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Applications of invariant operators in real parabolic geometries

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Abstract: In Riemannian geometry, the fundamental fact is that there exists a unique torsion-free connection (called the Levi-Civita connection) compatible with the Riemannian metric g, i.e. having the property $\nabla g = 0$. In projective geometry, the class of covariant derivatives defining the geometry is fixed and all these covariant derivatives have the same class of (non-parametrized) geodesics. Old (and non-trivial) problem is to find whether these curves are geodesics of a (pseudo-)Riemannian metric. Such projective structures are called metrizable. Surprisingly enough, U. Dini and R. Liuoville found in 19th century that the metrizability problem leads to a system of linear PDE's. In the last years, there were several papers dealing with these problems. The projective geometry is a representative example of the so called parabolic geometries (for full description, see the recent monograph by A. Čap and J. Slovák). It was realized recently that the corresponding linear metrizability operator is a special example of the so called first BGG operator. The flat model of projective geometry is the (real) projective space.

In this more general context, the metrizability problem for (pseudo-)Riemannian geometries is naturally generalized to the sub-Riemannian situation. In the recent preprint, D.Calderbank, J. Slovák and V. Souček are discussing the classification of (real) *irreducible* parabolic geometries for which the linearisation method can be applied. A part of the classification is the case of complex simple Lie algebras considered as real Lie algebras.

The aim of this thesis is to formulate the linearisation method in a full generality and to classify completely the cases of complex simple Lie algebras where the linearisation method is applicable. In Sect. 2, there is a summary of description of invariant differential operators on parabolic geometries and comments how to use it for real cases. A general discussion of the linearisation method is contained in Sect.3. The classification result for the case of complex simple Lie algebras is presented in Sect.5. Some examples of explicit solutions are contained in Sect. 6. There are several Appendices summarizing results used in the thesis.

Keywords: metrizability of covariant derivatives, the first BGG operator, sub-Riemannian geometry

Contents

1. Introduction

1.1 Motivation

In Riemannian geometry, the fundamental fact is that there exists a unique torsion-free connection ∇ (called the Levi-Civita connection) compatible with the Riemannian metric g, i.e., having the property $\nabla g = 0$. Projective geometry is an old and classical subject. In its modern form, projective structure on a smooth manifold is an equivalence class $[\nabla]$ of torsion-free covariant derivatives having the same geodesic curves. It can be shown that two covariant derivatives ∇ and $\tilde{\nabla}$ are in the same equivalence class if and only if $\tilde{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b + \delta_a^b \Upsilon_c X^c$ for some one-form Υ and for all vector fields X, [14].

Old (and non-trivial) problem is to find whether these curves are geodesics of Levi-Civita connection of (pseudo-)Riemannian metric. Such a projective structures are called metrizable. Surprisingly enough, U. Dini ([13]) and R. Liouville ([20]) found in 19th century that the metrizability problem leads to a system of linear PDE's. In the last years, there were (among others) papers [15], [24], [21] which dealt with these problems. Similar problems in dimension two were considered in [5].

It is well known that projective geometry is a representative example of the so called parabolic geometries (for full description, see [10]). It was realized recently that the corresponding linear metrizability operator is a special example of the so called first BGG operator. The flat model of projective geometry is the (real) projective space. A broader class of projective geometries (having as a model complex, quaternionic projective spaces, the Cayley octonian plane and the conformal sphere) were studied recently by D. Calderbank and G. Frost. It contained a systematic description of this class of parabolic geometries and the system of linear PDE's describing the non-degenerate solutions of the metrizability problem.

The method of linearisation of the metrizability problem used by D. Calderbank and G. Frost is not limited to the class of projective geometries. Here the class of the preferred covariant derivatives giving the projective structure is replaced by the class of Weyl covariant derivatives canonically associated with given parabolic geometry. In this more general context, the metrizability problem for parabolic geometries is naturally generalized to the sub-Riemannian situation in the sense of metrizability of subbundles. This is closely connected with the fact that invariant first order differential operators on parabolic geometries factorize through restricted jets (see chapter two for a more detailed discussion). Such a generalization of the problem has a very good potential for applications to optimal control theory (see [1], [2]).

In the recent preprint, D. Calderbank, J. Slovák and V. Souček are discussing the classification of (real) irreducible parabolic geometries for which the linearisation method can be applied. A part of the classification is the case of complex simple Lie algebras considered as real Lie algebras. The first example of such parabolic geometry treated with full details is the case of c-projective structure [8].

The aim of this thesis is to formulate the linearisation method in a full generality and to classify completely the cases of complex simple Lie algebras where the linearisation method is applicable. A general discussion of the linearisation method is contained in chapter three. The classification result for the case of complex simple Lie algebras is presented in chapter five.

1.2 Formulation of the problem

Now we will formulate the main problem of this thesis. Let $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ be a graded Lie algebra of G and let $(\mathcal{G} \to M)$ be a P-principal bundle over M, where P is a parabolic subgroup in G and let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be the Cartan connection. Let E be the bundle which is associated to a P-representation $\mathbb{E}$ and let us write $\mathcal{H}^*$ for the bundle $\mathcal{G} \times_P \mathfrak{g}_1$. We are looking for a sections $\rho \in \Gamma(E)$ such that there exists a Weyl covariant derivative ∇ for which $\nabla|_{\mathcal{H}}\rho = 0$, where $\mathcal{H} = \mathcal{G} \times_P \mathfrak{g}_{-1}$ by definition and $\nabla|_{\mathcal{H}}$ is the covariant derivative induced by a Weyl covariant derivative which acts in directions of $\mathcal{H}$ or more formally $\nabla|_{\mathcal{H}} : \Gamma(E) \to \Gamma(\mathcal{H}^* \otimes E)$ instead of $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$.

1.3 Method of finding a solution

In this chapter we describe the method we use to find a covariantly constant tensor fields.

Let us suppose E and F are vector bundles. We are looking for a Weyl covariant derivative ∇^W or equivalently a Weyl structure such that there are $\eta \in \Gamma(E)$ satisfying $\nabla^W|_{\mathcal{H}}\eta = 0$. For this purpose we use invariant first order differential operators. These operators are of the form $\mathcal{D} := \pi_F \circ \nabla^W : \Gamma(E) \to \Gamma(F)$ and do not depend on a choice of Weyl covariant derivative, where $E \otimes \mathcal{H}^* = F \oplus F'$. To say $\eta \in Ker\,\mathcal{D}$, which means $\pi_F \circ \nabla\eta = 0$, is equivalent to existence an element $X^\nabla \in \Gamma(F')$ such that $\nabla|_{\mathcal{H}}\eta = \iota(X^\nabla)$, where $\iota : \Gamma(F') \to \Gamma(E \otimes \mathcal{H}^*)$ is the inclusion. Our aim is to choose a covariant derivative ∇ in such a way that $\nabla|_{\mathcal{H}}\eta = 0$. From appendix C we know the effect of change of Weyl strucure on covariant derivative. Two covariant derivatives $\nabla, \tilde{\nabla}$ are related as $\tilde{\nabla}_Z\eta = \nabla_Z\eta + [\![\Upsilon, Z]\!]\eta$, where $\Upsilon \in \mathcal{H}^*$ and $Z \in \mathcal{H}$. This bracket $[\![,]\!]$ is linear in second input hence we can define map $F_\eta : \mathcal{H}^* \to E \otimes \mathcal{H}^*$ as $\Upsilon \mapsto [\![\Upsilon, \cdot]\!]\eta$. If we suppose $Im(F_\eta) \supset F'$ then there exists a Weyl covariant derivative ∇ such that $\nabla|_{\mathcal{H}}\eta = 0$. If we suppose F_η is linear, which is natural, we get necessary condition $dim(\mathcal{H}^*) \geq dim(F')$. In this procedure we are looking for a big F, where the best choice is the Cartan component.

Example:

Now suppose $E \subset \mathcal{G} \times_P \odot^2\mathfrak{g}_1$ which corresponds to sub-Riemannian metrics. By an easy example we show that previous procedure will fail but there is a way how to use this procedure to obtain good results in the same example of sub-Riemannian metrics. We will compute with representations only and we consider associated natural bundles and their sections. Consider complex projective geometry and let $\mathfrak{g}$ has rank n. In this case $\mathfrak{g}_0^{ss}$ acts on $\mathfrak{g}_1$ with the highest weight ω_{n-1}. The space $\odot^2\mathfrak{g}_1$ consist of one component only. This component has the highest weight $2\omega_{n-1}$, hence we put $E = E_{2\omega_{n-1}}$. Now we look at decomposition of $E \otimes \mathcal{H}^*$. On the level of representations it decomposes into Cartan component and the representation which is given by the highest weight $\omega_{n-2} + \omega_{n-1}$. This representation is bigger then $\mathfrak{g}_1$ hence we can not use previous procedure.

An important idea is to consider nondegenerated metrics and to go to the inverse metrics. Let us consider an inverse metric. If κ is a metric, the inverse metric η is defined by the equation $\kappa_{ab}\eta^{bc} = \delta_a^c$. The metric is nondegenerated if and only if the inverse metric is nondegenerated. Therefore we can transform the problem of finding covariantly constant metric to the problem of finding covariantly constant inverse metric in nondegenerated case. So we are interested in bundles $E \subset \mathcal{G} \times_P \odot^2 \mathfrak{g}_{-1}$. Let us compute the example with complex projective geometry. The $\mathfrak{g}_0^{ss}$ acts on $\mathfrak{g}_{-1}$ by the highest weight ω_1 and on the one-piece symmetric part of $\otimes^2 \mathfrak{g}_{-1}$ by $2\omega_1$. Decomposition of $E \otimes \mathcal{H}^*$ on the level of representations is Cartan component and representation with the highest weight ω_1 which is exactly the highest weight for $\mathfrak{g}_{-1}$. Let us note the Cartan component give rise to bundle F and the other representation give rise to F'. In this situation we can use the contraction (raising indices by the inverse metric) $E \otimes \mathcal{H}^* \to \mathcal{H}$ given by $\eta^{ab}\Upsilon_b$ to get isomorphism between $\mathcal{H}^*$ and $\mathcal{H} = F'$. Therefore $Im(F_\eta) = F'$.

There is one complication with the last paragraph. Every bundle E is given by a $\mathfrak{g}_0$-representation but we can construct an invariant first order differential operator for special action of $\mathfrak{z}(\mathfrak{g}_0)$ only. We need slightly deform the bundle E in the following sense. We multiply E by a line bundle L in such way that final bundle $E \otimes L$ will have correct action of center $\mathfrak{z}(\mathfrak{g}_0)$. In this case we constructed covariantly constant section η of $E \otimes L$ which is not an inverse metric. So we need to find covariantly constant section σ of the line bundle L^{-1} such that product η with σ is the inverse metric. The bundle L^{-1} can be constructed by using wedge product on the bundle $E \otimes L$ until we get a line bundle $\hat{L}$ and then by manipulating with $\hat{L}$. Sections of this bundle have to be covariantly constant, by Leibniz rule.

1.4 Contents

In the second chapter, we recall construction of an invariant first order differential operators on parabolic geometries. We follow [25]. First section deals with characterization of an invariant operators in complex case. In the second section, we are investigating an invariant operators on real geometries.

Chapter three describes the algebraic linearisation condition (ALC) mainly. This chapter contains an important lemma which deals with structure of $\mathfrak{g}_0$-representations $\mathfrak{g}_{\pm 1}$.

In the chapter four, we prove theorem on classification of parabolic geometries of (G, P)-type for irreducible $\mathfrak{g}_{-1}$-part, where the Lie algebra $\mathfrak{g}$ of G is of A-type. This theorem is proved with many details and it serves as leading example through classification of all parabolic geometries with irreducible $\mathfrak{g}_{-1}$-part which can be found in preprint of D. Calderbank, J. Slovák and V. Souček.

The fifth chapter is the core of this thesis. It contains new results. We classify all parabolic geometries of (G, P)-type with irreducible $\mathfrak{g}_{-1}$-part which satisfy the ALC, where the Lie algebra $\mathfrak{g}$ of G is complex Lie algebra which is considered as the real Lie algebra.

In the sixth chapter, we compute four examples of metric bundles, solutions of the first BGG equations and covariantly constant metrics.

The last chapter is dealing with generalization of the ALC to the case of parabolic geometries with reducible $\mathfrak{g}_{-1}$-part. First, we describe a way how the ALC can be generalized and then we illustrate this method on a few examples.

2. Invariant differential operators of first order

In this chapter we recall construction of an invariant differential operators on parabolic geometries. We will follow [25].

2.1 Complex case

In complex situation, standard parabolic subalgebras are in bijective correspondence with $|k|$-graded algebras which are in bijective correspondence with subsets of nodes in Dynkin diagram. We indicate these subsets as crosses in diagram. By a complex geometry with one cross or one-cross complex geometry, we always mean a complex parabolic geometry $(\mathcal{G} \to M, \omega)$ of a type (G, P) for which $\mathfrak{g}$ is the Lie algebra of G and P corresponds to a cross. By a real one-cross geometry we mean a real parabolic geometry such that in the complexification it is one-cross complex geometry. When it can not cause confusion we omit prefix real/complex. We use similar notions for geometries which arise from k-crosses in Dynkin diagram and its real versions.

Definition 1. *Let V_λ be a P-representation with the highest weight λ. We define the $\mathfrak{p}$-module J^1V_λ as $V_\lambda \oplus (\mathfrak{g}_-^* \otimes V_\lambda)$ with the action given by*

$$Z \cdot (v, \phi) = (\lambda(Z)v, \lambda(Z) \circ \phi - \phi \circ ad_-(Z) + \lambda(ad_{\mathfrak{p}}(Z)(\cdot))v) \tag{2.1}$$

Now consider an arbitrary principal P-bundle $\mathcal{G}$ with Cartan connection ω. The P-module V_λ gives rise to the associated bundle $\mathcal{V}(\lambda) := \mathcal{G} \times_P V_\lambda$ and its first jet prolongation $J^1\mathcal{V}(\lambda)$.

Theorem 1. *The invariant differentiation ∇^ω defines the mapping*

$$\iota : C^\infty(\mathcal{G}, V_\lambda)^P \to C^\infty(\mathcal{G}, J^1V_\lambda)^P, \quad \iota(s)(u) = (s(u), (X \mapsto \nabla^\omega s(u)(X))),$$

which yields diffeomorphism $J^1\mathcal{V}(\lambda) \simeq \mathcal{G} \times_P J^1V_\lambda$.

In J^1V_λ there is $\mathfrak{p}$-invariant subspace $\{0\} \oplus (\mathfrak{p}_+^2 \otimes V_\lambda)$, where $\mathfrak{p}_+^2 = [\mathfrak{p}_+, \mathfrak{p}_+]$. We define the restricted jets $J^1_{\mathcal{R}}V_\lambda$ as factor $\mathfrak{p}$-module $J^1V_\lambda/(\{0\} \oplus (\mathfrak{p}_+^2 \otimes V_\lambda))$. These jets are describing partial derivatives in some directions only.

According to appendix C, we can use a Weyl structure to get a linear connection on TM. A choice of the Weyl structure is not canonical.

Definition 2. *A differential operator which does not depend on a choice of Weyl structure will be called an invariant differential operator.*

Lemma 2. *Let E and F be irreducible P-modules. Then a G_0-module homomorphism $\Psi : J^1E \to F$ is a P-module homomorphism if and only if Ψ factors through $J^1_{\mathcal{R}}E$ and for all $Z \in \mathfrak{g}_1$*

$$\Psi(\sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha] \cdot v_0) = 0,$$

where η^α, ξ_α is dual basis of $\mathfrak{g}_{\pm 1}$.

Consider the endomorphism $\Phi(Z\otimes v) := \sum_\alpha \eta^\alpha \otimes [Z,\xi_\alpha]\cdot v_0$ on $\mathfrak{g}_1\otimes E$. By the lemma above we are interested in image of Φ.

Lemma 3. *Let E be an irreducible complex representation of $\mathfrak{g}_0$ characterized by $\lambda \in \mathfrak{h}^*$ and let $\mathfrak{g}_1 = \bigoplus_j \mathfrak{g}_1^j$ be a decomposition of $\mathfrak{g}_1$ into irreducible $\mathfrak{g}_0$-modules. Highest weights of individual components $\mathfrak{g}_1^j$ will be denoted by α_j. Suppose that $\mathfrak{g}_1\otimes E = \bigoplus_j \bigoplus_{\mu_i(j)} E^j_{\mu_i(j)}$ be a decomposition of the product into irreducible $\mathfrak{g}_0$-modules and π_{λ,μ_i} be the corresponding projections. Let ρ_0 be the half sum of positive roots for $\mathfrak{g}_0^{ss}$.*

Then for all $v\in E$,

$$\Phi(Z\otimes v)(X) = [Z,X]\cdot v = \sum_j\sum_{\mu_i} c_{\lambda,\mu_i,\alpha_j}\pi_{\lambda,\mu_i}(Z\otimes v)(X),$$

where

$$c_{\lambda,\mu_i,\alpha_j} = \frac{1}{2}[(\mu_i(j),\mu_i(j)+2\rho_0) - (\lambda,\lambda+2\rho_0) - (\alpha_j,\alpha_j+2\rho_0)].$$

Now we are able to characterize first order invariant differential operators.

Theorem 4. *The operator $\mathcal{D}_{j,\mu_i} = \pi_{\lambda,\mu_i}\circ\nabla^\omega$ is an invariant differential operator of first order if and only if $c_{\lambda,\mu_i,\alpha_j} = 0$. Moreover, all first order invariant operators acting on sections of E are obtained in such way.*

The condition $c_{\lambda,\mu,\alpha} = 0$ can be slightly reformulated. Every weight θ on $\mathfrak{g}_0$ decomposes into weight θ' on $\mathfrak{g}_0^{ss}$ and weight θ^0 on $\mathfrak{z}(\mathfrak{g}_0)$. If we split definition of $c_{\lambda,\mu,\alpha}$ into weights on semisimple part and on commutative part we get $0 = c_{\lambda,\mu,\alpha} = c_{\lambda',\mu'\alpha'} + (\lambda^0,\alpha^0)$ by orthogonality of Killing form on a reductive algebra. Therefore, if we have k-cross geometry with $dim\,\mathfrak{z}(\mathfrak{g}_0) = k$, we can construct k invariant first order differential operators. There are k linear equations on a central weights.

Remark. Let us note one more remark about number of invariant first order differential operators. Since simple roots α_i of $\mathfrak{g}$ form a basis of Cartan subalgebra in $\mathfrak{g}_0$, their parts α_i^0 form a basis on $\mathfrak{z}(\mathfrak{g}_0)$. Therefore the linear equations $0 = c_{\lambda',\mu'\alpha_i'} + (\lambda^0,\alpha_i^0)$ have unique solution for λ_0. So, if $B_{\lambda'}$ is $\mathfrak{g}_0^{ss}$-representation and $\mathfrak{g}_1 = \bigoplus^k \mathfrak{g}_1^j$ and let $V_{\mu_i}\subset B\otimes\mathfrak{g}_1^i$ be an irreducible component then there exist unique central weight λ^0 such that there exists a first order invariant differential operator acting from $B_{\lambda'+\lambda^0}$ to sum $\bigoplus^k V_{\mu_i}$.

An invariant differential operator of first order is composition of following maps: $\nabla^\omega : \Gamma(\mathcal{G}\times_P B) \to \Gamma(\mathcal{G}\times_P J^1B)$, $\pi_1 : \Gamma(\mathcal{G}\times_P J^1B) \to \Gamma(\mathcal{G}\times_P(\mathfrak{g}_1\otimes B))$, $\pi_2 : \Gamma(\mathcal{G}\times_P(\mathfrak{g}_1\otimes B)) \to \Gamma(\mathcal{G}\times_P E_\mu)$, where E_μ is an irreducible component in $\mathfrak{g}_1\otimes B$. Shortly, invariant first order differential operators are of the form $\pi_\mu\circ\nabla^\omega : \Gamma(\mathcal{G}\times_P B) \to \Gamma(\mathcal{G}\times_P E_\mu) \subset \Gamma(\mathcal{G}\times_P(\mathfrak{g}_1\otimes B))$.

Now we explain the convention of dual weights. Let V be irreducible representation with the highest weight λ. This representation will be denoted by V_μ, where μ is the highest weight of dual representation V^*. Now consider the highest weight λ as n-tuple $(\lambda_1,\cdots,\lambda_n)$ over Dynkin diagram. In dual convention, the highest weight $\lambda = (\lambda_1,\cdots,\lambda_n)$ will be displayed over Dynkin diagram as n-tuple $(\mu_1, \cdots, \mu_n)$, where $\mu = (\mu_1, \cdots, \mu_n)$ is the highest weight of dual module. As an illustration, let us consider $\mathfrak{g} \simeq sl(n+1, \mathbb{C})$. Let V be the irreducible $\mathfrak{g}$-representation with the highest weight ω_1. In convention of dual weights, we denote this representation as V_{ω_n}.

Lemma 5. *Let $\mathcal{D}$ be the first BGG operator given by a representation B. The operator $\mathcal{D}$ is of first order if and only if the number over crosses in $\mathfrak{g}_0$-representation B are zero.*

Proof. First we will analyse the case of one-cross geometry. We will use convention of dual weights. Denote by λ the highest weight of the $\mathfrak{g}_0$-representation B. Consider first cohomology group which is given by representation above. From the Kostant's version of Bott-Borel-Weil theorem we know how this group looks. Clearly it has only one irreducible component for which we compute the highest weight. In the Hasse diagram $W^{\mathfrak{p}}$ there is the unique element w of height one and $w = s_{\alpha_k}$, where α_k corresponds to a crossed node. Now, $w \cdot \lambda = s_{\alpha_k}(\lambda + \delta) - \delta = \lambda + \delta - (\lambda_k + 1)\alpha_k - \delta = \lambda - (\lambda_k + 1)\alpha_k = \lambda + (\lambda_k + 1)(-\alpha_k)$. We see the weight $w \cdot \lambda$ corresponds to $(\lambda_k + 1)$-th Cartan product of V with $\mathfrak{g}_1$, by the remark above theorem 7 in chapter three. If we want an operator of first order we need $\lambda_k = 0$. Otherwise a values of an operator will be at least in the associated bundle given by the representation $(B \circledcirc \mathfrak{g}_1) \circledcirc \mathfrak{g}_1$ which can not be an invariant subspace in $B \otimes \mathfrak{g}_1$. We see if the first BGG operator has order one it is an invariant first order differential operator with values in Cartan product and vice versa. It works similarly in k-cross geometry. $\square$

The advantage of the first BGG operator is that its values are in the biggest component, the Cartan component. The first BGG equation is the overdetermined system of PDEs. In the case of homogeneous model the dimension of solutions of the first BGG equation is finite and nonzero but in the case of curved geometries it may happen there is only a trivial solution.

Now we explain how to obtain an invariant first order differential operator on vector bundle $\mathcal{G} \times_P V_\lambda$, where V_λ is an irreducible $\mathfrak{g}_0$-module. For simplicity we suppose $\mathfrak{g}_0$ is an reductive algebra with dimension of centre $\mathfrak{z}(\mathfrak{g}_0)$ equal to one. Typically, centre $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ acts nontrivialy in the sense that a number over a cross in Dynkin diagram is nonzero. If we want to get an invariant first order differential operator with values in natural bundle given by representation $V_\lambda \circledcirc \mathfrak{g}_1$, we need such a weight λ which has zero over a cross. This can be ensure by multiplying bundle $\mathcal{G} \times_P V_\lambda$ by a natural line bundle $\mathcal{L}$, which is induced by a representation L such that $\mathfrak{z}(\mathfrak{g}_0)$ acts on L by the opposite number as in V_λ. This procedure works similarly in the real case.

Example: Let $\mathfrak{g} \simeq sl(n+1, \mathbb{C})$ and consider the first node to be crossed. This is complex projective geometry. Let $\lambda = \delta = \sum^n \omega_i$ where ω_i are the fundamental weights. If we want to find an invariant first order differential operator $\mathcal{D} : \Gamma(\mathcal{G} \times_P V_\lambda) \to \Gamma(\mathcal{G} \times_P (V_\lambda \circledcirc \mathfrak{g}_1))$ we need to construct a line bundle $\mathcal{G} \times_P L_\mu$ given by representation L_μ where $\mu = -\omega_1$. Then there exists an invariant first order differential operator $\mathcal{D}' : \Gamma(\mathcal{G} \times_P (V_\lambda \otimes L_\mu)) \to \Gamma(\mathcal{G} \times_P ((V_\lambda \otimes L_\mu) \circledcirc \mathfrak{g}_1))$.

2.2 Real case

There is an obvious way how to define an invariant operator in a real geometry. First we look at complexified version of the geometry and then we find an operator in such a way that bundles on which it acts are complexified versions of a real bundles.

Let V be a real $\mathfrak{g}_0^{ss}$-irreducible representation and let $\mathcal{G} \times_P V$ be real natural bundle induced by V. We are looking for a real invariant first order differential operator. Consider the complexification of the underlying representations and algebras. The natural bundle given by the representation V goes to $\mathcal{G} \times_P V^{\mathbb{C}}$. Now it depends on the complexification $V^{\mathbb{C}}$ and number of crosses in geometry how we will proceed. We will explain one and two cross real geometry only because the other cases are simple consequences of these types. There are two possibilities. Complexification of a real irreducible representation V is either one irreducible complex representation V_0 or a sum of two irreducible complex representations $V_1 \oplus V_2$. We denote the real representation V as $[V_0]_{\mathbb{R}}$ or $[V_1 \oplus V_2]_{\mathbb{R}}$, respectively. In other words, $V = [V_0]_{\mathbb{R}}$ or $V = [V_1 \oplus V_2]_{\mathbb{R}}$ if and only if $V^{\mathbb{C}} = V_0$ or $V^{\mathbb{C}} = V_1 \oplus V_2$, respectively.

Let us note that complex Lie algebras which are considered as real Lie algebras are two-cross real geometries. In the following we discuss possibilities how to construct real invariant first order differential operators. We relax notation in the following sense. An operator $\mathcal{D} : \Gamma(\mathcal{G} \times_P V) \to \Gamma(\mathcal{G} \times_P W)$ will be denoted by $\mathcal{D} : \Gamma(V) \to \Gamma(W)$. In the following, by the existence of an invariant operator $\mathcal{D}$ which acts on sections of natural bundle given by $\mathfrak{g}_0^{ss}$-representation $V_{\lambda'}$ we mean that there exists weight λ^0 on the center $\mathfrak{z}(\mathfrak{g}_0)$ such that $\mathcal{D}$ acts on sections of natural bundle given by $V_{\lambda'+\lambda^0}$.

The **one-cross geometry** case is almost same as the complex case. First, suppose $V^{\mathbb{C}} = V_0$. In this geometry $\mathfrak{g}_{\pm1}$ and $\mathfrak{g}_{\pm1}^{\mathbb{C}}$ are irreducible real and complex representations, respectively. Let $V_0 \otimes \mathfrak{g}_1^{\mathbb{C}}$ decomposes as $\bigoplus^m U_i$. Then there exists a real invariant differential operator $\mathcal{D}_i : \Gamma(V) \to \Gamma([U_i]_{\mathbb{R}})$ for such an i that U_i is the complexification of real irreducible representation.

Second, suppose $V^{\mathbb{C}} = V_1 \oplus V_2$. In this situation $V^{\mathbb{C}} \otimes \mathfrak{g}_1^{\mathbb{C}} = V_1 \otimes \mathfrak{g}_1^{\mathbb{C}} \oplus V_2 \otimes \mathfrak{g}_1^{\mathbb{C}}$ and suppose further decompositions $V_1 \otimes \mathfrak{g}_1^{\mathbb{C}} = \bigoplus^m U_i^1$, $V_2 \otimes \mathfrak{g}_1^{\mathbb{C}} = \bigoplus^m U_i^2$. Clearly, there is always a pair of representations which together form one irreducible real representation. A pair of representations consists of either the same representations or it consists of complex conjugated representations. For $i \in \{1,2\}, j \in \{1,\cdots,m\}$ there exists a complex invariant first order differential operator $\mathcal{D}_{i,j} : \Gamma(V_i) \to \Gamma(U_j^i)$. Let $U_r^1 \oplus U_p^2$ be the complexification of a real representation. There exists real invariant operator $\mathcal{D}^{r,p} : \Gamma(V) \to \Gamma([U_r^1 \oplus U_p^2]_{\mathbb{R}})$ which is defined as $\mathcal{D}^{r,p} = [\mathcal{D}_{1,r} \oplus \mathcal{D}_{2,p}]_{\mathbb{R}}$. Let us note, if $\eta \in \Gamma(\mathcal{G} \times_P V_1)$ and $\bar{\eta} \in \Gamma(\mathcal{G} \times_P V_2)$ then $\eta \in Ker\, \mathcal{D}_{1,r}$ if and only if $\bar{\eta} \in Ker\, \mathcal{D}_{2,p}$, by the argument that representations V_i are either the same or dual.

In **two-cross geometry**, the $\mathfrak{g}_{\pm1}^{\mathbb{C}}$-part has two components $\mathfrak{g}_{\pm1}^1 \oplus \mathfrak{g}_{\pm1}^2$, by the Kostant's version of Bott-Borel-Weil theorem or by lemma 6 in chapter three. These are either the complexification of one irreducible real representation or the complexification of two irreducible real representations. Again, there are two possibilities how an irreducible real representation V occurs in the complexification. We will use the notation from one-cross geometry case.

First, suppose $V^{\mathbb{C}} = V_0$ and $V_0 \otimes \mathfrak{g}_1^{\mathbb{C}} = V_0 \otimes \mathfrak{g}_1^1 \oplus V_0 \otimes \mathfrak{g}_1^2 = \bigoplus^m U_i^1 \oplus \bigoplus^n U_j^2$. Let $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$ come from two real irreducible representations. If U_j^i is the complexification of real representation, then there exists real invariant operator $\mathcal{D}_{i,j} : \Gamma(V) \to \Gamma([U_j^i]_{\mathbb{R}})$. And if U_l^1 and $U_{l'}^2$ are the complexification of two real representations, then there exists real invariant operator $\mathcal{D}_{l,l'} : \Gamma(V) \to \Gamma([U_l^1]_{\mathbb{R}} \oplus [U_{l'}^2]_{\mathbb{R}})$. Now let $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$ come from one real irreducible representation. Then we have the decomposition $V_0 \otimes \mathfrak{g}_1^{\mathbb{C}} = \bigoplus^m U_i^1 \oplus \bigoplus^m U_j^2$ in which a real irreducible representation occurs as a sum of two complex irreducible representations. There exists a real invariant first order differential operator $\mathcal{D}_i : \Gamma(V) \to \Gamma([U_i^1 \oplus U_{j(i)}^2]_{\mathbb{R}})$ for every $i \in \{1, \cdots, m\}$.

Second, suppose $V^{\mathbb{C}} = V_1 \oplus V_2$ and $V^{\mathbb{C}} \otimes \mathfrak{g}_1^{\mathbb{C}} = (V_1 \oplus V_2) \otimes (\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2) = V_1 \otimes \mathfrak{g}_1^1 \oplus V_2 \otimes \mathfrak{g}_1^1 \oplus V_1 \otimes \mathfrak{g}_1^2 \oplus V_2 \otimes \mathfrak{g}_1^2 = (\bigoplus^m U_j^1) \oplus (\bigoplus^n U_j^2) \oplus (\bigoplus^r W_j^1) \oplus (\bigoplus^p W_j^2)$. Let $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$ come from two real irreducible representations. Then $m = n$, $r = p$ and without loss of generality we can suppose $U_j^1 \oplus U_j^2$ is the complexification of a real irreducible representation and similarly $W_j^1 \oplus W_j^2$. There are following ways how to a real operators arise:

- $\mathcal{D}_i : \Gamma(V) \to \Gamma([U_i^1 \oplus U_i^2]_{\mathbb{R}})$
- $\mathcal{D}_i : \Gamma(V) \to \Gamma([W_i^1 \oplus W_i^2]_{\mathbb{R}})$
- $\mathcal{D}_{i,j} \oplus \mathcal{D}'_{i,j} : \Gamma(V) \to \Gamma(R)$, where $\mathcal{D}_{i,j} : \Gamma(V_1) \to \Gamma(U_i^1 \oplus W_j^1)$ and $\mathcal{D}'_{i,j} : \Gamma(V_2) \to \Gamma(U_i^2 \oplus W_j^2)$ and $R := [U_i^1 \oplus U_i^2]_{\mathbb{R}} \oplus [W_j^1 \oplus W_j^2]_{\mathbb{R}}$.

Now suppose $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$ come from one real irreducible representation. If we want to make real operator we need both components $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$. In this case, we can use last node above to get real invariant operator.

3. Algebraic linearisation condition - irreducible $\mathfrak{g}_{-1}$-part

We state and derive the algebraic linearisation condition (ALC) in the case of parabolic geometries with irreducible $\mathfrak{g}_{-1}$-part.

Metric and algebraic linearisation condition

Definition 3. *Let us use the symbol $\circledcirc$ between complex representations for usual Cartan product. Let V, W be an irreducible real representations of real reductive Lie algebra $\mathfrak{g}$. We define a real Cartan component $V \circledcirc W$ of tensor product $V \otimes W$ in the following cases:*

- *The complefixications of V and W are irreducible complex representations $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$. We define a real Cartan component as $[V_{\mathbb{C}} \circledcirc W_{\mathbb{C}}]_{\mathbb{R}}$.*
- *One of the representations occurs in the complexification as a sum of two irreducible modules, say $V_{\mathbb{C}} = V_1 \oplus V_2$, and the second representation is in complexification one irreducible module $W_{\mathbb{C}}$. Then we define a real Cartan component as a $[V_1 \circledcirc W_{\mathbb{C}} \oplus V_2 \circledcirc W_{\mathbb{C}}]_{\mathbb{R}}$.*
- *Both of the representations are in the complexification sum of two irreducible pieces, $V_{\mathbb{C}} = V_1 \oplus V_2$ and $W_{\mathbb{C}} = W_1 \oplus W_2$. We define a real Cartan component as $[V_1 \circledcirc W_2 \oplus V_2 \circledcirc W_1]_{\mathbb{R}} \oplus [V_1 \circledcirc W_1 \oplus V_2 \circledcirc W_2]_{\mathbb{R}}$.*

Now we state the ALC in the case of geometries with irreducible $\mathfrak{g}_{-1}$-part.

Definition 4. *Let $(\mathcal{G} \to M, \omega)$ be a real parabolic geometry of a type (G, P) and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{g}_1$ be irreducible $\mathfrak{g}_0$-module and B be $\mathfrak{g}_0$-irreducible subspace of $\odot^2 \mathfrak{g}_{-1}$ and let it contain nondegenerate elements. An invariant subspace B satisfies the algebraic linearisation condition if and only if $B \otimes \mathfrak{g}_1 \simeq B \circledcirc \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$.*

First we will analyse $\mathfrak{g}_{\pm 1}$-component for complex geometries. From appendix B we know both $\mathfrak{g}_{\pm 1}$ are isomorphic to the first cohomology group with trivial coefficients $H^1(\mathfrak{g}_-, \mathbb{C})$. Kostant's version of Bott-Borel-Weil theorem gives us that the number of an irreducible components of $H^*(\mathfrak{p}_+, \mathbb{C}) \simeq H^*(\mathfrak{g}_-, \mathbb{C})$ is the same as cardinality of the set $W^{\mathfrak{p}}$ and the isotypical component of a weight ν_w is contained in cohomology group of $\ell(w)$-order. We see that in the cohomology group $H^1(\mathfrak{p}_+, \mathbb{C})$, there can be only isotypical components corresponding to simple reflections which lie in $W^{\mathfrak{p}}$, because only these reflections have order one.

Lemma 6. *Let $\mathfrak{g}$ be a complex $|k|$-graded semisimple Lie algebra.*

- *The number of irreducible components of $\mathfrak{g}_{\pm 1}$ is exactly the number of crossed nodes in Dynkin diagram.*
- *Consider the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$. The highest weights of an irreducible components of $\mathfrak{g}_{-1}$ are exactly minus the simple roots which correspond to the crossed nodes.*

Proof. According to considerations above the lemma, for the first part of the lemma it is sufficient to prove that in $W^{\mathfrak{p}}$ there is exactly m reflections with length one, where m is the number of crossed nodes. If we use theorem 20 on Hasse diagram in appendix B we need to find the orbit of $\delta^{\mathfrak{p}}$ but for the proof it is enough to find a weights on orbit which are on the first level. So we are interesting in mappings $s_{\alpha_i}(\lambda) = \lambda - \frac{2<\lambda,\alpha_i>}{<\alpha_i,\alpha_i>}\alpha_i$. The result follows from the computations $s_{\alpha_i}(\lambda) = \lambda - \frac{2<\lambda,\alpha_i>}{<\alpha_i,\alpha_i>}\alpha_i = \lambda - \lambda_i\alpha_i$, by definition of fundamental weights ω_j, where $\lambda = \sum^n \lambda_i\omega_i$. From this it is easy to see every such reflection is not identity if and only if the i-th node is crossed. This can be easily generalized to an arbitrary number of crosses in Dynkin diagram.

Now we prove the second part of the lemma. From the gradation by Σ-height, minus simple roots which correspond to the crossed nodes have eigenspaces in $\mathfrak{g}_{-1}$-component. So, these generate isotypical components which are irreducible by Kostant theorem. From the first part, we know the number of components is exactly the number of crosses. □

Remark. One more observation about representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{\pm1}$ in the case of complex simple Lie algebra $\mathfrak{g}$. Obviously these two representations are dual. The fact is if we express j-th simple root as linear combination of fundamental weights the coefficients in this expression are exactly the coefficients in j-th column in Cartan matrix (or row, depending on definition of Cartan matrix). So if the highest weight of a representation is given by a simple root coefficients can be found in a Cartan matrix. Indeed, if we write $\alpha_j = \sum l_k\omega_k$ for α_j a simple root where ω_k are a fundamental weights and $l_k \in \mathbb{C}$, and multiply $\frac{2\alpha_i}{<\alpha_i,\alpha_i>}$ by α_j

$$a_{ij} = \frac{2 < \alpha_i, \alpha_j >}{< \alpha_i, \alpha_i >} = \sum 2\bar{l_k}\frac{< \alpha_i, \omega_k >}{< \alpha_i, \alpha_i >} = \sum \bar{l_k}\delta_{ik} = \bar{l_i} = l_i$$

by definition of a term a_{ij} in Cartan matrix and by definition of a fundamental weights.

Theorem 7. *Let B enjoys the ALC. Then there exists a line bundle $\mathcal{L}$ associated to a 1-dimensional module L such that there is a first order invariant operator $\mathcal{D}$ from $\Gamma(\mathcal{G}\times_P(B\otimes\mathfrak{g}_1\otimes L))$ to $\Gamma(\mathcal{G}\times_P(B\circledcirc\mathfrak{g}_1\otimes L))$. Moreover, for any nondegenerate solution η of this equation, there exists a Weyl covariant derivative ∇ such that $\nabla|_{\mathcal{H}}\eta = 0$. The operator $\mathcal{D}$ is the first operator in BGG complex which is given by $\mathfrak{g}$-representation $\mathbb{B}$, where $\mathbb{B}$ is obtained from B in such way that we put zeros over crosses in Dynkin diagram which correspond to $\mathfrak{g}_0$-representation B. In homogeneous case, $dim(Ker\,\mathcal{D}) = dim(\mathbb{B})$.*

Proof. Result about the dimension of the kernel of the operator in homogeneous models is stated in appendix D. Discussion about BGG operators can be found there, too.

First we show the theorem holds in one-cross complex geometries.

Let $c : \mathfrak{g}_1 \otimes \odot^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ be the natural contraction and let B be an invariant subspace of $\odot^2\mathfrak{g}_{-1}$ which contains nondegenerate elements. Let $b : \mathfrak{g}_1 \otimes B \to \mathfrak{g}_{-1}$ be the restriction of c. If the ALC is satisfied we may write $\mathfrak{g}_1 \otimes B = Ker\, b \oplus \zeta(\mathfrak{g}_{-1}) \simeq (B \circledcirc \mathfrak{g}_1) \oplus \mathfrak{g}_{-1}$ where $\zeta : \mathfrak{g}_{-1} \to \mathfrak{g}_1 \otimes B$ is a $\mathfrak{g}_0$-invariant map with $b \circ \zeta = id_{\mathfrak{g}_{-1}}$.

We want to consider invariant operator on natural bundle which is given by B with values in natural bundle which is given by representation $B \circledcirc \mathfrak{g}_1$. It is shown in the chapter two, we get such an operator if and only if the number over a crossed node in a defining representation for the first order invariant differential operator is zero. The way how to do this is to multiply the bundle $\mathcal{G} \times_P B$ by a line bundle $\mathcal{L}$ which is associated to the chosen central weight as we explain in chapter two (above real case - 2.1). So there is an invariant first order differential operator $\mathcal{D} : \Gamma((\mathcal{G} \times_P B) \otimes \mathcal{L}) \to \Gamma((\mathcal{G} \times_P (B \circledcirc \mathfrak{g}_1)) \otimes \mathcal{L})$, where $\mathcal{D} = \pi \circ \nabla|_{\mathcal{H}}$ for any Weyl covariant derivative ∇. According to these considerations we are working with tensor field η as with sections of $(\mathcal{G} \times_P B) \otimes \mathcal{L}$. Let us note the number of an irreducible components of $B \otimes \mathfrak{g}_1 \otimes L$, where L is a representation which induces the bundle $\mathcal{L}$, is the same as the number of components of $B \otimes \mathfrak{g}_1$ because $dim(L) = 1$.

Now, it is obvious that solutions of $\mathcal{D}\eta = 0$ can be characterized by an existence of a section X^∇ of $\mathcal{H} \otimes \mathcal{L}$ such that $\nabla|_{\mathcal{H}}\eta = \zeta(X^\nabla)$. Let $\tilde{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ be another covariant derivative, where $\Upsilon \in \mathcal{H}^*$. Then for any $Z \in \Gamma(\mathcal{H})$, $\tilde{\nabla}_Z\eta = \nabla_Z\eta + [\![\Upsilon, Z]\!] \cdot \eta$ and $[\![\Upsilon, \cdot]\!] \cdot \eta$ is nonzero only on image of ζ by the invariance of $\mathcal{D}$, where the bracket $[\![,]\!]$ is as in appendix C.

Let us note, there is a map

$$A : \mathcal{H}^* \otimes (\mathcal{G} \times_P B) \to \mathcal{H}^* \otimes (\mathcal{G} \times_P B)$$

$$\Upsilon \otimes \eta \mapsto (Z \mapsto [\![\Upsilon, Z]\!] \cdot \eta)$$

and by Schur's lemma we get

$$[\![\Upsilon, \cdot]\!] \cdot \eta = (\zeta \circ b)([\![\Upsilon, \cdot]\!] \cdot \eta) = (\zeta \circ b)(\ell \Upsilon \otimes \eta)$$

where ℓ is a scalar. Note that non-degeneration of η implies that the contraction $\sharp_\eta(\Upsilon) = b(\ell\Upsilon \otimes \eta)$ is surjective. We get $\tilde{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \zeta(\sharp_\eta(\Upsilon))$. Now if η is a nondegenerate solution of $\mathcal{D}\eta = 0$, with $\nabla|_{\mathcal{H}}\eta = \zeta(X^\nabla)$ for some Weyl covariant derivative ∇ and $X^\nabla \in \Gamma(\mathcal{H} \otimes \mathcal{L})$, we may take $\Upsilon = -\sharp_\eta^{-1}(X^\nabla)$ to obtain $\tilde{\nabla}|_{\mathcal{H}}\eta = \zeta(X^\nabla) - \zeta(\sharp_\eta(\Upsilon)) = 0$. Hence η is a covariantly constant tensor field.

Now we consider one-cross real geometry which satisfy the ALC. Almost whole procedure is same up to one exception. There was a step where we use Schur lemma. But if the equation holds in complex case it must be true for any real form.

In the following we suppose two-cross real geometry which satisfy the ALC. In the complexification there is the isomorphism $B^{\mathbb{C}} \otimes \mathfrak{g}_1^{\mathbb{C}} \simeq B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^1 \oplus B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^2 \oplus \mathfrak{g}_{-1}^1 \oplus \mathfrak{g}_{-1}^2$, where $\mathfrak{g}_{\pm1}^{\mathbb{C}} = \mathfrak{g}_{\pm1}^1 \oplus \mathfrak{g}_{\pm1}^2$ and $B^{\mathbb{C}}$ decomposes into either one complex irreducible representation or into two irreducible complex representations.

Let us consider the case in which $B^{\mathbb{C}}$ is one irreducible complex component, therefore it is self-dual representation. Clearly, $B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^1$ and $B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^2$ are dual representations. We have invariant operator $\mathcal{D}$ acting between sections of natural bundles which are given by representations $B^{\mathbb{C}} \to B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^1$. This operator can be corrected (in the sense above) on $\mathfrak{g}_{-1}^2$-component. Hence, there exists Weyl covariant derivative ∇ such that $\pi \circ \nabla\eta = 0$ if $\eta \in Ker\, \mathcal{D}$, where π is the bundle map induced by natural projection $B^{\mathbb{C}} \otimes \mathfrak{g}_1 \to B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^1 \oplus \mathfrak{g}_{-1}^2$. Because of duality between representations, the equation $\nabla\eta = 0$ holds for every $\eta \in Ker\, \mathcal{D}$. In other words, there is dual operator between sections of natural bundles which are given by $B^{\mathbb{C}} \to B^{\mathbb{C}} \circledcirc \mathfrak{g}_1^2 \oplus \mathfrak{g}_{-1}^1$ as we explain in section 2.2.

Next, let $B^{\mathbb{C}} = B_1 \oplus B_2$ and let $B_1 \subset \odot^2\mathfrak{g}_{-1}^1$ and $B_2 \subset \odot^2\mathfrak{g}_{-1}^2$. We choose invariant operator which corresponds to $B_1 \to B_1 \circledcirc \mathfrak{g}_1^1 \oplus B_1 \circledcirc \mathfrak{g}_1^2$. According to the ALC, we make correction of the operator on $\mathfrak{g}^1_{-1}$-part. Again, we consider dual operator which corresponds to representations $B_2 \to B_2 \circledcirc \mathfrak{g}^1_1 \oplus B_2 \circledcirc \mathfrak{g}^2_1 \oplus \mathfrak{g}^2_{-1}$. $\square$

4. Verification of the ALC for some geometries with irreducible $\mathfrak{g}_{-1}$-part

In this chapter we will verify the ALC for some parabolic geometries with irreducible $\mathfrak{g}_{-1}$-part. Whole classification of parabolic geometries which satisfy the ALC can be found in preprint of D. Calderbank, J. Slovák and V. Souček.

First we will study a complexified versions of these geometries and then we discuss its real forms. By convention we display a dual weights over a Dynkin diagrams. Computation algorithms using in this chapter can be found in the book [12].

We will consider geometries with at most two crosses in a Dynkin diagram only. In the case of two crosses they have to be placed in a symmetric way in a diagram. There is a reason for these two restrictions. One way how to obtain irreducible $\mathfrak{g}_{-1}$-part is to take only one cross in diagram. Now, if we make two crosses in a diagram in a symmetric way it makes it possible to turn $\mathfrak{g}_{-1}$-part into an irreducible real component. In the complexification $\mathfrak{g}_{-1}^{\mathbb{C}}$ has two complex irreducible components. These components could come from one real irreducible representation only if they have same dimensions. This can be ensure by placed crosses in a symmetric way. An assymetric position of crosses will never result in one real irreducible representation. More than two crosses creates more than one real irreducible component.

Now we start study the ALC in A_n-algebra case.

A_n case

We will study the ALC in the case when k-th node in Dynkin diagram of A_n-type is crossed or when two nodes in a symmetric way are crossed in diagram. The grading of the algebra looks like

$$\begin{array}{cc} & \begin{array}{cc} k & n{-}k{+}1 \end{array} \\ \begin{array}{c} k \\ n{-}k{+}1 \end{array} & \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array}$$

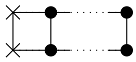
in the case of one cross. And in the case of two crosses the gradation can be obtained by superposition of a two gradations

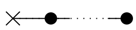
$$\begin{array}{cc} & \begin{array}{cc} k & n{-}k{+}1 \end{array} \\ \begin{array}{c} k \\ n{-}k{+}1 \end{array} & \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array} + \begin{array}{cc} & \begin{array}{cc} n{-}k{+}1 & k \end{array} \\ \begin{array}{c} n{-}k{+}1 \\ k \end{array} & \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array} = \left(\begin{array}{c|c|c} 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline -2 & -1 & 0 \end{array}\right),$$

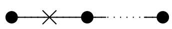
where k is in $\{1, ..., n/2\}$, if n is even, and in $\{1, ..., (n+1)/2\}$, if n is odd.

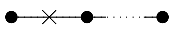
In the following theorem, the first part will be proved in theorem 11, chapter 5.

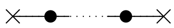
Theorem 8. *Let $\mathfrak{g}$ be a Lie algebra of A_n-type with irreducible $\mathfrak{g}_{-1}$-part and let a $B \subseteq \odot^2\mathfrak{g}_{-1}$ satisfies the ALC. The following list gives us a classification of all such cases.*

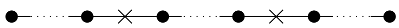
1. , $\mathfrak{g} = sl(n+1, \mathbb{C}), n > 1$ *where* $\mathfrak{g}$ *is considered as the real Lie algebra. The geometry is called c-projective geometry.*

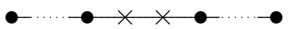
2. , $\mathfrak{g} = sl(n+1, \mathbb{R}), n > 1$. *This is the case of projective geometry on n-dimensional manifold M and B is the space of all (pseudo) Riemannian metrics on M.*

3. , $\mathfrak{g} = sl(n+1, \mathbb{R}), n \geq 4$. *The geometry is the almost Grassmannian structure on $2n$-manifold M. The tangent space $\mathcal{H} = TM$ can be identified with the tensor product $E \otimes F$ of auxiliary bundles E and F with dimensions 2 and n respectively. The metrics are of the form of tensor products of volume form on E and antisymmetric forms on F.*

4. , $\mathfrak{g} = sl(p, \mathbb{H}), n+1 = 2p, n \geq 5$. *The geometry is the almost quaternionic geometry on manifold of dimension $4n$. The metrics are real parts of quaternionic hermitian forms.*

5. , $g \simeq su(p,q), 1 \leq p \leq q, p+q = n+1, n \geq 3$. *These are CR geometries.*

6.  $g \simeq su(p,q), k \leq p \leq q, p+q = n+1$, *where the first cross is at the k-th place and the second in the symmetric position.*

7.  $g \simeq su(p, p+1), 2p = n$, *where the first cross is at p-th place. These geometries are CR geometries.*

Proof. **One cross**
Now we compute the highest weights of the dual adjoint representations of semisimple part $\mathfrak{g}_0^{ss}$ on $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$. To aim that we compute

$$\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & AC - CB \\ 0 & 0 \end{pmatrix}$$

We skip the case $k = 1$. In that case $\mathfrak{g}_0^{ss}$ reduces to a simple algebra and verification of the ALC is a little bit easier. Let $k \in \{2, ..., n/2\}$ if n is even or $k \in \{2, ..., (n+1)/2\}$ if n is odd. We can consider $C = u \otimes v$ where u and v are column and row, respectively. If we define a representations of $sl(k, \mathbb{C})$ and $sl(n-k+1, \mathbb{C})$ as $A \cdot u = Au$ and $B \cdot v = -vB$, respectively, it follows that the representation of $sl(k, \mathbb{C}) \oplus sl(n-k+1, \mathbb{C})$ is exactly what we are looking for. Now we see the representation of A on u is the first fundamental representation of $sl(k, \mathbb{C})$ therefore the highest weight is the first fundamental weight. It is easily seen that the representation of B-term is dual representation of the first fundamental representation of $sl(n-k+1, \mathbb{C})$, so the highest weight is the last fundamental weight. As we mentioned in the beginning of this chapter, we are displaying a highest weights of a dual representations over Dynkin diagrams.

According to these considerations, the representation of $\mathfrak{g}_0^{ss}$ on $\mathfrak{g}_1$ is given by $\bullet\cdots\overset{1}{\bullet}-\times-\overset{1}{\bullet}\cdots\bullet$. Analogously we can find representation of $\mathfrak{g}_0^{ss}$ on $\mathfrak{g}_{-1}$, which is given by $\overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet}$.

We just note changes in the case $k = 2$. Representations of $\mathfrak{g}_0^{ss}$ on $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are $\overset{1}{\bullet}-\times-\overset{1}{\bullet}\cdots\bullet$ and $\overset{1}{\bullet}-\times-\bullet\cdots\overset{1}{\bullet}$ respectively.

Now we would like to find a decomposition of the second symmetric tensor power of $\mathfrak{g}_{-1}$. Again, let us consider the $\mathfrak{g}_{-1}$-part as product $u \otimes v$.

We begin with a computations about decomposition of a representations of $sl(k,\mathbb{C})$ and $sl(n-k+1,\mathbb{C})$ on the tensor product $v\otimes v$ and $u \otimes u$, respectively, given by $\overset{1}{\bullet}\cdots\bullet$ and $\bullet\cdots\overset{1}{\bullet}$ respectively. We claim that the representation of $sl(k,\mathbb{C})$ on $v \otimes v$ given by $\overset{1}{\bullet}\cdots\bullet$ decomposes to $\overset{2}{\bullet}\cdots\bullet \oplus \bullet-\overset{1}{\bullet}\cdots\bullet$. For the verification we use Klimyk's formula, theorem 32 in appendix E. Let us try if these weights, namely $2\omega_{k-1}$ and ω_{k-2}, are included in the Klimyk's sum. For this purpose we have to compute these two equations,

$$\{\mu_1 + \omega_{k-1} + \rho\} = \rho + 2\omega_{k-1}$$

$$\{\mu_2 + \omega_{k-1} + \rho\} = \rho + \omega_{k-2}$$

In both equations on the right side we have dominant weights. So, on the left side there had to be dominant weights. By argument, if two dominant weights are conjugate by an element of a Weyl group they are same, we get simplifications,

$$\mu_1 + \omega_{k-1} + \rho = \rho + 2\omega_{k-1}$$

$$\mu_2 + \omega_{k-1} + \rho = \rho + \omega_{k-2},$$

which are equivalent to

$$\mu_1 = \omega_{k-1}$$

$$\mu_2 = \omega_{k-2} - \omega_{k-1}$$

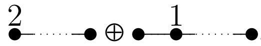
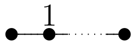
Now we compute multiplicities of these weights in original representation of $sl(k,\mathbb{C})$ on vector v characterized by $\overset{1}{\bullet}\cdots\bullet$. Let us note that weights which are conjugate by an element of a Weyl group have same multiplicities. In general, we can focus on a dominant weights of a representation only and compute their multiplicities by Freudental's formula. As we see, the weight μ_1 is dominant and in fact the highest weight, so it has multiplicity one. To obtain a result about multiplicity of μ_2 we will find the dominant weight to which is conjugate.

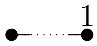
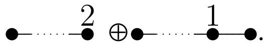
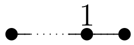
$$\omega_{k-2} - \omega_{k-1} = \mu_2 \mapsto \mu_2 + \alpha_{k-1} = \omega_{k-1}$$

So the multiplicity for μ_2 is same as for $\omega_{k-1} = \mu_1$ and thus one. We proceed to evaluation of the function s from the Klimyk's sum. We use the fact, stabilizer W_λ of a weight λ in the Weyl group W is generated by a simple reflections r_{α_i}, where $\alpha_i \in \Delta$ (set of simple roots) satisfies $< \lambda, \alpha_i >= 0$. Our aim is to show $s(\mu_i + \omega_{k-1} + \rho) \neq 0$, $i \in \{1,2\}$. We compute

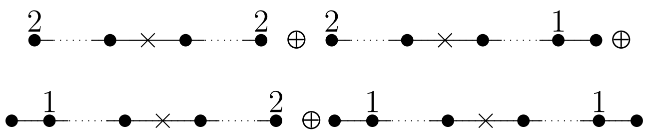
$$< 2\omega_{k-1} + \rho, \alpha_j >= 1$$

$$< \omega_{k-2} + \rho, \alpha_j >= \begin{cases} 1, & \text{if } j+1 < k-1 \\ 2, & \text{if } j+1 = k-1 \end{cases}$$

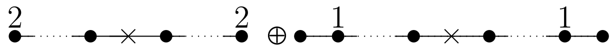
where $j \in \{1, ..., k-1\}$. Thus terms with the weights $2\omega_{k-1}$ and ω_{k-2} are included in Klimyk's sum. By dimensionality issues, using Weyl dimension formula, it can be proved decomposition of $v \otimes v$ is exactly . Finally we use the fact that characters in this Klimik's sum are different. The representation  is $(k-2)$-th fundamental representation and thus antisymmetric.

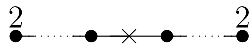
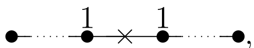
By an analogous approach it can be shown the representation of $sl(n-k+1, \mathbb{C})$ on space $u \otimes u$ given by  decomposes into . , where again  is antisymmetric part.

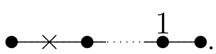
We have representation of $sl(k, \mathbb{C}) \oplus sl(n-k+1, \mathbb{C})$ on $(v \otimes v) \otimes (u \otimes u) \cong \otimes^2 \mathfrak{g}_{-1}$. Obviously this representation decomposes to



Now again by dimensionality issues one obtain

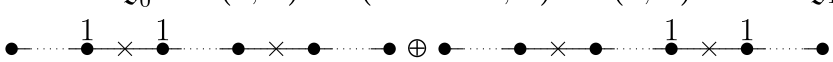
$\odot^2 \mathfrak{g}_{-1} =$  $=: B' \oplus B$

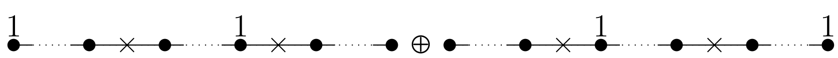
The last procedure of verification of the ALC is to know the number of a components of $B' \otimes \mathfrak{g}_1$ and $B \otimes \mathfrak{g}_1$. Let $k > 2$. For instance, we take $B' \otimes \mathfrak{g}_1$. We know $sl(k, \mathbb{C}) \oplus sl(n-k+1, \mathbb{C})$ acts on $B' \otimes \mathfrak{g}_1$ as  $\otimes$ , where the left parts of the crossed diagrams correspond to a representations of $sl(k, \mathbb{C})$ and right parts to $sl(n-k+1, \mathbb{C})$. We claim both of these representations (left parts and right parts in diagram) decompose into two components. So representation of $\mathfrak{g}_0^{ss}$ on $B' \otimes \mathfrak{g}_1$ has four irreducible components and it is too many to satisfy the ALC. In the same way it can be proved the representation of $\mathfrak{g}_0^{ss}$ on $B \otimes \mathfrak{g}_1$ has four components. So ALC is not satisfied in a case of crossed k-th node in Dynkin diagram of A_n-type for $k > 2$.

Let us show the decomposition in the case $k = 2$. For the algebra of rank $n = 3$ there are too many components in decomposition of $B \otimes \mathfrak{g}_1$. For $n \geq 4$ we get one B which satisfy the ALC. Representation of $\mathfrak{g}_0^{ss}$ on B is given by diagram . The real forms of the algebra corresponds either to $sl(n, \mathbb{R})$ or $sl(n, \mathbb{H})$. In the case of Grassmannian geometry ($\mathfrak{g} = sl(n+1, \mathbb{R})$), it is easy to see metrics are tensor product of volume form with antisymmetric forms.

Two crosses

According to observations and remark in chapter 3 above theorem 7, it is easy to see that $\mathfrak{g}_0^{ss} = sl(k, \mathbb{C}) \oplus sl(n-2k+1, \mathbb{C}) \oplus sl(k, \mathbb{C})$ acts on $\mathfrak{g}_1$ as

 and on $\mathfrak{g}_{-1}$ as

. A computations are straightforward but long. Everything can be computed as in one-cross case.

We state only important results.

$\odot^2(\mathfrak{g}_{-1}) \supset B = \overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet}$

$B \otimes \mathfrak{g}_1 = (\ \overset{1}{\bullet}\cdots\overset{1}{\bullet}-\times-\overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet} \oplus \overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet}-\times-\overset{1}{\bullet}\cdots\overset{1}{\bullet}\ )\oplus$

$\oplus\ (\bullet\cdots\bullet-\times-\overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet} \oplus \overset{1}{\bullet}\cdots\bullet-\times-\bullet\cdots\overset{1}{\bullet}-\times-\bullet\cdots\bullet)$

The representations in brackets comes from one real irreducible representation. We see that ALC is satisfied.

Let us make a note about cases which we skipped. Representations of a semisimple part $\mathfrak{g}_0^{ss}$ on spaces $\mathfrak{g}_{\pm1}$ are clear. For the geometry which arise from $\bullet\cdots\bullet-\times-\times-\bullet\cdots\bullet$ the ALC is satisfied for $B = \overset{1}{\bullet}\cdots\bullet-\times-\times-\bullet\cdots\overset{1}{\bullet}$. For the geometry of the type $\times-\bullet\cdots\bullet-\times$ the ALC is satisfied for $B = \times-\overset{1}{\bullet}\cdots\overset{1}{\bullet}-\times$

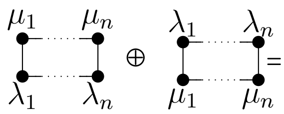
□

Remark. Let us make a remark about case $k = 1$ in one-cross geometry. The real form of $sl(n,\mathbb{C})$ can be $sl(n,\mathbb{R})$ only and there is exactly one component B and it satisfies ALC. The component B has representation $\times-\bullet\cdots\overset{2}{\bullet}$. Obviously metrics corresponds to Riemannian metric.

5. Complex Lie algebras as real Lie algebras

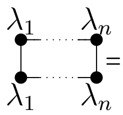
Every complex Lie algebra $\mathfrak{g}$ can be considered as a real Lie algebra $\mathfrak{g}(\mathbb{R})$. For such algebras, we will compute the ALC for an irreducible $\mathfrak{g}_{-1}(\mathbb{R})$-part. First we focus on some observations, which will be helpful in the sequel.

Representation theory for real simple Lie algebras can be found in [23]. Let $\mathfrak{g}(\mathbb{R})$ be one of algebras from the previous paragraph. The complexification $(\mathfrak{g}(\mathbb{R}))^{\mathbb{C}}$ of $\mathfrak{g}(\mathbb{R})$ is isomorphic to $\mathfrak{g} \oplus \bar{\mathfrak{g}}$. Similarly, complexification of complex vector space W, which is considered as a real vector space $W(\mathbb{R})$ is isomorphic to $W \oplus \bar{W}$. Irreducible real representations V of $\mathfrak{g}(\mathbb{R})$ are of two types: either the complexification $V^{\mathbb{C}}$ is irreducible and has form $V_\lambda \bar{V}_\lambda := V_\lambda \otimes \bar{V}_\lambda$, or it is a sum of two dual representations $V_\lambda \bar{V}_\mu \oplus V_\mu \bar{V}_\lambda$, where unbarred representations correspond to algebra $\mathfrak{g}$ and barred representations correspond to algebra $\bar{\mathfrak{g}}$. In the notation from previous chapter, $V = [V_\lambda \bar{V}_\lambda]_{\mathbb{R}}$ or $V = [V_\lambda \bar{V}_\mu \oplus V_\mu \bar{V}_\lambda]_{\mathbb{R}}$. We will denote these representation by a double Dynkin diagram with corresponding highest weights. For example, let $\mathfrak{g} \simeq sl(n+1, \mathbb{C})$ considered as real Lie algebra $\mathfrak{g}(\mathbb{R})$, the complexification of a real irreducible representation is either of the form



$=: V_\mu \bar{V}_\lambda \oplus V_\lambda \bar{V}_\mu$,

or



$=: V_\lambda \bar{V}_\lambda$.

A real irreducible $\mathfrak{g}_{-1}(\mathbb{R})$-part occurs in the complexification as a sum of two complex representations, namely $\mathfrak{g}_{-1} \oplus \overline{\mathfrak{g}_{-1}}$. We will always consider geometries with one cross only. The reason for this restriction is the following. If we consider geometries with two or more crosses the $\mathfrak{g}_{-1}(\mathbb{R})$-part will not be irreducible. If we want to check the ALC we need to find $\odot^2(\mathfrak{g}_{-1} \oplus \overline{\mathfrak{g}_{-1}}) = \odot^2(\mathfrak{g}_{-1}) \oplus [\mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}}] \oplus \odot^2(\overline{\mathfrak{g}_{-1}})$. The first and the last components decompose into irreducible parts in such a way that together (in pairs) give complexification of irreducible real representations. The middle component is always complexification of a real irreducible representation. So a choices for $B^{\mathbb{C}}$ (from the ALC) are either $\mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}}$ or sum of two conjugated representations.

Lemma 9. *Let $B^{\mathbb{C}} \simeq \mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}}$. The ALC is satisfied if and only if $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} = C \oplus \mathbb{C}$, where C is the Cartan component and $\mathbb{C}$ is the trivial representation.*

Proof. We work with complexified representations. Let us look at decomposition

$$B^{\mathbb{C}} \otimes (\mathfrak{g}_1 \oplus \bar{\mathfrak{g}}_1) = [\mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}} \otimes \mathfrak{g}_1] \oplus [\mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}} \otimes \bar{\mathfrak{g}}_1] \tag{5.1}$$

Clearly, first and second summands have the same decompositions up to conjugation. These representations are nontrivial hence decomposition of every summand is nontrivial.

Suppose the ALC is satisfied. We get $B^{\mathbb{C}} \otimes \mathfrak{g}_1^{\mathbb{C}} = B^{\mathbb{C}} \circledcirc (\mathfrak{g}_1 \oplus \bar{\mathfrak{g}}_1) \oplus \mathfrak{g}_{-1} \oplus \overline{\mathfrak{g}_{-1}} = (\mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}}) \circledcirc (\mathfrak{g}_1 \oplus \bar{\mathfrak{g}}_1) \oplus \mathfrak{g}_{-1} \oplus \overline{\mathfrak{g}_{-1}} = (\mathfrak{g}_{-1} \circledcirc \mathfrak{g}_1 \oplus \mathbb{C}) \otimes \bar{\mathfrak{g}}_{-1} \oplus \mathfrak{g}_{-1} \otimes (\mathbb{C} \oplus \bar{\mathfrak{g}}_{-1} \circledcirc \bar{\mathfrak{g}}_1)$. Comparing with previous decomposition, it follows that $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} = C \oplus \mathbb{C}$.

The opposite direction is trivial. It can be seen from the decomposition (5.1) at the beginning of the proof.

□

Remark. Defining bilinear form for a Lie algebras of B_n-type and D_n-type is symmetric and B_n-algebra is self-dual which implies that contracted tensors in second tensor power $\otimes^2 V$ of a representation V are in symmetric part $\odot^2 V$ of decomposition of this second tensor power. This result holds in the case of D_{2n}-algebras and for self-dual representations of D_{2n+1} too. Together with Cartan component the contracted tensors are in symmetric part of $\otimes^2 V$, hence something else must be in antisymmetric part $\wedge^2 V$. Dimension of a space of contracted tensors is one and there is only one representation with dimension one, the trivial representation. It is easy consequence of the Weyl dimension formula.

All together, for these algebras second tensor power of representations decomposes into at least three components. Actually, decomposition of second tensor power for self-dual algebras contains at least three components, because of contracted tensors. If we consider C_n-algebras these are self-dual and their defining bilinear form is anti-symmetric so contracted tensors are in anti-symmetric part $\wedge^2 V$ of tensor product $\otimes^2 V$ for arbitrary irreducible representation V. By dimensionality issues product $\otimes^2 V$ always decomposes into at least three components. Obviously, this holds for every selfdual representation.

Corollary. If the decomposition $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} = C \oplus \mathbb{C}$ holds then semisimple part of the Levi part $\mathfrak{g}_0$ is simple and of A, D_{2n+1}, E_6 - type.

Proof. We prove this corollary by contradiction. Suppose that the algebra $\mathfrak{g}_0$ is not simple, then it acts nontrivialy by its simple subalgebras on $\mathfrak{g}_{\pm 1}$. Therefore in decomposition of $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$ there is at least four components hence the algebra has to be simple. If $\mathfrak{g}_0$ is of different type then A, D_{2n+1}, E_6 then it is self-dual and decomposition of $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$ has at least three components, by the remark above. □

Lemma 10. *Let the complexification $B^{\mathbb{C}}$ of B be a sum of two conjugated representations, $B^{\mathbb{C}} = B_1 \oplus B_2$. The ALC holds if and only if $B_1 \otimes \mathfrak{g}_1 = C \oplus \mathfrak{g}_{-1}$.*

- *Let $\mathfrak{g}_0$ be a semisimple Lie algebra and not simple. In this case $\mathfrak{g}_0$ contains two or three simple algebras. If the ALC is satisfied then only one of simple algebras of $\mathfrak{g}_0$ acts nontrivialy on B.*

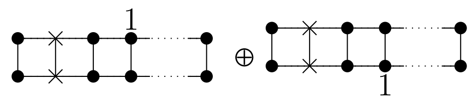
Proof. The first statement on equivalence of the ALC and decomposition $B_1 \otimes \mathfrak{g}_1$ is clear from definition of the ALC.

- Let $\mathfrak{g}_0 = \bigoplus_i^3 \mathfrak{g}_0^i$ be semisimple and not simple Lie algebra. If at least two subalgebras $\mathfrak{g}_0^i$ acts on B nontrivialy then $B_1 \otimes \mathfrak{g}_1$ decomposes into at least four components.

□

Let us introduce a new notation. In the following it will be important to know where a cross in Dynkin diagram is placed. Naturally, all representations are considered to be $\mathfrak{g}_0^{ss}$-representations. Hence every complex irreducible representation is characterized by ℓ-tuple of highest weights which correspond to simple subalgebras in the Levi part. Let $\mathfrak{g}_0 = \bigoplus^{\ell} \mathfrak{r}_j$ be decomposition of Levi part into simple subalgebras. Let $\mathfrak{g}_0$ corresponds to cross at the k-place in Dynkin diagram and let V be $\mathfrak{g}_0^{ss}$-representation. By superscript V^k we denote position of cross in Dynkin diagram. By subscripts $V^k_{(\lambda_1,\cdots,\lambda_\ell)}$ we denote highest weights which corresponds to simple algebras $\mathfrak{r}_j$.

We illustrate the notation using following example. Let $\mathfrak{g}$ be a complex Lie algebra of A_n-type which is considered as a real Lie algebra and let parabolic subalgebra corresponds to crossed second node in Dynkin diagram. The representation $V^2_{(0,\omega_2)}\bar{V}^2_{(0,0)} \oplus V^2_{(0,0)}\bar{V}^2_{(0,\omega_2)}$ corresponds to



Remark. Let us make one more remark about D_{2n+1}. Let $\mathfrak{g}_0^{ss}$ be of D_{2n+1}-type. Up to two exceptions, $\mathfrak{g}_0$-representations $\mathfrak{g}_{\pm 1}$ are self-dual. The exceptions are in E_6 and E_8 algebras, when the fifth and seventh node is crossed, respectively.

We proceed to find geometries which satisfy the ALC. The labelling of roots is as in [10].

Theorem 11. *Let $\mathfrak{g}$ be a complex simple Lie algebra considered as the real Lie algebra. The ALC is satisfied in the following cases:*

1. *A_n-type*
   - $B^{\mathbb{C}} = V^1_{\omega_1}\bar{V}^1_{\omega_1}$, *so called c-projective geometry.*
   - $B^{\mathbb{C}} = V^2_{(0,\omega_2)}\bar{V}^2_{(0,0)} \oplus V^2_{(0,0)}\bar{V}^2_{(0,\omega_2)}$
   - $B^{\mathbb{C}} = V^1_{2\omega_1}\bar{V}^1_0 \oplus V^1_0\bar{V}^1_{2\omega_1}$
2. *B_n-type*
   - $B^{\mathbb{C}} = V^n_{\omega_{n-1}}\bar{V}^n_{\omega_{n-1}}$
   - $B^{\mathbb{C}} = V^k_{(2\omega_{k-1},0)}\bar{V}^k_{(0,0)} \oplus V^k_{(0,0)}\bar{V}^k_{(2\omega_{k-1},0)}$ *for* $2 \leq k \leq n$
3. *C_4-type*
   - $B^{\mathbb{C}} = V^2_{(0,\omega_2)}\bar{V}^2_{(0,0)} \oplus V^2_{(0,0)}\bar{V}^2_{(0,\omega_2)}$
4. *C_n-type*
   - $B^{\mathbb{C}} = V^k_{(\omega_{k-2},0)}\bar{V}^k_{(0,0)} \oplus V^k_{(0,0)}\bar{V}^k_{(\omega_{k-2},0)}$ *for* $3 \leq 2k \leq n-1$
5. *D_n-type*
   - $B^{\mathbb{C}} = V^k_{(2\omega_{k-1},0)}\bar{V}^k_{(0,0)} \oplus V^k_{(0,0)}\bar{V}^k_{(2\omega_{k-1},0)}$ *for* $2 \leq k \leq n-3$

- $B^{\mathbb{C}} = V^{n-2}_{(2\omega_{n-3},0,0)}\bar{V}^{n-2}_{(0,0,0)} \oplus V^{n-2}_{(0,0,0)}\bar{V}^{n-2}_{(2\omega_{n-3},0,0)}$

6. *E_6-type*

   - $B^{\mathbb{C}} = V^1_{\omega_1}\bar{V}^1_0 \oplus V^1_0\bar{V}^1_{\omega_1}$

7. *G_2-type*

   - $B^{\mathbb{C}} = V^1_{\omega_1}\bar{V}^1_{\omega_1}$
   - $B^{\mathbb{C}} = V^1_{2\omega_1}\bar{V}^1_0 \oplus V^1_0\bar{V}^1_{2\omega_1}$

Proof. First, we make some restrictions. According to previous lemmas and considerations we should consider one cross only. If we choose $B^{\mathbb{C}} = \mathfrak{g}_{-1} \otimes \overline{\mathfrak{g}_{-1}}$ necessary condition to satisfy the ALC is $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 = C \oplus \mathbb{C}$. The only way how to get this decomposition is that $\mathfrak{g}_0$ is simple and is of correct type. Restrictions for $B^{\mathbb{C}} = B_1 \oplus B_2$ are clear from the lemma 10 above.

We start with $B^{\mathbb{C}} = \mathfrak{g}_{-1} \otimes \bar{\mathfrak{g}}_{-1}$. In each case we describe which node is crossed. From now, we work with weights instead of vector spaces labelled by weights.

- A_n **- first node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_1 \otimes \omega_{n-1} = C \oplus 0$ The ALC is satisfied.
- B_n **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_{n-1} \otimes \omega_1$. The ALC is satisfied.
- C_n **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq 2\omega_{n-1} \otimes 2\omega_1 = C \oplus (\omega_1 + \omega_{n-1}) \oplus 0$
- D_n **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_{n-2} \otimes \omega_2 = C \oplus (\omega_1 + \omega_{n-1}) \oplus 0$
- E_6 **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_3 \otimes \omega_3 = C \oplus (\omega_1 + \omega_5) \oplus (\omega_2 + \omega_4) \oplus 0$
- E_6 **- first node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_4 \otimes \omega_5 = C \oplus \omega_2 \oplus 0$
- E_7 **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_4 \otimes \omega_3 = C \oplus (\omega_2 + \omega_5) \oplus (\omega_1 + \omega_6) \oplus 0$
- E_7 **- first node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_1 \otimes \omega_5 = C \oplus \omega_6 \oplus 0$
- E_8 **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_3 \otimes \omega_5 = C \oplus (\omega_2 + \omega_6) \oplus (\omega_1 + \omega_7) \oplus 0$
- E_8 **- seventh node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_7 \otimes \omega_8 = C \oplus \omega_4 \oplus \omega_2 \oplus 0$
- G_2 **- first node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq \omega_1 \otimes \omega_1 = C \oplus 0$ The ALC is satisfied.
- G_2 **- last node:** $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \simeq 3\omega_1 \otimes 3\omega_1 = C \oplus 4\omega_1 \oplus 2\omega_1 \oplus 0$

Now we proceed to $B^{\mathbb{C}} = B_1 \oplus B_2$. Because B_1 and B_2 are conjugated it is sufficient to consider only one part, say B_1. We split choice of B into two cases. First, when $\mathfrak{g}_0$ is simple algebra and second when it is not simple.
Let $\mathfrak{g}_0$ be a simple Lie algebra.

- A_n **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1$, $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_{n-1} = C \oplus \omega_1$ The ALC is satisfied.
- B_n **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1 \oplus 0$, $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_1 = C \oplus (\omega_1 + \omega_2) \oplus \omega_1$

- B_n **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_{n-1} = 2\omega_{n-1}$, $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_{n-1} \otimes \omega_1 = C \oplus \omega_1$ The ALC is satisfied.

- C_n **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1$, $B_1 \otimes \mathfrak{g}_1 = 2\omega_1 \otimes \omega_1 = C \oplus (\omega_1 + \omega_2) \oplus \omega_1$

- C_n **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2 2\omega_{n-1} = 4\omega_{n-1} \oplus 2\omega_{n-2}$,

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 4\omega_{n-1} \otimes 2\omega_1 = C \oplus (\omega_1 + 3\omega_{n-1}) \oplus 2\omega_{n-1}$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_{n-2} \otimes 2\omega_1 = C \oplus (\omega_1 + \omega_{n-2} + \omega_{n-1}) \oplus 2\omega_{n-1}$. It works similarly for C_3-algebra.

- D_n **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1 \oplus 0$, $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_1 = C \oplus (\omega_1 + \omega_2) \oplus \omega_1$

- D_n **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_{n-2} = 2\omega_{n-2} \oplus \omega_{n-4}$,

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_{n-2} \otimes \omega_2 = C \oplus (\omega_1 + \omega_{n-2} + \omega_{n-1}) \oplus \omega_{n-2}$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_{n-4} \otimes \omega_2 = C \oplus (\omega_1 + \omega_{n-3}) \oplus \omega_{n-2}$

- E_6 **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_4 = 2\omega_4 \oplus \omega_1$,

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_4 \otimes \omega_5 = C \oplus (\omega_2 + \omega_4) \oplus \omega_4$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_1 \otimes \omega_5 = C \oplus \omega_4$. The ALC is satisfied.

- E_6 **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_3 = 2\omega_3 \oplus (\omega_1 + \omega_5)$,

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_3 \otimes \omega_3 = C \oplus (\omega_2 + \omega_3 + \omega_4) \oplus (\omega_1 + \omega_3 + \omega_5) \oplus \omega_3$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_1 + \omega_5) \otimes \omega_3 = C \oplus (\omega_1 + \omega_2) \oplus (\omega_4 + \omega_5) \oplus \omega_3$

- E_7 **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1 \oplus \omega_5$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_5 = C \oplus (\omega_1 + \omega_6) \oplus \omega_1$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_5 \otimes \omega_5 = C \oplus \omega_4 \oplus \omega_1$

- E_7 **- sixth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_5 = 2\omega_5 \oplus \omega_2$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_5 \otimes \omega_5 = C \oplus (\omega_4 + \omega_5) \oplus (\omega_2 + \omega_5) \oplus \omega_5$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_2 \otimes \omega_5 = C \oplus (\omega_1 + \omega_6) \oplus \omega_5$

- E_7 **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_3 = 2\omega_3 \oplus (\omega_1 + \omega_5)$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_3 \otimes \omega_4 = C \oplus (\omega_2 + \omega_3 + \omega_5) \oplus (\omega_1 + \omega_3 + \omega_6) \oplus \omega_3$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_1 + \omega_5) \otimes \omega_4 = C \oplus (\omega_1 + \omega_3 + \omega_6) \oplus 2\omega_5 \oplus (\omega_4 + \omega_6) \oplus (\omega_1 + \omega_2) \oplus \omega_3$

- E_8 **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1 \oplus \omega_6$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_1 = C \oplus (\omega_1 + \omega_2) \oplus (\omega_1 + \omega_6) \oplus \omega_1$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_1 \otimes \omega_6 = C \oplus \omega_7 \oplus \omega_1$

- E_8 **- sevent node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_6 = 2\omega_6 \oplus \omega_3$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_6 \otimes \omega_7 = C \oplus (\omega_4 + \omega_6) \oplus (\omega_2 + \omega_6) \oplus \omega_6$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq \omega_3 \otimes \omega_7 = C \oplus (\omega_2 + \omega_6) \oplus (\omega_1 + \omega_7) \oplus \omega_6$

- E_8 **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_5 = 2\omega_5 \oplus (\omega_3 + \omega_7)$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_5 \otimes \omega_3 = C \oplus (\omega_2 + \omega_5 + \omega_6) \oplus (\omega_1 + \omega_5 + \omega_7) \oplus \omega_5$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_3 + \omega_7) \otimes \omega_3 = C \oplus (\omega_2 + \omega_4 + \omega_7) \oplus (\omega_1 + \omega_5 + \omega_7) \oplus (\omega_2 + \omega_3) \oplus (\omega_1 + \omega_4) \oplus (\omega_6 + \omega_7) \oplus \omega_5$

- G_2 **- first node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2\omega_1 = 2\omega_1$, $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes \omega_1 = 3\omega_1 \oplus \omega_1$ The ALC is satisfied.

- G_2 **- last node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2 3\omega_1 = 6\omega_1 \oplus 2\omega_1$

  1. $B_1 \otimes \mathfrak{g}_1 \simeq 6\omega_1 \otimes 3\omega_1 = 9\omega_1 \oplus 7\omega_1 \oplus 5\omega_1 \oplus 3\omega_1$
  2. $B_1 \otimes \mathfrak{g}_1 \simeq 2\omega_1 \otimes 3\omega_1 = 5\omega_1 \oplus 3\omega_1 \oplus \omega_1$

Now suppose $\mathfrak{g}_0$ is not simple. In this case, irreducible representations of $\mathfrak{g}_0^{ss}$ are tensor product of irreducible representations of simple algebras from which $\mathfrak{g}_0^{ss}$ consist of. Let us suppose, $\mathfrak{g}_0^{ss}$ consist of two simple algebras. Following fact will be helpful in some computations, $\odot^2(A \otimes B) = \odot^2(A) \otimes \odot^2(B) \oplus \wedge^2(A) \otimes \wedge^2(B)$, where A and B are irreducible representations of simple factors. By this decomposition one can easily find whether admissible representation occurs in $\odot^2(A \otimes B)$, where by an admissible representation we mean a representation on which one simple subalgebra acts nontrivialy only.

- A_n **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1, \omega_1) = (2\omega_1, 2\omega_1) \oplus (0, \omega_2)$, $B_1 \otimes \mathfrak{g}_1 \simeq (0, \omega_2) \otimes (\omega_1, \omega_{n-2}) = (\omega_1, C) \oplus (\omega_1, \omega_1)$ The ALC is satisfied.

- A_n **-** k**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{k-1}, \omega_1) = (2\omega_{k-1}, 2\omega_1) \oplus (\omega_{k-2}, \omega_2)$

- B_n **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1, \omega_1) = (2\omega_1, 2\omega_1) \oplus (2\omega_1, 0) \oplus (0, \omega_2)$,

  1. $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_1, 0) \otimes (\omega_1, \omega_1) = (C, \omega_1) \oplus (\omega_1, \omega_1)$ The ALC is satisfied.
  2. $B_1 \otimes \mathfrak{g}_1 \simeq (0, \omega_2) \otimes (\omega_1, \omega_1) = (\omega_1, C) \oplus (\omega_1, \omega_3) \oplus (\omega_1, \omega_1)$

- B_n **-** k**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{k-1}, \omega_1) = (2\omega_{k-1}, 2\omega_1) \oplus (2\omega_{k-1}, 0) \oplus (\omega_{k-2}, \omega_2)$, $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_{k-1}, 0) \otimes (\omega_1, \omega_1) = (C, \omega_1) \oplus (\omega_{k-1}, \omega_1)$. The ALC is satisfied.

- B_n **-** $(n-1)$**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{n-2}, 2\omega_1) = (2\omega_{n-2}, 4\omega_1) \oplus (2\omega_{n-2}, 0) \oplus (\omega_{n-3}, 2\omega_1)$
  $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_{n-2}, 0) \otimes (\omega_1, 2\omega_1) = (C, 2\omega_1) \oplus (\omega_{n-2}, 2\omega_1)$. The ALC is satisfied.

- C_n **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1, \omega_1) = (2\omega_1, 2\omega_1) \oplus (0, \omega_2) \oplus (0, 0)$, $B_1 \otimes \mathfrak{g}_1 \simeq (0, \omega_2) \otimes (\omega_1, \omega_1) = (\omega_1, C) \oplus (\omega_1, \omega_3) \oplus (\omega_1, \omega_1)$. The ALC is satisfied in the case of C_4-algebra because of decomposition $B_1 \otimes \mathfrak{g}_1 \simeq (0, \omega_2) \otimes (\omega_1, \omega_1) = (\omega_1, \omega_2 + \omega_1) \oplus (\omega_1, \omega_1)$.

- C_n **-** k**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{k-1}, \omega_1) = (2\omega_{k-1}, 2\omega_1) \oplus (\omega_{k-2}, \omega_2) \oplus (\omega_{k-2}, 0)$, $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_{k-2}, 0) \otimes (\omega_1, \omega_1) = (C, \omega_1) \oplus (\omega_{k-1}, \omega_1)$. According to the fact above, we choose $\wedge^2(A) \otimes \wedge^2(B)$. In the case $k = 2l + 1$, there is no nondegenerated antisymmetric metric.

- C_n **-** $(n-1)$**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{n-2}, \omega_1) = (2\omega_{n-2}, 2\omega_1) \oplus (\omega_{n-3}, 0)$, $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_{n-3}, 0) \otimes (\omega_1, \omega_1) = (C, \omega_1) \oplus (\omega_{n-2}, \omega_1)$. The ALC is satisfied.

- D_n **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1,\omega_1) = (2\omega_1,2\omega_1) \oplus (2\omega_1,0) \oplus (0,\omega_2)$, $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_1,0) \otimes (\omega_1,\omega_1) = (C,\omega_1) \oplus (\omega_1,\omega_1)$. The ALC is satisfied.

- D_n **-** k**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{k-1},\omega_1) = (2\omega_{k-1},2\omega_1) \oplus (2\omega_{k-1},0) \oplus (\omega_{k-2},\omega_2)$ $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_{k-1},0) \otimes (\omega_1,\omega_1) = (C,\omega_1) \oplus (\omega_{k-1},\omega_1)$. The ALC is satisfied.

- D_n **-** $(n-2)$**-th node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_{n-3},\omega_1,\omega_1) = (2\omega_{n-3},2\omega_1,2\omega_1) \oplus (2\omega_{n-3},0,0) \oplus (\omega_{n-4},2\omega_1,0) \oplus (\omega_{n-4},0,2\omega_1)$. Clearly, if more that one algebra acts nontrivialy there will be too many components.
  $B_1 \otimes \mathfrak{g}_1 \simeq (2\omega_{n-3},0,0) \otimes (\omega_1,\omega_1,\omega_1) = (C,\omega_1,\omega_1) \oplus (\omega_{n-3},\omega_1,\omega_1)$. The ALC is satisfied.

- E_6 **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1,\omega_3) = (2\omega_1,2\omega_3) \oplus (2\omega_1,\omega_1) \oplus (0,\omega_2+\omega_4)$, $B_1 \otimes \mathfrak{g}_1 \simeq (0,\omega_2+\omega_4) \otimes (\omega_1,\omega_2) = (\omega_1,C) \oplus (\omega_1,\omega_1+\omega_3+\omega_4) \oplus (\omega_1,\omega_1+\omega_2) \oplus (\omega_1,2\omega_4) \oplus (\omega_1,\omega_3)$

- E_6 **- third node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_2,\omega_1,\omega_2) = (2\omega_2,2\omega_1,2\omega_2) \oplus (2\omega_2,0,\omega_1) \oplus (\omega_1,0,2\omega_2) \oplus (\omega_1,2\omega_1,\omega_1)$

- E_7 **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1,\omega_4) = (2\omega_1,2\omega_4) \oplus (2\omega_1,\omega_1) \oplus (0,\omega_3)$ $B_1 \otimes \mathfrak{g}_1 \simeq (0,\omega_3) \otimes (\omega_1,\omega_5) = (\omega_1,\omega_3+\omega_5) \oplus (\omega_1,\omega_2+\omega_4) \oplus (\omega_1,\omega_1+\omega_5) \oplus (\omega_1,\omega_4)$

- E_7 **- third node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_2,\omega_3) = (2\omega_2,2\omega_3) \oplus (2\omega_2,\omega_1) \oplus (\omega_1,\omega_2+\omega_4)$

- E_7 **- fourth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_3,\omega_1,\omega_2) = (2\omega_3,2\omega_1,2\omega_2) \oplus (2\omega_3,0,\omega_1) \oplus (\omega_2,2\omega_1,\omega_1) \oplus (\omega_2,0,2\omega_2)$

- E_7 **- fifth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_4,\omega_1) = (2\omega_4,2\omega_1) \oplus (\omega_2,2\omega_1) \oplus (\omega_3+\omega_5,0)$, $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_3+\omega_5,0) \otimes (\omega_2,\omega_1) = (\omega_2+\omega_3+\omega_5,\omega_1) \oplus (\omega_1+\omega_4+\omega_5,\omega_1) \oplus (\omega_1+\omega_3,\omega_1) \oplus (2\omega_5,\omega_1) \oplus (\omega_4,\omega_1)$

- E_8 **- sixth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1,\omega_5) = (2\omega_1,2\omega_5) \oplus (2\omega_1,\omega_3) \oplus (0,\omega_4+\omega_6)$ $B_1 \otimes \mathfrak{g}_1 \simeq (0,\omega_4+\omega_6) \otimes (\omega_1,\omega_2) = (\omega_1,C) \oplus (\omega_1,\omega_1+\omega_5+\omega_6) \oplus (\omega_1,\omega_1+\omega_4) \oplus (\omega_1,2\omega_6) \oplus (\omega_1,\omega_5)$

- E_8 **- fifth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_2,\omega_1,\omega_4) = (2\omega_2,2\omega_1,2\omega_4) \oplus (2\omega_2,0,2\omega_4) \oplus (2\omega_2,0,\omega_3) \oplus (\omega_1,0,2\omega_4)$

- E_8 **- fourth node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_3,\omega_3) = (2\omega_3,2\omega_3) \oplus (\omega_2+\omega_4,\omega_2) \oplus (\omega_1,2\omega_3)$

- E_8 **- third node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_4,\omega_2) = (2\omega_4,2\omega_2) \oplus (\omega_3,\omega_1) \oplus (\omega_1,2\omega_2)$

- E_8 **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_5,\omega_1) = (2\omega_5,2\omega_1) \oplus (\omega_1,2\omega_1) \oplus (\omega_4,0)$ $B \otimes \mathfrak{g}_1 \simeq (\omega_4,0) \otimes (\omega_1,\omega_1) = (\omega_1+\omega_4,\omega_1) \oplus (\omega_5+\omega_6,\omega_1) \oplus (\omega_2,\omega_1) \oplus (\omega_5,\omega_1)$

- F_4 **- second node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(\omega_1,\omega_2) = (2\omega_1,2\omega_2) \oplus (0,\omega_1)$
  $B_1 \otimes \mathfrak{g}_1 \simeq (0,\omega_1) \otimes (\omega_1,\omega_1) = (\omega_1,C) \oplus (\omega_1,\omega_2)$. There is a complication with this result. Clearly, B corresponds to $\wedge^2(A) \otimes \wedge^2(B)$, according to the fact above. Since $dim(B)$ is three, as $\mathfrak{g}_0$-representation, all antisymmetric forms are degenerated.

- F_4 **- third node:** $\odot^2\mathfrak{g}_{-1} \simeq \odot^2(2\omega_2,\omega_1) = (4\omega_2,2\omega_1) \oplus (\omega_1+2\omega_2,0) \oplus (2\omega_1,2\omega_1)$ $B_1 \otimes \mathfrak{g}_1 \simeq (\omega_1+2\omega_2,0) \otimes (2\omega_1,\omega_1) = (C,\omega_1) \oplus (\omega_1+3\omega_2,\omega_1) \oplus (2\omega_1+\omega_2,\omega_1) \oplus (2\omega_2,\omega_1) \oplus (\omega_1,\omega_1)$

$\square$

Some decompositions can be computed by LiE online service.

6. Examples of metric bundles, solutions and metrics

In this chapter we show a few examples how to compute metric tractor bundle and its $\mathfrak{g}_0^{ss}$-decomposition, solutions of the first BGG equation and the corresponding metric.

Definition 5. *By a metric tractor bundle we understand a tractor bundle on which an invariant operator of first order acts.*

Remark. Let the ALC holds, $B \otimes \mathfrak{g}_1 \simeq B \circledcirc \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$. According to construction of the ALC, the invariant operator goes from B to $B \circledcirc \mathfrak{g}_1$ hence it is the first BGG operator.

Remark. Suppose a reductive algebra $\mathfrak{g}_0$ acts on V by the highest weight $\lambda = \lambda' + \lambda^0$, where decomposition is into action of semisimple part and commutative part. An action of one-dimensional center $\mathfrak{z}(\mathfrak{g}_0)$ on l-th tensor power of representation V_λ is given by $l\lambda^0$.

Suppose, $\mathcal{D} : \Gamma(\mathcal{G} \times_P B) \to \Gamma(\mathcal{G} \times_P V)$ is an invariant operator and let $\eta \in Ker\,\mathcal{D}$. If $\mathfrak{z}(\mathfrak{g}_0)$ acts nontrivialy on B, then by using exterior powers we can find covariantly constant section $\wedge^{dim(B)}\eta$ of a line bundle on which $\mathfrak{z}(\mathfrak{g}_0)$ acts nontrivialy. Then by operations with line bundles we can construct a line bundle $\mathcal{L}$ with covariantly constant sections such that action of $\mathfrak{z}(\mathfrak{g}_0)$ on $\mathcal{L}$ is as prescribed.

Let us suppose that $\eta \in Ker\,\mathcal{D}$, where $\mathcal{D} : \Gamma(\mathcal{G} \times_P (B_\lambda \otimes L_c)) \to \Gamma(\mathcal{G} \times_P V)$ and L_c is one dimensional representation described by action of grading element $E \in \mathfrak{z}(\mathfrak{g}_0)$ on L, $E \cdot \ell = c\ell$, $\ell \in L$. If we find a line bundle L_{-c} with covariantly constant section σ we can form covariantly constant section $\eta \otimes \sigma \in \Gamma(\mathcal{G} \times_P B)$. The last section is covariantly constant inverse metric if B is as in the ALC.

We use a symbol $\mathbb{V}$ for $\mathfrak{g}$-representation and V for $\mathfrak{g}_0$-representation.

Example 1 - projective geometry

Let us consider the projective geometry, $\mathfrak{g} \simeq sl(n+1, \mathbb{R})$ with first crossed node. An invariant differential operator of first order is acting on natural bundle which is given by the $\mathfrak{g}$-representation $B = \bullet\!\!-\!\cdots\!-\!\!\overset{2}{\bullet}$ as the first BGG operator. If we consider this representation as $\mathfrak{g}_0$-representation it is given by the highest weight $\lambda = -4\omega_1 + 2\omega_2$. It is symmetric part of second tensor power of defining representation. We compute its $\mathfrak{g}_0^{ss}$-decomposition. Let us start with defining representations and their decompositions:

$$\mathbb{V}_{\omega_1} \otimes \mathbb{V}_{\omega_1} = \begin{pmatrix} c_1 \\ V_\alpha \end{pmatrix} \otimes \begin{pmatrix} c_2 \\ V_\beta \end{pmatrix} = \begin{pmatrix} c \\ V_\alpha \oplus V_\beta \\ V_{\alpha\beta} \end{pmatrix}$$

Now we can define coordinates on metric tractor bundle $\mathcal{V}_B = \begin{pmatrix} c \\ v^a \\ \tau^{ab} \end{pmatrix}$, where $\tau^{ab} = \tau^{(ab)}$ and v^a is in symmetric part. We know from appendix D how to find normal solutions in normal coordinates. In a suitable chart a solution is the algebraic projection of R from metric tractor bundle to the zeroth cohomology, where $R := f(\phi(X)) = exp(-X) \cdot v_0$ and $X \in \mathfrak{g}_-$ by appendix D. In our case $R = v - \rho(X)v + \rho(X)^2 v/2$, where $v \in \mathcal{V}_B$ and ρ is standard representation on symmetric tensor power given by defining representation. An easy computation shows $\rho(X)v = \begin{pmatrix} 0 \\ 2cX^a \\ X^a v^b \end{pmatrix}$ and $\rho(X)^2 v = 2\begin{pmatrix} 0 \\ 0 \\ cX^a X^b \end{pmatrix}$. Now the solution is $\eta^{ab}(X) = \tau^{ab} - X^{(a}v^{b)} + cX^{(a}X^{b)}$.

In the following we want to show that there exists the inverse metric which correspond to η. To aim this it is sufficient to show the grading element E acts nontrivialy on B_λ which is equivalent to evaluation $\lambda(E) \neq 0$. From definition of grading element E it is easy to show

$$E = \begin{pmatrix} \frac{n}{n+1} & 0 & \cdots & 0 \\ 0 & \frac{-1}{n+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-1}{n+1} \end{pmatrix}$$

Since $\lambda(E) = -2$ there exists the metric which correspond to solution η.

Example 2. - 2-Grassmannian geometry

Similarly we do this process for 2-Grassmannian and almost quaternion geometry. Let us recall that the metric bundle is given by $B = \bullet \cdots\cdots \overset{1}{\bullet} - \bullet$ and as $\mathfrak{g}_0$-representation its weight is the $\lambda = -4\omega_2 + \omega_4$. It is the second fundamental representation hence antisymmetric part of second tensor power. Again, we start with defining representations:

$$\mathbb{V}_{\omega_1} \otimes \mathbb{V}_{\omega_1} = \begin{pmatrix} U_\alpha \\ V_\alpha \end{pmatrix} \otimes \begin{pmatrix} U_\beta \\ V_\beta \end{pmatrix} = \begin{pmatrix} U_{\alpha\beta} \\ U_\alpha V_\beta \oplus V_\alpha U_\beta \\ V_{\alpha\beta} \end{pmatrix}$$

We define coordinates as $\mathcal{V}_B = \begin{pmatrix} \nu \\ \tau^{AA'} \\ \zeta^{AB} \end{pmatrix}$, where $\zeta^{AB} = \zeta^{[AB]}$ and τ^{ab} is in antisymmetric part. Again by an elementary computations, we get $\rho(X)v = \begin{pmatrix} 0 \\ - \\ \tau^{A'[A}X^{K]}_{A'} \end{pmatrix}$ and $\rho(X)^2 v = 2\begin{pmatrix} 0 \\ 0 \\ X^{[K}_{[L}X^{K']}_{L']}\nu \end{pmatrix}$. So solutions will have the form $\eta^{AB}(X) = \zeta^{AB} - \tau^{A'[A}X^{B]}_{A'} + X^{[A}_{[A'}X^{B]}_{B']}\nu$.

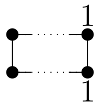
The grading element $E = diag(\frac{n-1}{n+1}, \frac{n-1}{n+1}, \frac{2}{n+1}, \cdots, \frac{2}{n+1})$ gives us $\lambda(E) = \frac{-6n+2}{n+1}$, hence there exists the metric which correspond to solution η.

Example 3. - almost quaternionic geometry

If we consider an almost quaternionic geometry computations will be similar. The only difference is that index A' disappear. It is caused by dimension of complexified quaternions. Coordinates on metric tractor bundle will be $\mathcal{V}_B = \begin{pmatrix} \nu \\ \tau^A \\ \zeta^{AB} \end{pmatrix}$

where ν is a quaternion τ^A is a quaternionic vector which lies in antisymmetric part and $\zeta^{AB} = \zeta^{[AB]}$. Solution is $\eta^{AB}(X) = \zeta^{AB} - \tau^{[A}X^{B]} + 2X^{[A}X^{B]}\nu$.

Example 4. - c-projective geometry.
Let us look at metric tractor bundle. It is A_n-algebra with representation which is given by diagram:



$= V_{\omega_1\overline{\omega_1}}$

It is tensor product of conjugated defining representations, so

$$\mathbb{V}_{\omega_1\overline{\omega_1}} = \mathbb{V}_{\omega_1} \otimes \mathbb{V}_{\overline{\omega_1}} = \begin{pmatrix} c_1 \\ V_\alpha \end{pmatrix} \otimes \begin{pmatrix} c_2 \\ V_{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} c \\ V_\alpha \oplus V_{\bar{\alpha}} \\ V_{\alpha\bar{\alpha}} \end{pmatrix}$$

In the metric tractor bundle $\mathcal{V}_B$, we will use following coordinates $\begin{pmatrix} c \\ v^a \\ \tau^{ab} \end{pmatrix}$. Now we want to describe solutions of the corresponding equation $\mathcal{D}\eta = 0$ for $\mathcal{D} : G \times_P B \to G \times_P (B \circledcirc \mathfrak{g}_1)$. Computations are identical with projective case, $\rho(X)u = \begin{pmatrix} 0 \\ (c_1 + c_2)X^a \\ v^a X^b \end{pmatrix}$ and $\rho^2(X)u = \begin{pmatrix} 0 \\ 0 \\ 2cX^{ab} \end{pmatrix}$, where $u \in \mathcal{V}_B$. Therefore $\eta^{ab}(X) = \tau^{ab} - v^a X^b + cX^{ab}$.

Computations for nontrivial action of the center is almost the same as in projective case. The only exception is that in c-projective geometry we use semisimple algebra $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ instead of simple algebra, hence grading element is of the form (E, E).

7. Reducible $\mathfrak{g}_{-1}$-cases

In this chapter, we show a way how to the ALC can be generalized to reducible $\mathfrak{g}_{\pm 1}$-part. This procedure is similar to the end of the proof of theorem 7 in chapter 3 and to construction of invariant operators in section 2.2.

Let us suppose complex k-cross geometry. In this case we have decompositions of $\mathfrak{g}_{\pm 1}$-parts into $\bigoplus_{i=1}^{k} \mathfrak{g}_{\pm 1}^{i}$, by the Kostant's version of the Bott-Borel-Weil theorem. Let V be complex irreducible $\mathfrak{g}_0$-representation. We can find an invariant first order differential operator $\mathcal{D} : \Gamma(\mathcal{G} \times_P V) \to \Gamma(\mathcal{G} \times_P \bigoplus_{i}^{k} (V \circledcirc \mathfrak{g}_1^i))$. Let us recall $\nabla|_{\mathcal{H}} : \Gamma(\mathcal{G} \times_P V) \to \Gamma(\mathcal{G} \times_P (V \otimes \mathfrak{g}_1))$. If we consider $\eta \in Ker\, \mathcal{D}$ we have $\pi \circ \nabla|_{\mathcal{H}} \eta = 0$ for every Weyl covariant derivative ∇, where π is bundle map between natural bundles which arise from natural projection $V \otimes \mathfrak{g}_1 \to \bigoplus_{i}^{k} (V \circledcirc \mathfrak{g}_1^i)$ and $\mathcal{H} = \mathcal{G} \times_P \mathfrak{g}_{-1}$. If we suppose that V is not trivial representation and $V \subset \odot^2 \mathfrak{g}_{-1}^{i_0}$ for some $i_0 \in \{1, \cdots, k\}$, then it is possible to choose Weyl covariant derivative ∇ in such a way that $\pi' \circ \nabla|_{\mathcal{H}} \eta = 0$ for $\eta \in Ker\, \mathcal{D}$, where π' is the map between natural bundles which is induced by natural projection $V \otimes \mathfrak{g}_1 \to \bigoplus_{i}^{k} (V \circledcirc \mathfrak{g}_1^i) \oplus \mathfrak{g}_{-1}^{i_0}$.

Now, we suppose $B_l \subset \odot^2 \mathfrak{g}_{-1}^{l}$ is nontrivial irreducible $\mathfrak{g}_0$-representation and $B_l \otimes \mathfrak{g}_1 = \bigoplus_{i}^{k} (B_l \circledcirc \mathfrak{g}_1^i) \oplus \mathfrak{g}_{-1}^{l}$ for all $l \in \{1, \cdots, k\}$. Clearly, $B := \bigoplus_{i}^{k} B_i$ corresponds to nondegenerated inverse metrics. By symbol $\mathcal{D}_l$ we denote first order invariant differential operator which corresponds to B_l in the sense as above. If $\eta \in Ker\, \mathcal{D}$ then there exists Weyl covariant derivative ∇ such that $\nabla|_{\mathcal{H}} \eta = 0$, where $\mathcal{D} = \bigoplus \mathcal{D}_l$. Let us mention, if decomposition of $B_l \otimes \mathfrak{g}_1$ has less then $k+1$ components it does not make a problem. These operators exist after a suitable choice of action of $\mathfrak{z}(\mathfrak{g}_0)$ which was explained in chapter 2. Therefore we skip details concerning manipulation with suitable line bundles.

By appropriate manipulation with invariant operators, this construction can be applied in real k-cross geometries, too.

Now, we illustrate this method in a few examples. We will consider $\mathfrak{g}_0^{ss}$-representations only. In the following, it is easy to see that $\mathfrak{g}_0^{ss}$-representation B which is trivial has nontrivial action of center $\mathfrak{z}(\mathfrak{g}_0)$.

A_n-case:
Let $\mathfrak{g}$ be a Lie algebra of A_n-type for $n \geq 3$. Let us consider complex two-cross geometry, which arise from two crosses placed together in the left of Dynkin diagram. For $n = 3$ both $\mathfrak{g}_{\pm 1}$-parts have the highest weight $0 \oplus \omega_1$. Clearly, $\odot^2 \mathfrak{g}_{-1} = 0 \oplus 2\omega_1 \oplus \omega_1$. According to the considerations above, we have $B = B_1 \oplus B_2 = 0 \oplus 2\omega_1$ and B corresponds to nondegenerated metrics. Tensor product $B \otimes \mathfrak{g}_1 = (B_1 \otimes \mathfrak{g}_1) \oplus (B_2 \otimes \mathfrak{g}_1)$ decomposes to $(0 \oplus \omega_1) \oplus (2\omega_1 \oplus 3\omega_1 \oplus \omega_1)$. Therefore, we can construct two invariant operators $\mathcal{D}_1 : \Gamma(\mathcal{G} \times_P B_1) \to \Gamma(\mathcal{G} \times_P V_{\omega_1})$ and $\mathcal{D}_2 : \Gamma(\mathcal{G} \times_P B_2) \to \Gamma(\mathcal{G} \times_P (V_{2\omega_1} \oplus V_{3\omega_1}))$ such that $\mathcal{D}_1$ can be corrected on bundle which is given by the trivial representation and $\mathcal{D}_2$ can be corrected on $\Gamma(\mathcal{G} \times_P V_{\omega_1})$. So, there exists a Weyl covariant derivative ∇ such that $\nabla|_{\mathcal{H}} \eta = 0$ for $\eta \in Ker\, (\mathcal{D}_1 \oplus \mathcal{D}_2)$.

The case $n > 3$ is similar but with different representations. Namely, represen-

tation $\mathfrak{g}_{-1}$ has the highest weight $0 \oplus \omega_1$ and the highest weight of representation $\mathfrak{g}_1$ is $0 \oplus \omega_{n-2}$. Second symmetric power of $\mathfrak{g}_{-1}$ decomposes into $0 \oplus 2\omega_1 \oplus \omega_1$. We choose $B = B_1 \oplus B_2 = 0 \oplus 2\omega_1$. Now, tensor product $B \otimes \mathfrak{g}_1 = (B_1 \otimes \mathfrak{g}_1) \oplus (B_2 \otimes \mathfrak{g}_1)$ decomposes into $(0 \oplus \omega_{n-2}) \oplus (2\omega_1 \oplus (2\omega_1 + \omega_{n-2}) \oplus \omega_1)$. Again, we see there are two invariant operators which can be corrected.
Now, suppose that geometry arise from Dynkin diagram in which two crosses are placed together in such a way that $\mathfrak{g}_0^{ss}$ is of $(A_k \oplus A_{n-k-2})$-type. First nonsymmetric position is in Dynkin diagram of A_n-type with $k = 1$ and $n \geq 5$. In this case we have following list of informations:

- $\mathfrak{g}_{-1} \simeq (\omega_1, 0) \oplus (0, \omega_1)$
- $\mathfrak{g}_1 \simeq (\omega_1, 0) \oplus (0, \omega_{n-3})$
- $\odot^2 \mathfrak{g}_{-1} \simeq (2\omega_1, 0) \oplus (\omega_1, \omega_1) \oplus (0, 2\omega_1)$, $B = B_1 \oplus B_2 \simeq (2\omega_1, 0) \oplus (0, 2\omega_1)$
- $B \otimes \mathfrak{g}_1 = [B_1 \otimes \mathfrak{g}_1] \oplus [B_2 \otimes \mathfrak{g}_1] \simeq [(3\omega_1, 0) \oplus (\omega_1, 0) \oplus (2\omega_1, \omega_{n-3})] \oplus [(\omega_1, 2\omega_1) \oplus (0, 2\omega_1 + \omega_{n-3}) \oplus (0, \omega_1)]$

We can get two invariant operators and make corrections of both operators on components of $\mathfrak{g}_{-1}$-part.

Last way how to obtain reducible $\mathfrak{g}_{-1}$-part is to put pair of crosses together in Dynkin diagram in such a way that $k \geq 2$, $n - k - 2 \geq 2$ and $k \neq n - k - 2$. The first example of such diagram is A_7 with $k = 2$. Again, we compute generalized ALC for geometries which arise from diagrams as described above. We have the following list:

- $\mathfrak{g}_{-1} \simeq (\omega_k, 0) \oplus (0, \omega_1)$
- $\mathfrak{g}_1 \simeq (\omega_1, 0) \oplus (0, \omega_{n-k-2})$
- $\odot^2 \mathfrak{g}_{-1} \simeq (2\omega_k, 0) \oplus (\omega_k, \omega_1) \oplus (0, 2\omega_1)$, $B = B_1 \oplus B_2 \simeq (2\omega_k, 0) \oplus (0, 2\omega_1)$
- $B \otimes \mathfrak{g}_1 = [B_1 \otimes \mathfrak{g}_1] \oplus [B_2 \otimes \mathfrak{g}_1] \simeq [(\omega_1 + 2\omega_k, 0) \oplus (\omega_k, 0) \oplus (2\omega_k, \omega_{n-k-2})] \oplus [(\omega_1, 2\omega_1) \oplus (0, 2\omega_1 + \omega_{n-k-2}) \oplus (0, \omega_1)]$

As in previous case, we can form two invariant operators which can be corrected on components of $\mathfrak{g}_{-1}$-part.

This procedure can be mimic in different types of parabolic geometries, i.e. we can consider Dynkin diagram of different type than A and make crosses as above.

8. Appendix A - Homogeneous spaces

All definitions and informations in this appendix can be found in [10]. So proofs and more careful descriptions are contained there.

Klein geometry

Let G be a Lie group and let $H \subset G$ be a closed subgroup. Then H is a Lie subgroup of G and a space of a cosets G/H is canonically a smooth manifold endowed with a transitive left action of G.

Definition 6. *Let G be a Lie group and let $H \subset G$ be a closed subgroup. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h}$ be the Lie algebra of H. The pair (G, H) is called Klein geometry if the group of the automorphisms of G/H is exactly the set $\{\ell_g | g \in G\}$ where ℓ_g is a left action by an element of G. The kernel $K \subset G$ of the Klein geometry is the set $\{g \in G | \ell_g = id_{G/H}\}$. A Klein geometry is called*

- *effective, if the action of G on G/H is effective.*
- *infinitesimally effective, if the kernel K is discrete.*
- *reductive, if there is an H-invariant subspace $\mathfrak{n} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as an H-module.*
- *split, if there is a Lie algebra $\mathfrak{g}_- \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ as vector spaces.*

It is easy to see that $K \subset H$ and K is normal subgroup of G.

Lemma 12. *Let (G, H) be a Klein geometry. Then the kernel K is the maximal normal subgroup of G which is contained in H. Moreover, the Klein geometry is effective if and only if there is no nontrivial normal subgroup of G which is contained in H.*

Homogeneous bundles

A homogeneous bundles are objects associated to homogeneous spaces in the sense of a Klein geometries. Let (G, H) be a Klein geometry. There is the canonical projection $p : G \to G/H$ which is an H-principal bundle. By $\ell : G \times G/H \to G/H$ we denote the canonical left action $\ell(g, g'H) := gg'H$.

Definition 7.
- *A homogeneous bundle over G/H is a locally trivial fiber bundle $\pi : E \to G/H$ together with a left G-action $\tilde{\ell} : G \times E \to E$ which lifts the action on G/H.*
- *A homogeneous vector bundle over G/H is a homogeneous bundle $\pi : E \to G/H$, which is a vector bundle and such that for each element $g \in G$ the bundle map $\tilde{\ell}_g : E \to E$ is a vector bundle homomorphism.*

- *A homogeneous principal bundle is a homogeneous bundle $\pi : E \to G/H$ which is a principal bundle and such that for all $g \in G$ the bundle map $\tilde{\ell}_g$ is a homomorphism of principle bundles.*

- *A morphism of homogeneous bundles (respectively homogeneous vector bundles or homogeneous principal bundles) is a G-equivariant bundle map (respectively G-equivariant homomorphism of vector bundles or principal bundles) which covers the identity on G/H. This property is sometimes called intertwining property and morphism is called intertwining operator on homogeneous bundles.*

Theorem 13. *Every associated bundle to a homogeneous (vector) bundle is again a homogeneous (vector) bundle, respectively.*

As an example of the last theorem the tangent bundle $T(G/H)$ of G/H is the homogeneous vector bundle corresponding to a H-representation on $\mathfrak{g}/\mathfrak{h}$. This H-representation is the restriction of the adjoint representation of G. By the naturality of the correspondence between homogeneous vector bundles and H-representations, this implies that the cotangent space $T^*(G/H)$ corresponds to the dual representation $(\mathfrak{g}/\mathfrak{h})^*$. Again, by naturality, the tensor bundle $\otimes^k T(G/H) \otimes \otimes^l T^*(G/H)$ corresponds to the representation $\otimes^k \mathfrak{g}/\mathfrak{h} \otimes \otimes^l (\mathfrak{g}/\mathfrak{h})^*$.

There is a way how to describe the tangent bundle $T(G/H)$ as a first order G-structure which use the Maurer-Cartan form as a fundamental tool. This description leads to the notion of the Cartan geometry. But first we state a special case of the geometric version of Frobenius reciprocity, which connect a sections of homogeneous bundles and an invariant elements.

Theorem 14. *Let $E \to G/H$ be a homogeneous bundle with a standard fiber E_0 (viewed as H-space). Then there is a natural bijection between the set $\Gamma(E)^G$ of G-invariant sections of E and the set $(E_0)^H$ of H-invariant elements in the standard fiber.*

There is at least one interesting example as a consequence of the theorem. Namely, it allows us to reduce questions about the existence of an invariant tensor fields (e.g. Riemannian metric) to problems of finite-dimensional representation theory. Since $\otimes^k T(G/H) \otimes \otimes^l T^*(G/H)$ is homogeneous vector bundle with standard fibre $\otimes^k \mathfrak{g}/\mathfrak{h} \otimes \otimes^l (\mathfrak{g}/\mathfrak{h})^*$, it follows from the Frobenius reciprocity theorem that G-invariant tensors fields are in bijective correspondence with H-invariant elements of $\otimes^k \mathfrak{g}/\mathfrak{h} \otimes \otimes^l (\mathfrak{g}/\mathfrak{h})^*$.

The following theorem is a result on classification of homogeneous principal bundles.

Theorem 15. *Let G and K be a Lie groups and let $H \subset G$ be a closed subgroup. Let $\mathcal{P} \to G/H$ be a homogeneous principal bundle with structure group K. Then there is a smooth homomorphism $i : H \to K$ such that $\mathcal{P} \simeq G \times_H K$, where the action of H on K is given by $h \cdot k = i(h)k$ for $h \in H$ and $k \in K$.*

The bundles corresponding to two homomorphisms $i, \hat{i} : H \to K$ are isomorphic (over $id_{G/H}$) if and only if i and $\hat{i}$ are conjugate, i.e., $\hat{i}(h) = ki(h)k^{-1}$ for some fixed $k \in K$ and for all $h \in H$.

Let us recall the definition of an invariant differential operators in the case of a homogeneous spaces.

Definition 8. *Let E, F be a homogeneous vector bundles over G/H. An invariant differential operator is a differential operator $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ which is G-equivariant, i.e. such that $\mathcal{D}(g \cdot s) = g \cdot \mathcal{D}(s)$ for all $g \in G$ and for all $s \in \Gamma(E)$.*

Now we explain how jet prolongation help us to characterize linear differential operators. Let $E \to M$ be arbitrary vector bundle over smooth manifold M, then for $k \in \mathbb{N}$ we have the k-jet prolongation $J^k E$ which is again vector bundle over M. If F is another vector bundle over M, then a differential operator $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ is of order $\leq k$ if and only if for any two sections $s, t \in \Gamma(E)$ and any point $x \in M$, the equation $j^k_x s = j^k_x t$ implies $\mathcal{D}(s)(x) = \mathcal{D}(t)(x)$. If $\mathcal{D}$ is such an operator, then we get an iduced bundle map $\tilde{\mathcal{D}}^k_J E \to F$ over the identity on M, defined by $\tilde{\mathcal{D}}(j^k_x s) := \mathcal{D}(s)(x)$. Conversely, this formula associates a differential operator to any bundle map $\tilde{\mathcal{D}}$ $\mathcal{D}$. Clearly, $\mathcal{D}$ is linear if and only if $\tilde{\mathcal{D}}$ is a homomorphism of vector bundles.

9. Appendix B - Parabolic geometry

First we need a few concepts concerning on the Cartan geometry and then we will define a parabolic geometry. We are following the book [10].

Cartan geometry

Definition 9. *Let $H \subset G$ be a Lie subgroup of a Lie group G and let $\mathfrak{g}$ and $\mathfrak{h}$ be a Lie algebra of G and H, respectively. A Cartan geometry of type (G, H) on manifold M is a principal fibre bundle $p : \mathcal{P} \to M$ with structure group H which is endowed with a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(M, \mathfrak{g})$, called Cartan connection, such that:*

$$(r^h)^*\omega = Ad(h^{-1}) \circ \omega, \ \forall h \in H$$

$$\omega(\zeta_X(u)) = X, \ \forall X \in \mathfrak{h}$$

$$\omega(u) : T_u\mathcal{P} \to \mathfrak{g} \text{ is a linear isomorphism for all } u \in \mathcal{P}$$

where ζ_X are fundamental vector fields.

The homogeneous model for the Cartan geometry of type (G, H) is the Klein geometry of type (G, H) endowed with canonical Cartan connection which is Maurer-Cartan form.

Parabolic geometry

Before we define a parabolic geometry we need a few notions and relations. In a parabolic geometries there is an important notion of $|k|$-graded semisimple Lie algebra.

Definition 10. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $k > 0$ be an integer. A $|k|$-grading on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ into direct sum of subspaces such that*

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, *where* $\mathfrak{g}_i = \{0\}$ *if* $|i| > k$
- *the subalgebra* $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ *is generated (as a Lie algebra) by* $\mathfrak{g}_{-1}$
- $\mathfrak{g}_{-k} \neq \{0\}$ *and* $\mathfrak{g}_k \neq \{0\}$

Lemma 16. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra. Then $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is nilpotent ideal in $\mathfrak{p}$. Similarly, the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is nilpotent.*

The $\mathfrak{g}_0$-part is a subalgebra and the adjoint action makes each $\mathfrak{g}_i$ into a $\mathfrak{g}_0$-module such that the bracket $[,] : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ is a $\mathfrak{g}_0$-homomorphism.

Parallel important notion with grading is filtration.

Definition 11. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra. We define filtration of $\mathfrak{g}$ by $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$.*

Among other things there is one important element in the centre of $\mathfrak{g}_0$.

Lemma 17. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be nondegenerate invariant bilinear form. Then*

- *There is a unique element $E \in \mathfrak{g}$, called the grading element, such that $[E, X] = jX$ for all $X \in \mathfrak{g}_j$, $j = -k, \cdots, k$. The element E lies in the center of the subalgebra $\mathfrak{g}_0$.*
- *The isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ provided by B is compatible with the filtration and the grading of $\mathfrak{g}$. In particular, B induces dualities of $\mathfrak{g}_0$-modules between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ and the filtration component $\mathfrak{g}^i$ is exactly the annihilator (with respect to B) of $\mathfrak{g}^{-i+1}$. Hence, B induces a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $\mathfrak{g}^i$ and in particular between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$.*

Now we recall some consequences on the group level.

Lemma 18.

- *The Lie subalgebras $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$ can be characterized as*

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} |\, ad(X)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i = -k, \cdots, k\},$$

$$\mathfrak{p} = \{X \in \mathfrak{g} |\, ad(X)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for all } i = -k, \cdots, k\},$$

- *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then $P := \cap_{i=-k}^{k} N_G(\mathfrak{g}^i) \subset G$ is a closed subgroup with a Lie algebra $\mathfrak{p}$, where $N_G(A) := \{g \in G |\, Ad(g)(A) \subset A\}$ is a stabilizer.*

Definition 12. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra and let G be a Lie group with Lie algebra $\mathfrak{g}$.*

- *A parabolic subgroup of G corresponding to the given $|k|$-grading is a subgroup $P \subset G$ which lies between $\cap_{i=-k}^{k} N_G(\mathfrak{g}^i)$ and its connected component of the identity.*
- *Given a parabolic subgroup $P \subset G$, we define the Levi subgroup $G_0 \subset P$ by*

$$G_0 = \{g \in P |\, Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i = -k, \cdots, k\}$$

Note that any parabolic subgroup has Lie algebra $\mathfrak{p}$ and by definition it is a closed subgroup. Finally we can state the definition of parabolic geometry.

Definition 13. *A parabolic geometry is a Cartan geometry of type (G, P), where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup corresponding to some $|k|$-grading of the Lie algebra $\mathfrak{g}$ of G.*

Obviously, a homogeneous model for parabolic geometries are Klein geometries of a given type. Namely if we have a parabolic geometry of type (G, P), the Klein geometry of type (G, P) is homogeneous bundle with semisimple group G and parabolic subgroup P and on algebra level we have corresponding grading. So we can use theory of homogeneous spaces in the case of homogeneous model for parabolic geometry.

Now we state a few theorems about representation theory which are closely related to parabolic geometries.

Definition 14. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie algebra that contains a Borel subalgebra $\mathfrak{b}$ which can be characterized as $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ where $\mathfrak{h}$ is a Cartan subalgebra and Δ^+ is the set of positive roots.*

There is a natural question how to describe all standard parabolic subalgebras of $\mathfrak{g}$.

Theorem 19. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra, Δ the corresponding system of roots and Δ^0 the set of simple roots. Then standard parabolic subalgebras $\mathfrak{p}$ are in bijective correspondence with subsets $\Sigma \subset \Delta^0$.*

Similar theorems hold in real versions but there is a few complications with real representation theory. Now we state theorem about $\mathfrak{p}$-dominant weights. These are important in constructing Hasse diagram which in turn is important for BGG operators. First, let us explain notation. The Weyl group of $\mathfrak{g}$ will be denoted by $W_{\mathfrak{g}}$. The set of those elements of $W_{\mathfrak{g}}$ which send $\mathfrak{g}$-dominant weights to $\mathfrak{p}$-dominant weights is denoted by $W^{\mathfrak{p}}$.

Theorem 20. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{p}$ be the standard parabolic subalgebra corresponding to a set Σ of simple roots. Let $\delta^{\mathfrak{p}}$ be the sum of all fundamental weights corresponding to elements of Σ. Then the map $\omega \mapsto \omega^{-1}(\delta^{\mathfrak{p}})$ restricts to a bijection between $W^{\mathfrak{p}}$ and the orbit of $\delta^{\mathfrak{p}}$ under $W_{\mathfrak{g}}$.*

There is a fundamental theorem which has key consequences to our computations. It is Kostant's version of the Bott-Borel-Weil theorem.

Theorem 21. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ a standard parabolic subalgebra, W the Weyl group of $\mathfrak{g}$, $W^{\mathfrak{p}}$ the set of the elements of W which send $\mathfrak{g}$-dominant weights to $\mathfrak{p}$-dominant weights and δ the sum of all fundamental weights of $\mathfrak{g}$. For a finite-dimensional complex irreducible representation V of $\mathfrak{g}$ with highest weight λ, consider the Lie algebra cohomolohy $H^*(\mathfrak{p}_+, V)$ and for $\mathfrak{g}_0$-dominant weight ν, let $H^*(\mathfrak{p}_+, V)^\nu$ be the isotypical component of highest weight ν for the natural $\mathfrak{g}_0$-representation on the cohomology. Then we have:*

1. *$H^*(\mathfrak{p}_+, V)^\nu \neq \{0\}$ if and only if there is an element $w \in W^{\mathfrak{p}}$ such that $\nu = \nu_w := w(\lambda + \delta) - \delta$.*
2. *For any $w \in W^{\mathfrak{p}}$, the isotypical component $H^*(\mathfrak{p}_+, V)^{\nu_w}$ is irreducible and even the multiplicity of ν_w as a weight of $C^*(\mathfrak{p}_+, V)$ is one, where $C^*(\mathfrak{p}_+, V)$ stands for co-chains in cohomology. In particular, the set of irreducible components of $H^*(\mathfrak{p}_+, V)$ is in bijective correspondence with $W^{\mathfrak{p}}$.*
3. *For $w \in W^{\mathfrak{p}}$, the isotypical component $H^*(\mathfrak{p}_+, V)^{\nu_w}$ is contained in the space $H^{\ell(w)}(\mathfrak{p}_+, V)$ where $\ell(w)$ is the length of w.*

A simple computation shows $H^1(\mathfrak{g}_-, \mathbb{K}) \simeq (\mathfrak{g}_{-1})^* \simeq \mathfrak{g}_1$ where $\mathbb{K}$ is a field of complex or real numbers.

10. Appendix C - Weyl structures

Here we describe a notion of a Weyl structure and a Weyl connection. Again, more details are in [10].

Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$, let $P \subset G$ be a parabolic subgroup for the given grading and $G_0 \subset P$ the Levi subgroup. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and consider the underlying principal G_0-bundle $p_0 : \mathcal{G}_0 \to M$, which is by definition $\mathcal{G}/P_+$. So there is a natural projection $\pi : \mathcal{G} \to \mathcal{G}_0$ which is a principal bundle with structure group P_+.

Definition 15. *A (local) Weyl structure for the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is a (local) smooth G_0-equivariant section of $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$.*

Fundamental result about Weyl structures is that there always exists a global Weyl structure. Moreover the space of all Weyl structures is in bijective correspondence with sections of graded cotangent space. This result is similar to properties of covariant derivatives and affine vector spaces.

Theorem 22. *For any parabolic geometry $(p : \mathcal{G} \to M, \omega)$ there exists a global Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$.*

Fixing one Weyl structure σ there is a bijective correspondence between the set of all Weyl structures and the space $\Gamma(gr(T^*M))$ of smooth sections of the associated graded of the cotangent bundle. Explicitly, this correspondence is given by mapping $\Upsilon \in \Gamma(gr(T^*M))$ with corresponding functions $\Upsilon_i : \mathcal{G}_0 \to \mathfrak{g}_i$ for $i = 1, \cdots, k$ to the Weyl structure $\hat{\sigma}(u) := \sigma(u)exp(\Upsilon_1(u))\cdots exp(\Upsilon_k(u))$.

There is a obvious way how to connect a Cartan connection ω and a Weyl structure σ. We can consider pullback $\sigma^*\omega \in \Omega(\mathcal{G}_0, \mathfrak{g})$. Now we can use the grading of algebra $\mathfrak{g}$ and decompose this one-form according to grading, namely $\sigma^*\omega = \sigma^*\omega_{-k} + \cdots + \sigma^*\omega_k$. Let us note every component of this one-form is G_0-equivariant.

Theorem 23. *Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure on a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then we have:*

- *The component $\sigma^*\omega_0 \in \Omega(\mathcal{G}_0, \mathfrak{g}_0)$ defines a principal connection on the bundle $p_0 : \mathcal{G}_0 \to M$.*
- *The components $\sigma^*\omega_{-k}, \cdots, \sigma^*\omega_{-1}$ can be interpreted as defining element of $\Omega(M, gr(TM))$. This form determines an isomorphism $TM \to gr(TM)$ which is a splitting of the filtration of TM. This means that for each $i = -k, \cdots, -1$ the subbundle T^iM is mapped to $\bigoplus_{j \geq i} gr_j(TM)$ and the component in $gr_i(TM)$ is given by the canonical surjection $T^iM \to T^iM/T^{i+1}M$.*
- *The components $\sigma^*\omega_1, \cdots, \sigma^*\omega_k$ can be interpreted as a one-form $\mathrm{P} \in \Omega(M, gr(T^*M))$.*

Definition 16. *Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure on a parabolic geometry $(p : \mathcal{G} \to M, \omega)$.*

- *The principal connection $\sigma^*\omega_0$ on the bundle $\mathcal{G}_0 \to M$ is called the Weyl connection associated to the Weyl structure σ.*
- *The $gr(TM)$-valued one-form on M determined by the negative components of $\sigma^*\omega$ is called soldering form associated to the Weyl structure σ.*
- *The one-form $\mathrm{P} \in \Omega(M, gr(T^*M))$ induced by the positive components of $\sigma^*\omega$ is called the Rho tensor associated to the Weyl structure σ.*

Now we will proceed to the theorem about Weyl connections on associated natural bundles.

Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let $\ell : P \times S \to S$ be a smooth left action of the group P on a smooth manifold S. We can restrict this action to a smooth left action $\underline{\ell} : G_0 \times S \to S$. We can form the associated bundle $\mathcal{G} \times_P S \to M$ via the action ℓ, while $\underline{\ell}$ gives rise to a fiber bundle $\mathcal{G}_0 \times_{G_0} S \to M$. We know that we have a Weyl connection on the associated bundle $\mathcal{G}_0 \times_{G_0} S \to M$.

Theorem 24. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let S be a smooth manifold endowed with a smooth left P-action. Then choosing a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ induces an isomorphism $\mathcal{G} \times_P S \simeq \mathcal{G}_0 \times_{G_0} S$ and thus gives rise to a connection on the natural bundle $\mathcal{G} \times_P S$. In the case of a natural vector bundle this connection is automatically earilinear.*

We need last observation about effect of a change of a Weyl structures on Weyl connection. First we need a few new notions. A tractor bundle is an associated bundle which correspond to restricted representation of G to P. Let V be a G-module then it is P-module too and we can form an associated bundle $\mathcal{G} \times_P V$ which is a tractor bundle. There is one tractor bundle, called the adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$, which is of great importance, where the action of P on $\mathfrak{g}$ is the restricted adjoint action of G. There is an action $\mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$ which is realized by derivative ρ' if $\rho : G \to GL(V)$ is a representation. There is an obvious filtration on $\mathcal{A}M$ which rise from the filtration of the Lie algebra $\mathfrak{g}$, namely $\mathcal{A}M = \mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^{k}M$. As usual we have a grading of the adjoint tractor bundle, $gr_i(\mathcal{A}M) = \mathcal{A}^iM/\mathcal{A}^{i+1}M$.

Now we focus on slightly different set-up. Let V be a complete reducible representation of P and let $\mathcal{V}M = \mathcal{G} \times_P V = \mathcal{G}_0 \times_{G_0} V$. From the above there is an action $gr_0(\mathcal{A}M) \times \mathcal{V}M \to \mathcal{V}M$ which we denote by $\bullet$. Let $\underline{i} = (i_1, \cdots, i_k)$ where i_j are nonnegative integers for $j = 1, \cdots, k$. Let us denote $||\underline{i}|| := i_1 + 2i_2 + \cdots + ki_k$ and $(-1)^{\underline{i}} := (-1)^{i_1 + \cdots + i_k}$ and $\underline{i}! := i_1! \cdots i_k!$.

Theorem 25. *Let σ and $\hat{\sigma}$ be two Weyl structures related by*

$$\hat{\sigma}(u) = \sigma(u) exp(\Upsilon_1(u)) \cdots \exp(\Upsilon_k(u)),$$

with corresponding section $\Upsilon = (\Upsilon_1, \cdots, \Upsilon_k)$ of $gr(T^*M)$. For a smooth section ν of a bundle $\mathcal{V}M$ associated to a completely reducible representation of P, the Weyl covariant derivatives ∇ and $\hat{\nabla}$ are related by

$$\hat{\nabla}_\xi \nu = \nabla_\xi \nu + [\![\Upsilon, \xi]\!] \cdot \nu$$

where $(\xi)_\sigma = (\xi_{-k}, \cdots, \xi_{-1})$ *and*

$$[\![\Upsilon, \xi]\!] \cdot \nu = \sum_{||\underline{i}||+j=0} \frac{(-1)^{\underline{i}}}{\underline{i}!} \left(ad(\Upsilon_k)^{i_k} \circ \cdots \circ ad(\Upsilon_1)^{i_1}(\xi_j) \right) \bullet \nu$$

11. Appendix D - BGG resolutions

In this appendix we use constructions of [6],16, [9].

Lie algebra $\mathfrak{p}_+$-homology with values in W

Let us consider a decomposition of a Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ as vector spaces. Now we define the space $C_k(\mathfrak{p}_+, W) = \wedge^k \mathfrak{p}_+ \otimes W$ of k-chains on $\mathfrak{p}_+$ with values in W where W is a $\mathfrak{g}$-module. This spaces carries a natural action of $\mathfrak{p}$, the action on W is the restriction of the $\mathfrak{g}$ action and $\xi \in \mathfrak{p}$ acts on $\beta \in W$ ordinary as $\xi \cdot \beta = \sum_i [\xi, e^i] \wedge (e_i \lrcorner \beta)$ where e^i is a basis of $\mathfrak{p}_+$ and e_i is dual basis.

Now we define the boundary operator or codifferential $\delta : C_k(\mathfrak{p}_+, W) \to C_{k-1}(\mathfrak{p}_+, W)$ as $\delta(\beta \otimes w) = \sum_i (\frac{1}{2} e^i \cdot (e_i \lrcorner \beta) \otimes w + e_i \lrcorner \beta \otimes e^i \cdot w)$.

Lemma 26. *The boundary operator δ has property $\delta^2 = 0$ and for $\alpha \in \mathfrak{p}_+$, $c \in C_k(\mathfrak{p}_+, W)$ $\delta(\alpha \cdot c) = \alpha \cdot \delta(c)$ holds.*

Definition 17. *The cycles Z_k, boundaries B_k and homology H_k of $\delta : C_k(\mathfrak{p}_+) \to C_{k-1}(\mathfrak{p}_+, W)$ are given by:*

$$Z_k(\mathfrak{p}_+, W) := Ker\, \delta$$

$$B_k(\mathfrak{p}_+, W) := Im\, \delta$$

$$H_k(\mathfrak{p}_+, W) := Z_k(\mathfrak{p}_+, W) / B_k(\mathfrak{p}_+, W)$$

All these spaces are $\mathfrak{p}$-modules by construction. It can be shown $\mathfrak{p}_+$ maps $Z_k(\mathfrak{p}_+, W)$ into $B_k(\mathfrak{p}_+, W)$ and hence acts trivially on the homology $H_k(\mathfrak{p}_+, W)$.

Lie algebra $\mathfrak{g}_-$-cohomology with values in W

Now denote $C^k(\mathfrak{g}_-, W) = \wedge^k \mathfrak{p}_+ \otimes W$ the space of k-cochains on $\mathfrak{g}_-$ with values in W. It carries a natural $\mathfrak{p}^*$-action where the action of $\chi \in \mathfrak{p}^*$ on $\beta \in \wedge^k \mathfrak{p}_+$ is $\chi \cdot \beta = \sum_i e^i \wedge ([e^i, \chi] \lrcorner \beta) = \sum_i [\chi, e^i]_{\mathfrak{p}_+} \wedge (e_i \lrcorner \beta)$ where $[,]_{\mathfrak{p}_+}$ is Lie bracket projected to $\mathfrak{p}_+$ part. The coboundary operator or differential $d : C^k(\mathfrak{g}_-, W) \to C^{k+1}(\mathfrak{g}_-, W)$ is defined by $d(\beta \otimes w) = \sum_i (\frac{1}{2} e^i \wedge (e_i \cdot \beta) \otimes w + e^i \wedge \beta \otimes e_i \cdot w)$.

Lemma 27. *The equation $d = -(\delta)^*$ holds for δ as above for which k-chains are $C_k(\mathfrak{g}_-, W^*) = C^k(\mathfrak{g}_-, W)^*$.*

There is possibility to define quabla operator $\Box = d\delta + \delta d$ which provides $\mathfrak{g}_0$-isomorphisms $H^k(\mathfrak{g}_-, W) \simeq Ker\, \Box \simeq H^k(\mathfrak{p}_+, W)$. Let us remind cohomology spaces are naturally $\mathfrak{p}^*$-modules but with trivial action of $\mathfrak{g}_-$.

Bernstein-Gel̆fand-Gel̆fand (BGG) sequences

Let $(\mathcal{G}, M)$ be a Cartan geometry of a type (G, P) and let V be a tractor bundle (corresponding to $\mathfrak{g}$-module $\mathbb{V}$) over M with a covariant derivative ∇ and the exterior covariant derivative $d^\nabla : \Omega^k(V) \to \Omega^{k+1}(V)$ where space $\Omega^k(V)$ stands for k-forms on M with values in bundle V. This space is isomorphic to the natural bundle which is generated by P-module $\wedge^k \mathfrak{p}_+ \otimes V$. It allows us to construct sequence of bundles which are derived from homology.

Theorem 28. *Let $(\Omega^k(V))^j$ denote the filtration on $\Omega^k(V)$ and let $gr(\Omega^k(V))$ denote the associated graded bundle, similarly for $\Omega^{k+1}(V)$. Let E_k be a filtration preserving differential operator from $\Omega^k(V) \to \Omega^{k+1}(V)$ with the property that the associated graded map coincides with $gr(d)$.*

Then for every $\sigma \in H_k$, there exists a unique element $s \in Ker\,\delta$ with properties:

1. *$\Pi_k(s) = \sigma$, where $\Pi_k : Ker\,\delta \subset \Omega^k(V) \to H_k$ is the natural projection*
2. *$E_k(s) \in Ker\,\delta$.*

Moreover, the mapping L_k defined by $\sigma \mapsto L_k(\sigma) := s$ is given by a differential operator. The corresponding operator D_k is the defined by $D_k := \Pi_{k+1} \circ E_k \circ L_k : H_k \to H_{k+1}$.

The sequence of operators $\{D_k\}$ is called the BGG sequence determined by $\mathfrak{g}$-module $\mathbb{V}$ and the equation $D_0(\sigma) = 0$ is called the first BGG equation.

Theorem 29. *Let $L_{k+1} \circ D_k = E_k \circ L_k$ holds. Then Π_k and L_k restrict to inverse isomorphisms between $Ker\,E_k \cap Ker\,\delta$ and $Ker\,D_k$. In particular, if $E_0 = \nabla$ and $E_1 = d^\nabla$ we have isomorphism between parallel sections of ∇ and kernel of the first BGG operator D_0.*

Let (G, P) be a parabolic Klein geometry and let $\mathcal{D}$ be the first BGG operator given by a representation $\mathbb{V}$. We claim $dim(Ker\,\mathcal{D}) = dim(\mathbb{V})$. According to theorem there is an isomorphism between $Ker\,\mathcal{D}$ and parallel sections. Covariant constant sections are given by a value in one point, by the invariance of an operator. Therefore the result about equality of dimensions.

Now we define a special class of solutions of a first BGG equation.

Definition 18. *Let $\mathcal{V}$ be the tractor bundle associated to a $\mathfrak{g}$-module $\mathbb{V}$. Let $\{D_k\}$ be the BGG sequence determined by $\mathbb{V}$ and let $\Pi : \mathcal{V} \to \mathcal{H}_0$ be the natural projection, where the bundle $\mathcal{H}_0$ is the associated bundle to the zeroth cohomology. A solution of the first BGG equation determined by $\mathbb{V}$ is called normal if it is of the form $\Pi(s)$ for a parallel section s of the tractor bundle $\mathcal{V}$.*

The following theorems ensure in some chart a normal solution of a first BGG equation can be written as a polynomial. But first we need more considerations and definitions.

Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of a type (G, P). Fix a point $u_0 \in \mathcal{G}$ and put $x_0 := p(u_0) \in M$. For $X \in \mathfrak{g}_-$ we can consider the constant vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ which is characterized by $\omega(\tilde{X})(u) = X$ for all $u \in \mathcal{G}$. There is an open neighbourhood $V \subset \mathfrak{g}_-$ of zero such that the flow $Fl_t^{\tilde{X}}(u_0)$ through u_0 is defined up to time $t = 1$ for all $X \in V$. Then $\Phi(X) := Fl_1^{\tilde{X}}(u_0)$ defines a smooth map $\Phi : V \to \mathcal{G}$ and we define $\phi = p \circ \Phi : V \to M$.

By construction, $\phi(0) = x_0$ and the derivative $T_0\phi : \mathfrak{g}_- \to T_{x_0}M$ is given by $X \mapsto T_{u_0} p \omega_{u_0}^{-1}(X)$. Since $\mathfrak{g}_-$ is complementary to $\mathfrak{p}$ this is a linear isomorphism by defining properties of a Cartan connection. Hence we can shrink V in such a way that ϕ defines a diffeomorphism from V onto an open neighbourhood U of x_0 in M.

Definition 19. *Let* $(p : \mathcal{G} \to M, \omega)$ *be a parabolic geometry of a type* (G, P) *and let* $\mathbb{W}$ *be a* P*-representation. We will use notation of previous two paragraphs.*

- *The normal chart determined by* u_0 *is the diffeomorphism* $\phi^{-1} : U \to V \subset \mathfrak{g}_-$*. Choosing a basis in* $\mathfrak{g}_-$ *we get induced local coordinates on* M *called the normal coordinates determined by* u_0*.*
- *The normal section* σ *of* $\mathcal{G}$ *determined by* u_0 *is the smooth map* $\sigma : U \to \mathcal{G}$ *characterized by* $\sigma(p(\Phi(X))) = \Phi(X)$ *for all* $X \in V$*.*
- *The normal trivialisation of the associated bundle* $\mathcal{G} \times_P \mathbb{W}$ *determined by* u_0 *is the trivialisation induced by the normal section determined by* u_0*.*
- *A normal frame for* $\mathcal{G} \times_P \mathbb{W}$ *determined by* u_0 *is a frame obtained from a basis of* $\mathbb{W}$ *via a normal trivialisation.*

There is a useful description of the trivialisations determined by σ in terms of smooth functions. Recall that smooth sections of the bundle $\mathcal{G} \times_P \mathbb{W} \to M$ over $U \subset M$ are in bijective correspondence with smooth maps $f : p^{-1}(U) \to \mathbb{W}$ which are P-equivariant. In the local trivialisation determined by σ this section is given by $x \to (x, f(\sigma(x)))$. So it corresponds to the function $f \circ \sigma : U \to \mathbb{W}$.

Lemma 30. *Let* $(p : \mathcal{G} \to M, \omega)$ *be a parabolic geometry of some fixed type* (G, P) *and consider tractor bundle* $\mathcal{V} \to M$ *corresponding to a representation* $\mathbb{V}$ *of* G*. Fix a point* $u_0 \in \mathcal{G}$*, write* $x_0 = p(u_0) \in M$ *and consider the normal section* $\sigma : U \to \mathcal{G}$ *centred at* x_0 *which is determined by* u_0*.*

If $s \in \Gamma(\mathcal{V})$ *is parallel for the canonical tractor connection, then the function* $f : U \to \mathbb{V}$ *which describes* s *in the given normal trivialisation is given by* $f(\phi(X)) = exp(-X) \cdot (x_0)$*.*

Theorem 31. *Let* $(p : \mathcal{G} \to M, \omega)$ *be a parabolic geometry of a type* (G, P) *and let* $\mathbb{V}$ *be a representation of* G *with natural grading* $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$*, and suppose that* $\alpha \in \Gamma(\mathcal{H}_0)$ *is a normal solution to the first BGG operator determined by* $\mathbb{V}$*.*

Then for any normal section σ*, the coefficients of* α *in a normal frame are polynomials of degree at most* N *in the normal coordinates determined by* σ*.*

Because of grading property and construction of normal coordinates for any element $X \in \mathfrak{g}_i$ with $i < 0$, the corresponding linear map $\rho(X) : \mathbb{V} \to \mathbb{V}$ has the property $\rho(X)^{N+1} = 0$, where ρ is the representation of G on $\mathbb{V}$. It follows, $f(\phi(X)) = exp(-X) \cdot v_0 = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \rho(X)^k v_0$.

12. Appendix E - Representation theory

We state a few theorems which we use in the thesis with references.

Let us define a function $s : P \to \{0, \pm 1\}$, where P stands for an integral functions on dual of Cartan subalgebra.

$$s(\mu) = \begin{cases} 0 & \text{, if there exists } 1 \neq g \in W \text{ such that } g(\mu) = \mu \\ sgn(g) & \text{, where } g \in W \text{ is unique element such that } g(\mu) \in P_+ \end{cases},$$

where we denote ordinary Weyl group by symbol W and P_+ are those functions from P which are dominant. For $\mu \in P$ we denote the unique element of P_+ to which μ is conjugate by $\{\mu\}$.

Theorem 32 (Klimyk's formula, [12])**.** *Let λ_1, λ_2 be the highest weights of the irreducible highest-weight modules V_1 and V_2 respectively. Len χ_1, χ_2 be the characters of V_1, V_2 respectively. Let m^1_μ denote the multiplicity of μ in V_1. Then*

$$\chi_1 \chi_2 = \sum_{\mu \in P} m^1_\mu s(\mu + \lambda_2 + \rho) \chi_{\{\mu + \lambda_2 + \rho\} - \rho}$$

where ρ is the Weyl vector.

Now we state the PRV theorem.

Theorem 33 (PRV theorem)**.** *For any dominant weights λ, μ the irreducible module $V_{\{\lambda + w_0 \mu\}}$ occurs with multiplicity one in the tensor product $V_\lambda \otimes V_\mu$ where w_0 is the longest element of Weyl group W and notation $\{\lambda + w_0 \mu\}$ stands for unique dominant weight which is on the orbit of $\lambda + w_0 \mu$.*

Proof. Proof can be found in [19], Theorem 5.1. □

In particular, loosely speaking decomposition of tensor product has at least two components.

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