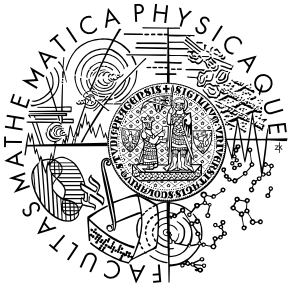

MATEMATICKO-FYZIKÁLNÍ FAKULTA
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DIPLOMOVÁ PRÁCE

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Frakcionální Brownův pohyb ve financích

Katedra pravděpodobnosti a matematické statistiky

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Frakcionální Brownův pohyb ve financích

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Abstrakt: Tato práce se zabývá stochastickým integrálem vůči gaussovským procesům, které se dají vyjádřit ve tvaru $B_t = \int_0^t K(t,s)dWs$, kde W je Wienerův proces a K je kvadraticky integrovatelné volterrovské jádro. Tyto procesy představují zobecnění frakcionálního Brownova pohybu. Protože se nejedná o semimartingaly, je nutné k zavedení stochastického integrálu použít jiné metody, než poskytuje Itôův kalkulus. V této práci jsou popsány dvě metody: za předpokladu dostatečné regularity integrandu a procesu B lze definovat integrál po trajektoriích zobecněním Lebesgue-Stieltjesova integrálu. Druhý způsob využívá metody Malliavinova počtu a integrál definuje jako adjungovaný operátor k Malliavinově derivaci. Jako aplikace obou metod je v závěru práce řešena stochastická diferenciální rovnice $dS_t = \mu S_t dt + \sigma S_t dB_t$, jejíž pomocí se modeluje chování ceny akcie. Následuje stručná diskuse vlastností tohoto modelu.

Klíčová slova: frakcionální Brownův pohyb, Malliavinův počet, Itôovo lemma, stochastická integrace, volterrovské procesy, arbitráž

Title: Fractional Brownian Motion in Finance

Author: Bc. Matěj Kratochvíl

Department: Department of Probability and Mathematical Statistics

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Abstract: This thesis deals with the stochastic integral with respect to Gaussian processes, which can be expressed in the form $B_t = \int_0^t K(t,s)dWs$. Here W stands for a Brownian motion and K for a square integrable Volterra kernel. Such processes generalize fractional Brownian motion. Since these processes are not semimartingales, Itô calculus cannot be used and other methods must be employed to define the stochastic integral with respect to these proceses. Two ways are considered in this thesis. If both the integrand and the process B are regular enough, it is possible to define the integral in the pathwise sense as a generalization of Lebesgue-Stieltjes integral. The other method uses the methods of Malliavin calculus and defines the integral as an adjoint operator to the Malliavin derivative. As an application, the stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dB_t$, which is used to model price of a stock, is solved. Implications of such a model are briefly discussed.

Keywords: Fractional Brownian Motion, Malliavin Calculus, Itô lemma, Stochastic Integration, Volterra Processes, Arbitrage

Obsah

Introduction

The first sentence in a basic reference text book of stochastic analysis is a quote by Michel Loève: "Two of the most fundamental concepts in the theory of stochastic processes are the *Markov property* and the *martingale property*".

The field of stochastic analysis went through remarkable growth over the last few decades and attracts more and more attention. It plays a vital role in both pure and applied mathematics. Great deal of credit for this rapid development can be undoubtedly attributed to its numerous contributions in Economics and Finance. Probably, the best-known result is the Black-Scholes options pricing model [10] and the corresponding Samuelson model for a price of a stock [54].

An important object in the study of stochastic processes is the Brownian motion. Despite some unpleasant properties, in particular infinite total variation and non-differentiability of sample paths, plays central role in the Itô stochastic integration theory with respect to semimartingales.

On the other hand, it has been recently argued, that its properties make it unsuitable for modeling certain phenomena. In particular, observed market data suggested presence of long range dependence, which can't be captured by Brownian motion which is a process with independent increments.

A process that was suggested to replace the Brownian motion in order to capture this long range dependence is called fractional Brownian motion. It is a generalization of the standard Brownian motion with one additional parameter called *Hurst index*. Same as the Brownian motion, this process is Gaussian, self-similar and has stationary increments. However, it is neither martingale nor Markov process. As a consequence, the Itô calculus can't be used and other approaches to stochastic integration with respect to the fractional Brownian motion must be adopted.

In this thesis, we introduce *Volterra processes* B that further generalize the fractional Brownian motion. The Skorohod and pathwise stochastic integrals are defined with respect to these processes. For the Skorohod integration, we employ the theory of the Malliavin calculus, and the pathwise integral is based on a generalization of Lebesgue-Stieltjes integral defined in terms of fractional integrals. As an application, we solve the Black-Scholes stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

for both Skorohod and pathwise integrals with respect to the process B. The fractional Brownian motion serves as an example throughout the whole thesis.

1. Preliminaries

1.1 Stochastic Preliminaries

In this section we present some basic results from stochastic analysis. We only present those results which are used in the following chapters. In particular, we present basic existence theorems such as the Daniell-Kolmogorov Theorem and the Kolmogorov-Chentsov Theorem and a short review of the Itô stochastic calculus.

1.1.1 Existence and Continuity of Stochastic Processes

Definition 1.1.1 ([29], p.49)**.** Let T be the set of finite sequences $\tilde{t} = (t_1, \ldots, t_n)$ of distinct, nonnegative numbers, where the length n of these sequences ranges over the set of positive integers. Suppose that for each $\tilde{t}$ of length n, we have a probability measure $Q_{\tilde{t}}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then the collection $\{Q_{\tilde{t}}, \tilde{t} \in T\}$ is called a *family of finite-dimensional distributions*. This family is said to be *consistent* provided that the following two conditions are satisfied:

1. if $\tilde{s} = (t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ is a permutation of $\tilde{t} = (t_1, t_2, \ldots, t_n)$, then for any $A_i \in \mathcal{B}(\mathbb{R}), i = 1, \cdots, n$, we have that

$$Q_{\tilde{t}}\left(A_1 \times A_2 \times \cdots \times A_n\right) = Q_{\tilde{s}}\left(A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n}\right);$$

2. if $\tilde{t} = (t_1, t_2, \ldots, t_n)$, with $n \geq 1$, $\tilde{s} = (t_1, t_2, \ldots, t_{n-1})$, and $A \in \mathcal{B}(\mathbb{R}^{n-1})$, then

$$Q_{\tilde{t}}(A \times \mathbb{R}) = Q_{\tilde{s}}(A).$$

Denote by $\mathbb{R}^{[0,\infty)}$ the set of real-valued functions on $\mathbb{R}_+$. Given a probability measure $\mathbb{P}$ on $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)})\right)$, it is possible to define a consistent family of finite-dimensional distributions by

$$Q_{\tilde{t}}(A) = \mathbb{P}\left\{\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\right\},$$

where $A \in \mathcal{B}(\mathbb{R}^n)$ and $\tilde{t} = (t_1, t_2, \ldots, t_n) \in T$. The converse is also true as stated by the following theorem.

Theorem 1.1.2 (Daniell-Kolmogorov, [29], p.50)**.** *Suppose that $\{Q_{\tilde{t}}, \tilde{t} \in T\}$ is a consistent family of finite-dimensional distributions. Then there is a probability measure $\mathbb{P}$ on $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)})\right)$, such that*

$$Q_{\tilde{t}}(A) = \mathbb{P}\left\{\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{R}^n)$$

holds for every $\tilde{t} \in T$.

Daniell-Kolmogorov theorem can be used for the construction of important processes. In the next section the Brownian and the fractional Brownian motion are presented. More generally, we are interested in Gaussian processes, their definition is now given.

Definition 1.1.3 (Gaussian process)**.** A stochastic process $X = (X_t, t \geq 0)$ is *Gaussian* with a covariance function $R : \mathbb{R}^2_{+,0} \to \mathbb{R}$ if all finite-dimensional distributions of X are Gaussian and $\operatorname{cov}(X_s, X_t) = R(s,t)$ for $s, t \in \mathbb{R}_{+,0}$. The process X is called *centered* if $\mathbb{E}X_t = 0$ for all $t \geq 0$.

Definition 1.1.4. A function $f : \mathbb{R}^2_{+,0} \to \mathbb{R}$ is called *positive semidefinite* if for all $n \in \mathbb{N}$, for all $\tilde{t} = (t_1, \ldots, t_n)$ such that $0 \leq t_1 < t_2 < \cdots < t_n$ and for all $c_1, \ldots, c_n \geq 0$ the following holds:

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k f(t_j, t_k) \geq 0.$$

As a corollary of the Daniell-Kolmogorov theorem, Theorem 1.1.2, for every positive semidefinite function φ there exists a Gaussian process $X = (X_t, t \geq 0)$ with φ as a covariance function. The following existence theorem is given in [20], p.72.

Theorem 1.1.5 (Existence of Gaussian processes)**.** *For every real, symmetric, and positive semidefinite function* $\varphi : \mathbb{R}^2_+ \to \mathbb{R}$ *there is a real, centered Gaussian process* $X = (X_t, t \geq 0)$ *such that*

$$\mathbb{E}\left(X_t X_s\right) = \varphi(t,s), \quad s, t \in \mathbb{R}_+.$$

The following theorem provides a condition under which the existence of a continuous version of a stochastic process is assured. This condition is easily verified for Gaussian processes.

Theorem 1.1.6 (Kolmogorov-Chentsov, [29], p.53)**.** *Let* $T > 0$. *Suppose that a process* $X = (X_t, t \in [0,T])$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfies the condition*

$$\mathbb{E}\left|X_t - X_s\right|^\alpha \leq C|t-s|^{1+\beta}, \quad 0 \leq s, t \leq T, \tag{1.1}$$

for some positive constants α, β *and* C. *Then there exists a continuous version* $\tilde{X} = (\tilde{X}_t, t \in [0,T])$ *of* X, *which is locally Hölder-continuous with exponent* γ *for every* $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$, *i.e.*

$$\mathbb{P}\left\{\omega \in \Omega : \sup_{\substack{0<t-s<h(\omega);\\ s,t\in[0,T]}} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{|t-s|^\gamma} \leq \delta\right\} = 1,$$

where $h(\omega)$ is a $\mathbb{P}$-almost surely positive random variable and $\delta > 0$ is an appropriate constant.

1.1.2 Itô Calculus

Here we present few basic results in the field of stochastic analysis. We start with a definition and basic properties of the Brownian motion, then we continue with a brief review of the Itô integral with respect to the Brownian motion and we conclude this section with the Itô lemma.

Brownian motion

Definition 1.1.7 (Stochastic basis)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A *filtration* $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$ is non-decreasing sequence of σ-algebras such that $\mathcal{F}_t \subset \mathcal{F}$ for each $t \geq 0$. Given a probability measure $\mathbb{P}$ on $\mathcal{F}$, the *stochastic basis* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 1.1.8 (Brownian motion, [29], p.47)**.** The *standard Brownian motion* is a continuous $(\mathcal{F}_t)$-adapted process $W = (W_t, \mathcal{F}_t, t \geq 0)$, defined on some stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, with the following properties:

(i) $W_0 = 0, \quad \mathbb{P} - \text{a.s.},$

(ii) for $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and is normally distributed with zero mean and variance $t - s$.

The condition (ii) stating the independence of increments of W means that the whole system of increments is independent, i.e., for all $n \in \mathbb{N}$ and for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ the random variables $\left(W_{t_{i+1}} - W_{t_i}, i = 1, \ldots, n-1\right)$ are independent. The explicit formula for the covariance function of W follows from Definition 1.1.8. The joint distribution of $(W_{t_1}, \ldots, W_{t_n})$ is normal, $\text{var} W_{t_i} = t_i$ and $\text{cov}\left(W_{t_i}, W_{t_j}\right) = t_i \wedge t_j$. Since this defines a positive semidefinite function, we have that such a process exists by Theorem 1.1.5. Further, the Kolmogorov-Chentsov Theorem, Theorem 1.1.6 guarantees the existence of a continuous version of W on $[0, T]$ for all $T > 0$. Indeed, from the properties of the Normal distribution follows that the condition (1.1) is satisfied with $\alpha = 4$, $\beta = 1$ and $C = 3$.

Further, some basic properties of the standard Brownian motion are presented.

Theorem 1.1.9 (Strong law of large numbers, [29], p.104)**.** *Let $W = (W_t, t \geq 0)$ be the standard Brownian motion. Then*

$$\lim_{t\to\infty} \frac{W_t}{t} = 0, \quad \mathbb{P} - \text{a.s.}$$

A more precise result on rate of growth is provided by the Law of Iterated Logarithm.

Theorem 1.1.10 (Law of Iterated Logarithm, [29], p.112)**.**

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = 1\right) = 1. \tag{1.2}$$

The following theorem is useful to define the stochastic integral in the sense of Itô. It states that the integral with respect to a quadratic variation of the standard Brownian motion is simply an integral with respect to the Lebesgue measure.

Theorem 1.1.11 (Quadratic variation, [29], p.105)**.** *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of partitions of the interval $[0,t]$ for some $t > 0$ such that the norm of these partitions goes to zero as $n \to \infty$. Then*

$$V_t^{(2)}(\Delta_n) = \sum_{k=1}^{m_n} \left| W_{t_k^{(n)}} - W_{t_{k-1}^{(n)}} \right|^2 \overset{L^2(\mathbb{P})}{\underset{n\to\infty}{\longrightarrow}} t,$$

where m_n denotes the number of points of the partition Δ_n.

For almost all trajectories of the standard Brownian motion it holds that their total variation over interval $[0,t]$ is infinite for all $t \geq 0$. This property makes it impossible to define the integral with respect to the standard Brownian motion pathwise in the standard Lebesgue-Stieltjes sense.

Brownian motion presents an example of a mapping that produces continuous yet nowhere differentiable functions.

Theorem 1.1.12 (Non-differentiability,[29], p.110)**.** *For almost every $\omega \in \Omega$, the Brownian sample path $t \mapsto W_t(\omega)$ is nowhere differentiable.*

By the Kolmogorov-Chentsov Theorem, Theorem 1.1.6, the trajectories of the standard Brownian motion are locally $\left(\frac{1}{2} - \varepsilon\right)$-Hölder continuous for all $\varepsilon \in \left(0, \frac{1}{2}\right)$.

Itô Integral

In general, the Itô integral can be defined for semimartingales as integrators. However, we do not attempt such a construction here and only introduce the Itô stochastic integral with respect to the standard Brownian motion. For a complete overview of this topic we refer to various monographs such as [29], [42], [47], [49]. For the applications of the theory of stochastic calculus in mathematical finance we recommend the books of Steele ([59]) and Shreve ([57]). Stochastic calculus in mathematical finance on a very sound mathematical level can also be found in Part 3 of Dupačová, Hurt, Štěpán [22].

Denote by $L^2_{\text{ad}}([0,T]\times\Omega)$ the space of $(\mathcal{F}_t)$-progressively-measurable stochastic

processes $u = (u_t, t \in [0,T])$ on $[0,T]$ satisfying $\mathbb{E}\int_0^T |u_t|^2 \mathrm{d}t < \infty$. Following [32], we present a construction of an Itô stochastic integral of the process $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$ with respect to the standard Brownian motion, $\int_0^t u_s dW_s$.

Denote by $\mathscr{S}$ the class of simple processes on $[0,T]$, that is, processes of the form

$$G_t = \sum_{i=0}^{m-1} \xi_i \mathbb{1}_{[t_i,t_{i+1})}(t),$$

where ξ_s is $\mathcal{F}_s$-measurable random variable such that $\mathbb{E}|\xi_s|^2 < \infty$, $s = 0,\dots,m-1$, and $0 = t_0 < t_1 < \cdots < t_m = T$, $m \in \mathbb{N}$.

For a simple process $G \in \mathscr{S}$, we define the stochastic integral $\int_0^T G_s dW_s$ as

$$\int_0^T G_s dW_s = \sum_{i=0}^{m-1} \xi_i (W_{t_{i+1}} - W_{t_i}).$$

This integral is obviously zero-mean and satisfies

$$\int_0^T (aG_s + bH_s) dW_s = a\int_0^T G_s dW_s + b\int_0^T H_s dW_s \quad \mathbb{P}-a.s.,$$

for any $a, b \in \mathbb{R}$ and $G, H \in \mathscr{S}$. The two following important results allow us to extend the integral to the space $L^2_{\mathrm{ad}}([0,T]\times\Omega)$.

Lemma 1.1.13 (Itô isometry, [32], p.44)**.** *Let $G \in \mathscr{S}$ be a simple process. Then*

$$\mathbb{E}\left[\int_0^T G_s dW_s\right]^2 = \mathbb{E}\left[\int_0^T G_s^2 \mathrm{d}s\right]. \tag{1.3}$$

Lemma 1.1.14 ([32], Lemma 4.3.3, p.45)**.** *Let $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$. Then there exists a sequence of simple processes $\{u^{(n)}, n \in \mathbb{N}\} \subset \mathscr{S}$ such that*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |u_s - u_s^{(n)}|^2 \mathrm{d}s = 0. \tag{1.4}$$

Now, for any $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$, by Lemma 1.1.14, there exists a sequence $\{u^{(n)}, n \in \mathbb{N}\}$ in $\mathscr{S}$ such that (1.4) holds. Then we define the Itô integral of u as

$$\int_0^T u_s dW_s = \lim_{n\to\infty} \int_0^T u_s^{(n)} dW_s, \tag{1.5}$$

where the convergence is in $L^2(\mathbb{P})$. Note that by Lemma 1.1.14, the sequence $\{u^{(n)}, n \in \mathbb{N}\}$ is Cauchy in $L^2_{\mathrm{ad}}([0,T]\times\Omega)$ and by the Itô isometry, Lemma 1.1.13, the sequence $\int_0^T u_s^{(n)} dW_s$ is Cauchy in $L^2(\mathbb{P})$. To see this let us write

$$\mathbb{E}\left[\int_0^T u_s^{(n)} dW_s - \int_0^T u_s^{(m)} dW_s\right]^2 = \mathbb{E}\left[\int_0^T (u_s^{(n)} - u_s^{(m)})^2 \mathrm{d}s\right] \to 0, \quad n,m \to \infty.$$

The space $L^2(\mathbb{P})$ is a complete metric space (see for example [50]), hence the limit in (1.5) exists in $L^2(\mathbb{P})$. Moreover, the limit does not depend on the choice of the approximating sequence. Indeed, for two sequences $\{u^{(n)}\}, \{v^{(n)}\} \subset \mathscr{S}$, which both satisfy (1.4), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T u_s^{(n)} dW_s - \int_0^T v_s^{(m)} dW_s\right]^2 = \mathbb{E}\left[\int_0^T (u_s^{(n)} - v_s^{(m)})^2 \mathrm{d}s\right] \\
&\leq 2\left(\mathbb{E}\left[\int_0^T (u_s^{(n)} - u_s)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T (u_s - v_s^{(m)})^2 \mathrm{d}s\right]\right) \to 0, \quad n, m \to \infty,
\end{aligned}$$

thus the integral is well-defined.

Remark 1.1.15. The linearity of the integral and Itô isometry (1.3) remain valid also for any $u \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$. Further, the Itô isometry has the following corollary:

$$\mathbb{E}\left[\int_0^T u_s dW_s \int_0^T v_s dW_s\right] = \mathbb{E}\left[\int_0^T u_s v_s \mathrm{d}s\right], \quad u, v \in L^2_{\mathrm{ad}}([0,T] \times \Omega). \tag{1.6}$$

△

Now let $u \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$. Then for any $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t u_s^2 \mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^T u_s^2 \mathrm{d}s\right] < \infty,$$

hence for each $t \in [0,T]$ the stochastic integral $\int_0^t u_s dW_s$ is defined.

Remark 1.1.16. Observe that this integral is not defined pathwise for fixed $\omega \in \Omega$ as a Riemann-Stieltjes or Lebesgue-Stieltjes integral. Therefore, the continuity (or even measurability) of it is not obvious. △

Theorem 1.1.17 ([32], Theorem 4.6.2, p.55)**.** *Let $u \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$. Then the stochastic process*

$$X_t = \int_0^t u_s dW_s, \quad t \in [0,T],$$

admits a continuous version, namely, almost all of its sample paths are continuous functions on the interval [0,T].

Now we state a change-of-variables formula for the Itô integral.

Theorem 1.1.18 (Itô lemma, [32], Theorem 7.4.3, p.103)**.** *Let $u \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$ and*

$$X_t = \int_0^t u_s dW_s.$$

Let $F(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$, that is F and its partial derivatives $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}$

are continuous. Then for $t \in [0,T]$

$$\begin{aligned} F(t,X_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial t}(s,X_s)\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(s,X_s)u_s dW_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)u_s^2\mathrm{d}s, \quad \mathbb{P}-a.s. \end{aligned} \tag{1.7}$$

Theorem 1.1.19 (Girsanov, [32], Theorem 8.9.4, p.143)**.** *Let* $h \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$*, where*

$$\mathcal{E}(h_t) = \exp\left\{\int_0^t h_s dW_s - \frac{1}{2}\int_0^t h_s^2\mathrm{d}s\right\}, \quad 0 \le t \le T. \tag{1.8}$$

and assume that $\mathbb{E}\mathcal{E}(h_t) = 1$ *for all* $t \in [0,T]$*. Then the stochastic process*

$$\tilde{W}_t = W_t - \int_0^t h_s\mathrm{d}s, \quad 0 \le t \le T, \tag{1.9}$$

is a Brownian motion with respect to the probability measure $\tilde{\mathbb{P}}$ *on* $(\Omega,\mathcal{F})$ *given by*

$$\tilde{\mathbb{P}}(A) = \int_A \mathcal{E}(h_T)\mathrm{d}\mathbb{P}, \quad A \in \mathcal{F}. \tag{1.10}$$

The Itô integral can be viewed as a limit of Riemann sums.

Theorem 1.1.20 ([32], Theorem 4.7.1, p.57)**.** *Let* $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$ *and assume that* $\mathbb{E}[u_t u_s]$ *is a continuous function of* t *and* s*. Then*

$$\lim_{|\pi_n|\to 0}\sum_{i=0}^{n-1} u_{t_i}(W_{t_{i+1}} - W_{t_i}) = \int_0^T u_t dW_t, \quad in L^2(\Omega),$$

where $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ *and* $|\pi_n| = \max_{1\le i\le n}(t_i - t_{i-1})$*.*

1.2 Fractional Calculus

Deterministic fractional calculus will be useful while working with the fractional Brownian motion. In the first part of this section, we recall basic definitions and results which are used to define and investigate the extension of the Stieltjes integral in the second part. Exhaustive survey of the theory of fractional integrals can be found in [53]. Classical results can also be found in Hardy's and Littlewood's work [25] and [26].

We start by the well-known Cauchy's formula for iterated integral.

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} f(t_n)\mathrm{d}t_n\cdots\mathrm{d}t_1 = \frac{1}{(n-1)!}\int_0^t \frac{f(s)}{(t-s)^{1-n}}\mathrm{d}s. \tag{1.11}$$

Since $(n-1)! = \Gamma(n)$, the right hand side of (1.11) makes sense even for non-

integer n. This motivates the following definition.

Definition 1.2.1 (Left and right-sided fractional Riemann-Liouville integral)**.** Suppose that $f \in L^1([0,T])$, where $T > 0$ and let $\alpha > 0$. The *left and right-sided fractional Riemann-Liouville integral* of f *of order* α on $(0,T)$ are given for almost all $t \in (0,T)$ by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \tag{1.12}$$

and

$$I_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) \mathrm{d}s, \tag{1.13}$$

respectively. It can be easily verified that the following composition formula holds for any α, $\beta > 0$:

$$I_{0+}^{\alpha} \left(I_{0+}^{\beta} f \right) = I_{0+}^{\alpha+\beta} f,$$
$$I_{T-}^{\alpha} \left(I_{T-}^{\beta} f \right) = I_{T-}^{\alpha+\beta} f.$$

Now as we have defined fractional integral, we can look for an analogue of the classical differential operator which is an inverse of the left(right)-sided fractional Riemann-Liouville integral. Denote by $I_{0+}^{\alpha}(L^p([0,T]))$ the image of $L^p([0,T])$ under the operator I_{0+}^{α}, and $I_{T-}^{\alpha}(L^p([0,T]))$ the image of $L^p([0,T])$ under the operator I_{T-}^{α}.

Definition 1.2.2 (Left-sided fractional Riemann-Liouville derivative)**.** Suppose that $f \in I_{0+}^{\alpha}(L^p([0,T]))(f \in I_{T-}^{\alpha}(L^p([0,T])))$ where $T > 0$, $p > 1$ and $\alpha \in (0,1)$. The *left and right-sided fractional Riemann-Liouville derivatives of* f *of order* α are given for almost all $t \in (0,T)$ by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} \mathrm{d}s \tag{1.14}$$

and

$$D_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^T \frac{f(s)}{(s-t)^{\alpha}} \mathrm{d}s. \tag{1.15}$$

In this case, the fractional derivatives $D_{0+}^{\alpha} f$ and $D_{T-}^{\alpha} f$ admit the Weyl representation

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} \mathrm{d}s \right), \tag{1.16}$$

and

$$D_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} \mathrm{d}s \right), \tag{1.17}$$

where the integrals in (1.16) and (1.17) converge point-wise for almost all $t \in (0,T)$ if $p = 1$ and in L^p-sense if $p > 1$, see e.g. [42], p.355. By construction, it is

clear that for a function $f \in I^{\alpha}_{0+}(L^p([0,T]))$ we have that

$$I^{\alpha}_{0+}\left(D^{\alpha}_{0+}f\right) = f,$$

and, for a general function $f \in L^1([a,b])$, we have that

$$D^{\alpha}_{0+}\left(I^{\alpha}_{0+}f\right) = f.$$

Similarly, there exist inversion formulas for the operators I^{α}_{T-} and D^{α}_{T-}.

1.2.1 Fractional Integrals

Following [61] and [62], we define fractional integral of f with respect to g. This integral extends the Lebesgue-Stieltjes integral to integrators of unbounded variation via fractional calculus. This construction will later be used for pathwise stochastic integral with respect to strictly regular vanishing Volterra process. See also related works on extension of Stieltjes integral of Vallois and Russo [52] and Dudley and Norvaiša [21].

Let us denote

$$f_{0+}(t) = (f(t) - f(0+))\mathbb{1}_{(0,T)}(t),$$
$$g_{T-}(t) = (g(t) - g(T-))\mathbb{1}_{(0,T)}(t),$$

where $+$ denotes the limit from the left and $-$ the limit from the right.

Definition 1.2.3. Suppose that f and g are functions such that $f(0+), g(0+)$ and $g(T-)$ exist, $f_{0+} \in I^{\alpha}_{0+}(L^p)$ and $g_{T-} \in I^{1-\alpha}_{T-}$ for some $p,\ q \geq 1,\ 1/p + 1/q \leq 1$ and $0 < \alpha < 1$. Then the integral of f with respect to g is defined by

$$\int_0^T f\mathrm{d}g = \int_0^T D^{\alpha}_{0+}f_{0+}(t)D^{1-\alpha}_{T-}g_{T-}(t)\mathrm{d}t + f(0+)(g(T-) - g(0+)). \tag{1.18}$$

It is shown in [61], Proposition 2.1, p.340, that the definition is independent of the choice of α. Further we present basic properties of such defined integral.

Proposition 1.2.4 ([61], Proposition 2.2, p.341)**.** *Let $(c,d] \subset (0,T)$. If g is λ-Hölder continuous on $(0,T)$ for some $\lambda > 0$ then we have*

(i) $\displaystyle\int_0^T \mathbb{1}_{(c,d]}(t)\mathrm{d}g(t) = g(d) - g(c)$

(ii) $\displaystyle\int_0^T \mathbb{1}_{(c,T)}(t)\mathrm{d}g(t) = g(T-) - g(c).$

This result extends to step functions by linearity.

If g is of bounded variation then, under some additional assumptions on f, this

integral agrees with the classical Lebesgue-Stieltjes integral.

Theorem 1.2.5 ([61], Theorem 2.4, p.343)**.** *Suppose that f, g are as in the Definition 1.2.3 and g has bounded variation with variation measure μ. If either*

$$\int_0^T I_{0+}^{\alpha}(|D_{0+}^{\alpha} f_{0+}|)(t)\mu(\mathrm{d}t) < \infty$$

or f is bounded and right (or left-)continuous at μ-a.a. points then

$$\int_0^T f(t)\mathrm{d}g(t) = (\mathrm{L}-\mathrm{S})\int_0^T f(t)\mathrm{d}g(t).$$

As a special case, for any continuous f, we obtain equivalence with the Riemann-Stieltjes integral:

$$\int_0^T f(t)\mathrm{d}g(t) = (\mathrm{R}-\mathrm{S})\int_0^T f(t)\mathrm{d}g(t),$$

where

$$(\mathrm{R}-\mathrm{S})\int_0^T f(t)\mathrm{d}g(t) = \lim_{|\Delta|\to 0}\sum_i f(t_i^*)(g(t_{i+1}) - g(t_i)), \tag{1.19}$$

and the convergence holds uniformly in all finite partitions $\Delta = \{0 = t_0 \le t_0^* \le t_1 \le \cdots \le t_n \le t_n^* \le t_{n+1} = T\}$. The integral is also an additive function of the boundary ([61], Theorem 2.5., p.345).

Next we consider the case when f and g are Hölder continuous of summed order greater than one.

Theorem 1.2.6 ([61], Theorem 4.2.1, p.349)**.** *If $f \in C^{\lambda}(0,T)$ and $g \in C^{\mu}(0,T)$ are Hölder continuous with $\lambda + \mu > 1$, then the Riemann-Stieltjes integral (1.19) exists and agrees with $\int_0^T f\mathrm{d}g$.*

The integral as a function of the upper or lower boundary is Hölder continuous of order μ.

Proposition 1.2.7 ([61], Proposition 4.4.1, p.351)**.** *Let $f \in C^{\lambda}(0,T)$ and $g \in C^{\mu}(0,T)$ with $\lambda + \mu > 1$. Then*

$$\int_0^{\cdot} f\mathrm{d}g \in C^{\mu}(0,T).$$

Now we state the change-of-variables formula.

Theorem 1.2.8 ([62], Theorem 3.1, p.159)**.** *Let $0 < \alpha < 1/2$ $f \in I_{0+}^{\alpha}(L^2([0,T])$ be bounded, $g_{T-} \in I_{T-}^{1-\alpha}(L^2([0,T]))$ and*

$$h(t) = h(0) + \int_0^t f(s)\mathrm{d}g(s), \quad t \in (0,T].$$

Then for any function $F \in C^{1,2}([0,T] \times \mathbb{R})$ *and for any* $0 \leq t_0 \leq t \leq T$*:*

$$F(t,h(t)) - F(t_0,h(t_0)) = \int_{t_0}^{t} \frac{\partial F}{\partial t} F(s,h(s))\mathrm{d}s + \int_{t_0}^{t} \frac{\partial F}{\partial x}(s,h(s))f(s)\mathrm{d}g(s). \quad (1.20)$$

Remark 1.2.9. The formula holds under the more restrictive assumptions that g is Hölder continuous of order μ, f is Hölder continuous of order $\lambda > 1 - \mu$, and $\frac{\partial F}{\partial x}(\cdot, h(\cdot))$ is Hölder continuous of order λ ([62]). △

1.3 Fractional Brownian Motion

This section contains the definition and basic properties of the standard fractional Brownian motion which represents a generalisation of the standard Brownian motion. For a thorough survey and a comprehensive overview of the standard fractional Brownian motion and the stochastic calculus defined with respect to it, we refer to [7] or [36]. Other references include the original paper of Mandelbrot and Van Ness [35], where a stochastic integral representation for the fractional Brownian motion was given. For further reading see also [16] and [40].

Definition 1.3.1 (Fractional Brownian Motion)**.** Let $T > 0$. We say that a real-valued stochastic process $B^H = \left(B_t^H, t \in [0,T]\right)$ is the *standard fractional Brownian motion with the Hurst parameter* $H \in (0,1)$ if it is a centered Gaussian process, $B_0^H = 0 \quad \mathbb{P}-$a.s., and if its covariance function is given by

$$R^H(t,s) = \mathrm{cov}\left(B_s^H, B_t^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \quad (1.21)$$

It follows from the definition that the standard fractional Brownian motion is a process with stationary increments, i.e. the probability law of $B_{t+s}^H - B_t^H$ is the same as the law of B_s^H for $s,t \in [0,T]$. The existence of such a process follows, by similar arguments as for the standard Brownian motion, from Theorem 1.1.5 (for a proof that the function $R^H(t,s)$ in (1.21) is positive semidefinite, see for example [39], p.8).

The regularity of sample paths of the fractional Brownian motion also depends on the Hurst index H. This property follows from the Kolmogorov-Chentsov Theorem, Theorem 1.1.6. We have

$$\mathbb{E}\left[B_t^H - B_s^H\right]^k = c_k |t-s|^{kH},$$

where $c_k = \mathbb{E}X^k$ with $\mathrm{Law}(X) = N(0,1)$. Hence, by the Kolmogorov-Chentsov Theorem, the process B^H has a version with Hölder continuous sample paths of any order less then H. In the sequel, we only consider this version. On the other hand, we also have the Law of Iterated Logarithm for the fractional Brownian

motion([7], p.11, or [4] for a general result).

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{B_t^H}{t^H\sqrt{\log\log t}} = d_H\right) = 1, \tag{1.22}$$

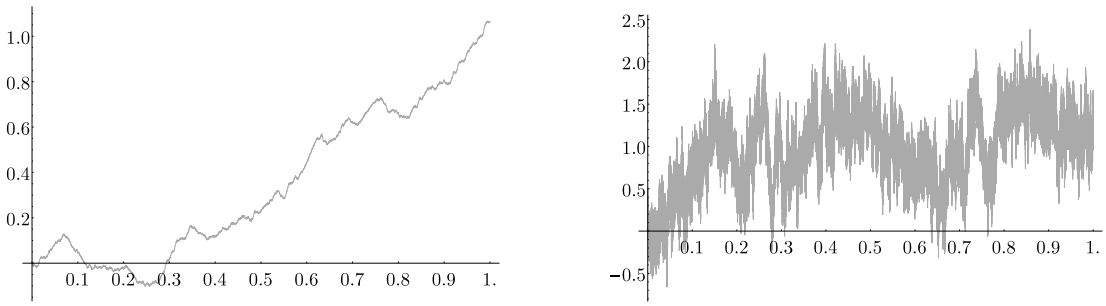
where d_H is a suitable constant.

The standard fractional Brownian process with the Hurst parameter H is an *H-self-similar* process, that is, for every $a > 0$ it holds that

$$\text{Law}\left(B_{at}^H, t \geq 0\right) = \text{Law}\left(a^H B_t^H, t \geq 0\right).$$

This is an immediate consequence of the fact, that the covariance function fullfils $R^H(at, as) = a^{2H} R^H(t, s)$ for every $a > 0$.

Depending on the value of the Hurst parameter H, we recognize three different cases. If $H = \frac{1}{2}$ it is the standard Brownian motion, for $H < \frac{1}{2}$ we have the singular (or antipersistent) case and the regular (or persistent) case when $H > \frac{1}{2}$. Only in the case $H = \frac{1}{2}$ are the increments of the process B^H independent. If $H \neq \frac{1}{2}$, it follows by (1.21) that the increments $B_{t+h}^H - B_t^H$ and $B_{t+2h}^H - B_{t+h}^H$ are correlated negatively in the singular case and positively in the regular case. This phenomenon is depicted in Figure 1.1.



$H = 0.75$ $H = 0.25$

Obrázek 1.1: Simulated trajectories of fractional Brownian motion for different values of Hurst parameter

The fractional Brownian motion with $H > \frac{1}{2}$ is a process with long memory ([7], p.9), by which we mean that the increments $X_k = B_k^H - B_{k-1}^H, k \in \mathbb{N}$ possess the long-range dependence property.

Definition 1.3.2. A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if the autocovariance function $\rho(n) = \text{cov}(X_k, X_{k+n})$ satisfies

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1$$

for some constant c and $\alpha \in (0,1)$. In this case the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Since almost all sample paths of the standard Brownian motion are nowhere differentiable, such a property cannot be expected in the case of the standard fractional Brownian motion which is its immedieate extension.

Theorem 1.3.3 (Differentiablity of sample paths of fBm, [35], p.427)**.** *The sample path $B_t^H(\omega)$ is $\mathbb{P}$-almost surely not differentiable; in fact,*

$$\limsup_{t \to t_0} \left| \frac{B_t^H(\omega) - B_{t_0}^H(\omega)}{t - t_0} \right| = \infty$$

with probability one.

The fractional Brownian motion with the Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale. In fact, it is shown in [7], p.12, that B^H has finite and positive $\frac{1}{H}$-variation; see also [48] for original result. As a consequence we cannot use Itô stochastic calculus to define the stochastic integral with respect to the fractional Brownian motion.

1.3.1 Liouville fractional Brownian motion

The Liouville fractional Brownian motion is a process whose properties reflect the properties of the standard fractional Brownian motion, but its formulation is simpler and, therefore, it does not pose such a technical challenge. Only the definition is presented here. Note that unlike the fractional Brownian motion, this process generally doesn't have stationary increments. For a stochastic integration with respect to the Liouville fractional Brownian motion, see [12].

Definition 1.3.4 (Liouville fractional Brownian motion, [12], p.7)**.** Let

$$R_L^H(s,t) = \frac{1}{\left(\Gamma\left(H + \frac{1}{2}\right)\right)^2} \int_0^{s \wedge t} (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathrm{d}u, \quad s,t, \in [0,T].$$

A *Liouville fractional Brownian motion of order* $0 < H < 1$ is a Gaussian process $L^H = (L_t^H, t \in [0,T])$, such that

$$\mathbb{E}\left(L_s^H L_t^H\right) = R_L^H(s,t), \quad s,t \in [0,T].$$

2. Malliavin Calculus

In this chapter we review the basics of Malliavin Calculus. First we define isonormal Gaussian processes, which can be viewed as a collection of random variables indexed by elements of a separable Hilbert space. Then we proceed with definition of the Malliavin derivative and present some of its basic properties. Finally we introduce a stochastic integral as an adjoint to the derivative operator. Main reference for this chapter is Nualart's book [42]. For further reading see also [45], [46], [55], [43] and [28].

2.1 Isonormal Gaussian processes

The general framework

Definition 2.1.1 ($\mathbb{H}$-isonormal Gaussian process)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be complete probability space and $\mathbb{H}$ be a real, separable Hilbert space. We say that a stochastic process $X = (X(h), h \in \mathbb{H})$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is an $\mathbb{H}$-*isonormal Gaussian process* (or a *Gaussian process on* $\mathbb{H}$) if X is a centered Gaussian family of random variables such that

$$\mathbb{E}\left(X(h)X(g)\right) = \langle h, g\rangle_{\mathbb{H}}, \quad h, g \in \mathbb{H}. \tag{2.1}$$

Here, the mapping $h \mapsto X(h)$ is linear, as the above relationship between scalar products yields

$$\mathbb{E}\left[X(\lambda h + \mu g) - \lambda X(h) - \mu X(g)\right]^2 = 0.$$

Therefore, the mapping provides a linear isometry between $\mathbb{H}$ and a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ denoted by $\mathcal{H}_1$, which consisting of centered Gaussian random variables. Indeed, by (2.1) we have

$$\|X(h)\|^2_{L^2(\Omega)} = \mathbb{E}\left(X(h)\right)^2 = \|h\|^2_{\mathbb{H}}.$$

Example 2.1.2 (The isonormal Gaussian process associated to the standard Brownian motion). Let $W = (W_t, t \in [0, T])$ be the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Having set $\mathbb{H} = L^2([0, T])$, we denote

$$X(h) = \int_0^T h(s)dW_s, \quad h \in \mathbb{H},$$

where the stochastic integral is defined in the sense of Itô. Then X is a zero-mean Gaussian random variable with variance $\int_0^T h^2(s)\mathrm{d}s$, $h \in \mathbb{H}$, and from the Itô isometry (1.6) we obtain

$$\mathbb{E}\left[\int_0^T h_1(s)dW_s \cdot \int_0^T h_2(s)dW_s\right] = \int_0^T h_1(s)h_2(s)\mathrm{d}s = \langle h_1, h_2\rangle_{\mathbb{H}}$$

for all $h_1, h_2 \in \mathbb{H}$. Since $\mathbb{1}_{[0,t]} \in L^2([0,T])$ for all $t \in [0,T]$, we clearly have that

$$X(\mathbb{1}_{[0,t]}) = \int_0^T \mathbb{1}_{[0,1]}(s)dW_s = W_t - W_0 = W_t.$$

Thus, the isonormal Gaussian process on $L^2([0,T])$ corresponds to the standard Brownian motion (except for the fact that we have not proved that it has continuous paths; however, there exists a continuous modification by Kolmogorov-Chentsov Theorem). Note that for all $s, t \in [0,T]$, we have that

$$\mathbb{E}(W_s W_t) = \mathbb{E}\left(X(\mathbb{1}_{[0,s]})X(\mathbb{1}_{[0,t]})\right) = \left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]}\right\rangle_{\mathbb{H}} = s \wedge t.$$

$\diamondsuit$

Example 2.1.3 (The isonormal Gaussian process associated to the fractional Brownian motion). Let us recall that fractional Brownian motion with Hurst parameter $H \in (0,1)$, $B^H = (B_t^H, t \in [0,T])$, is a centered Gaussian process with a covariance function

$$R^H(t,s) = \mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

Let us denote by $\mathcal{E}$ the step functions on the interval $[0,T]$ and let $\mathscr{H}^H$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]}\right\rangle_{\mathscr{H}^H} = R^H(t,s).$$

The mapping $\mathbb{1}_{[0,t]} \mapsto B_t^H$ can be extended to isometry between $\mathscr{H}^H$ and the Gaussian space $\mathcal{H}_1^H$ associated with B^H. We denote this isometry $\varphi \mapsto B^H(\varphi)$. Then $\{B^H(\varphi), \varphi \in \mathscr{H}^H\}$ is an $\mathscr{H}^H$-isonormal Gaussian process. For $B^H(\varphi)$ we also use the notation $B^H(\varphi) = \int_0^T \varphi(s)\mathrm{d}B_s^H$. A more general case of this example will be discussed in Section 3.2. $\diamondsuit$

2.2 The Malliavin Derivative

The purpose of this section is to introduce the derivative operator DF of a random variable $F : \Omega \to \mathbb{R}$ with respect to the parameter $\omega \in \Omega$.

Let $X = (X(h), h \in \mathbb{H})$ be an isonormal Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is generated by X and let $p \geq 1$. We start

with a definition of the derivative on a dense subset of $L^p(\Omega, \mathcal{F}, \mathbb{P})$ (=: $L^p(\Omega)$). Further let $C_p^\infty(\mathbb{R}^n)$ ($C_b^\infty(\mathbb{R}^n)$, respectively) be the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its derivatives have at most polynomial growth (are bounded, respectively). Denote by $\mathcal{S}$ the set of *smooth random variables* of the form

$$F = f(X(h_1), \dots, X(h_n)), \tag{2.2}$$

where $n \geq 1$, $f \in C^\infty(\mathbb{R}^n)$ and $h_1, \dots, h_n \in \mathbb{H}$.

We further denote by $\mathcal{S}_b$ the set of smooth random variables for the case when $f \in C_b^\infty(\mathbb{R}^n)$ in (2.2). We call $F \in \mathcal{S}$ *smooth* random variable. Clearly $\mathcal{S}_b \subset \mathcal{S}$. Note that $\mathcal{S}_b$ and hence $\mathcal{S}$ are dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (for proof see e.g. [46], p.13).

Definition 2.2.1. The *Malliavin derivative* of a smooth random variable $F \in \mathcal{S}$ of the form (2.2) is the $\mathbb{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \partial_i f(X(h_1), \dots, X(h_n)) h_i. \tag{2.3}$$

The *directional derivative* of $F \in \mathcal{S}$ in the direction $h \in \mathbb{H}$ is defined as $D^h F = \langle DF, h \rangle_{\mathbb{H}}$.

Remark 2.2.2. The definition of the Malliavin derivative does not depend of a particular representation of the random variable F in (2.2). This can be seen from the proof of Theorem 2.2.7. △

To get acquainted with this notion, let us compute a simple example.

Example 2.2.3. Let $X = W$ be the standard Brownian motion and $\mathbb{H} = L^2([0, T])$. Let $h = \mathbb{1}_{[0,t]}$. Then $W(h) = \int_0^T \mathbb{1}_{[0,t]}(s) dW_s = W_t$. Consider the following functions f in (2.2): $f(x) = x$, $f(x) = x^2$. The Malliavin derivative of W_t is $DW_t = DW(\mathbb{1}) = \mathbb{1}_{[0,t]}$ and $D(W_t)^2 = 2W_t \mathbb{1}_{[0,t]}$. ♢

In order to extend the derivative to a larger class of random variables, our next aim is to show that D is a closable operator from $L^p(\Omega)$ to $L^p(\Omega; \mathbb{H})$. That is, if $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ converges to zero in $L^2(\Omega)$ and the sequence of derivatives $\{DF_n\}_{n \in \mathbb{N}}$ converges to G in $L^p(\Omega; \mathbb{H})$, then $G = 0$. In order to show this, we make use of the following integration by parts formula.

Lemma 2.2.4 ([42], Lemma 1.2.1, p.25)**.** *Let $F \in \mathcal{S}$ and $h \in \mathbb{H}$. Then*

$$\mathbb{E}\left[\langle DF, h \rangle_{\mathbb{H}}\right] = \mathbb{E}\left[FX(h))\right]. \tag{2.4}$$

The previous Lemma together with the fact that

$$D(FG) = FDG + GDF, \quad F, G \in \mathcal{S},$$

yields a formula for integration-by-parts.

Lemma 2.2.5. *Let $F, G \in \mathcal{S}$ and $h \in \mathbb{H}$. Then*

$$\mathbb{E}\left[G\left\langle DF, h\right\rangle_{\mathbb{H}}\right] = \mathbb{E}\left[-F\left\langle DG, h\right\rangle_{\mathbb{H}}\right] + \mathbb{E}\left[GFX(h)\right]. \tag{2.5}$$

Now we can prove the closability of the derivative, but before doing so, let us remind the definition of vector valued L^p spaces.

Remark 2.2.6. Let $\mathbb{B}$ be a Banach space. A function $f : \Omega \to H$ is *(strongly) measurable* if it is the pointwise limit of simple functions $\sum_{k=1}^{n} x_k \mathbb{1}_{A_k}$, where $x_k \in \mathbb{B}$ and $A_k \in \mathcal{F}$. A measurable function f is p-integrable, if $\mathbb{E}\|f\|_{\mathbb{B}}^p < \infty$. We then write $f \in L^p(\Omega; \mathbb{B})$. As ususal with working with L^p spaces, we identify functions which are equal almost everywhere. △

Theorem 2.2.7 (Closability of D, [42], Proposition 1.2.1, p.26)**.** *The operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; \mathbb{H})$ for any $p \geq 1$.*

Důkaz. Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $F_n \to 0$ in $L^p(\Omega)$ and $DF_n \to \eta$ in $L^p(\Omega; \mathbb{H})$ as $n \to \infty$. Let $h \in \mathbb{H}$ and $F \in \mathcal{S}_b$ such that $FX(h)$ is bounded. Then, using Lemma 2.2.5, we obtain

$$\begin{aligned}\mathbb{E}\left[\langle \eta, h\rangle_{\mathbb{H}} F\right] &= \lim_{n\to\infty} \mathbb{E}\left[\langle DF_n, h\rangle_{\mathbb{H}} F\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[-F_n\left\langle DF, h\right\rangle_{\mathbb{H}}\right] + \mathbb{E}\left[GFX(h)\right] = 0,\end{aligned}$$

because $F_n \to 0$, for $n \to \infty$, and the other terms are bounded. □

Definition 2.2.8. The closure $\bar{D}$ of the operator D in $L^p(\Omega)$ is also denoted be D. The domain of D in $L^p(\Omega)$ is the space $\mathbb{D}^{1,p}$ defined as the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,p}$:

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|DF\|_{\mathbb{H}}^p. \tag{2.6}$$

For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with respect to the scalar product $\langle F, G\rangle_{1,2} = \mathbb{E}\left(FG\right) + \mathbb{E}\left\langle DF, DG\right\rangle_{\mathbb{H}}$.

Note that directional derivative $D^h = \langle DF, h\rangle_{\mathbb{H}}$ in the direction $h \in \mathbb{H}$ is also closable operator from $L^p(\Omega)$ to $L^p(\Omega)$ and its domain is denoted $\mathbb{D}^{h,p}$.

Remark 2.2.9. Consider now $\mathbb{H} = L^2([0, T])$. We can identify $L^2(\Omega; \mathbb{H}) \cong L^2([0, T] \times \Omega$. We then have that the derivative of a random variable $F \in \mathbb{D}^{1,2}$ is a square integrable stochastic process $(D_t F, t \in [0, T])$ (defined almost everywhere $\lambda \times \mathbb{P}$, where λ denotes Lebesgue measure) and $\mathbb{E}\|DF\|_{\mathbb{H}}^2 = \mathbb{E}\int_0^T (D_t F)^2 \mathrm{d}t$. △

Example 2.2.10. Consider the fractional Brownian motion from Example 2.1.3.

Let us compute

$$\|B_t^H\|_{1,2}^2 = \mathbb{E}(B_t^H)^2 + \left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathscr{H}^H} = \mathbb{E}(B_t^H)^2 + R^H(t,t) = 2\mathbb{E}(B_t^H)^2$$

and

$$\begin{aligned}\|f(B_t^H)\|_{1,2}^2 &= \mathbb{E}f(B_t^H)^2 + \mathbb{E}\left\langle f'(B_t^H)\mathbb{1}_{[0,t]}, f'(B_t^H)\mathbb{1}_{[0,t]} \right\rangle_{\mathscr{H}^H} \\ &= \mathbb{E}f(B_t^H)^2 + \mathbb{E}f'(B_t^H)^2\mathbb{E}(B_t^H)^2.\end{aligned}$$

♢

The next result is a chain rule for the Malliavin derivative D.

Lemma 2.2.11 (Chain rule, [42], Proposition 1.2.3, p.28)**.** *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \dots, F^m)$ is a random vector with components in $\mathbb{D}^{1,p}$. Then $\varphi(F)$ is in $\mathbb{D}^{1,p}$ and*

$$D(\varphi(F)) = \sum_{i=1}^m \partial_i \varphi(F) DF^i.$$

As a next step, we define higher order Malliavin derivatives and extend the Malliavin derivative to smooth random variables with values in a real separable Hilbert space V. Let us first recall the notion of tensor product of Hilbert spaces.

Remark 2.2.12 (Tensor product). Let $(\mathbb{H}_1, \langle\cdot,\cdot\rangle_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \langle\cdot,\cdot\rangle_{\mathbb{H}_2})$ be real separable Hilbert spaces. Their *tensor product* is a Hilbert space $\mathbb{H}_1 \otimes \mathbb{H}_2$ together with a bilinear mapping $\otimes : \mathbb{H}_1 \times \mathbb{H}_2 \to \mathbb{H}_1 \otimes \mathbb{H}_2 : \ (h_1, h_2) \mapsto h_1 \otimes h_2$, such that

1. $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathbb{H}_1 \otimes \mathbb{H}_2} = \langle f_1, g_1 \rangle_{\mathbb{H}_1} \langle f_2, g_2 \rangle_{\mathbb{H}_2}$,
2. and the set $\{h_1 \otimes h_2, h_1 \in \mathbb{H}_1, h_2 \in \mathbb{H}_2\}$ forms a total subset of $\mathbb{H}_1 \otimes \mathbb{H}_2$.

△

Remark 2.2.13. Let $\mathcal{V}$ be a real separable Hilbert space. The space $L^2(\mathcal{V}; \mathbb{H})$ of square integrable functions $\mathcal{V} \to \mathbb{H}$ is isomorphic to $L^2(\mathcal{V}) \otimes \mathbb{H}$. The isomorphism maps $f(x) \otimes \varphi \in L^2(\mathcal{V}) \otimes \mathbb{H}$ to $f(x)\varphi \in L^2(\mathcal{V}; \mathbb{H})$. △

For a real separable Hilbert space V denote by $\mathcal{S}_V$ the V-valued smooth random variables of the form

$$u = \sum_{j=1}^n F_j v_j, \quad v_j \in V, \ F_j \in \mathcal{S}, \ j = 1, \dots, n, \ n \in \mathbb{N}. \tag{2.7}$$

Definition 2.2.14. We define the *Malliavin derivative* of $u \in \mathcal{S}_V$ as the $\mathbb{H} \otimes V$-valued random variable

$$Du = \sum_{j=1}^n (DF_j) \otimes v_j.$$

Similarly as above, the Malliavin derivative is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ to $L^p(\Omega; \mathbb{H} \otimes V)$. Denote $\mathbb{D}^{1,p}(V)$ the completion of $\mathcal{S}_V$ with respect to the norm $\|\cdot\|_{1,p,V}$, where

$$\|u\|_{1,p,V}^p = \mathbb{E}\|u\|_V^p + \mathbb{E}\|Du\|_{\mathbb{H}\otimes V}^p.$$

Consequently, we define higher derivatives by iteration as

$$D^k F = DD^{k-1}F, \quad k \geq 1.$$

If $F \in \mathcal{S}$ is a smooth random variable and we apply the derivative operator on DF again, then we get a random variable with values in $\mathbb{H} \otimes \mathbb{H}$. By induction, $D^k F$ is a random variable with values in $\mathbb{H}^{\otimes k}$. Similarly to $\|\cdot\|_{1,p}$, we use the following notation for any $k \in \mathbb{N}$ and $p \geq 1$:

$$\|F\|_{k,p}^p = \mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^j F\|_{\mathbb{H}^{\otimes j}}^p.$$

Then as above, the operator D^k is closable from $\mathcal{S}$ into $L^p(\Omega; \mathbb{H}^{\otimes k})$.

Note that for $p = 2$ and $k \geq 1$, the space $\mathbb{D}^{k,2}$ is a Hilbert space with respect to the scalar product

$$\langle F, G\rangle_{k,2} = \mathbb{E}\,(FG) + \sum_{j=1}^{k} \mathbb{E}\left(\left\langle D^j F, D^j G\right\rangle_{\mathbb{H}^{\otimes j}}\right).$$

Remark 2.2.15 ([46], Corollary 1.32, p.19). If $u \in \mathbb{D}^{1,2}(\mathbb{H})$, then $\langle u, h\rangle_{\mathbb{H}}$ belongs to $\mathbb{D}^{1,2}$ for all $h \in \mathbb{H}$ and $D(\langle u, h\rangle_{\mathbb{H}}) = \langle Du, h\rangle_{\mathbb{H}} = D^h u$. △

Remark 2.2.16 ([42], Corollary 1.2.1). Suppose $\mathbb{H} = L^2([0,T])$. Let $[a,b] \in \mathcal{B}([0,T])$, where $\mathcal{B}([0,T])$ denotes the Borel σ-algebra on the interval $[0,T]$. Let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_{[a,b]^c}$-measurable, where $\mathcal{F}_{[a,b]} = \sigma\{W_s, s \in [a,b]\}$. Then $D_t F = 0$, $(\lambda \times \mathbb{P})$-for almost every $(t,\omega) \in [a,b] \times \Omega$. △

2.3 The Divergence Operator

In this section, we introduce the divergence operator defined as the adjoint of the derivative operator. Let us recall that the derivative operator D is a closed unbounded operator with values in $L^2(\Omega; \mathbb{H})$ defined on the dense subset $\mathbb{D}^{1,2}$ of $L^2(\Omega)$.

Definition 2.3.1. The *divergence* operator δ is defined as the adjoint of the operator D. That is, δ is an unbounded operator on $L^2(\Omega; \mathbb{H})$ with values in $L^2(\Omega)$ such that:

1. The domain of δ, denoted by dom δ, is the set of $\mathbb{H}$-valued random variables $u \in L^2(\Omega;\mathbb{H})$ satisfying

$$|\mathbb{E}\left[\langle DF, u\rangle_{\mathbb{H}}\right]| \leq c\|F\|_{L^2(\Omega)} \quad \forall F \in \mathbb{D}^{1,2},$$

   where c is a constant depending on u.

2. If $u \in \text{dom}\,\delta$ then $\delta(u)$ is the random variable in $L^2(\Omega)$ satisfying for any $F \in \mathbb{D}^{1,2}$

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathbb{H}}\right]. \tag{2.8}$$

Denote by $\mathcal{S}_{\mathbb{H}} \subset L^2(\Omega;\mathbb{H})$ the class of $\mathbb{H}$-valued smooth random variables of the form

$$u = \sum_{j=1}^{n} F_j h_j, \tag{2.9}$$

where $F_j \in \mathcal{S}$ is a smooth random variable and $h_j \in \mathbb{H}$ for all $j = 1, \ldots, n$.

The divergence operator has the following properties:

Proposition 2.3.2 (Properties of δ)**.**

1. The operator $\delta : \text{dom}\,\delta \to L^2(\Omega)$ is linear and closed.

2. If $u \in \text{dom}\,\delta$, then

$$\mathbb{E}\left[\delta(u)\right] = 0. \tag{2.10}$$

3. If $u \in \mathcal{S}_{\mathbb{H}}$, then $u \in \text{dom}\,\delta$ and

$$\delta(u) = \sum_{j=1}^{n} \left(F_j X(h_j) - \langle DF_j, h_j\rangle_{\mathbb{H}}\right). \tag{2.11}$$

4. Let $u \in \mathcal{S}_{\mathbb{H}}$ and $h \in \mathbb{H}$. Then

$$D^h\delta(u) = \langle D(\delta(u)), h\rangle_{\mathbb{H}} = \langle u, h\rangle_{\mathbb{H}} + \delta\left(\sum_{j=1}^{n} \langle DF_j, h\rangle_{\mathbb{H}}\, h_j\right) = \langle u, h\rangle_{\mathbb{H}} + \delta D^h u. \tag{2.12}$$

Důkaz.

1. δ is the adjoint of a linear and densely defined operator, hence also linear and closed.

2. Take $F = 1$ in (2.8).

3. We use Lemma 2.2.5 and get

$$\begin{aligned}|\mathbb{E}\left[\langle DF, u\rangle_{\mathbb{H}}\right]| &= \left|\sum_{j=1}^{n} \mathbb{E}\left[F_j \langle DF, h_j\rangle_{\mathbb{H}}\right]\right| \\ &\leq \sum_{j=1}^{n} \left(\left|\mathbb{E}\left[F \langle DF_j, h_j\rangle_{\mathbb{H}}\right]\right| + |\mathbb{E}\left[FF_j X(h_j)\right]|\right) \\ &\leq c_u \|F\|_{L^2(\Omega)},\end{aligned}$$

so $u \in \operatorname{dom} \delta$. Now, Lemma 2.2.5 is used again together with (2.8) in order to get for all $F \in \mathbb{D}^{1,2}$

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathbb{H}}\right] = \mathbb{E}\left[F\left(\sum_{j=1}^{n} F_j X(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_{\mathbb{H}}\right)\right].$$

4. For simplicity, let us consider u in the form $u = G \cdot g$ where $G \in \mathcal{S}$ and $g \in \mathbb{H}$. We use (2.11) in order to get

$$\begin{aligned}\langle D\delta(u), h\rangle_{\mathbb{H}} &= \langle D(GX(g) - \langle DG, g\rangle_{\mathbb{H}}), h\rangle_{\mathbb{H}} \\ &= \langle u, h\rangle_{\mathbb{H}} + X(g)\langle DG, h\rangle_{\mathbb{H}} - \langle D\left(\langle DG, h\rangle_{\mathbb{H}}\right), g\rangle_{\mathbb{H}} \\ &= \langle u, h\rangle_{\mathbb{H}} + \delta(\langle DG, h\rangle_{\mathbb{H}}),\end{aligned}$$

which follows from the fact that $\langle D(\langle DG, h\rangle_{\mathbb{H}}), g\rangle_{\mathbb{H}} = \langle D(\langle DG, g\rangle_{\mathbb{H}}), h\rangle_{\mathbb{H}}$ and then applied (2.11) once more.

□

The next result gives us a class of $\mathbb{H}$-valued random variables included in the domain of the divergence operator.

Proposition 2.3.3 ([42], Proposition 1.3.1, p.37)**.** *The space* $\mathbb{D}^{1,2}(\mathbb{H})$ *is included in* $\operatorname{dom} \delta$*. If* $u, v \in \mathbb{D}^{1,2}(\mathbb{H})$*, then*

$$\langle \delta(u), \delta(v)\rangle_{L^2(\Omega)} = \langle u, v\rangle_{L^2(\Omega;\mathbb{H})} + \langle Du, Dv\rangle_{L^2(\Omega;\mathbb{H}\otimes\mathbb{H})}. \tag{2.13}$$

Proposition 2.3.4 ([42], Proposition 1.3.2, p.39)**.** *Let* $u \in \mathbb{D}^{1,2}(\mathbb{H})$*, and suppose that* $D^h u$ *belongs to* $\operatorname{dom} \delta$ *for* $h \in \mathbb{H}$*. Then* $\delta(u) \in \mathbb{D}^{h,2}$*, and*

$$D^h(\delta(u)) = \langle u, h\rangle_{\mathbb{H}} + \delta(D^h u). \tag{2.14}$$

Proposition 2.3.5 ([42], Proposition 1.3.3, p.39)**.** *Let* $F \in \mathbb{D}^{1,2}$ *and* u *be in the domain of* δ *such that* $Fu \in L^2(\Omega;\mathbb{H})$*. Then* Fu *belongs to the domain of* δ *and we obtain the equality*

$$\delta(Fu) = F\delta(u) - \langle u, DF\rangle_{\mathbb{H}}, \tag{2.15}$$

provided that the right hand side is square integrable.

Důkaz. For any smooth random variable G we have:

$$\begin{aligned}\mathbb{E}\left[G\delta(Fu)\right] &= \mathbb{E}\left\langle DG, Fu\right\rangle_{\mathbb{H}} \\ &= \mathbb{E}\left[\left\langle u, D(FG) - GDF\right\rangle_{\mathbb{H}}\right] \\ &= \mathbb{E}\left[\left(\delta(u)F - \left\langle u, DF\right\rangle_{\mathbb{H}}\right)G\right].\end{aligned}$$

□

The following proposition asserts that the divergence operator enjoys the local property.

Proposition 2.3.6 ([42], Proposition 1.3.15, p.47)**.** *Let $u \in \mathbb{D}^{1,2}(\mathbb{H})$ and $A \in \mathcal{F}$, such that $u = 0$ on A. Then $\delta(u) = 0$ $\mathbb{P}$-a.s. on A.*

Skorohod integral

Let us consider $\mathbb{H} = L^2([0,T])$ and $W(h) = \int_0^T h(s)dW_s$ for $h \in L^2([0,T])$. In this case the elements of $\operatorname{dom}\delta \subset L^2([0,T]\times\Omega)$ are square integrable processes and the divergence $\delta(u)$ of the random process $u \in \operatorname{dom}\delta$ is called the *Skorohod integral* of the process u. We will also denote the Skorohod integral by

$$\delta(u) = \int_0^T u_t \delta W_t.$$

The space $\mathbb{L}^{1,2} = \mathbb{D}^{1,2}(L^2([0,T]))$ coincides with the class of processes $u \in L^2([0,T]\times\Omega)$ such that $u_t \in \mathbb{D}^{1,2}$ for almost all t, and such that there exists a measurable version of the two-parameter process $D_s u_t$ satisfying $\mathbb{E}\int_0^T\int_0^T (D_s u_t)^2 \mathrm{d}s\mathrm{d}t < \infty$. This space is included in $\operatorname{dom}\delta$ by Proposition 2.3.3. Equation (2.13) can be rewritten as

$$\mathbb{E}\left[\delta(u)\delta(v)\right] = \mathbb{E}\int_0^T u_t v_t \mathrm{d}t + \mathbb{E}\int_0^T\int_0^T D_s u_t D_t v_s \mathrm{d}s\mathrm{d}t. \tag{2.16}$$

Suppose now that the processes u and v are adapted to the filtration generated by the Brownian motion W. Then by Remark 2.2.16 we have that $D_s u_t = 0$ for almost all (s,t) such that $s > t$. Hence the second term in (2.16) is zero, and we get the usual Itô isometry.

The next result shows that if $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$, then the Skorohod integral coincides with the Itô integral with respect to the Brownian motion.

Proposition 2.3.7 ([46], Proposition 1.65, p.34)**.** *If $u \in L^2_{\mathrm{ad}}([0,T]\times\Omega)$ then $u \in \operatorname{dom}\delta$ and the Skorohod integral $\delta(u)$ coincides with the Itô integral with*

respect to the Brownian motion,

$$\delta(u) = \int_0^T u_s dW_s.$$

Now we can reformulate Proposition 2.3.4 which will allow us to compute the derivative of the Skorohod integral.

Proposition 2.3.8 ([42], Proposition 1.3.8, p.43)**.** *Suppose that* $u \in \mathbb{L}^{1,2}$*. Assume that for almost all* $t \in [0,T]$ *the process* $(D_t u_s, s \in [0,T])$ *is in* $\operatorname{dom} \delta$ *and that there exists a version of the process* $(\int_0^T D_t u_s \delta W_s, t \in [0,T])$ *which is in* $L^2([0,T] \times \Omega)$*. Then* $\int_0^T u_s \delta W_s \in \mathbb{D}^{1,2}$ *and for all* $t \in [0,T]$

$$D_t(\int_0^T u_s \delta W_s) = u_t + \int_0^T D_t u_s \delta W_s. \tag{2.17}$$

3. Volterra Processes

In this chapter, we introduce a broad class of Gaussian processes which are natural generalization of the fractional Brownian motion. We develop stochastic integral with respect to these processes and show some of its main properties. These processes are determined by their covariance function R defined in terms of a kernel K. Further, we introduce the class of regular processes by imposing some conditions on the kernel K and for these processes we present the Itô formula. We follow the paper [1] and develop the stochastic calculus using the techniques of Malliavin calculus. A stochastic calculus for processes of the so-called covariance measure structure is developed in a similar way in [30]. For a different approach we refer to [15] where the integral is based on convergence of discrete sums. For yet another approach to stochastic integration with respect to an even broader class of Gaussian processes in the white noise setting, we refer to [33]. Let us also mention the paper [60] by Čoupek and Maslowski on stochastic evolution equations driven by (non-Gaussian) Volterra processes.

3.1 Definition and Basic Properties

Definition 3.1.1 (Volterra process)**.** Let $T > 0$ and $W = (\mathrm{W}_t, t \in [0,T])$ be the standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $K : [0,T]^2 \to [0,\infty)$ be a function such that

(i) $K(t,s) = 0$ for $s > 0$, and $K(0,s) = 0$ for all $s \in [0,T]$.

(ii) K is a kernel satisfying

$$\sup_{t\in[0,T]} \int_0^t K(t,s)^2 \mathrm{d}s < \infty. \tag{3.1}$$

Then the process B given by

$$B_t = \int_0^t K(t,s) dW_s, \tag{3.2}$$

is called *Gaussian Volterra process*. The considered integral is in the sense of Itô.

Clearly B is Gaussian with the covariance function

$$R(t,s) = \mathbb{E}[B_t, B_s] = \int_0^{t\wedge s} K(t,r)K(s,r)\mathrm{d}r. \tag{3.3}$$

Denote by $\mathcal{E}$ the set of step functions on $[0,T]$, that is

$$\mathcal{E} = \Big\{ \varphi;\ \varphi(s) = \sum_{i=1}^{n} \varphi_i \mathbb{1}_{(t_i, t_{i+1}]}(s), \\ \varphi_i \in \mathbb{R},\ i \in \{1,\dots,n\},\ 0 = t_1 \le t_2 \le \cdots \le t_{n+1} = T,\ n \in \mathbb{N} \Big\}.$$

$\mathcal{E}$ is clearly a dense subset of $L^2([0,T])$ (see [51]). Suppose further that the kernel K satisfies the following condition:

(K) *The function* $K(\cdot, s) : (s,T] \to [0,\infty)$ *is continuous and of bounded variation on any interval* $(u,T]$, *for each* $u > s$, *where* $0 < s < T$.

By the condition (K) we can, for each $s \in (0,T)$, decompose the function $K(\cdot, s)$ into a difference of two non-decreasing functions K^+ and K^-. Thus, we can define a Lebesgue-Stieltjes measure $\mathcal{K}(\cdot, s)$ on $\mathcal{B}((s,T])$, using the decomposition of K into these non-decreasing components. For a precise construction we refer to [34], p.186. We have that

$$\mathcal{K}((s,t], s) = K(t-, s) - K(s+, s),$$

where the $+$ and $-$ denote the limit from the right and left-hand side, respectively. Consider the operator $\mathscr{K}^* : \mathcal{E} \to L^2([0,T])$ defined by

$$(\mathscr{K}^* \varphi)(s) = \varphi(s) K(T,s) + \int_s^T (\varphi(t) - \varphi(s))\, \mathcal{K}(\mathrm{d}t, s). \tag{3.4}$$

Further, the relationship between the operator $\mathscr{K}^*$ and the kernel K is discussed. Note that for $\varphi \in \mathcal{E}$, the function $(\mathscr{K}^* \varphi)(\cdot)$ takes the form

$$(\mathscr{K}^* \varphi)(s) = \sum_{i=1}^{n} \phi_i \mathbb{1}_{(s_i, s_{i+1}]}(s) K(T,s) + \\ + \sum_{i=1}^{n-1} \mathbb{1}_{(s_i, s_{i+1}]}(s) \sum_{j=i+1}^{n} (\phi_j - \phi_i)\, (K(s_{j+1}, s) - K(s_j, s)), \tag{3.5}$$

where $s \in [0,T]$, for $T > 0$. The expression (3.5) can be found in [1], p.771. In particular, we have

$$\left(\mathscr{K}^* \mathbb{1}_{[0,t]}\right)(s) = K(t,s) \mathbb{1}_{[0,t]}(s).$$

In order to avoid degenerate cases, we make the following assumption:

(I) $\mathscr{K}^* : \mathcal{E} \to L^2([0,T])$ is injective.

Example 3.1.2. Let $T > 0$ and $B^H = \left(B_t^H, t \in [0,T]\right)$ be the standard fractional Brownian motion with the Hurst parameter $H \in (0,1)$. The covariance function R^H of B^H is given by (1.21). Let B denote the Beta function. The covariance

function R^H can be expressed as

$$R^H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)\mathrm{d}r, \quad t,s \in [0,T],$$

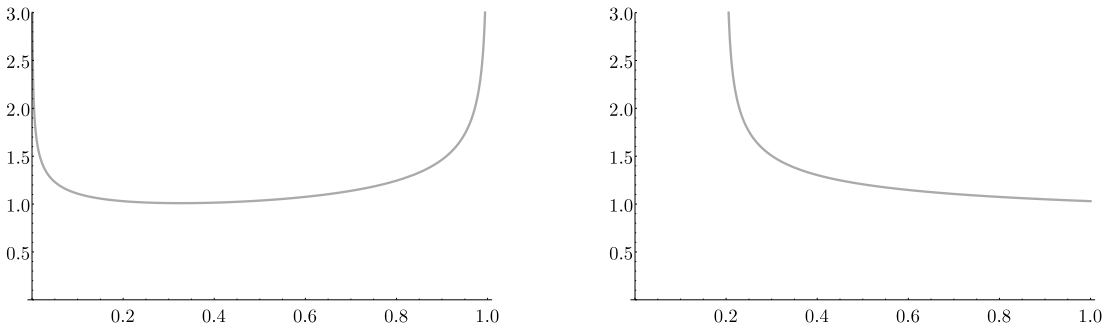
with a square integrable kernel K_H given, for $H < \frac{1}{2}$, by ([7], p.25)

$$\begin{aligned} K_H(t,s) = b_H \Bigg[&\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ &- \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \mathrm{d}u \Bigg], \end{aligned} \tag{3.6}$$

where b_H is a constant

$$b_H = \left(\frac{2H}{(1-2H)\mathrm{B}(1-2H, H+1/2)}\right)^{\frac{1}{2}}.$$

The behavior of the kernel K_H with $H < 1/2$ is illustrated in Figure 3.1.



$K_H(1,s),\ s \in (0,1)$ $\qquad$ $K_H(t,0.2),\ t \in (0.2,1]$

Obrázek 3.1: Kernel K_H of a fractional Brownian motion with the Hurst index $H = 0.25$.

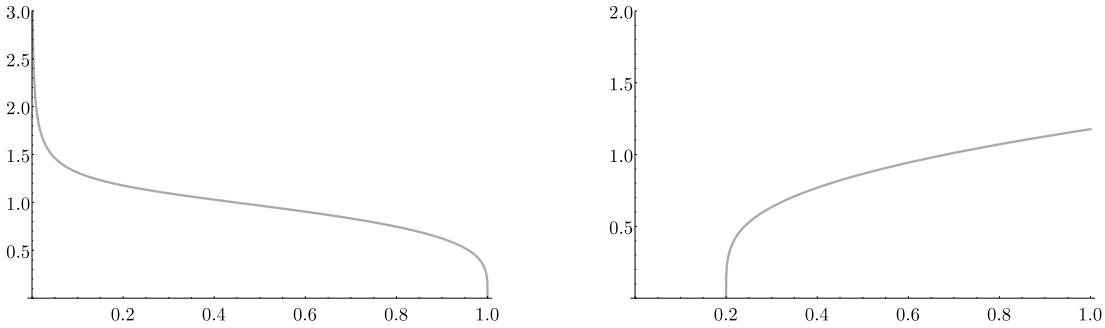
If $H > \frac{1}{2}$ then the kernel K_H takes a simpler form ([7], p.24)

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, \tag{3.7}$$

with

$$c_H = \left(\frac{H(2H-1)}{\mathrm{B}(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}.$$

This kernel is depicted in Figure 3.2.



$K_H(1,s),\ s\in(0,1]$ $K_H(t,0.2),\ t\in[0.2,1]$

Obrázek 3.2: Kernel K_H of a fractional Brownian motion with the Hurst index $H = 0.75$.

Now we present the explicit form of the operator $\mathscr{K}^*$. Let $\varphi \in \mathcal{E}$. Exploiting the notion of fractional calculus, the operator $\mathscr{K}_H^*$ can be expressed for $H < \frac{1}{2}$ as

$$\mathscr{K}_H^*\varphi(s) = b_H\Gamma\left(H+\frac{1}{2}\right)s^{1/2-H}\left(D_{T-}^{1/2-H}u^{H-1/2}\varphi(u)\right)(s), \tag{3.8}$$

and for $H > \frac{1}{2}$ as

$$\mathscr{K}_H^*\varphi(s) = c_H\Gamma\left(\frac{1}{2}-H\right)s^{1/2-H}\left(I_{T-}^{H-1/2}u^{H-1/2}\varphi(u)\right)(s). \tag{3.9}$$

♢

Example 3.1.3. Let $T > 0$ and $L^H = \left(L_t^H, t\in[0,T]\right)$ be the Liouville fractional Brownian motion. By the definition 1.3.4, the covariance function R_L^H can be expressed in the form (3.3) with the kernel

$$K_H^L(t,s) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}(t-s)^{H-\frac{1}{2}}\mathbb{1}(t<s). \tag{3.10}$$

♢

3.2 Integrable functions

In order to develop stochastic calculus with respect to the process B, the space $\mathscr{H}$ of integrable functions is defined in this section. Let us define the inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$. Set

$$\left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\right\rangle_{\mathscr{H}} = R(t,s), \quad (t,s)\in[0,T]^2. \tag{3.11}$$

By linearity, we can extend this inner product to $\mathcal{E}$ and thus $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ forms a pre-Hilbert space. The completion of $\mathcal{E}$ with respect to the inner product (3.11)

will be denoted by $\mathscr{H}$.

The operator $\mathscr{K}^*$ provides an isometry between $\mathcal{E}$ and closed subspace of $L^2([0,T])$ spanned by the functions $\{K(t,\cdot)\mathbb{1}_{[0,t]}, t\in[0,T]\}$. To see this let us write

$$\begin{aligned}\left\langle \mathscr{K}^*\mathbb{1}_{[0,t]}, \mathscr{K}^*\mathbb{1}_{[0,s]}\right\rangle_{L^2([0,T])} &= \left\langle K(t,\cdot)\mathbb{1}_{[0,t]}, K(s,\cdot)\mathbb{1}_{[0,s]}\right\rangle_{L^2([0,T])}\\ &= \int_0^{t\wedge s} K(t,u)K(s,u)\mathrm{d}u\\ &= R(t,s)\\ &= \left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\right\rangle_{\mathscr{H}}.\end{aligned}$$

Hence

$$\mathscr{H} = (\mathscr{K}^*)^{-1}(L^2([0,T])). \tag{3.12}$$

Let $\varphi\in\mathcal{E}$. Then we define the *Wiener integral* of φ with respect to the process B as

$$\int_0^T \varphi(t)\mathrm{d}B_t = \sum_{i=1}^n \varphi_i(B_{t_{i+1}} - B_{t_i}). \tag{3.13}$$

On the other hand, by the formulas (3.2) and (3.5) we have

$$\begin{aligned}\int_0^T \varphi(t)\mathrm{d}B_t &= \varphi_n B_T - \sum_{i=1}^{n-1}(\varphi_{i+1}-\varphi_i)\,B_{t_{i+1}}\\ &= \int_0^T\left(\varphi_n K(T,s) - \sum_{i=1}^{n-1}(\varphi_{i+1}-\varphi_i)\,\mathbb{1}_{(0,t_{i+1}]}(s)K(t_{i+1},s)\right)dW_s\\ &= \int_0^T\Bigg(\sum_{i=1}^n \varphi_i\mathbb{1}_{(t_i,t_{i+1}]}(s)K(T,s)\\ &\qquad + \sum_{i=1}^{n-1}(\varphi_{i+1}-\varphi_i)\,\mathbb{1}_{(0,t_{i+1}]}(s)\,(K(T,s)-K(t_{i+1},s))\Bigg)dW_s\\ &= \int_0^T\Bigg(\varphi(s)K(T,s)\\ &\qquad + \sum_{i=1}^{n-1}\mathbb{1}_{(t_i,t_{i+1}]}(s)\sum_{j=i+1}^{n}(\varphi_{i+1}-\varphi_i)\,(K(t_{j+1},s)-K(t_j,s))\Bigg)dW_s\\ &= \int_0^T (\mathscr{K}^*\varphi)(s)dW_s.\end{aligned}$$

Hence we can write

$$\int_0^T \varphi(t)\mathrm{d}B_t = \int_0^T (\mathscr{K}^*\varphi)(s)dW_s, \quad \varphi\in\mathcal{E}. \tag{3.14}$$

It follows that the operator $\mathscr{K}^*$ may be extended into $\mathscr{H}$, $\mathscr{K}^*:\mathscr{H}\to L^2([0,T])$ is injective and the equality (3.14) holds for $\varphi\in\mathscr{H}$.

Remark 3.2.1. Note, that the process $\left(\int_0^T \varphi(t)\mathrm{d}B_t, \varphi \in \mathscr{H}\right)$ is an $\mathscr{H}$-isonormal Gaussian process in the sense of Definition 2.1.1, since it is clearly zero-mean, Gaussian and

$$\begin{aligned}\mathbb{E}\left[\int_0^T \varphi(t)\mathrm{d}B_t \cdot \int_0^T \psi(t)\mathrm{d}B_t\right] =& \mathbb{E}\left[\int_0^T \left(\mathscr{K}^*\varphi\right)(t)dW_t \cdot \int_0^T \left(\mathscr{K}^*\psi\right)(t)dW_t\right] \\ &= \int_0^T \left(\mathscr{K}^*\varphi\right)(s) \cdot \left(\mathscr{K}^*\psi\right)(s)\mathrm{d}s \\ &= \left\langle \mathscr{K}^*\varphi, \mathscr{K}^*\psi\right\rangle_{L^2([0,T])} \\ &= \left\langle \varphi, \psi\right\rangle_{\mathscr{H}},\end{aligned}$$

by the Itô isometry (1.6). △

Example 3.2.2. In this example, the domain of the operator $\left(\mathscr{K}_H^L\right)^*$ associated to the Liouville fractional Brownian motion. The kernel K_H^L is of the form (3.10) and the operator $\left(\mathscr{K}_H^L\right)^*$, by its definition (3.4), can be expressed as

$$\left(\left(\mathscr{K}_H^L\right)^*\varphi\right)(s) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left(\frac{\varphi(s)}{(T-s)^{\frac{1}{2}-H}} - \left(\frac{1}{2}-H\right)\int_s^T \frac{\varphi(s)-\varphi(t)}{(t-s)^{\frac{3}{2}-H}}\mathrm{d}t\right),$$

which is, however, precisely the Weyl representation (1.17) of the right-sided fractional derivative. This means that

$$\left(\left(\mathscr{K}_H^L\right)^*\varphi\right)(s) = \left(D_{T-}^{\frac{1}{2}-H}\varphi\right)(s).$$

The inverse operator defined in terms of the left-sided Riemann-Liouville fractional integral exists for all functions $\varphi \in I_{T-}^{\frac{1}{2}-H}\left(L^p\right)$. Therefore, we have that

$$\left(\left(\mathscr{K}_H^L\right)^*\right)^{-1} = I_{T-}^{\frac{1}{2}-H},$$

with the domain

$$\mathrm{dom}\left(\left(\left(\mathscr{K}_H^L\right)^*\right)^{-1}\right) = I_{T-}^{\frac{1}{2}-H}\left(L^p\right).$$

♢

In the following, we denote by $D_B, \delta_B, \mathbb{D}_B^{k,p}$ the operators and spaces associated with the process B and by $D, \delta, \mathbb{D}^{k,p}$ the operators and spaces associated with the Wiener process W. For the convenience of the reader, let us recall definitions of these symbols. Let F be smooth random variable of the form (2.2), that is

$$F = f(B(h_1), \ldots, B(h_n)),$$

for some $f \in C_b^\infty(\mathbb{R}^n)$, $h_i \in \mathscr{H}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. We denote the set of smooth random variables by $\mathcal{S}$. The derivative of $F \in \mathcal{S}$ is the $\mathscr{H}$-valued random

variable

$$D_B F = \sum_{i=1}^{n} \partial_i f(B(h_1), \dots, B(h_n)) h_i.$$

The space $\mathbb{D}_B^{k,p}$ is defined as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{B,k,p}^p = \mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D_B^j F\|_{\mathscr{H}^{\otimes j}}^p.$$

The divergence operator δ_B is then the adjoint of the derivative D_B. The domain of δ_B in $L^2(\Omega)$ is denoted by dom δ_B. If u is in dom δ_B then $\delta_B(u)$ is defined by the duality relationship

$$\mathbb{E}\left[F\delta_B(u)\right] = \mathbb{E}\left[\langle D_B F, u\rangle_{\mathbb{H}}\right], \quad F \in \mathbb{D}_B^{1,2}.$$

By Proposition 2.3.3 we have that the space $\mathbb{D}_B^{1,2}(\mathscr{H})$ of $\mathscr{H}$-valued random variables is included in dom δ_B.

The equality (3.12) implies that

$$\mathbb{D}^{1,2}(\mathscr{H}) = (\mathscr{K}^*)^{-1}(\mathbb{L}^{1,2}), \tag{3.15}$$

where we denote $\mathbb{L}^{1,2} = \mathbb{D}^{1,2}(L^2([0,T]))$.

The next proposition gives us a relationship between the derivative D_B with respect to B and the derivative D with respect to the standard Brownian motion W.

Proposition 3.2.3. *For any F in $\mathbb{D}_B^{1,2}$ we have*

$$\mathscr{K}^* D_B F = DF. \tag{3.16}$$

Důkaz. Let $F = f(B(t))$ for $f \in C^1$, and let $u \in L^2(\Omega; \mathscr{H})$. Then

$$\begin{aligned}
\mathbb{E}\left[\langle u, D_B F\rangle_{\mathscr{H}}\right] &= \mathbb{E}\left[\left\langle \mathscr{K}^* u, \mathscr{K}^* D_B F\right\rangle_{L^2([0,T])}\right] \\
&= \mathbb{E}\left[\left\langle \mathscr{K}^* u, f'(B(t))\mathscr{K}^* \mathbb{1}_{[0,t]}\right\rangle_{L^2([0,T])}\right] \\
&= \mathbb{E}\left[\left\langle \mathscr{K}^* u, f'(B(t))K(t,\cdot)\mathbb{1}_{[0,t]}\right\rangle_{L^2([0,T])}\right] \\
&= \mathbb{E}\left[\left\langle \mathscr{K}^* u, f'(W(K(t,\cdot)\mathbb{1}_{[0,t]})K(t,\cdot)\mathbb{1}_{[0,t]}\right\rangle_{L^2([0,T])}\right] \\
&= \mathbb{E}\left[\left\langle \mathscr{K}^* u, DF\right\rangle_{L^2([0,T])}\right],
\end{aligned}$$

and the general case follows by the density argument. □

Thus, we have

$$\text{dom } \delta_B = (\mathscr{K}^*)^{-1}(\text{dom } \delta), \tag{3.17}$$

and $\delta_B(u) = \delta(\mathscr{K}^* u)$.

3.3 Processes with Regular Kernel

In this section, we introduce the conditions for a regular kernel and investigate properties of Volterra processes with regular kernels. We develop stochastic integral with respect to these processes and present the Itô formula.

Definition 3.3.1 (Regular process)**.** Suppose that $B = (B_t, t \in [0,T])$ is a zero-mean Gaussian Volterra process with a kernel K. We say that the process B is *regular* or with *a regular kernel* on $[0,T]$, if it satisfies the condition

(R) *For all* $s \in [0,T)$, $K(\cdot, s)$ *has bounded variation on the interval* $(s,T]$*, and*

$$\int_0^T |\mathcal{K}|\left((s,T],s\right)^2 \mathrm{d}s < \infty.$$

Let us denote $K(s^+,s) = K(T,s) - \mathcal{K}((s,T],s)$. By the conditions (R) and (3.1) we have that $K(s^+,s)$ is square integrable in $[0,T]$. In this case the operator $\mathscr{K}^*$ defined by (3.4) can be expressed as

$$\left(\mathscr{K}^*\varphi\right)(s) = \varphi(s)K(s^+,s) + \int_s^T \varphi(t)\mathcal{K}(\mathrm{d}t,s), \tag{3.18}$$

for $\varphi \in \mathcal{E}$.

Consider the following seminorm on $\mathcal{E}$

$$\|\varphi\|^2_{\mathscr{H}_R} = \int_0^T |\varphi(s)|^2 K(s^+,s)^2 \mathrm{d}s + \int_0^T \left(\int_s^T |\varphi(t)||\mathcal{K}|(\mathrm{d}t,s)\right)^2 \mathrm{d}s. \tag{3.19}$$

Denote by $\mathscr{H}_R$ the completion of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathscr{H}_R}$. Then $\mathscr{H}_R$ is continuously embedded in $\mathscr{H}$, because, for $\varphi \in \mathcal{E}$, we have that

$$\begin{aligned}
\|\varphi\|^2_{\mathscr{H}} &= \|\mathscr{K}^*\varphi\|^2_{L^2([0,T])} \\
&= \int_0^T \left| K(s^+,s)\varphi(s) + \int_s^T \varphi(t)|\mathcal{K}|(\mathrm{d}t,s)\right|^2 \mathrm{d}s \\
&\le 2\int_0^T \left(K(s^+,s)|\varphi(s)|\right)^2 \mathrm{d}s + 2\int_0^T \left(\int_s^T |\varphi(t)||\mathcal{K}|(\mathrm{d}t,s)\right)^2 \mathrm{d}s,
\end{aligned}$$

which is

$$\|\varphi\|^2_{\mathscr{H}} \le 2\|\varphi\|^2_{\mathscr{H}_R}.$$

Example 3.3.2. In the case of fractional Brownian motion with $H > \frac{1}{2}$, we have

(see [2], p.5 and computations on p.4)

$$\|\varphi\|^2_{\mathscr{H}_R^H} = H(2H-1)\int_0^T\int_0^T |\varphi(t)||\varphi(t)||t-s|^{2H-2}\mathrm{d}t\mathrm{d}s.$$

In particular, $L^{\frac{1}{H}}([0,T]) \subset \mathscr{H}_R^H$ (see e.g. [7], p.33). ♢

Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in $\mathbb{D}^{1,2}(\mathscr{H}_R)$. That means, that u satisfies

$$\mathbb{E}\|u\|^2_{\mathscr{H}_R} < \infty \tag{3.20}$$

and

$$\mathbb{E}\int_0^T \|D_r u\|^2_{\mathscr{H}_R}\mathrm{d}r < \infty. \tag{3.21}$$

These conditions imply that $\mathscr{K}^* u \in \mathbb{L}^{1,2}$ and hence $u \in \operatorname{dom}\delta_B$. For a process $u \in \mathbb{D}^{1,2}(\mathscr{H}_R)$ we denote $\delta_B(u)$ also by $\int_0^T u_s \delta B_s = \int_0^T (\mathscr{K}^* u)_s \delta W_s$.

Remark 3.3.3 (Indefinite integral). If u satisfies the above conditions, so does $u\mathbb{1}_{[0,t]}$ for any $t \in [0,T]$. Then the operator $\mathscr{K}_t^*$ on $[0,T]$ is given by

$$(\mathscr{K}_t^* u)_s = u_s K(s^+, s) + \int_{(s,t]} u_r \mathcal{K}(\mathrm{d}r, s),$$

and we define the indefinite integral of $u \in \mathbb{D}^{1,2}(\mathscr{H}_R)$ as

$$\int_0^t u_s \delta B_s = \int_0^t (\mathscr{K}_t^* u)_s \delta W_s = \int_0^t u_s K(s^+, s)\delta W_s + \int_0^t\int_s^t u_r \mathcal{K}(\mathrm{d}r, s)\delta W_s. \tag{3.22}$$

△

Let us denote $R_t = R(t,t) = \mathbb{E}B_t^2$.

Theorem 3.3.4 (Itô formula,[1], Theorem 2, p.778)**.** *Let $F \in C^{1,2}([0,T]\times\mathbb{R})$ be a continuous function satisfying the growth condition*

$$\max\left\{|F(t,x)|, \left|\frac{\partial F}{\partial t}(t,x)\right|, \left|\frac{\partial F}{\partial x}(t,x)\right|, \left|\frac{\partial^2 F}{\partial x^2}(t,x)\right|\right\} \le c\exp\left\{\lambda|x|^2\right\}, \tag{3.23}$$

where c and λ are positive constants such that $\lambda < \frac{1}{4}\left(\sup_{0\le t\le T} R_t\right)^{-1}$.

Let $B = \{B_t, t \in [0,T]\}$ be a Gaussian Volterra process satisfying the condition (R). Then $\frac{\partial F}{\partial x}(t, B_t) \in \mathbb{D}^{1,2}(\mathscr{H}_R)$ and for each $t \in [0,T]$ the following formula holds:

$$\begin{aligned} F(t,B_t) = \; & F(0,0) + \int_0^t \frac{\partial F}{\partial s}(s,B_s)\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(s,B_s)\delta B_s \\ & + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x}(s,B_s)\mathrm{d}R_s \quad \mathbb{P}-a.s. \end{aligned}$$

Remark 3.3.5.

1. The previous theorem is givenin [1] for F not explicitly depending on t. Although the proof can be readily modified in order to allow for time dependent function F, it is not presented here due to the length of the original proof.

2. The growth restriction (3.23) can be dropped thanks to the local property of Skorohod integral (Proposition 2.3.6). The localization argument is discussed e.g. in [45] and is carried out in the proof of Theorem 5.1.2. on p.193.

△

3.3.1 Strictly Regular processes

Now we impose some more conditions on the kernel K. This will allow us to obtain Hölder continuity of the sample paths of Volterra processes and the Itô formula for the integrals driven by Volterra processes.

By (3.22), we can decompose the process B as

$$B_t = \int_0^t K(s^+, s)\delta W_s + \int_0^t \int_s^t \mathcal{K}(\mathrm{d}r, s)\delta W_s, \tag{3.24}$$

where $K(s^+, s)$ is square integrable by (R), hence the first term in (3.24) is a martingale. Therefore we will often assume that $K(s^+, s) = 0$ in which case we will call the kernel *vanishing*.

Remark 3.3.6. The condition $K(s^+, s) = 0$ for all $0 < s < T$, is natural for it is satisfied in the case of the fractional Brownian motion B^H with $H > \frac{1}{2}$, since by the Dominated Convergence Theorem, we have

$$\lim_{t\to s+} K_H(t, s) = c_H \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \lim_{t\to s+} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u = 0.$$

Similarly, also the kernel of Liouville fractional Brownian motion is vanishing. △

Definition 3.3.7 (Strictly regular process)**.** Suppose that $B = (B_t, t \in [0, T])$ is a Gaussian Volterra process with a kernel K satisfying the condition (K). The process B is *strictly regular on* $[0, T]$, if, for some $\gamma \in \left(0, \frac{1}{2}\right)$, the kernel K satisfies the following:

(C) For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T] \to \mathbb{R}$ is differentiable in the interval (s, T) and both $K(t, s)$ and its derivative $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.

(SR) There exists a constant $c > 0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leq c(t-s)^{\gamma-1}\left(\frac{s}{t}\right)^{-\gamma}, \tag{3.25}$$

$$\int_s^t K(t,r)^2 \mathrm{d}r \leq c(t-s)^{2\gamma+1} \tag{3.26}$$

for $0 \leq s < t \leq T$.

Remark 3.3.8. Let us note that if K satisfies (C) and (SR) then also (R) holds, that is, if B is strictly regular kernel then it is also regular. △

The next example shows that the fractional Brownian motion with the Hurst parameter $H > 1/2$ satisfies the conditions of strict regularity.

Example 3.3.9. Let $T > 0$ and let $B^H = (B_t^H, t \in [0,T])$ be the standard fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$. Then the kernel K_H is of the form (3.7), that is

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u.$$

Differentiating $K_H(t,s)$ in the first variable gives

$$\frac{\partial K}{\partial t}(t,s) = c_H \left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}. \tag{3.27}$$

Thus, the conditions (C) and (SR) hold with $\gamma = H - \frac{1}{2}$. Therefore, the standard fractional Brownian motion is a strictly regular Gaussian Volterra process. ♢

Proposition 3.3.10 (Hölder continuity of strictly regular processes)**.** *Let $T > 0$ and $B = (B_t, t \in [0,T])$ be a Gaussian Volterra process satisfying the condition (SR) for some $0 < \gamma < \frac{1}{2}$. Then there exists a version of B with Hölder continuous paths up to the order $\frac{1}{2} + \gamma$.*

Důkaz. We will estimate $\mathbb{E}\left[B_t - B_s\right]^2$ in order to use the Kolmogorov-Chentsov Theorem 1.1.6.

$$\begin{aligned}\mathbb{E}\left[B_t - B_s\right]^2 =& \mathbb{E}\left[\int_0^s (K(t,r) - K(s,r))\, dW_r + \int_s^t K(t,r) dW_r\right]^2 \\ =& \int_0^s (K(t,r) - K(s,r))^2 \,\mathrm{d}r + \int_s^t K(t,r)^2 \mathrm{d}r.\end{aligned}$$

The second term can be directly estimated by (3.26) as

$$\int_s^t K(t,r)^2 \mathrm{d}r \leq c(t-s)^{2\gamma+1}.$$

In the following, the constant c can differ from line to line. For the first term let

us write

$$\begin{aligned}
&\int_0^s (K(t,r)-K(s,r))^2\,\mathrm{d}r \\
&= \int_0^s \left(\int_s^t \frac{\partial}{\partial u}K(u,r)\mathrm{d}u\right)^2 \mathrm{d}r \\
&= \int_0^s \left(\int_s^t \frac{\partial}{\partial u}K(u,r)\mathrm{d}u\right)\left(\int_s^t \frac{\partial}{\partial u}K(v,r)\mathrm{d}v\right)\mathrm{d}u \\
&\leq \int_0^s \int_s^t \int_s^t \left(\left|\frac{\partial}{\partial u}K(u,r)\right|\left|\frac{\partial}{\partial v}K(v,r)\right|\right)\mathrm{d}u\mathrm{d}v\mathrm{d}r \\
&= \int_s^t \int_s^t \int_0^{u\wedge v} \left(\left|\frac{\partial}{\partial u}K(u,r)\right|\left|\frac{\partial}{\partial v}K(v,r)\right|\right)\mathrm{d}r\mathrm{d}u\mathrm{d}v \\
&\leq c\int_s^t \int_s^t (uv)^\gamma \int_0^{u\wedge v} (u-r)^{\gamma-1}(v-r)^{\gamma-1}r^{-2\gamma}\mathrm{d}r\mathrm{d}u\mathrm{d}v \\
&\leq c\int_s^t \int_s^t |u-v|^{2\gamma-1}\mathrm{d}u\mathrm{d}v \\
&\leq c\,(t-s)^{2\gamma+1}\,.
\end{aligned}$$

□

Now we will state the Itô formula for Skorohod integral.

Theorem 3.3.11 ([1], Theorem 4, p.791)**.** *Let F be a function of class $C_b^{1,2}([0,T]\times\mathbb{R})$. Suppose that B is a zero-mean vanishing strictly regular Volterra process. Let u be an adapted process bounded in the norm of the space $\mathbb{D}^{2,4}(\mathscr{H}_R)$. Set $X_t = \int_0^t u_s\delta B_s$. Then the process $\frac{\partial F}{\partial x}(t,X_t)$ belongs to $\mathbb{D}^{1,2}(\mathscr{H}_R)$ and for each $t\in[0,T]$ the following formula holds:*

$$\begin{aligned}
F(t,X_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial t}(s,X_s)\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(s,X_s)u_s\delta B_s \\
&+ \int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)u_s\left(\int_0^s \frac{\partial K}{\partial s}(s,r)\left(\int_0^s D_r(\mathscr{K}_s^* u)_v\delta W_v\right)\mathrm{d}r\right)\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)\frac{\partial}{\partial s}\left(\int_0^s (\mathscr{K}_s^* u)_r^2\,\mathrm{d}r\right)\mathrm{d}s, \quad \mathbb{P}-a.s.
\end{aligned} \tag{3.28}$$

Remark 3.3.12. As with the previous Itô formula, we omit the proof due to its lengthiness since it is straightforward to alter the proof given in [1] for F not being dependent on t. By localization argument, the assumption of boundedness of function F can be dropped. △

Pathwise Integral

The results from Section 1.2.1 can now be used to define a stochastic integral with respect to strictly regular Volterra processes. Let B be strictly regular vanishing Volterra process. Hölder continuity ensures the pathwise existence of the integral

$$\int_0^t u_s dB_s, \quad t \in [0,T], \tag{3.29}$$

in sense of Definition 1.2.3 with probability one for any random process u such that $u_{0+} \in I^{\alpha}_{0+}(L^1(0,T))$ with probability one for some $\alpha > \frac{1}{2} - \gamma$.

If moreover $u \in C^{\lambda}(0,T)$ with probability one for some $\lambda > \frac{1}{2} - \gamma$, we may understand the integral (3.29) in terms of the Riemann sums (1.19). We will call this integral the *pathwise integral* and the corresponding differential will be denoted as $\mathrm{d}B_t$ in order to distinguish from the Skorohod differential which we denote by δB_t, and Itô differential which is denoted (for Wiener process) by dW_t.

The Theorem 1.2.8 can now be stated as follows.

Theorem 3.3.13 (Itô lemma for pathwise integral)**.** *Let B be a strictly regular vanishing Gaussian Volterra process. Let $\alpha > \frac{1}{2} - \gamma$, $u \in I^{\alpha}_{0+}(L^2([0,T])$ with probability 1 be bounded and*

$$Y_t = \int_0^t u_s dB_s, \quad t \in (0,T].$$

Then for any function $F(t,x)$, $F \in C^{1,2}([0,T] \times \mathbb{R})$ and for any $0 \le t_0 \le t \le T$:

$$F(t,Y_t) - F(t_0,Y_{t_0}) = \int_{t_0}^t \frac{\partial F}{\partial t} F(s,Y_s)\mathrm{d}s + \int_{t_0}^t \frac{\partial F}{\partial x}(s,Y_s) u_s dB_s, \quad \mathbb{P}-a.s. \tag{3.30}$$

Let $u = (u_t, t \in [0,T])$ be a process bounded in the norm of the space $\mathbb{D}^{1,p}(\mathscr{H}_R)$ for some $p \ge 2$. Then $u \in \mathbb{D}^{1,p}(\mathscr{H}_R)$ and we have

$$(\mathscr{K}_t^* u)_s = \int_s^t u_r \frac{\partial K}{\partial r}(r,s)\mathrm{d}r.$$

Remark 3.3.14. Let B be a vanishing strictly regular Volterra process, $u \in \mathbb{D}^{1,2}(\mathscr{H}_R)$ be a process with λ-Hölder continuous trajectories with $\lambda > \frac{1}{2} - \gamma$ and suppose that

$$\int_0^t \left(\int_s^t |D_s u_r| |\mathcal{K}|(\mathrm{d}r, s) \right) \mathrm{d}s < \infty, \quad \mathbb{P}-a.s.$$

The following relationship between pathwise and Skorohod integral can be proved similarly as in the case of fractional Brownian motion (see e.g. [2], [42], Chapter

5, or [15])

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + \int_0^t \left(\int_s^t D_s u_r \mathcal{K}(\mathrm{d}r, s) \right) \mathrm{d}s. \tag{3.31}$$

△

Remark 3.3.15. We know by (2.10) that the expectation of Skorohod integral is zero. This is in general not true for the pathwise integral as can be seen from the equation (3.31). △

Approximation by Riemann sums

We already know from Theorem 1.1.20 and Remark 3.3.1, that both Itô and pathwise integrals can be viewed as limits of Riemann sums. Here we show that the Skorohod integral can be viewed as a limit of certain Riemann sums, where the ordinary product is replaced by the *Wick product*.

Without going into much detail, we define the Wick exponential of a zero-mean Gaussian random variable ξ by $\varepsilon(\xi) = \exp\left\{\xi - \frac{1}{2}\mathbb{E}|\xi|^2\right\}$. For a standard Brownian motion W we obtain for $t_1 < t_2 < t_3$

$$\varepsilon(B_{t_3} - B_{t_2})\varepsilon(B_{t_2} - B_{t_1}) = \varepsilon(B_{t_3} - B_{t_1}).$$

To enforce this property for all, possibly correlated random variables ξ, η, we introduce the Wick product for Wick exponentials by

$$\varepsilon(\xi) \diamond \varepsilon(\eta) = \varepsilon(\xi + \eta).$$

The Wick product can then be extended to a larger class of random variables, for more on Wick products see e.g. [31]. Note that for smooth random variable $F \in \mathcal{S}$ and $h \in \mathscr{H}$ the Wick product of F and $B(h)$ is given by

$$F \diamond B(h) = FB(h) - \langle D_B F, h \rangle_{\mathbb{H}}.$$

Proposition 3.3.16 ([1], p.795)**.** *Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0, T]$. Suppose that B is Gaussian Volterra process with a regular kernel K and u is a process continuous in the norm of $\mathbb{D}^{1,2}(\mathscr{H}_R)$. Then*

$$\lim_{|\pi| \to 0} \sum_{i=0}^{n-1} u_{t_i} \diamond (B_{t_{i+1}} - B_{t_i}) = \int_0^T u_t \delta B_t, \tag{3.32}$$

where the convergence is in $L^2(\Omega)$.

Note that if B was a Brownian motion and u was adapted process, then the Skorohod integral coincides with the Itô integral and the Wick product in the previous proposition would be equal to the ordinary product, for a brief discussion see [41], p.1551.

4. Financial Application

In this chapter, we apply the theory from previous chapters to the stochastic differential equation (4.2), where the Wiener process W is replaced by a strictly regular vanishing Volterra process B. As an introduction, basic financial terms and the standard Black-Scholes model is presented in the first part. Further, pathwise and Skorohod Volterra analogues of the Black-Scholes model are examined.

4.1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete probability space. The standard Black-Scholes market model based on a Brownian motion W consists (in the simplest form) of a money-market account A (riskless asset) and a stock S (risky asset). The money-market account A_t is given by

$$\begin{aligned} \mathrm{d}A_t &= rA_t\mathrm{d}t, \quad r \in \mathbb{R} \\ A_0 &= 1. \end{aligned} \tag{4.1}$$

The dynamics of a risky asset S_t is modeled

$$\begin{aligned} dS_t &= \mu S_t\mathrm{d}t + \sigma S_t dW_t \\ S_0 &= s_0, \end{aligned} \tag{4.2}$$

where $\mu \in \mathbb{R}$ and $\sigma, s_0 > 0$ are constants. The μ is called *drift* and σ the *volatility*. The interpretation of this model is following: The stock follows an exponential growth which is randomly perturbated by the Itô integral.

Remark 4.1.1. This market model was originally proposed by the economist Paul Samuelson in [54]. △

It can be shown by the Itô formula, Theorem 1.1.18 that the (unique) solution of (4.2) is given by

$$S_t = s_0 \exp\left\{\mu t - \frac{\sigma^2}{2}t + \sigma W_t\right\}. \tag{4.3}$$

Definition 4.1.2. A *portfolio* or a *trading strategy* is a pair of $\mathcal{F}_t$-adapted processes $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t\in[0,T]} = (u_t, v_t)_{t\in[0,T]}$, where u_t represents the amount of money stored in the riskless asset at time t and v_t number of stock shares at time t. The *value* of the portfolio $\boldsymbol{\pi}$ at time t is given by

$$V_t^{\pi} = u_t A_t + v_t S_t. \tag{4.4}$$

Let us first consider a simple buy and hold strategy $\boldsymbol{\pi} = (1_{(\tau,T]}F, 1_{(\tau,T]}G)$, where τ is a stopping time, $0 \le \tau \le T$ and F, G are $\mathcal{F}_\tau$-measurable random variables.

We say that this portfolio is *self-financing* if its value satisfies

$$V_t^\pi = V_0^\pi + F(A_t - A_{\tau\wedge t}) + G(S_t - S_{\tau\wedge t}). \tag{4.5}$$

This condition means that we don't insert or withdraw money.

For general portfolios, the condition (4.5) is translated to

$$V_t^\pi = V_0^\pi + \int_0^t u_s \mathrm{d}A_s + \int_0^t v_s dS_s, \tag{4.6}$$

where the second integral is understood as

$$\int_0^t v_s dS_s = \int_0^t v_s \mu S_s \mathrm{d}s + \int_0^t v_s \sigma S_s dW_s. \tag{4.7}$$

In order to exclude doubling strategies (see e.g. [6], Theorem 5.6.), we further assume that the value of the portfolio is bounded from below.

Definition 4.1.3. A portfolio $\boldsymbol{\pi} = (u, v)$ is called *admissible* if

1. All the corresponding integrals exist, that is $(v \cdot S) \in L^2_{\mathrm{ad}}([0,T] \times \Omega)$ and

$$\int_0^T |u_s| \mathrm{d}As < \infty, \quad \int_0^T |v_s S_s| \mathrm{d}s < \infty, \quad \mathbb{P} - a.s.$$

2. There exists a constant $c \geq 0$ such that

$$\inf_{t\in[0,T]} V_t^\pi \geq -c, \quad \mathbb{P} - a.s.$$

Intuitively, the admissibility requirement means that the investor can only implement strategies that respect a finite credit line.

Definition 4.1.4 (Arbitrage)**.** We say that admissible self-financing portfolio $\boldsymbol{\pi}$ is an *arbitrage* if its corresponding value V^π satisfies

(i) $V_0^\pi \leq 0$,

(ii) There exists $t \in [0,T]$ such that $\mathbb{P}(V_t^\pi \geq 0) = 1$ and $\mathbb{P}(V_t^\pi > 0) > 0$.

First we show that the standard Black-Scholes model does not admit arbitrage. Let us define a *discount process* by

$$\begin{aligned} \mathrm{d}d_t &= -r d_t \mathrm{d}t, \\ d_0 &= 1. \end{aligned}$$

Definition 4.1.5. A probability measure $\tilde{\mathbb{P}}$ is said to be *risk-neutral* if $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ and the discounted stock price $d_t S_t$ is a martingale under $\tilde{\mathbb{P}}$.

Theorem 4.1.6 (First Fundamental Theorem of Asset Pricing, [57], p.231)**.** *If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*

Remark 4.1.7. General version of the First Fundamental Theorem of Asset Pricing by Delbaen and Schachermayer [18] links the existence of a risk-neutral measure to the existence of weaker form of arbitrage called *free lunch with vanishing risks*. Besides other things, they show that if the price process S is a bounded semi-martingale, then a risk-neutral probability measure exists if and only if S does not admit free lunch with vanishing risks. For a comprehensive text on arbitrage theory see also [19]. △

Hence, we only need to show that there exists an equivalent probability measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ under which the discounted stock price process $d_t S_t$ is a martingale. By the Itô lemma, Theorem 1.1.18, it follows that the discounted stock price process satisfies

$$\begin{aligned} d(d_t S_t) &= (\mu - r) d_t S_t \mathrm{d}t + \sigma d_t S_t dW_t \\ &= \sigma d_t S_t (M \mathrm{d}t + dW_t), \end{aligned}$$

where

$$M = \frac{\mu - r}{\sigma}.$$

Now from the Girsanov Theorem 1.1.19 it follows that the process

$$\tilde{W}_t = W_t + Mt$$

is a standard Brownian motion under the measure $\tilde{\mathbb{P}}$. Thus the process $d_t S_t$ satisfies

$$d_t S_t = s_0 + \int_0^t \sigma d_s S_s d\tilde{W}_s,$$

and hence is a martingale.

4.2 Pathwise Model

In this section, we replace the Brownian motion in the equation (4.2) with a strictly regular vanishing Volterra process B and all the corresponding stochastic integrals are understood in a pathwise sense, so that the stock follows the stochastic differential equation

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t \mathrm{d}B_t, \\ S_0 &= s_0, \end{aligned} \tag{4.8}$$

with $s_0 > 0$, a continuous function $\mu \in [0, T]$ and $\sigma \in C^\lambda(0, T)$ be λ-Hölder continuous with $\lambda > \frac{1}{2} - \gamma$, where γ is from Proposition 3.3.10.

This means that the price process S is given by

$$S_t = s_0 \exp\left\{\int_0^t \alpha_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s\right\}, \tag{4.9}$$

as can be easily computed if we use the Itô formula, Theorem 3.3.13, on the function

$$F(t,y) = s_0 \exp\left\{\int_0^t \alpha_s \mathrm{d}s + y\right\}.$$

In what follows, we restrict to the case of (4.8) with constant coefficients μ, σ. The following example of arbitrage by Shiryaev can be found in [56] where it is discussed in the case of fractional Brownian motion with $H > \frac{1}{2}$.

Example 4.2.1. Let $\boldsymbol{\pi} = (u, v)$ with

$$\begin{aligned} u_t &= 1 - \exp\left\{-2rt + 2\mu t + 2\sigma B_t\right\}, \\ v_t &= \frac{2}{s_0}\left(\exp\left\{-rt + \mu t + \sigma B_t\right\} - 1\right). \end{aligned}$$

Then the value of this portfolio is

$$\begin{aligned} V_t^\pi &= u_t A_t + v_t S_t \\ &= \exp\left\{rt\right\} + \exp\left\{-rt + 2\mu t + 2\sigma B_t\right\} \\ &\quad - 2\exp\left\{\mu t + \sigma B_t\right\} \\ &= \mathrm{e}^{rt}\left(\exp\left\{\sigma B_t + (\mu - r)t\right\} - 1\right)^2, \end{aligned}$$

which is almost surely positive. Now we want to check the condition (4.6), i.e. that the portfolio is self-financing. To do this, we apply Itô formula to V_t^π and get

$$\begin{aligned} V_t^\pi &= \int_0^t r\mathrm{e}^{rs}\left(1 - \exp\left\{-2rs + 2\mu s + 2\sigma B_s\right\}\right)\mathrm{d}s \\ &\quad + \int_0^t 2\mu\left(\exp\left\{-rs + \mu s + \sigma B_s\right\} - 1\right)\exp\left\{\mu s + \sigma B_s\right\}\mathrm{d}s \\ &\quad + \int_0^t 2\sigma\left(\exp\left\{-rs + \mu s + \sigma B_s\right\} - 1\right)\exp\left\{\mu s + \sigma B_s\right\}\mathrm{d}B_s \\ &= \int_0^t rA_s u_s \mathrm{d}s + \int_0^t \mu v_s S_s \mathrm{d}s + \int_0^t \sigma v_s S_s \mathrm{d}B_s \\ &= \int_0^t u_s \mathrm{d}A_s + \int_0^t v_s dS_s. \end{aligned}$$

We have shown that the portfolio is self-financing, almost surely positive, and $V_0^\pi = 0$. Consequently, it is arbitrage. ♢

Remark 4.2.2.

1. The existence of arbitrage in a market model involving fractional Brownian

motion was first pointed out by Rogers in [48]. He was working with a fractional equivalent of the so-called Bachelier model, where the price process is given by

$$S_t = s_0 + \mu t + \sigma B_t^H.$$

He constructed arbitrage as a combination of buy-and-hold strategies. For his construction he needed the whole history of the process B_t^H from the time $-\infty$. His ideas were further refined by Cheridito in [14], where arbitrage strategy was constructed in the model (4.8) and on the interval $[0, T]$. Cheridito further proved that arbitrage can be eliminated by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions, i.e. continuous trading is renounced. The arbitrage is also excluded in the presence of transaction costs as shown by Guasoni in [24].

2. On the other hand, the pathwise integral is sensible from economic point of view. Let $0 = t_0 < t_1 < \cdots < t_n = T$ and $\left\{v_{t_k} \mathbb{1}_{[t_k, t_{k+1})}; k = 0, \ldots, n-1\right\}$, be a piecewise buy-and-hold strategy of investment in stock. Then the capital earned by this strategy is naturally $\sum_{k=0}^{n-1} v_{t_k}(S_{t_{k+1}} - S_{t_k})$. It can be shown ([37], [3]) that it converges $\mathbb{P}$-a.s. to $\int_0^t v_s \mathrm{d}S_s$ as $\max |t_{k+1} - t_k| \to 0$, which was defined as

$$\int_0^t v_s \mathrm{d}S_s = \int_0^t v_s \mu S_s \mathrm{d}s + \int_0^t v_s \sigma S_s \mathrm{d}B_s.$$

△

4.3 Skorohod Model

At first glance, Skorohod integral may appear to be better suited for replacement of Itö integral in the standard Black-Scholes model. We know that in the case of Brownian motion and adapted integrands it coincides with Itö integral. It also has zero expectation and the Itö formula looks similar, too. However, difficulties arise with the use of the Skorohod integral.

Let us now suppose that the price process S follows a stochastic differential equation

$$\begin{aligned} \delta S_t &= \mu_t S_t dt + \sigma_t S_t \delta B_t, \\ S_0 &= s_0, \end{aligned} \tag{4.10}$$

where $s_0 > 0$, and μ, σ are continuous deterministic functions.

Set

$$w(t) = \frac{\partial}{\partial t}\left(\int_0^t \left(\mathscr{K}_t^* \sigma\right)^2 (r)\mathrm{d}r\right), \tag{4.11}$$

and suppose that $w \in C(\mathbb{R}_+)$. Let us denote $X_t = \int_0^t \sigma_s \delta B_s$. In order to derive an explicit form of a solution of (4.10), we apply the Theorem 3.3.11 to the function

$$F(t,x) = s_0 \exp\left\{\int_0^t \mu_s \mathrm{d}s + x - \frac{1}{2}\int_0^t w(s)\mathrm{d}s\right\}.$$

We have

$$\begin{aligned}
\frac{\partial F}{\partial t}(t,x) &= F(t,x)(\mu_t - \frac{1}{2}w(t)),\\
\frac{\partial F}{\partial x}(t,x) &= F(t,x),\\
\frac{\partial^2 F}{\partial x^2}(t,x) &= F(t,x).
\end{aligned}$$

Then

$$\begin{aligned}
F(t,X_t) = &\ F(0,0) + \int_0^t F(s,X_s)(\mu_s - \frac{1}{2}w(s))\mathrm{d}s + \int_0^t F(s,X_s)\sigma_s \delta B_s\\
+ &\ \frac{1}{2}\int_0^t F(s,X_s)w(s)\mathrm{d}s.
\end{aligned}$$

Hence, the solution to the equation (4.10) is

$$S_t = s_0 \exp\left\{\int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \delta B_s - \frac{1}{2}\int_0^t w(s)\mathrm{d}s\right\}. \tag{4.12}$$

Remark 4.3.1. If σ is constant, then

$$w(s) = \frac{\partial}{\partial t}\int_0^t \sigma^2 K(t,s)^2 \mathrm{d}s = \sigma^2 \frac{\partial}{\partial t}\mathbb{E}(B_t)^2 = \sigma^2 \frac{\partial}{\partial t} R_t.$$

△

Let us now consider that μ and σ in the equation (4.10) are constant. The stock price process S is then

$$S_t = s_0 \exp\left\{\mu t + \sigma B_t - \frac{\sigma^2}{2}R_t\right\}. \tag{4.13}$$

Let us further suppose that the integral in the definition of self-financing condition (4.6) is also understood in the Skorohod sense,

$$V_t^\pi = V_0^\pi + \int_0^t u_s \mathrm{d}A_s + \int_0^t v_s \mu S_s \mathrm{d}s + \int_0^t v_s \sigma S_s \delta B_s. \tag{4.14}$$

We call this condition *Skorohod self-financing*. For a portfolio to be admissible (Definition 4.1.3), it is now required that $(v \cdot S) \in \mathbb{D}^{1,2}(\mathscr{H}_R)$. The absence of arbitrage follows basically from the fact that the expectation of Skorohod integral is zero. In the following suppose that $r = \mu = 0$ (see Remark 4.3.2). Then the equation (4.14) has a simpler form

$$V_t^\pi = V_0^\pi + \int_0^t v_s \sigma S_s \delta B_s.$$

Assume now that $V_0^\pi \leq 0$. Then $\mathbb{E}\left[V_t^\pi\right] = V_0^\pi \leq 0$. Hence, if $V_t^\pi \geq 0, , \mathbb{P}$-a.s., then by the previous inequality $V_t^\pi = 0$ $\mathbb{P}$-a.s. As a consequence, $\boldsymbol{\pi}$ is not arbitrage.

Remark 4.3.2.

1. The general case with $r \neq \mu \neq 0$ would require a Girsanov Theorem for Volterra processes, which, to the best of our knowledge, does not exist. Modest enquiry in this area revealed that such a result can be derived from the Girsanov Theorem for Brownian motion provided that the inverse operator $\mathbb{K}^{-1}$ of the integral operator $\mathbb{K} : (\mathbb{K}h)(t) = \int_0^t K(t,s)h(s)\mathrm{d}s$ is *causal*. Unfortunately, no conditions in terms of the kernel K that would ensure this property are known.

2. The Girsanov Theorem is known for fractional Brownian motion and can be found in various publications (see e.g. [38], [17], [11], [5], [27], [23]). In that case, the fractional Girsanov Theorem and Itö formula are used by the very same procedure as in Section 4.1 to show that in the Skorohod model arbitrage does not exist.

3. This model was proposed in Wick calculus in [27] and [23]. It was later noted by several authors ([58], [5], [9] and [44], to name a few) that the use of Skorohod integral in the definition of self-financing portfolios lacks economic meaning, and for example Björk and Hult in [9] give an example of a simple buy-and-hold strategy that fails to be Skorohod self-financing. Other curiousity that is worth mentioning is that in their no-arbitrage argument did not require the adaptedness of the portfolio, so it is not possible to construct arbitrage even with the knowledge of future prices of the stock.

△

We can also obtain (4.13) as a solution to the pathwise equation

$$dS_t = \mu S_t dt + \sigma S_t \mathrm{d}B_t - \frac{1}{2}\sigma^2 \frac{\partial}{\partial t} R_t \mathrm{d}t, \tag{4.15}$$

this is straightforward application of the Itô lemma for pathwise integral, but let us show this by means of the relation between Skorohod and pathwise integral

(3.31). We have

$$\int_0^t S_s \delta B_s = \int_0^t S_s \mathrm{d}B_s - \int_0^t \left(\int_s^t D_s S_r \mathcal{K}(\mathrm{d}r, s) \right) \mathrm{d}s. \tag{4.16}$$

By Proposition 3.2.3 and chain rule 2.2.11 it follows that the Malliavin derivative in the second term is equal to

$$D_s S_r = \mathscr{K}^* D_s^B S_r = \sigma S_r K(r, s),$$

hence the second term in (4.16) can be written as

$$\begin{aligned}
\int_0^t \left(\int_s^t D_s u_r \mathcal{K}(\mathrm{d}r, s) \right) \mathrm{d}s &= \sigma \int_0^t \left(\int_s^t S_r K(r, s) \frac{\partial}{\partial r} K(r, s) \mathrm{d}r \right) \mathrm{d}s \\
&= \sigma \int_0^t \int_0^r S_r K(r, s) \frac{\partial}{\partial r} K(r, s) \mathrm{d}s \mathrm{d}r \\
&= \sigma \frac{1}{2} \int_0^t S_r \frac{\partial}{\partial r} R(r, r) \mathrm{d}r \\
&= \sigma \frac{1}{2} \int_0^t S_r \mathrm{d}R_r.
\end{aligned}$$

Unfortunately, the situation did not change for the better, as with the pathwise integral arbitrage will be present again. An example of arbitrage strategy similar to the one from Example 4.2.1 is shown in the following example.

Example 4.3.3. Let us suppose that the stock price process is given by (4.13) and assume the pathwise integral, in particular, the self-financing condition is given by

$$V_t^\pi = V_0^\pi + \int_0^t u_s \mathrm{d}A_s + \int_0^t v_s \mu S_s \mathrm{d}s + \int_0^t v_s \sigma S_s dW_s - \frac{1}{2} \sigma^2 \int_0^t v_s S_s \mathrm{d}R_s.$$

Then by the same way as in the Example 4.2.1 it can be shown, that the portfolio π given by

$$\begin{aligned}
u_t &= 1 - \exp\left\{ -2rt + 2\mu t + 2\sigma B_t - \sigma^2 R_t \right\}, \\
v_t &= \frac{2}{s_0} \left(\exp\left\{ -rt + \mu t + \sigma B_t - \frac{\sigma^2}{2} R_t \right\} - 1 \right),
\end{aligned}$$

is arbitrage. ♢

Example 4.3.4. This exemple shows the limit behavior of the process S_t given by (4.13) in the case of fractional Brownian motion with $H > \frac{1}{2}$. We have

$$S_t^H = s_0 \exp\left\{ \mu t + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H} \right\}.$$

The expectation and variance of S_t^H follow from the properties of log-normal

distribution and are equal to

$$\mathbb{E}S_t^H = s_0 \exp\{t\mu\}, \quad \operatorname{var}(S_t^H) = {s_0}^2 \exp\{2\mu t\} \left(\exp\{\sigma^2 t^{2H}\} - 1\right).$$

For the limit behavior we use the Law of Iterated Logarithm for fractional Brownian motion (1.22). For $H > \frac{1}{2}$, we have that $S_t^H \to 0 \quad \mathbb{P}$-a.s., $t \to \infty$. Thus, the mean value of the process S_t^H converges to infinity, whereas almost all of its trajectories converge to zero. This can be explained by the fact that S_t^H is not uniformly integrable. ♢

4.4 Summary

As we have seen, simple replacement of the Wiener process in the Black-Scholes model (4.2) by a fractional Brownian motion or more generally by a Volterra process does not produce satisfactory results. Furthermore, the properties of the resulting process depend on the chosen integration theory. The use of the pathwise integral is sensible from economic perspective, as it can be understood as a limit of simple predictable trading strategies. For a discussion we refer to [8], [58], [3]). This approach, however, leads to existence of arbitrage. On the other hand, with Skorohod integration the model does not admit arbitrage for the price of no economic meaning of such a model.

More promising could be to study *mixed models* which replace the Wiener process in (4.2) a linear combination of Wiener and Volterra processes. Such models are already studied in the case of fractional Brownian motion with satisfactory results; in particular, under various assumptions no arbitrage can be obtained. Let us mention works of Cheridito [13], Mishura and Valkeila [37] or [63].

It could also be of interest to consider Volterra processes in models of stochastic volatility, i.e. models where σ in (4.2) would follow a stochastic differential equation driven by Volterra process. This is, however, out of scope

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