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Mathematical modelling of selected problems in cryogenic fluid mechanics

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Zde bych chtěl poděkovat svojí milované ženě, rodině a přátelům za nehynoucí podporu, bez které by tato práce sotva vznikla. Za odbornou pomoc bych chtěl potom poděkovat v první řadě Marcovi La Mantia PhD. za trpělivé vedení diplomové práce a podporu v oblasti anglického jazyka, panu Doc. Mgr. Milanu Pokornému, Ph.D za cenné rady během psaní existenční teorie, panu prof. Ing. Miroslavu Tůmovi a Mgr. Peteru Frankovi PhD. za konzultace numerické části a nakonec Mgr. Danielu Dudovi a Mgr. Emilu Vargovi za podnětné fyzikální poznámky.

Title: Matematické modelování vybraných problémů v mechanice kryogenních tekutin

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Abstract: The dynamics of low-temperature fluids, such as superfluid helium 4, is an open scientific problem. The experimental study of similarities and differences between quantum (superfluid) and classical (viscous) flows is specifically an active research field, which already led to significant progress in our phenomenological understanding of the underlying physics. It also revealed that a comprehensive theoretical description is still missing, as, for example, in the case of the observed behaviour of moving bodies in quantum flows. The work aim is to derive the existence theory for the weak solution of a relevant system of equations based on the Landau model of superfluid helium 4 and appropriate numerical schemes to solve these equations.

Keywords: superfluidity, Landau model, weak solution, numerical methods

Contents

Introduction

The thesis is divided into three parts. The first part is a compilation of previous knowledge, which has the aim of introducing the physical background of the studied problem, related to the analysis of quantum flows of superfluid helium 4. It also includes a general description of the mathematical tools used to model the chosen problem. The second part presents the existence theory of the non-stationary PDE system describing the Landau two-fluid model of superfluid helium. In it I specifically prove the existence of a weak solution of this system and its convergence to the initial conditions. The last part is focused on a number of numerical simulations of this system of equations, based on actual experimental conditions, that is, the described numerical methods are applied on the solution of a real problem.

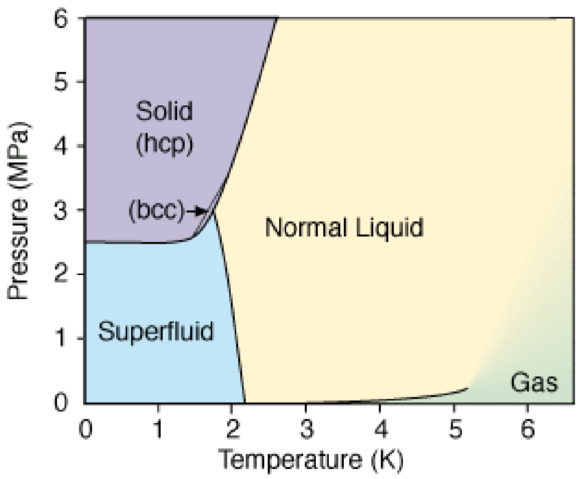
The introduction, as just mentioned, is motivated by two goals: one is to introduce the reader to some of the physical problems connected with superfluid helium flows, including the existence of quantized vortices; the other is focused on the description of the mathematical tools used in the second part, that is, the existence theory. The description of the physics of superfluid helium is divided into three parts. The first one is focused on general properties, the second part is on quantized vortices and the third one introduces the two-fluid Landau model. The thesis is written either for a mathematician or for a physicist, who has basic knowledge of mathematical and functional analysis.

Helium 4 – normal fluid and superfluid

Helium is a chemical element. It was discovered in the spectrum of sunlight in 1868 and it was named by Lockyer from *Helios,* which means sun in Greek. Helium, which was isolated from uranium minerals by Ramsay in 1895, is, at room temperature and atmospheric pressure, a transparent gas. We know two isotopes, ${}^{3}_{2}He$ and ${}^{4}_{2}He$, which have different physical properties. The atoms of ${}^{3}_{2}He$ are fermions while ${}^{4}_{2}He$ ones are bosons. This thesis is focused only on ${}^{4}_{2}He$. To obtain more information on ${}^{3}_{2}He$, see, for example, [1].

Helium was first liquefied in 1908 in Leiden by the Dutch physicist H. Kamerlingh Onnes. He was awarded the Nobel prize in 1913 for his investigations of the low-temperature properties of matter, which led, among other results, to the production of liquid helium. The phase diagram of ${}^{4}_{2}He$ is shown in Figure 1.

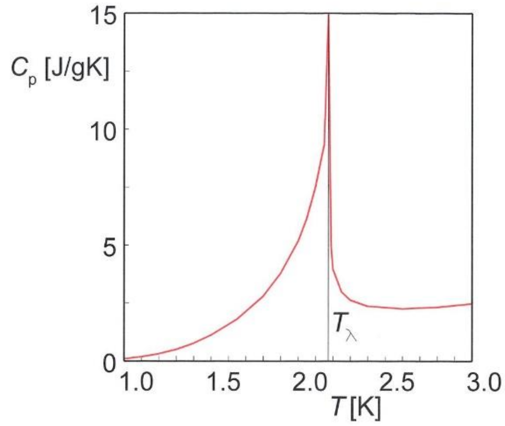
Figure 1: Phase diagram of Helium 4



We can see that there is no triple point. Another line of equilibrium connects the solid with the gas and the normal (viscous) fluid (the latter is often called He I). The superfluid can be assumed to be an inviscid fluid, behaving, in certain conditions, as a *Bose-Einstein condensate.* Additionally, its properties can be described as a macroscopical manifestation of superfluidity, which will be discussed below in more detail.

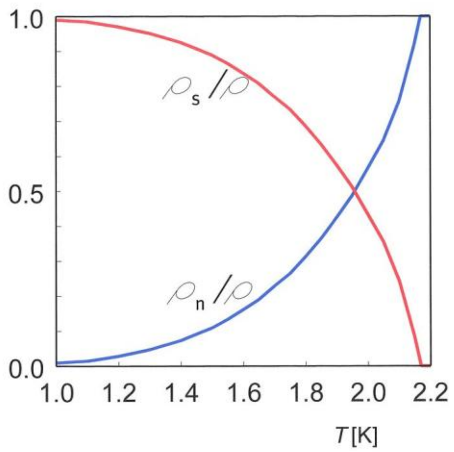
We can observe that He II behaves (above ca. 1 K) like a mixture of two fluids: one is the superfluid component and the other is the normal fluid. The behavior of this mixture will be described below, in the chapter on the Landau model. The most important base property of ${}^{4}_{2}He$ is probably the temperature dependency of its heat capacity (see Figure 2). This dependence was measured for the first time in 1930 by Keesom. The peak in the diagram was called the *Lambda-peak* due to its form, resembling the Greek letter λ.

Figure 2: Heat capacity C_p of ${}^4_2He$, plotted as a function of temperature T.



The Lambda peak corresponds to the phase transition from the normal fluid to the superfluid. As already mentioned, He II is made of two components, between ca. 1 K and 2.17 K (the latter is the transition temperature, at the saturated vapour pressure, between He I and He II). The sum of the densities of both components is almost temperature independent, see Figure 3.

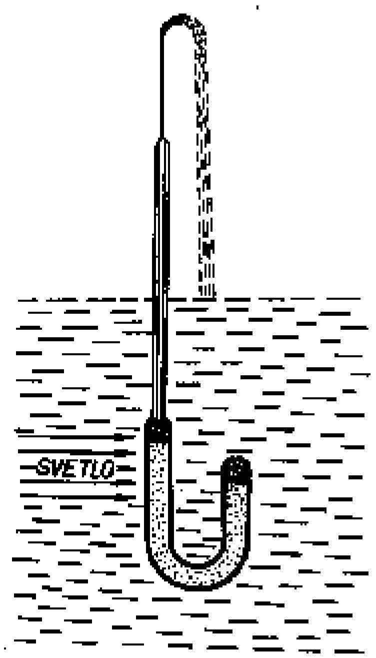
Figure 3: Density of the normal and superfluid components of He II, plotted as a function of temperature.



We can observe in Figure 3 that, at temperatures below 1 K, He II is made almost entirely by the superfluid component and that the normal fluid component vanishes. The most important mechanical property of He II is called superfluidity, which was discovered by Kapica in 1937. He submerged in the liquid a pile of thin discs, suspended vertically, and observed that, above the lambda point, the disc motions had a certain resistance, as expected. However, after decreasing the temperature, there was less resistance to the motion of the discs, indicating that the liquid became less viscous. This property of He II will be discussed below in more detail, in the derivation of the Landau model.

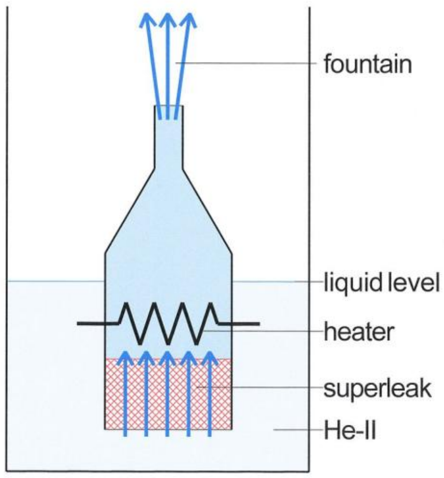
Additionally, we also mention here another property of He II, which is called thermo-mechanical. In 1937 Allen and Jones performed the following experiment: a test tube was immersed in the liquid and closed on both ends by extremely porous plugs (often called superleaks). Consequently, only the superfluid component can go out of the tube (on relevant time scales). One end of the test tube was connected to a vertical pipe terminating above the bath, while the other end was entirely immersed in the liquid. The test tube was illuminated by light, which caused the heating of the fluid inside the tube, leading to the generation of pressure and, consequently, to a liquid fountain, flowing outside the vertical pipe into the bath, see Figure 4. The outcome can be derived from equations 15.a) and 15.b) below, by using adequate assumptions.

Figure 4: Fountain effect, obtained by illuminating the helium bath.



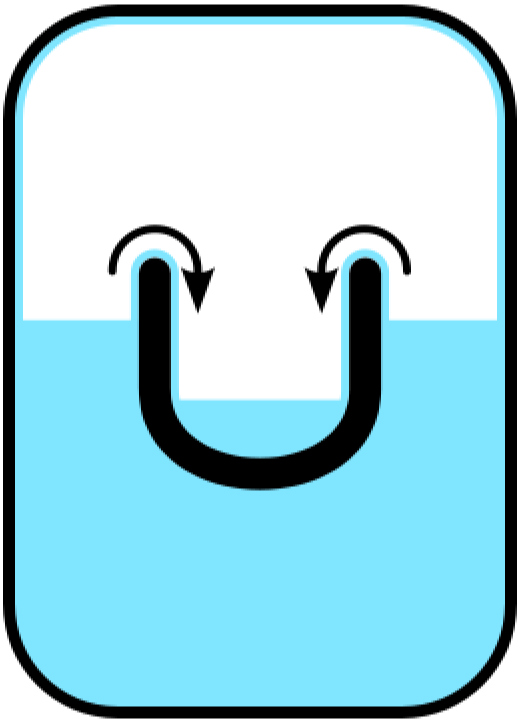
The same effect can be observed if the helium bath is heated differently, see Figure 5. Similarly, it also occurs in the opposite way, that is, a pressure gradient in the bath can cause a temperature gradient.

Figure 5: Fountain effect obtained by using a heater.



Another remarkable property of this liquid is related to the generation of this films on solid surfaces in contact with He II, see Figure 6, due to its extremely low viscosity.

Figure 6: Helium II film.



Quantum vortices

Let's now focus on the properties of the superfluid and on its non-classical behaviour. As mentioned above, the superfluid can be seen, in certain conditions, as a Bose-Einstein (BE) condensate, that is, the entire fluid behaviour can be represented by one wave function (a BE gas has zero viscosity, infinity heat conductivity and low heat capacity). However, the superfluid does not behave as an ideal BE gas due to the mutual interactions between the superfluid particles. Consequently, we need to describe the superfluid by using the so-called quasiparticles that enable us to employ the one-particle description, as in the case of the ideal BE gas.

Additionaly, the superfluid has a critical velocity, above which its non-dissipative flow is thought to break down and whose value was experimentally measured [1]. The existence of such a critical velocity can be derived theoretically and it is due to the generation of quantized vortices, which are lines singularities within the superfluid.

More specifically, the energy of the superfluid moving without excitations is given by the following equation:

$$E_0 = \frac{1}{2}Mv^2 = \frac{P^2}{2M}, \qquad 1)$$

where E_0 is the initial fluid energy, M its mass, v its velocity and $\vec{P}$ its momentum. Now we can calculate the increase of the fluid energy, by adding one excitation, and we assume a linear addition of momentum, that is:

$$\overrightarrow{P'} = \vec{P} + \vec{p}. \qquad 2.a)$$

We calculate the new energy as:

$$E = \frac{P\prime^2}{2M} + E(p) = \frac{\left(\vec{P}+\vec{p}\right)^2}{2M} + E(p) = E_0 + \vec{p}\cdot\vec{v} + \frac{p^2}{2M} + E(p), \qquad 2.b)$$

where $E(p)$ is the excitation energy. The third term on the right hand side can be neglected because we assume a low-momentum excitation and we obtain the following equation for the energy:

$$E = E_0 + \vec{p}\cdot\vec{v} + E(p). \qquad 2.c)$$

The necessary condition for the formation of an excitation is the decrease of the total energy. So the following inequality holds:

$$E < E_0. \qquad 3.a)$$

We input equations 1) into 2.c) and we obtain this condition:

$$v > \frac{E(p)}{p} = f(p). \qquad 3.b)$$

for the critical velocity estimate. Here we also assume that the angle between the fluid velocity and the momentum of the excitation is 180°, so that the corresponding scalar product is a minimum. Now the minimum of f(p) corresponds to set its first derivative to zero. This gives us the condition:

$$\frac{dE(p)}{dp} = \frac{E(p)}{p}. \quad 4)$$

The explanation of the excitations as rotons and phonons (see [1] for further details) failed because the experimental value of the critical velocity for rotons is too low and that for phonons is too large. There must be another source. It was experimental confirmed that this source can be found in the so-called quantum vortices.

Let us assume $\psi(\vec{r};t)$ to be a solution of the Schroedinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_4}\Delta\psi + \overline{V(\vec{r})}\psi, \quad 5)$$

where m_4 is the mass of one helium atom (the other symbols are customary). Then we can calculate its momentum as an eigenvalue of the momentum operator:

$$\hat{p}\psi = \hbar\nabla\psi, \quad 6)$$

the solution of which is, after dividing by the mass:

$$\overrightarrow{u_s} = \frac{\hbar}{m_4}\nabla\psi, \quad 7)$$

where $\overrightarrow{u_s}$ is the superfluid velocity. This implies that the superfluid flow (often called superfluid) is potential, that is:

$$rot\overrightarrow{u_s} = 0 \quad 8)$$

We define the *circulation* as the line integral:

$$\Gamma = \oint_L \overrightarrow{u_s}dl = \frac{\hbar}{m_4}\oint_L \nabla\psi\, dl. \quad 9.a)$$

where L is a curve and S is an area bounded by L. Then we obtain, by using *the Stokes theorem*:

$$\Gamma = \frac{\hbar}{m_4}\oint_L \nabla\psi\, dl = \int_S rot\overrightarrow{u_s}dS. \quad 9.b)$$

For a simple connected region the circulation must be equal to zero, as we see from 9.b). For multiple connected regions it must be quantized, so we obtain the Bohr-Sommerfeld condition:

$$\Gamma = n\frac{2\pi\hbar}{m_4} = n\kappa, \quad 10)$$

So the circulation is quantized. Now we assume the existence of one vortex. We can see from 10) that for the superfluid velocity it holds:

$$\overrightarrow{u_s} = \frac{n\kappa}{2\pi r}, \quad 11)$$

where r is the distance from the core of the vortex and κ is the circulation quantum. We define the characteristic length b which indicates either the dimension of the considered volume or the distance between two vortices. Then the energy of one vortex is equal to:

$$\varepsilon_V = \int_{a_0}^{b}\pi\rho_s v_s^2 r dr = \frac{n^2\rho_s\kappa^2}{4\pi}\ln\left(\frac{b}{a_0}\right), \quad 12)$$

where a_0 is a radius of the core of the vortex. This core can be seen as a hole in the superfluid because the existence of such singularities in the domain S follows from the discussion above. This can explain, for example, the conservation of superfluidity.

Note also that the most general form of motion of superfluids is called quantum turbulence and that the latter is mostly due to the dynamics of the quantized vortex tangle within the two-component (in the case of He II above 1 K) flow.

Thermal counterflow and the Landau model

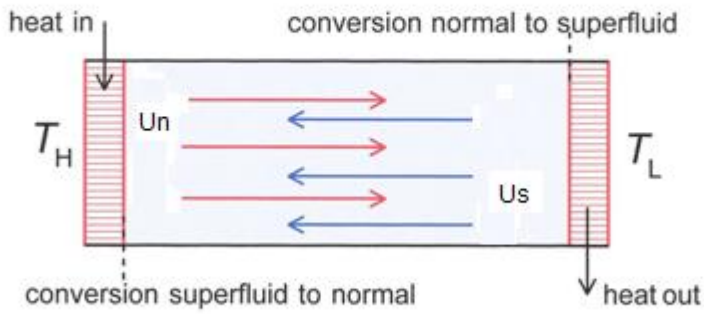
Hall and Vinen (see [1], page 136) postulated the existence of a force, called mutual friction force, between the normal fluid and the superfluid components of He II, given by the equation:

$$\overrightarrow{F_{ns}} = B\frac{\rho_s\rho_n}{\rho}\vec{\Omega}\times\left[\vec{\Omega}\times(\overrightarrow{u_s}-\overrightarrow{u_n})\right] + B'\vec{\Omega}\times(\overrightarrow{u_s}-\overrightarrow{u_n}), \qquad 13)$$

where $\vec{\Omega} = rot\overrightarrow{u_s}$ and B, B' are tabulated constants, depending on temperature [1]; $\overrightarrow{u_s}$ and $\overrightarrow{u_n}$ are the velocities of the superfluid and normal fluid components, respectively.

The implications of equation 13) can be suitably studied experimentally in thermal counterflow, which is a thermally generated flow of He II where the two fluid components flow, on large enough scales, larger than the average distance between the vortices, in opposite directions, see Figure 7.

Figure 7: Thermal counterflow.



The heat source is usually placed at the bottom of a channel (inside a cryostat) filled with He II. The normal component flows away from it (upward), while the superfluid moves downward (toward the heater), in order to conserve the null mass flow rate.

A simplified form of equation 13), will be used below:

$$\vec{f_{ns}} = -\frac{B}{2}|rot\overrightarrow{u_s}|(\overrightarrow{u_n}-\overrightarrow{u_s}). \qquad 14)$$

L.D. Landau derived the following equations for the two components of He II (see [1], page 124):

$$\frac{\partial\overrightarrow{u_n}}{\partial t} + \overrightarrow{u_n}\cdot\nabla\overrightarrow{u_n} = -\frac{1}{\rho_n}\nabla p + \nu\Delta\overrightarrow{u_n} - \frac{\rho_s}{\rho_n}S\nabla T - \frac{\rho_s}{\rho}\nabla(\overrightarrow{u_n}-\overrightarrow{u_s}) + \overrightarrow{f_{ns}} + \vec{g} \qquad 15.a)$$

$$\frac{\partial\overrightarrow{u_s}}{\partial t} + \overrightarrow{u_s}\cdot\nabla\overrightarrow{u_s} = -\frac{1}{\rho_s}\nabla p - \overrightarrow{f_{ns}} + \vec{g} + S\nabla T + \frac{\rho_n}{\rho}\nabla(\overrightarrow{u_n}-\overrightarrow{u_s}). \qquad 15.b)$$

In the case of isothermal flow the terms including temperature and enthropy can be neglected (this will be assumed in the following). We will also assume small gradients of counterflow velocities and neglect the terms including the gradient of the difference between the velocities. The assumption of no quantum turbulence (no vortices) can be given by 8).

For low velocity values we assume the condition:

$$div(\overrightarrow{u_n} + \overrightarrow{u_s}) = 0. \qquad 16)$$

In the case of large velocities, we will assume the non-divergence condition for each one of the components, that is:

$$div\overrightarrow{u_n} = 0; \quad div\overrightarrow{u_s} = 0. \qquad 17)$$

The conditions 17) will be automatically assumed in case of non-zero vorticity of the superfluid; the condition 16) is instead assumed in the case of zero vorticity of the superfluid velocity. We define the *Reynolds number:*

$$Re \coloneqq \frac{VL}{\nu}, \qquad 18)$$

where V is a typical (suitably defined) velocity of the flow, L is a characteristic flow dimension (for example, the diameter of a pipe or the largest size of an obstacle) and ν is the kinematic viscosity of the fluid.

The just introduced physical notions have the aim of define the background for the problem studied below, that is, their relevance is clarified in the following.

Tools

This part of the thesis is devoted to a general introduction of the employed mathematical techniques: we will specifically define the functional vector spaces (see [2] for more details, especially for the proofs; the reader should also have a basic knowledge of standard Lebesgue spaces and measure theory).

Definition (Def.) 1:

a) X is a vector space. A set of all linear maps on X is called a *dual space,* we denote it X'. If X is a normed space, we can introduce a norm on X' in this way:

$$f \in X' : \|f\|' := sup\{|f(x)|; x \in X;\ \|x\| \leq 1\}. \qquad 18)$$

b) The dual space of X' is called a *bidual space* of X and we denote it X''. The space X is called *reflective,* if and only if (iff) a map $J: X \to X''$ exists and if the latter is an isomorphismus. All Hilbert spaces are reflective. X is called separable, iff a dense subspace of X exists.
c) The map J is, for Lebesgue spaces, given by theformula:

$$\langle f; g \rangle = \int_\Omega fg dx, \qquad 19)$$

where g is a function. This formula can be generalized for Sobolev spaces as follows:

$$\langle f; g \rangle = \int_\Omega fg + \nabla f \cdot \nabla g dx. \qquad 20)$$

Derivations can be defined in the sense of distributions too, see Def. 3. It's customary to call this map *a duality.*

Def. 2: Let us define $\alpha = (\alpha_1; \dots \alpha_N)$, where the numbers $\alpha_1 - \alpha_N$are nonnegative integer numbers. α is then called *multiindex.* Its norm is defined as: $|\alpha| = \sum_{k=1}^N \alpha_k$ and we denote the *multiindex derivation* as:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \dots \partial x^{\alpha_N}}. \qquad 21)$$

Def. 3: Let us define the functions $u, v \in L^1_{loc}(\Omega)$, where the latter denote the space of all local integrable functions on an open set Ω. v is then called the α^{th} *weak derivation of u*, iff

$$\int_\Omega uD^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx. \qquad 22)$$

$\forall \varphi \in C_c^\infty(\Omega)$ (infinitely differentiable functions with compact support).

Def. 4: We call $W^{k,p}(\Omega)$ a *Sobolev space,* iff it consists of all locally integrable functions $u: \Omega \to \mathbb{R}$, in such a way that for each multiindex $\alpha, |\alpha| \leq k$, it exists $D^\alpha u$ in the weak sense and belongs to $L^p(\Omega)$.

Theorem 1: $W^{k,p}(\Omega)$ is a Banach space $\forall p \in [0, \infty]$, separable $\forall p \in [0,\infty)$, reflective $\forall p \in (0, \infty)$ and a Hilbert space for p=2. We define the norm of $W^{k,p}(\Omega)$ as:

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} esssup_\Omega |D^\alpha u| & p = \infty \end{cases}. \qquad 23)$$

The scalar product on $W^{k,2}(\Omega)$ is defined as:

$$(u,v)_{W^{k,2}(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha f D^\alpha g dx. \qquad 24)$$

Note: we often denote $W^{k,2}(\Omega)$ as $H^k(\Omega)$. $W_0^{k,p}(\Omega)$ indicates the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. It's then customary to write: $W_0^{k,2}(\Omega) = H_0^k(\Omega)$. We know that an orthonormal basis exists in each Hilbert space and this will be discussed below in the theoretical part of thesis. We use for the norm of a Lebesgue space $L^p(\Omega)$ the notation $\|f\|_p$.

Def. 5: Let Ω be an open set. We say that

$$u \in C^{0,1}(\overline{\Omega}), \text{ iff: } u \in C(\overline{\Omega}), \exists c > 0: \forall\, x, y \in \Omega, x \neq y: \frac{|u(x)-u(y)|}{|x-y|} \le c, \qquad 25)$$

where u is called a Lipschitz continuous function. We say that Ω is a Lipschitz domain, iff it is possible to describe its boundary by a Lipschitz continuous function.

Def. 6:

a) Let $\{u_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$. We define convergence in $W^{k,p}(\Omega)$ as:

$$u_n \to u \iff \lim_{n\to\infty} \|u_n - u\| = 0 \quad .$$
26.a)

b) As $\{u_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ the weak convergence is defined as:

$$u_n \rightharpoonup u \iff \lim_{n\to\infty} \langle u_n - u; v\rangle = 0 \qquad 26.b)$$

Theorem 2: $\lim_{n\to\infty}\|u_n\| = \|u\|$ holds. Then b) implies a).

Theorem 3: the following inequalities hold:

a) As $f \in W^{k,p}(\Omega) \cap L^q(\Omega)$ for $1 \le q < \infty, p < M$, where M is a dimension of a domain in such a way that $\Omega \subset R^M$; $\Omega \in C^{0,1}$. Let $\alpha \in (0,1)$. Then $f \in L^r(\Omega)$ and it holds:

$$\|f\|_r \le C\|f\|_{1,p}^\alpha \|f\|_q^{1-\alpha}, \qquad 27)$$

where C is a constant independent on the function f.

b) Poincaré inequality: $p: 1 \le p < \infty$ and Ω is a subset with at least one bound. Then $C = C(p,\Omega)$ and the following holds:

$$\|f\|_p \le C\|\nabla f\|_p. \qquad 28)$$

More common inequalities (especially Minkowski, Hölder, Young etc.) are not reported here as we assume their knowledge, see [2,3].

Now we write two fundamental theorems for the existence of a solution. A is an operator: $A: X \to X'$. We focus on an operator equation in the form:

$$Au = f, \qquad 29.a)$$

where f is an element of X‘. By using the fact that X‘ is a dual space of X, we reformulate the problem in this way:

$$\langle Au; v\rangle_{X,X'} = \langle f; v\rangle \;\forall v \in X. \qquad 29.b)$$

The following two theorems hold: one is called the (generalized) Lax-Milgram theorem and the other is the main theorem of monotone operators. We state directly the generalized Lax-Milgram theorem. The special version, which holds for eliptical operators, is given below and is its direct consequence.

Def. 7: A is an operator. Then we say that A is:

a) Coercive, iff: $\lim_{\|u\|\to\infty} \frac{\langle Au;u\rangle}{\|u\|} = \infty$.

b) A is a (strictly) monotone operator, iff $\forall u, v\colon u \neq v\ \langle A(u) - A(v); u - v\rangle \geq 0$, $(\langle A(u) - A(v); u - v\rangle > 0)$.

c) A is strongly monotone, iff it exists a constant C>0, such that:

$$\forall u, v\colon u \neq v\ \langle A(u) - A(v); u - v\rangle \geq C\|u - v\|_X^2. \qquad 30)$$

d) A is bounded, iff it exists a constant M>0: $|\langle A(u); v\rangle| \leq M\|v\|_X \;\forall v \in X$.

e) Let A be a symmetrical operator. Then A is eliptic, iff it exists a constant: $\alpha > 0$:

$$\langle A(u); u\rangle \geq \alpha\|u\|_X^2. \qquad 31)$$

f) A is a weak continous operator iff it holds $\langle A(u_n); v\rangle \to \langle A(u); v\rangle \;\forall v \in X$ whenever $u_n \to u$.

Theorem 4 (uniquness): A is strictly monotone. Then $\forall f \in X'$ it exists at most one solution of 29.b).

Theorem 5 (generalized Lax-Milgram): X is a Hilbert space. Def 7c) holds and hence A is a Lipschitz map. Then it exists a unique solution of 29.b) $\forall f \in X'$.

Special case (Lax-Milgram): A is a bilinear mapping and X is a Hilbert space. Def 7d) and 7e) hold. Then it exists a solution of 29.b) $\forall f \in X'$. This solution is unique.

Theorem 6 (the main theorem of monotone operators): let Def 7a), 7b) and 7f) hold. Then it exists a solution of 29.b) $\forall f \in X'$.

Now we focus on non-stationary problems. We introduce the Bochner spaces and remind two important theorems, that is, the Gelfand triplet and the Aubin-Lions lemma. We will not define here basic terms such as simple function or Bochner integrable functions, because they are natural generalization of the same terms from the standard Lebesgue spaces and measure theory, see [2,3].

As a first step, we define $L^p(\mathrm{I};\mathrm{X})$ spaces:

Def. 8: X is a Banach space, with $1 \leq p \leq \infty$, $I \subset R$. Then $L^p(\mathrm{I};\mathrm{X})$ is the set of all strong measurable functions $f\colon I \to X$, such that:

$$\int_I \|f\|_X^p\, dt < \infty, 1 \leq p < \infty \qquad 32.a)$$

$$esssup_I \|f\|_X < \infty; p = \infty. \qquad 32.b)$$

Note: it's usual to use (0,T) instead of I to denote a time interval, because we often solve the given problem for some finite T>0.

Theorem 7: $L^p(\mathrm{I};\mathrm{X})$ is a Banach space, whose norm is defined as:

$$\|f\|_{L^p(\mathrm{I};\mathrm{X})} = \left(\int_I \|f\|_X^p\, dt\right)^{1/p}, 1 \leq p < \infty \qquad 33.a)$$

$$\|f\|_{L^p(\mathrm{I};\mathrm{X})} = esssup_I \|f\|_X; p = \infty. \qquad 33.b)$$

X' is dual of X. Then each functional is represented in the following way:

$$\langle \varphi; f \rangle_{(L^p(\mathrm{I};\mathrm{X}))^*(L^p(\mathrm{I};\mathrm{X}))} := \int_I \langle \varphi; f \rangle_{X';X} dt\, ; f \in L^p(\mathrm{I};\mathrm{X}), \varphi \in L^{p'}(\mathrm{I};\mathrm{X}). \qquad 33.c)$$

These spaces are reflective for $1 < p < \infty$ and separable for $1 \leq p < \infty$. Additionaly, p' denotes the conjugated coefficient:

$$\frac{1}{p} + \frac{1}{p'} = 1. \qquad 34)$$

Now we must formulate the definitions of continous and compact embeddings:

Def. 9: X is a Banach space and Y is its subspace, $\| \quad \|_X, \| \quad \|_Y$are the corresponding norms. Then we say that Y is *continuously* embedded in X, iff it holds $\forall f \in X, Y$: $\|f\|_Y \leq \|f\|_X$ and we denote $Y \hookrightarrow X$. We say that Y is *compactly* embedded in X, iff Y is continuously embedded in X and every bounded sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that has its limit in X and we indicate $Y \hookrightarrow\hookrightarrow X$.

Now we can formulate *the Gelfand triplet lemma*:

Theorem 8: X is a reflective Banach space, X' its dual and H is a Hilbert space. $X \hookrightarrow H = H' \hookrightarrow X'$ densely. As $u \in L^p(\mathrm{I};\mathrm{X}), u' \in L^{p'}(\mathrm{I};\mathrm{X}')$, 1<p<∞, it follows that u is almost everywhere equal to a smooth function on $(0, T)$, which is defined on $[0, T]$ in H, and there the following holds:

$$\frac{d}{dt}\|u\|_H^2 = 2\langle u', u \rangle_{V',V}, \qquad 35)$$

in $D'(0, T)$ (a space of all functionals from the space of all smooth functions of R).

Finally, we formulate *the Aubin-Lions lemma*, which is very important in the theory of weak solutions of the nonstationary problems. We define a needed function space first:

Def 10: $W := W_{X_0,X_1}^{p,q} = \{v \in L^p(0, T; \mathrm{X}); v' \in L^q(0, \mathrm{T}; \mathrm{X})\}$, where$X_0$ and X_1 are Banach spaces.

X_0, X_1, X are three Banach spaces and it holds: $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$, with X_0, X_1 reflective. As $1 < p < \infty$ and $1 < q < \infty$, then for $1 < T < \infty$, it holds: $W \hookrightarrow\hookrightarrow L^p(0, T; \mathrm{X})$.

Theoretical part of the thesis

The governing equations:

We consider the following system of partial differential equations:

$$\frac{\partial \overrightarrow{u_n}}{\partial t} + \overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u_n} + \overrightarrow{f_{ns}} + \vec{g} \qquad 1.a)$$

$$\frac{\partial \overrightarrow{u_s}}{\partial t} + \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p - \overrightarrow{f_{ns}} + \vec{g}, \qquad 1.b)$$

where 1.a) describes a normal fluid and 1.b) describes a superfluid. We consider our two-fluid system isothermal, additionally we neglect the influence of S and T, and we assume that both equations are bounded by the condition:

$$\vec{f}_{ns} = -\frac{B}{2} \left| rot \vec{u}_s \right| \left(\vec{u}_n - \vec{u}_s \right). \qquad 1.c)$$

If both move fast enough, in such a way that they are decoupled, we can write:

$$div \vec{u}_n = 0 \qquad div \vec{u}_s = 0. \qquad 1.d)$$

And for a low fluid velocity, it holds:

$$div(\vec{u}_n + \vec{u}_n) = 0. \qquad 1.e)$$

We consider for simplicity that the velocity of the superfluid is small enough if and only if its vorticity is zero, that is:

$$rot \vec{u}_s = 0. \qquad 2)$$

The force formula 1.c) is simplified. For a more general expression, see [1], equation (3). The boundary conditions are discussed below and we finally assume that the only physical reason of the existence of the volume forces are either the quantized vortices or the gravitation, which give the different signs in equations 1.a) and 1.b).

It is possible to consider a slip boundary condition and no superfluid vorticity, because of the small thickness of the boundary layer. The vortices in the layer might possibly have the same effect as the classical viscosity, so we can introduce the boundary conditions for both components in the same way. The densities of both components are space independent.

With respect to the solved tasks we have to prove the existence and uniqness of the weak solution on the outer domains and find a numerical solution using the FEM and discontinous Galerkin methods for the system. We will use the special property of the system, following [1]. Because of the assumed special property of the volume force we will get no dependence of the apriori estimates on the force.

The goal is to prove the existence of the weak solution of the following problems, physically different. So finally we obtain two tasks to solve. The existence theory and the numerical simulations of the following equation systems:

Without quantized vortices:

$$\frac{\partial \overrightarrow{u_n}}{\partial t} + \overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u_n} + \vec{g} \qquad 3.a)$$

$$\frac{\partial \overrightarrow{u_s}}{\partial t} + \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p + \vec{g} \qquad 3.b)$$

$$rot\vec{u}_s = 0 \qquad 3.c)$$

$$div(\overrightarrow{u^n} + \overrightarrow{u^s}) = 0. \qquad 3.d)$$

And the system including quantized vortices:

$$\frac{\partial \overrightarrow{u_n}}{\partial t} + \overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u_n} + \overrightarrow{f_{ns}} + \vec{g} \qquad 4.a)$$

$$\frac{\partial \overrightarrow{u_s}}{\partial t} + \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p - \overrightarrow{f_{ns}} + \vec{g} \qquad 4.b)$$

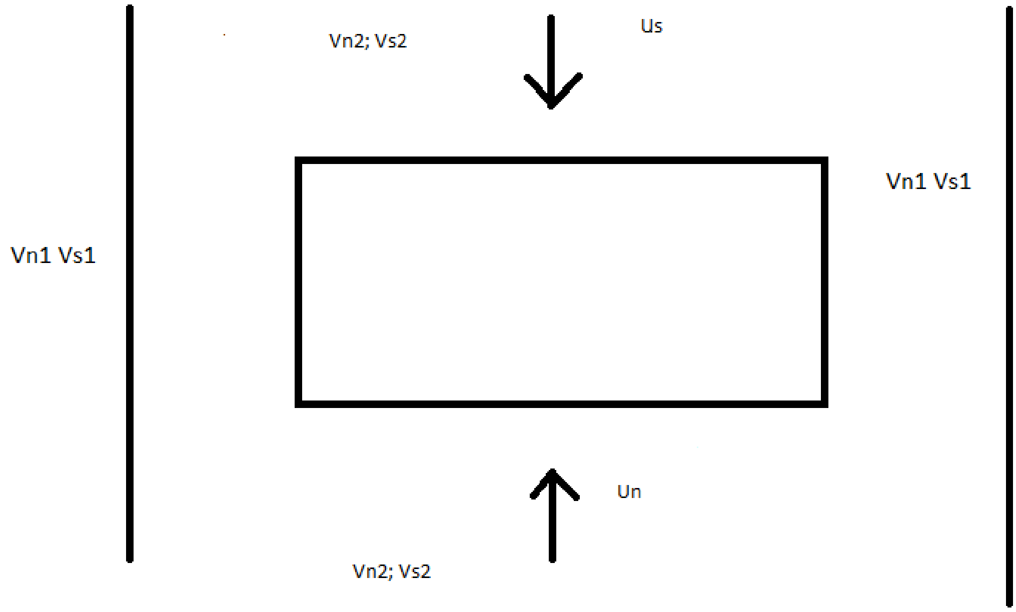
$$\vec{f}_{ns} = -\frac{B}{2} \left| rot\vec{u}_s \right| \left(\vec{u}_n - \vec{u}_s \right) \qquad 4.c)$$

$$div\vec{u}_n = 0 \qquad div\vec{u}_s = 0. \qquad 4.d)$$

The geometry of the task and the boundary conditions

The geometry of the problem is as follows: an obstacle, of rectangular cross-section, is placed in the middle part of our experimental volume (in the inner part of cryostat). Around the obstacle the two-component fluid moves: the normal component flows from below, that is, upward, while the superfluid flows downward. We assume the no-slip boundary condition on the inner sides of the cryostat. These are caused for the normal component by the non-zero value of the viscosity and for the superfluid by the quantized vortices in the boundary layer. This can be justified by the non-zero viscosity of the normal component and by the existence of quantized vortices for the superfluid, due to the boundary roughness. Moreover, the thickness of the boundary layer is considered small, compared to the experimental volume size. We also assume that both components have a parabolical velocity profile.

Figure 1: geometry of the problem.



It then follows that the velocity of the obstacle is equal to the velocity of the fluid around the surfaces parallel with the flow direction. The problem is solved in the reference system of the obstacle.

We consider a parabolical profile on the top and the bottom of the domain, where the problem is solved. We consider the upper and the lower parts of the experimental volume to be far enough from the obstacle, so that the latter do not affect it. U_0 indicates the obstacle velocity and V_n and V_s are fluid velocities, depending on temperature, given by the equations:

$$V_n = \frac{1}{\pi R^2 S T \rho} \frac{dQ}{dt} \qquad 1.a)$$

$$V_s = \frac{1}{\pi R^2 S T \rho_s} \frac{dQ}{dt}. \qquad 1.b)$$

Q is the thermal energy of the heating, R is the dimension of the cylindrical cryostat with square cross section, S is the enthropy of the normal liquid. The density ρ is the sum of the densities of both components and depends weakly on temperature. The densities of the two components have a stronger temperature dependence and are tabulated. r, x and y are the coordinates, in the cartesian system, describing the obstacle. r indicates the obstacle position inside the cryostat. a and b are the dimensions of the obstacle, perpedicular and parallel to the flow. We consider no condition on pressure on the boundary.

The boundary conditions are marked as Vn1; Vs1 – Vn2; Vs2 (in the existence theory they will be marked with the more general Vn and Vs) and are:

$$V_{n1} = U_0 \quad 2.a)$$

$$V_{s1} = U_0 \quad 2.b)$$

$$V_{n2} = \frac{U_0 - V_n}{R^2} r^2 + V_n \quad 2.c)$$

$$V_{s2} = -\frac{U_0 + V_n}{R^2} r^2 + V_s. \quad 2.d)$$

We can include the oscillations of the obstacle. This can be obtained easily if we put $U_0 cos\omega t$ instead of only U_0. So, we can write for the nonstationary problem:

$$V_{n1} = U_0 cos\omega t \quad 3.a)$$

$$V_{s1} = U_0 cos\omega t \quad 3.b)$$

$$V_{n2} = \frac{U_0 - V_n}{R^2} r^2 + V_n \quad 3.c)$$

$$V_{s2} = -\frac{U_0 + V_n}{R^2} r^2 + V_s. \quad 3.d)$$

These boundary conditions hold only for the flow without gravitation (only caused by heating in the lower part of the cryostat). The influence of the gravitation (or of other volume forces) can be obtained using the compatibility condition (derived below):

$$0 = \nu \int_{\partial\Omega} \vec{e} \cdot \nabla\overrightarrow{v_n} ds + \int_{\Omega} \vec{g}\vec{e} \, \mathrm{dx}, \quad 4)$$

which is the way for a simple derivation of the velocity gradient boundary conditions. We consider a zero velocity gradient on the left and the right wall of the cryostat and nonzero on the upper and lower parts. We assume a nonzero gradient component only in the direction of the z axis so:

$$\nabla\overrightarrow{v_n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial v_n^z}{\partial z} \end{pmatrix}. \quad 5)$$

So we obtain from 5) this equation:

$$0 = \nu \frac{\partial v_n^z}{\partial z} 2\pi R^2 - g\pi R^2 h \quad 6.a)$$

$$\frac{gh}{2\nu} = \frac{\partial v_n^z}{\partial z} \quad 6.b)$$

The nonstationary PDE system, describing the Landau model

The weak solution theory of the nonstationary PDE system without the vortices:

1.1 Definition: The weak solution of the Landau model:

Let $\Omega \in R^M$, with $M = 2,3$; $\overrightarrow{u_{n,0}} - \overrightarrow{v_{n,0}}$ and $\overrightarrow{u_{s,0}} - \overrightarrow{v_{s,0}} \in L_0^2(\Omega)$; $\overrightarrow{v_n}, \overrightarrow{v_s}$ are obtained from continous functions, defined on the two boudaries, which are the inner and outer Ω domain boundaries, which do not penetrate or touch each other. Then the functions satisfying

$$\overrightarrow{u_n} - \overrightarrow{v_n}, \qquad \overrightarrow{u_s} - \overrightarrow{v_s} \in L^2\left(0, T; \left(W_0^{1,2}(\Omega)\right)^M\right) \cap L^\infty\left(0, T; \left(L^2\ (\Omega)\right)^M\right);$$

$$\frac{\partial \overrightarrow{u_n}}{\partial t}, \frac{\partial \overrightarrow{u_s}}{\partial t} \in L^1\left(0, T; \left(\left(W_0^{1,2}(\Omega)\right)^M\right)^*\right),$$

are called *a weak solution* of the system 1.a) –1.c) without quantized vortices, formulated as:

$$\langle \frac{\partial \overrightarrow{u_n}}{\partial t}, \vec{v} \rangle_{\left(W_0^{1,2}(\Omega)\right)^*, \left(W_0^{1,2}(\Omega)\right)} + \int_\Omega (\overrightarrow{u_n} \cdot \nabla)\overrightarrow{u_n}\vec{v}dx + \nu \int_\Omega \nabla\overrightarrow{u_n} : \nabla\vec{v}dx + \int_\Omega p div \overrightarrow{u_n} dx = \nu \int_{\partial\Omega} \vec{v} \cdot \nabla\overrightarrow{v_n} ds + \int_\Omega \vec{g} \cdot \vec{v}dx \qquad 1.a)$$

$$\langle \frac{\partial \overrightarrow{u_s}}{\partial t}, \vec{v} \rangle_{\left(W_0^{1,2}(\Omega)\right)^*, \left(W_0^{1,2}(\Omega)\right)} + \int_\Omega (\overrightarrow{u_s} \cdot \nabla)\overrightarrow{u_s}\vec{v}dx + \int_\Omega p div \vec{v}dx = \int_\Omega \vec{g} \cdot \vec{v}dx \qquad 1.b)$$

$$\int_\Omega (\overrightarrow{u_n} + \overrightarrow{u_s}) \cdot \nabla\varphi dx = 0 \qquad 1.c)$$

$$\int_\Omega \overrightarrow{u_s} rot \vec{v} dx = 0 \qquad 1.d)$$

For all $\vec{v} \in \left(L^2(\Omega)\right)^M, \varphi \in L^2(\Omega)$.

The preliminaries of the existence theory

Here we use the Green theorem, applied on the Laplace operator, as it is customary in the theory of the N-S equations. In the first step we derive the energetic inequality. We sum both PDEs and, as a test function, we use a sum of the solutions and obtain:

$$\frac{\partial(\overrightarrow{u_n}+\overrightarrow{u_s})}{\partial t}\cdot(\overrightarrow{u_n}+\overrightarrow{u_s})-\frac{1}{2}\int_\Omega(div\overrightarrow{u_n}|\overrightarrow{u_n}|^2+div\overrightarrow{u_s}|\overrightarrow{u_s}|^2)dx-\frac{1}{2}\int_\Omega(|\overrightarrow{u_s}|^2div\overrightarrow{u_n}+|\overrightarrow{u_n}|^2div\overrightarrow{u_s})\,dx=$$

$$\underbrace{\int_\Omega pdiv(\overrightarrow{u_n}+\overrightarrow{u_s})dx}_{=0}-\nu\|\nabla\overrightarrow{u_n}\|_2^2+\int_{\partial\Omega}\vec{n}\cdot(\overrightarrow{u_n}+\overrightarrow{u_s})(|\overrightarrow{u_s}|^2+|\overrightarrow{u_n}|^2)+\nu\nabla\overrightarrow{u_n}(\overrightarrow{u_n}+\overrightarrow{u_s})\vec{n}ds. \qquad 2)$$

We see that the member with pressure vanishes and we use the Gelfand triplet for the time-derivation member. The volume-integrated members vanish, using the divergence equation. The nonlinear members are consequently equal to zero and we obtain:

$$\frac{d}{dt}\|\overrightarrow{u_n}+\overrightarrow{u_s}\|_2^2+2\nu\|\nabla\overrightarrow{u_n}\|_2^2=\int_{\partial\Omega}(|\overrightarrow{u_n}|^2+|\overrightarrow{u_s}|^2)\cdot(\overrightarrow{u_s}+\overrightarrow{u_n})\vec{n}\,ds, \qquad 3)$$

which we integrate from 0 to t and get

$$\|\overrightarrow{u_n}+\overrightarrow{u_s}\|_2^2(t)+2\nu\int_0^t\|\nabla\overrightarrow{u_n}\|_2^2\,dt=\|\overrightarrow{u_n}+\overrightarrow{u_s}\|_2^2(0)+\int_{\partial\Omega}(|\overrightarrow{u_n}|^2+|\overrightarrow{u_s}|^2)\cdot(\overrightarrow{u_s}+\overrightarrow{u_n})\vec{n}\,ds. \qquad 4)$$

This equality is not as useful as we would need, because it gives no estimate for the norm of both functions, but only for the normal component. We will derive one equality of another type, which will be more helpful. We can use the Bernoulli equation, deriving the apriori estimates for the superfluid component:

$$\frac{\partial\overrightarrow{u_s}}{\partial t}+(\overrightarrow{u_s}\cdot\nabla)\overrightarrow{u_s}+\nabla\mathrm{p}=0 \qquad 5.a)$$

and integrate it over a stream line. This is well defined and gives:

$$\int_\lambda\frac{\partial\overrightarrow{u_s}}{\partial t}dl+\int_\lambda(\overrightarrow{u_s}\cdot\nabla)\overrightarrow{u_s}\,dl+\mathrm{p}(t)=p(0). \qquad 5.b)$$

The substitution theorem, with $d\vec{l}=\vec{u}_s$dt, gives us for the time-derived member the integration over time from 0 to t and we obtain finally the Bernoulli equation for the superfluid:

$$\int_\lambda\frac{\partial\overrightarrow{u_s}}{\partial t}\cdot\overrightarrow{u_s}dt+\int_\lambda\nabla|\overrightarrow{u_s}|^2\cdot dl+\mathrm{p}(t)=p(0). \qquad 6)$$

Now we can evaluate the second integral and integrate over Ω:

$$\int_\lambda\left\langle\frac{\partial\overrightarrow{u_s}}{\partial t},\overrightarrow{u_s}\right\rangle_{\left(W_0^{1,2}(\Omega)\right)^*\left(W_0^{1,2}(\Omega)\right)}dt+\int_\Omega|\overrightarrow{u_s}|^2(t)dx+\int_\Omega p(t)dx=\int_\Omega p(0)dx+\int_\Omega|\overrightarrow{u_s}|^2(0)dx. \qquad 7)$$

Using the Gelfand triplet we obtain:

$$\frac{3}{2}\int_\Omega|\overrightarrow{u_s}|^2(t)dx+\int_\Omega p(t)dx=\int_\Omega p(0)dx+\frac{3}{2}\int_\Omega|\overrightarrow{u_s}|^2(0)dx. \qquad 8)$$

But there is no estimate for the norm of the gradient, so we have no estimate for the norm in the Sobolev space. We must find it throught the limit of the Galerkin approximation, as it will be shown.

The existence theory without superfluid vorticity

At first, we use equation 2.c). The zero rotation is equivalent to the theorem:

$$\exists\Phi: \quad \overrightarrow{u_s} = \nabla\Phi, \tag{9}$$

and equation 2.d) gives:

$$\exists\vec{A}: \quad \overrightarrow{u_n} + \overrightarrow{u_s} = rot\vec{A} \qquad 10.a)$$

$$\overrightarrow{u_n} = rot\vec{A} - \nabla\Phi. \qquad 10.b)$$

So this shows the existence and uniqness of the solution of the problem, because it is now given by an explicit formula.

Let us solve now a problem for the pressure. It is now simply given by the formula 8):

$$\mathrm{p}(t) = p(0) - \int_\lambda \frac{\partial\overrightarrow{u_s}}{\partial t}\cdot\overrightarrow{u_s}dt - \int_\lambda \nabla|\overrightarrow{u_s}|^2\cdot dl, \qquad 11.a)$$

where the velocity terms give the classical Bernoulli equation as a non-integral formula:

$$\mathrm{p}(t) = p(0) - \int_\lambda \frac{\partial\overrightarrow{u_s}}{\partial t}\cdot\overrightarrow{u_s}dt - \frac{1}{2}|\overrightarrow{u_s}|^2(t) + \frac{1}{2}|\overrightarrow{u_s}|^2(0). \qquad 11.b)$$

And we rewrite the time term to get a non-integral form:

$$\mathrm{p}(t) = p(0) - \frac{3}{2}|\overrightarrow{u_s}|^2(t) + \frac{3}{2}|\overrightarrow{u_s}|^2(0) \qquad 11.c)$$

The formulae 11.c, 9 and 10.b will be used for the numerical simulations too.

The weak solution theory of the system, including the superfluid vorticity:

1.2 Definition: The weak solution of the Landau model:

Let $\Omega \in R^M$, with $M = 2,3; \overrightarrow{u_{n,0}} - \overrightarrow{v_{n,0}}$ and $\overrightarrow{u_{s,0}} - \overrightarrow{v_{s,0}} \in L_0^2(\Omega); \overrightarrow{v_n}, \overrightarrow{v_s}$ are obtained from the continous functions, defined on the two boundaries, which are the inner and outer Ω domain boundaries, which do not penetrate or touch each other. Let B>0. Then the functions satisfying:

$\overrightarrow{u_n} - \overrightarrow{v_n}$,

$\overrightarrow{u_s} - \overrightarrow{v_s} \in L^2\left(0,T;\left(W_0^{1,2}(\Omega)\right)^M\right) \cap L^\infty\left(0,T;\left(L^2\ (\Omega)\right)^M\right); \frac{\partial \overrightarrow{u_n}}{\partial t}, \frac{\partial \overrightarrow{u_s}}{\partial t} \in L^2\left(0,T;\left(\left(W_0^{1,2}(\Omega)\right)^M\right)^*\right)$,

are called *a weak solution* of the system 1.a) –1.c) with quantized vortices, formulated as:

$$\langle \frac{\partial \overrightarrow{u_n}}{\partial t}, \vec{v} \rangle_{\left(W_0^{1,2}(\Omega)\right)^*, \left(W_0^{1,2}(\Omega)\right)} + \int_\Omega (\overrightarrow{u_n} \cdot \nabla)\overrightarrow{u_n}\vec{v}dx + \nu \int_\Omega \nabla\overrightarrow{u_n} : \nabla\vec{v}dx + \int_\Omega p div\vec{v}dx = \nu \int_{\partial\Omega} \vec{v} \cdot \nabla\overrightarrow{v_n}ds + \int_\Omega \overrightarrow{f_{ns}}\vec{v}dx + \int_\Omega \vec{g}\vec{v}\,\mathrm{dx} \qquad 12.a)$$

$$\langle \frac{\partial \overrightarrow{u_s}}{\partial t}, \vec{v} \rangle_{\left(W_0^{1,2}(\Omega)\right)^*, \left(W_0^{1,2}(\Omega)\right)} + \int_\Omega (\overrightarrow{u_s} \cdot \nabla)\overrightarrow{u_s}\vec{v}dx + \int_\Omega p div\vec{v}dx = - \int_\Omega \overrightarrow{f_{ns}}\vec{v}dx + \int_\Omega \vec{g}\vec{v}\,\mathrm{dx} \qquad 12.b)$$

$$\int_\Omega \overrightarrow{u_n} \cdot \nabla\varphi dx = 0; \int_\Omega \overrightarrow{u_s} \cdot \nabla\varphi dx = 0 \qquad 12.c)$$

$$\vec{f}_{ns} = -\frac{B}{2}|rot\vec{u}_s|(\vec{u}_n - \vec{u}_s) \qquad 12.d)$$

For all $\vec{v} \in \left(L^2(\Omega)\right)^N, \varphi \in L^2(\Omega)$.

Setting $\vec{v} = \vec{e}$ we obtain the compatibility condition:

$$0 = \nu \int_{\partial\Omega} \vec{e} \cdot \nabla\overrightarrow{v_n}ds + 2\int_\Omega \vec{g}\vec{e}\,\mathrm{dx} \qquad 12.e)$$

in such a way that the right hand side vanishes in equations 12.a) and 12.b). Then we sum both and both vorticity terms will be subsequently subtracted from each other.

The construction of the Galerkin approximation

Before we start to derive the Galerkin approximation, we summarize what we know about the solution of the Stokes problem in the function space $W^{1,2}_{0,div}(\Omega)$.

1.3 Definition: The operator Λ:

$$\Lambda: L^2_{0,div}(\Omega) \to W^{1,2}_{0,div}(\Omega) \subset \left(W^{1,2}_0(\Omega)\right)^M. \qquad 13.a)$$

It follows that

$$\Lambda\vec{f} = \vec{u}, \qquad 13.b)$$

where **u** is a weak solution of the Stokes problem:

$$\int_\Omega \nabla\vec{u}:\nabla\vec{v}dx = \langle\vec{f},\vec{v}\rangle,\ \ for\ \vec{f} \in L^2_{0,div}(\Omega), for\ all\ \vec{v} \in W^{1,2}_{0,div}(\Omega), \vec{u} \in W^{1,2}_{0,div}(\Omega). \qquad 13.c)$$

1.4 Theorem: Ω is a bounded domain. Then the eigenfunctions of the operator Λ build an orthonormal basis of the space $W^{1,2}_{0,div}(\Omega)$.

These facts enable us to define the Galerkin approximation very effectively. We take $\{\vec{w}_i\}_{i=1}^{\infty}$ as an orthonormal basis from 1.4.

1.5 Definition: We define the Galerkin approximation with an artificial viscosity:

The functions $\overrightarrow{u_n^N} := \sum_{i=1}^N c_n^i\vec{w}_i$; $\overrightarrow{u_s^N} := \sum_{i=1}^N c_s^i\vec{w}_i$ are called the Galerkin approximations of the solution of the problem 1.a – 1.c, if they fulfill:

$$\int_\Omega \frac{\partial\overrightarrow{u_n^N}}{\partial t}\cdot\overrightarrow{w_i}\,dx + \int_\Omega \overrightarrow{u_n^N}\cdot\nabla\overrightarrow{u_n^N}\cdot\overrightarrow{w_i}dx + \nu\int_\Omega \nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{w_i}dx + \int_\Omega p\,div\overrightarrow{w_i}dx = \nu\int_{\partial\Omega}\overrightarrow{u_n^N}\cdot\nabla\overrightarrow{w_i}ds - \frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)\overrightarrow{w_i}dx + \int_\Omega \vec{g}\cdot\overrightarrow{w_i}\,dx \qquad 14.a)$$

$$\int_\Omega \frac{\partial\overrightarrow{u_s^N}}{\partial t}\cdot\overrightarrow{w_i}\,dx + \int_\Omega \overrightarrow{u_s^N}\cdot\nabla\overrightarrow{u_s^N}\cdot\overrightarrow{w_i}dx =$$
$$-\nu(N)\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{w_i}\,dx + \int_\Omega p\,div\overrightarrow{w_i}dx + +\frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)\overrightarrow{w_i}dx + \int_\Omega \vec{g}\cdot\overrightarrow{w_i}\,dx \qquad 14.b)$$

$$div\left(\overrightarrow{u_s^N}\right) = 0 \qquad 14.c)$$

$$div\left(\overrightarrow{u_n^N}\right) = 0 \qquad 14.d)$$

for all $\vec{w}_i \in \{\vec{w}_i\}_{i=1}^{\infty}$, which represents the ortogonal system from Definition 1.3. The initial conditions are expressed as:

$$\overrightarrow{u_n^N}(0) = \sum_{i=1}^N \overrightarrow{c_i^n}(0)\overrightarrow{w_i}\ ,\overrightarrow{u_s^N}(0) = \sum_{i=1}^N \overrightarrow{c_i^s}(0)\overrightarrow{w_i} \qquad 15)$$

Note that we consider that the Euler PDE is a special limit of the N-S Equations, describing the superfluid on finite-dimensional subspaces of $W^{1,2}_{0,div}(\Omega)$. The use of the eigenfunctions ensures that the pressure members give zero values. The boundary conditions will be expressed using the integration of the Galerkin approximation over the domain boundary.

The existence of the Galerkin approximations

First we must prove the existence of the Galerkin approximations. Let's take the definition of the Galerkin approximation for both equations and we obtain from 14.c) and 14.d) the equality of the pressure terms to zero. Because of the properties of the chosen basis of the function space, we obtain from this system of equations:

$$\int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t} \cdot \overrightarrow{w_i}\, dx + \int_\Omega \overrightarrow{u_n^N} \cdot \nabla \overrightarrow{u_n^N} \cdot \overrightarrow{w_i} dx + \nu \int_\Omega \nabla \overrightarrow{u_n^N} : \nabla \overrightarrow{w_i} dx = \nu \int_{\partial\Omega} \overrightarrow{u_n^N} \cdot \nabla \overrightarrow{w_i} ds - \frac{B}{2} \int_\Omega \left| rot \overrightarrow{u_s^N} \right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N} \right) \overrightarrow{w_i} dx + \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx \qquad 16.a)$$

$$\int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t} \cdot \overrightarrow{w_i}\, dx + \int_\Omega \overrightarrow{u_s^N} \cdot \nabla \overrightarrow{u_s^N} \cdot \overrightarrow{w_i} dx = -\nu(N) \int_\Omega \nabla \overrightarrow{u_s^N} : \nabla \overrightarrow{w_i}\, dx + \frac{B}{2} \int_\Omega \left| rot \overrightarrow{u_s^N} \right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N} \right) \overrightarrow{w_i} dx + \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx. \qquad 16.b)$$

For the time-derivation terms we obtain:

$$\int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t} \cdot \overrightarrow{w_i}\, dx = \int_\Omega \frac{\partial \sum_{i=1}^N c_n^i \overrightarrow{w_i}}{\partial t} \cdot \overrightarrow{w_i}\, dx = \int_\Omega \dot{c_n^i} |\overrightarrow{w_i}|^2 dx = \dot{c_n^i};\ \int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t} \cdot \overrightarrow{w_i}\, dx = \dot{c_s^i}, \qquad 17.a)$$

which we derived from the orthonormality of the basis. For the Laplace term we obtain:

$$\int_\Omega \nabla \overrightarrow{u_n^N} : \nabla \overrightarrow{w_i}\, dx = c_n^i \lambda_i;\ \int_\Omega \nabla \overrightarrow{u_s^N} : \nabla \overrightarrow{w_i}\, dx = c_s^i \lambda_i, \qquad 17.b)$$

where λ_i is an eigenvalue of the Laplace operator. The convection terms can be treated as follows:

$$\int_\Omega \overrightarrow{u_n^N} \cdot \nabla \overrightarrow{u_n^N} \cdot \overrightarrow{w_i} dx = c_n^k(t) c_n^j(t) \int \left(\overrightarrow{w_k} \cdot \nabla \overrightarrow{w_j} \right) \cdot \overrightarrow{w_i} dx. \qquad 17.c)$$

The vorticity term can be written as:

$$\frac{B}{2} \int_\Omega \left| rot \overrightarrow{u_s^N} \right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N} \right) \overrightarrow{w_i} dx = \frac{B}{2} (c_i^n - c_i^s) \int_\Omega \left| rot \overrightarrow{u_s^N} \right| |\overrightarrow{w_i}|^2 dx \leq$$
$$\frac{B}{2} (c_i^n - c_i^s) \int_\Omega \left| rot \overrightarrow{u_s^N} \right| dx \int_\Omega |\overrightarrow{w_i}|^2\, dx = \frac{B}{2} (c_i^n - c_i^s) c_i^s \int_\Omega \left| rot \overrightarrow{w_i} \right| dx \leq \frac{B}{2} (c_i^n - c_i^s) c_i^s \left\| \nabla \overrightarrow{w_i} \right\|_2 \qquad 17.d)$$

We then obtain these equations:

$$\dot{c_n^i} = -c_n^k(t) c_n^j(t) \int \left(\overrightarrow{w_k} \cdot \nabla \overrightarrow{w_j} \right) \cdot \overrightarrow{w_i} dx - \nu c_n^i \lambda_i + \nu \int_{\partial\Omega} \overrightarrow{v_n^N} \cdot \nabla \overrightarrow{w_i} ds + \frac{B}{2} (c_i^n - c_i^s) c_i^s \int_\Omega |rot \overrightarrow{w_i}| |\overrightarrow{w_i}|^2 dx + \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx \qquad 18.a)$$

$$\dot{c_s^i} = -c_s^k(t) c_s^j(t) \int \left(\overrightarrow{w_k} \cdot \nabla \overrightarrow{w_j} \right) \cdot \overrightarrow{w_i} dx - \nu c_s^i \lambda_i + \nu(N) \int_{\partial\Omega} \overrightarrow{v_n^N} \cdot \nabla \overrightarrow{w_i} ds + \frac{B}{2} (c_i^n - c_i^s) c_i^s \int_\Omega |rot \overrightarrow{w_i}| |\overrightarrow{w_i}|^2 dx + \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx, \qquad 18.b)$$

with these estimates:

$$\dot{c_n^i} \leq -c_n^k(t)c_n^j(t)\int \left(\overrightarrow{w_k}\cdot\nabla\overrightarrow{w_j}\right)\cdot\overrightarrow{w_i}dx - \nu c_n^i\lambda_i + \nu\int_{\partial\Omega}\overrightarrow{\mathrm{v}_n^N}\cdot\nabla\overrightarrow{w_i}ds + \frac{B}{2}(c_i^n - c_i^s)\mathrm{c}_i^s\left\|\overrightarrow{\nabla \mathrm{w}_i}\right\|_2 + \|\vec{g}\|_2^2 \qquad 18.c)$$

$$\dot{c_s^i} \leq -c_s^k(t)c_s^j(t)\int \left(\overrightarrow{w_k}\cdot\nabla\overrightarrow{w_j}\right)\cdot\overrightarrow{w_i}dx - \nu c_s^i\lambda_i + \nu(N)\int_{\partial\Omega}\overrightarrow{\mathrm{v}_n^N}\cdot\nabla\overrightarrow{w_i}ds + \frac{B}{2}(c_i^n - c_i^s)\mathrm{c}_i^s\left\|\overrightarrow{\nabla \mathrm{w}_i}\right\|_2 + \|\vec{g}\|_2^2. \qquad 18.d)$$

It is possible to apply the Carathéodory on this system, because the right hand side is continous in coefficients and is measurable in eact; additionally, it is constant and the boundary conditions are continous in time. The right hand side is Lebesgue measurable too, so the assumptions on the ODE systems are satisfied. It follows that the Galerkin approximations: $\overrightarrow{u_n^N} \coloneqq \sum_{i=1}^N c_n^i\overrightarrow{w}_i$; $\overrightarrow{u_s^N} \coloneqq \sum_{i=1}^N c_s^i\overrightarrow{w}_i$, exist in the sense of the definition 14.a) – 14.d).

The apriori estimates for the Galerkin approximations

We can derive these apriori estimates for the problem 14.a) – 14.d). Now we can multiply the equations by the corresponsing coefficients for each function from the orthogonal system; for example, equation 14.a) can be written as:

$$\int_\Omega \frac{\partial \overrightarrow{\mathrm{u}_n^N}}{\partial t}\cdot c_i^n\overrightarrow{w_\iota}\,dx+\int_\Omega \overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}\cdot c_i^n\overrightarrow{w_\iota}dx+\nu\int_\Omega \nabla\overrightarrow{\mathrm{u}_n^N}:\nabla c_i^n\overrightarrow{w_\iota}dx+\int_\Omega p\,div\,c_i^n\overrightarrow{w_\iota}dx=\nu\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n^N}\cdot\nabla c_i^n\overrightarrow{w_\iota}ds-\frac{B}{2}\int_\Omega\left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)c_i^n\overrightarrow{w_\iota}dx+\int_\Omega \vec{g}\cdot c_i^n\overrightarrow{w_\iota}\,dx. \qquad 19)$$

We then sum these equations for i=1…N and we obtain the equation for the Galerkin solution with artificial viscosity tested by their solution:

$$\int_\Omega \frac{\partial \overrightarrow{\mathrm{u}_n^N}}{\partial t}\cdot\overrightarrow{\mathrm{u}_n^N}\,dx+\int_\Omega \overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}\cdot\overrightarrow{\mathrm{u}_n^N}dx+\nu\int_\Omega \nabla\overrightarrow{\mathrm{u}_n^N}:\nabla\overrightarrow{\mathrm{u}_n^N}dx+\int_\Omega p\,div\overrightarrow{\mathrm{u}_n^N}dx=\nu\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}ds-\frac{B}{2}\int_\Omega\left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)\overrightarrow{\mathrm{u}_n^N}dx+\int_\Omega \vec{g}\cdot\overrightarrow{\mathrm{u}_n^N}\,dx. \qquad 20.a)$$

Similarly, for the other equation we get:

$$\int_\Omega \frac{\partial \overrightarrow{\mathrm{u}_s^N}}{\partial t}\cdot\overrightarrow{\mathrm{u}_s^N}\,dx+\int_\Omega \overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{\mathrm{u}_s^N}\cdot\overrightarrow{\mathrm{u}_s^N}dx+\nu(N)\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{\mathrm{u}_s^N}\,dx=\int_\Omega p\,div\overrightarrow{\mathrm{u}_s^N}dx+\frac{B}{2}\int_\Omega\left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)\overrightarrow{\mathrm{u}_s^N}dx+\int_\Omega \vec{g}\cdot\overrightarrow{\mathrm{u}_s^N}\,dx \qquad 20.b)$$

We now sum both equations to neglect the quantized vorticity term. Before it, we need to make some calculations to simplify the work. Let us take the convective terms and calculate:

$$\int_\Omega \overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}\cdot\overrightarrow{\mathrm{u}_n^N}dx=\int_\Omega \overrightarrow{\mathrm{u}_n^N}\nabla|\mathrm{u}_n^N|^2dx=-\frac{1}{2}\int_\Omega div\overrightarrow{\mathrm{u}_n^N}|\mathrm{u}_n^N|^2\,dx+\frac{1}{2}\int_{\partial\Omega}|\mathrm{u}_n^N|^2\,\overrightarrow{\mathrm{u}_n^N}\cdot\vec{n}ds=\frac{1}{2}\int_{\partial\Omega}\left|\mathrm{v}_n\right|^2\overrightarrow{\mathrm{v}_n}\cdot\vec{n}ds. \qquad 21)$$

The same we obtain in the case of the equation for the superfluid:

$$\int_\Omega \overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{\mathrm{u}_s^N}\cdot\overrightarrow{\mathrm{u}_s^N}dx=\frac{1}{2}\int_{\partial\Omega}\left|\mathrm{v}_s\right|^2\overrightarrow{\mathrm{v}_s}\cdot\vec{n}ds \qquad 22)$$

The sign of this term depends on the boundary conditions. In this case we obtain zero. We derive the consistency condition for the volume forces and for the boundary conditions below. The pressure terms vanish because of the zero-divergence conditions:

$$\int_\Omega p\,div\overrightarrow{\mathrm{u}_n^N}dx=0;\ \int_\Omega p\,div\overrightarrow{\mathrm{u}_s^N}dx=0. \qquad 23)$$

Together with the calculations for the convection and pressure terms we obtain, after summing the equations 17.a) and 17.b):

$$\int_\Omega \frac{\partial \overrightarrow{\mathrm{u}_n^N}}{\partial t}\cdot\overrightarrow{\mathrm{u}_n^N}\,dx+\int_\Omega \frac{\partial \overrightarrow{\mathrm{u}_s^N}}{\partial t}\cdot\overrightarrow{\mathrm{u}_s^N}\,dx+\nu\int_\Omega \nabla\overrightarrow{\mathrm{u}_n^N}:\nabla\overrightarrow{\mathrm{u}_n^N}dx+\nu(N)\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{\mathrm{u}_s^N}\,dx+\frac{1}{2}\int_{\partial\Omega}\left|\mathrm{v}_s\right|^2\overrightarrow{\mathrm{v}_s}\vec{n}ds+\frac{1}{2}\int_{\partial\Omega}\left|\mathrm{v}_n\right|^2\overrightarrow{\mathrm{v}_n}\vec{n}ds=\nu\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}ds+\nu(N)\int_{\partial\Omega}\overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{\mathrm{u}_s^N}ds-\frac{B}{2}\int_\Omega\left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)^2dx+\int_\Omega \vec{g}\cdot\left(\overrightarrow{\mathrm{u}_n^N}+\overrightarrow{\mathrm{u}_s^N}\right)dx. \qquad 24)$$

The quantized vorticity term is negative and we can put it to the left hand side. We then neglect the boundary terms derived from the Laplace member because we assume zero velocity gradient on the boundaries.

$$\int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t}\cdot \overrightarrow{u_n^N}\,dx + \int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t}\cdot \overrightarrow{u_s^N}\,dx + \nu \int_\Omega \nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{u_n^N} dx + \nu(N)\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{u_s^N}\,dx$$

$$+ \; + \frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)^2 dx =$$

$$\int_\Omega \vec{g}\cdot\left(\overrightarrow{u_n^N} + \overrightarrow{u_s^N}\right)dx - \frac{1}{2}\int_{\partial\Omega} |v_s|^2\, \overrightarrow{v_s}\,\vec{n}ds - \frac{1}{2}\int_{\partial\Omega} |v_n|^2\, \overrightarrow{v_n}\,\vec{n}ds. \qquad 25)$$

The right hand side is bounded and we obtain:

$$\int_\Omega \vec{g}\cdot\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)dx - \frac{1}{2}\int_{\partial\Omega} |v_s|^2\, \overrightarrow{v_s}\,\vec{n}ds - \frac{1}{2}\int_{\partial\Omega} |v_n|^2\, \overrightarrow{v_n}\,\vec{n}ds \le \frac{C(\nu(N))}{2}\|\vec{g}\|^2_{(L^2(\Omega))^M} + \frac{\nu(N)}{2}\left\|\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M} + C(\overrightarrow{v_n};\overrightarrow{v_s}) \le \frac{2C(\nu(N))}{3}\|\vec{g}\|^2_{(L^2(\Omega))^M} + C(\overrightarrow{v_n};\overrightarrow{v_s}) + \frac{\nu(N)}{2}\left\|\nabla\overrightarrow{u_n^N} + \nabla\overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M}. \qquad 26)$$

The estimate for the gradient norm term is:

$$\frac{\nu(N)}{3}\left\|\nabla\overrightarrow{u_n^N} - \nabla\overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M} \le \frac{\nu(N)}{3}\left(\left\|\nabla\overrightarrow{u_n^N}\right\|^2_{(L^2(\Omega))^M} + 2\left\|\nabla\overrightarrow{u_n^N}\right\|_{(L^2(\Omega))^M}\left\|\nabla\overrightarrow{u_s^N}\right\|_{(L^2(\Omega))^M} + \left\|\nabla\overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M}\right) \le \frac{2\nu(N)}{3}\left(\left\|\nabla\overrightarrow{u_n^N}\right\|^2_{(L^2(\Omega))^M} + \left\|\nabla\overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M}\right). \qquad 27)$$

Consequently, the whole estimate of the right hand side can be written as:

$$\int_\Omega \vec{g}\cdot\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)dx - \frac{1}{2}\int_{\partial\Omega} |v_s|^2\, \overrightarrow{v_s}\,\vec{n}ds - \frac{1}{2}\int_{\partial\Omega} |v_n|^2\, \overrightarrow{v_n}\,\vec{n}ds \le \frac{2\nu(N)}{3}\left(\left\|\nabla\overrightarrow{u_n^N}\right\|^2_{(L^2(\Omega))^M} + \left\|\nabla\overrightarrow{u_s^N}\right\|^2_{(L^2(\Omega))^M}\right) + \frac{2C(\nu(N))}{3}\|\vec{g}\|^2_{(L^2(\Omega))^M} + C(\overrightarrow{v_n};\overrightarrow{v_s}). \qquad 28)$$

So, we can write the following estimates for the left hand side:

$$\int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t}\cdot \overrightarrow{u_n^N}\,dx + \int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t}\cdot \overrightarrow{u_s^N}\,dx + \nu\int_\Omega \nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{u_n^N}dx + \nu(N)\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{u_s^N}\,dx + \frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right)^2 dx \le \int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t}\cdot \overrightarrow{u_n^N}\,dx + \int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t}\cdot \overrightarrow{u_s^N}\,dx + \nu\|\nabla\overrightarrow{u_n^N}\|_2^2 + \nu(N)\|\nabla\overrightarrow{u_s^N}\|_2^2. \qquad 29)$$

And, by using the Gelfand triplet, we obtain:

$$\int_\Omega \frac{\partial \overrightarrow{u_n^N}}{\partial t}\cdot \overrightarrow{u_n^N}\,dx = \frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{u_n^N}\right\|_2^2;\ \int_\Omega \frac{\partial \overrightarrow{u_s^N}}{\partial t}\cdot \overrightarrow{u_s^N}\,dx = \frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{u_s^N}\right\|_2^2. \qquad 30)$$

It follows that the whole inequality is of the form:

$$\frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2+\nu\|\nabla\overrightarrow{\mathrm{u}_n^N}\|_2^2+\nu(N)\|\nabla\overrightarrow{\mathrm{u}_s^N}\|_2^2\leq\frac{2\nu(N)}{3}\left(\left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2+\left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2\right)+\frac{2C(\nu(N))}{3}\|\vec{g}\|_2^2+C(\overrightarrow{v_n};\overrightarrow{v_s}), \qquad 31.a)$$

and we integrate it:

$$\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(t)+\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t)+2\int_0^t\left(\nu\|\nabla\overrightarrow{\mathrm{u}_n^N}\|_2^2+\nu(N)\|\nabla\overrightarrow{\mathrm{u}_s^N}\|_2^2\right)d\tau\leq\frac{4\nu(N)}{3}\int_0^t\left(\left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2+\left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2\right)d\tau+\frac{4C(\nu(N))}{3}\int_0^t\|\vec{g}\|_2^2d\tau+2\int_0^t C(\overrightarrow{v_n};\overrightarrow{v_s})d\tau+\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(0)+\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0). \qquad 32.b)$$

The series $\{C(\nu(N))\}_{N=1}^{\infty}$; $\{\nu(N)\}_{N=1}^{\infty}$ are bounded, because $\nu(N)$ converges to zero. So, finally, we have these apriori estimates:

$$\sup_{t\in[0,T]}\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(t)+\sup_{t\in[0,T]}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t)+\int_0^T\left(\left(2\nu-\frac{4\nu(N)}{3}\right)\left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2+\frac{2}{3}\nu(N)\left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2\right)d\tau$$

$$\leq\frac{4C(\nu(N))}{3}\int_0^T\|\vec{g}\|_2^2d\tau+2\int_0^T C(\overrightarrow{v_n};\overrightarrow{v_s})d\tau+\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(0)+\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0)$$

$$\leq K(\overrightarrow{v_n};\overrightarrow{v_s};\overrightarrow{u_n}(0);\overrightarrow{u_s}(0);\vec{g};\nu). \qquad 33.c)$$

So it is proven: the series $\left\{\overrightarrow{u_n^N}\right\}_{N=1}^{\infty}$, $\left\{\overrightarrow{u_s^N}\right\}_{N=1}^{\infty}$ are bounded in the space of functions $L^2\left(0,T;\left(W_0^{1,2}(\Omega)\right)^N\right)\cap L^{\infty}\left(0,T;\left(L^2\ (\Omega)\right)^N\right)$.

The time derivation apriori estimates and the limit passage

Now let us estimate the time derivation terms. We calculate them for both components, in order not to have to do the same thing two times.

As the function $\varphi \in L^2\left(0,T;\left(\left(W^{1,2}_{0,div}(\Omega)\right)^M\right)\right)$, it follows:

$$\left(\left\|\frac{\partial\overrightarrow{u_n^N}}{\partial t}\right\| + \left\|\frac{\partial\overrightarrow{u_s^N}}{\partial t}\right\|\right)_{L^2\left(0,T;\left(\left(W^{1,2}_{0,div}(\Omega)\right)^N\right)^*\right)} := \sup_{\|\varphi\|\leq 1}\left(\left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_n^N}}{\partial t}\cdot\vec{\varphi}dxdt\right| + \left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_s^N}}{\partial t}\cdot\vec{\varphi}dxdt\right|\right) =$$

$$\sup_{\|\varphi\|\leq 1}\left(\left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_n^N}}{\partial t}\cdot\overrightarrow{\varphi^N}dxdt\right| + \left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_s^N}}{\partial t}\cdot\overrightarrow{\varphi^N}dxdt\right|\right). \qquad 34.a)$$

We do not write the maximum and make the estimates for the norm of the time derivative terms:

$$\left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_n^N}}{\partial t}\cdot\overrightarrow{\varphi^N}dxdt\right| + \left|\int_0^T\int_\Omega \frac{\partial\overrightarrow{u_s^N}}{\partial t}\cdot\overrightarrow{\varphi^N}dxdt\right| = \left|-\int_0^T\int_\Omega \overrightarrow{u_n^N}\cdot\nabla\overrightarrow{u_n^N}\cdot\overrightarrow{\varphi^N}dx - \nu\int_\Omega \nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{\varphi^N}dx - \right.$$
$$\int_\Omega p\,div\overrightarrow{\varphi^N}dx + \nu\int_{\partial\Omega}\overrightarrow{u_n^N}\cdot\nabla\overrightarrow{\varphi^N}ds - \frac{B}{2}\int_\Omega\left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N}-\overrightarrow{u_s^N}\right)\overrightarrow{\varphi^N}dx + \left.\int_\Omega \vec{g}\cdot\overrightarrow{\varphi^N}\,dxdxdt\right| +$$
$$\left|-\int_0^T\int_\Omega \overrightarrow{u_n^N}\cdot\nabla\overrightarrow{u_n^N}\cdot\overrightarrow{\varphi^N}dx - \nu\int_\Omega \nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{\varphi^N}dx - \int_\Omega p\,div\overrightarrow{\varphi^N}dx + \nu\int_{\partial\Omega}\overrightarrow{u_s^N}\cdot\nabla\overrightarrow{\varphi^N}ds - \right.$$
$$\frac{B}{2}\int_\Omega\left|rot\overrightarrow{u_s^N}\right|\left(\overrightarrow{u_n^N}-\overrightarrow{u_s^N}\right)\overrightarrow{\varphi^N}dx + \left.\int_\Omega \vec{g}\cdot\overrightarrow{\varphi^N}\,dxdxdt\right| \leq$$
$$\int_0^T\int_\Omega\left|-\overrightarrow{u_n^N}\cdot\nabla\overrightarrow{u_n^N}\cdot\overrightarrow{\varphi^N}\right|dx + \nu\int_\Omega\left|\nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{\varphi^N}\right|dx + \nu\int_{\partial\Omega}\left|\overrightarrow{v_n^N}\cdot\nabla\overrightarrow{\varphi^N}\right|ds + B\int_\Omega\left|rot\overrightarrow{u_s^N}\right|\left|\left(\overrightarrow{u_n^N}-\right.\right.$$
$$\left.\left.\overrightarrow{u_s^N}\right)\overrightarrow{\varphi^N}\right|dx + 2\int_\Omega\left|\vec{g}\cdot\overrightarrow{\varphi^N}\right|dxdt + \int_0^T\int_\Omega\left|-\overrightarrow{u_s^N}\cdot\nabla\overrightarrow{u_s^N}\cdot\overrightarrow{\varphi^N}\right|dx + \nu(N)\int_\Omega\left|\nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{\varphi^N}\right|dx +$$
$$\nu(N)\int_{\partial\Omega}\left|\overrightarrow{v_s^N}\cdot\nabla\overrightarrow{\varphi^N}\right|ds \leq 2\|\vec{g}\|_2^2 + B\left\|\nabla\overrightarrow{u_s^N}\right\|_2^2\left\|\left(\overrightarrow{u_s^N}-\overrightarrow{u_n^N}\right)\overrightarrow{\varphi^N}\right\|_2^2 + \int_\Omega\left|\overrightarrow{u_s^N}\cdot\left(\overrightarrow{u_s^N}\cdot\nabla\overrightarrow{\varphi^N}\right)\right|dx +$$
$$\int_\Omega\left|\overrightarrow{u_n^N}\cdot\left(\overrightarrow{u_n^N}\cdot\nabla\overrightarrow{\varphi^N}\right)\right|dx + C\left(\overrightarrow{v_n^N};\overrightarrow{v_s^N}\right) + \nu\int_\Omega\left|\nabla\overrightarrow{u_n^N}:\nabla\overrightarrow{\varphi^N}\right|dx + \nu(N)\int_\Omega\left|\nabla\overrightarrow{u_s^N}:\nabla\overrightarrow{\varphi^N}\right|dx \leq$$
$$\int_0^T\left(2\|\vec{g}\|_2^2 + +\nu\left\|\nabla\overrightarrow{u_n^N}\right\|_2^2 + \nu(N)\left\|\nabla\overrightarrow{u_s^N}\right\|_2^2\right)\|\varphi^N\|_2^2 + \left(B\left\|\nabla\overrightarrow{u_s^N}\right\|_2^2\left\|\overrightarrow{u_s^N}\right\|_4^2 + B\left\|\nabla\overrightarrow{u_s^N}\right\|_2^2\left\|\overrightarrow{u_n^N}\right\|_4^2 +\right.$$
$$\left.\left\|\overrightarrow{u_n^N}\right\|_4^2 + \left\|\overrightarrow{u_s^N}\right\|_4^2\right)\|\nabla\varphi^N\|_2^2 dt.$$

34.b)

We obtain, using the apriori estimates 33.c), the estimates for the time derivatives:

$$\left(\left\|\frac{\partial\overrightarrow{u_n^N}}{\partial t}\right\| + \left\|\frac{\partial\overrightarrow{u_s^N}}{\partial t}\right\|\right)_{L^{z(M)}\left(0,T;\left(\left(W^{1,2}_{0,div}(\Omega)\right)^N\right)^*\right)} \leq C\left(\overrightarrow{v_n};\overrightarrow{v_s};\overrightarrow{u_n}(0);\overrightarrow{u_s}(0);\vec{g};\nu\right) \qquad 35)$$

where $z(M)$ is a function, for M=2, equal to 2, and, for M=3, equal to 4/3 (M is a dimension).

The existence of the weak solution:

The estimates above give us more results for the studied PDE system than derived in [1], where better results for the solution uniqueness are presented.

Note that the quality of the solution does not depend on the convective members. Let us use the Aubin-Lions lemma:

As $X_0:=W_0^{1,2}(\Omega)$; $X:=L_0^2(\Omega)$; $X_1:=\left(W_0^{1,2}(\Omega)\right)^*$; $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$; it follows that all sequences have their subsequences, converging in the corresponding function spaces. This is enough for the linear members but not for the nonlinear convective one. We multiply equations 14.a) and 14.b) by $\vec{w_i}$ and investigate if they converge weakly to the equations:

$$\int_\Omega \overrightarrow{\frac{\partial \mathrm{u}_n}{\partial t}} \cdot \vec{w_i}\, dx + \int_\Omega \overrightarrow{\mathrm{u}_n} \cdot \nabla\overrightarrow{\mathrm{u}_n} \cdot \vec{w_i} dx + \nu \int_\Omega \nabla\overrightarrow{\mathrm{u}_n} : \nabla\vec{w_i} dx =$$
$$\nu \int_{\partial\Omega} \overrightarrow{\mathrm{u}_n} \cdot \nabla\vec{w_i} ds \; - \frac{B}{2}\int_\Omega \left|rot\overrightarrow{\mathrm{u}_s}\right| \left(\overrightarrow{\mathrm{u}_n} - \overrightarrow{\mathrm{u}_s}\right)\vec{w_i} dx + \int_\Omega \vec{g} \cdot \vec{w_i}\, dx \qquad 36.a)$$

$$\int_\Omega \overrightarrow{\frac{\partial \mathrm{u}_s}{\partial t}} \cdot \vec{w_i}\, dx + \int_\Omega \overrightarrow{\mathrm{u}_s} \cdot \nabla\overrightarrow{\mathrm{u}_s} \cdot \vec{w_i} dx = \frac{B}{2}\int_\Omega \left|rot\overrightarrow{\mathrm{u}_s}\right| \left(\overrightarrow{\mathrm{u}_n} - \overrightarrow{\mathrm{u}_s}\right)\vec{w_i} dx + \int_\Omega \vec{g} \cdot \vec{w_i}\, dx. \qquad 36.b)$$

For the linear members, as we already said, it is enought the Aubin-Lions lemma. For the second right-hand side member in 19.b) it is enough the convergence $\nu(N) \to 0$; similarly for the Laplace member of the superfluid. So, we obtain from the Aubin-Lions lemma these convergences:

$$\overrightarrow{u^{n_k}} \overset{*}{\rightharpoonup} u \; in\; L^\infty\left(0,T;\left(L^2(\Omega)\right)^M\right) \qquad 37.a)$$

$$\overrightarrow{u^{n_k}} \rightharpoonup u \; in\; L^2\left(0,T;\left(W^{1,2}_{0,div}(\Omega)\right)\;\right) \qquad 37.b)$$

$$\overrightarrow{\frac{\partial u^{n_k}}{\partial t}} \rightharpoonup \frac{\partial u}{\partial t} \; in\; L^{z(M)}\left(0,T;\left(W^{1,2}_{0,div}(\Omega)\right)^*\right), \qquad 37.c)$$

and using the boundness of $\overrightarrow{u^{n_k}}$ in $L^\infty\left(0,T;\left(L^2(\Omega)\right)^M\right)$ and $L^2\left(0,T;\left(W^{1,2}_{0,div}(\Omega)\right)\;\right)$ we obtain the strong convergence:

$$\overrightarrow{u^{n_k}} \to u \; in\; L^q\left(0,T;\left(L^2(\Omega)\right)^M\right)\; \forall q < \infty \qquad 37.d)$$

$$\overrightarrow{u^{n_k}} \to u \; in\; L^2\left(0,T;\left(L^p(\Omega)\right)^M\right)\; M = 2{:}\; \forall p < \infty; M = 3{:}\; \forall p < 6. \qquad 37.e)$$

This give us the convergence of the time derivatives and of the Laplace operator. For both nonlinear terms it is necessary to prove the convergence using some estimates. Now we prove the convergence of the convective terms, so we want to prove that the following holds:

$$\int_\Omega \left[(\overrightarrow{u_n} \cdot \nabla)\overrightarrow{u_n} - \overrightarrow{\mathrm{u}_n^N} \cdot \nabla\overrightarrow{\mathrm{u}_n^N}\right]\vec{w_i} dx \to 0 \qquad 38.a)$$

$$\int_\Omega \left((\overrightarrow{u_s} \cdot \nabla)\overrightarrow{u_s} - \overrightarrow{\mathrm{u}_s^N} \cdot \nabla\overrightarrow{\mathrm{u}_s^N}\right)\vec{w_i} dx \to 0. \qquad 38.b)$$

We multiply both equations by $\psi \in C^{\infty}(\Omega)$ and integrate over (0,T). We then obtain:

$$\int_0^T \int_\Omega \left[(\overrightarrow{u_n} \cdot \nabla)\overrightarrow{u_n} - \overrightarrow{u_n^N} \cdot \nabla \overrightarrow{u_n^N}\right] \overrightarrow{w_i} dx\, \psi dt = \int_0^T \int_\Omega \left[\left(\overrightarrow{u_n^N} \cdot \nabla \overrightarrow{w_i}\right) \overrightarrow{u_n^N} - (\overrightarrow{u_n} \cdot \nabla \overrightarrow{w_i})\overrightarrow{u_n}\right] \psi dx\, dt + \int_0^T \int_{\partial\Omega} \left[(\overrightarrow{v_n} \cdot \nabla)\overrightarrow{v_n} - \overrightarrow{v_n^N} \cdot \nabla \overrightarrow{v_n^N}\right] \overrightarrow{w_i} \psi ds\, dt \leq \|\vec{u}_n - \vec{u}_n^N\|_{L^2\left(0,T;\left(L^3(\Omega)\right)^N\right)} \|\nabla \overrightarrow{w_i}\|_{\left(L^2(\Omega)\right)^{N^2}} \|\psi\|_{L^\infty(0,T)} + \int_{\partial\Omega} c_i^2 \nabla |\vec{w}_i|^2 \overrightarrow{w_i} ds - \int_{\partial\Omega} c_i^2 \nabla |\vec{w}_i|^2 \overrightarrow{w_i} ds. \qquad 39)$$

The first estimate term was derived as in [3]. The rewriting of the boundary conditions is due to the orthonormality of the system $\{\vec{w}_i\}_{i=1}^{\infty}$. Similarly we can do these calculations for the convective member of the superfluid: it is identic. Now let us calculate it for the vorticity term too:

$$\left|\int_0^T \left(\frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx - \frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s}\right| \left(\overrightarrow{u_n} - \overrightarrow{u_s}\right) \overrightarrow{w_i} dx\right) \psi dt \right| \leq \frac{B}{2}\int_0^T \left(\left|\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx - \int_\Omega \left|rot\overrightarrow{u_s}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx\right| + \left|\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx - \int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n} - \overrightarrow{u_s}\right) \overrightarrow{w_i} dx\right|\right) |\psi| dt. \qquad 40)$$

We use the fact that it is possible to estimate the norm of the rotation from the norm of the gradient, using the equivalence of the norms of the divergence and of the rotation, and we obtain:

$$\frac{B}{2}\int_0^T \left(\left|\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx - \int_\Omega \left|rot\overrightarrow{u_s}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx\right|\right) |\psi| dt \leq \frac{B}{2}\int_0^T \left(\left\|\nabla \left(\overrightarrow{u_s^N} - \overrightarrow{u_s}\right)\right\|_2^2 \left\|\overrightarrow{c_n^i} - \overrightarrow{c_s^i}\right\|_2^2\right) |\psi| dt \leq B \cdot C(data) \|\vec{u}_n - \vec{u}_n^N\|_{L^2\left(0,T;\left(W^{1,2}_{div,0}(\Omega)\right)^M\right)} \|\psi\|_{L^\infty(0,T)} \to 0 \qquad 41)$$

$$\int_0^T \left|\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{w_i} dx - \int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n} - \overrightarrow{u_s}\right) \overrightarrow{w_i} dx\right| |\psi| dt \leq \int_0^T \left\|\nabla\overrightarrow{u_s^N}\right\|_3^2 \left(\left\|\overrightarrow{u_n^N} - \overrightarrow{u_n}\right\|_6^2 + \left\|\overrightarrow{u_s^N} - \overrightarrow{u_s}\right\|_6^2\right) \|\nabla \overrightarrow{w_i}\|_2 \, |\psi| dt \leq \left(\|\vec{u}_n - \vec{u}_n^N\|_{L^2\left(0,T;\left(L^3(\Omega)\right)^M\right)} + \|\vec{u}_s - \vec{u}_s^N\|_{L^2\left(0,T;\left(L^3(\Omega)\right)^M\right)}\right) \|\nabla \overrightarrow{w_i}\|_{\left(L^2(\Omega)\right)^{N^2}} \|\psi\|_{L^\infty(0,T)}. \qquad 42)$$

We used above the weak convergence of the Galerkin approximations and their convergence to their norm in order to obtain the strong convergence, used in equation 42).We then used the Friedrichs inequality for the basis function. So, it holds that the system of equations 14) converges to the system 36.a) and 36.b).

Let us now prove the energy inequality. We take equations 20.a) and 20.b) and rewrite them in the same way as during the derivation of the apriori estimates. We obtain:

$$\frac{d\left\|\overrightarrow{u_n^N}\right\|_2^2}{dt} + \int_{\partial\Omega} \overrightarrow{u_n^N} \cdot \vec{n} \left|\overrightarrow{u_n^N}\right|^2 ds + \nu \int_\Omega \nabla\overrightarrow{u_n^N} : \nabla\overrightarrow{u_n^N} dx = \nu \int_{\partial\Omega} \overrightarrow{u_n^N} \cdot \nabla\overrightarrow{u_n^N} ds - \frac{B}{2}\int_\Omega \left|rot\overrightarrow{u_s^N}\right| \left(\overrightarrow{u_n^N} - \overrightarrow{u_s^N}\right) \overrightarrow{u_n^N} dx + \int_\Omega \vec{g} \cdot \overrightarrow{u_n^N}\, dx. \qquad 43.a)$$

Similarly, for the other equation:

$$\frac{d\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2}{dt} + \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \nu(N) \int_{\Omega} \nabla \overrightarrow{u_s^N} : \nabla \overrightarrow{\mathrm{u}_s^N}\, dx = \nu \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \nabla \overrightarrow{\mathrm{u}_s^N} ds \ + \frac{B}{2} \int_{\Omega} \left|rot\overrightarrow{\mathrm{u}_s^N}\right| \left(\overrightarrow{\mathrm{u}_n^N} - \overrightarrow{\mathrm{u}_s^N}\right) \overrightarrow{\mathrm{u}_s^N} dx + \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_s^N}\, dx. \tag{43.b}$$

We then sum both equations:

$$\frac{d\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2}{dt} + \frac{d\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2}{dt} + \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \nu(N) \int_{\Omega} \nabla \overrightarrow{u_s^N} : \nabla \overrightarrow{\mathrm{u}_s^N}\, dx =$$
$$\nu \int_{\partial\Omega} \overrightarrow{\mathrm{u}_n^N} \cdot \nabla \overrightarrow{\mathrm{u}_n^N} ds \ + \nu(N) \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \nabla \overrightarrow{\mathrm{u}_s^N} ds \ - \frac{B}{2} \int_{\Omega} \left|rot\overrightarrow{\mathrm{u}_s^N}\right| \left(\overrightarrow{\mathrm{u}_n^N} - \overrightarrow{\mathrm{u}_s^N}\right)^2 dx + \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_s^N}\, dx + \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_n^N}\, dx. \tag{44}$$

We now integrate over (0,t) and we obtain:

$$\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t) +$$
$$\int_0^t \int_{\partial\Omega} \left(\overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds\right) d\tau + \int_0^t \left(\nu(N) \left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2 + \nu \left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2\right) d\tau =$$
$$\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0) + \int_0^t \left(\nu \int_{\partial\Omega} \overrightarrow{\mathrm{u}_n^N} \cdot \nabla\overrightarrow{\mathrm{u}_n^N} ds \ + \nu(N) \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \nabla\overrightarrow{\mathrm{u}_s^N} ds\ \right) d\tau -$$
$$\int_0^t \frac{B}{2} \int_{\Omega} \left|rot\overrightarrow{\mathrm{u}_s^N}\right| \left(\overrightarrow{\mathrm{u}_n^N} - \overrightarrow{\mathrm{u}_s^N}\right)^2 dx\, d\tau \ + 2\int_0^t \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_s^N}\, dx + \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_n^N}\, dx\, d\tau. \tag{45}$$

To prove the energy inequality we multiply 45) by $\psi \in C_0^{\infty}(\Omega)$; $\psi \geq 0$ and integrate over (0,T). So, we want to investigate the convergence of this equation:

$$\int_0^T \left[\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t) +\right.$$
$$\left.\int_0^t \int_{\partial\Omega} \left(\overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \vec{n} \left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds\right) d\tau + \int_0^t \left(\nu(N) \left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2 + \nu \left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2\right) d\tau\right] \psi dt =$$
$$\int_0^T \left[\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0) + \int_0^t \left(\nu \int_{\partial\Omega} \overrightarrow{\mathrm{u}_n^N} \cdot \nabla\overrightarrow{\mathrm{u}_n^N} ds \ + \nu(N) \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \nabla\overrightarrow{\mathrm{u}_s^N} ds\ \right) d\tau -\right.$$
$$\left.\int_0^t \frac{B}{2} \int_{\Omega} \left|rot\overrightarrow{\mathrm{u}_s^N}\right| \left(\overrightarrow{\mathrm{u}_n^N} - \overrightarrow{\mathrm{u}_s^N}\right)^2 dx\, d\tau \ + 2\int_0^t \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_s^N}\, dx + \int_{\Omega} \vec{g} \cdot \overrightarrow{\mathrm{u}_n^N}\, dx\, d\tau\right] \psi dt. \tag{46}$$

Everything has been prepared to investigate the convergence. The terms multiplied by the artificial viscosity vanish, because the integrals are bounded and due to the definition of the artificial viscosity. It then holds:

$$\nu(N) \int_{\partial\Omega} \overrightarrow{\mathrm{u}_s^N} \cdot \nabla\overrightarrow{\mathrm{u}_s^N} ds \ \to 0 \tag{47.a}$$

$$\nu(N) \int_{\Omega} \nabla\overrightarrow{\mathrm{u}_s^N} \cdot \nabla\overrightarrow{\mathrm{u}_s^N} dx \ \to 0. \tag{47.b}$$

The norm of the initial conditions of the Galerkin approximation converges to the exact initial conditions because of the comprehensiveness of the orthonormal basis and of the Parseval equality.

The same holds for the boundary conditions. For the last term it is enough the weak convergence and we use for the diffusion term the lower weak convergence of the norm and the Fatou lemma:

$$\lim_{n\to\infty} inf \int_0^T \int_0^t \int_\Omega \left|\nabla \overrightarrow{u_n^N}\right|^2 dx\, d\tau\psi dt \geq \int_0^T \lim_{n\to\infty} inf \int_0^t \int_\Omega \left|\nabla \overrightarrow{u_n^N}\right|^2 dx\, d\tau\psi dt \geq$$
$$\int_0^T \int_0^t \int_\Omega \left|\overrightarrow{\nabla u_n}\right|^2 dx\, d\tau\psi dt. \qquad 48)$$

For the vorticity term it is not enough the strong convergence of the solution of the Galerkin approximations to the exact solution 37.d)-37.e) but this term is positive on the left side and we then obtain the inequality:

$$\int_0^T \Big[\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(t) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(t) +$$
$$\int_0^t \int_{\partial\Omega}\left(\overrightarrow{\mathrm{u}_s^N}\cdot\vec{n}\left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds + \int_{\partial\Omega}\overrightarrow{\mathrm{u}_s^N}\cdot\vec{n}\left|\overrightarrow{\mathrm{u}_s^N}\right|^2 ds\right)d\tau + \int_0^t\left(\nu(N)\left\|\nabla\overrightarrow{\mathrm{u}_s^N}\right\|_2^2 + \nu\left\|\nabla\overrightarrow{\mathrm{u}_n^N}\right\|_2^2\right)d\tau -$$
$$\int_0^t\left(\nu\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}ds + \nu(N)\int_{\partial\Omega}\overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{\mathrm{u}_s^N}ds\right)d\tau - 2\int_0^t\int_\Omega \vec{g}\cdot\overrightarrow{\mathrm{u}_s^N}\,dxd\tau\Big]\psi dt \leq \int_0^T\Big[\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s^N}\right\|_2^2(0) +$$
$$\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_n^N}\right\|_2^2(0)\Big]\psi dt. \qquad 49)$$

Using the regularization kernel as test function and the convergence $\varepsilon \to 0^+$, we obtain:

$$\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s}\right\|_2^2(t) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_n}\right\|_2^2(t) +$$
$$\int_0^t\int_{\partial\Omega}\left(\overrightarrow{\mathrm{u}_s}\cdot\vec{n}\left|\overrightarrow{\mathrm{u}_s}\right|^2 ds + \int_{\partial\Omega}\overrightarrow{\mathrm{u}_s}\cdot\vec{n}\left|\overrightarrow{\mathrm{u}_s}\right|^2 ds\right)d\tau + \int_0^t\left(\nu(N)\left\|\nabla\overrightarrow{\mathrm{u}_s}\right\|_2^2 + \nu\left\|\nabla\overrightarrow{\mathrm{u}_n}\right\|_2^2\right)d\tau \leq$$
$$\frac{1}{2}\left\|\overrightarrow{\mathrm{u}_s}\right\|_2^2(0) + \frac{1}{2}\left\|\overrightarrow{\mathrm{u}_n}\right\|_2^2(0) + \int_0^t\left(\nu\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n}\cdot\nabla\overrightarrow{\mathrm{u}_n}\,ds + \nu(N)\int_{\partial\Omega}\overrightarrow{\mathrm{u}_s}\cdot\nabla\overrightarrow{\mathrm{u}_s}\,ds\right)d\tau + \int_0^t\langle\vec{g},\overrightarrow{\mathrm{u}_s}\rangle d\tau +$$
$$\int_0^t\langle\vec{g},\overrightarrow{\mathrm{u}_n}\rangle d\tau\,. \qquad 50)$$

Now let us investigate the convergence of the solution to the initial conditions. We take the equations 16.a) and 16.b) and multiply them by an element of the orthonormal basis and by a function $\psi \in C^\infty[0,T], \psi(T) = 0$. We integrate the equations over the cylinder Ω x [0,T] and obtain:

$$-\int_0^T\int_\Omega \overrightarrow{\mathrm{u}_n^N}\cdot\overrightarrow{w_\iota}\frac{\partial\psi}{\partial t}dxdt - \int_\Omega \overrightarrow{\mathrm{u}_n}(0)\cdot\overrightarrow{w_\iota}\psi(0)dx + \int_0^T\int_\Omega \overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{\mathrm{u}_n^N}\cdot\overrightarrow{w_\iota}\psi dx\,dt +$$
$$\nu\int_0^T\int_\Omega \nabla\overrightarrow{\mathrm{u}_n^N}:\nabla\overrightarrow{w_\iota}\psi dx\,dt = \nu\int_0^T\int_{\partial\Omega}\overrightarrow{\mathrm{u}_n^N}\cdot\nabla\overrightarrow{w_\iota}\psi ds\;dt - \frac{B}{2}\int_0^T\int_\Omega \left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)\overrightarrow{w_\iota}\psi dx\,dt +$$
$$\int_0^T\int_\Omega \vec{g}\cdot\overrightarrow{w_\iota}\psi dxdt \qquad 51.\ a)$$

$$-\int_0^T\int_\Omega \overrightarrow{\mathrm{u}_s^N}\cdot\overrightarrow{w_\iota}\frac{\partial\psi}{\partial t}dxdt - \int_\Omega \overrightarrow{\mathrm{u}_s}(0)\cdot\overrightarrow{w_\iota}\psi(0)dx + \int_0^T\int_\Omega \overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{\mathrm{u}_s^N}\cdot\overrightarrow{w_\iota}\psi dx\,dt +$$
$$\nu(N)\int_0^T\int_\Omega \nabla\overrightarrow{\mathrm{u}_s^N}:\nabla\overrightarrow{w_\iota}\psi dx\,dt =$$
$$\nu(N)\int_0^T\int_{\partial\Omega}\overrightarrow{\mathrm{u}_s^N}\cdot\nabla\overrightarrow{w_\iota}\psi ds\;dt - \frac{B}{2}\int_0^T\int_\Omega\left|rot\overrightarrow{\mathrm{u}_s^N}\right|\left(\overrightarrow{\mathrm{u}_n^N}-\overrightarrow{\mathrm{u}_s^N}\right)\overrightarrow{w_\iota}\psi dx\,dt + \int_0^T\int_\Omega \vec{g}\cdot\overrightarrow{w_\iota}\psi dxdt.$$
$$51.\ b)$$

Thanks to the comprehensiveness of the orthonormal basis, using the limit passage, we obtain the exact equations:

$-\int_0^T \int_\Omega \overrightarrow{\mathrm{u}_n} \cdot \vec{\varphi} \frac{\partial \psi}{\partial t} dxdt - \int_\Omega \overrightarrow{\mathrm{u}_n}(0) \cdot \vec{\varphi}\psi(0)dx + \int_0^T \int_\Omega \overrightarrow{\mathrm{u}_n} \cdot \nabla \overrightarrow{\mathrm{u}_n} \cdot \vec{\varphi}\psi dx\, dt +$
$\nu \int_0^T \int_\Omega \nabla \overrightarrow{\mathrm{u}_n} : \nabla \vec{\varphi}\psi dx\, dt = \nu \int_0^T \int_{\partial\Omega} \overrightarrow{\mathrm{u}_n} \cdot \nabla \vec{\varphi}\psi ds\ dt - \frac{B}{2} \int_0^T \int_\Omega \left| rot \overrightarrow{\mathrm{u}_s} \right| \left(\overrightarrow{\mathrm{u}_n} - \overrightarrow{\mathrm{u}_s} \right) \vec{\varphi}\psi dx\, dt +$
$\int_0^T \int_\Omega \vec{g} \cdot \vec{\varphi}\, \psi dxdt$ 52. a)

$-\int_0^T \int_\Omega \overrightarrow{\mathrm{u}_s} \cdot \vec{\varphi} \frac{\partial \psi}{\partial t} dxdt - \int_\Omega \overrightarrow{\mathrm{u}_s}(0) \cdot \vec{\varphi}\psi(0)dx + \int_0^T \int_\Omega \overrightarrow{\mathrm{u}_s} \cdot \nabla \overrightarrow{\mathrm{u}_s} \cdot \vec{\varphi}\psi dx\, dt =$
$-\frac{B}{2} \int_0^T \int_\Omega \left| rot \overrightarrow{\mathrm{u}_s} \right| \left(\overrightarrow{\mathrm{u}_n} - \overrightarrow{\mathrm{u}_s} \right) \vec{\varphi}\psi dx\, dt + \int_0^T \int_\Omega \vec{g} \cdot \vec{\varphi}\, \psi dxdt.$ 52. b)

Now we focus on the time derivative term. We make the calculation:

$\int_0^T \langle \frac{\partial \overrightarrow{u_n}}{\partial t}; \vec{\varphi} \rangle \psi dt = \int_0^T \frac{d\langle \overrightarrow{u_n}; \vec{\varphi} \rangle}{dt} \psi dt = -\int_0^T \langle \overrightarrow{u_n}; \vec{\varphi} \rangle \frac{d\psi}{dt} dt = -\int_0^T \left(\int_\Omega \overrightarrow{u_n} \cdot \vec{\varphi} dx \right) \frac{d\psi}{dt} dt - \int_\Omega \overrightarrow{u_n}(0) \cdot$
$\vec{\varphi} dx \psi(0).$ 53)

We know that $\int_\Omega \overrightarrow{u_n}(0) \cdot \vec{\varphi} dx \psi(0) \in C\left([0,T]; \left(L_w^2(\Omega)\right)^M\right)$. Setting $\psi(0) \neq 0$, we obtain the equality:

$$\int_\Omega \overrightarrow{u_n}(0) \cdot \vec{\varphi} dx = \int_\Omega \overrightarrow{u_{n,0}} \cdot \vec{\varphi} dx, \qquad 54)$$

and we calculate the same for the superfluid terms. So, finally, we obtain these limit passages:

$$\overrightarrow{u_n} \to \overrightarrow{u_{n,0}};\ \overrightarrow{u_s} \to \overrightarrow{u_{s,0}} \qquad 55)$$

in $\left(L^2(\Omega)\right)^M$ for $t \to 0^+$. Especially, we obtain:

$$\lim_{t\to 0^+} inf \|\overrightarrow{u_n}(t)\|_2^2 \geq \left\|\overrightarrow{u_{n,0}}\right\|_2^2 \qquad 56.a)$$

56.a)

$$\lim_{t\to 0^+} inf \|\overrightarrow{u_s}(t)\|_2^2 \geq \left\|\overrightarrow{u_{s,0}}\right\|_2^2,$$

56.b)

which is called weakly lower semicontinuity and is a consequence of the weak continuity. We use it to obtain the convergence to the initial conditions. As a consequence of the inequalities 56.a) and 56.b) we obtain (from the triangle inequality and lineratity of a weak convergence):

$$\lim_{t\to 0^+} inf \|\overrightarrow{u_s}(t) + \overrightarrow{u_n}(t)\|_2^2 \geq \left\|\overrightarrow{u_{s,0}} + \overrightarrow{u_{n,0}}\right\|_2^2. \qquad 57)$$

It then holds from the energetical inequality:

$$\lim_{t\to 0^+} inf\ \|\overrightarrow{u_s}(t) + \overrightarrow{u_n}(t)\|_2^2 \leq \left\|\overrightarrow{u_{s,0}} + \overrightarrow{u_{n,0}}\right\|_2^2. \qquad 58)$$

So, we obtain the convergence of the solution to the initial conditions in the form:

$$\lim_{t\to 0^+} \left\|\overrightarrow{u_s}(t) + \overrightarrow{u_n}(t) - \overrightarrow{u_{s,0}} - \overrightarrow{u_{n,0}}\right\|_2^2 \to 0. \qquad 59)$$

The numerical solution with the zero vorticity term

The stationary problem

As a first step, we discretize the stationary problem, with time-independent boundary conditions, by using the amplitude of the oscillations of the obstacle. The boundary conditions will be indicated in the same way as in the part about the existence theory, by the symbols $\overrightarrow{v_n}$ and $\overrightarrow{v_s}$. The solution of the stationary problem will then be used as an initial condition for the non-stationary problem. To solve the stationary problem we use the so called *upwind method*, see [5],[6]. We discretize the stationary problem as follows:

$$\overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u^n} + \vec{g} \qquad 1.a)$$

$$\overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p + \vec{g} \qquad 1.b).$$

We use the boundary conditions given in the second chapter, equations 2.a) - 2.d). These are considered time-independent ones. We expect the solution as a six-component vector, consisting of the three components of the normal fluid and of three components of the superfluid. We use the recomended value of the *artificial viscosity* from [6] to stabilize the solution. Besides, it holds the incompressibility condition:

$$div(\overrightarrow{u_s} + \overrightarrow{u_n}) = 0 \qquad 1.c)$$

and we assume zero vorticity of superfluid:

$$rot\overrightarrow{u_s} = 0. \qquad 1.d)$$

We consider stationary boundary conditions as written on pages 15 and 16 above.

As recommended in [5], page 117, we introduce the artificial viscosity for the Euler equation and we approximate equation 1.b) as*:*

$$\int_\Omega \overrightarrow{\mathrm{u}_s^N} \cdot \nabla \overrightarrow{\mathrm{u}_s^N} \cdot \overrightarrow{w_i} dx + \nu(N) \int_\Omega \nabla \overrightarrow{u_s^N} : \nabla \overrightarrow{w_i}\, dx = \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx. \qquad 2.a)$$

In a similar way, we discretize the stationary Navier-Stokes problem:

$$\int_\Omega \overrightarrow{\mathrm{u}_n^N} \cdot \nabla \overrightarrow{\mathrm{u}_n^N} \cdot \overrightarrow{w_i} dx + \nu \int_\Omega \nabla \overrightarrow{u_n^N} : \nabla \overrightarrow{w_i}\, dx = \int_\Omega p div \vec{v}\, dx + \int_\Omega \vec{g} \cdot \overrightarrow{w_i}\, dx. \qquad 2.b)$$

We now introduce an elegant formalism. We define the following equalities:

$$a(\vec{u}; \vec{v}) = \int_\Omega \nabla \vec{u} \cdot \nabla \vec{v}\, dx \qquad 3.a)$$

$$n(\vec{u}, \vec{w}, \vec{v}) = \int_\Omega \vec{v} \cdot (\vec{w})\, \vec{u} dx \qquad 3.b)$$

$$b(\vec{v}, p) = -\int_\Omega p div \vec{v}\, dx. \qquad 3.c)$$

We then consider the weak formulation of the problem 1.a) - 1.c), by using these equalities.

Def.1*:* Let $\nu > 0; \vec{f} \in H^{-1}(\Omega)^M$; $\overrightarrow{v_n}, \overrightarrow{v_s} \in H^{1/2}(\Omega)^M$. These equations hold:

$$\nu a(\overrightarrow{u_n}; \vec{v}) + n(\overrightarrow{u_n}, \overrightarrow{u_n}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}; \vec{v} \rangle \qquad 4.a)$$

$$n(\overrightarrow{u_s}, \overrightarrow{u_s}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}; \vec{v} \rangle \qquad 4.b)$$

$$\int_\Omega (\overrightarrow{u_s} + \overrightarrow{u_n}) \nabla \varphi = 0 \qquad 4.c)$$

$$\int_\Omega \vec{v} \cdot rot \overrightarrow{u_s} \, dx = 0. \qquad 4.d)$$

We then call $p, \overrightarrow{u_n}, \overrightarrow{u_s}$ a *weak solution* of 1.a) – 1.d).

The existence of the solution of the problem is to be proven by using the existence of a potential as a consequence of 4.d), in such a way that, for the superfluid, it holds that a function Φ exists:

$$\overrightarrow{u_s} = \nabla \Phi. \qquad 5)$$

We then obtain from 4.c) the existence of the normal component in the form:

$$\overrightarrow{u_n} = -\nabla \Phi + \mathrm{rot} \vec{A}, \qquad 6)$$

where $\vec{A}$ is an unspecified vector function. Equation 5) enables us to derive the Bernoulli equation for the superfluid:

$$\frac{\rho_s}{2} |\overrightarrow{u_s}|^2(t) + p(t) + g\rho_s x(t) = p(0) + \frac{\rho_s}{2} |\overrightarrow{u_s}|^2(0) + g\rho_s x(0), \qquad 7)$$

where we consider as the only volume force $\vec{f}$ the gravitation field, denoted with $\vec{g}$.

Now we start to build the solver. We can define the approximate solution in the following way. We consider an approximate solution from a finite dimensional subspace of the vector space $H_0^1(\Omega)^M$, where the dimension of the function space is indicated by N and the finite dimensional subspace is denoted as V^N. In a similar way it holds for the space of the pressure: $Q^N \hookrightarrow\hookrightarrow L_0^2(\Omega)$. We denote the approximation of the boundary conditions as $\overrightarrow{v_n^N}$ and $\overrightarrow{v_s^N}$ for the normal liquid and superfluid, respectively. The approximations are defined as extensions of the boundary conditions, defined on the domain Ω, satisfying the condition on the boundary:

$$\overrightarrow{v_n} = \overrightarrow{v_n^N} + O(h^{N+1}), \overrightarrow{v_s} = \overrightarrow{v_s^N} + O(h^{N+1}), \qquad 8)$$

where h is the diameter of either a circle (2D) or a sphere (3D), defined by the shape of the cell of the mesh. We will generate such a mesh for the whole geometry where we solve the given problems.

Now we can define the approximate (discrete) solution of the problem 2.a)-2.b).

Def.2*:* The functions $\overrightarrow{u_n^N}, \overrightarrow{u_s^N} \in H_0^1(\Omega)^M$ and $p^N \in Q^N$ are discrete solutions of the problem 4.a) -4.c), *iff:*

$$\overrightarrow{u_n^N} - \overrightarrow{v_n^N}, \overrightarrow{u_s^N} - \overrightarrow{v_s^N} \in V^N \qquad 9.a)$$

$$\nu a\left(\overrightarrow{u_n^N} - \overrightarrow{v_n^N}; \vec{v}\right) + n\left(\overrightarrow{u_n^N}, \overrightarrow{u_n^N}, \vec{v}\right) + b(\vec{v}, p^N) = \langle \vec{f}; \vec{v} \rangle \qquad 9.b)$$

$$\nu(N) a\left(\overrightarrow{u_s^N} - \overrightarrow{v_s^N}; \vec{v}\right) + n\left(\overrightarrow{u_s^N}, \overrightarrow{u_s^N}, \vec{v}\right) + b(\vec{v}, p^N) = \langle \vec{f}; \vec{v} \rangle \qquad 9.c)$$

$$\int_\Omega \overrightarrow{v_h} \cdot rot\overrightarrow{u_s^N}\, dx = 0 \qquad 9.d)$$

$$\int_\Omega \left(\overrightarrow{u_n^N} + \overrightarrow{u_s^N}\right) \nabla q dx = 0 \qquad 9.e)$$

$\forall\, \vec{v} \in V^N, q \in Q^N$.

Standard methods for the solution of the incompressible stationary N-S equations [6] cannot be used here because of equation 9.e). But we can assume low Reynolds numbers in such a way that the methods discussed in [6],[7] can be employed. As there is only one dominating flow caused by the movement of the obstacle, we choose the Oseen method (see below). The calculated velocity of the normal liquid is then used to calculate the potential of the superfluid, which is then used, by the means of the Bernoulli equation, to calculate the pressure. We write initially the iterative Oseen scheme for 8.b). It is based on a relative scheme found in [5] but it does not consider incompressibility:

1) Choose $\overrightarrow{u_n^{N,0}}, p^{N,0}$ (an initial iterative step).
2) Calculate the next step, by satisfying the equation:

$$\nu a\left(\overrightarrow{u_n^{N,1}} - \overrightarrow{v_n^N}; \vec{v}\right) + n\left(\overrightarrow{u_n^{N,0}}, \overrightarrow{u_n^{N,1}}, \vec{v}\right) + b(\vec{v}, p^{N,1}) = \langle \vec{f}; \vec{v} \rangle. \qquad 10)$$

As a generalization we can write it for each iterative step: $\overrightarrow{u_n^{N,k}}, p^{N,k}$, where N and k are positive finite integers. Then the step k+1 is obtained as:

1) Choose $\overrightarrow{u_n^{N,k}}, p^{N,k}$ (calculated).
2) Calculate the next step, by satisfying the equation:

$$\nu a\left(\overrightarrow{u_n^{N,k+1}} - \overrightarrow{v_n^N}; \vec{v}\right) + n\left(\overrightarrow{u_n^{N,k}}, \overrightarrow{u_n^{N,k+1}}, \vec{v}\right) + b\left(\vec{v}, p^{N,k+1}\right) = \langle \vec{f}; \vec{v} \rangle. \qquad 11)$$

This iterative process is called *Oseen iterative process*. If we know $\overrightarrow{u_n}$ in the whole space, we can calculate the superflluid potential as a solution of the Laplace equation:

$$\Delta\varphi = -div\overrightarrow{u_n}, \nabla\varphi|_{\partial\Omega} = \overrightarrow{v_s}, \qquad 12)$$

and this can be input into the Bernoulli equation to calculate the pressure:

$$\frac{v_s^2}{2} + \frac{p}{\rho_s} + gh = const. \qquad 13)$$

We now define the so-called Triangulation (that is, the mesh) of the finite elements and the base functions. We will use the same mesh for both velocity fields and for the pressure.

Def. 3: The *Triangulation of the computational domain* is a decomposition of the computational domain Ω, denoted as T_N, such as that, for all $T \in T_N$, the following holds:

1) T is closed and its interior is non-empty and connected.
2) The boundary ∂T is, for any T, Lipschitz-continous.
3) $\overline{\Omega} = \cup_{T \in T_N} T$.
4) The intersection of the interiors of any two different sets of T_N is empty.

A finite element is called a triplet (T, V^N, Σ), where T is a bounded closed subset of R^M, with a nonempty interior and a Lipschitz-continous boundary.

1) V^N is a finite dimensional subspace of real functions, defined on T, with $\dim V^N = N$.
2) Σ is a set of N linear forms Φ_i, with $i = 1 \dots N$, such as that V^N is uni-solvent:

$$\forall \alpha_1 \dots \alpha_N \in R \; \exists! \, p \in V^N \colon \Phi_i(p) = \alpha_i, i = 1 \dots N. \qquad 14.a)$$

Thus we have:

$$p = \sum_{i=1}^{N} \Phi_i(p) p_i \, , \forall p \in V^N. \qquad 14.b)$$

Note: we call the points where more than two T touch *nodes*.

Def. 4: a base function in the node number l is a polynom of order $\sqrt[M]{N}$ and is called *a base function,* iff it holds: $p(l) = 1$, and p is equal to zero outside the outlying T.

It is possible to write an approximate solution in the form:

$$\vec{u}_n^N = \sum_{i=1}^{N} c_i^n \overrightarrow{\varphi^{\iota,d}} \qquad 15.a)$$

$$\vec{u}_s^N = \sum_{i=1}^{N} c_i^s \overrightarrow{\varphi^{\iota,d}} \qquad 15.b)$$

$$p^N = \sum_{i=1}^{N} c_i^p \varphi^i, \qquad 15.c)$$

where φ^i are base functions. The functions $\overrightarrow{\varphi^{\iota,d}}$ are then vector base functions, defined as $\overrightarrow{\varphi^{\iota,d}} = (\varphi^i; 0; 0)$ for d=1; $\overrightarrow{\varphi^{\iota,d}} = (0; \varphi^i; 0)$ for d=2; and, similarly, for the third component. We assume that we know the base functions that we need in order to calculate the coefficients to obtain the approximate solution. We choose such base functions in order to have $\int_\Omega \varphi^i \varphi^j dx = \delta^{ij}$, where δ^{ij} is the standard *Kronecker delta.* We assume that the system of vector base functions is defined in the following way: for the system $\{\varphi^i\}$ we define a system of $\{\overrightarrow{\varphi^{\iota,d}}\}$ such that $\{\overrightarrow{\varphi^{\iota,1}}\} := \{(\varphi^i, 0,0)\}_{i=1}^N$, $\{\overrightarrow{\varphi^{\iota,2}}\} := \{(0, \varphi^i, 0)\}_{i=1}^N$ and $\{\overrightarrow{\varphi^{\iota,3}}\} := \{(0,0, \varphi^i)\}_{i=1}^N$.

Now we can formulate the numerical scheme:

1) Choose $\overrightarrow{u_n^{N,k}}, p^{N,k}$ (calculated).

2) Calculate the next step satisfying the equations:

$$\nu a\left(\overrightarrow{u_n^{N,k+1}} - \overrightarrow{v_n^N}; \vec{v}\right) + n\left(\overrightarrow{u_n^{N,k}}, \overrightarrow{u_n^{N,k+1}}, \vec{v}\right) + b\left(\vec{v}, p^{N,k}\right) = \langle \vec{f}; \vec{v} \rangle. \tag{16.a}$$

3) Calculate $div\overrightarrow{u_n^{N,k+1}}$.

4) Calculate the potential $\varphi^{N,k+1}$ of an approximative solution $\overrightarrow{u_s^{N,k+1}}$ by using the application of FEM (finite element method, described below) on the Laplace equation:

$$\Delta\varphi^{N,k+1} = -div\overrightarrow{u_n^{N,k+1}}, \nabla\varphi^{N,k+1}\big|_{\partial\Omega} = \overrightarrow{v_s^{N,k+1}}. \tag{16.b}$$

5) Calculate the superfluid velocity: $\overrightarrow{u_s^{N,k+1}} = \nabla\varphi^{N,k+1}$.

6) Use the Bernoulli equation to calculate the pressure values in each cell of the mesh:

$$p^{N,k+1} = p^{N,k} + \rho_s \frac{\left|\overrightarrow{u_s^{N,k}}\right|_2^2 - \left|\overrightarrow{u_s^{N,k+1}}\right|_2^2}{2}, \tag{16.c}$$

where $|\;\;|_2$ denotes the standard Euclid vector norm. The number of iterations depends on the concrete case and it can be adjusted. Each one step 1)-6) is detail rewritten below.

Let us now focus on steps 2) – 5). The error of the used method can be estimated then using by this equation:

$$\left\|\overrightarrow{u_n^N} - \overrightarrow{u_n}\right\|_{W^{1,2}(\Omega)} + \left\|p^N - p\;\;\right\|_{W^{1,2}(\Omega)} \leq Ch, \tag{17.a}$$

where h is the diameter of the used mesh and C depends on the chosen approximation of the data, see [4], page 62.

It holds for the potential of the superfluid velocity the Céan Lemma:

$$\left\|\varphi^N - \varphi\;\;\right\|_{W^{2,2}(\Omega)} \leq C \cdot inf_{v^N \in V^N}\left\|v^N - v\right\|_{W^{2,2}(\Omega)}, \tag{17.b}$$

see [4], pages 22/23.

2) The problem is rewritten as:

$$\left[A'_x\left(\vec{u}_n^{N,k}\right) + A'_y\left(\vec{u}_n^{N,k}\right) + A'_z\left(\vec{u}_n^{N,k}\right)\right]\vec{u}_n^{N,k+1} + \nu B\left(\overrightarrow{u_n^{N,k+1}} - \overrightarrow{v_n^N}\right) = F\left(p^{N,k}\right) + G, \tag{18}$$

where A'_m, B, F and G are matrices, and m=x,y or z. Let us describe them. G is non-zero only for the z component, because it is a discretization of the gravitational field. We assume the solution in this form:

$$\vec{u}_n^{N,k} = \sum_{i=1}^N c_{i,k}^n \overrightarrow{\varphi^i} \tag{19.a}$$

$$\vec{u}_s^{N,k} = \sum_{i=1}^{N} c_{i,k}^{s} \overrightarrow{\varphi^i} \qquad 19.b)$$

$$p^{N,k} = \sum_{i=1}^{N} c_{i,k}^{p} \varphi^i. \qquad 19.c)$$

We input 19.a) and 19.c) into 16.a) and multiply by the base function φ_m. We denote:

$$U_n^{N,k} = \left(c_{i,k}^{n}\right)_{i=1}^{N};\ U_n^{N,k+1} = \left(c_{i,k+1}^{n}\right)_{i=1}^{N} \qquad 20.a)$$

$$U_s^{N,k} = \left(c_{i,k}^{s}\right)_{i=1}^{N};\ U_s^{N,k+1} = \left(c_{i,k+1}^{s}\right)_{i=1}^{N} \qquad 20.b)$$

$$P^{N,k} = \left(c_{i,k}\right)_{i=1}^{N};\ P^{N,k+1} = \left(c_{i,k+1}\right)_{i=1}^{N}. \qquad 20.c)$$

Then the matrices $A_x \coloneqq \{a_{x,imj}\}, A_y \coloneqq \{a_{y,imj}\}, A_z \coloneqq \{a_{z,imj}\}$ exist in such a way that:

$$a_{x,imj} \coloneqq \int_\Omega \varphi^i \varphi^m \frac{\partial \varphi^j}{\partial x} dx;\ a_{y,imj} \coloneqq \int_\Omega \varphi^i \varphi^m \frac{\partial \varphi^j}{\partial y} dx; a_{z,imj} \coloneqq \int_\Omega \varphi^i \varphi^m \frac{\partial \varphi^j}{\partial z} dx. \qquad 21.a)$$

It then holds for a x-component equation:

$$\left(c_{i,k}^{n,x} a_{x,imj} + c_{i,k}^{n,y} a_{y,imj} + c_{i,k}^{n,z} a_{z,imj} + \nu D_{mj}\right) c_{i,k+1}^{n,x} = c_{i,k} \int_\Omega \varphi_m \frac{\partial \varphi_j}{\partial x} dx + \int_\Omega \overrightarrow{v_n^N} \varphi^m dx, \qquad 21.b)$$

where $c_{i,k}^{n,x}$ are the coefficients multiplied in 19.a) by the base functions $(\varphi^j; 0; 0)$. In this way we define the other coefficient vectors $c_{i,k}^{n,y}$ and $c_{i,k}^{n,z}$. The matrix D_{mj} can be calculated as:

$$D_{mj} = \int_\Omega \nabla \varphi^m \nabla \varphi^j \, dx. \qquad 21.c)$$

We used in equation 21.b) the standard *Einstein convention.* We can write this schematically, using 20.a) – 20.c):

$$U_{n,x}^{N,k} = \left(c_{i,k}^{n,x}\right)_{i=1}^{N}; U_{n,y}^{N,k} = \left(c_{i,k}^{n,y}\right)_{i=1}^{N};\ U_{n,z}^{N,k} = \left(c_{i,k}^{n,z}\right)_{i=1}^{N} \qquad 22.a)$$

$$U_{n,x}^{N,k+1} = \left(c_{i,k+1}^{n,x}\right)_{i=1}^{N}; U_{n,y}^{N,k+1} = \left(c_{i,k+1}^{n,y}\right)_{i=1}^{N};\ U_{n,z}^{N,k+1} = \left(c_{i,k+1}^{n,z}\right)_{i=1}^{N}. \qquad 22.b)$$

And now we can write schematically 21.b):

$$\left(U_{n,x}^{N,k} A_x + U_{n,y}^{N,k} A_y + U_{n,z}^{N,k} A_z + \nu D\right) U_{n,x}^{N,k+1} = P^{N,k} R + \nu B\left(\overrightarrow{v_n^N}\right), \qquad 22.c)$$

where the matrix R is defined as:

$$R \coloneqq \left\{\int_\Omega \varphi_m \frac{\partial \varphi_j}{\partial x} dx\right\} \qquad 22.d)$$

and $B\left(\overrightarrow{v_n^N}\right)$ is defined as:

$$\mathrm{B} :=\left\{\int_\Omega \overrightarrow{v_n^N} \varphi^m dx\right\}. \qquad 22.e)$$

This problem can be solved using Gauss elimination. We define the matrix A:

$$U_{n,x}^{N,k} A_x + U_{n,y}^{N,k} A_y + U_{n,z}^{N,k} A_z + \nu D = A, \tag{23)}$$

and then we rewrite 22.c) as:

$$A U_{n,x}^{N,k+1} = P^{N,k} R + B. \tag{24)}$$

This system of linear algebraic equations can be solved using the method of preconditioned conjugated gradients.

3) We obtained from the step 2) $U_n^{N,k+1}$. We specifically estimated the coefficients $\left(c_{i,k}^{n}\right)_{i=1}^{N}$, which enable us to calculate the coefficients $\left(c_{i,k}^{s}\right)_{i=1}^{N}$, using 16.b). The discretization $div\overrightarrow{u_n^{N,k+1}}$ can be calculated using the base functions:

$$div\overrightarrow{u_n^{N,k+1}} = \sum_{i=1}^{N} c_{i,k}^{n} \mathrm{tr} \nabla \overrightarrow{\varphi^{\iota,d}} \tag{25)}$$

4) Now, we can use equation 16.b) and solve it by using the FEM method:

$$\Delta\varphi^{N,k+1} = -div\overrightarrow{u_n^{N,k+1}}, \nabla\varphi^{N,k+1}\big|_{\partial\Omega} = \overrightarrow{v_s^{N,k+1}}. \tag{26.a)}$$

We multiply 26.a) by φ^j, integrate over the whole domain and obtain:

$$A U_s^{N,k+1} = B\left(\overrightarrow{v_s^{N}}\right) + \sum_{i=1}^{N}\left(c_{i,x}^{n}; c_{i,y}^{n}\right)\nabla\overrightarrow{\varphi^{\iota,d}}, \tag{26.b)}$$

where A denotes a matrix (usually called the *stiffness matrix)*, which is defined as:

$$S := \left\{\int_\Omega \nabla\varphi^i : \nabla\varphi^j \, dx\right\}. \tag{26.c)}$$

We see that it holds:

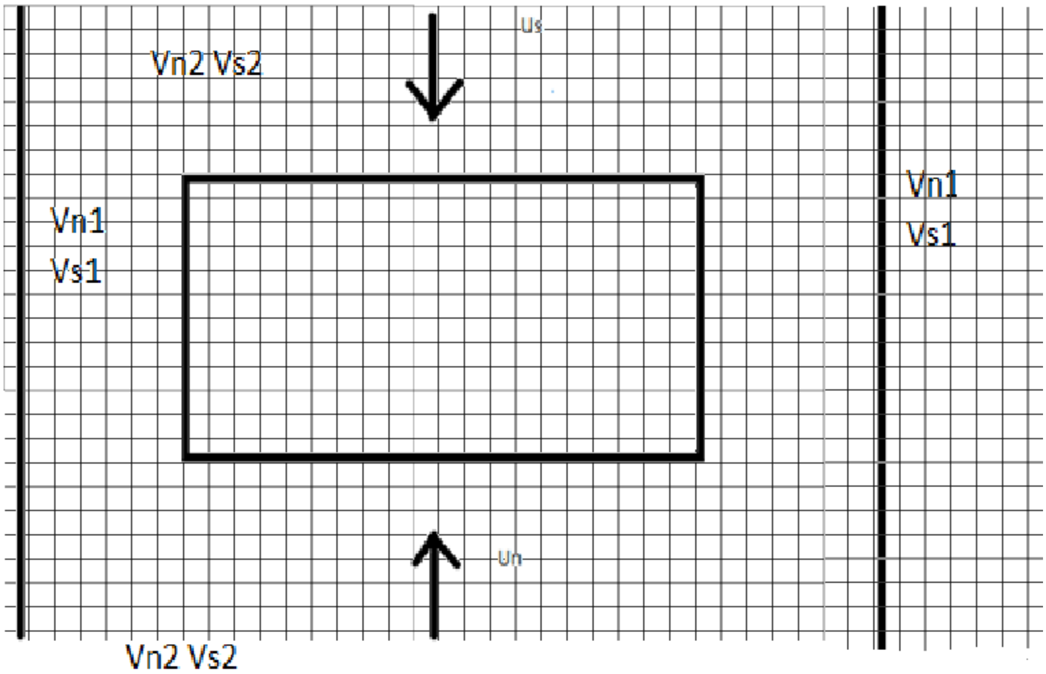
$$S = D. \tag{27)}$$

The solution of 26.b) can be found in the same way as in the case of 24).

5) Using the calculated values of the coefficients $U_s^{N,k+1}$ we can combine them with the calculated superfluid velocity. It is meaningful to compute them now in order to obtain the value of pressure in the next step, by using formula 17).

To implement the derived numerical scheme we derive the base functions as the first step. We assume a *quadratical finite element*; the whole mesh will consist of regular quadrates, as we can see in *Figure 1*. To make it illustrative, we make all calculations in 2D:

Figure 1: Used mesh.



The drawn mesh is likely not enough fine. We made calculations with cells of 1x1 mm. It was not possible to store larger matrices in the computer.

We introduce *a reference element* in order to avoid the calculation of integrals in the previous numerical scheme. We evaluate the integrals only on a regular rectangle or on a square and use a transformation to each finite element. It is very simple in the case of a regular mesh due to the fact that we need only to translate the reference element on the considered cell. As we make calculations in the coordinate system of the oscillating obstacle, we can put the base functions in the area of the obstacle equal to zero, including the nodes on the boundary of the oscillating object. The scalar base functions on the reference element are shown in *Figure 2.* They are denoted using the angles of the reference element; the corresponding letter stays at that angle in which it is nonzero.

It is possible to derive that the functions must be of the form:

$$\varphi_1 = -xy + x \qquad 28.a)$$

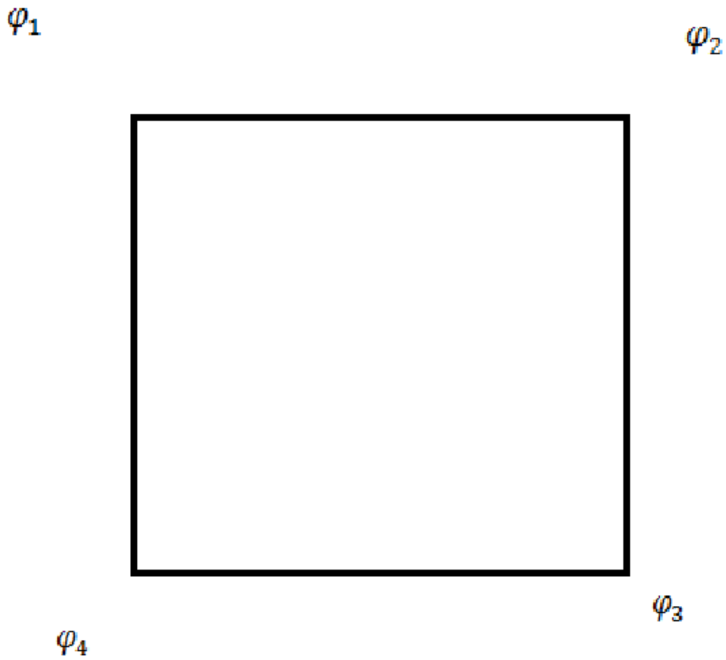
$$\varphi_2 = xy \qquad 28.b)$$

$$\varphi_3 = -xy + y \qquad 28.c)$$

$$\varphi_4 = xy - x - y + 1. \qquad 28.d)$$

The graph of the base functions is pyramide-like and its support is created by four elements, which are around the node, where the base function is equal to one.

Figure 2: Reference element.



Now we derive the form of the matrices. Because of the support of the base functions, we obtain them as *band matrices.* We see that they are tridiagonal. Now, we can write the form of matrices. It holds:

$$D=\begin{pmatrix}
\begin{matrix} -\frac{1}{6} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{6} & \frac{2}{3} \\ 0 & \frac{2}{3} & \ddots \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix} \\
\begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{matrix} \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \ddots \end{matrix} & \begin{matrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{matrix} & \begin{matrix} \ddots & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} \end{matrix} \\
\begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \ddots & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{6} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{6} \end{matrix}
\end{pmatrix}. \qquad 29.a)$$

We can see that D is a bound matrix and, for the components on the positions corresponding to the obstacle (rectangle in 2D), we obtain zeros. Because we consider a mesh of 24x24 cells and for the rectangle we use 3x10 (the original dimensions are 3x10 mm), we obtain the whole matrix as 576x576 and the "hole" including only zeros as a matrix 30x30. Those dimensions of matrices are implied by equations 21.a), 22.d), 22.e) and 26.c). Because the area of one finite element is 1x1=1 mm^2, it must be that all matrices are multiplied during the computation by this number (in meters) to obtain the velocity coefficients in the correct units. It is better to write the matrices without this scaling, for the sake of clarity.

There are two smaller matrices representing the no-slip boundary condition on the rectangle around the "hole", having 1/3 on the main diagonal. Let us have a look at the matrices discretizing the convection terms. The matrices are three block cubes of numbers. We obtain this cube 576x576x576 (the not displayed positions are identically zero):

$$A'_{x,y,z} = \left\{ \begin{pmatrix} -\frac{5}{12} & \frac{5}{12} & 0 \\ -\frac{1}{12} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{5}{12} \\ 0 & -\frac{1}{12} & -\frac{1}{4} \end{pmatrix} \dots \right\}. \qquad 29.b)$$

For the matrices on the right hand side we obtain:

$$B(\overrightarrow{v_n})_y = \left(V_n + \frac{U_0 - V_n}{3R} \quad \frac{1}{4}U \dots 0 \quad 0 \quad \dots \frac{1}{4}U \quad V_n + \frac{U_0 - V_n}{3R} \right)^T, \qquad 29.c)$$

and for the x component we obtain zeros. The matrix R is:

$$\begin{pmatrix} \begin{matrix} \frac{1}{12} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \ddots \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{matrix} -\frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \ddots \end{matrix} & \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} & \begin{matrix} \ddots & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{3} \end{matrix} \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \ddots & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{12} \end{matrix} \end{pmatrix}. \qquad 29.d)$$

Now we need only to subtract from the y-component the gravitation (-10N/kg), because for the 2D case this is the vertical axis. This method is iterative, so we will start with a parabolical profile of the velocity field and with the pressure value over the surface of liquid helium in the cryostat. The matrix

B depends on the physical conditions of the studied problem. We used the values from [8]. For U_0 we put $U_0 = 2\pi f a$, where f is the frequency of the oscillations of the obstacle and a is the amplitude of oscillations. $\nu = 1.66 \cdot 10^{-8} m^2/s$, which corresponds to $Re \approx 9500$, so we need to use an artificial viscosity to stabilize the matrix A. We calculate it as $\nu(N) = Uh$. I assume T=1.24 K, which corresponds to $\rho_n = 4.99\ kg/m^3$, $\rho_s = 3\ kg/m^3$, $S = 6.56 \cdot 10^{-2}\ J/K$, see [9]. As an initial velocity for the iterative process I used a constant velocity field, equal to the boundary conditions with parabolical profile with zero on the place of the obstacle. We assume absence of counterflow, so in the matrix B it is equal to zero. For the numerical results after 10 iterations, see *Figures 3.a)-3.c)* with the normal liquid velocity field, superfluid velocity field and with the pressure field:

Figure 3.a: Pressure field in the stationary case.

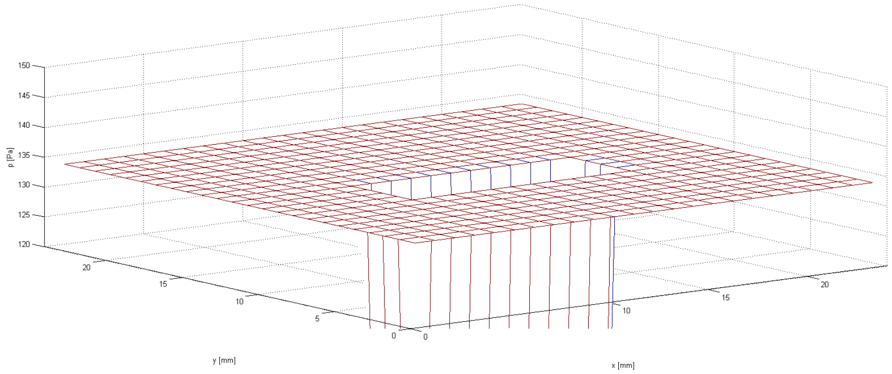


Figure 3.b: Velocity field of the normal liquid in the stationary case.

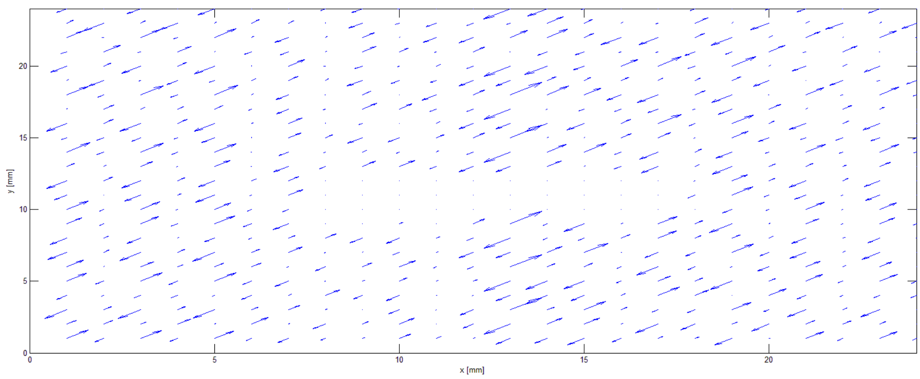
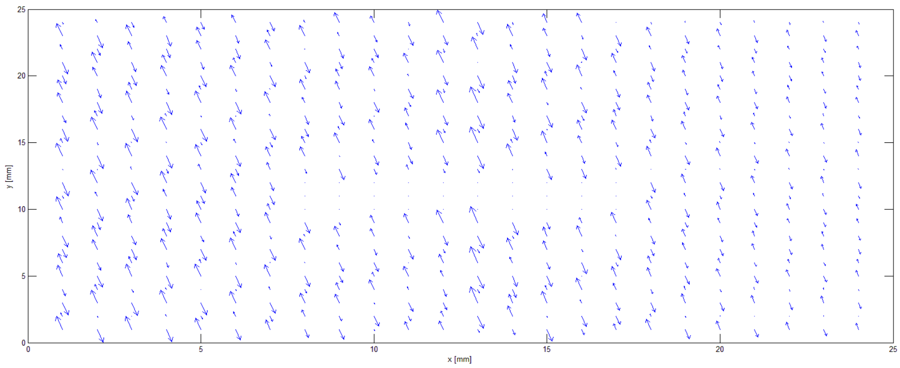


Figure 3.c: Velocity field of the superfluid in the stationary case.



The numbers on the axes correspond to the experiment dimensions and the length of the longest arrow in the figure of the velocity of the normal liquid is around 3 mm/s, while for the superfluid it is around 1.4 mm/s. We can see that the results are not physical because they are not symmetric. Additionally, this can be seen as a property of the Oseen scheme at high Reynolds numbers; so, it should not be used for flows over Re=1000. For higher pressure around 1 atmosphere we obtained those figures:

Figure 4.a: Pressure field in the stationary case, over the liquid p=101133.3 Pa.

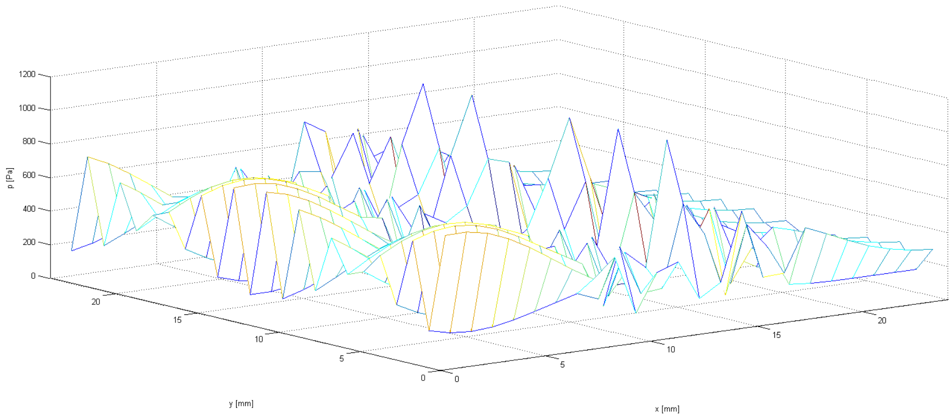


Figure 4.b: Velocity field of the normal liquid in the stationary case, over the liquid p=101133.3 Pa.

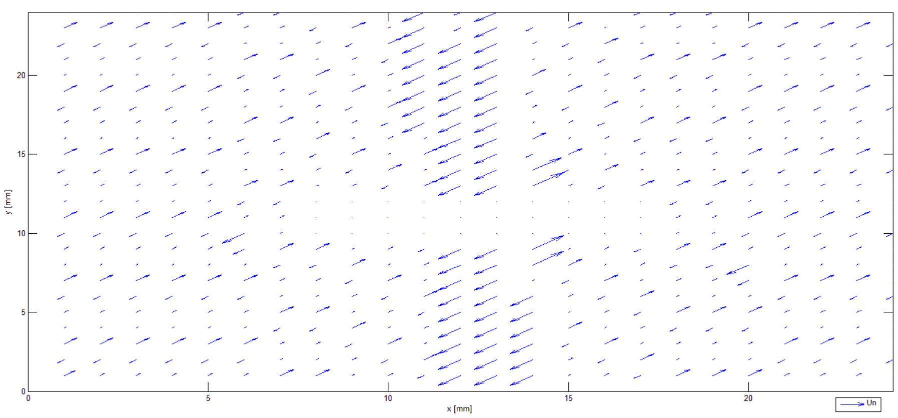
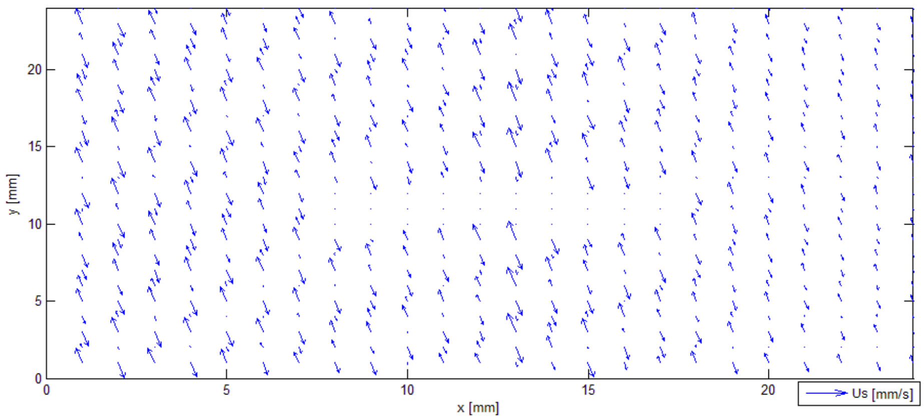


Figure 4.c: Velocity field of the superfluid in the stationary case, over the liquid p=101133.3 Pa.



In Figure 4.a we substracted 1 Bar for better clarity. The length of the longest arrow corresponds to around 0.3 mm/s, while for the superfluid it is 10 mm/s. We obtained in both cases solutions that are not physical for the reasons mentioned above.

The nonstationary problem

We have now the solution of the system of equations 9.a) - 9.e), which is an approximate solution of 4.a) – 4.d). We use it as an initial condition for the problem:

Def.5*:* Let $\nu > 0; \vec{f} \in L^2(0, T; H^{-1}(\Omega)^M)$; $\overrightarrow{v_n}, \overrightarrow{v_s} \in L^2(0, T; H^1(\Omega)^M)$; $0 < T < \infty$; $\overrightarrow{u_{n,0}}, \overrightarrow{u_{s,0}} \in L^2(\Omega)$. These equations hold:

$$\langle \frac{\partial \overrightarrow{u_n}}{\partial t}; \vec{v} \rangle + \nu a(\overrightarrow{u_n}; \vec{v}) + n(\overrightarrow{u_n}, \overrightarrow{u_n}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}; \vec{v} \rangle \qquad 30.a)$$

$$\langle \frac{\partial \overrightarrow{u_s}}{\partial t}; \vec{v} \rangle + n(\overrightarrow{u_s}, \overrightarrow{u_s}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}; \vec{v} \rangle \qquad 30.b)$$

$$\int_\Omega (\overrightarrow{u_s} + \overrightarrow{u_n}) \nabla \varphi = 0 \qquad 30.c)$$

$$\int_\Omega \vec{v} \cdot rot \overrightarrow{u_s}\, dx = 0, \qquad 30.d)$$

then we call $p, \overrightarrow{u_n}, \overrightarrow{u_s}$ a *weak solution* of 1.a) – 1.d).

To find a numerical solution we change the stationary solution. We define *a partition* and *a time step*:

Def.6: Let T>0 and $[0, T]$ is an interval. Then we call the finite sequence of numbers $\{t_i\}_{i=1}^n; \forall i \in I$, with $0 < t_1 < t_2 < \cdots < T$, a partition. Let define τ in such a way that it holds $\tau = |t_i - t_{i-1}|$ for all i. Then we call τ *a time step* and the partition *an equidistant partition.*

Def.7*:* Let $0 < T < \infty$; $\overrightarrow{u_{n,0}^N}, \overrightarrow{u_{s,0}^N} \in L^2(\Omega)$. Then the functions $\overrightarrow{u_n^N}, \overrightarrow{u_s^N} \in C^1(0, T; H_0^1(\Omega)^M)$ and $p^N \in L^2(0, T; Q^N)$ are a *semidiscrete solution* of the problem 4.a) -4.c), *iff:*

$$\overrightarrow{u_n^N} - \overrightarrow{v_n^N}, \overrightarrow{u_s^N} - \overrightarrow{v_s^N} \in 2(0, T; V^N) \qquad 29.a)$$

$$\langle \frac{\partial \overrightarrow{u_n^N}}{\partial t}; \vec{v} \rangle + \nu a\left(\overrightarrow{u_n^N} - \overrightarrow{v_n^N}; \vec{v}\right) + n\left(\overrightarrow{u_n^N}, \overrightarrow{u_n^N}, \vec{v}\right) + b(\vec{v}, p^N) = \langle \vec{f}; \vec{v} \rangle \qquad 29.b)$$

$$\langle \frac{\partial \overrightarrow{u_s^N}}{\partial t}; \vec{v} \rangle + \nu(N) a\left(\overrightarrow{u_s^N} - \overrightarrow{v_s^N}; \vec{v}\right) + n\left(\overrightarrow{u_s^N}, \overrightarrow{u_s^N}, \vec{v}\right) + b(\vec{v}, p^N) = \langle \vec{f}; \vec{v} \rangle \qquad 29.c)$$

$$\int_\Omega \overrightarrow{v_h} \cdot rot \overrightarrow{u_s^N}\, dx = 0 \qquad 29.d)$$

$$\int_\Omega \left(\overrightarrow{u_n^N} + \overrightarrow{u_s^N}\right) \nabla q dx = 0 \qquad 29.e)$$

$\forall\, \vec{v} \in L^2(0, T; V^N), q \in L^2(0, T; Q^N)$.

We consider $\vec{f} = \vec{g}$, like in the stationary case. The boundary conditions are discretized in the same way as in the stationary case. The only difference is that we assume time-dependent coefficients; so, we assume an approximate solution in this form:

$$\vec{u}_n^N(t) = \sum_{i=1}^N c_i^n(t) \overrightarrow{\varphi^i} \qquad 30.a)$$

$$\vec{u}_s^N(t) = \sum_{i=1}^N c_i^s(t)\overrightarrow{\varphi^i} \quad 30.b)$$

$$p^N(t) = \sum_{i=1}^N c_i^p(t)\varphi^i. \quad 30.c)$$

We assume the time dependent coefficients to be one-time differentiable. For the discretization we denote the value of the coefficient c_i at the time step k as $c_{i,k}$ in such a way that it holds for the pressure and both velocity fields at the step k:

$$\vec{u}_n^{N,k} = \sum_{i=1}^N c_{i,k}^n\overrightarrow{\varphi^i} \quad 31.a)$$

$$\vec{u}_s^{N,k} = \sum_{i=1}^N c_{i,k}^s\overrightarrow{\varphi^i} \quad 31.b)$$

$$p^{N,k} = \sum_{i=1}^N c_{i,k}^p\varphi^i. \quad 31.c)$$

For the stationary case we denoted k an iteration step; for the nonstationary scheme k corresponds to a time step. We consider that we know the initial values of the coefficients, so that we can solve the system as a system of ordinary differential equations.

Now we can formulate the numerical scheme:

1) Choose $\overrightarrow{u_n^{N,k}}, p^{N,k}$ (calculated in a previous time step).

2) Calculate the next step, satisfying the equations:

$$\overrightarrow{u_n^{N,k+1}} = \overrightarrow{u_n^{N,k}} - \tau\nu a\left(\overrightarrow{u_n^{N,k}} - \overrightarrow{v_n^N};\vec{v}\right) - \tau n\left(\overrightarrow{u_n^{N,k}},\overrightarrow{u_n^{N,k}},\vec{v}\right) - \tau b\left(\vec{v},p^{N,k}\right) + \tau\langle\vec{f};\vec{v}\rangle.32.a)$$

3) Calculate $div\overrightarrow{u_n^{N,k+1}}$ in the same way as in the stationary case.

4)Calculate the potential $\varphi^{N,k+1}$ of the approximate solution $\overrightarrow{u_s^{N,k+1}}$, by using the application of the FEM (finite element method, described below) on the Laplace equation:

$$\Delta\varphi^{N,k+1} = -div\overrightarrow{u_n^{N,k+1}}, \nabla\varphi^{N,k+1}\big|_{\partial\Omega} = \overrightarrow{v_s^{N,k+1}}. \quad 32.b)$$

5) 5)Calculate the superfluid velocity: $\overrightarrow{u_s^{N,k+1}} = \nabla\varphi^{N,k+1}$.

6) 6)Use the Bernoulli equation to calculate the pressure values in each cell of the mesh:

$$p^{N,k+1} = p^{N,k} + 3\rho_s\frac{\left|\overrightarrow{u_s^{N,k}}\right|_2^2 - \left|\overrightarrow{u_s^{N,k+1}}\right|_2^2}{2}, \quad 32.c)$$

The unsteady numerical algorithm can be obtained from the stationary one. The steps 1), 3)-5) are not changed, because they discretize the same equations as in the stationary non-vorticity case. The step 6) is only adapted by adding a multiplicative constant in the second term on the right side of equation 32.c). It is caused by a different form of the Euler equation in the nonstationary case. The step 2) gives this matrix:

$$U_{n,x}^{N,k}A_xU_{n,x}^{N,k} + U_{n,y}^{N,k}A_yU_{n,x}^{N,k} + U_{n,z}^{N,k}A_zU_{n,x}^{N,k} + \nu D = A \quad 33.a)$$

$$U_{n,x}^{N,k+1} = U_{n,x}^{N,k} - \tau A - \tau P^{N,k} R + \tau B + \tau G. \qquad 33.b)$$

The matrices are defined in the same way as in the stationary case, excluding the matrix A, which is defined by 33.a). The numerical scheme 33.a)-33.b) is usually called *one-step Euler forward method*; for more numerical ODE solution techniques see [4]. As test functions the same functions used in the stationary case, defined as constant functions in time, are chosen. The error estimation can be calculated by using the standard formula for Euler methods for each cell of the mesh:

$$e_k = \frac{e^{k\tau L} - 1}{L} N\tau, \qquad 33.c)$$

where e_k is the error after k steps of length τ, L is the Lipschitz constant of the right hand side and N is the Lipschitz constant of its approximation. The time step was set to $\tau = 10^{-6} s$. Equation 33.c) gives us the error rapid increase as a function of time. The solution is then not enough exact and another method must be used, that is, *the finite volume method*, see below. It does not allow long enough time intervals. It exists at least one positive eigenvalue of the matrix:

$$M = A - P^{N,k} R + B + G, \qquad 33.d)$$

which implies that the solution is not stable. The results of the calculations after 100 iterations are shown below in the Figures:

Figure 5.a: Pressure field in the unsteady case.

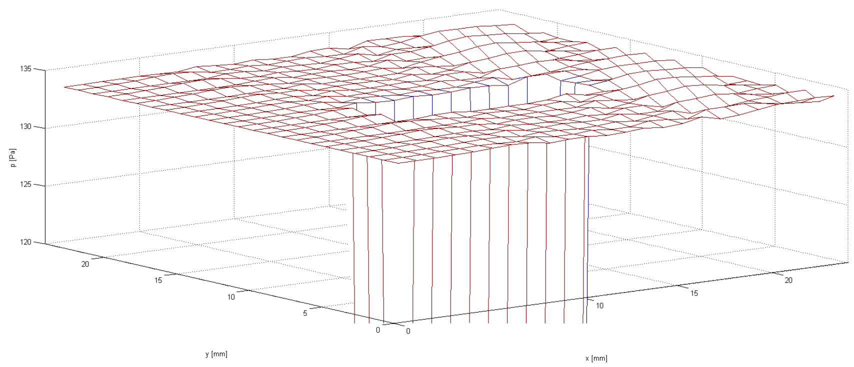


Figure 5.b: Velocity field of the normal liquid in the unsteady case.

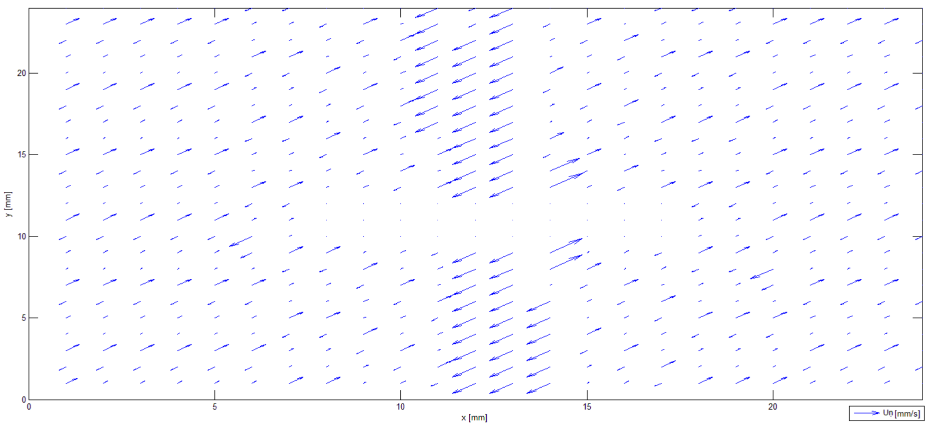
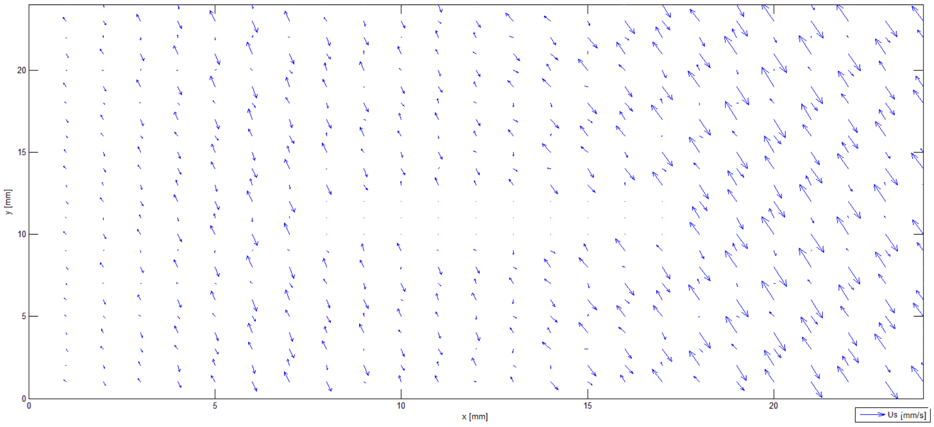


Figure 5.c: Velocity field of the superfluid in the unsteady case.



The pressure of saturated helium vapour was used as the pressure value over helium as in the previous case. We used the previous calculations as an initial condition for this case, which leads to the propagation of the mistake of the Oseen scheme. We chose such small number of iterations because we observed a high nonstability. The velocity fields are in an area of 24 x 24 mm as in the previous case. The longest arrow corresponds to 0.49 m/s in case of the normal liquid velocity and to 258.80 m/s in the case of the velocity of superfluid.

Figure 6.a): Pressure at atmospherical pressure over fluid.

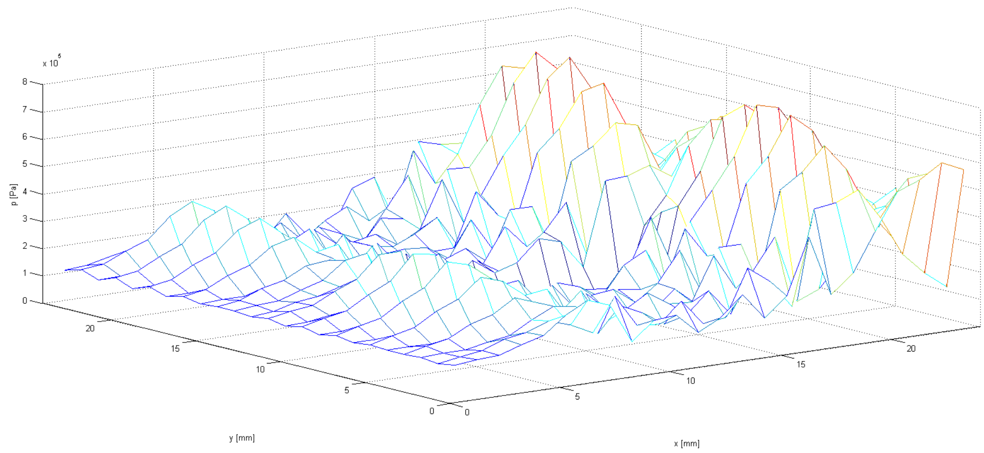


Figure 6.b): normal liquid velocity at atmospherical pressure over fluid.

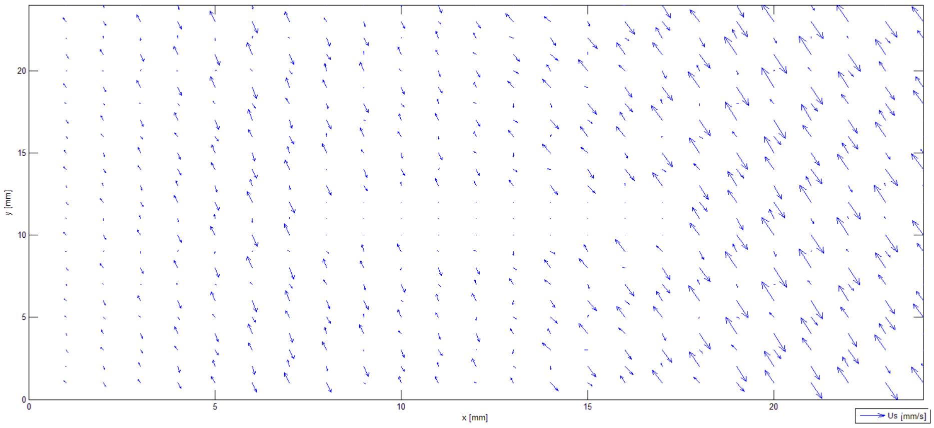
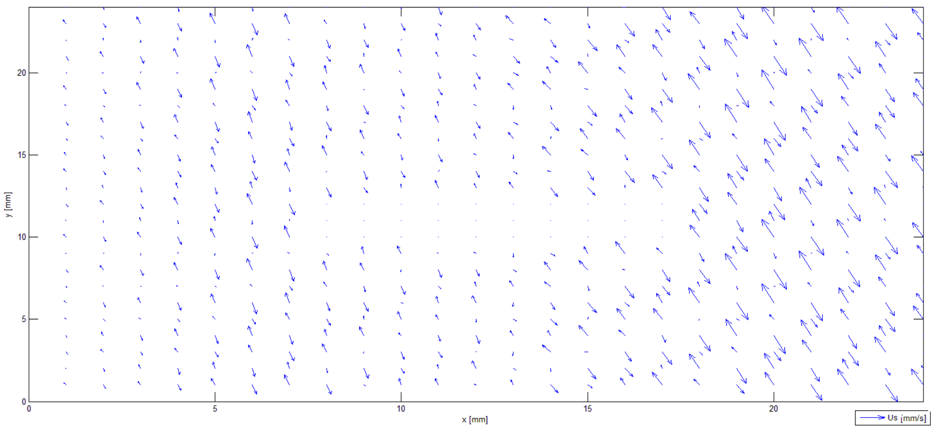


Figure 6.c): the superfluid velocity at atmospherical pressure over fluid.



We can observe that the pressure value reaches 8 Bar, which is nonphysical. More iterations (around 1000) would give such numbers that the computer cannot process. The velocity fields are in an area of 24 x 24 mm as in the previous case. The longest arrow corresponds to 28.73 m/s in the case of the normal liquid velocity and to 37.65 m/s in the case of the velocity of the superfluid. We can observe that the method of lines cannot be used for those flows with such a high Reynolds number.

Numerical model including the nonzero superfluid vorticity

Existence of the weak solution of the stationary problem:

We proved that it exists a solution of the system with given boundary conditions:

$$\frac{\partial \overrightarrow{u_n}}{\partial t} + \overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u_n} + \overrightarrow{f_{ns}} + \vec{g} \qquad 1.a)$$

$$\frac{\partial \overrightarrow{u_s}}{\partial t} + \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p - \overrightarrow{f_{ns}} + \vec{g} \qquad 1.b)$$

$$\vec{f}_{ns} = -\frac{B}{2} \left| rot \vec{u}_s \right| \left(\vec{u}_n - \vec{u}_s \right) \qquad 1.c)$$

$$div \vec{u}_n = 0 \qquad div \vec{u}_s = 0. \qquad 1.d)$$

We used it in the last chapter in this way: we calculated a solution of the stationary problem and used the solution as an initial condition. It is impossible to use it here because of the equivalence of the norms of rotation and divergence, combined with the condition 1.d). We want now to apply the Main existence theorem, which is very general tool for stationary problems. It functions for the normal fluid velocity equation 1.a) without the time-derivative term, but not for equation 1.b), reduced of the time derivative term. We write the stationary system as a first step:

$$\overrightarrow{u_n} \cdot \nabla \overrightarrow{u_n} = -\frac{1}{\rho_n} \nabla p + \nu \Delta \overrightarrow{u_n} + \overrightarrow{f_{ns}} + \vec{g} \qquad 2.a)$$

$$\overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} = -\frac{1}{\rho_s} \nabla p - \overrightarrow{f_{ns}} + \vec{g} \qquad 2.b)$$

$$\vec{f}_{ns} = -\frac{B}{2} \left| rot \vec{u}_s \right| \left(\vec{u}_n - \vec{u}_s \right) \qquad 2.c)$$

$$div \vec{u}_n = 0 \qquad div \vec{u}_s = 0. \qquad 2.d)$$

We can formulate equation 2.b) as a functional, defined as a standard duality on a reflective Lebesgue space:

$$\langle \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} + \frac{1}{\rho_s} \nabla p + \overrightarrow{f_{ns}}; \overrightarrow{u_s} \rangle = \int_\Omega \overrightarrow{u_s} \cdot \nabla \overrightarrow{u_s} \overrightarrow{u_s} dx - \int_\Omega p div \overrightarrow{u_s}\, dx - \frac{B}{2} \int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx \quad 3.a)$$

$$\int_{\partial\Omega} |\overrightarrow{u_s}|^2 (\overrightarrow{u_s} \cdot \vec{n}) dS - \int_\Omega |\overrightarrow{u_s}|^2 div \overrightarrow{u_s}\, dx - \frac{B}{2} \int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx =$$
$$C(Boundary\ conditions) - \frac{B}{2} \int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx = C - \frac{B}{2} \int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx, \quad 3.b)$$

where C is a constant, depending only on the boundary conditions. And if we input this result to the definition of coercivity, we obtain:

$$\lim_{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} \frac{C - \frac{B}{2} \int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx}{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} = -\frac{B}{2} \lim_{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} \frac{\int_\Omega |rot \overrightarrow{u_s}| (\overrightarrow{u_n} - \overrightarrow{u_s}) \overrightarrow{u_s} dx}{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} = 0,$$

3.c)

which holds because of the equivalence of the norms of the divergence and rotation operators. It follows:

$$-\frac{B}{2}\lim_{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} \frac{\int_\Omega |div\overrightarrow{u_s}|(\overrightarrow{u_n}-\overrightarrow{u_s})\overrightarrow{u_s}dx}{\|\overrightarrow{u_s}\|_{W_0^{1,2}(\Omega)}} = 0, \qquad 3.d)$$

which implies 3.c). So, there is no chance for the Main existence theorem. It does not say that there exists no stationary solution, but only that we would have to use another existence theorem or technique.

Solver algorithm

We assume that the vorticity will be generated during time. So, we will use both boundary conditions for both cases and the assumption if the vorticity of the superfluid is zero is to be decided by the user. We will solve it using an *explicite finite volume method (FVM):*

We divide the domain into cells and assume constant values of velocity fields and pressure in the cells, denoted as D_{ij}, which fulfill the condition $\overline{\Omega} = \bigcup D_{ij}$. We call D_{ij} *finite volumes.* The system of finite volumes is called *finite volume mesh*, if they are polygons. Two finite volumes are either disjoint or their intersection is created by a common boundary. The boundaries of two neighbouring finite volumes are called *faces.* We will use the same mesh of squares as in the previous case and an equidistant time step. We assume a 2D problem but the program can calculate 3D problems too (be careful that it requires a powerful computer).

The derivation of the general finite volume method is written for example in [6], page 55. The approximate solution will be searched as a constant on a finite volume, as an integral middle value of the velocity fields or pressure. We write directly the result for equation 1.a)-1.d):

$$\overrightarrow{w^{N,k}} := \begin{pmatrix} \overrightarrow{u_n^{N,k}} \\ \overrightarrow{u_s^{N,k}} \end{pmatrix};\ U := \frac{1}{2}\begin{pmatrix} \overrightarrow{u_n^{N,k}} \otimes \overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n} I & 0 \\ 0 & \overrightarrow{u_s^{N,k}} \otimes \overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s} I \end{pmatrix};\ T := \begin{pmatrix} \nu \nabla \overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N) \nabla \overrightarrow{u_s^{N,k}} \end{pmatrix} \qquad 4.a)$$

$$\overrightarrow{w_{D_{ij}}^{N,k+1}} = \overrightarrow{w_{D_{ij}}^{N,k}} + \frac{1}{|D_{ij}|}\int_{t_k}^{t_{k+1}} \int_{\partial D_{ij}} (T-U)\vec{n}\, dS\, dt + \tau \begin{pmatrix} \vec{g} \\ \vec{g} \end{pmatrix} - \frac{B}{2|D_{ij}|}\int_{t_k}^{t_{k+1}} \int_{D_{ij}} \left|\overrightarrow{\Omega^k}\right| \begin{pmatrix} \overrightarrow{u_n^{N,k}} - \overrightarrow{u_s^{N,k}} \\ \overrightarrow{u_s^{N,k}} - \overrightarrow{u_n^{N,k}} \end{pmatrix} dV dt, \qquad 4.b)$$

where $\vec{n} := (\vec{\nu}; \vec{\nu})^T$, if $\vec{\nu}$ is a standard normal vector of the surface of the finite volume and $\overrightarrow{\Omega^k} = rot\overrightarrow{u_s^{N,k}}$. We assume the approximation of all functions as piecewise constant, on each finite volume equal to one value. As *nonviscid flux* we call the matrix U and as *viscid flux* we call the matrix T. Equation 4.b) is then rewritten in a discretized form:

$$\overrightarrow{w_{D_{ij}}^{N,k+1}} = \overrightarrow{w_{D_{ij}}^{N,k}} + \frac{\tau}{|D_{ij}|}\int_{\partial D_{ij}} (T-U)\vec{n}\, dS + \tau \begin{pmatrix} \vec{g} \\ \vec{g} \end{pmatrix} - \frac{B}{2|D_{ij}|}\tau \int_{D_{ij}} \left|\overrightarrow{\Omega^k}\right| \begin{pmatrix} \overrightarrow{u_n^{N,k}} - \overrightarrow{u_s^{N,k}} \\ \overrightarrow{u_s^{N,k}} - \overrightarrow{u_n^{N,k}} \end{pmatrix} dV. \qquad 4.c)$$

Now we evaluate the integrals. We start with evaluating the first one. We approximate the differential operators T and $\overrightarrow{\Omega^k}$. $|D_{ij}|$ indicates the volume of a cell, $|\partial D_{ij}|$ is its surface and D is the length of the edge of a cubic cell in the mesh.

The circulation can be calculated in 2D by using the Stokes formula in this way:

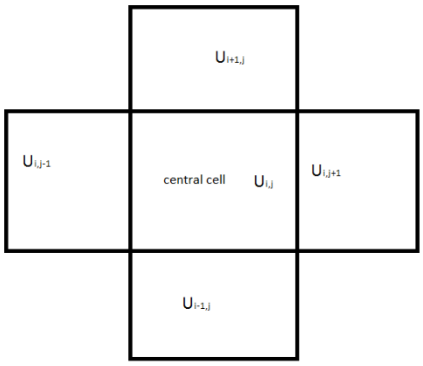
$$\left|\overrightarrow{\Omega^k}\right| = \frac{1}{|\partial D_{ij}|}\int_l \overrightarrow{u_s^{N,k}} dl = \frac{2}{D}\left(u_{s,x}^{N,k} + u_{s,y}^{N,k}\right). \qquad 5)$$

Before we use the given initial condition for this scheme, we must approximate it by a piecewise constant function on the finite volumes, that is:

$$\overrightarrow{w_{D_{ij}}^{N,0}} := \frac{1}{|D_{ij}|}\begin{pmatrix} \int_{D_{ij}} \overrightarrow{u_n^{N,k}}\, dx \\ \int_{D_{ij}} \overrightarrow{u_s^{N,k}} dx \end{pmatrix}. \qquad 6)$$

We assume initially both velocity fields equal to zero. The vorticity and the boundary values of velocity change with time (the same boundary conditions as in the previous two cases without vorticity). To calculate the values of T and U at the time iteration k+1 we will need not only the values of the velocity fields and of the pressure in the central cell but also those in the neighbouring cells, see *Figure 1*:

Figure 1:



The viscid flux can be approximated by using a curve integral along the line connecting the centers of two opposite faces of a cube of the mesh:

$$\int_l \; T\, d\vec{l} = \int_l \begin{pmatrix} \nu\nabla\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\nabla\overrightarrow{u_s^{N,k}} \end{pmatrix} d\vec{l} = \begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(A) - \begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(B), \qquad 7)$$

where A and B are two edges of a cell and the integration is calculated for each component. We denote this matrix F. The direction of the integration curve corresponds to the derivative variable in the velocity gradient. So we obtain this approximation:

$$T_{D_{ij}}\vec{n} = \frac{1}{D}\left[\begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(A) - \begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(B)\right]\vec{n} = \frac{1}{D}F\vec{n}. \qquad 8)$$

To approximate the nonviscid flux we must evaluate the integral over the surface of a cell. We must evaluate an integral of a piecewise constant function, whose discontinuity is exactly on the integration set. We solve it using an adjustable weight parameter λ, which is given by the user before the calculations. We obtain for the surface integral of the nonviscid flux this approximation:

$\int_{\partial D_{ij}} U_{ij}\vec{n}dS :=$

$$\int_{\partial D_{ij}} \frac{\lambda}{2}\begin{pmatrix} \overrightarrow{u_n^{N,k}}\otimes\overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n}I & 0 \\ 0 & \overrightarrow{u_s^{N,k}}\otimes\overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s}I \end{pmatrix}_{ij} - \frac{1-\lambda}{8}\left[\begin{pmatrix} \overrightarrow{u_n^{N,k}}\otimes\overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n}I & 0 \\ 0 & \overrightarrow{u_s^{N,k}}\otimes\overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s}I \end{pmatrix}_{i-1,j} + \begin{pmatrix} \overrightarrow{u_n^{N,k}}\otimes\overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n}I & 0 \\ 0 & \overrightarrow{u_s^{N,k}}\otimes\overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s}I \end{pmatrix}_{i,j+1} + \begin{pmatrix} \overrightarrow{u_n^{N,k}}\otimes\overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n}I & 0 \\ 0 & \overrightarrow{u_s^{N,k}}\otimes\overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s}I \end{pmatrix}_{i+1,j} + \begin{pmatrix} \overrightarrow{u_n^{N,k}}\otimes\overrightarrow{u_n^{N,k}} + \frac{p^k}{\rho_n}I & 0 \\ 0 & \overrightarrow{u_s^{N,k}}\otimes\overrightarrow{u_s^{N,k}} + \frac{p^k}{\rho_s}I \end{pmatrix}_{i,j-1}\right] dS, \quad 9)$$

where i,j are the positions of the cell. The first term is given by the value of nonviscid flux in the previous time step and during the integration process vanishes because we integrate a constant function over the opposite faces of the same measure, whose normal vectors are oriented in opposite directions. We obtain this approximation of U:

$$\int_{\partial D_{ij}} U\vec{n}dS = D^2\frac{1-\lambda}{8}\Sigma\vec{n}, \qquad 9)$$

where Σ signs a sum of U in the neighbouring cells at the previous time iteration around the central one. The approximation of the first integral is then:

$$\frac{\tau}{|D_{ij}|}\int_{\partial D_{ij}} (T-U)\vec{n}\,dS = D\left[\begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(A) - \begin{pmatrix} \nu\overrightarrow{u_n^{N,k}} & 0 \\ 0 & \nu(N)\overrightarrow{u_s^{N,k}} \end{pmatrix}(B)\right]\vec{n} - D^2\frac{1-\lambda}{8}\Sigma\vec{n}. \qquad 10)$$

We already approximated the vorticity of the superfluid. As the term including the mutual friction is a volume integral, we need only the values of both velocity fields in the central cell. We calculate it directly:

$$\frac{B}{2|D_i|}\tau\int_{D_{ij}} \left|\overrightarrow{\Omega^k}\right| \begin{pmatrix} \overrightarrow{u_n^{N,k}} - \overrightarrow{u_s^{N,k}} \\ \overrightarrow{u_s^{N,k}} - \overrightarrow{u_n^{N,k}} \end{pmatrix} dV = \frac{B}{2}\tau\left|\overrightarrow{\Omega^k}\right| \begin{pmatrix} \overrightarrow{u_n^{N,k}} - \overrightarrow{u_s^{N,k}} \\ \overrightarrow{u_s^{N,k}} - \overrightarrow{u_n^{N,k}} \end{pmatrix}. \qquad 11)$$

So, finally we obtain the numerical scheme:

$$\overrightarrow{w_{D_i}^{N,k+1}} = \overrightarrow{w_{D_i}^{N,k}} + \frac{6\tau}{D}\left[F - \frac{1-\lambda}{8}\Sigma\right]\vec{n} + \tau\begin{pmatrix}\vec{g} \\ \vec{g}\end{pmatrix} - \frac{B}{2}\tau\left|\overrightarrow{\Omega^k}\right| \begin{pmatrix} \overrightarrow{u_n^{N,k}} - \overrightarrow{u_s^{N,k}} \\ \overrightarrow{u_s^{N,k}} - \overrightarrow{u_n^{N,k}} \end{pmatrix}. \qquad 12)$$

The initial condition on the pressure is equal to the pressure of saturated vapour, because we neglect a depth of the cryostat. From equation 12) we cannot obtain the pressure values but we can assume that each cell is so small that there it holds the *nonstationary Bernoulli equation* from the previous chapters*:*

$$p^{N,k+1} = p^{N,k} + 3\rho_n\frac{\left|\overrightarrow{u_n^{N,k}}\right|_2^2 - \left|\overrightarrow{u_n^{N,k+1}}\right|_2^2}{2}. \qquad 13)$$

If we assume a flow in such a small scale like a cell in the mesh, we obtain a lower Reynolds number than in the global case; so, we can assume a flow in one cell as laminar; it follows that equation 13) holds.

We say that the finite volume scheme is stable, if there exists such a constant C that:

$$\left\|\overrightarrow{w_{D_{ij}}^{N,0}}\right\|_{W_0^{1,2}(\Omega)} C \geq \left\|\overrightarrow{w_{D_{ij}}^{N,k}}\right\|_{W_0^{1,2}(\Omega)}. \qquad 14.a)$$

The initial condition for the velocity fields is equal to zero and the rectangle starts moving from rest.

The necessary stability (CFL) condition must fulfill this inequality:

$$\frac{\tau c|\partial D_{ij}|}{|D_{ij}|} \leq 1. \qquad 14.b)$$

where c denotes the sound propagation speed, which can be calculated using the thermodynamics and is written in [9]. For more information about the CFL condition, see [6], page 58. Here we considered the same temperature as in zero vorticity case, so the physical values are the same as in the previous chapters. We choose a time step to fulfill the CFL condition 13.b) to avoid a blow up in time.

For the results of the numerical simulation after 100 iterations, see *Figures 3-a) -3-c).* We assumed as an initial pressure 133.3 Pa everywhere. We started from the initial condition, shown in *Figure 2-a)* and 2-b). The length of time interval is set to 0.0001s. The velocity corresponding to the longest arrow is 1 mm/s in all images below*.*

Figure 2-a: The normal liquid initial condition.

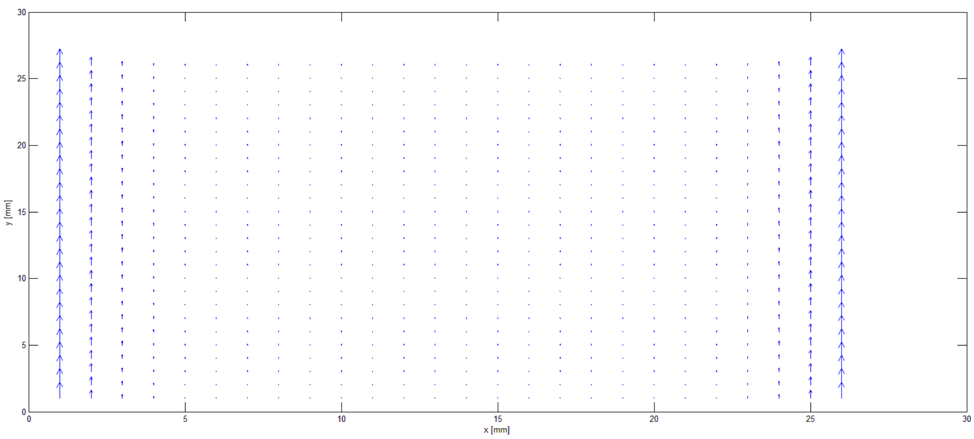


Figure 2-b: The superfluid initial condition.

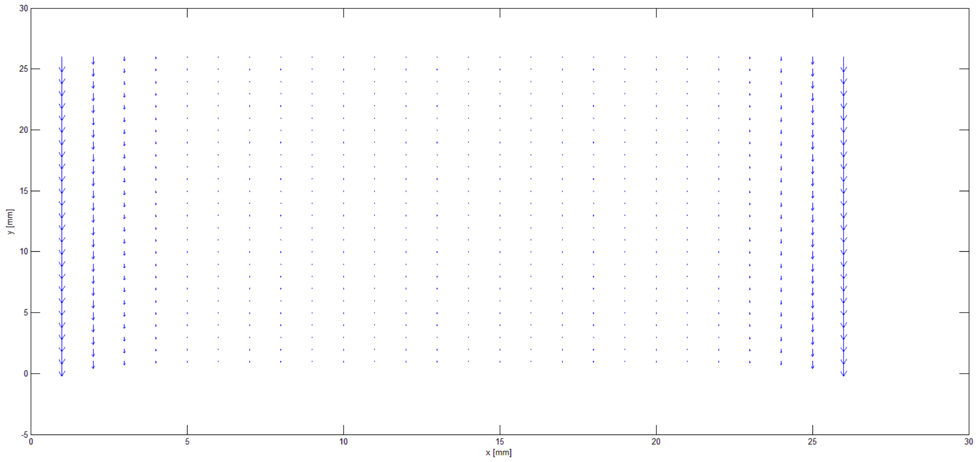


Figure 3-a: The normal liquid after 1000 steps.

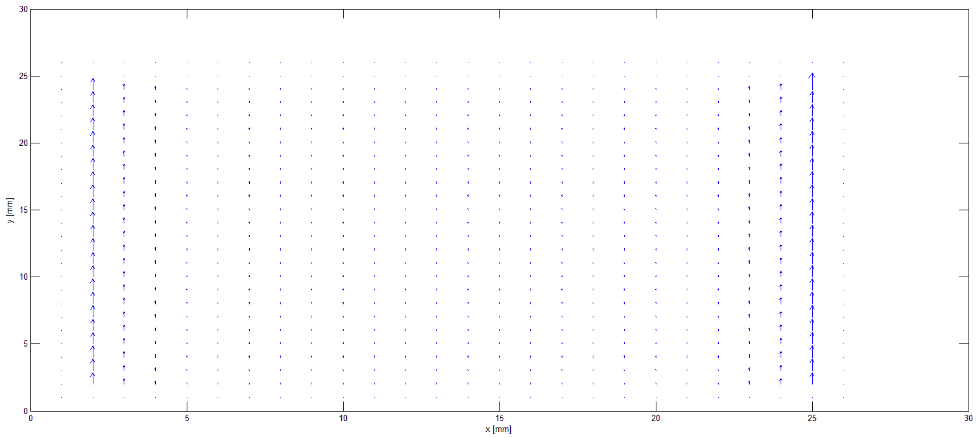


Figure 3-b: The superfluid after 1000 steps.

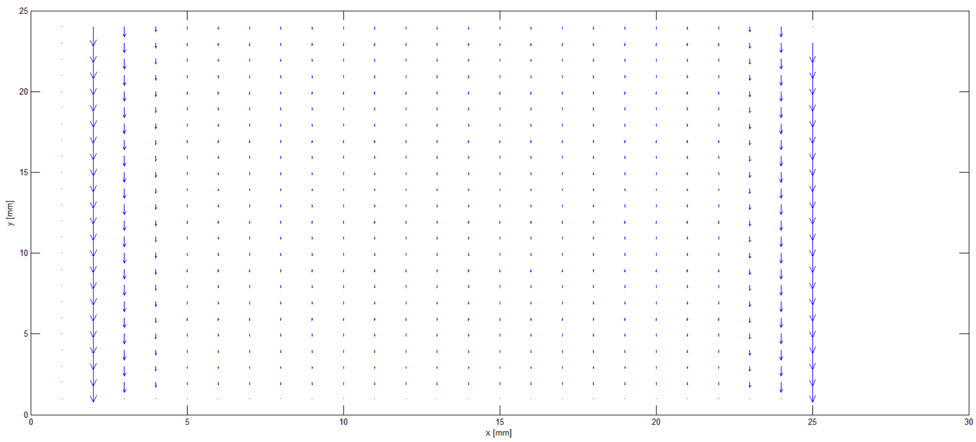
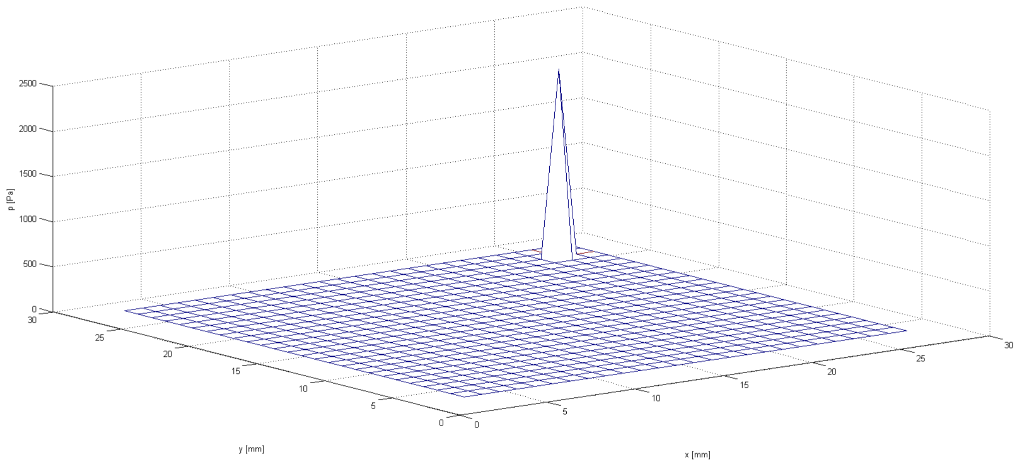


Figure 3-c: Pressure after 1000 steps



The solution obtained by using the finite volume method is better then the one obtained for higher Reynolds numbers by using the finite element method, because it is symmetric. We can, however, say that it is not good enough because there is too slow velocity propagation for the x component, if it is equal to zero for zero time. It may be that the pressure remained constant. For longer times (around 10000 iterations) the solution does not converge.We can observe a large numerical error in the pressure field, which can be caused by an initial nonstability.

Conclusion:

The work aim is to derive (i) the existence theory for the weak solution of a system of equations based on the Landau model of superfluid helium 4 and (ii) appropriate numerical schemes to solve these equations. We used succesfully an analogous way to prove the existence of the weak solution of the Landau model, as it is customary in the case of the classic Navier-Stokes equations, and proved some of its properties.

We showed that it is suitable to use potential methods if the Landau model does not include quantized vortices. We derived the Bernoulli equation for the stationary and nonstationary cases. We also derived apriori estimates of the Landau model, including quantized vortices, and proved the existence of the weak solution, using Galerkin approximations. Moreover, we demonstrated the convergence of the weak solution to initial conditions. We derived the consistence conditions for the studied systems of equations.

We designed numerical schemes for three cases, which solve the Landau model without quantized vortices (stationary and unsteady) and including them (unsteady). We wrote three scripts in Mathlab to demonstrate the solution behavior, based on the designed schemes. We observed that the finite element method including the Oseen scheme does not work for too low pressure and for too high Reynolds numbers. We demonstrated that the method of lines is also not suitable, because of its nonstability. We showed that it is necessary to use the finite volume method to solve the Landau model including the vortices.

For the future we would like to focus on the uniqueness of the Landau model for the 2D case. We would like also to create a compact solver included into some existing solver platform, such as OpenFOAM, for the zero superfluid vorticity and for the system including quantum vortices. It would also be suitable if it includes some models of turbulence, customary used in CFD.

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