+l) + 2\pi\,\delta(k-l).$$

Therefore we have

$$\int_0^\infty \mathrm{d}u\,|A|^2 \sin pu \sin p'u = |A|^2 2\pi\,\delta(p-p').$$

The term $\delta(p+p')$ does not contribute, since p, p' are both positive. Thus

$$|A| = \frac{1}{\sqrt{2\pi}} \quad \Rightarrow \quad A = \frac{e^{i\Phi}}{\sqrt{2\pi}},$$

where Φ is a general phase not affecting physics.

Finally, the normalized states look like

$$\psi_p(u) = \frac{1}{\sqrt{2\pi}} \frac{\sin pu}{\sinh u} e^{i\Phi}. \tag{3.15}$$

We have constructed a family of self-adjoint operators labeled by α, namely $\hat{C}_1^\alpha$ with domain $\mathrm{D}(\hat{C}_1^\alpha) = D_{\alpha\, 0}$. Corresponding eigenstates are, for $j=0$, (3.15). These states do not depend on α.

For more mathematical details see [Rudin, 1987].

We have managed to find the eigenstates of generators of gauge group. We do not, however, use the states in next chapters in group averaging procedure. We are going to use different approach, that proved to be much more convenient. We only note that even in case we used results of this chapter, we would not need to know, what the eigensolution of $\hat{C}_1$ looks like for general $j \neq 0$. It can be shown that we would only need states with $j=0$, only such states contrinute.

4. Strategy of refined algebraic quantization

Let us finish the partial quantization of the system of ECT introduced in previus chapters. There has always been issues concerning canonical quantization of gauge systems, or systems possesing first class constraints. The Hamiltonian of ECT *purely* consists of constraints (explicitly given in [Nikolič, 1995]), therefore we need to use different approach to quantization. Dirac proposed an approach ([Dirac, 1950]), in which the classical constraints C_i are represneted by self-adjoint operators $\hat{C}_i$[1]. The physical states are then states annihilated by the constraints, that is, satisfying $\hat{C}_i\,\psi_{phys} = 0$. There are, however, some flaws of this approach. It is not clear, for example, on which space the constraint should act[2]. Another question is how one should impose an appropriate inner product on the space of solutions of the constraits.

In this chapter we discuss procedure called *refined algebraic quantization*, or RAQ, as a possible solution to the issues mentioned above. There is still an open problem how to quantize more general systems. However, in our case[3] RAQ works perfectly fine.

The basic idea of RAQ is at first to ignore the constraints (as in Dirac's procedure) and construct an auxiliary Hilbert space $\mathcal{H}_{aux}$ (representing the observables by self-adjoint operators on $\mathcal{H}_{aux}$). Further, one represents the constraint (let us have only one constraint in this example, for simplicity) as self-adjoint operator on $\mathcal{H}_{aux}$. To find solutions to $\hat{C}\,\psi_{phys}$ one realizes that the condition of annihilating the physical states is equivalent to stating that the physical states are invariant under one-parametr group $U(t) = e^{i\,t\,\hat{C}}$, generated by the constraint $\hat{C}$. This equivalence is simply proven by formally expanding the exponential in Taylor series

$$e^{i\,t\,\hat{C}} = \hat{1} + i\,t\,\hat{C} + \frac{(i\,t)^2}{2!}\,\hat{C}\,\hat{C} + \dots$$

and we immediately see that on physical states, only the unity operaor contributes. The solutions are then found by means of *group averaging techniques*: states $\bar{\phi} := \int \mathrm{d}t\, U(t)\,|\phi>$ are group invariant (ϕ is a general state from $\mathcal{H}_{aux}$).

Proof. First we realize that dt is Haar measure on the group given by $U(t)$, that

[1] That is possible since the constraints are real functions.

[2] recall the covariant quantization of Maxwell theory, where naive application of Lorentz condition $\widehat{\partial A}\psi_{phys} = 0$, for ψ_{phys} being states from the original Hilbert space constructed for the Maxwell theory, leads to contradictions.

[3] such as in other examples, like linear gravity, minisuperspace models of gravity or free Maxwell field.

is translations[4]. That means $\mathrm{d}(t \pm t') = \mathrm{d}t$. The rest is trivial

$$\begin{aligned} U(t')\,\bar{\phi} &= \int \mathrm{d}t\, e^{i\,(t+t')\,\hat{C}}\,|\phi> \\ &= \ldots [t + t' = \bar{t}] \cdots = \int \mathrm{d}(\bar{t} - t')\, e^{i\,\bar{t}\,\hat{C}}\,|\phi> \\ &= \int \mathrm{d}\bar{t}\, e^{i\,\bar{t}\,\hat{C}}\,|\phi> = \bar{\phi}. \end{aligned}$$

□

It is important to note that in case of more general averaging integral than the one in this example, we must also use Haar measure (bi-invariant volume of the group). We saw in the construction above that this was the cruical point, which makes states defined by averaging group invariant.

A common issue with $\bar{\phi}$ is that, typically, it does not belong to $\mathcal{H}_{aux}$. However, it usually has a well defined action on some dense subset Φ of $\mathcal{H}_{aux}$, in the sense that $\bar{\phi} \circ \psi = \int \mathrm{d}t \langle\phi|\, U(t) \circ |\psi\rangle$ is defined for all $\psi \in \Phi$. Thus, solutions to constraints lie in a dual to Φ (denoted Φ'), which is naturally larger then $\mathcal{H}_{aux}$. One can further introduce an inner product on the space of solutions Φ' simply by $\langle\bar{\phi}_1 \,|\, \bar{\phi}_2\rangle := \bar{\phi}_1 \circ \phi_2$, $\phi_2 \in \Phi$. Then we can construct a physical Hilbert space in $\mathcal{H}_{aux}$ by choice of a suitable dense subspace Φ in $\mathcal{H}_{aux}$ (each element of Φ is mapped to an element of Φ' by group averaging).

It is important to realize that group averaging *can*, in principle, lead to solutions of the constraints. It may not be always so.

Let us now outline the strategy of RAQ in more detail. We will follow the paper [Ashtekar et al., 1995]. Considering a classical system with first class constraints $C_i = 0$ and a phase space Γ, we follow the following steps.

Step 1. Select a subspace **S** of vector space of all smooth, complex-valued functions on Γ with the following properties:

- **a** **S** should be large enough so that any regular function on Γ can be expressed via elements in **S**.
- **b** **S** should be closed under Poisson brackets.
- **c** **S** should be closed under complex conjugation.

Elements in **S** are regarded as elementary classical variables with unambiguous quantum analogue. An example of functions in **S** is the six functions of canonical pair of position and momenta (x, y, z, p_x, p_y, p_z).

Step 2. Associate with each element F in **S** an abstract operator $\hat{F}$, construct algebra generated by these operators and impose canonical commutation relations $[\hat{F}, \hat{G}] = i\hbar\widehat{\{F, G\}}$ (or anticommutation relations, if necessary) and denote the resulting algebra $\mathcal{B}_{aux}$.

Step 3. Introduce an involution operation $\star$ on this algebra: if for F, G in **S** holds $F^* = G$, then $\hat{F}^\star = \hat{G}$ in $\mathcal{B}_{aux}$. Denote the resulting $\star$-algebra $\mathcal{B}^\star_{aux}$.

[4] in this case, Haar measure and Lebesgue measure coinccide.

Step 4. Construct a linear $\star$-representation R of $\mathcal{B}^\star_{aux}$ on $\mathcal{H}_{aux}$ via linear operators, such that $R(\hat{F}^\star) = R(\hat{F})^\dagger$, for all $\hat{F} \in \mathcal{B}^\star_{aux}$, where $\dagger$ denotes Hermitian conjugation with respect to inner product on $\mathcal{H}_{aux}$.

The remaining steps use $\mathcal{H}_{aux}$ as a space, where the constraints 'live', and where they act (on some subspace of $\mathcal{H}_{aux}$). Then we construct the physical Hilbert space $\mathcal{H}_{phys}$ from the solutions to the constraints. In general, $\mathcal{H}_{phys}$ is not a subspace of $\mathcal{H}_{aux}$.

Step 5a. Represent the constraints C_i as self-adjoint operators $\hat{C}_i$ on $\mathcal{H}_{aux}$.

As we previously discussed, solutions to the constraints $\hat{C}_i$ lie in an topological dual Φ' of some dense subspace Φ of $\mathcal{H}_{aux}$. Φ' and Φ will be used to construct the physical Hilbert space.

Step 5b. Choose a dense subspace Φ in $\mathcal{H}_{aux}$, which is left invariant by the constraints $\hat{C}_i$ (invariant states found by group averaging) and let $\mathcal{B}^\star_{phys}$ be the $\star$-algebra of operators on $\mathcal{H}_{aux}$ commuting with $\hat{C}_i$ and such that both $\hat{F}$, $\hat{F}^\dagger$ ($\hat{F} \in \mathcal{B}^\star_{phys}$ are defined on Φ and map Φ to itself.

The space Φ should be large enough, so $\mathcal{B}^\star_{phys}$ contains sufficient number of physicaly interesting operators. On the other hand, it should be small enough, so the space Φ' contains sufficient number of physical states.

The key idea in construction of $\mathcal{H}_{phys}$, is to find an appropriate map $\eta : \Phi \to \Phi'$ such that $\eta(\phi)$ is a solution of the constraint for all $\phi \in \Phi$.

Step 5c. Find an anti-linear map $\eta : \Phi \to \Phi'$ satisfying:

a $\eta(\phi_1)$ is a soluiton of the constraint for every $\phi_1 \in \Phi$, that is, for any $\phi_2 \in \Phi$, $0 = \left(\hat{C}_i(\eta\phi_1)\right)[\phi_2] := (\eta\phi_1)[\hat{C}_i\phi_2]$.

b For any $\phi_1, \phi_2 \in \Phi$, η is real: $(\eta\phi_1)[\phi_2] = ((\eta\phi_2)[\phi_1])^*$ and positive: $(\eta\phi_1)[\phi_1] \geq 0$

c η commutes with action of every $\hat{F} \in \mathcal{B}^\star_{phys}$: for every ϕ_1, $\phi_2 \in \Phi$, $(\eta\phi_1)[\hat{F}\phi_2] = (\eta\hat{F}^\dagger\phi_1)[\phi_2]$.

Step 5d. The vectors $\eta\phi$ span a vector space $\mathcal{V}_{phys}$ of solutions of the constraints. The inner product on $\mathcal{V}_{phys}$ is defined $< \eta\phi_1|\eta\phi_2 >_{phys} := (\eta\phi_2)[\phi_1]$. The completion of $\mathcal{V}_{phys}$ with respect to this scalar product, is the physical Hilbert space $\mathcal{H}_{phys}$.

The last statments is justified by Step 5c (b), which says, that the scalar product on $\mathcal{V}_{phys}$ is well defined, Hermitian and positive definite.

A natural candidate for the map η is provided by group averaging. Indeed, we define

$$\eta|\phi> := \left(\int_G \mathrm{d}g\hat{U}(g)|\phi>\right)^\dagger = \int_G \mathrm{d}g < \phi|\hat{U}^{-1}(g),$$

where $\hat{U}(g)$ is an exponentiated form of all constraints $\hat{C}_i$ and defines an unitary action of a gauge group G on $\mathcal{H}_{aux}$, $\mathrm{d}g$ is a Haar measure on the group.

The scalar product is then

$$\langle \eta\phi_1 | \eta\phi_2 \rangle_{phys} = \int\limits_G \mathrm{d}g \langle \phi_2 | \hat{U}^{-1}(g) | \phi_1 \rangle_{aux}.$$

In the last step of this program, we represent the physical operators on $\mathcal{V}_{phys}$.

Step 6 Operators $\hat{F} \in \mathcal{B}^{\star}_{phys}$ act naturally (by duality) on $\mathcal{V}_{phys}$, leaving this space invariant. This is used to induce a densely defined operators $\hat{F}_{phys}$ on $\mathcal{H}_{phys}$: $\hat{F}_{phys}(\eta\phi) = \eta(\hat{F}\phi)$.

The map $\mathcal{B}^{\star}_{phys} \to \mathcal{H}_{phys}$ and Step 5 lead to the anti-$\star$-representation of $\mathcal{B}^{\star}_{phys}$. We will aply this procedure, in the next chapter, to our case.

5. Refined algebraic quantization and group averaging procedure

In chapter 2 we have introduced Hilbert space $\mathcal{H} = \mathrm{L}^2(\Lambda, \frac{\mathrm{d}^4\lambda}{\lambda^4})$ and showed that it is invariant under $\mathrm{GL}^+(\mathbb{R}) \times \mathrm{SO}^+(1,3)$, which plays a role of a gauge group of the system. The classical phase space Γ is spanned by coordinates (λ^a, π_a). These (real) functions are the function from subspace $\mathbf{S}$ in Step 1. of the previous chapter. The Poisson brackets of the classical variables are $\{\lambda^a, \pi_b\} = \delta^a_b$ [Pilc, 2013a]. The representation of classical system on $\mathcal{H} = \mathrm{L}^2(\Lambda, \frac{\mathrm{d}^4\lambda}{\lambda^4})$[1] was found in chapter 2 by means of self-adjoint operators $\hat{\pi}, \hat{L}_{ab}$. Their algebra is given by (2.4) and constitutes the representation of algebra $\mathcal{B}^\star_{aux}$ in Step 3 and Step 4. The first class constraint (only single constraint is present in case of temporal part of ECT decomposition) is $\pi^a = 0$ [see Pilc, 2013b], thus is also represented by self-adjoint operators $\hat{\pi}, \hat{L}_{ab}$. Their exponentiation generates a unitary action of the gauge group $\mathrm{GL}^+(\mathbb{R}) \times \mathrm{SO}^+(1,3)$.

To find solutions to the constraint and construct the physical Hilbert space, we must average over the gauge group. To find the scalar product, we need to perform an integral

$$\langle \eta\psi|\eta\phi\rangle_{phys} = \int\limits_G \mathrm{d}g \langle \phi|\hat{U}^{-1}(g)|\psi\rangle_{aux},$$

where $G = \mathrm{GL}^+(\mathbb{R}) \times \mathrm{SO}^+(1,3)$ and η is the rigging map defined in Step 5c of the previous chapter.

First step is to construct Haar measures for gauge groups. For the group $\mathrm{GL}^+(\mathbb{R})$ it is easy. $\mathrm{GL}^+(\mathbb{R})$ is isomorphic to the group $(\mathbb{R}^+, \times)$. Each element of $\mathrm{GL}^+(\mathbb{R})$ is $m = e^\tau$ with $\tau \in \mathbb{R}$. Therefore the left-invariant measure would be[2]

$$\mathrm{d}M = m^{-1}\mathrm{d}m = e^{-\tau}\mathrm{d}e^\tau = \mathrm{d}\tau.$$

This coincides with right-invariant measure $\mathrm{d}m\, m^{-1}$, therefore $\mathrm{d}\tau$ is Haar measure of $\mathrm{GL}^+(\mathbb{R})$. We can equivalenty work with group $(\mathbb{R}^+, \times)$. Then each element is $e^\tau = N$ with $N \in \mathbb{R}^+$ and Haar measure is $\frac{\mathrm{d}N}{N}$.

Finding Haar measure $\mathrm{d}s$ on $\mathrm{SO}^+(1,3)$ will give us some work. From Cartan decomposition [see Naimark and Farahat, 1964] we know that every element $s \in \mathrm{SO}^+(1,3)$ can be written as $s = h\,k$, with $k \in \mathrm{SO}(3)$ and $h \in \mathrm{SO}^+(1,3)/\mathrm{SO}^+(3)$ (boost). $\mathrm{SO}^+(1,3) = \mathrm{SO}(3) \times \mathrm{SO}^+(1,3)/\mathrm{SO}(3)$, thus we can decompose the measure $\mathrm{d}s = \mathrm{d}h\,\mathrm{d}k$ ('volume of $\mathrm{SO}^+(1,3)$ = volume of SO(3) $\times$ volume of $\mathrm{SO}^+(1,3)/\mathrm{SO}(3)$'). To find the Haar measure $\mathrm{d}k$ on SO(3), we first use the fact [Fecko, 2006] that left-invariant volume forms ω_L are related to right-ivariant volume forms ω_R via the adjoint representation, in the following way $\omega_R(g) = \mathrm{detAd}_g\omega_L(g)$ with g being the element of the group and detAd_g the determinant of matrix of the element in adjoint representation. For SO(3) the adjoint representation is given by matrix S, characterising the rotation, wich acts on

[1] Which serves as the auxiliary Hilbert space $\mathcal{H}_{aux}$.

[2] According to [Fecko, 2006], left invariant one-forms on $\mathrm{GL}(\mathbb{R})$ are $\alpha_L = g^{-1}\mathrm{d}g$, right invariant forms are $\alpha_R = \mathrm{d}g\, g^{-1}$. We use this in more detail below in the text.

other matrices X by $\mathrm{Ad}_S X = S\,X\,S^{-1}$. The determinant of rotation matrices is of course 1, therefore $\omega_R(S) = \omega_L S$ and it is sufficient to find left-invariant form, to automatically get the bi-invariant Haar measure. To find the left-invariant forms, we again use a result from [Fecko, 2006], that a general left-invariant form α on a matrix group $\mathrm{GL}(n,\mathbb{R})$ is of the form $\alpha = x^{-1}\mathrm{d}x$, $x \in \mathrm{GL}(n,\mathbb{R})$. SO(3) is a (maximal compact) subgroup of $\mathrm{GL}(3,\mathbb{R})$ and in this case, the left-invariant forms are obtained via pull-back of forms in $\mathrm{GL}(3,\mathbb{R})$, iduced by embedding SO(3) in $\mathrm{GL}(3,\mathbb{R})$. Thus, for SO(3) we get the left-invariant 1-forms e^a from

$$S^{-1}\mathrm{d}S = e^a E_a \tag{5.1}$$

whith S being a general rotation matrix and E_a a basis in so(3) algebra. The basis is given by $(E_i)_{jk} = -\epsilon_{ijk}$, so for example

$$E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is useful to parametrize S by euler angles $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 2\pi)$. S then takes the form

$$S(\alpha\,\beta\,\gamma) = S_z(\alpha)\,S_y(\beta)\,S_z(\gamma),$$

where $S_z(\alpha)$, for example, stands for rotaion around z axis about an angle α. The right-hand side of (5.1) then looks like

$$\begin{aligned} S^{-1}\mathrm{d}S &= S_z^{-1}(\gamma)\,S_y^{-1}(\beta)\,S_z^{-1}(\alpha)\,\mathrm{d}[S_z(\alpha)\,S_y(\beta)\,S_z(\gamma)] \\ &= S_z^{-1}(\gamma)\,S_y^{-1}(\beta)\,S_z^{-1}(\alpha)\,\mathrm{d}[S_z(\alpha)]\,S_y(\beta)\,S_z(\gamma) + \\ &\quad S_z^{-1}(\gamma)\,S_y^{-1}(\beta)\,\mathrm{d}[S_y(\beta)]\,S_z(\gamma) + S_z^{-1}(\gamma)\,\mathrm{d}[S_z(\gamma)]. \end{aligned} \tag{5.2}$$

Using explicit matrices for rotations

$$\begin{aligned} S_z(\alpha) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S_y(\beta) &= \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \\ S_z(\gamma) &= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and the fact that the inverse is the same as transpose, we read off the left-invariant 1-forms on left-hand side of (5.1):

$$\begin{aligned} e^1 &= \sin\gamma\,\mathrm{d}\beta - \sin\beta\,\cos\gamma\,\mathrm{d}\alpha, \\ e^2 &= \cos\gamma\,\mathrm{d}\beta + \sin\beta\,\sin\gamma\,\mathrm{d}\alpha, \\ e^3 &= \mathrm{d}\gamma + \cos\beta\,\mathrm{d}\alpha. \end{aligned}$$

Haar measure $\mathrm{d}k = e^1 \wedge e^2 \wedge e^3$, on $SO(3)$, is then

$$\mathrm{d}k = \sin\beta \,\mathrm{d}\beta \,\mathrm{d}\alpha \,\mathrm{d}\gamma. \tag{5.3}$$

For Haar measure $\mathrm{d}h$ we will use different approach, which leads to the result quicker. We know already, that measure $\mathrm{d}^4\lambda$ is $\mathrm{SO}^+(1,3)$ bi-invariant. Further we know, that the upper sheet of hyperboloid is homogeneous under $\mathrm{SO}^+(1,3)/\mathrm{SO}(3)$ and can thus be parametrized by parameters of $\mathrm{SO}^+(1,3)/\mathrm{SO}(3)$. We can identify $\mathrm{SO}^+(1,3)/\mathrm{SO}(3)$ the upper sheet of hyperboloid. We have already transformed the measure into coordinates (3.3), it reads

$$\mathrm{d}^4\lambda = \lambda^3 \sinh^2 u \,\sin\theta \,\mathrm{d}\lambda \,\mathrm{d}u \,\mathrm{d}\theta \,\mathrm{d}\phi.$$

The coordinates $(u,\, \theta\, \phi)$ are coordinates on hyperboloid. The whole space is then "sliced" into hyperboloids, each of them labeled by given λ. λ itself is not affected by $\mathrm{SO}^+(1,3)$ at all. From the form of the metric $\mathrm{d}^4\lambda = \lambda^3 \mathrm{d}\lambda \,\mathrm{d}h$ we see that the Haar measure on hyperboloid is

$$\mathrm{d}h = \sinh^2 u \,\sin\theta \,\mathrm{d}u \,\mathrm{d}\theta \,\mathrm{d}\phi.$$

Before we proceed, we introduce normalization convention for Haar measures. For compact groups G we require

$$\int_G \mathrm{d}g = 1.$$

In our case, for the group SO(3) we have $\int \mathrm{d}k = 8\pi^2$. We devide the measure by this volume. We will not write the volume factor explicitly, for convenience. By $\mathrm{d}k$ we mean, untill the end of this chapter, $\frac{\mathrm{d}k}{8\pi^2}$. Same for the "compact part" of $\mathrm{d}h$, where we devide by the volume 4π.

We are going to work in different representation[3]. We change our Hilbert space to unitarily equivalent one

$$\mathrm{L}^2\left(\Lambda, \frac{\mathrm{d}^4\lambda}{\lambda^4}\right) \quad \to \quad \mathrm{L}^2\left(\Lambda, \mathrm{d}^4\lambda\right)$$

So far, the representation[4] was such, that the measure $\frac{\mathrm{d}^4\lambda}{\lambda^4}$ is invariant uder $\mathrm{GL}^+(\mathbb{R}) \times \mathrm{SO}^+(1,3)$, if we represent $\mathrm{GL}^+(\mathbb{R})$ by $U(\tau)\phi(\lambda^a) = \phi(e^{-\tau}\lambda^a)$. If we want to use measure $\mathrm{d}^4\lambda$ instead, we need to change the unitary representation of $\mathrm{GL}^+(\mathbb{R})$ (it is already $\mathrm{SO}^+(1,3)$ invariant). The change looks like $\bar{U}(\tau)\phi(\lambda^a) = e^{-2\tau}\phi(e^{-\tau}\lambda^a)$, then we have

$$\begin{aligned} ||\bar{U}(\tau)\phi(\lambda)|| &= \int \lambda^3 \mathrm{d}\lambda \, e^{-4\tau} |\phi(e^{-\tau}\lambda)|^2 = \ldots [e^{-\tau} = y] \cdots = \int y^3 \mathrm{d}y \, |\phi(y)|^2 \\ &= ||\phi(\lambda)||. \end{aligned}$$

We have used the part of $\mathrm{d}\lambda^4$ affected by $\mathrm{GL}^+(\mathbb{R})$.

[3]The reason is that it has turned out to be more convenient and the calculations more transparent. The representation used so far is better and more elegant when working out the eigenstates decomposition.

[4]Chapter 2 at the end.

We now get to the procedure of averaging. It shows that in averaging over $SO^+(1,3)$ it is convenient to work with distributional states $|x_1\rangle$, $|x_2\rangle$, that means states that satisfy $\langle x_1|x_2\rangle = \delta^4(x_1 - x_2)$. We avoid using λ instead of x, so there is no confusion with $\lambda = \sqrt{-\eta x^a x^b}$ and λ denoting a vector $(\lambda^0, \lambda^1, \lambda^2, \lambda^3)$.

We need to compute the integral

$$I\colon = \int_G \mathrm{d}g \int_\Lambda \mathrm{d}x_1\, \mathrm{d}x_2 \phi(x_1)\langle x_1|\hat{U}^{-1}(g)|x_2\rangle \psi^*(x_2), \tag{5.4}$$

where functions ϕ, ψ lie in some dense subspace of $\mathcal{H}$.

To integrate (5.4), we first average over $SO^+(1,3)$. For that, we compute

$$I_1\colon \int_{SO^+(1,3)} \mathrm{d}s\langle x_1|\hat{U}^{-1}(s)|x_2\rangle = \int \mathrm{d}h\, \mathrm{d}k\langle x_1|\hat{U}^{-1}(hk)|x_2\rangle.$$

The strategy now, is to integrate out over rotations. We first realize, that every boost can be decomposed [Naimark and Farahat, 1964] into a pure boost, and two SO(2) rotations: $h = k_2(\theta)\, k_1(\phi)\, b(u)$, where $b(u)$ is a hyperbolic rotation in plane (x^0, x^1) with rapidity u and $k_1(\phi)$, $k_2(\theta)$ are rotations in planes (x^1, x^2), (x^2, x^3) respectively. Now $h\, k$ becomes $k_2\, k_1\, b\, k$ with $k_2, k_1 \in SO(2)$ and $k \in SO(3)$. To be able to integrate out all the SO(2) rotations, we will add another SO(3) rotation into the integral by adding unity ($\int \mathrm{d}k'$) and using the fact that $k's = s'$ with $s, s' \in SO^+(1,3)$ (Lorentz transformation followed by rotation is, again, a Lorentz transformation). Doing so, the SO(2) can be absorbed into the SO(3) rotation: $k'\, k_1\, k_2 = \bar{k} \in SO(3)$. We then use the SO(3) invariance of measure $\mathrm{d}k'$ to get

$$I_1 = \int \mathrm{d}\bar{k}\, \mathrm{d}h\, \mathrm{d}k\langle x_1|\hat{U}^{-1}(\bar{k}\, b(u)\, k)|x_2\rangle.$$

We now can integrate out over SO(2)-part of $\mathrm{d}h$, which is normalized to unity and get the form

$$I_1 = \int \mathrm{d}\bar{k}\, \mathrm{d}k\, \sinh^2 u\, \mathrm{d}u\langle x_1|\hat{U}^{-1}(k)\hat{U}^{-1}(b(u))\hat{U}^{-1}(\bar{k})|x_2\rangle.$$

To proceed, we introduce two complete sets of states:

$$I_1 = \int \mathrm{d}^4x'\, \mathrm{d}^4x''\, \mathrm{d}\bar{k}\, \mathrm{d}k\, \sinh^2 u\, \mathrm{d}u\, \langle x_1|\hat{U}^{-1}(k)|x'\rangle\langle x'|\hat{U}^{-1}(b(u))|x''\rangle\langle x''|\hat{U}^{-1}(\bar{k})|x_2\rangle.$$

We can now perform averaging over SO(3). We compute following integral

$$I_{rot}\colon = \int_{SO(3)} \mathrm{d}k\langle x|\hat{U}^{-1}(k)|x'\rangle.$$

First, only the 3-dimensional space, orthogonal to temporal part[5], is affected, thus we factor this coordinate out, giving us a delta function $\langle x^0|(x^0)'\rangle = \delta(x^0 - (x^0)')$. Secondly, SO(3) acts on 2-dimensional sphere, so the remaining 3-dimensional space given by $|\vec{x}\rangle$ contains a 1-dimensional invariant subspace. That is given by

[5]We have chosen the temporal part to given by x^0 which serves as the axis of rotation.

the combination $\sum_{i=1}^{3} x^i =: r^2$. Since this subspace is not affected by rotation, it gives us another delta function: $\delta(r^2 - (r')^2)$. We parametrize the 2-dimensional space, affected by rotation, by two anles θ and ϕ. We want to compute

$$I_{rot} = \int_{\mathrm{SO}(3)} \mathrm{d}k \langle \theta\, \phi | \hat{U}^{-1}(k) | \theta'\, \phi' \rangle.$$

To do that, we introduce a complete set of states $|j\, m\rangle$, on which SO(3) has natural action via Wigner matrices (a completely reducible representation of rotation group):

$$I_{rot} = \int \mathrm{d}k \sum_{j=0,1,2\ldots} \sum_{m=-l}^{l} \sum_{j'=0,1,2\ldots} \sum_{m'=-l'}^{l'} \langle \theta\, \phi | j\, m \rangle \langle j\, m | \hat{U}'(k) | j'\, m' \rangle \langle j'\, m' | \theta'\, \phi' \rangle,$$

where we recognize spherical harmonics $Y_{jm} = \langle \theta\, \phi | j\, m \rangle$. Wigner matrix is defined as

$$\langle j\, m | \hat{U}^{-1}(k) | j'\, m' \rangle =: \delta_{jj'} D^{j'}_{mm'}(k).$$

In Euler angles, Wigner matrix factorizes as

$$D^{j}_{mm'}(\alpha\, \beta\, \gamma) = e^{i\, m\, \alpha} e^{i\, m'\, \gamma} d^{j}_{mm'}(\beta).$$

The integral now looks like

$$I_{rot} = \int \mathrm{d}k \sum_{jmm'} Y_{jm}(\theta\, \phi) Y_{jm'}(\theta'\, \phi') e^{i\, m\, \alpha} e^{i\, m'\, \gamma} d^{j}_{mm'}(\beta).$$

Writing the Haar measure dk explicitly as (5.3), we integrate over α and γ, using the standart formula

$$\int_0^{2\pi} \mathrm{d}\alpha e^{i\, m\, \alpha} = 2\pi\, \delta_{m0}$$

(integral is zero but when $m = 0$, then the value is 2π), we get

$$I_{rot} = \frac{1}{8\pi^2} 4\pi^2 \int \sin\beta\, \mathrm{d}\beta \sum_{j} Y_{j0}(\theta\, \phi) Y_{j0}(\theta'\, \phi') d^{j}_{00}(\beta),$$

where the numerical factor $8\pi^2$ comes from our convention of normalized Haar measure. The small Wigner matrix can be given explicitly

$$d^{j}_{00}(\beta) = P_j(\cos\beta),$$

where $P_j(x)$ is Legendre polynomial. We are now left with final integration in the form

$$I_{rot} = \frac{1}{2} \int_0^{\pi} \sin\beta\, \mathrm{d}\beta P_j(\cos\beta) = [\ldots x = \cos\beta \ldots] = \frac{1}{2} \int_{-1}^{1} \mathrm{d}x P_l(x). \qquad (5.5)$$

If we now use result from [Gradshteyn and Ryznik, 2007]

$$\int_{-1}^{1} \mathrm{d}x\, x^k P_l(x) = 0 \quad \text{for} \quad 0 \leq k < l,$$

we see that in our case $k = 0$, thus (5.5) is nonzero only for $j = 0$. But $P_0(x) = 1$, so the value of (5.5) is 2. Now we are left with

$$Y_{00}(\theta\, \phi)\, Y_{00}(\theta'\, \phi') = \frac{1}{4\pi},$$

becouse $Y_{00}(\theta\, \phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$. The final result of averaging over SO(3) is

$$I_{rot} = \int_{\mathrm{SO}(3)} \mathrm{d}k \langle x|\hat{U}^{-1}(k)|x'\rangle = \frac{1}{4\pi}\,\frac{1}{r}\,\delta(r^2 - (r')^2)\,\delta(x^0 - (x^0)').$$

The factor $\frac{1}{r}$ was added from dimensional reasons. The measure is dimensionless, while the matrix element has dimension of length^{-4} (time coordinate also has dimension of length in our units).

After the boost in plain (x^0, x^1) we get boosted coordinates, which we will for brievity denote with lower index $_u$, therefore x_u stands for

$$\begin{pmatrix} x_u^0 \\ x_u^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \cosh u + x^1 \sinh u \\ x^0 \sinh u + x^1 \cosh u \\ x^2 \\ x^3 \end{pmatrix}$$

Then our integral looks (we denote $c = \frac{1}{(4\pi)^2}$)

$$I_1 = \int \frac{c \sinh^2 u}{r_1\, r_2}\, \mathrm{d}u\, \mathrm{d}^4x'\, \mathrm{d}^4x'' \delta(r_1^2 - r'^2)\delta(x_1^0 - x'^0)\delta(x_u' - x'')\delta(r_2^2 - r''^2)\delta(x_2^0 - x''^0).$$

We now integrate over x'' to get

$$I_1 = \frac{c}{r_1\, r_2} \int \sinh^2 u\, \mathrm{d}u\, \mathrm{d}^4x' \delta(r_1^2 - r'^2)\delta(x_1^0 - x'^0)\delta(r_2^2 - r_u'^2)\delta(x_2^0 - x_u'^0).$$

Now we perform integration over x'^0 and substitute in the last delta function

$$I_1 = \frac{c}{r_1\, r_2} \int \sinh^2 u\, \mathrm{d}u\, \mathrm{d}^3x' \delta(r_1^2 - r'^2)\delta(r_2^2 - r_u'^2)\delta(x_2^0 - x_1^0 \cosh u - x'^1 \sinh u).$$

Using the following property of delta function

$$\delta f(x) = \frac{1}{|f'(x_0)|}\,\delta(x - x_0)$$

where now $'$ stands for derivative and x_0 is a solution to $f(x_0) = 0$. This yields

$$\delta(x_2^0 - x_1^0 \cosh u - x'^1 \sinh u) = \frac{1}{\sinh u}\,\delta(x'^1 - X'^1),$$

with

$$X'^1 = \frac{x_2^0 - x^0 \cosh u}{\sinh u}.$$

Another integration over x'^1 gives

$$I_1 = \frac{c}{r_1\, r_2} \int \sinh^2 u \, \mathrm{d}u \, \mathrm{d}^2 x' \delta(r_1^2 - A^2 - X'^1)\delta(r_2^2 - A^2 - (x_1^0 \sinh u + X'^1 \cosh u)^2), \tag{5.6}$$

Where we denoted $A^2 = (x^1{}')^2 + (x^2{}')^2$. Transforming $\mathrm{d}^2 x'$ according to general rule when using spherical coordinates

$$\mathrm{d}^n x = r^{n-1}\mathrm{d}r \times \text{surface of } (n-1) \text{ dimmensional sphere},$$

then gives

$$\mathrm{d}^2 x = 2\pi \, A \mathrm{d}A$$

The last term in second delta function in (5.6) looks

$$x_1^0 \sinh u + X'^1 \cosh u = \frac{x_2^0 \cosh u - x_1^0}{\sinh u}.$$

Performing now transformation of the second delta function in (5.6) with respect to A, we get

$$\delta(r_2^2 - A^2 - \left(\frac{x_2^0 \cosh u - x_1^0}{\sinh u}\right)^2) = \frac{1}{2A_0}[\delta(A - A_0) + \delta(A + A_0)], \tag{5.7}$$

where

$$A_0 = \frac{\sqrt{r_2^2 \sinh^2 u - (x_2^0 \cosh u - x_1^0)^2}}{\sinh u}$$

We realize, that the two terms in (5.7) contribute in the same way since we substitute into second power, where the sign does not matter. Putting all pieces together, we obtain

$$I_1 = \frac{2\pi\, c}{r_1\, r_2} \int \sinh^2 u \, \mathrm{d}u \, A_0 \, \frac{1}{A_0} \delta(r_1^2 - r_2^2 + (\frac{x_2^0 \cosh u - x_1^0}{\sinh u})^2 - (\frac{x_2^0 - x_1^0 \cosh u}{\sinh u})^2)$$

and after adjusting terms in delta funtion, we get

$$I_1 = 2\pi\, c \frac{\delta(s_1^2 - s_2^2)}{r_1\, r_2} \int \sinh^2 u \, \mathrm{d}u, \tag{5.8}$$

where we have introduced a space-time interval $s_i^2 = -(x_i^0)^2 + r_i^2$.

The last integral over u seems to be divergent, we show, it is not the case in time-like case. The integration domain is not correct. We must require that the term in squareroot in A_0 is positive. Denoting $y = \cosh u$ and setting

$$r_2^2 \sinh^2 u - (x_2^0 \cosh u - x_1^0)^2 \geq 0, \tag{5.9}$$

then using $\sinh^2 u = \cosh^2 u - 1$ we have

$$s^2 y^2 + 2 x_1^0 x_2^0 y - (r_1^2 + (x_2^0)^2) \geq 0.$$

The two roots are

$$y_\pm = \frac{x_1^0 x_2^0 \pm \sqrt{(x_1^0)^2(x_2^0)^2 + s_1^2(r_1^2 + (x_2^0)^2)}}{s_1^2}.$$

In time-like case when $s^2 = -\lambda^2 < 0$, we have that (5.9) is satisfied in interval $[y_-, y_+]$. The integral over u then evaluates to

$$\int_1^\infty \mathrm{d}y = \int_{x_-}^{x_+} \mathrm{d}y = \frac{2}{\lambda_1^2}\sqrt{(x_1^0)^2(x_2^0)^2 + (-\lambda_1^2)(r_1^2 + (x_2^0)^2)}.$$

We now adjust the term in the brackets, using also the term $r_1\,r_2$ in the denominator in (5.8) to get

$$\sqrt{\frac{(x_1^0)^2(x_2^0)^2 + (-\lambda_1^2)(r_1^2 + (x_2^0)^2)}{r_1^2\,r_2^2}} = \sqrt{\frac{-\lambda_1^2 + \lambda_2^2 + r_2^2}{r_2^2}} = 1$$

The last equality effectivelly holds due to the delta function $\delta(\lambda_1^2 - \lambda_2^2)$. We now finally get

$$I_1 = \frac{2}{8\pi}\frac{\delta(\lambda_1^2 - \lambda_2^2)}{\lambda_2^2}.$$

The averaging procedure over $\mathrm{SO}(1,3)$ leads to

$$I_1 = \frac{1}{4\pi}\frac{\delta(\lambda_1^2 - \lambda_2^2)}{\lambda_2^2}.$$

Group $\mathrm{GL}^+(\mathbb{R})$ affects only the 'radial' part of x given by λ. $\mathrm{SO}(1,3)$ affects only hyperboloid. That allows us to first average over $\mathrm{SO}(1,3)$ *independently* of $\mathrm{GL}^+(\mathbb{R})$ and then continue with the GL part. We thus do not need to introduce third complete set of states $|x\rangle$ to put between $\hat{U}(\tau)$ and $\hat{U}(s)$.

To finish the procedure, we compute

$$I = \frac{1}{4\pi}\int_{\mathbb{R}_0^+} \frac{\mathrm{d}N}{N}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\frac{\delta(\lambda_1^2 - \lambda_2^2)}{\lambda_2^2}\,\phi(x_1)\hat{U}^{-1}(N)\psi^*(x_2).$$

We have decided to use group $(\mathbb{R}_0^+, \times)$ isomorphic to $\mathrm{GL}^+(\mathbb{R})$. Its action is $\hat{U}(N)\psi(x) = N^{-2}\psi(N^{-1}x)$. Writing now the space-time measure as $\mathrm{d}^4x = \lambda^3\mathrm{d}\lambda\,\mathrm{d}h$, where $\mathrm{d}h$ denotes measure on hyperboloid and integrating over $\mathrm{d}\lambda_2$, we obtain

$$I = \frac{1}{4\pi}\int_{\mathbb{R}_0^+} \frac{\mathrm{d}N}{N}\int \lambda_1^3\mathrm{d}\lambda_1\,\mathrm{d}h_1\,\mathrm{d}h_2\,\phi(\lambda_1, h_1)\hat{U}^{-1}(N)\psi^*(\lambda_1, h_2),$$

where we have used the fact that $\lambda > 0$ therefore $\delta(\lambda_1^2 - \lambda_2^2) = \frac{\delta(\lambda_1 - \lambda_2)}{2\lambda_2}$. After evaluating the action of the group (acting as $\hat{U}^{-1}$) we get

$$I = \frac{1}{4\pi}\int_{\mathbb{R}_0^+} \frac{\mathrm{d}N}{N}\int \lambda_1^3\mathrm{d}\lambda_1\,\mathrm{d}h_1\,\mathrm{d}h_2\,N^2\phi(N\lambda_1, h_1)\,\psi^*(\lambda_1, h_2).$$

The functions ϕ and ψ are functions with compact support, therefore the integrals over hyperboloid are finite. We denote

$$\begin{aligned}\Phi(\lambda) : &= \int\limits_{\Lambda_h} \mathrm{d}h\, \phi(\lambda, h),\\ \Psi^*(\lambda) : &= \int\limits_{\Lambda_h} \mathrm{d}h\, \psi^*(\lambda, h).\end{aligned}$$

Then we have

$$I = \frac{1}{4\pi} \int\limits_{\Lambda_\lambda} \lambda_1 \mathrm{d}\lambda_1\, \Psi^*(\lambda_1) \int\limits_{\mathbb{R}_0^+} y \mathrm{d}y \Phi(y). \tag{5.10}$$

We have used the substitution $N\lambda_1 = y$, while keeping λ constant for the moment, so we integrate over λ later. We proceed by evaluating the integral over y

$$\int\limits_0^\infty y \mathrm{d}y\, \Phi(y) = \int\limits_K y \mathrm{d}y\, \Phi(y) =: \Phi \tag{5.11}$$

where $Phi < \infty$ is a constant since ϕ has compact support thus the integral is over compact domain. The remaining integral is

$$\int\limits_{\Lambda_\lambda} \lambda_1 \mathrm{d}\lambda_1\, \Psi^*(\lambda_1) =: \Psi^*,$$

with Ψ^* being some finite constant from the same reason as above. Putting all pieces together we get a result after averaging

$$\langle \eta\phi | \eta\psi \rangle_{phys} = \frac{\Phi\, \Psi^*}{4\pi}.$$

We see easily that the map η is positive and real in sense of step 5c in chapter 4.

We have thus managed to construct a physical Hilbert space. To see that, we look at the elements $\phi_{phys} := \eta\phi \in \mathcal{H}_{phys}$. The structure of states ϕ_{phys} can be seen from the construction of scalar product, namely from equations (5.10) and (5.11). The scalar product is actually an action of functional ϕ_{phys} on ψ

$$\langle \eta\psi | \eta\phi \rangle = [\phi_{phys}](\psi) \tag{5.12}$$

and

$$\phi_{phys} = \left(\int\limits_G \mathrm{d}g\, \hat{U}(g)\phi \right)^\dagger$$

equation (5.10) is precisely what (5.12) looks like in terms of *functions* $\psi(\Lambda)$ and $\Phi(\Lambda)$. Then vectors $\psi_{phys} \in \mathcal{H}_{phys}$ are just c-numbers therefore the overall result is

$$\mathcal{H}_{phys}(\lambda^a) = \mathbb{C}$$

It is proved in [Gomberoff and Marolf, 1999] that the map η (in case of $\mathrm{SO}^+(1,3)$ satisfies also the other conditions in Step 5c.

Conclusion

We have successfully managed to quantize the temporal part of ECT 3+1 decomposition. To do so, we had to deal with the fact that the Hamiltonian for ECT consists only of constraints, therefore we had to use Dirac‘s approach to quantization. The decomposition of ECT theory, finding new Poisson algebra in terms of generalized Dirac brackets and constructing a point-wise Hilbert space $L^2(\Lambda, d\Lambda) \otimes L^2(\mathcal{E}, d\mathcal{E})$ has been done in papers [Pilc, 2013b,a]. Unitary representation of gauge groups leaving the space $\Lambda \times \mathcal{E}$ is also given there. The gauge group of temporal part Λ is $GL^+(\mathbb{R}) \times SO^+(1,3)$

In this thesis we have used those results and focused on the part $L^2(\Lambda, d\Lambda)$. We computed the $GL^+(\mathbb{R}) \times SO^+(1,3)$ bi-invariant measure $d\Lambda$. On this Hilbert space we represented the generators of gauge group in terms of self-adjoint operators $\hat{\pi}$ (GL^+) and $\hat{L}_{ab}$ ($SO^+(1,3)$). We found its algebra and introduced two Casimir operators, one of which is identically zero in our representation. We discussed that there exist four quantum numbers labeling the basis in Hilbert space and found spectra and eigenstates of the four operators forming the complete set of commuting operators. In case of Casimir $\hat{C}_1 = \frac{1}{2}\hat{L}_{ab}\hat{L}^{ab}$ we were not able to normalize its eigenstates ψ_{pj} for general quantum number j[6]. However, we were able to do that for $j = 0$. In article [Huszár, 1985], the problem of eigenstates of $\hat{C}_1$ is solved in different coordinate system [7] and the solution is given in terms of “spherical harmonics” consisting of Bessel functions. The normalization is performed generally in the article.

We thus found the eigenstates of generators of the gauge groups GL^+ and $SO^+(1,3)$.

Further we had to deal with the fact that the system posseses firts class constraint given by $\pi = 0$. The gauge freedom of the system also needed to be dealt with. In order to do that, we had to use a method called Refined algebraic quantization. The method first quantizes the system regardless the constraints (constructing an auxiliary Hilbert space $\mathcal{H}_{aux}$). Then represents the constraints as self-adjoint operators on some dense subspace of $\mathcal{H}_{aux}$ and uses the physical states $|\Psi_{phys}\rangle$[8] to construct the physical Hilbert space $\mathcal{H}_{phys}$[9]. The scalar product is constructed via group averaging procedure. Namely

$$\langle\psi|\phi\rangle_{phys} := \int_G dg\, \langle\psi|\hat{U}^{-1}(g)|\phi\rangle_{aux}, \tag{5.13}$$

where G is the gauge group, dg its Haar measure and $\hat{U}(g)$ its unitary representation on $\mathcal{H}_{aux}$, is gauge group invariant. We have managed to find such scalar procuct on our system in chapter 5, thus completing the procedure of quantization.

We have shown that the integral in (5.13) is convergent in time-like case, however, it is not convergent for space-like case. In arcticle [Gomberoff and

[6] Where j is a quantum number associated to the square of angular momentum, the generator of SO(3).

[7] Which is not suitable for the other operators in our set. Therefore we did not use it.

[8] The states are solutions to the constraints $\hat{C}_i$ in the sense that $\hat{C}_i|\Psi_{phys}\rangle = 0$.

[9] The physical Hilbert space may not be a subspace of the auxiliary Hilbert space.

Marolf, 1999] the averaging over $\mathrm{SO}(1,n)$ is performed for general space dimension n and space-like case is also discussed in there.

In chapter 5 we have managed to fing a rigging map from Step 5c in chapter 4. Actually, we have found a scalar product on physical Hilbert space

$$\langle \eta\phi | \eta\psi \rangle = (\eta\psi)[\phi],$$

where η is the rigging map mentioned above. The physical Hilbert space is $\mathcal{H}_{phys} = \mathbb{C}$.

Further step to continue in the present thesis would be to focus on the spatial part of the 3+1 decomposition, where the gravity manifests itself. The steps would be similar, although more complicated and tidious. This is done in [Nikolič, 1995] in a different way.

The result $\mathcal{H}_{phys}(\lambda^a) = \mathbb{C}$ means that physical states do not depend on gauge in λ^a. When a Hilbert space of some system is just $\mathbb{C}$, it means there exist a single state. The whole physical Hilbert space is

$$\mathcal{H}_{phys}(\mathbf{e}^a) = \mathcal{H}_{phys}(\lambda^a) \otimes \mathcal{H}_{phys}(\mathbf{E}^a) = \mathbb{C} \otimes \mathcal{H}_{phys}(\mathbf{E}^a).$$

Classicly, the constraint $\pi^a = 0$ gives that the physics does not depend on λ^a[Pilc, 2013b]. We reconstructed this fact in chapter 5 by quantizing the system.

The variables λ^a are identified with the *lapse function* N and *shift vector field* $\vec{N}$ in ADM formalism [more generally on ADM formalism in Thiemann, 2007]. The relevant physics plays role on the spatial hypersurface Σ.

Appendix A

In this Appendix we briefly discuss how to proceed to obtain a Hamiltonian formulation of dynamics in case of systems with constraints. Dirac proposed this algorithm in his paper [Dirac, 1950], it is also nicely described in [Thiemann, 2007].

We are going work in finite number of degrees of freedom, for simplicity, which is sufficent for our purpose of introduction to this problem. Generalization to infinite number of degrees of freedom is straightforward. Suppose we have a system with m degrees of freedom and let L be a Lagrangian function on tangential bundle over a configuration manifold $\mathcal{M}$. We denote coordinates on $\mathcal{M}$ as q^a and corresponding velocities (in the tangential bundle $T\mathcal{M}$) as $\dot{q}^a =: v^a$ with $a = 1, \dots, m$. To proceed to Hamiltonian formalism, we introduce canonicall momenta defined by

$$p^a(q, v) := \frac{\partial L}{\partial v^a}(q, v);\, p_a \in T^*\mathcal{M},$$

where $T^*\mathcal{M}$ is the cotangential bundle, and perform a Legendre transformation in order to obtain Hamiltonian function

$$H := (p_a \dot{q}^a - L)_{\dot{q}^a = f(q,p)} .$$

Problems occur when the Lagrangian is singular which can be written in terms of Hessian as

$$\det\left(\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right)^m_{a,\,b=1}\right) = 0. \tag{5.14}$$

In that case we are not able to express all velocities in terms of q 's and p 's only. However, if matrix (5.14) has rank $m - r$ for some $r \in (0, m]$, we can solve for $m - r$ velocities as functions of q 's, p 's and v 's. That is for some index $A = 1, \dots, m - r$; $a = 1, \dots, m$ and $i = m - r + 1, \dots, m$ we have

$$p_A = \frac{\partial L}{\partial v^A}(q, v) \;\Rightarrow\; v^A = u^A(q^a, p_A, v^i). \tag{5.15}$$

We then get r remaining equations

$$p_i = \left(\frac{\partial L}{\partial v^i}(q, v)\right)_{v^A = u^A(q^a,\, p_A,\, v^j)} =: \pi_i(q^a, p_A), \tag{5.16}$$

where the right-hand side cannot depend on v^i, otherwise the rank of (5.14) would exceed $m - r$. Then we would be able to solve for another velocities which would contradict the assumption that we can only solve for $m - r$ velocities.

We then obtain r constraints

$$\Phi_i(q^a, p_a) := p_i - \pi_i(q^a, p_a) \tag{5.17}$$

that we call *primary constraints* and we define *primary Hamiltonian*

$$H'(q^a, p_a, v^i) = (p_a \dot{q}^a - L(q^a, \dot{q}^a))_{\dot{q}^A = u^A(q^a,\, p_A,\, v^i)} . \tag{5.18}$$

Primary Hamiltonian depends on v^i 's linearly. We can see that by differentiating (5.18) with respect to v^i. We have, using (5.15), (5.16) and (5.17)

$$\frac{\partial H'(q^a, p_a, v^j)}{\partial v^i} = (p_i - \pi_i(q^a, p_A)) = \Phi_i(q^a, p_a)$$

and we can write

$$H'(q^a, p_a, v^i) = \bar{H}(q^a, p_a) + v^i \Phi_i(q^a, p_a),$$

where $\bar{H}$ is Hamiltonian independent of v^i 's.

We have reduced the $2m$ dimensional phase space to a subspace with dimension $2m - r$. In order for the theory to be consistent, we require the reduced contrained surface to be invariant with respect to time evolution. Thus we require that on constrained surface $\bar{\mathcal{M}} := \mathcal{M}|_{\Phi=0}$

$$\dot{\Phi}_i|_{\Phi=0} = \{H',\, \Phi_i\}|_{\Phi=0} = \left(\{\bar{H}, \Phi_i\} + v^j\{\Phi_j, \Phi_i\}\right)_{\Phi=0} = 0$$

for all $j = 1, \ldots, r$. The symbol $\{,\}$ denotes Poisson brackets. The primary contraints fall into one of the following four categories:

Type I $\dot{\Phi}_i|_{\Phi=0} \equiv 0$ for $i = 1, \ldots, \alpha$ identically for all v^i

Type II $\dot{\Phi}_i|_{\Phi=0} \neq 0$ and $\{\Phi_j, \Phi_i\}|_{\Phi=0} = 0$ for all $j = 1, \ldots, r$ and $i = \alpha + 1, \ldots, \beta$

Type III $\dot{\Phi}_i|_{\Phi=0} \neq 0$ for generic v^i but the matrix $\{\Phi_j, \Phi_i\}|_{\Phi=0}$ with $j = 1, \ldots, r$ and $i = \beta + 1, \ldots, r$ has maximal rank $r - \beta$

Type IV as in Type III, but the matrix $\{\Phi_j, \Phi_i\}|_{\Phi=0}$ has rank less than $r - \beta$.

We can rule out Type IV since in that case we cannot find v^i to set $\dot{\Phi}_i|_{\Phi=0} = 0$, which means the theory would be inconsistent. Constraints in Type I are automatically consistent with the dynamics. Type III already fixes some of the multipliers v^i since we can solve for them. The remaining Type II gives rise to new constraints. We call the new constraints *secondary constraints*.

We iterate this procedure until we no longer generate any other independent secondary constraints, which happens for $\beta = \alpha$. The procedure stops at most after $2m - r$ steps becouse in worst case we get in each step one new constraint and for the new constraint we need to repeat the iteration. We either end up with Type I, type III (then we stop the iteration) or type II, which generates another contstraints. Since we started with $2m$ dimensional phase space and r constraints, we can get at most $2m - r$ new constraints. In each step we add the new secondary contraints to the primary constraints, that is, $\Phi_i = \dot{\Phi}_{i-r+\alpha}$ and rename r by $r \to r' = r + \beta - \alpha$. So now in each step the condition $\Phi = 0$ means $\Phi_i = 0$ for $i = 1, \ldots, r'$. It should be clear that in every iteration we only use primary constraints in H', we do not add the secondary constraints into H'.

In the end of the procedure we have $\dot{\Phi}_i = 0$ satisfied automatically for $i = 1, \ldots, \alpha$, $0 \leq \alpha \leq r'$ for arbitrary v^i. Also the matrix $\{\Phi_j, \Phi_i\}|_{\Phi=0}$ for $j = 1, \ldots, r$, $i = \alpha + 1, \ldots, r'$ has maximal rank $r' - \alpha \leq r$. We can now write $v^j = v_0^j + \lambda^\mu v_\mu^j$ where $v_0^j(q^a, p_a)$ is a particular solution of the inhomogeneous equation

$$\left(\{\bar{H}, \Phi_i\} + v^j\{\Phi_j, \Phi_i\}\right)_{\Phi=0} = 0$$

and $v^j_\mu(q^a,p_a)$, with $\mu = 1,\dots,r-(r'-\alpha)$, is a basis for a general solution of the homogeneous equation. So we see that in special case of maximal rank r we managed to solve for all the multipliers. In general, however, we solve only for some of them and for the rest we introduce new multipliers λ_μ and denote $\phi_\mu = v^j_\mu\Phi_j$ and $H = \bar{H} + v^j_0\Phi_j$. We define the *extended Hamiltonian*

$$H_\lambda = H + \lambda^\mu\phi_\mu.$$

Let us define a first class function. We say that function f is of first class if $\{\Phi_i, f\}|_{\Phi=0} = 0$ for *all* $i = 1,\dots,r'$, otherwise of second class. It is clear from construction that H_λ and ϕ_μ are of first class.

Physics should not depend on parameters λ. We need the time evolution, or $\{H_\lambda, f\}|_{\Phi=0}$, to be independent of any λ. This leads to a condition $\{\phi_\mu, f\}|_{\Phi=0} = 0$ for all μ. Since physics does not depend on λ, these parameters are just gauge and first class constraints may be considered as generators of gauge transformations. Since the reality is unaffected by first class constraints, we can afford to extend the first class ϕ_μ 's for $\mu = 1,\dots,r-(r'-\alpha)$, that we already have, to a maximal set C_μ, where $\mu = 1,\dots,k$ and $k \geq r-(r'-\alpha)$, with additional multipliers. The remaining independent constraints are second class constraints Φ_I for $I = 1,\dots,r'-k$. It shows in multiple examples that this extension to C_μ is justified and it also can be shown that unlike the previous ϕ_μ, the maximal set C_μ forms a closed constraint algebra [for more detail see Thiemann, 2007].

It is easy to show that the number of second-class constraints is always even and that the matrix $\{\Phi_I, \Phi_J\}|_{\Phi=0}$ is invertible. We denote its inverse as $c^{IJ} := ((\{\Phi_K,\Phi_L\})^{-1})^{IJ}$. We define modified poisson brackets, called *Dirac brackets*, in the following way:

$$\{f,g\}^* := \{f,g\} + \{\Phi_I, f\}c^{IJ}\{g,\Phi_J\},$$

where f, g are functions of q 's and p 's. It holds for first class constraints C_μ and Hamiltonian H_λ that for any f we have $\{C_\mu, f\}^*|_{\Phi=0} = \{C_\mu, f\}|_{\Phi=0}$ and $\{H_\lambda, f\}^*|_{\Phi=0} = \{H_\lambda, f\}|_{\Phi=0}$. That means that on the constraints surface the Dirac bracket define the same equations of motion as the original Poisson brackets. It, however, does not hold for general function f (that is not of first class), so in general, Dirac brackets change the symplectic structure. We also have for any second class constraint Φ_K that $\{f,\Phi_K\}^* = 0$ for any f especially when f is also a second class constraint. This is similar as first class constraints with respect to original poisson brackets. That means that if we use Dirac brackets, we can put all second class constraints identically to zero without affecting the dynamics. It also holds, if we set all second class constraints to zero (denoting the resulting reduced space as $\mathcal{M}'$), that

$$(\{f|_{\mathcal{M}'}, g|_{\mathcal{M}'}\}^*)\,|_{\mathcal{M}'} = (\{f,g\}^*)\,|_{\mathcal{M}'}.$$

That means that it does not matter wheather we set the second class constraints equal to zero before or after evaluating the Dirac brackets.

We see that we can solve for second class constraints and work with original Poisson brackets on the reduced phase space, or use Dirac brackets on the whole phase space[10]. Dirac brackets reduce the number of degrees of freedom to half

[10]It is our choice. It depends whether it is easy to solve the constraints or not.

of the number of second class constraints[11]. We can however not get rid of first class constraints as easily. It is possible by means of gauge fixing. That means we compute gauge orbits associated with each first class constraint and choose one representative from each orbit. That is done be introducing additional constraint $k_\mu = 0$. This procedure is however not always possible.

We have four types of spaces:

- The full phase space $\mathcal{M}$ with $2m$ dimensions,
- the reduced phase space $\mathcal{M}'$ with second class constraints eaqual to zero,
- the fully reduced constraint surface $\bar{\mathcal{M}}$ defined by both first class and second class constraints
- if we compute gauge orbits on $\bar{\mathcal{M}}$, we get so called fully reduced phase space $\hat{\mathcal{M}}$. Space $\hat{\mathcal{M}}$ is indeed a phase space, since first class constraints also remove even number of degrees of freedom (one for reduction to constraint surface and one for computing gauge orbit) which means the resulting dimension of $\hat{\mathcal{M}}$ is even.

It shows that in general it may be extremely difficult to compute $\bar{\mathcal{M}}$ or $\hat{\mathcal{M}}$.

[11]The dimension of the reduced phase space is $2m - N$ where N is the number of second-class constraints.

Appendix B

We prove the commutation relations (2.4). First relation is trivial. It is easy to prove the second one, we have

$$\begin{aligned}
[\hat{L}_{ab},\hat{\pi}] &= (\eta_{ac}\lambda^c\partial_b - \eta_{bc}\lambda^c\partial_a)\,\lambda^d\partial_d - \lambda^d\partial_d\,(\eta_{ac}\lambda^c\partial_b - \eta_{bc}\lambda^c\partial_a)\\
&= \eta_{ac}\lambda^c\partial_b - \eta_{bc}\lambda^c\partial_a + \eta_{ac}\lambda^c\lambda^d\partial_d\partial_b - \eta_{bc}\lambda^c\lambda^d\partial_d\partial_a\\
&\quad -\eta_{ad}\lambda^d\partial_b + \eta bd\lambda^d\partial_a - \eta_{ac}\lambda^c\lambda^d\partial_d\partial_b + \eta_{bc}\lambda^c\lambda^d\partial_d\partial_a\\
&= 0.
\end{aligned}$$

This result is obvious also from the fact that, as discussed in chapters 3 and 4, these operators act on different independent spaces.

Third relation is also straightforward:

$$[\hat{L}_{ab},\hat{L}_{cd}] = -[\eta_{k[a}\lambda^k\partial_{b]},\eta_{l[c}\lambda^l\partial_{d]}] = -(1)+(2)+(3)-(4)$$

$$\begin{aligned}
(1) &= [\eta_{ak}\lambda^k\partial_b,\eta_{cl}\lambda^l\partial_d]\\
&= \eta_{ak}\lambda^k\eta_{cb}\partial_d + \eta_{ak}\lambda^k\eta_{cl}\lambda^l\partial_b\partial_d\\
&\quad -\eta_{cl}\lambda^l\eta_{ad}\partial_b - \eta_{cl}\lambda^l\eta_{ak}\lambda^k\partial_b\partial_d\\
&= \eta_{ak}\lambda^k\eta_{cb}\partial_d - \eta_{cl}\lambda^l\eta_{ad}\partial_b\\
(2) &= [\eta_{ak}\lambda^k\partial_b,\eta_{dl}\lambda^l\partial_c]\\
&= \eta_{ak}\lambda^k\eta_{db}\partial_c - \eta_{dl}\lambda^l\eta_{ac}\partial_b\\
(3) &= [\eta_{bk}\lambda^k\partial_a,\eta_{cl}\lambda^l\partial_d]\\
&= \eta_{bk}\lambda^k\eta_{ca}\partial_d - \eta_{cl}\lambda^l\eta_{bd}\partial_a\\
(4) &= [\eta_{bk}\lambda^k\partial_a,\eta_{dl}\lambda^l\partial_c]\\
&= \eta_{bk}\lambda^k\eta_{da}\partial_c - \eta_{dl}\lambda^l\eta_{bc}\partial_a
\end{aligned}$$

$$\begin{aligned}
[\hat{L}_{ab},\hat{L}_{cd}] &= \eta_{bc}\left(\eta_{dk}\lambda^k\partial_a - \eta_{ak}\lambda^k\partial_d\right) + \eta_{ad}\left(\eta_{ck}\lambda^k\partial_b - \eta_{bk}\lambda^k\partial_c\right)\\
&= \eta_{bd}\left(\eta_{ak}\lambda^k\partial_c - \eta_{ck}\lambda^k\partial_a\right) + \eta_{ac}\left(\eta_{bk}\lambda^k\partial_d - \eta_{dk}\lambda^k\partial_b\right)\\
&= i\left(\eta_{ac}\hat{L}_{bd} + \eta_{bd}\hat{L}_{ac} - \eta_{ad}\hat{L}_{bc} - \eta_{bc}\hat{L}_{ad}\right).
\end{aligned}$$

We now show, that operators (3.1) are Casimir operators:

$$[\hat{C}_1,\hat{\pi}] = \frac{1}{2}\eta^{ac}\eta^{bd}[\hat{L}_{ab}\hat{L}_{cd},\hat{\pi}] = \frac{1}{2}\eta^{ac}\eta^{bd}\left(\hat{L}_{ab}[\hat{L}_{cd},\hat{\pi}] + [\hat{L}_{ab},\hat{\pi}]\hat{L}_{cd}\right) = 0$$

due to second comutation relations (2.4). $[\hat{C}_2,\hat{\pi}] = 0$ from the very same reason.

Next we have

$$
\begin{aligned}
[\hat{C}_1, \hat{L}_{ij}] &= \frac{1}{2}\eta^{ac}\eta^{bd}\left(\hat{L}_{ab}[\hat{L}_{cd}, \hat{L}_{ij}] + [\hat{L}_{ab}, \hat{L}_{ij}]\hat{L}_{cd}\right) \\
&= i\,\frac{1}{2}\eta^{ac}\eta^{bd}\left\{\hat{L}_{ab}\left(\eta_{ci}\hat{L}_{dj} + \eta_{dj}\hat{L}_{ci} - \eta_{cj}\hat{L}_{di} - \eta_{di}\hat{L}_{cj}\right)\right. \\
&\quad \left. + \left(\eta_{ai}\hat{L}_{bj} + \eta_{bj}\hat{L}_{ai} - \eta_{aj}\hat{L}_{bi} - \eta_{bi}\hat{L}_{aj}\right)\hat{L}_{cd}\right\} \\
&= i\,\frac{1}{2}\left\{\eta^{bd}\left[\hat{L}_{ab}(\delta^a_i\hat{L}_{dj} - \delta^a_j\hat{L}_{di}) + (\delta^c_i\hat{L}_{dj} - \delta^c_j\hat{L}_{di})\hat{L}_{cd}\right]\right. \\
&\quad \left. +\eta^{ac}\left[\hat{L}_{ab}(\delta^b_j\hat{L}_{ci} - \delta^b_i\hat{L}_{cj}) + (\delta^d_j\hat{L}_{si} - \delta^d_i\hat{L}_{sj})\hat{L}_{cd}\right]\right\} \\
&= i\,\frac{1}{2}\left\{\eta^{bd}\left[\hat{L}_{ib}\hat{L}_{dj} - \hat{L}_{jb}\hat{L}_{di} + \hat{L}_{jb}\hat{L}_{di} - \hat{L}_{ib}\hat{L}_{dj}\right]\right. \\
&\quad \left. +\eta^{ac}\left[\hat{L}_{aj}\hat{L}_{ci} - \hat{L}_{ai}\hat{L}_{cj} + \hat{L}_{ai}\hat{L}_{cj} - \hat{L}_{aj}\hat{L}_{ci}\right]\right\} = 0
\end{aligned}
$$

The last relation to prove is:

$$
\begin{aligned}
[\hat{C}_2, \hat{L}_{ij}] &= i\,\frac{1}{8}\epsilon^{abcd}\left\{\hat{L}_{ab}\left(\eta_{ci}\hat{L}_{dj} + \eta_{dj}\hat{L}_{ci} - \eta_{cj}\hat{L}_{di} - \eta_{di}\hat{L}_{cj}\right)\right. && (5.19) \\
&\quad \left. + \left(\eta_{ai}\hat{L}_{bj} + \eta_{bj}\hat{L}_{ai} - \eta_{aj}\hat{L}_{bi} - \eta_{bi}\hat{L}_{aj}\right)\hat{L}_{cd}\right\}. && (5.20)
\end{aligned}
$$

We now use the fact that the indices have only four possible values and in relation $\epsilon^{abcj}\hat{L}_{ab}\hat{L}_{ci}$ with four indices, only possibility is $j = i$, due to antisymmetry of ϵ^{abcd}. Thus we have

$$
\begin{aligned}
& i\,\frac{1}{8}\left\{\epsilon^{abci}\hat{L}_{ab}(\hat{L}_{ci} - \hat{L}_{ci}) + \epsilon^{abid}\hat{L}_{ab}(\hat{L}_{di} - \hat{L}_{di})\right. && (5.21) \\
& \left. +\epsilon^{aicd}(\hat{L}_{ai} - \hat{L}_{ai})\hat{L}_{cd} + \epsilon^{ibcd}(\hat{L}_{bi} - \hat{L}_{bi})\hat{L}_{cd}\right\} = 0. && (5.22)
\end{aligned}
$$

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