Universidad de Murcia — Facultad de Matemáticas

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Report of Mrs Hana Kulišová PhD dissertation

QUANTITATIVE PROPERTIES OF BANACH SPACES

The study of quantification of properties in Banach spaces and the quantitative versions of classic results has became an active research area during the last decade, although the origins of this branch can be traced back half a century. A typical example is Krein's theorem establishing that a the closed convex hull $\overline{conv}(A)$ of a relatively weakly compact subset $A \subset X$ of a Banach space is again weakly compact. Assume that we have a measure of non-weak compactness, like the following

$$\mathrm{wk}_X(A) = \sup\{\inf\{\|x^{**} - x\| : x \in X\} : x^{**} \in \overline{A}^{w^*}\}$$

where the weak* closure is taken in the bidual space X^{**}. It is not difficult to see that A is relatively weakly compact if and only if $\mathrm{wk}_X(A) = 0$. With that notation, the quantitative version of Krein's theorem takes this form

$$\mathrm{wk}_X(\overline{conv}(A)) \leq 2\,\mathrm{wk}_X(A)$$

due to Fabian, Hájek, Montesinos and Zizler, (Rev. Mat. Iberoamericana 21 (2005), no. 1, 237–248). Let me point out that I am motivating here the subject of the dissertation, since its very abridged introduction contains no concrete example.

The thesis consists of four papers concerned with quantitative properties of Banach spaces. The main properties whose quantification is studied along the dissertation are the Grothendieck property, the Banach-Saks property and the Pełczyński's (V) property. Note that all these properties are of sequential nature, which improves the thematic coherence of the dissertation. On the other hand, each of the properties needs special techniques: for instance, the study of the Grothendieck property is done with the help of tools that were developed around James' compactness theorem, the Banach-Saks property is related to the spreading models, and the last paper deals with the quantification of Pełczyński's property in C^*-algebras. In my opinion, the diversity of techniques used in the dissertation reveals the mastering of Mrs Kulišová of a rather large part of Banach space theory.

The first chapter is a paper published in J. Math. Anal. Appl. entitled *Quantitative Grothendieck property*. Let us recall that a Banach space X is Grothendieck (or has the Grothendieck property) if the weak and the weak* convergence of sequences in X^* coincide. The Grothendieck property is a well known tool in isomorphic theory of Banach spaces, being the more remarkable facts that its stable by quotients, $\ell_\infty(\Gamma)$ is Grothendieck and separable

Grothendieck spaces are reflexive. Mrs Krulišová proposes a quantification of Grothendieck property based on some measure of weak and weak* "non-cauchyness" of sequences in X^*. That allows her to introduce c-Grothendieck spaces for $c \geq 1$, turning out that the typical examples $\ell_\infty(\Gamma)$ are 1-Grothendieck, that is, the best possible constant (Theorem 1.1, Theorem 4.2). The proof is a consequence of the connection of Grothendieck property with (I)-envelopes and previous results of the advisor. Another remarkable fact is that the constant of grothendieckness is preserved by quotients (Lemma 3.3). On the other hand, not all the Grothendieck spaces admit the quantified Grothendieck property (Theorem 1.2). The example provided is a convenient direct sum of renormed ℓ_∞ spaces.
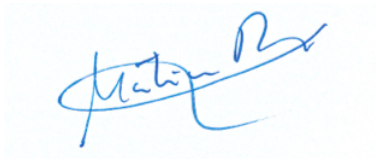
The second chapter is a paper coauthored with J. Spurný and the advisor O. Kalenda, published in J. Func. Anal. and entitled *Quantification of the Banach-Saks property*. A Banach space is (weakly) Banach-Saks if every bounded (weakly convergent) sequence has a subsequence which is Cesàro-convergent. The Banach-Saks property lies strictly between reflexivity and super-reflexivity and it is connected to ℓ_1-spreading models (a kind of representation of the space ℓ_1 involving sequences) after the work of B. Beauzamy (Math. Scand. 44 (1979), 357–384, not quoted among the references of the chapter). The quantification of the non-Banach-Saks property (stated for sets in a weaker form) is quite technical while the quantification for the spreading model is almost natural. The main result is Theorem 4.1 which shows the equivalence between the measure of being non-weakly Banach-Saks, to spread a model of ℓ_1 and a related sequential property. The application of the measure of non-weak Banach-Saks wbs to the unit ball of a Banach space leads to a dichotomy: either $wbs(B_X) = 2$ or $wbs(B_X) = 0$, the last one if and only if X is weakly Banach-Saks (Theorem 5.1). That allows the computation of the measures in several particular cases.

The third chapter entitled *Quantification of Pełczyński's property (V)* is not yet published. A Banach space has the Pełczyński's property (V) if any unconditionally converging operator defined on it is automatically weakly compact, and an operator is unconditionally converging if it takes weakly unconditionally Cauchy series to a (unconditionally) convergent series in the range space. The importance of this property comes from the fact that several classes of Banach spaces have it: $C(K)$ spaces (Pełczyński), L^1-preduals (Johnson-Zippin) and C^*-algebras (Pfitzner). The quantification of how much an operator is non-unconditionally converging is tightly related to how much the operator fixes a copy of c_0 in a certain sense. The quantitative versions of Pełczyński's theorem for $C_0(\Omega)$, where Ω is locally compact, depends on the measure of non-weak compactness used since De Blasi measure is not equivalent to others (Theorem 4.1). A similar quantified version for the theorem of Johnson-Zippin on L^1-preduals is given (Theorem 4.2). The last section is devoted to the relations between the quantified Pełczyński's property (V) and quantified versions of several properties of Banach spaces in relation with weak compactness of operators defined on them.

Finally, the fourth chapter with title C^*-*algebras have a quantitative version of Pełczyński's property (V)* is not published either. As the title shows, the chapter deals with the third class of Banach spaces having Pełczyński's property (V), namely the class of C^*-algebras. The ideas are based in the important paper of H. Pfitzner (Math. Ann. 298 (1994), no. 2, 349–371) where he showed that C^*-algebras enjoy the Grothendieck property and the Pełczyński's property (V). The original proof is far from being obvious and thus this chapter is extremely

technical, compared to the previous ones.

After my complain because of the short introduction, I have found each of the chapters well written and enough motivated, fully expectable as independent papers. I believe some aesthetic aspects are improvable, for instance, the lack of coherence in the style of names of authors in references. As a general remark supporting the importance of the results of the dissertation, I would like to point out that quantification of properties and results involving them is not a simple modification of known proofs. There are several different ways to quantify that a property is not fulfilled and not all the results are quantifiable. Moreover, the deep insight into the arguments of the known results in order to obtain their quantified versions sometimes leads to simple proofs. In this sense, the dissertation of Mrs Kulišová is plenty of interesting contributions, even for those who are just concerned with the classical (non quantified) results. For that reason, I believe that the dissertation presented by Mrs Kulišová is an excellent research work and therefore it should be more than enough in order that she achieves the PhD degree.

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