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Quantitative properties of Banach spaces

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Abstract: The present thesis consists of four research papers. Each article deals with quantifications of certain properties of Banach spaces. The first paper is devoted to the Grothendieck property. The main result is that the space ℓ^∞ enjoys its quantitative version. The second paper investigates quantifications of the Banach-Saks and the weak Banach-Saks property. The relationship of compact, weakly compact, Banach-Saks, and weak Banach-Saks sets is quantified, as well as some characterizatons of weak Banach-Saks sets. In the third article we discuss possible quantifications of Pełczyński's property (V), their characterizations and relations to quantitative versions of other properties of Banach spaces. The last paper is a continuation of the third one. We prove that C^*-algebras have a quantitative version of the property (V), which generalizes one of the results obtained in the previous paper. Moreover, we establish a relationship between quantitative versions of the property (V) and the Grothendieck property in dual Banach spaces.

Keywords: Banach space, quantification, Grothendieck property, Banach-Saks property, Pełczyński's property (V)

Název práce: Kvantitativní vlastnosti Banachových prostorů

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Abstrakt: Tato dizertační práce sestává ze čtyř odborných článků. Každý z nich se zabývá kvantifikacemi určitých vlastností Banachových prostorů. První článek je věnován Grothendieckově vlastnosti. Hlavním výsledkem je, že prostor ℓ^∞ má její kvantitativní verzi. Druhý článek zkoumá kvantifikace Banachovy–Saksovy a slabé Banachovy–Saksovy vlastnosti. Je zde kvantifikován vztah kompaktních, slabě kompaktních, Banachových–Saksových a slabě Banachových–Saksových množin, jakož i některé charakterizace slabě Banachových–Saksových množin. Ve třetím článku studujeme možné kvantifikace Pełczyńského vlastnosti (V), jejich charakterizace a vztahy ke kvantitativním verzím dalších vlastností Banachových prostorů. Poslední článek navazuje na třetí. Je v něm dokázáno, že C^*-algebry mají kvantitativní verzi vlastnosti (V), což zobecňuje jeden z výsledků dosažených v předchozím článku. Navíc zde popisujeme vztah mezi kvantitativními verzemi vlastnosti (V) a Grothendieckovy vlastnosti v duálních Banachových prostorech.

Klíčová slova: Banachův prostor, kvantifikace, Grothendieckova vlastnost, Banachova–Saksova vlastnost, Pełczyńského vlastnost (V)

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Contents

Introduction

The present thesis consists of four chapters, each of them corresponding to a research paper. All these articles contain original results concerning quantifications of certain properties of Banach spaces.

The beginnings of the study of analytic notions and properties from a quantitative point of view via miscellaneous moduli, indices, measures, and other quantities can be traced back to the late 19th century. This approach leads to deeper understanding of the notions under study and often gives rise to some stronger or weaker notions or properties. Let us mention some of the best-known examples. Continuity of a continuous function on a compact metric space K can be measured quantitatively for instance by the Lipschitz constant or the modulus of continuity. On the other hand, oscillation of a bounded real function f on K defined by

$$\operatorname{osc}(f) = \sup_{x \in K} \inf \left\{ \sup_{y,z \in U} |f(y) - f(z)| : \ U \text{ is a neighbourhood of } x \right\}$$

measures discontinuity of f. In fact, $\frac{1}{2}\operatorname{osc}(f)$ equals the distance of f to the space $C(K)$ of continuous real functions on K (see [8, Prop. 1.18]). Regarding convergence of sequences of real numbers, measuring the speed of convergence has proved to be very useful particularly in numerical analysis. Similarly to the case of functions, we can also consider oscillation

$$\operatorname{ca}(x_n) = \limsup_{n \to \infty} x_n - \liminf_{n \to \infty} x_n$$

of a bounded sequence (x_n) in $\mathbb{R}$, which measures how far is (x_n) from being convergent. An intention to quantify the notions of nowhere dense and meager set has naturally originated the notions of porous and σ-porous set (see for instance [13, 30]). Attempts to measure compactness in metric spaces resulted in the concept of entropy numbers (see e.g. [24]). A great deal of attention has been paid also to quantifications of non-compactness. Measures of non-compactness and weak non-compactness of sets and operators in Banach spaces have been studied by dozens of mathematicians and they have found plenty of applications, a few of them also in this thesis. The first measure of non-compactness was defined by K. Kuratowski [28] in 1930. In 1957, Gohberg, Goldenštein and Marcus [15] introduced the Hausdorff measure of non-compactness. Goldenštein and Marcus later used it to prove a quantitative version of Schauder's theorem about compact operators [16]. An analogue of the Hausdorff measure of non-compactness for measuring weak non-compactness is the de Blasi measure of weak non-compactness [12], considered first by de Blasi in 1977. Many other measures of weak non-compactness based on various characterizations of weakly compact sets have been introduced since then (see for example [1, 11] and the references given there).

A basic kind of application of such quantities is the following. Classical qualitative results can be strengthened by replacing an implication with an inequality between some suitable quantities. Let us explain this in more detail. Many results have the following form: "Under some assumptions the implication

(1) $$x \text{ satisfies } ♥ \quad \Longrightarrow \quad x \text{ satisfies } 🐈$$

holds for all x." A quantitative version of such a result aims to enrich it with some additional information: if x does not satisfy ♥ but is not far from it, then it cannot be far from satisfying 🐈. A way to say this mathematically is to replace the implication (1) by an inequality

$$m_{\text{🐈}}(x) \leq C \cdot m_{\text{♥}}(x),$$

where $C > 0$ is a constant which does not depend on x, and $m_{\text{♥}}(x)$, $m_{\text{🐈}}(x)$ are some quantities which have positive values for each x and vanish if and only if x satisfies ♥ or 🐈, respectively. These quantities measure for each x "how far is x from satisfying ♥ and 🐈".

Quantitative versions of theorems and properties in the Banach space theory of the form described above have been studied by many mathematicians recently. Although this research topic was brought to the forefront about ten years ago, first quantitative results are much older – let us mention for example a quantitative version of Schauder's theorem [16] proved by Goldenštein and Marcus in 1965 or Behrends' quantitative version of Rosenthal's ℓ^1–theorem [7] from 1996.

The number of recently published papers about quantifications of certain theorems and properties is quite large. Among all notions in the theory of Banach spaces weak compactness is maybe the one studied from a quantitative point of view the most intensively. Outputs of this research are papers concerning quantitative version of Krein's theorem [9, 14, 17, 19], the Eberlein-Šmulyan and the Gantmacher theorem [1], James' compactness theorem [11, 18], and many other publications, see for instance [4, 6, 12, 25, 26, 27]. Other properties whose quantifications have been studied during the last decade are for example the weak sequential completeness and the Schur property [21, 22], the Dunford-Pettis property [20], the reciprocal Dunford-Pettis property [23], or the Radon-Nikodym property [10]. Other publications dealing with quantifications are for instance [2, 3, 5].

Let us now briefly sum up the contribution of this thesis. The list of the presented papers is the following:

- Hana Bendová: Quantitative Grothendieck property, *J. Math. Anal. Appl.*, 412(2):1097–1104, 2014.
- Hana Bendová, Ondřej F. K. Kalenda, and Jiří Spurný: Quantification of the Banach-Saks property, *J. Funct. Anal.*, 268(7):1733–1754, 2015.
- Hana Krulišová: Quantification of Pełczyński's property (V), submitted (2015), preprint available at `http://arxiv.org/abs/1509.06610`.
- Hana Krulišová: C*-algebras have a quantitative version of Pełczyński's property (V), accepted to *Czechoslovak Math. J.* (2016), preprint available at `http://arxiv.org/abs/1605.04900`.

In the first paper the Grothendieck property is quantified. A Banach space X is *Grothendieck* if for each bounded sequence (x_n^*) in the dual space X^* the following implication hold:

$$(x_n^*) \text{ converges in the weak}^* \text{ topology} \quad \Longrightarrow \quad (x_n^*) \text{ converges in the weak topology.}$$

This implication is replaced by an inequality as described above to obtain a quantitative version of the Grothendieck property. We characterize it using (I)-envelopes and then use this characterization and some results of O. Kalenda to prove that the space ℓ^∞, known to have the Grothendieck property, enjoys also its quantitative version. This result is then further generalized. It is also shown that the Grothendieck property is not automatically quantitative, i.e. a Grothendieck space exists which is not quantitatively Grothendieck.

The second paper investigates quantifications of the Banach-Saks and the weak Banach-Saks property. It quantifies relationships of Banach-Saks, weak Banach-Saks, compact, and weakly compact sets. A bounded subset A of a Banach space X is

- a *Banach-Saks set* if each sequence in A has a Cesàro convergent subsequence,
- a *weak Banach-Saks set* if any weakly convergent sequence in A admits a Cesàro convergent subsequence.

For a bounded subset A of a Banach space we have the following relations:

$$\begin{gathered} A \text{ is relatively compact} \\ \Downarrow \\ A \text{ is Banach-Saks} \\ \Updownarrow \\ A \text{ is weak Banach-Saks and weakly compact.} \end{gathered}$$

Two of these implications are quantified, the remaining one is proven to be merely qualitative. We also quantify characterizations of weak Banach-Saks sets which use uniform weak convergence and ℓ_1-spreading models, and we prove an analogue of James' distortion theorem for ℓ^1-spreading models.

The last two papers study possible quantifications of Pełczyński's property (V). A Banach space X has Pełczyński's property (V) if for every Banach space Y and every bounded linear operator $T\colon X \to Y$

$$T \text{ is unconditionally converging} \quad \Longrightarrow \quad T \text{ is weakly compact,}$$

or equivalently (see [29, Proposition 1]) for every $K \subset X^*$

$$K \text{ satisfies the condition } (*) \text{ below} \quad \Longrightarrow \quad K \text{ is relatively weakly compact.}$$

$$(*) \qquad \lim_{n\to\infty} \sup_{x^*\in K} |x^*(x_n)| = 0 \text{ for every weakly unconditionally Cauchy series } \sum_{n=1}^{\infty} x_n \text{ in } X$$

In the third paper we prove that quantifications of these two conditions are equivalent, too. The latter one is then used to prove that $C_0(\Omega)$ spaces for a locally compact space Ω and real L^1 preduals enjoy a quantitative version of Pełczyński's property (V). The last paper generalizes one of the results obtained before – it contains a proof that all C^*-algebras have a quantitative version of the property (V). The third paper also quantifies a characterization of unconditionally converging operators as those which does not fix a copy of c_0. This gives rise to another characterization of a quantitative version of the property (V). Furthermore, in both papers we study a relationship between quantitative versions of the property (V) and several other properties of Banach spaces, including for example the reciprocal Dunford-Pettis property, the Dieudonné property, or the Grothendieck property.

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I. Quantitative Grothendieck property

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Abstract: A Banach space X is Grothendieck if the weak and the weak* convergence of sequences in the dual space X^* coincide. The space ℓ^∞ is a classical example of a Grothendieck space due to Grothendieck. We introduce a quantitative version of the Grothendieck property, we prove a quantitative version of the above-mentioned Grothendieck's result and we construct a Grothendieck space which is not quantitatively Grothendieck. We also establish the quantitative Grothendieck property of $L^\infty(\mu)$ for a σ-finite measure μ.

1 Introduction

A Banach space X is said to be *Grothendieck* if the weak and the weak* convergence of sequences in the dual space X^* coincide. The space ℓ_∞ is a classical example of a Grothendieck space due to Grothendieck [10]. Some other examples are $C(K)$ where K is an F-space [19], weak L^p spaces [18], and the Hardy space H^∞ [2]. R. Haydon has constructed a Grothendieck space which does not contain ℓ_∞ [11].

In this paper, we introduce a quantitative version of the Grothendieck property. Our inspiration comes from many recent quantitative results. Quite a few properties and theorems have been given in a quantitative form lately. Let us mention quantitative versions of Krein's theorem [5, 9, 7, 3], quantitative versions of the Eberlein–Šmulyan and the Gantmacher theorem [1], quantitative version of James's compactness theorem [4, 8], quantitative weak sequential continuity and quantitative Schur property [13, 14], quantification of Dunford–Pettis [12] and reciprocal Dunford–Pettis property [17].

The definition of the Grothendieck property can be rephrased as follows. A Banach space X is Grothendieck if every weak* Cauchy sequence in X^* is weakly Cauchy. The quantitative version is derived from this formulation in the following way. Let X be a Banach space and (x_n^*) be a bounded sequence in X^*. Define

$$\delta_w(x_n^*) = \sup_{x^{**}\in B_{X^{**}}} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x^{**}(x_k^*) - x^{**}(x_l^*)|$$

the "measure of weak non-cauchyness" of the sequence (x_n^*), and

$$\delta_{w^*}(x_n^*) = \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x_k^*(x) - x_l^*(x)|$$

the "measure of weak* non-cauchyness" of the sequence (x_n^*). The quantities $\delta_w(x_n^*)$ and $\delta_{w^*}(x_n^*)$ are equal to zero if and only if the sequence (x_n^*) is weakly and weak* Cauchy, respectively. We now replace the implication in the definition of the Grothendieck property by an inequality between these two quantities, which is a stronger condition.

Definition (Quantitative Grothendieck property)**.** Let $c \geq 1$. A Banach space X is *c-Grothendieck* if

$$\delta_w(x_n^*) \leq c\delta_{w^*}(x_n^*)$$

whenever (x_n^*) is a bounded sequence in X^*.

Section 2 establishes the relation between the quantitative Grothendieck property and (I)-envelopes of unit balls. It is then used to prove the following quantitative version of the above-mentioned Grothendieck's result.

Theorem 1.1. *The space ℓ_∞ is 1-Grothendieck.*

If X is c-Grothendieck for some $c \geq 1$, then it is Grothendieck. In Section 3 we show that the converse is not true.

Theorem 1.2. *There is a Grothendieck space which is not c-Grothendieck for any $c \geq 1$.*

Section 4 contains a generalization of Theorem 1.1 and its consequences.

2 Relation to (I)-envelopes

In this section, we characterize the quantitative Grothendieck property using (I)-envelopes. Some results on (I)-envelopes presented in [15] and [16] have been found extremely useful to us.

Definition. Let X be a Banach space and $B \subset X^*$. The (I)-envelope of B is defined by

$$\text{(I)-env}(B) = \bigcap \left\{ \overline{\operatorname{co} \bigcup_{n=1}^{\infty} \overline{\operatorname{co} C_n}^{w^*}}^{\|\cdot\|} \ : \ B = \bigcup_{n=1}^{\infty} C_n \right\}.$$

Any Banach space X is considered to be canonically embedded into its bidual X^{**}. If B is a set in a Banach space X, then B is regarded as a subset of X^{**} and so is the (I)-envelope of B. By $\overline{B}^{w^*}$ we mean the weak* closure of B in X^{**}.

The following lemma, proved by Kalenda [15, Lemma 2.3], provides the characterization of (I)-envelopes. It allows us to prove Proposition 2.2, which describes the relation between (I)-envelopes and the quantitative Grothendieck property.

Lemma 2.1. *Let X be a Banach space, $B \subset X$ be a closed convex set and $z^{**} \in \overline{B}^{w^*}$. Then the following conditions are equivalent:*

(1) $z^{**} \notin \text{(I)-env}(B)$*;*

(2) there is a sequence (ξ_n^*) in B_{X^*} such that

$$\sup_{x \in B} \limsup_{n \to \infty} \xi_n^*(x) < \inf_{n \in \mathbb{N}} z^{**}(\xi_n^*);$$

(3) there is a sequence (ξ_n^*) in B_{X^*} such that

$$\sup_{x \in B} \limsup_{n \to \infty} \xi_n^*(x) < \liminf_{n \to \infty} z^{**}(\xi_n^*);$$

(4) there is a sequence (ξ_n^*) in B_{X^*} such that

$$\sup_{x \in B} \limsup_{n \to \infty} \xi_n^*(x) < \limsup_{n \to \infty} z^{**}(\xi_n^*).$$

Proposition 2.2. *Let X be a Banach space and $c \geq 1$. Then X is c-Grothendieck if and only if* (I)-env$(B_X) \supset \frac{1}{c} B_{X^{**}}$.

Proof. Suppose that X is not c-Grothendieck. Find a bounded sequence (x_n^*) in X^* such that $\delta_w(x_n^*) > c\delta_{w^*}(x_n^*)$, i.e.

$$\sup_{x^{**}\in B_{X^{**}}} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x^{**}(x_k^*) - x^{**}(x_l^*)| > c \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x_k^*(x) - x_l^*(x)|.$$

There is no loss of generality in assuming that $x_n^* \in B_{X^*}$, $n \in \mathbb{N}$. Let $x^{**} \in B_{X^{**}}$ be such that

$$\inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x^{**}(x_k^*) - x^{**}(x_l^*)| > c \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x_k^*(x) - x_l^*(x)|,$$

and set $z^{**} = \frac{1}{c} x^{**}$. Then $z^{**} \in \frac{1}{c} B_{X^{**}}$, and

$$\begin{aligned} \limsup_{n\to\infty} z^{**}(x_n^*) - \liminf_{n\to\infty} z^{**}(x_n^*) &= \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |z^{**}(x_k^*) - z^{**}(x_l^*)| \\ &= \frac{1}{c} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x^{**}(x_k^*) - x^{**}(x_l^*)| \\ &> \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x_k^*(x) - x_l^*(x)| \\ &= \sup_{x\in B_X} \left(\limsup_{n\to\infty} x_n^*(x) - \liminf_{n\to\infty} x_n^*(x) \right). \end{aligned} \tag{1}$$

Find subsequences (y_k^*) and (z_k^*) of the sequence (x_n^*) for which $\limsup_{n\to\infty} z^{**}(x_n^*) = \lim_{k\to\infty} z^{**}(y_k^*)$, and $\liminf_{n\to\infty} z^{**}(x_n^*) = \lim_{k\to\infty} z^{**}(z_k^*)$. Set $\xi_k^* = \frac{1}{2}(y_k^* - z_k^*)$, $k \in \mathbb{N}$. Then (ξ_k^*) is a sequence in B_{X^*}, and

$$\begin{aligned} \lim_{k\to\infty} z^{**}(\xi_k^*) &= \frac{1}{2} \left(\lim_{k\to\infty} z^{**}(y_k^*) - \lim_{k\to\infty} z^{**}(z_k^*) \right) \\ &= \frac{1}{2} \left(\limsup_{n\to\infty} z^{**}(x_n^*) - \liminf_{n\to\infty} z^{**}(x_n^*) \right) \\ &\overset{(1)}{>} \frac{1}{2} \sup_{x\in B_X} \left(\limsup_{n\to\infty} x_n^*(x) - \liminf_{n\to\infty} x_n^*(x) \right) \\ &\geq \frac{1}{2} \sup_{x\in B_X} \left(\limsup_{k\to\infty} y_k^*(x) - \liminf_{k\to\infty} z_k^*(x) \right) \\ &\geq \frac{1}{2} \sup_{x\in B_X} \limsup_{k\to\infty} (y_k^*(x) - z_k^*(x)) \\ &= \sup_{x\in B_X} \limsup_{k\to\infty} \xi_k^*(x). \end{aligned}$$

By Lemma 2.1, $z^{**} \notin$ (I)-env(B_X), and so (I)-env$(B_X) \not\supset \frac{1}{c} B_{X^{**}}$.

Now suppose that X is c-Grothendieck and fix arbitrary $z^{**} \in \frac{1}{c} B_{X^{**}}$. Let (x_n^*) be a sequence in B_{X^*}. Then $\delta_w(x_n^*) \leq c\delta_{w^*}(x_n^*)$, that is

$$\sup_{x^{**}\in B_{X^{**}}} \left(\limsup_{n\to\infty} x^{**}(x_n^*) - \liminf_{n\to\infty} x^{**}(x_n^*) \right) \leq c \sup_{x\in B_X} \left(\limsup_{n\to\infty} x_n^*(x) - \liminf_{n\to\infty} x_n^*(x) \right).$$

Since $cz^{**} \in B_{X^{**}}$, it follows that

$$\begin{aligned} \limsup_{n\to\infty} z^{**}(x_n^*) - \liminf_{n\to\infty} z^{**}(x_n^*) &= \frac{1}{c} \left(\limsup_{n\to\infty} cz^{**}(x_n^*) - \liminf_{n\to\infty} cz^{**}(x_n^*) \right) \\ &\leq \sup_{x\in B_X} \left(\limsup_{n\to\infty} x_n^*(x) - \liminf_{n\to\infty} x_n^*(x) \right). \end{aligned}$$

For $k \in \mathbb{N}$ find an $x_k \in B_X$ satisfying

$$\limsup_{n\to\infty} x_n^*(x_k) - \liminf_{n\to\infty} x_n^*(x_k) > \limsup_{n\to\infty} z^{**}(x_n^*) - \liminf_{n\to\infty} z^{**}(x_n^*) - \frac{2}{k}.$$

Then either $\limsup_{n\to\infty} x_n^*(x_k) > \limsup_{n\to\infty} z^{**}(x_n^*) - \frac{1}{k}$ or $\liminf_{n\to\infty} x_n^*(x_k) < \liminf_{n\to\infty} z^{**}(x_n^*) + \frac{1}{k}$. If the former inequality holds for infinitely many $k \in \mathbb{N}$, then $\limsup_{n\to\infty} z^{**}(x_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} x_n^*(x)$. Otherwise the latter holds for infinitely many $k \in \mathbb{N}$, and $\liminf_{n\to\infty} z^{**}(x_n^*) \geq \inf_{x\in B_X} \liminf_{n\to\infty} x_n^*(x)$, which gives $\limsup_{n\to\infty} -z^{**}(x_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} x_n^*(x)$.

So far we have shown that whenever (x_n^*) is a sequence in B_{X^*}, either

$$\limsup_{n\to\infty} z^{**}(x_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} x_n^*(x) \quad \text{or} \quad \limsup_{n\to\infty} -z^{**}(x_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} x_n^*(x).$$

Consider now an arbitrary sequence (x_n^*) in B_{X^*}. Set $(y_n^*)_n = (x_1^*, -x_1^*, x_2^*, -x_2^*, \ldots)$. From what has already been proved, we obtain

$$\limsup_{n\to\infty} z^{**}(y_n^*) = \limsup_{n\to\infty} -z^{**}(y_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} y_n^*(x).$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} z^{**}(x_n^*) &\leq \limsup_{n\to\infty} z^{**}(y_n^*) \leq \sup_{x\in B_X} \limsup_{n\to\infty} y_n^*(x)\\ &= \sup_{x\in B_X} \max\{\limsup_{n\to\infty} x_n^*(x), \limsup_{n\to\infty} -x_n^*(x)\}\\ &= \sup_{x\in B_X} \limsup_{n\to\infty} x_n^*(x).\end{aligned}$$

Lemma 2.1 gives $z^{**} \in \text{(I)-env}(B_X)$, which shows that $\frac{1}{c}B_{X^{**}} \subset \text{(I)-env}(B_X)$. □

We are now able to prove Theorem 1.1. It is a trivial consequence of Proposition 2.2 and Kalenda's theorem [15, Example 4.1], which says that $\text{(I)-env}(B_{\ell_\infty}) = B_{(\ell_\infty)^{**}}$.

3 The relation between Grothendieck property and its quantitative version

We have already mentioned that the quantitative Grothendieck property is stronger than its original qualitative version. This section is devoted to the construction of a Banach space which is Grothendieck but not c-Grothendieck for any $c \geq 1$.

The following proposition is a strengthening of Kalenda's theorem [16, Theorem 2.2], and its proof is a modification of the original one.

Proposition 3.1. *Let X be a nonreflexive Banach space and $c \geq 1$. Then there exists an equivalent norm $\|\!|\cdot|\!\|$ on X such that $(X, \|\!|\cdot|\!\|)$ is not c-Grothendieck.*

Proof. If X is separable, then $\text{(I)-env}(B_X) = B_X$ (see [15, Remark 1.1(ii)]). By nonreflexivity, $\frac{1}{c}B_{X^{**}} \not\subset B_X$ for any $c \geq 1$, so the assertion follows from Proposition 2.2. Renorming is not necessary.

Suppose that X is nonseparable. Find a separable subspace $Y \subset X$ which is not reflexive. Let $x^* \in S_{X^*}$ be such that $x^*|_Y = 0$, and fix $x_0 \in X$ with $x^*(x_0) = 1$. Obviously, $\|x_0\| \geq 1$. The bidual Y^{**} can be canonically identified with the w^*-closure of Y in X^{**}, and $Y = Y^{**} \cap X$. Thus we can find some $y^{**} \in S_{Y^{**}} \setminus X$. Set $Z = \operatorname{span}(Y \cup \{x_0\})$. Since $y^{**} \in Z^{**} \setminus Z$, $y^{**}|_{B_{Z^*}}$ is not weak* continuous. Clearly, Z is separable, thus (B_{Z^*}, w^*) is metrizable, hence $y^{**}|_{B_{Z^*}}$ is not even weak* sequentially continuous. Therefore there exists a sequence $(\widetilde{x_n^*})$ in B_{Z^*} weak* converging to 0 and $\eta \in (0, 1]$ such that $y^{**}(\widetilde{x_n^*}) \geq \eta$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ extend $\widetilde{x_n^*}$ to $x_n^* \in B_{X^*}$ by the Hahn-Banach theorem.

Define

$$B = \left\{x \in X : \|x - x^*(x)x_0\| \leq 1 \text{ and } |x^*(x)| + \operatorname{dist}(x - x^*(x)x_0, Y) \leq \frac{\eta}{c}\right\}.$$

Then B is a closed absolutely convex set. Moreover, we show that

$$\frac{\eta}{c(2 + \|x_0\|)} B_X \subset B \subset \left(1 + \frac{\eta}{c}\right)\|x_0\| B_X.$$

For $x \in B$ we have

$$\|x\| \leq 1 + |x^*(x)|\|x_0\| \leq \|x_0\| + \frac{\eta}{c}\|x_0\|,$$

which proves the second inclusion. To prove the first one let $x \in B_X$. Then

$$|x^*(x)| \leq 1,$$

$$\|x - x^*(x)x_0\| \leq 1 + \|x_0\|,$$

$$\operatorname{dist}(x - x^*(x)x_0, Y) \leq \|x - x^*(x)x_0\| \leq 1 + \|x_0\|.$$

Hence for $z = \frac{\eta x}{c(2+\|x_0\|)}$ we have

$$\|z - x^*(z)\, x_0\| = \frac{\eta}{c(2 + \|x_0\|)}\|x - x^*(x)x_0\| \leq \frac{\eta}{c}\frac{1 + \|x_0\|}{2 + \|x_0\|} \leq 1,$$

and

$$\begin{aligned}|x^*(z)| + \operatorname{dist}(z - x^*(z)\, x_0, Y) &\leq \frac{\eta}{c(2 + \|x_0\|)} + \frac{\eta}{c(2 + \|x_0\|)}(1 + \|x_0\|) \\ &\leq \frac{\eta}{c}\frac{1 + 1 + \|x_0\|}{2 + \|x_0\|} = \frac{\eta}{c}.\end{aligned}$$

Thus B is the unit ball of an equivalent norm on X. According to Proposition 2.2, we shall have established the proposition if we show that $\frac{1}{c}\overline{B}^{w^*} \not\subset \text{(I)-env}(B)$.

Set $z^{**} = \frac{1}{c}(\frac{\eta}{c}x_0 + y^{**})$. Let (y_ν) be a net in B_Y weak* converging to y^{**}. Then $\frac{\eta}{c}x_0 + y_\nu$ weak* converges to $\frac{\eta}{c}x_0 + y^{**}$. Furthermore, $\frac{\eta}{c}x_0 + y_\nu \in B$ since

$$x^*\left(\frac{\eta}{c}x_0 + y_\nu\right) = \frac{\eta}{c}x^*(x_0) + x^*(y_\nu) = \frac{\eta}{c},$$

$$\left\|\frac{\eta}{c}x_0 + y_\nu - x^*\left(\frac{\eta}{c}x_0 + y_\nu\right)x_0\right\| = \|y_\nu\| \leq 1,$$

$$\operatorname{dist}\left(\frac{\eta}{c}x_0 + y_\nu - x^*\left(\frac{\eta}{c}x_0 + y_\nu\right)x_0, Y\right) = \operatorname{dist}(y_\nu, Y) = 0.$$

Therefore $z^{**} \in \frac{1}{c}\overline{B}^{w^*}$. It remains to prove that $z^{**} \notin \text{(I)-env}(B)$. Define $\xi_n^* = x^* + x_n^*$, $n \in \mathbb{N}$. Then (ξ_n^*) is a bounded sequence in X^*, and

$$\begin{aligned}\liminf_{n\to\infty} z^{**}(\xi_n^*) &= \frac{1}{c}\liminf_{n\to\infty}\left(x^*\left(\frac{\eta}{c}x_0\right) + x_n^*\left(\frac{\eta}{c}x_0\right) + y^{**}(x^*) + y^{**}(x_n^*)\right) \\ &= \frac{1}{c}\left(\frac{\eta}{c} + \frac{\eta}{c}\lim_{n\to\infty} x_n^*(x_0) + y^{**}(x^*) + \liminf_{n\to\infty} y^{**}(x_n^*)\right) \\ &\geq \frac{1}{c}\left(\frac{\eta}{c} + 0 + 0 + \eta\right) = \frac{c+1}{c}\frac{\eta}{c} > \frac{\eta}{c}.\end{aligned}$$

In the last inequality, we have used the following two facts. Firstly, $x_n^*(x_0) \to 0$, as $x_0 \in Z$. Secondly, $y^{**}(x^*) = 0$, since $y^{**} \in \overline{Y}^{w^*}$ and $x^*|_Y = 0$. On the other hand, if $x \in B$, $y \in Y$ are arbitrary, then

$$\begin{aligned}
\limsup_{n\to\infty} \xi_n^*(x) &= x^*(x) + \limsup_{n\to\infty}(x_n^*(x - x^*(x)x_0 - y) + x_n^*(x_0)x^*(x) + x_n^*(y)) \\
&= x^*(x) + \limsup_{n\to\infty} x_n^*(x - x^*(x)x_0 - y) + \lim_{n\to\infty} x_n^*(x_0)x^*(x) + \lim_{n\to\infty} x_n^*(y) \\
&\leq x^*(x) + \limsup_{n\to\infty} \|x_n^*\|\|x - x^*(x)x_0 - y\| + 0 + 0 \\
&\leq x^*(x) + \limsup_{n\to\infty} \|x - x^*(x)x_0 - y\|,
\end{aligned}$$

because $x_0, y \in Z$, and $x_n^*(z) \to 0$ for all $z \in Z$. Hence for every $x \in B$

$$\limsup_{n\to\infty} \xi_n^*(x) \leq |x^*(x)| + \operatorname{dist}(x - x^*(x)x_0, Y) \leq \frac{\eta}{c}.$$

We thus obtain

$$\liminf_{n\to\infty} z^{**}(\xi_n^*) > \frac{\eta}{c} \geq \sup_{x\in B} \limsup_{n\to\infty} \xi_n^*(x).$$

Lemma 2.1 yields $z^{**} \notin$ (I)-env(B), which completes the proof. □

Lemma 3.2. *Suppose that X_n, $n \in \mathbb{N}$, are Grothendieck spaces. Then the space $X = \oplus_{\ell_2} X_n$ is also Grothendieck.*

Proof. The dual space X^* and the bidual space X^{**} can be represented as $\oplus_{\ell_2} X_n^*$ and $\oplus_{\ell_2} X_n^{**}$, respectively. Let (x_k^*) be a sequence in X^* which weak* converges to $x^* \in X^*$. For $x \in X$ we have $x_k^*(x) \to x^*(x)$, that is

$$\sum_{n=1}^{\infty} x_k^*(n)(x(n)) \to \sum_{n=1}^{\infty} x^*(n)(x(n)),\ k \to \infty.$$

Let $n \in \mathbb{N}$. If $x_n \in X_n$, then $\bar{x}_n = (0, \dots, 0, x_n, 0, 0, \dots) \in X$, and so

$$x_k^*(n)(x_n) = x_k^*(\bar{x}_n) \to x^*(\bar{x}_n) = x^*(n)(x_n),\ k \to \infty.$$

Hence the sequence $(x_k^*(n))_k$ converges to $x^*(n)$ in the weak* topology, and by the Grothendieck property even in the weak topology.

To prove that x_k^* weakly converges to x^*, fix arbitrary $x^{**} \in X^{**}$. Then

$$x^{**}(n)(x_k^*(n)) \to x^{**}(n)(x^*(n)), \quad n \in \mathbb{N}.$$

We need to establish

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} x^{**}(n)(x_k^*(n)) = \lim_{k\to\infty} x^{**}(x_k^*) = x^{**}(x^*) = \sum_{n=1}^{\infty} x^{**}(n)(x^*(n)),$$

so the proof is completed by showing that the sum $\sum_{n=1}^{\infty} x^{**}(n)(x_k^*(n))$ is uniformly convergent with respect to $k \in \mathbb{N}$.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be arbitrary. If $j \in \mathbb{N}$, then

$$\begin{aligned}\Big|\sum_{n=j}^{\infty} x^{**}(n)(x_k^*(n))\Big| &\leq \sum_{n=j}^{\infty} \|x^{**}(n)\|\|x_k^*(n)\| \\ &\leq \Big(\sum_{n=j}^{\infty} \|x^{**}(n)\|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=j}^{\infty} \|x_k^*(n)\|^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{n=j}^{\infty} \|x^{**}(n)\|^2\Big)^{\frac{1}{2}}\|x_k^*\|_{X^*}.\end{aligned}$$

The sequence $(x_k^*)_k$ is bounded by the uniform boundedness principle. Hence $M > 0$ can be found such that $\|x_k^*\|_{X^*} \leq M$, $k \in \mathbb{N}$. As $x^{**} \in \oplus_{\ell_2} X_n^{**}$, the sum $\sum_{n=1}^{\infty} \|x^{**}(n)\|^2$ is convergent. Thus we can choose $j_0 \in \mathbb{N}$ such that for $j \geq j_0$

$$\sum_{n=j}^{\infty} \|x^{**}(n)\|^2 \leq \frac{\varepsilon^2}{M^2}.$$

Then for all $j \geq j_0$

$$\Big|\sum_{n=j}^{\infty} x^{**}(n)(x_k^*(n))\Big| \leq \left(\frac{\varepsilon^2}{M^2}\right)^{\frac{1}{2}} \cdot M = \varepsilon,$$

which is the desired conclusion. □

Lemma 3.3. *Let X be a Banach space and $c \geq 1$. If X is c-Grothendieck, and Y is a quotient of X, then Y is c-Grothendieck.*

Proof. Let $q\colon X \to Y$ be a quotient map. It is easily seen that the dual operator $q^*\colon Y^* \to X^*$ is an isometric embedding. Consequently, $q^{**}\colon X^{**} \to Y^{**}$ satisfy $q^{**}(B_{X^{**}}) = B_{Y^{**}}$. Indeed, for $x^{**} \in B_{X^{**}}$

$$\|q^{**}x^{**}\| = \|x^{**} \circ q^*\| \leq \|x^{**}\|\|q^*\| = \|x^{**}\| \leq 1,$$

thus $q^{**}x^{**} \in B_{Y^{**}}$. Let $y^{**} \in B_{Y^{**}}$ be arbitrary. Define a linear functional x^{**} on $q^*(Y^*) \subset X^*$ by $x^{**}(q^*y^*) = y^{**}(y^*)$, $y^* \in Y^*$, and extend it to a linear functional on X^* with the same norm by the Hahn-Banach theorem. Obviously, $\|x^{**}\| = \|y^{**}\|$ and $q^{**}x^{**} = y^{**}$.

Let (y_n^*) be a bounded sequence in Y^*. Then

$$\begin{aligned}\delta_w(q^*y_n^*) &= \sup_{x^{**}\in B_{X^{**}}} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |x^{**}(q^*y_k^*) - x^{**}(q^*y_l^*)| \\ &= \sup_{x^{**}\in B_{X^{**}}} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |q^{**}x^{**}(y_k^*) - q^{**}x^{**}(y_l^*)| \\ &= \sup_{y^{**}\in B_{Y^{**}}} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y^{**}(y_k^*) - y^{**}(y_l^*)| = \delta_w(y_n^*),\end{aligned} \tag{2}$$

and

$$\begin{aligned}\delta_{w^*}(q^*y_n^*) &= \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |q^*y_k^*(x) - q^*y_l^*(x)| \\ &= \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y_k^*(qx) - y_l^*(qx)| \\ &= \sup_{x\in X,\|x\|<1} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y_k^*(qx) - y_l^*(qx)| \\ &= \sup_{y\in Y,\|y\|<1} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y_k^*(y) - y_l^*(y)| \\ &= \sup_{y\in B_Y} \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y_k^*(y) - y_l^*(y)| = \delta_{w^*}(y_n^*),\end{aligned} \tag{3}$$

where the fourth equality follows from the fact that q is a quotient map. Since X is c-Grothendieck, $\delta_w(q^* y_n^*) \leq c\delta_{w^*}(q^* y_n^*)$. Together with (2) and (3), it yields $\delta_w(y_n^*) \leq c\delta_{w^*}(y_n^*)$, so Y is c-Grothendieck. $\square$

Proof of Theorem 1.2. By Proposition 3.1, for each $n \in \mathbb{N}$ we can find an equivalent norm $\|\cdot\|_n$ on ℓ_∞ such that the space $X_n = (\ell_\infty, \|\cdot\|_n)$ is not n-Grothendieck. Set $X = \oplus_{\ell_2} X_n$. Then X is Grothendieck by Lemma 3.2, for all X_n, $n \in \mathbb{N}$, are Grothendieck spaces. Moreover, each X_n is a quotient of X. Suppose that there is some $c \geq 1$ such that X is c-Grothendieck. Find $n \in \mathbb{N}$, $n > c$. Then X is n-Grothendieck and, by Lemma 3.3, X_n should also be n-Grothendieck, which is a contradiction. $\square$

4 More general results

Kalenda's theorem [15, Example 4.1], which we have used to prove a quantitative version of Grothendieck's theorem (Theorem 1.1), can be generalized and then applied in the same way to obtain more general quantitative results.

Theorem 4.1. *Let Γ be a set and $E = \ell_\infty(\Gamma)$. Then* (I)-env $B_E = B_{E^{**}}$.

Proof. The proof of [15, Example 4.1] works here as well. It suffices to replace $\mathbb{N}$ by Γ in the right places. Lemmata 4.4 and 4.5 remain unchanged. We prove Lemma 4.6 for sequences of measures on Γ by substituting $\mathbb{N}$ with Γ wisely. Then we use it to prove Propositions 4.3 and 4.2 for measures on Γ just as in the original proof. $\square$

Theorem 4.2. *The space $\ell_\infty(\Gamma)$ is* 1*-Grothendieck for each set* Γ.

Proof. It follows from Theorem 4.1 and Proposition 2.2. $\square$

Corollary 4.3. *Let μ be a σ-finite measure on a measurable space X. Then $L^\infty(\mu)$ is 1-Grothendieck.*

Proof. If μ is σ-finite, then $L^\infty(\mu)$ is 1-injective (see for instance [6, (5.91)]) and thus 1-complemented in $\ell_\infty(\Gamma)$ for some set Γ, which is a 1-Grothendieck space by Theorem 4.2. By Lemma 3.3, $L^\infty(\mu)$ is 1-Grothendieck. $\square$

The space in Corollary 4.3 is 1-complemented in $\ell_\infty(\Gamma)$. In fact, not only 1-complemented subspaces but all quotients of $\ell_\infty(\Gamma)$ are 1-Grothendieck.

Corollary 4.4. *Let Γ be an arbitrary set. Each quotient of the space $\ell_\infty(\Gamma)$ is* 1*-Grothendieck.*

Proof. It is a consequence of Theorem 4.2 and Lemma 3.3. $\square$

Let us remark finally that we do not know whether the other spaces with the Grothendieck property mentioned in the Introduction enjoy the quantitative version as well.

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II. Quantification of the Banach-Saks property

(with Ondřej F. K. Kalenda and Jiří Spurný)

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Abstract: We investigate possible quantifications of the Banach-Saks property and the weak Banach-Saks property. We prove quantitative versions of relationships of the Banach-Saks property of a set with norm compactness and weak compactness. We further establish a quantitative version of the characterization of the weak Banach-Saks property of a set using uniform weak convergence and ℓ_1-spreading models. We also study the case of the unit ball and in this case we prove a dichotomy which is an analogue of the James distortion theorem for ℓ_1-spreading models.

1 Introduction

A bounded subset A of a Banach space X is said to be a *Banach-Saks set* if each sequence in A has a Cesàro convergent subsequence. A Banach space X is said to have the *Banach-Saks property* if each bounded sequence in X has a Cesàro convergent subsequence, i.e., if its closed unit ball B_X is a Banach-Saks set.

This property goes back to S. Banach and S. Saks who proved in [7] that, in the modern terminology, the spaces $L^p(0,1)$ for $p \in (1,+\infty)$ enjoy the Banach-Saks property. Any Banach space with the Banach-Saks property is reflexive [28] and there are reflexive spaces without the Banach-Saks property [6]. On the other hand, superreflexive spaces enjoy the Banach-Saks property (S. Kakutani showed in [21] that uniformly convex spaces have the Banach-Saks property and by [13] any superreflexive space admits a uniformly convex renorming).

A localized version of the mentioned result of [28] says that any Banach-Saks set is relatively weakly compact (see [26, Proposition 2.3]). This inspires the definition of the *weak Banach-Saks property* – a Banach space X is said to have this property if any weakly compact subset of X is a Banach-Saks set, i.e., if any weakly convergent sequence in X admits a Cesàro convergent subsequence. There are nonreflexive spaces enjoying the weak Banach-Saks property, for example c_0 or $L^1(\mu)$ [17, 30].

In the present paper we investigate possibilities of quantifying the Banach-Saks property. This is inspired by a large number of recent results on quantitative versions of various theorems and properties of Banach spaces, see, e.g., [3, 2, 11, 10, 14, 19, 20, 8]. Another approach to quantification of the Banach-Saks property and some related properties was followed by A. Kryczka in a recent series of papers [22, 23, 24, 25]. More precisely, in the quoted papers mainly quantitative versions of Banach-Saks and weak Banach-Saks operators are investigated.

The quantification means, roughly speaking, to replace implications between some notions by inequalities between certain quantities. Let us now introduce the basic quantities we will use.

Let (x_k) be a bounded sequence in a Banach space. Following [20] we set

$$\begin{aligned}
&(1) \qquad & \mathrm{ca}(x_k) &= \inf_{n\in\mathbb{N}} \sup\{\|x_k - x_l\| : k, l \geq n\}, \\
&(2) \qquad & \widetilde{\mathrm{ca}}\,(x_k) &= \inf\{\mathrm{ca}\,(x_{k_n}) : (x_{k_n}) \text{ is a subsequence of } (x_k)\}.
\end{aligned}$$

The first quantity measures how far the sequence is from being norm Cauchy. Clearly, $\mathrm{ca}(x_k) = 0$ if and only if the sequence (x_k) is norm Cauchy (hence norm convergent).

Since we are interested in Cesàro convergence of sequences, it is natural to define

$$\begin{aligned}
&(3) \qquad & \mathrm{cca}\,(x_k) &= \mathrm{ca}\left(\frac{1}{k}(\sum_{i=1}^{k} x_i)\right), \\
&(4) \qquad & \widetilde{\mathrm{cca}}\,(x_k) &= \inf\{\mathrm{cca}\,(x_{k_n})) : (x_{k_n}) \text{ is a subsequence of } (x_k)\}.
\end{aligned}$$

Let us remark that the quantities $\mathrm{cca}\,()$ and $\widetilde{\mathrm{cca}}\,()$ behave differently than the quantities $\mathrm{ca}()$ and $\widetilde{\mathrm{ca}}\,()$. More precisely, the quantity $\mathrm{ca}()$ decreases when passing to a subsequence but it is not the case of $\mathrm{cca}\,()$. Indeed, a subsequence of a Cesàro convergent sequence need not be Cesàro convergent, in fact, any bounded sequence is a subsequence of a Cesàro convergent sequence.

For a bounded set A in a Banach space X we introduce the following two quantities:

$$\begin{aligned}
&(5) \qquad & \mathrm{bs}\,(A) &= \sup\{\widetilde{\mathrm{cca}}\,(x_k) : (x_k) \subset A\}, \\
&(6) \qquad & \mathrm{wbs}\,(A) &= \sup\{\widetilde{\mathrm{cca}}\,(x_k) : (x_k) \subset A \text{ is weakly convergent}\}.
\end{aligned}$$

The first one measures how far is A from being Banach-Saks. Indeed, $\mathrm{bs}\,(A) = 0$ if and only if A is Banach-Saks by Corollary 4.3 below. Further, the same statement yields that $\mathrm{wbs}\,(A) = 0$ if and only if any weakly convergent sequence in A has a Cesàro convergent subsequence (let us stress that the limit could be outside A). The sets with the latter property will be called *weak Banach-Saks sets*.

2 Preliminaries

We use mostly a standard notation. If X is a Banach space, B_X denotes its closed unit ball. If A is any set, we denote by $\#A$ the cardinality of A. (We use this notation mainly for finite sets).

We investigate, among others, quantifications of the relationship of the Banach-Saks property to compactness and weak compactness. To formulate such results we need some quantities measuring non-compactness and weak non-compactness. There are several ways how to do it. We will use the notation from [20]. Let us recall the basic quantities.

If A and B are two nonempty subsets of a Banach space X, we set

$$\begin{aligned}
\mathrm{d}(A, B) &= \inf\{\|a - b\| : a \in A, b \in B\}, \\
\widehat{\mathrm{d}}(A, B) &= \sup\{\mathrm{d}(a, B) : a \in A\}.
\end{aligned}$$

Hence, $\mathrm{d}(A, B)$ is the ordinary distance of the sets A and B and $\widehat{\mathrm{d}}(A, B)$ is the non-symmetrized Hausdorff distance (note that the Hausdorff distance of A and B is equal to $\max\{\widehat{\mathrm{d}}(A, B), \widehat{\mathrm{d}}(B, A)\}$).

Let A be a bounded subset of a Banach space X. Then the Hausdorff measure of non-compactness of A is defined by

$$\chi(A) = \inf\{\widehat{\mathrm{d}}(A, F) : \emptyset \neq F \subset X \text{ finite}\} = \inf\{\widehat{\mathrm{d}}(A, K) : \emptyset \neq K \subset X \text{ compact}\}.$$

This is the basic measure of non-compactness. We will need one more such measure:

$$\beta(A) = \sup\{\widetilde{\mathrm{ca}}\,(x_k) : (x_k) \text{ is a sequence in } A\}.$$

It is easy to check that for any bounded set A we have

$$\chi(A) \leq \beta(A) \leq 2\chi(A),$$

thus these two measures are equivalent. (And, of course, they equal zero if and only if the respective set is relatively compact.)

An analogue of the Hausdorff measure of non-compactness for measuring weak non-compactness is the de Blasi measure of weak non-compactness

$$\omega(A) = \inf\{\widehat{\mathrm{d}}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact}\}.$$

Then $\omega(A) = 0$ if and only if A is relatively weakly compact. Indeed, the 'if' part is obvious and the 'only if' part follows from [12, Lemma 1].

There is another set of quantities measuring weak non-compactness. Let us mention two of them:

$$\begin{aligned} \mathrm{wk}_X(A) &= \widehat{\mathrm{d}}(\overline{A}^{w^*}, X), \\ \mathrm{wck}_X(A) &= \sup\{\, \mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X) : (x_k) \text{ is a sequence in } A\}. \end{aligned}$$

By $\overline{A}^{w^*}$ we mean the weak* closure of A in X^{**} (the space X is canonically embedded in X^{**}) and $\mathrm{clust}_{X^{**}}(x_k)$ is the set of all weak* cluster points in X^{**} of the sequence (x_k). It follows from [3, Theorem 2.3] that for any bounded subset A of a Banach space X we have

$$\begin{gathered} \mathrm{wck}_X(A) \leq \mathrm{wk}_X A \leq 2\,\mathrm{wck}_X(A), \\ \mathrm{wk}_X(A) \leq \omega(A). \end{gathered}$$

So, putting together these inequalities with the measures of norm non-compactness we obtain the following diagram:

$$\begin{array}{ccccc} & & \chi(A) & \leq & \beta(A) & \leq & 2\chi(A) \\ & & \vee\!| & & & & \\ & & \omega(A) & & & & \\ & & \vee\!| & & & & \\ \mathrm{wck}_X(A) & \leq & \mathrm{wk}_X(A) & \leq & 2\,\mathrm{wck}_X(A) & & \end{array}$$

Let us remark that the inequality $\omega(A) \leq \chi(A)$ is obvious and that the quantities $\omega(\cdot)$ and $\mathrm{wk}_X(\cdot)$ are not equivalent, see [5, 3].

Some quantities related to the Banach-Saks property were defined and used in [24]. Let us recall them. If (x_k) is a bounded sequence in X, we define the *arithmetic separation* of (x_k) by

$$\mathrm{asep}(x_k) = \inf\left\{\left\|\frac{1}{\#F}\left(\sum_{n\in F} x_n - \sum_{n\in H} x_n\right)\right\| : F, H \subset \mathbb{N}, \#F = \#H, \max F < \min H\right\}.$$

Further, for any bounded set $A \subset X$ define

$$\varphi(A) = \sup\{\operatorname{asep}(x_k) : (x_k) \text{ is a sequence in } A\},$$
$$\varphi'(A) = \sup\{\operatorname{asep}(x_k) : (x_k) \text{ is a weakly convergent sequence in } A\}.$$

The quantities asep and φ are from [24], the quantity φ' is an obvious modification.

3 Quantitative relation to compactness and weak compactness

Since any convergent sequence is also Cesàro convergent, relatively norm compact sets are Banach-Saks. Further, a set is Banach-Saks if and only if it is weakly Banach-Saks and relatively weakly compact. In this section we investigate quantitative versions of these relationships. Positive results are summed up in the following theorem.

Theorem 3.1. *Let A be a bounded subset of a Banach space X. Then*

$$\max\{\operatorname{wck}_X(A), \operatorname{wbs}(A)\} \le \operatorname{bs}(A) \le \beta(A). \tag{7}$$

The second inequality in the theorem quantifies the implication

A is relatively norm compact $\Rightarrow$ A is Banach-Saks,

the first one quantifies the implication

A is Banach-Saks $\Rightarrow$ A is weakly Banach-Saks and relatively weakly compact.

We point out that the latter implication cannot be quantified by using the de Blasi measure ω and that the converse implication is purely qualitative. This is illustrated by the following two examples.

Example 3.2. *There exists a separable Banach space X such that*

$$\forall \varepsilon > 0\ \exists A \subset B_X\colon\ \operatorname{bs}(A) < \varepsilon\ \&\ \omega(A) > \frac{1}{2}.$$

Example 3.3. *There exists a separable Banach space X such that*

$$\forall \varepsilon > 0\ \exists A \subset B_X\colon\ \operatorname{bs}(A) = 2\ \&\ \operatorname{wbs}(A) = 0\ \&\ \omega(A) < \varepsilon.$$

The rest of this section will be devoted to the proofs of these results. The proof of Example 3.3 will be postponed to the next section since we will make use of Theorem 4.1.

Proof of Theorem 3.1. We start by the second inequality. It immediately follows from the following lemma which is a quantitative version of the well-known fact that any convergent sequence is Cesàro convergent.

Lemma 3.4. *Let (x_k) be a bounded sequence in a Banach space X. Then*

$$\operatorname{cca}(x_k) \le \operatorname{ca}(x_k).$$

Proof. Set $M = \sup_k \|x_k\|$ and fix any $c > \mathrm{ca}(x_k)$. Further, set $y_m = \frac{1}{m}\sum_{i=1}^m x_i$ for $m \in \mathbb{N}$.

We find $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < c$ for each $n, m \geq n_0$. Let $\varepsilon > 0$ be given. We find $n_1 \geq n_0$ such that $\frac{Mn_0}{n_1} < \varepsilon$. Then we have for $n_1 \leq n \leq m$ inequalities

$$\begin{aligned}
\|y_m - y_n\| &= \left\| \frac{1}{m}\sum_{i=1}^m x_i - \frac{1}{n}\sum_{j=1}^n x_j \right\| = \left\| \sum_{i=1}^m \sum_{j=1}^n \frac{x_i - x_j}{mn} \right\| \\
&\leq \sum_{i=1}^m \sum_{j=1}^{n_0} \frac{\left\|x_i - x_j\right\|}{mn} + \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} \frac{\left\|x_i - x_j\right\|}{mn} + \sum_{i=n_0+1}^{m} \sum_{j=n_0+1}^{n} \frac{\left\|x_i - x_j\right\|}{mn} \\
&\leq \frac{2Mmn_0}{mn} + \frac{2Mn_0(n-n_0)}{mn} + \frac{(m-n_0)(n-n_0)c}{mn} \\
&\leq \frac{2Mn_0}{n_1} + \frac{2Mn_0}{n_1} + c < c + 4\varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\mathrm{cca}\,(x_k) = \mathrm{ca}(y_m) \leq c$. □

Since the inequality $\mathrm{bs}\,(A) \geq \mathrm{wbs}\,(A)$ is obvious, it remains to prove $\mathrm{bs}\,(A) \geq \mathrm{wck}_X(A)$. To do that we first prove the following lemma using an auxiliary quantity γ_0 defined by the formula

$$\begin{aligned}
\gamma_0(A) \quad = \quad \sup\{\, |\lim_m \lim_n x_m^*(x_n)| : \\
&(x_m^*) \text{ is a sequence in } B_{X^*} \text{ weak* converging to } 0, \\
&(x_n) \text{ is a sequence in } A \\
&\text{and all the involved limits exist}\}.
\end{aligned}$$

Lemma 3.5. *Let A be a bounded set in a Banach space X. Then*

$$\gamma_0(A) \leq \mathrm{bs}\,(A)\,.$$

Proof. Let $\gamma_0(A) > c$. Then there exists a sequence (x_k) in A and a sequence (x_j^*) in B_{X^*} weak* converging to 0 such that $\lim_j \lim_k x_j^*(x_k) > c$. Without loss of generality we may assume that

$$\forall j \in \mathbb{N}\colon\ \lim_k x_j^*(x_k) > c.$$

Let $y_k = \frac{1}{k}(x_1 + \cdots + x_k)$, $k \in \mathbb{N}$. Then

$$\forall j \in \mathbb{N}\colon\ \lim_k x_j^*(y_k) = \lim_k x_j^*(x_k) > c.$$

Let $\varepsilon > 0$ be arbitrary. Fix $k \in \mathbb{N}$. Since (x_j^*) weak* converges to zero, there exists $j \in \mathbb{N}$ such that $x_j^*(y_k) < \varepsilon$. Then we find $l > k$ such that $x_j^*(y_l) > c$. Then

$$\|y_l - y_k\| \geq x_j^*(y_l - y_k) > c - \varepsilon.$$

Hence $\mathrm{cca}\,(x_k) = \mathrm{ca}(y_k) \geq c$. Further, any subsequence of (x_k) has the same properties, hence $\widetilde{\mathrm{cca}}\,(x_k) \geq c$. Therefore $\mathrm{bs}\,(A) \geq c$ and the proof is complete. □

Let us now complete the proof of the remaining inequality.

Assume first that X is separable. Then (B_{X^*}, w^*) is metrizable, and thus angelic. By [11, Theorem 6.1], $\gamma_0(A) = \mathrm{wck}_X(A)$, and thus $\mathrm{wck}_X(A) \leq \mathrm{bs}\,(A)$ by the previous lemma.

Assume now that X is arbitrary and $\mathrm{wck}_X(A) > c$ for some $c > 0$. Let (x_k) be a sequence in A with $\mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X) > c$. Set $Y = \overline{\mathrm{span}}\{x_k \colon k \in \mathbb{N}\}$. Then Y is a separable subspace of X and $\mathrm{d}(\mathrm{clust}_{Y^{**}}(x_k), Y) > c$.

Indeed, let $y^{**} \in \mathrm{clust}_{Y^{**}}(x_k)$ be arbitrary. Set $x^{**}(x^*) = y^{**}(x^*|_Y)$, $x^* \in X^*$. Then $x^{**} \in \mathrm{clust}_{X^{**}}(x_k)$ and for each $y \in Y$ we have due to the Hahn-Banach theorem

$$\begin{aligned}\|y^{**} - y\|_Y &= \sup_{y^* \in B_{Y^*}} |y^{**}(y^*) - y^*(y)| \\ &= \sup_{x^* \in B_{X^*}} |(y^{**}(x^*|_Y) - x^*(y)| \\ &= \|x^{**} - y\|_X .\end{aligned}$$

Therefore

$$\mathrm{d}_{Y^{**}}(y^{**}, Y) = \mathrm{d}_{X^{**}}(x^{**}, Y) \geq \mathrm{d}_{X^{**}}(x^{**}, X),$$

hence

$$\mathrm{d}(\mathrm{clust}_{Y^{**}}(x_k), Y) \geq \mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X) > c.$$

Hence $\mathrm{wck}_Y(A \cap Y) > c$, by the separable case we get

$$\mathrm{bs}\,(A) \geq \mathrm{bs}\,(A \cap Y) \geq \mathrm{wck}_Y(A \cap Y) > c,$$

which concludes the proof of Theorem 3.1. □

Proof of Example 3.2. Let us fix $\alpha > 0$ and consider an equivalent norm $|\cdot|_\alpha$ on ℓ_1 given by the formula

$$|x|_\alpha = \max\{\alpha\|x\|_1, \|x\|_\infty\}$$

and let

$$X = \left(\oplus_{n=1}^{\infty} (\ell_1, |\cdot|_{1/n})\right)_{\ell_1}.$$

It is clear that X is a separable Banach space. Further, X has the Schur property as it is an ℓ_1-sum of spaces with the Schur property (this follows by a straightforward modification of the proof that ℓ_1 has the Schur property, see [16, Theorem 5.19]).

Further, let us define the following elements of X:

$$x_k^n = (0, \dots, 0, \overset{n\text{-th}}{e_k}, 0, \dots), \qquad n, k \in \mathbb{N},$$

where e_k is the k-th canonical basic vector in ℓ_1. Fix $n \in \mathbb{N}$ and set

$$A_n = \{x_k^n \colon k \in \mathbb{N}\}.$$

Since $\|x_k^n\| = 1$ for $k \in \mathbb{N}$, we get $A_n \subset B_X$. Further, $\|x_k^n - x_{k'}^n\| \geq 1$ whenever $k \neq k'$, so $\beta(A_n) \geq 1$ and hence $\chi(A_n) \geq \frac{1}{2}$. Since X has the Schur property, we get $\omega(A_n) = \chi(A_n) \geq \frac{1}{2}$.

Let (z_k) be an arbitrary sequence in A_n. If it has a constant subsequence, then $\widetilde{\mathrm{cca}}\,(z_k) = 0$. Otherwise there is a one-to-one subsequence (z_{k_i}). It is clear that

$$\left\|\frac{1}{m}(z_{k_1} + \cdots + z_{k_m})\right\| = \left\|\frac{1}{m}(x_1^n + \cdots + x_m^n)\right\| = \max\left\{\frac{1}{n}, \frac{1}{m}\right\} \quad \text{for } m \in \mathbb{N},$$

hence $\mathrm{cca}\,(z_{k_i}) \leq \frac{2}{n}$. So $\mathrm{bs}\,(A_n) \leq \frac{2}{n}$.

This completes the proof. □

4 Quantitative characterization of weak Banach-Saks sets

It follows from the results summarized in [26, Section 2] that the following assertions are equivalent for a subset A of a Banach space:

- A is a weak Banach-Saks set.
- Each weakly convergent sequence in A admits a uniformly weakly convergent subsequence.
- No weakly convergent sequence in A generates an ℓ_1-spreading model.

More precisely, in the quoted paper the authors formulate characterizations of Banach-Saks sets, adding to the other assertions the assumption that A is relatively weakly compact. In this section we will prove a quantitative version of these characterizations. To formulate it, we need to introduce some natural quantities related to the above-mentioned properties:

Let (x_k) be a sequence which weakly converges to some $x \in X$. This sequence is said to be *uniformly weakly convergent* if for each $\varepsilon > 0$

$$\exists n \in \mathbb{N}\ \forall x^* \in B_{X^*}: \#\{k \in \mathbb{N}\colon\ |x^*(x_k - x)| > \varepsilon\} \leq n.$$

The quantity $\mathrm{wu}\,(x_k)$ is then defined to be the infimum of all $\varepsilon > 0$ satisfying this condition. Further, we set

$$\widetilde{\mathrm{wu}}\,(x_k) \quad = \quad \inf\{\mathrm{wu}\,(x_{k_n}) : (x_{k_n}) \text{ is a subsequence of } x_k\}.$$

Finally, for a bounded set A we define

$$\mathrm{wus}\,(A) \quad = \quad \sup\{\widetilde{\mathrm{wu}}\,(x_k) : (x_k) \subset A \text{ is weakly convergent}\}.$$

We continue by a definition related to spreading models. Let (x_k) be a bounded sequence. We say that it *generates an ℓ_1-spreading model with $\delta > 0$* if

$$\forall F \subset \mathbb{N}, \#F \leq \min F\ \forall (\alpha_i)_{i\in F} \in \mathbb{R}^F: \left\| \sum_{i\in F} \alpha_i x_i \right\| \geq \delta \sum_{i\in F} |\alpha_i|\,.$$

The sequence (x_k) *generates an ℓ_1-spreading model* if it generates an ℓ_1-spreading model with some $\delta > 0$.

For a bounded set A we set

$$\begin{aligned}\mathrm{sm}\,(A) = \sup\{\delta > 0\colon\ &\exists (x_k) \subset A, x_k \xrightarrow{w} x,\\ &(x_k - x) \text{ generates an } \ell_1\text{-spreading model with } \delta\}.\end{aligned}$$

Now we are ready to formulate the promised quantitative characterizations.

Theorem 4.1. *Let A be a bounded set in a Banach space X. Then*

$$\mathrm{sm}\,(A) \leq \frac{1}{2}\,\mathrm{wbs}\,(A) \leq \mathrm{wus}\,(A) \leq 2\,\mathrm{sm}\,(A)\,. \tag{8}$$

Remark 4.2. *For any bounded set $A \subset X$ have*

$$\text{(9)} \qquad \operatorname{sm}(A) \le \frac{1}{2}\varphi'(A) \le \operatorname{wbs}(A)\,.$$

Indeed, it is easy to check that $\operatorname{cca}(x_k) \ge \frac{1}{2}\operatorname{asep}(x_k)$ *for any bounded sequence* (x_k). *Since the quantity* asep *cannot decrease when we pass to a subsequence, we get* $\widetilde{\operatorname{cca}}(x_k) \ge \frac{1}{2}\operatorname{asep}(x_k)$, *hence the second inequality in* (9) *follows. The first inequality follows from Lemma 4.6 below.*

Hence, by (9) *and Theorem 4.1 the quantity φ' inspired by [24] is equivalent to our quantities. Further, since clearly $\varphi' \le \varphi$, we get*

$$\operatorname{sm}(A) \le \frac{1}{2}\varphi'(A) \le \frac{1}{2}\varphi(A) \le \operatorname{bs}(A)\,.$$

The last inequality follows from the previous paragraph. It seems not to be clear whether the quantity φ is equivalent to bs () *also for sets which are not relatively weakly compact.*

As a consequence of Theorem 4.1 we get that the introduced quantities wbs () and bs () really measure the failure of the weak Banach-Saks (Banach-Saks, respectively) property of a set.

Corollary 4.3. *Let A be a bounded set in a Banach space X.*

- *If* $\operatorname{wbs}(A) = 0$, *then A is a weak Banach-Saks set.*
- *If* $\operatorname{bs}(A) = 0$, *then A is a Banach-Saks set.*

To prove the corollary we will use the following lemma, which also proves the inequality $\operatorname{wbs}(A) \le 2\operatorname{wus}(A)$ from Theorem 4.1.

Lemma 4.4. *Let (x_k) be a sequence in a Banach space X which weakly converges to some $x \in X$. Then* $\operatorname{cca}(x_k) \le 2\operatorname{wu}(x_k)$.

Proof. Let $M = \sup_k \|x_k\|$. Fix an arbitrary $c > \operatorname{wu}(x_k)$. Let $N \in \mathbb{N}$ be such that

$$\text{(10)} \qquad \forall x^* \in B_{X^*}: \#\{k \in \mathbb{N}\colon\ |x^*(x_k - x)| > c\} \le N.$$

Set $z_k = \frac{1}{k}(x_1 + \cdots + x_k)$, $k \in \mathbb{N}$. Given $\varepsilon > 0$, we find $n_0 \in \mathbb{N}$ such that $\frac{2MN}{n_0} < \varepsilon$.

Now, for any couple of indices $n_0 \le n < m$ and $x^* \in B_{X^*}$ we obtain from (10)

$$\begin{aligned}
|x^*(z_m - z_n)| &= |x^*(z_m - x) + x^*(x - z_n)| \\
&= \left|\left(\frac{1}{m} - \frac{1}{n}\right)(x^*(x_1 - x) + \cdots + x^*(x_n - x))\right. \\
&\qquad \left. + \frac{1}{m}\,(x^*(x_{n+1} - x) + \cdots + x^*(x_m - x))\right| \\
&\le \frac{2NM}{n} + c\left(n\left(\frac{1}{n} - \frac{1}{m}\right) + \frac{m-n}{m}\right) \\
&\le \varepsilon + c\left(2 - 2\frac{n}{m}\right) \\
&\le 2c + \varepsilon.
\end{aligned}$$

Since $x^* \in B_{X^*}$ is arbitrary,

$$\|z_m - z_n\| \le 2c + \varepsilon$$

for each $n_0 \le n < m$. Thus $\operatorname{cca}(x_k) = \operatorname{ca}(z_k) \le 2c$, which completes the proof. □

Proof of Corollary 4.3. Suppose that $\mathrm{wbs}(A) = 0$. By Theorem 4.1 we deduce $\mathrm{wus}(A) = 0$. Let (x_k) be a weakly convergent sequence in A. Then we can construct by induction sequences $(y_k^n)_{k=1}^\infty$ for $n \in \mathbb{N}$ such that

- $y_k^1 = x_k$, $k \in \mathbb{N}$;
- (y_k^{n+1}) is a subsequence of (y_k^n);
- $\mathrm{wu}\left(y_k^{n+1}\right) < \frac{1}{n+1}$.

Consider the diagonal sequence $(z_k) = (y_k^k)$. Then (z_k) is a subsequence of (x_k), $\mathrm{wu}(z_k) = 0$ and hence $\mathrm{cca}(z_k) = 0$ (by Lemma 4.4), so (z_k) is Cesàro convergent. This completes the proof that A is a weak Banach-Saks set.

For the second part, suppose that $\mathrm{bs}(A) = 0$. Hence $\mathrm{wbs}(A) = 0$, so by the first part, A is a weak Banach-Saks set. Further, by Theorem 3.1 we get $\mathrm{wck}_X(A) = 0$, hence A is relatively weakly compact. Therefore A is a Banach-Saks set. □

We continue with the inequality $2\,\mathrm{sm}(A) \le \mathrm{wbs}(A)$. It follows immediately from the following lemma.

Lemma 4.5. *Let (x_k) be a bounded sequence in a Banach space X and $x \in X$. Suppose that $(x_k - x)$ generates an ℓ_1-spreading model with a constant c. Then* $\widetilde{\mathrm{cca}}(x_{k^3}) \ge 2c$.

Proof. Since $\mathrm{cca}(x_k - x) = \mathrm{cca}(x_k)$, we may without loss of generality suppose that $x = 0$. Let $M = \sup_k \|x_k\|$. Let (z_k) be any subsequence of (x_{k^3}). We will show that $\mathrm{cca}(z_k) \ge 2c$. To this end we notice that, for $F \subset \mathbb{N}$ satisfying $\#F \le (\min F)^3$, we have $\|\sum_{i\in F} \alpha_i z_i\| \ge c \sum_{i\in F} |\alpha_i|$ whenever (α_i) are arbitrary scalars.

Let $N \in \mathbb{N}$ be given. We set $n = N + N^2$ and $m = N + N^3$. Then the set $F = \{N+1, \dots, m\}$ satisfies $\#F \le (\min F)^3$, which implies

$$\begin{aligned}
&\left\|\frac{1}{m}(z_1 + \dots + z_m) - \frac{1}{n}(z_1 + \dots + z_n)\right\| \\
&\quad = \left\|\left(\frac{1}{m} - \frac{1}{n}\right)(z_1 + \dots + z_n) + \frac{1}{m}(z_{n+1} + \dots + z_m)\right\| \\
&\quad \ge \left\|\left(\frac{1}{m} - \frac{1}{n}\right)(z_{N+1} + \dots + z_n) + \frac{1}{m}(z_{n+1} + \dots + z_m)\right\| \\
&\qquad\qquad - \left\|\left(\frac{1}{m} - \frac{1}{n}\right)(z_1 + \dots + z_N)\right\| \\
&\quad \ge c\left(\frac{1}{n} - \frac{1}{m}\right)(n - N) + \frac{c}{m}(m - n) - MN\left(\frac{1}{n} - \frac{1}{m}\right) \\
&\quad = c\frac{N^2(N^3 - N^2)}{(N + N^3)(N + N^2)} + c\frac{N^3 - N^2}{N^3 + N} - M\frac{N(N^3 - N^2)}{(N + N^3)(N + N^2)}.
\end{aligned}$$

Since the last term converges to $2c$ as N tends to infinity, $\mathrm{cca}(z_k) \ge 2c$, which completes the proof. □

The following lemma provides the promised proof of the first inequality in (9). It is an easier variant of the previous lemma.

Lemma 4.6. *Let (x_k) be a bounded sequence in a Banach space X and $x \in X$. Suppose that $(x_k - x)$ generates an ℓ_1-spreading model with a constant strictly greater than c. Then there is $n \in \mathbb{N}$ such that* $\mathrm{asep}((x_{k^2})_{k \ge n}) > 2c$.

Proof. Fix $d > c$ such that $(x_k - x)$ generates an ℓ_1-spreading model with the constant d. Let $M = \sup_k \|x_k\|$. Fix $n \in \mathbb{N}$ such that $n \geq 2$ and

$$2d - \sqrt{\frac{2}{n}}(d + 2M) > 2c.$$

If we show that

$$\operatorname{asep}((x_{k^2})_{k \geq n}) \geq 2d - \sqrt{\frac{2}{n}}(d + 2M),$$

the proof will be completed. To do that fix any $m \in \mathbb{N}$ and indices

$$n \leq p_1 < p_2 < \cdots < p_m < q_1 < q_2 < \cdots < q_m.$$

If $m \leq \frac{1}{2}n^2$, then $2m \leq n^2$ and therefore

$$\left\|\frac{1}{m}\left(\sum_{i=1}^m x_{p_i^2} - \sum_{i=1}^m x_{q_i^2}\right)\right\| = \left\|\frac{1}{m}\left(\sum_{i=1}^m (x_{p_i^2} - x) - \sum_{i=1}^m (x_{q_i^2} - x)\right)\right\| \geq d \cdot \frac{1}{m} \cdot 2m = 2d.$$

Finally, suppose that $m > \frac{1}{2}n^2$. Let $j \in \{1, \dots, m\}$ be the smallest number satisfying $p_j^2 \geq 2m$. Such a number exists since $p_m^2 \geq m^2 \geq 2m$ (as $m > \frac{1}{2}n^2 \geq 2$). Moreover, necessarily $j \leq \sqrt{2m} + 1$. Indeed, if $p_i^2 < 2m$, then $i^2 \leq p_i^2 < 2m$ and hence $i < \sqrt{2m}$. We have

$$\begin{aligned}
\left\|\frac{1}{m}\left(\sum_{i=1}^m x_{p_i^2} - \sum_{i=1}^m x_{q_i^2}\right)\right\| &= \left\|\frac{1}{m}\left(\sum_{i=1}^m (x_{p_i^2} - x) - \sum_{i=1}^m (x_{q_i^2} - x)\right)\right\| \\
&\geq \left\|\frac{1}{m}\left(\sum_{i=j}^m (x_{p_i^2} - x) - \sum_{i=1}^m (x_{q_i^2} - x)\right)\right\| - \left\|\frac{1}{m}\sum_{i=1}^{j-1} (x_{p_i^2} - x)\right\| \\
&\geq d \cdot \frac{1}{m} \cdot (2m - \sqrt{2m}) - \frac{1}{m} \cdot 2M \cdot \sqrt{2m} \\
&= 2d - \sqrt{\frac{2}{m}}(d + 2M) \geq 2d - \sqrt{\frac{2}{n}}(d + 2M).
\end{aligned}$$

This completes the proof. □

Finally we will show $\operatorname{wus}(A) \leq 2\operatorname{sm}(A)$. This follows from the following lemma. The lemma essentially follows from [26, Theorem 2.1]. However, since this theorem is not proved in [26] and we were not able to completely recover it from the references therein and since we, moreover, need to know precise constants, we decided to give here a complete proof of the lemma using the results of [4, 27].

Lemma 4.7. *Let $c > 0$ and let (x_k) be a weakly null sequence with $\widetilde{\operatorname{wu}}(x_k) > c$. Then there is a subsequence of (x_k) generating an ℓ_1-spreading model with the constant $\frac{c}{2}$.*

Proof. Fix $\delta \in (0, 1)$ such that $\widetilde{\operatorname{wu}}(x_k) > c + 3\delta$. We begin by showing that without loss of generality (up to passing to a subsequence) we may suppose that for any finite set $F \subset \mathbb{N}$ the following holds:

$$\tag{11} \begin{aligned}
&\Big(\exists x^* \in B_{X^*}\ \forall k \in F\colon x^*(x_k) \geq c + 3\delta\Big) \\
&\qquad \Longrightarrow \exists y^* \in B_{X^*}\Big(\forall k \in F : y^*(x_k) > c + 2\delta \text{ and } \sum_{k \in \mathbb{N}\setminus F} |y^*(x_k)| < \delta\Big).
\end{aligned}$$

To this end we will use [4, Lemma 2.4.7]. The set

$$D = \{(x^*(x_k))_{k=1}^\infty : x^* \in B_{X^*}\}$$

is a convex symmetric weakly compact subset of c_0. (Indeed, the mapping $x^* \mapsto (x^*(x_k))_{k=1}^\infty$ is a mapping of B_{X^*} onto D which is continuous from the weak* topology to the weak topology of c_0.) Further, fix $\varepsilon \in (0,1)$ such that

$$(1-\varepsilon)(c+3\delta) > c+2\delta \text{ and } \varepsilon(c+3\delta) < \delta.$$

Then the quoted lemma applied to the set D, the constant $c+3\delta$ in place of δ and ε yields an infinite set $M \subset \mathbb{N}$ such that for any finite set $F \subset M$ (11) is satisfied with M in place of $\mathbb{N}$. Up to passing to a subsequence we may suppose that $M = \mathbb{N}$, therefore without loss of generality (11) holds for any finite set $F \subset \mathbb{N}$.

Further, set

$$\mathcal{K}_0 = \{\{k \in \mathbb{N} : x^*(x_k) \geq c+3\delta\} : x^* \in B_{X^*}\},$$

let $\mathcal{K}$ be the family of those subsets of $\mathbb{N}$ which are contained in an element of $\mathcal{K}_0$.

To complete the proof we will use the following lemma which is an easier variant of [27, Theorem 2] and was suggested to us by the referee:

Lemma 4.8. *Let $\mathcal{F}$ be a nonempty hereditary family of finite subsets of $\mathbb{N}$. Then there is an infinite set $M \subset \mathbb{N}$ such that one of the following conditions is satisfied:*

(a) There is some $d \in \mathbb{N} \cup \{0\}$ such that $\mathcal{F} \cap \mathcal{P}(M) = [M]^{\leq d}$.

(b) There is a strictly increasing mapping $f : M \to \mathbb{N}$ such that $\{F \subset M : \#F \leq f(\min F)\} \subset \mathcal{F}$.

(A family $\mathcal{F}$ is heredirary if $B \in \mathcal{F}$ whever $B \subset A$ and $A \in F$. Furhter, by $\mathcal{P}(M)$ we denote the power set of M, by $[M]^{\leq d}$ the family of all subsets of M of cardinality at most d, by $[M]^d$ the family of all subsets of M of cardinality exactly d.)

Proof. Suppose that there is an infinite set $M \subset \mathbb{N}$ and $d \in \mathbb{N}$ such that $\mathcal{F} \cap [M]^d = \emptyset$. Let $d_0 \in \mathbb{N}$ be the minimal number with this property. If $d_0 = 1$, then $\mathcal{F} \cap \mathcal{P}(M) = \{\emptyset\} = [M]^{\leq 0}$. Suppose that $d_0 > 1$. By the classical Ramsey theorem there is an infinite set $N \subset M$ such that either $\mathcal{F} \cap [N]^{d_0-1} = \emptyset$ or $[N]^{d_0-1} \subset \mathcal{F}$. The first possibility cannot occur due to the minimality of d_0. The second one implies $\mathcal{F} \cap \mathcal{P}(N) = [N]^{\leq d_0-1}$ (as $\mathcal{F}$ is hereditary and $\mathcal{F} \cap [N]^{d_0} = \emptyset$). Hence, the condition (a) is satisfied.

Next suppose that such an infinite set $M \subset \mathbb{N}$ and $d \in \mathbb{N}$ do not exist. It means that $\mathcal{F} \cap [M]^d \neq \emptyset$ for each inifinite $M \subset \mathbb{N}$ and each $d \in \mathbb{N}$. Using the classical Ramsey theorem we deduce that for any infinite $M \subset \mathbb{N}$ and any $d \in \mathbb{N}$ there is an infinite set $N \subset M$ such that $[N]^d \subset \mathcal{F}$. Since $\mathcal{F}$ is hereditary, automatically $[N]^{\leq d} \subset F$. Therefore we can by induction construct a sequence (M_n) with the following properties.

- $\mathbb{N} \supset M_1 \supset M_2 \supset \cdots$
- M_n is infinite for each $n \in \mathbb{N}$.
- $[M_n]^{\leq n} \subset \mathcal{F}$ for each $n \in \mathbb{N}$.

Choose a strictly increasing sequence (m_n) of natural numbers such that $m_n \in M_n$ for each $n \in \mathbb{N}$. Set $M = \{m_n : n \in \mathbb{N}\}$ and define $f : M \to \mathbb{N}$ by $f(m_n) = n$. Then f witnesses that (b) is satisfied for M. ◻

Now we continue the proof of Lemma 4.7. We apply the preceding lemma to the family $\mathcal{K}$. We observe that the case (a) cannot occur. Indeed, suppose that $M \subset \mathbb{N}$ is infinite and $d \in \mathbb{N}$ such that $\mathcal{K} \cap \mathcal{P}(M) = [M]^{\leq d}$. Then, in particular, $\mathcal{K}_0 \cap \mathcal{P}(M) \subset [M]^{\leq d}$, hence for each $x^* \in B_{X^*}$ we have

$$\begin{aligned}\#\{k \in M : |x^*(x_k)| \geq c + 3\delta\} = \#\{k \in M : x^*(x_k) \geq c + 3\delta\} \\ + \#\{k \in M : (-x^*)(x_k) \geq c + 3\delta\} \leq 2d,\end{aligned}$$

hence $\widetilde{\mathrm{wu}}\,(x_k) \leq c + 3\delta$, a contradiction with the choice of δ.

Thus the case (b) must occur. Fix the relevant set M and function f. Up to passing to a subsequence we may suppose that $M = \mathbb{N}$. Now we are going to check that the sequence (x_k) (which was made from the original one by passing twice to a subsequence) generates an ℓ_1-spreading model with the constant $\frac{c}{2}$. To do that let $F \subset \mathbb{N}$ be a subset satisfying $\#F \leq \min F$ and $(\alpha_i)_{i \in F}$ be any choice of scalars. Set $F^+ = \{i \in F : \alpha_i > 0\}$ and $F^- = \{i \in F : \alpha_i < 0\}$. Without loss of generality we may suppose that $\sum_{i \in F^+} \alpha_i \geq \sum_{i \in F^-}(-\alpha_i)$ (otherwise we can pass to $(-\alpha_i)$). Since $\#F^+ \leq \#F \leq \min F \leq f(\min F) \leq f(\min F^+)$, we get $F^+ \in \mathcal{K}$. Therefore using (11) we can find $x^* \in B_{X^*}$ such that

$$x^*(x_i) > c + 2\delta \text{ for } i \in F^+ \qquad \text{and} \qquad \sum_{i \in \mathbb{N} \setminus F^+} |x^*(x_i)| < \delta.$$

Then

$$\begin{aligned}\left\| \sum_{i \in F} \alpha_i x_i \right\| &\geq \sum_{i \in F} \alpha_i x^*(x_i) \geq \sum_{i \in F^+} \alpha_i (c + 2\delta) - \sum_{i \in F \setminus F^+} |\alpha_i| |x^*(x_i)| \\ &\geq \frac{1}{2}(c + 2\delta) \sum_{i \in F} |\alpha_i| - \delta \sum_{i \in F} |\alpha_i| = \frac{c}{2} \sum_{i \in F} |\alpha_i|,\end{aligned}$$

which completes the argument. □

Remark 4.9. *The proof of Theorem 4.1 is inspired by the proof of [26, Theorem 2.4]. More precisely, Lemma 4.4 is straightforward. Lemma 4.5 is a more elementary and more precise version of the argument in [26, p. 2256, second paragraph]. The quoted approach would yield* $\mathrm{wbs}\,(A) \geq \frac{1}{4}\,\mathrm{sm}\,(A)$*, with a little more care* $\mathrm{wbs}\,(A) \geq \frac{1}{2}\,\mathrm{sm}\,(A)$*. Our approach is more elementary, we use just the triangle inequality and not the possibility to extract a basic subsequence, and we obtain the best possible inequality. Finally, Lemma 4.7 is a more precise version of the proof of [26, Theorem 2.4(a) $\Longrightarrow$ (b)].*

We finish this section by giving the last missing proof from the previous section.

Proof of Example 3.3. Let B be the Baernstein space, i.e., the separable reflexive space constructed in [6] and let (b_n) denotes its canonical basis. Let $X = B \oplus_\infty \ell_1$ and (e_n) be the standard basis of ℓ_1. For $\varepsilon \in [0, 1]$ set $A_\varepsilon = \{(b_n, \varepsilon e_n) : n \in \mathbb{N}\}$. Then $A_\varepsilon \subset B_X$. Since (b_n) converges weakly to zero, the set A_0 is weakly compact and hence $\omega(A_\varepsilon) \leq \widehat{\mathrm{d}}(A_\varepsilon, A_0) \leq \varepsilon$.

Fix $\varepsilon \in (0, 1]$. It is clear that the sequence $(b_n, \varepsilon e_n)$ is equivalent to the ℓ_1 basis and hence A_ε contains no nontrivial weakly convergent sequences, and thus trivially $\mathrm{wbs}\,(A_\varepsilon) = 0$. Finally, by the very definitions we get $\mathrm{bs}\,(A_\varepsilon) \geq \mathrm{bs}\,(A_0)$. By the construction of B in [6] we know that (b_n) is weakly null and generates an ℓ_1-spreading model with $\delta = 1$. Thus $\mathrm{bs}\,(A_0) \geq 2$ by Theorem 4.1 and hence $\mathrm{bs}\,(A_\varepsilon) \geq 2$. This completes the proof. □

5 Quantities applied to the unit ball

In this section we investigate possible values of the quantities bs () and wbs () when applied to the unit ball of a Banach space. There are two main results in this section. The first one is a dichotomy for the quantity wbs (), the second one deals with the quantity bs () in nonreflexive spaces.

Theorem 5.1. *Let X be a Banach space. Then*

$$\operatorname{wbs}(B_X) = \begin{cases} 0 & \text{if } X \text{ has the weak Banach-Saks property}, \\ 2 & \text{otherwise}. \end{cases}$$

In particular:

- *There is a separable reflexive Banach space X with* $\operatorname{bs}(B_X) = \operatorname{wbs}(B_X) = 2$.
- *There is a nonreflexive Banach space X with separable dual with* $\operatorname{bs}(B_X) = \operatorname{wbs}(B_X) = 2$.
- *If $X = C[0,1]$, then* $\operatorname{bs}(B_X) = \operatorname{wbs}(B_X) = 2$.

Theorem 5.2.

1. *Let X be a Banach space containing an isomorphic copy of ℓ_1. Then* $\operatorname{bs}(B_X) = 2$. *In particular,* $\operatorname{bs}(B_{\ell_1}) = 2$ *and* $\operatorname{wbs}(B_{\ell_1}) = 0$.
2. *Let X be a nonreflexive Banach space containing no isomorphic copy of ℓ_1. Then* $\operatorname{bs}(B_X) \in [1,2]$. *In particular,* $\operatorname{bs}(B_{c_0}) = 1$, $\operatorname{bs}(B_c) = 2$ *and* $\operatorname{wbs}(B_{c_0}) = \operatorname{wbs}(B_c) = 0$, *where c denotes the space of convergent sequences equipped with the supremum norm.*

The key ingredient of the proof of Theorem 5.1 is the following lemma which can be viewed as a variant of the James distortion theorem for spreading models.

Lemma 5.3. *Let X be a Banach space. Then* $\operatorname{sm}(B_X) \in \{0,1\}$.

Proof. Assume that $\operatorname{sm}(B_X) > 0$. Then there is a sequence (x_k) in B_X weakly converging to some $x \in B_X$ and $\delta > 0$ such that the sequence $(x_k - x)$ generates an ℓ_1-spreading model with $\delta > 0$, i.e.,

$$\forall F \subset \mathbb{N}, \#F \leq \min F \ \forall (\alpha_i)\colon \left\| \sum_{i \in F} \alpha_i (x_i - x) \right\| \geq \delta \sum_{i \in F} |\alpha_i| \, .$$

We will show that, for any $\omega > 0$, there exists a weakly null sequence (y_k) in B_X which generates an ℓ_1-spreading model with $1 - \omega$.

The first step is to replace the sequence $(x_k - x)$ by a normalized weakly null sequence generating an ℓ_1-spreading model. Since $(x_k - x)$ generates an ℓ_1-spreading model, no subsequence is norm-convergent and hence $\inf_k \|x_k - x\| > 0$. Thus, if we set $u_k = \frac{x_k - x}{\|x_k - x\|}$, then (u_k) is a normalized weakly null sequence. Moreover, since $\|x_k - x\| \leq 2$ for each k, the sequence (u_k) generates an ℓ_1-spreading model with $\frac{\delta}{2}$. By [1, Proposition 1.5.4] we can suppose (up to passing to a subsequence) that (u_k) is a basic sequence. Moreover, by [15, Theorem 6.6] we may assume (by passing to a further subsequence if necessary) that there exists a Banach space $(Y, |\cdot|)$ with a (sub-symmetric) basis (e_k) such that

$$(12)\qquad \begin{gathered}\forall \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists m(\varepsilon, N) \in \mathbb{N}\colon m(\varepsilon, N) \le k_1 < k_2 < \cdots < k_N \implies \\ \forall(\alpha_i)\colon (1-\varepsilon)\left|\sum_{i=1}^{N} \alpha_i e_i\right| \le \left\|\sum_{i=1}^{N} \alpha_i u_{k_i}\right\| \le (1+\varepsilon)\left|\sum_{i=1}^{N} \alpha_i e_i\right|.\end{gathered}$$

Since (u_k) generates an ℓ_1-spreading model, the sequence (e_k) is equivalent to the ℓ_1-basis, so we may suppose that Y is the space ℓ_1 with an equivalent norm $|\cdot|$ and (e_k) is the canonical basis. Hence, if we set

$$\beta = \inf\left\{\left|\sum_{i=1}^{\infty} \alpha_i e_i\right| : \sum_{i=1}^{\infty} |\alpha_i| = 1\right\},$$

then $\beta > 0$ and

$$\forall(\alpha_i) \in \ell_1\colon \left|\sum_{i=1}^{\infty} \alpha_i e_i\right| \ge \beta \sum_{i=1}^{\infty} |\alpha_i|.$$

To complete the proof choose an arbitrary $\omega > 0$. We can fix $\eta > 0$ such that $\frac{1-\eta}{(1+\eta)^2} > 1-\omega$. Using the density of c_{00} in ℓ_1 we find $n \in \mathbb{N}$ and scalars $(\alpha_i)_{i=1}^n$ such that $\sum_{i=1}^n |\alpha_i| = 1$ and $\left|\sum_{i=1}^n \alpha_i e_i\right| < (1+\eta)\beta$. Set $m_0 = m(\eta, n)$. For every $k \in \mathbb{N}$ we set

$$y_k = \frac{1}{(1+\eta)^2\beta} \sum_{i=1}^{n} \alpha_i u_{m_0+kn+i}.$$

From (12) and the choice of $(\alpha_i)_{i=1}^n$ we obtain

$$\|y_k\| \le \frac{1}{(1+\eta)^2\beta}(1+\eta)\left|\sum_{i=1}^{n} \alpha_i e_i\right| \le \frac{(1+\eta)^2\beta}{(1+\eta)^2\beta} = 1,$$

hence y_k are elements of B_X. Further, the sequence (y_k) weakly converges to zero.

Let $N \in \mathbb{N}$ be fixed. Let $k_1 < k_2 < \cdots < k_N$ be indices, where $k_1 n \ge m(\eta, nN)$. If $(\beta_j)_{j=1}^N$ are scalars, then we have from (12) and from the definition of β estimates

$$\begin{aligned}\left\|\sum_{j=1}^{n} \beta_j y_{k_j}\right\| &= \frac{1}{(1+\eta)^2\beta}\left\|\sum_{j=1}^{N} \beta_j\left(\sum_{i=1}^{n} \alpha_i u_{m_0+k_j n+i}\right)\right\| \\ &\ge \frac{1-\eta}{(1+\eta)^2\beta}\left|\sum_{j=1}^{N} \beta_j\left(\sum_{i=1}^{n} \alpha_i e_{(j-1)n+i}\right)\right| \\ &\ge \frac{1-\eta}{(1+\eta)^2} \sum_{j=1}^{N} \left|\beta_j\right|\left(\sum_{i=1}^{n} |\alpha_i|\right) \\ &= \frac{1-\eta}{(1+\eta)^2} \sum_{j=1}^{N} \left|\beta_j\right| \ge (1-\omega)\sum_{j=1}^{N} \left|\beta_j\right|.\end{aligned}$$

Hence we have shown the following statement for the sequence (y_k):

$$\forall N \in \mathbb{N}\colon \left(\frac{m(\eta, nN)}{n} \le k_1 < k_2 < \cdots < k_N \implies \forall(\beta_j)\colon \left\|\sum_{j=1}^{N} \beta_j y_{k_j}\right\| \ge (1-\omega)\sum_{j=1}^{N} \left|\beta_j\right|\right).$$

To finish the proof it is enough to extract a further subsequence from (y_k) satisfying the definition of the ℓ_1-spreading model with $1-\omega$. To this end, we choose an increasing sequence (n_j) of indices satisfying $n_j \geq \frac{m(\eta,nj)}{n}$ and set $z_j = y_{n_j}$, $j \in \mathbb{N}$. Let now $N \in \mathbb{N}$ and let $N \leq k_1 < k_2 < \cdots < k_N$ be indices and (α_i) be scalars. Then

$$\left\| \sum_{i=1}^{N} \alpha_j z_{k_j} \right\| = \left\| \sum_{i=1}^{N} \alpha_j y_{n_{k_j}} \right\| \geq (1-\omega) \sum_{i=1}^{N} |\alpha_i| ,$$

because $\frac{m(\eta,nN)}{n} \leq n_N \leq n_{k_1} < n_{k_2} < \cdots < n_{k_N}$. □

Proof of Theorem 5.1. The equality follows from Lemma 5.3 and Theorem 4.1.

The first example of a separable reflexive space without the Banach-Saks property is constructed in [6].

As a nonreflexive space with separable dual which fails the weak Banach-Saks property one can take the Schreier space described for example in [9, Construction 0.2]. It is not reflexive since it contains a copy of c_0 by [9, Proposition 0.7], it has separable dual since it has an unconditional basis and does not contain a copy of ℓ_1 [9, Proposition 0.4(iii) and Proposition 0.5]. It fails the weak Banach-Saks property since the basis is weakly null and generates an ℓ_1-spreading model due to [9, Proposition 0.4(iv)].

The space $C[0,1]$ fails the weak Banach-Saks property since it contains any separable space as a subspace. A direct proof is contained already in [29].

□

Proof of Theorem 5.2. (1) Let (x_k) be a bounded sequence and $\delta > 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq \delta \sum_{i=1}^{n} |\alpha_i| , \qquad n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{R}. \tag{13}$$

Then $\operatorname{cca}(x_k) \geq 2\delta$. Indeed, let $m \geq n$. Then

$$\begin{aligned} \left\| \frac{1}{m} \sum_{i=1}^{m} x_i - \frac{1}{n} \sum_{i=1}^{n} x_i \right\| &= \left\| \left(\frac{1}{m} - \frac{1}{n} \right) \sum_{i=1}^{n} x_i + \frac{1}{m} \sum_{i=n+1}^{m} x_i \right\| \\ &\geq \delta \left(n \left(\frac{1}{n} - \frac{1}{m} \right) + \frac{1}{m}(m-n) \right) = 2\delta \left(1 - \frac{n}{m} \right). \end{aligned}$$

For any fixed n the latter expression has limit 2δ when $m \to +\infty$. This shows that $\operatorname{cca}(x_k) \geq 2\delta$. Since any subsequence of (x_k) satisfies (13) as well, we get even $\widetilde{\operatorname{cca}}(x_k) \geq 2\delta$.

Let X be a Banach space containing an isomorphic copy of ℓ_1. By the James distortion theorem there is, given $\varepsilon \in (0,1)$, a normalized sequence (x_k) in X which satisfies (13) with the constant $1-\varepsilon$ in place of δ. It follows that $\operatorname{bs}(B_X) \geq \widetilde{\operatorname{cca}}(x_k) \geq 2(1-\varepsilon)$. Since $\varepsilon \in (0,1)$ is arbitrary, $\operatorname{bs}(B_X) \geq 2$. The converse inequality is obvious.

Finally, the equality $\operatorname{wbs}(B_{\ell_1}) = 0$ follows from the Schur property of ℓ_1.

(2) The inequality $\operatorname{bs}(B_X) \leq 2$ is trivial. The inequality $\operatorname{bs}(B_X) \geq 1$ follows from Theorem 3.1 since for a nonreflexive space X one has $\operatorname{wck}_X(B_X) = 1$ (this follows for example from [18, Theorem 1] and [11, Proposition 2.2]).

The spaces c_0 and c are isomorphic and, moreover, they enjoy the weak Banach-Saks property by [17]. Therefore $\operatorname{wbs}(B_{c_0}) = \operatorname{wbs}(B_c) = 0$.

To show that $\operatorname{bs}(B_c) = 2$ define a sequence (x_k) in B_c by the formula

$$x_k(i) = \begin{cases} 1, & 1 \le i \le k, \\ -1, & i \ge k+1. \end{cases}$$

Let (x_{k_n}) be any subsequence of (x_k). Denote by $y_n = \frac{1}{n}\sum_{j=1}^{n} x_{k_j}$ for $n \in \mathbb{N}$. Let $m < n$ be two natural numbers. Then

$$y_m(k_m+1) = -1 \quad \text{and} \quad y_n(k_m+1) = \frac{1}{n}(m\cdot(-1) + (n-m)\cdot 1) = 1 - \frac{2m}{n}.$$

Hence $\|y_m - y_n\| \ge 2 - \frac{2m}{n}$. For any fixed m this value has limit 2 for $n \to \infty$, thus $\operatorname{cca}(x_{k_n}) \ge 2$. It follows that $\widetilde{\operatorname{cca}}(x_k) \ge 2$, so $\operatorname{bs}(B_c) \ge 2$.

Finally, it remains to show that $\operatorname{bs}(B_{c_0}) \le 1$. To do this let us fix a sequence (x_k) in B_{c_0}. Up to passing to a subsequence we may suppose that the sequence (x_k) pointwise converges to some $x \in B_{\ell^\infty}$. Fix an arbitrary $\varepsilon \in (0,1)$. We will construct increasing sequences of natural numbers (k_n) and (p_n) using the following inductive procedure:

- $k_1 = 1$;
- $|x_k(i)| < \varepsilon$ for $k \le k_n$ and $i \ge p_n$;
- $|x_k(i) - x(i)| < \varepsilon$ for $i \le p_n$ and $k \ge k_{n+1}$.

It is clear that this construction can be performed, using the facts that the points x_k belong to c_0 and that the sequence (x_k) pointwise converges to x. Let us consider the sequence (x_{k_n}) and set $y_n = \frac{1}{n}(x_{k_1} + \cdots + x_{k_n})$ for $n \in \mathbb{N}$. Fix an arbitrary $i \in \mathbb{N}$. Let $m \in \mathbb{N}$ be the minimal number such that $i \le p_m$. By the construction we have $\left|x_{k_n}(i)\right| < \varepsilon$ for $n < m$ and $\left|x_{k_n}(i) - x(i)\right| < \varepsilon$ for $n > m$, thus

$$\left|x_{k_n}(i) - \frac{x(i)}{2}\right| < \frac{1}{2}|x(i)| + \varepsilon \text{ for } n \in \mathbb{N} \setminus \{m\}.$$

It follows that for any $N \in \mathbb{N}$ we have

$$\begin{aligned} \left|y_N(i) - \frac{x(i)}{2}\right| &\le \frac{1}{N}\sum_{n=1}^{N}\left|x_{k_n}(i) - \frac{x(i)}{2}\right| \le \frac{1}{N}\left((N-1)\left(\frac{1}{2}|x(i)| + \varepsilon\right) + \frac{3}{2}\right) \\ &\le \frac{1}{2}|x(i)| + \varepsilon + \frac{3}{2N}. \end{aligned}$$

Therefore for any $M, N \in \mathbb{N}$ we have

$$\|y_N - y_M\| \le \|x\| + 2\varepsilon + \frac{3}{2N} + \frac{3}{2M},$$

so clearly $\operatorname{cca}(x_{n_k}) = \operatorname{ca}(y_k) \le \|x\| + 2\varepsilon$. Since $\varepsilon \in (0,1)$ is arbitrary, we get $\widetilde{\operatorname{cca}}(x_k) \le \|x\| \le 1$. This completes the proof. $\square$

6 Final remarks and open problems

It is natural to ask whether the inequalities in our results are optimal and which inequalities may become strict.

Let us start by Theorem 3.1.

- If $X = C[0,1]$ and $A = B_X$, then $\operatorname{wbs}(A) = \operatorname{bs}(A) = \beta(A) = 2$, hence we have equalities. Indeed, $\operatorname{wbs}(A) = 2$ by Theorem 5.1 and obviously $\beta(A) \leq 2$.
- If $X = \ell_1$ and $A = B_X$, then $\operatorname{wbs}(A) = 0$ by the Schur property of ℓ_1, obviously $\operatorname{wck}_X(A) \leq 1$ (in fact, $\operatorname{wck}_X(A) = 1$ since ℓ_1 is not reflexive) and $\operatorname{bs}(A) = \beta(A) = 2$ by Theorem 5.2, hence the first inequality is strict, the second one becomes equality.
- If $X = c_0$ and $A = B_X$, then $\operatorname{wbs}(A) = 0$ by [17], $\operatorname{wck}_X(A) = 1$ since c_0 is not reflexive (in this concrete case the equality can be verified directly by the use of the sequence (x_k) where $x_k = e_1 + \cdots + e_k$), $\operatorname{bs}(A) = 1$ by Theorem 5.2 and $\beta(A) = 2$ (the constant 2 is attained by the sequence $x_k = e_1 + \cdots + e_k - e_{k+1}$), hence the first inequality becomes equality and the second one is strict.

So, it seems that Theorem 3.1 is optimal.

Further, let us focus on Theorem 4.1. The first inequality may become equality – it is the case if $A = B_X$ by Theorem 5.1 and Lemma 5.3. However, we know no example when the first inequality is strict. The second inequality may become equality, for example if X is the Baernstein space from [6] and A is the canonical basis of X. In this case the last inequality is strict. We do not know any example when the second inequality is strict. So, we can ask the following question.

Question. *Let X be a Banach space and $A \subset X$ a bounded set. Is it necessarily true that*

$$\operatorname{wbs}(A) = 2\operatorname{sm}(A) = 2\operatorname{wus}(A) \quad ?$$

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III. Quantification of Pełczyński's property (V)

(Submitted to Studia Mathematica, preprint available at `http://arxiv.org/abs/1509.06610`*.)*

Abstract: A Banach space X has Pełczyński's property (V) if for every Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact. In 1962, Aleksander Pełczyński showed that $C(K)$ spaces for a compact Hausdorff space K enjoy the property (V), and some generalizations of this theorem have been proved since then. We introduce several possibilities of quantifying the property (V). We prove some characterizations of the introduced quantitative versions of this property, which allow us to prove a quantitative version of Pelczynski's result about $C(K)$ spaces and generalize it. Finally, we study the relationship of several properties of operators including weak compactness and unconditional convergence, and using the results obtained we establish a relation between quantitative versions of the property (V) and quantitative versions of other well known properties of Banach spaces.

1 Introduction

A Banach space X is said to have *Pełczyński's property (V)* if for every Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact. Recall that a linear operator $T\colon X \to Y$ is *weakly compact* if the image under T of the unit ball of X is a relatively weakly compact set in Y. We say that a bounded linear operator $T\colon X \to Y$ is *unconditionally converging* if $\sum_n T x_n$ is an unconditionally convergent series in Y whenever $\sum_n x_n$ is a weakly unconditionally Cauchy series in X.

Spaces known to enjoy the property (V) are for example $C(K)$ for a compact Hausdorff space K; this result from 1962 is due to A. Pełczyński [24]. Several generalizations of Pełczyński's theorem have been proved since then. W. B. Johnson and M. Zippin have shown that all real L^1 preduals have the property (V) (see [16]). H. Pfitzner has proved that all C^*-algebras enjoy it as well (see [26]).

The aim of this paper is to explore some possibilities of quantifying Pełczyński's property (V). Our inspiration comes from plenty of recently published quantitative results. Let us mention for example quantitative versions of Krein's theorem [10, 14, 12, 6], the Eberlein-Šmulyan and the Gantmacher theorem [2], James' compactness theorem [7, 13], weak sequential continuity and the Schur property [19, 20], the Dunford-Pettis [18] and the reciprocal Dunford-Pettis property [21], the Grothendieck property [4], and the Banach-Saks property [5].

The main idea of quantifying an existing qualitative result is simple – to replace an implication by an inequality. In case of the property (V) we will attempt to replace the implication

$$T \text{ is unconditionally converging } \Rightarrow T \text{ is weakly compact} \tag{1}$$

by an inequality

$$\begin{aligned}&\text{measure of weak non-compactness of } T\\ &\qquad\qquad \le C \cdot \text{measure of } T \text{ not being unconditionally converging,}\end{aligned}$$

where C is some positive constant depending only on X. These two measures should be positive numbers for each operator T and should equal zero if and only if T is weakly compact or unconditionally converging, respectively. This inequality then trivially includes the original implication, but it says even more.

In Section 2 we explain how to define the above mentioned measures and we introduce a quantitative version of the property (V). Section 3 is devoted to characterizations of a quantitative version of the property (V). Using these characterizations, in Section 4 we prove quantitative versions of the above-mentioned theorem of Pełczyński and that of Johnson and Zippin. Section 5 describes a relationship of various properties of operators including weak compactness and unconditional convergence. These relationships are quantified, which enables us to establish a relation between a quantitative version of the property (V) and quantitative versions of some other well known properties of Banach spaces.

Throughout the paper, all Banach spaces can be considered either real or complex (most of the results are valid in both cases), unless stated otherwise. By an operator we always mean a bounded linear operator. If X is a Banach space, we denote by B_X its closed unit ball $\{x \in X : \|x\| \leq 1\}$ and by U_X its open unit ball $\{x \in X : \|x\| < 1\}$. Every Banach space X is considered canonically embedded into its bidual X^{**}.

2 Quantification of Pełczyński's property (V)

In this section we remind the definition of the property (V). Then we define a few related quantities, which allow us to quantify the property (V). We first focus on a quantity which measures how far is an operator from being unconditionally converging. Then we remind some well known measures of weak non-compactness of sets and operators. Eventually, we introduce a quantitative version of the property (V).

2.1 Unconditionally converging operators and related quantities

Definition. A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is

- *unconditionally convergent* if the series $\sum_{n=1}^{\infty} t_n x_n$ converges whenever (t_n) is a bounded sequence of scalars,
- *weakly unconditionally Cauchy* (*wuC* for short) if for all $x^* \in X^*$ the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ converges.

Definition. Let X, Y be Banach spaces. An operator $T\colon X \to Y$ is *unconditionally converging (uc)* if for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X the series $\sum_{n=1}^{\infty} T x_n$ is unconditionally convergent.

It is easy to see that an operator T is unconditionally converging if and only if for every weakly unconditionally Cauchy series $\sum x_n$ in X the series $\sum T x_n$ is convergent. Indeed, the "only if implication" is trivial since every unconditionally convergent series is convergent. Suppose that T sends wuC series to convergent series. If $\sum x_n$ is a wuC series in X and (t_n) is a bounded sequence of scalars, then $\sum t_n x_n$ is also wuC and hence $\sum t_n x_n$ converges. Therefore T is uc.

Let (x_n) be a bounded sequence in a Banach space X. Set

$$\operatorname{ca}((x_n)) = \inf_{n \in \mathbb{N}} \sup\{\|x_k - x_l\| : k, l \in \mathbb{N}, k, l \geq n\}.$$

This quantity is a measure of non-cauchyness of the sequence (x_n). More precisely, $\operatorname{ca}((x_n))$ is a positive number for every bounded sequence (x_n) and it is equal to zero if and only if (x_n) is Cauchy. Since we deal with Banach spaces only, the quantity ca measures non-convergence of sequences.

We are now prepared to define a quantity which measures how far is an operator T from being unconditionally converging. Let $T: X \to Y$ be an operator between Banach spaces X and Y. We set

$$\operatorname{uc}(T) = \sup\left\{\operatorname{ca}\left(\left(\sum_{i=1}^{n} Tx_i\right)_n\right) : \ (x_n) \subset X, \ \sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |x^*(x_n)| \leq 1\right\}.$$

Clearly, $\operatorname{uc}(T) = 0$ provided T is unconditionally converging. On the other hand, if $\sum x_n$ is a wuC series in X, then the sets

$$M_k = \left\{x^* \in X^* : \ \sum_{n=1}^{\infty} |x^*(x_n)| \leq k\right\}, \ k \in \mathbb{N},$$

are closed, and $\bigcup_{k=1}^{\infty} M_k = X^*$. If we use Baire's theorem, it is not difficult to find a constant $C > 0$ such that $\sum_{n=1}^{\infty} |x^*(x_n)| \leq C$ for all $x^* \in B_{X^*}$. From this we see that $\operatorname{uc}(T) = 0$ if and only if T is unconditionally converging.

2.2 Measuring non-compactness and weak non-compactness of sets and operators

We will use the following notation. For A, B subsets of a Banach space X we set

$$\begin{aligned}
\operatorname{dist}(a, B) &= \inf\{\|a - b\| : \ a \in A, \ b \in B\}, \\
\hat{\mathrm{d}}(A, B) &= \sup\{\operatorname{dist}(a, B) : \ a \in A\}.
\end{aligned}$$

The former is the ordinary distance between the sets A and B, the latter is the non-symetrized Hausdorff distance from A to B.

Let A be a bounded subset of a Banach space X. The Hausdorff measure of non-compactness of the set A is defined by

$$\begin{aligned}
\chi(A) &= \inf\{\hat{\mathrm{d}}(A, K) : \emptyset \neq K \subset X \text{ is compact}\} \\
&= \inf\{\hat{\mathrm{d}}(A, F) : \emptyset \neq F \subset X \text{ is finite}\}.
\end{aligned}$$

It is easy to see that $\chi(A) = 0$ if and only if the set A is relatively compact.

There are many ways of measuring weak non-compactness. The de Blasi measure of weak non-compactness of the set A, which is an analogue of the Hausdorff measure of non-compactness, is defined by

$$\omega(A) = \inf\{\hat{\mathrm{d}}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact}\}.$$

Clearly, $\omega(A) = 0$ for any relatively weakly compact set A. De Blasi has proved (see [8]) that $\omega(A) = 0$ if and only if A is relatively weakly compact. For every bounded subset A of a Banach space X the inequality

$$\omega(A) \leq \chi(A) \tag{2}$$

trivially holds.

Other most commonly used quantities measuring weak non-compactness are

$$\begin{aligned}
\mathrm{wk}_X(A) &= \hat{\mathrm{d}}(\overline{A}^{w^*}, X),\\
\mathrm{wck}_X(A) &= \sup\{\mathrm{dist}(\mathrm{clust}_{w^*}(x_n), X) :\ (x_n) \text{ is a sequence in } A\},\\
\gamma(A) &= \sup\{|\lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n)| :\ (x_n) \text{ is a sequence in } A,\\
&\qquad\qquad (x_m^*) \text{ is a sequence in } B_{X^*},\ \text{ and the limits exist}\}.
\end{aligned}$$

Here $\overline{A}^{w^*}$ stands for the weak* closure of the set A in the bidual space X^{**} and $\mathrm{clust}_{w^*}(x_n)$ is the set of all weak* cluster points of the sequence (x_n) in X^{**}. The quantity wck_X is related to the Eberlein-Šmulyan theorem and the quantity γ to the Grothendieck double limit criterion for weak compactness.

The above defined quantities are studied for example in [2] and the following relationships between them are proved there [2, Theorem 2.3]. For every bounded subset A of a Banach space X

$$\mathrm{wck}_X(A) \leq \mathrm{wk}_X(A) \leq \gamma(A) \leq 2\,\mathrm{wck}_X(A), \tag{3}$$

$$\mathrm{wk}_X(A) \leq \omega(A). \tag{4}$$

Moreover, all these quantities are measures of weak non-compactness in the sense that they are equal to zero if and only if the set A is relatively weakly compact. The estimates (3) say that the measures wk_X, wck_X, and γ are equivalent. The quantity ω is, however, not equivalent to the other three (see [2, Corollary 3.4]), i.e. a Banach space X exists such that there is no constant C satisfying for every bounded $A \subset X$ the inequality $\omega(A) \leq C\,\mathrm{wk}_X(A)$.

An operator $T\colon X \to Y$ between Banach space X and Y is *weakly compact* if the image $T(B_X)$ of the unit ball of X under T is relatively weakly compact. A natural way to measure how far is an operator $T\colon X \to Y$ from being weakly compact is to measure weak non-compactness of $T(B_X)$. We do it using the above defined measures of weak non-compactness of sets. Let us denote $\omega(T(B_X))$ simply by $\omega(T)$. Analogously $\gamma(T)$, $\mathrm{wk}_Y(T)$, and $\mathrm{wck}_Y(T)$ stand for $\gamma(T(B_X))$, $\mathrm{wk}_Y(T(B_X))$, and $\mathrm{wck}_Y(T(B_X))$, respectively.

The Gantmacher theorem states that an operator $T\colon X \to Y$ between Banach spaces X and Y is weakly compact if and only if the dual operator $T^*\colon Y^* \to X^*$ is weakly compact. This theorem has a quantitative version [2, Theorem 3.1]. It says that for any operator T

$$\gamma(T) \leq \gamma(T^*) \leq 2\gamma(T). \tag{5}$$

The analogous result with the quantity ω in place of γ does not hold (see [3, Theorem 4]).

2.3 Quantitative version of Pełczyński's property (V)

Definition. Let X be a Banach space. We say that X has *Pełczyński's property (V)* if for every Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact.

The property (V) can be now quantified as follows.

Definition. We say that a Banach space X has a quantitative version of Pełczyński's property (V) – let us denote it by (V_q) – if there exists a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\gamma(T) \leq C \cdot \mathrm{uc}(T). \tag{6}$$

If X has a quantitative version of Pełczyński's property (V_q), then it also enjoy the original qualitative property (V). Indeed, for any uc operator T we have $\mathrm{uc}(T) = 0$, hence $\gamma(T) = 0$ which means that T is weakly compact.

One may ask what would happen if we use a different measure of weak non-compactness in (6). By replacing γ with wk_X or wck_X we achieve nothing new since these quantities are equivalent. However, if we use ω instead of γ, we obtain a stronger assertion. Proposition 4.3 (ii) shows that this quantification is really different.

Definition. We say that a Banach space X has the property $(V_q)_\omega$ if there exists a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\omega(T) \leq C \cdot \mathrm{uc}(T).$$

There are other possibilities of quantifying the property (V). As we will see later, it sometimes seems to be more natural to quantify the inequality

$$T \text{ is uc} \;\Rightarrow\; T^* \text{ is weakly compact}$$

which is equivalent to (1) by Gantmacher's theorem.

Definition. We say that a Banach space X has the property $(V_q)^*_\omega$ if there exists a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\omega(T^*) \leq C \cdot \mathrm{uc}(T).$$

Here we have no choice concerning the measure of weak non-compactness. If we used $\gamma(T^*)$ in place of $\omega(T^*)$, it would only yield a reformulation of the property (V_q) by the quantitative Gantmacher theorem (5).

3 Characterizations of a quantitative Pełczyński's property (V)

Pełczyński's property (V) has multiple different characterizations. It turns out that some of these characterizations can also be quantified. We will show that their quantitative versions are equivalent to a quantitative version of Pełczyński's property (V).

3.1 Characterization through subsets of the dual space

Proposition 3.1. *Let X be a Banach space. The following assertions are equivalent.*

1. *X has Pełczyński's property (V).*
2. *Every $K \subset X^*$ which satisfies the condition $(*)$ below is relatively weakly compact.*

$$(*) \qquad \lim_{n\to\infty} \sup_{x^*\in K} |x^*(x_n)| = 0 \text{ for every wuC series } \sum_{n=1}^{\infty} x_n \text{ in } X.$$

This proposition, proven by Pełczyński [24, Proposition 1], has its quantitative analogue. We have already explained in the previous section how to reformulate the former assertion quantitatively. We now define a quantity which is essential for quantifying the latter one, and then we prove that also quantitative versions of the assertions (1) and (2) are equivalent.

Let X be a Banach space and K be a bounded subset of X^*. We set

$$\eta(K) = \sup\Big\{\limsup_n \sup_{x^*\in K} |x^*(x_n)| : (x_n) \subset X, \sup_{x^*\in B_{X^*}} \sum_{n=1}^{\infty} |x^*(x_n)| \le 1\Big\}.$$

This quantity measures to what extent K fails to satisfy the condition $(*)$ from the Proposition 3.1 (2). Obviously, $\eta(K)$ is positive for every bounded $K \subset X^*$ and equals zero if and only if K satisfies the condition $(*)$.

Proposition 3.2. *Let X be a Banach space. The following assertions are equivalent.*

(1_q) X has the property (V_q), i.e. there exists $C > 0$ such that for any Banach space Y and any operator $T : X \to Y$

$$\gamma(T) \le C \cdot \mathrm{uc}(T).$$

($1'_q$) There exists $C > 0$ such that for every operator $T : X \to \ell^\infty$

$$\gamma(T) \le C \cdot \mathrm{uc}(T).$$

(2_q) There exists $C > 0$ such that for each bounded $K \subset X^*$

$$\gamma(K) \le C \cdot \eta(K).$$

This proposition follows immediately from the next one and the quantitative version of Gantmacher's theorem (5). The preceding and the following proposition are much alike, in the latter one $\gamma(T)$ is replaced by $\gamma(T^*)$. Then the three assertions are equivalent "with the same constant" $C > 0$. Thus the quantification of the property (V) of the form

$$\gamma(T^*) \le C \cdot \mathrm{uc}(T)$$

seems to be more natural here.

Proposition 3.3. *Let X be a Banach space and $C > 0$. The following assertions are equivalent.*

$(1_q)_C$ For any Banach space Y and any operator $T : X \to Y$

$$\gamma(T^*) \le C \cdot \mathrm{uc}(T).$$

$(1'_q)_C$ For every operator $T : X \to \ell^\infty$

$$\gamma(T^*) \le C \cdot \mathrm{uc}(T).$$

$(2_q)_C$ For each bounded $K \subset X^*$

$$\gamma(K) \le C \cdot \eta(K).$$

Proof. We follow Pełczyński's original proof [24, Prop. 1], it only needs to be done more carefully. The implication $(1_q)_C \Rightarrow (1'_q)_C$ is obvious.

Let us prove $(1'_q)_C \Rightarrow (2_q)_C$. Let K be a bounded subset of X^* and $\delta < \gamma(K)$. From the definition of γ it is easily seen that a sequence (x_n^*) in K exists such that $\gamma(\{x_n^* : n \in \mathbb{N}\}) > \delta$. Let us define $T\colon X \to \ell^\infty$ by $T(x)(n) = x_n^*(x)$, $n \in \mathbb{N}$, $x \in X$. For each $n \in \mathbb{N}$ set $p_n((a_k)) = a_n$, $(a_k) \in \ell^\infty$. Then $p_n \in (\ell^\infty)^*$, $\|p_n\| = 1$. Moreover, $T^* p_n = x_n^*$, because for $x \in X$ we have $T^* p_n(x) = p_n(Tx) = x_n^*(x)$. Thus

$$\gamma(T^*) = \gamma(T(B_{(\ell^\infty)^*}) \geq \gamma(\{T^* p_n : n \in \mathbb{N}\}) = \gamma(\{x_n^* : n \in \mathbb{N}\}) > \delta.$$

From $(1'_q)_C$ it follows that $\operatorname{uc}(T) > \frac{\delta}{C}$. By the definition of the quantity uc there is a wuC series $\sum x_n$ in X with $\sup_{x^* \in B_{X^*}} \sum |x^*(x_n)| \leq 1$ such that $\operatorname{ca}\Big(\big(\sum_{i=1}^n Tx_i\big)_n\Big) > \frac{\delta}{C}$. The definition of ca gives indices $k_1 < l_1 < k_2 < l_2 < \dots$ such that for each $n \in \mathbb{N}$

$$\tag{7} \begin{aligned} \frac{\delta}{C} &< \left\| \sum_{i=k_n}^{l_n} Tx_i \right\|_{\ell^\infty} = \sup_{m \in \mathbb{N}} \left| \sum_{i=k_n}^{l_n} T(x_i)(m) \right| \\ &= \sup_{m \in \mathbb{N}} \left| \sum_{i=k_n}^{l_n} x_m^*(x_i) \right| \leq \sup_{x^* \in K} \left| \sum_{i=k_n}^{l_n} x^*(x_i) \right|. \end{aligned}$$

Let us define $\widetilde{x}_n = \sum_{i=k_n}^{l_n} x_i$, $n \in \mathbb{N}$. Then the series $\sum_n \widetilde{x}_n$ is wuC since $\sum_i x_i$ is wuC and for every $x^* \in X^*$

$$\sum_{n=1}^\infty |x^*(\widetilde{x}_n)| = \sum_{n=1}^\infty \left| \sum_{i=k_n}^{l_n} x^*(x_i) \right| \leq \sum_{n=1}^\infty \sum_{i=k_n}^{l_n} |x^*(x_i)| \leq \sum_{i=1}^\infty |x^*(x_i)|.$$

Moreover,

$$\sup_{x^* \in B_{X^*}} \sum_{n=1}^\infty |x^*(\widetilde{x}_n)| \leq \sup_{x^* \in B_{X^*}} \sum_{i=1}^\infty |x^*(x_i)| \leq 1.$$

From (7) we have for each $n \in \mathbb{N}$

$$\sup_{x^* \in K} |x^*(\widetilde{x}_n)| = \sup_{x^* \in K} \left| \sum_{i=k_n}^{l_n} x^*(x_i) \right| > \frac{\delta}{C},$$

and so

$$\limsup_{n \in \mathbb{N}} \sup_{x^* \in K} x^*(\widetilde{x}_n) > \frac{\delta}{C}.$$

Hence $\eta(K) > \frac{\delta}{C}$. As $\delta < \gamma(K)$ has been chosen arbitrarily, we obtain $\gamma(K) \leq C \cdot \eta(K)$.

It remains to prove the implication $(2_q)_C \Rightarrow (1_q)_C$. Let Y be a Banach space and $T\colon X \to Y$ an operator. Let us fix $\delta < \gamma(T^*) = \gamma(T^*(B_{Y^*}))$. Set $K = \gamma(T^*(B_{Y^*}))$. Then K is a bounded subset of X^* and from $(2_q)_C$ we have $C \cdot \eta(K) \geq \gamma(K) > \delta$. By the definition of η there is a wuC series $\sum x_n$ in X with $\sup_{x^* \in B_{X^*}} \sum |x^*(x_n)| \leq 1$ such that

$$\begin{aligned} \frac{\delta}{C} &< \limsup_n \sup_{x^* \in K} |x^*(x_n)| = \limsup_n \sup_{y^* \in B_{Y^*}} |T^* y^*(x_n)| \\ &= \limsup_n \sup_{y^* \in B_{Y^*}} |y^*(Tx_n)| = \limsup_n \|Tx_n\|. \end{aligned}$$

Thus $\operatorname{ca}\Big(\big(\sum_{i=1}^n Tx_i\big)_n\Big) > \frac{\delta}{C}$ and hence $\operatorname{uc}(T) \geq \frac{\delta}{C}$. Since $\delta < \gamma(T^*)$ is arbitrary, we conclude that $\gamma(T^*) \leq C \cdot \operatorname{uc}(T)$. $\square$

The following proposition provides an analogous characterization of the property $(\mathrm{V}_q)^*_\omega$.

Proposition 3.4. *Let X be a Banach space and $C > 0$. The following assertions are equivalent.*

$(1_q^\omega)_C$ For any Banach space Y and any bounded linear operator $T\colon X \to Y$

$$\omega(T^*) \leq C \cdot \mathrm{uc}(T).$$

$(2_q^\omega)_C$ For each bounded $K \subset X^*$

$$\omega(K) \leq C \cdot \eta(K).$$

Proof. This proposition has the "same" proof as the previous one. The implication $(2_q^\omega)_C \Rightarrow (1_q^\omega)_C$ can be proven exactly the same way, we simply substitute ω for γ.

As for the converse implication, suppose that $(1_q^\omega)_C$ holds, and let K be a bounded subset of X^* and $\delta < \omega(K)$. Let us define $T\colon X \to \ell^\infty(K)$ by $Tx(x^*) = x^*(x)$, $x^* \in K$, $x \in X$. For each $x^* \in K$ set $F_{x^*}(f) = f(x^*)$, $f \in \ell^\infty(K)$. Then $F_{x^*} \in (\ell^\infty(K))^*$, $\|F_{x^*}\| = 1$, and $T^*F_{x^*} = x^*$, $x^* \in X^*$. Hence

$$\omega(T^*) = \omega(T^*(B_{(\ell^\infty(K))^*}) \geq \omega(\{T^*F_{x^*} : x^* \in K\}) = \omega(K) > \delta.$$

By $(1_q^\omega)_C$, $\mathrm{uc}(T) > \frac{\delta}{C}$. We then continue just as in the proof of the implication $(1'_q)_C \Rightarrow (2_q)_C$ in the previous proposition to get $(2_q^\omega)_C$. □

From the estimates (3) and (4) it follows that if some Banach space X satisfies the condition $(1_q^\omega)_C$ from the previous proposition 3.4, then it also satisfies the condition $(1_q)_{2C}$ from Proposition 3.3.

Propositions 3.2, 3.3, and 3.4 characterize only the properties (V_q) and $(\mathrm{V}_q)^*_\omega$. We do not have a similar characterization of the property $(\mathrm{V}_q)_\omega$.

3.2 Characterization of uc operators and its consequence

The following theorem is a well known characterization of unconditionally converging operators due to Pełczyński (see e.g. [9, p. 54, Exercise 8]). It gives rise to another characterization of the property (V). Pełczyński's result has its quantitative version (Theorem 3.6 below), which yields another characterization of a quantitative version of the property (V).

Theorem 3.5. *Let X, Y be Banach spaces and $T\colon X \to Y$ an operator. Then T is unconditionally converging if and only if it does not fix any copy of c_0, i.e. there is no subspace $X_0 \subset X$ isomorphic to c_0 such that $T\restriction_{X_0}$ is an isomorphism.*

To quantify this proposition we will need the quantity fix_{c_0} which measures the failure of the condition that T does not fix a copy of c_0. For a bounded linear operator $T\colon X \to Y$ we set

$$\mathrm{fix}_{c_0}(T) = \sup\Big\{(\|U\|\|V\|)^{-1} : \begin{array}{l} \exists X_0 \subset X \text{ such that } T\restriction_{X_0} \text{ is an isomorphism} \\ \text{onto } T(X_0)\text{, and } (T\restriction_{X_0})^{-1} = U \circ V \text{ for some} \\ \text{onto isomorphisms } U\colon c_0 \to X_0,\ V\colon T(X_0) \to c_0 \end{array}\Big\}.$$

If the set on the right is empty, we set $\mathrm{fix}_{c_0}(T) = 0$. This happens if and only if T does not fix a copy of c_0, for otherwise the set contains $(\|U\|\|V\|)^{-1}$, where $U\colon c_0 \to X_0 \subset X$ is an isomorphism onto X_0 such that $T \restriction_{X_0}$ is an isomorphism, and $V = (T \circ U)^{-1}$.

Let us explain why may this quantity be considered a measure of the failure of the condition that T does not fix a copy of c_0. First of all, note that $\mathrm{fix}_{c_0}(cT) = c\ \mathrm{fix}_{c_0}(T)$, $c > 0$. This is important, for we need fix_{c_0} to be positively homogeneous like all the other quantities that we use. Now, suppose that T is an operator of norm 1 which fixes a copy of c_0. Let X_0 be a subspace of X isomorphic to c_0 such that $T \restriction_{X_0}$ is an isomorphism onto $T(X_0) \subset Y$. If we wanted to measure how "nice" is this isomorphism, we would have to take a closer look at $\|(T \restriction_{X_0})^{-1}\|$. If it equals 1, then $T \restriction_{X_0}$ is an isometry. The greater is $\|(T \restriction_{X_0})^{-1}\|$, the more "deforming" is the isomorphism $T \restriction_{X_0}$. We thus see that $\|(T \restriction_{X_0})^{-1}\|^{-1}$ is a natural measure of "niceness" of $T \restriction_{X_0}$. In our case, we would like to measure how nice is the isomorphism $T \restriction_{X_0}$ and how nice copy of c_0 is X_0 in X simultaneously. The operator $(T \restriction_{X_0})^{-1}$ factors through c_0 in a way that there are isomorphisms U and V like in the definition of fix_{c_0} such that $(T \restriction_{X_0})^{-1} = U \circ V$. So we replace $\|(T \restriction_{X_0})^{-1}\|$ by $\|U\|\|V\|$. The quantity $(\|U\|\|V\|)^{-1}$ not only measures "niceness" of $U \circ V$, but it also takes into account the isomorphism $U\colon c_0 \to X_0$ itself. Eventually, the supremum over all suitable X_0, U and V is taken to measure how nicest an isomorphism on some nice copy of c_0 can we get.

The following theorem is a quantitative version of Theorem 3.5. Both implications of the equivalence are replaced by inequalities between relevant measures.

Theorem 3.6. *Let X be a Banach space. For every Banach space Y and every bounded linear operator $T\colon X \to Y$*

$$\frac{1}{2}\,\mathrm{uc}(T) \leq \mathrm{fix}_{c_0}(T) \leq \mathrm{uc}(T).$$

Proof. Let us start with the second inequality. If $\mathrm{fix}_{c_0}(T) = 0$, it holds trivially. Suppose that $\mathrm{fix}_{c_0}(T) > 0$, i.e. T fixes a copy of c_0. Take X_0 a subspace of X isomorphic to c_0 and $U\colon c_0 \to X_0$, $V\colon T(X_0) \to c_0$ onto isomorphisms which satisfy $(T \restriction_{X_0})^{-1} = U \circ V$. Is it enough to show that $\mathrm{uc}(T) \geq (\|U\|\|V\|)^{-1}$.

For the series $\sum e_n$ in c_0 we have

$$\sup_{x^* \in B_{(c_0)^*}} \sum |x^*(e_n)| = \sup_{(a_n) \in B_{\ell^1}} \sum |a_n| = 1$$

and $\mathrm{ca}\Big(\big(\sum_{i=1}^n e_i\big)_n\Big) = 1$. Set $f_n = \frac{1}{\|U\|} U e_n$, $n \in \mathbb{N}$. Then $\sum f_n$ is a wuC series in $X_0 \subset X$, since $\sum e_n$ is wuC and U is continuous. We have even

$$\sup_{x^* \in B_{X^*}} \sum |x^*(f_n)| = \sup_{x^* \in B_{X^*}} \sum \left| \Big(\tfrac{1}{\|U\|} x^* \circ U\Big)(e_n) \right| \leq \sup_{y^* \in B_{(c_0)^*}} \sum |y^*(e_n)| = 1.$$

Moreover,

$$\begin{aligned}
\mathrm{ca}\left(\left(\sum_{i=1}^{n} Tf_i\right)_n\right) &= \inf_{n\in\mathbb{N}} \sup_{k>l\geq n} \left\| \sum_{i=l+1}^{k} T\left(\tfrac{1}{\|U\|} Ue_i\right)\right\| \\
&= \frac{1}{\|U\|\|V\|} \inf_{n\in\mathbb{N}} \sup_{k>l\geq n} \|(T\circ U)^{-1}\| \left\| \sum_{i=l+1}^{k} (T\circ U)e_i \right\| \\
&\geq (\|U\|\|V\|)^{-1} \inf_{n\in\mathbb{N}} \sup_{k>l\geq n} \left\| \sum_{i=l+1}^{k} e_i \right\| \\
&= (\|U\|\|V\|)^{-1}\, \mathrm{ca}\left(\left(\sum_{i=1}^{n} e_i\right)_n\right) \\
&= (\|U\|\|V\|)^{-1}.
\end{aligned}$$

It follows that $\mathrm{uc}(T) \geq (\|U\|\|V\|)^{-1}$, which is what we need.

We proceed to show the inequality $\mathrm{uc}(T) \leq 2\, \mathrm{fix}_{c_0}(T)$. It is trivial if $\mathrm{uc}(T) = 0$. Suppose that $\mathrm{uc}(T) > 0$ and fix $0 < \delta < \mathrm{uc}(T)$. First we find $\varepsilon > 0$ satisfying $\mathrm{uc}(T) > \delta(1+\varepsilon)$, and we set $\delta' = \delta(1+\varepsilon)$. The definition of $\mathrm{uc}(T)$ gives a wuC series $\sum x_n$ in X with $\sup_{x^*\in B_{X^*}} \sum |x^*(x_n)| \leq 1$ such that $\mathrm{ca}\left(\left(\sum_{i=1}^{n} Tx_i\right)_n\right) > \delta'$. By the definition of the quantity ca we find indices $k_1 < l_1 < k_2 < l_2 < \dots$ such that $\left\|\sum_{i=k_n}^{l_n} Tx_i\right\| > \delta'$, $n\in\mathbb{N}$. Let us set $\widetilde{x}_n = \sum_{i=k_n}^{l_n} x_i$, $n\in\mathbb{N}$. Then $\sum \widetilde{x}_n$ is a wuC series in X with

$$\sup_{x^*\in B_{X^*}} \sum_{n=1}^{\infty} |x^*(\widetilde{x}_n)| \leq \sup_{x^*\in B_{X^*}} \sum_{n=1}^{\infty} |x^*(x_n)| \leq 1.$$

For each $n\in\mathbb{N}$ we have $\|T\widetilde{x}_n\| > \delta'$, and so $\|\widetilde{x}_n\| > \frac{\delta'}{\|T\|} > 0$. The series $\sum \widetilde{x}_n$ is wuC and therefore $\widetilde{x}_n \to 0$ weakly. By [1, Proposition 1.5.4], there is a subsequence $(\widetilde{x}_{n_k})$ of $(\widetilde{x}_n)$ which is basic. Since $T\widetilde{x}_{n_k} \to 0$ weakly by the continuity of T, and $\inf\{\|T\widetilde{x}_{n_k}\| : k\in\mathbb{N}\} \geq \delta' > 0$, we can use theorem [1, Proposition 1.5.4] again to obtain a subsequence (z_m) of $(\widetilde{x}_{n_k})$ such that (Tz_m) is a basic sequence in Y with a basic constant $\mathrm{bc}(Tz_m) < 1+\varepsilon$.

Since (z_n) is a basic sequence in X for which $\sum z_n$ is wuC, and $\inf\{\|z_n\| : n\in\mathbb{N}\} > 0$, (z_n) is equivalent to the canonical basis of c_0 by [23, Theorem 6.6]. For the same reason the sequence (Tz_n) in Y is also equivalent to the canonical basis of c_0. Hence both $\overline{\mathrm{span}}\{z_n : n\in\mathbb{N}\}$ and $\overline{\mathrm{span}}\{Tz_n : n\in\mathbb{N}\}$ are isomorphic to c_0, and $T\restriction_{\overline{\mathrm{span}}\{z_n:\, n\in\mathbb{N}\}}$ is an isomorphism onto $\overline{\mathrm{span}}\{Tz_n : n\in\mathbb{N}\}$.

Let us set $X_0 = \overline{\mathrm{span}}\{z_n : n\in\mathbb{N}\}$ and define $U\colon c_0 \to X_0$ by $U(e_n) = z_n$, $n\in\mathbb{N}$. Then U is an onto isomorphism. Further, set $V = (T\circ U)^{-1}$. We will prove that $(\|U\|\|V\|)^{-1} \geq \frac{\delta}{2}$. For $(a_n)\in c_0$ we have

$$\begin{aligned}
\left\|U((a_n))\right\| &= \left\|\sum_{n=1}^{\infty} a_n z_n\right\| = \sup_{x^*\in B_{X^*}} \left| x^*\left(\sum_{n=1}^{\infty} a_n z_n\right)\right| \\
&\leq \sup_{x^*\in B_{X^*}} \sum_{n=1}^{\infty} |a_n||x^*(z_n)| \leq \sup_{n\in\mathbb{N}} |a_n| \sup_{x^*\in B_{X^*}} \sum_{n=1}^{\infty} |x^*(z_n)| \\
&\leq \|(a_n)\| \sup_{x^*\in B_{X^*}} \sum_{i=1}^{\infty} |x^*(\widetilde{x}_i)| \leq \|(a_n)\|,
\end{aligned}$$

and hence $\|U\| \leq 1$. If $(a_n) \in c_0$, we also have for each $n \in \mathbb{N}$

$$\begin{aligned}
\delta' |a_n| &\leq |a_n| \|Tz_n\| = \|a_n T z_n\| = \left\| \sum_{i=1}^{n} a_i T z_i - \sum_{i=1}^{n-1} a_i T z_i \right\| \\
&\leq \left\| \sum_{i=1}^{n} a_i T z_i \right\| + \left\| \sum_{i=1}^{n-1} a_i T z_i \right\| \leq 2\,\mathrm{bc}(Tz_k) \left\| \sum_{k=1}^{\infty} a_k T z_k \right\| \\
&= 2\,\mathrm{bc}(Tz_k) \left\| (T \circ U)((a_k)) \right\|,
\end{aligned}$$

which gives

$$\begin{aligned}
\|(a_n)\| = \sup_{n \in \mathbb{N}} |a_n| &\leq \frac{2\,\mathrm{bc}(Tz_n)}{\delta'} \left\| (T \circ U)((a_n)) \right\| \\
&\leq \frac{2(1+\varepsilon)}{\delta(1+\varepsilon)} \left\| (T \circ U)((a_n)) \right\| = \frac{2}{\delta} \left\| (T \circ U)((a_n)) \right\|.
\end{aligned}$$

Hence $\|V\| = \left\| (T \circ U)^{-1} \right\| \leq \frac{2}{\delta}$, and we thus obtain $(\|U\|\|V\|)^{-1} \geq 1 \cdot \frac{\delta}{2} = \frac{\delta}{2}$. Consequently, $\mathrm{fix}_{c_0}(T) \geq \frac{\delta}{2}$. This yields the desired inequality $\mathrm{uc}(T) \leq 2\ \mathrm{fix}_{c_0}(T)$. □

4 Quantitative version of Pełczyński's theorem and its generalizations

A theorem of A. Pełczyński from 1962 asserts that the space $C(K)$ of continuous real functions on a compact Hausdorff space K has the property (V) (see [24, Theorem 1]). Using a characterization of a quantitative Pełczyński's property (V) from the section 3 we prove a quantitative strengthening of this theorem. The proof is inspired by Pełczyński's original proof and it uses some results of [21].

Theorem 4.1. *Let Ω be a locally compact space. Then the space $C_0(\Omega)$ enjoys the quantitative property $(V_q)^*_\omega$ (and hence also (V_q)). More precisely, for every Banach space Y and every operator $T\colon C_0(\Omega) \to Y$*

$$\omega(T^*) \leq \pi\,\mathrm{uc}(T).$$

In the real case (i.e. if $C_0(\Omega)$ are real functions) the constant π can be replaced by 2*.*

Remark. It might seem that the quantification with ω in this theorem is stronger than the quantification through the inequality $\gamma(T^*) \leq C\ \mathrm{uc}(T)$ (which is equivalent to (V_q)), but it is not. In fact, by [18, Theorem 7.5] the quantities ω, wk_X, and wck_X coincide on $\mathcal{M}(\Omega)$. Therefore the properties (V_q) and $(\mathrm{V}_q)^*_\omega$ are equivalent for $C_0(\Omega)$.

Proof. Throughout the proof we identify the dual of $C_0(\Omega)$ with the space $\mathcal{M}(\Omega)$ of all finite complex (or signed in the real case) Radon measures on Ω. By Proposition 3.4 it suffices to show that for every $K \subset (C_0(\Omega))^* = \mathcal{M}(\Omega)$ bounded $\omega(K) \leq \pi\eta(K)$. Let K be a bounded subset of $\mathcal{M}(\Omega)$. From [21, Proposition 5.2] it follows that

$$\frac{1}{\pi}\omega(K) \leq \sup \left\{ \limsup_{k \to \infty} \sup_{\mu \in K} |\mu(U_k)| :\ U_k \subset \Omega,\ k \in \mathbb{N},\ \text{pairwise disjoint, open} \right\}$$

(in the real case $\frac{1}{\pi}$ can be replaced by $\frac{1}{2}$).

Let us fix an arbitrary $\delta < \omega(K)$. Using the above inequality we find a sequence (U_n) of pairwise disjoint open subsets of Ω and a sequence (μ_n) in K such that $|\mu_n(U_n)| > \frac{\delta}{\pi}$. For each $n \in \mathbb{N}$ we find a continuous function f_n on Ω with a compact support such that $\|f_n\| = 1$, $f_n = 0$ outside U_n, and

$$|\mu_n(f_n)| = \left|\int_\Omega f_n \mathrm{d}\mu_n\right| > \frac{\delta}{\pi}. \tag{8}$$

Then for every $\mu \in (C_0(\Omega))^*$ and $n \in \mathbb{N}$

$$\sum_{i=1}^{n} |\mu(f_i)| \leq \sum_{i=1}^{n} |\mu|(|f_i|) = |\mu|\left(\sum_{i=1}^{n} |f_i|\right) \leq |\mu|(1) = \|\mu\|,$$

hence $\sum f_n$ is a wuC series in $C_0(\Omega)$, and $\sup_{\mu\in B_{(C_0(\Omega))^*}} \sum_{i=1}^{\infty} |\mu(f_i)| \leq 1$. By (8) we have

$$\limsup_n \sup_{\mu\in K} \left|\int f_n \mathrm{d}\mu\right| \geq \limsup_n \left|\int f_n \mathrm{d}\mu_n\right| \geq \frac{\delta}{\pi}.$$

From this we conclude that $\eta(K) \geq \frac{\delta}{\pi}$, and since $\delta < \omega(K)$ has been chosen arbitrarily, $\omega(K) \leq \pi\eta(K)$. In the real case we obtain the similar inequality with 2 instead of π. □

Recall that a Banach space X is an L^1 *predual*, if the dual space X^* is isometrical to a space $L^1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ). In 1973 Johnsson and Zippin proved that every real L^1 predual has the property (V) (see [16, Corrollary (i)]). We prove a quantitative version of this theorem using results of their paper and the quantitative version of Pełczyński's theorem.

Theorem 4.2. *Let X be a real L^1 predual. Then X has the quantitative properties (V_q) and $(V_q)^*_\omega$.*

Proof. Let Y be a Banach space and $T\colon X \to Y$ be an operator. We prove that $\gamma(T) \leq 4\,\mathrm{uc}(T)$, that is, X enjoys (V_q). From this is follows that $\gamma(T^*) \leq 8\,\mathrm{uc}(T)$ by the quantitative version of the Gantmacher theorem (5). But the quantities γ and ω are equivalent on X^* – by [18, Theorem 7.5] and (3) we obtain $\omega(T^*) \leq 16\,\mathrm{uc}(T)$, which means that X has $(\mathrm{V}_q)^*_\omega$.

Let us fix $\delta < \gamma(T) = \gamma(T(B_X))$. By the definition of γ we can find a sequence (x_n) in B_X for which $\gamma(\{Tx_n : n \in \mathbb{N}\}) > \delta$. The space $\overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\}$ is a closed separable subspace of the L^1 predual X, hence by [22, § 23, Lemma 1] we can find a separable L^1 predual Z such that $\overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\} \subset Z \subset X$.

By [16], Z is a quotient of $C(\Delta)$, where $\Delta = \{0,1\}^{\mathbb{N}}$ is the Cantor space. Let $q\colon C(\Delta) \to Z$ be a quotient map, i.e. $q(U_{C(\Delta)}) = U_Z$. Then $T \circ q\colon C(\Delta) \to Y$ is a bounded linear operator, and

$$\begin{aligned}2\omega((T\circ q)^*) &\overset{(3),(4)}{\geq} \gamma((T\circ q)^*) \overset{(5)}{\geq} \gamma(T\circ q) = \gamma\left(T\circ q\left(B_{C(\Delta)}\right)\right) = \gamma\left(T\left(q\left(U_{C(\Delta)}\right)\right)\right)\\ &= \gamma\left(T\left(U_Z\right)\right) = \gamma\left(T\left(B_Z\right)\right) \geq \gamma\left(\{Tx_n : n \in \mathbb{N}\}\right) > \delta.\end{aligned}$$

Since Δ is compact, Theorem 4.1 gives $\omega((T\circ q)^*) \leq 2\,\mathrm{uc}(T\circ q)$, and we thus get $\mathrm{uc}(T\circ q) > \frac{\delta}{4}$. Hence we can find a wuC series $\sum f_n$ in $C(\Delta)$ with $\sup_{\mu\in B_{(C(\Delta))^*}} \sum |\mu(f_n)| \leq 1$ such that $\mathrm{ca}\left(\left(\sum_{i=1}^n T(qf_i)\right)_n\right) > \frac{\delta}{4}$.

We set $z_n = q(f_n)$, $n \in \mathbb{N}$. Then $\sum z_n$ is a wuC series in $Z \subset X$ with

$$\sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |x^*(z_n)| = \sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |(x^* \circ q)(f_n)| \leq \sup_{\mu \in B_{(C(\Delta))^*}} \sum |\mu(f_n)| \leq 1.$$

Furthermore, $\operatorname{ca}\left(\left(\sum_{i=1}^{n} T z_i\right)_n\right) > \frac{\delta}{4}$. Hence $\operatorname{uc}(T) > \frac{\delta}{4}$. This inequality holds for every $\delta < \gamma(T)$, therefore $\gamma(T) \leq 4 \operatorname{uc}(T)$. □

Proposition 4.3.

(i) If Ω is a scattered locally compact space, then $C_0(\Omega)$ has the properties (V_q), $(V_q)^*_\omega$, and also $(V_q)_\omega$.

(ii) If Ω is an uncountable separable metrizable locally compact space, then $C_0(\Omega)$ has the properties (V_q) and $(V_q)^*_\omega$, but it does not enjoy the property $(V_q)_\omega$.

Proof. Let Ω be a scattered locally compact space. The space $C_0(\Omega)$ has the properties (V_q) and $(\mathrm{V}_q)^*_\omega$ by Theorem 4.1. Let Y be a Banach space and $T\colon C_0(\Omega) \to Y$ an operator. Since for Ω scattered $C_0(\Omega)^*$ is isometric to $\ell^1(\Omega)$, we have $\omega(T) \leq 2\omega(T^*)$ by [18, Theorem 8.2]. Combining it with Theorem 4.1 we obtain $\omega(T) \leq 2\pi \operatorname{uc}(T)$, that is, $C_0(\Omega)$ has the property $(\mathrm{V}_q)_\omega$.

The second statement is proved in Section 5.2. □

The proposition above shows that $(\mathrm{V}_q)_\omega$ differs from the other two quantifications, but we do not know whether there is any difference between the properties (V_q) and $(\mathrm{V}_q)^*_\omega$.

Question 4.4. *Is there a Banach space which has the property (V_q) but not the property $(V_q)^*_\omega$?*

There is one even more interesting open question whether the Pelczynski's property (V) is automatically quantitative or not.

Question 4.5. *Is there a Banach space which has Pełczyński's property (V) but not the quantitative version (V_q)?*

5 Some other properties of Banach spaces, their quantification and relationship to the property (V)

In this section we remind the definitions of some known properties of operators between Banach spaces, relationships between them, and their relation to unconditionally converging operators. These relationships are then quantified. The introduced properties of operators give rise to some properties of Banach spaces which are related to Pełczyński's property (V). These properties can be quantified in the same way as the property (V). Using the proved quantitative relationships between different kinds of operators we establish the relation between quantitative versions of relevant properties of Banach spaces, including the property (V). Finally, we apply these results and those of [18] to some $C_0(\Omega)$ spaces.

5.1 Some properties of operators, their relation to unconditionally converging operators, and their quantification

Let X be a Banach space. We will denote by ρ the topology of uniform convergence on weakly compact subsets of X^*. This topology is called the *Right topology* and it is the restriction to X of the Mackey topology $\mu(X^{**}, X^*)$ on X^{**} with respect to the dual pair (X^{**}, X^*). An operator from X into a Banach space Y is weakly compact if and only if it is Right-to-norm continuous (see [25]).

We say that an operator between Banach spaces is

- *completely continuous (cc)* if it is weak-to-norm sequentially continuous,
- *pseudo weakly compact (pwc)* if it is Right-to-norm sequentially continuous,
- *weakly completely continuous (wcc)* if it maps weakly Cauchy sequences to weakly convergent sequences,
- *Right completely continuous (Rcc)* if it maps Right-Cauchy sequences to Right-convergent sequences.

M. Kačena has proved in [17, § 3] (using also [25]) that for every operator T between Banach spaces the following implications hold:

$$(9)\qquad \begin{tikzcd} & T \text{ is } w\text{-compact} \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "/" marking] & T \text{ is pwc} \arrow[dr, Rightarrow] & & \\ T \text{ is compact} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & T \text{ is Rcc} \arrow[r, Rightarrow] & T \text{ is uc} \\ & T \text{ is cc} \arrow[r, Rightarrow] & T \text{ is wcc} \arrow[ur, Rightarrow] & & \end{tikzcd}$$

Some of these implications have already been quantified in [18, § 3,4]. In this section we quantify the rest.

Let X, Y be Banach spaces and $T\colon X \to Y$ an operator. We set

$$\begin{aligned} \mathrm{cc}(T) &= \sup\{\, \mathrm{ca}\,((Tx_n)) : (x_n) \text{ is a weakly Cauchy sequence in } B_X\}, \\ \mathrm{cc}_\rho(T) &= \sup\{\, \mathrm{ca}\,((Tx_n)) : (x_n) \text{ is a Right-Cauchy sequence in } B_X\}. \end{aligned}$$

The former quantity measures how far is T from being completely continuous, the latter one measures how far is T from being pseudo weakly compact.

As for the other properties mentioned above, let us first remind that a bidual space X^{**} is complete with respect to both the weak* and the Mackey topology and that the weak* topology is coarser than the Mackey topology. Therefore every weakly Cauchy sequence in a Banach space X is weak*-convergent in X^{**}, every Right-Cauchy sequence in a Banach space X is $\mu(X^{**}, X^*)$-convergent and hence also weak*-convergent in X^{**}. Each bounded linear operator, which is by definition norm-to-norm continuous, is also weak-to-weak continuous and Right-to-Right continuous (see [25, Lemma 12]). Let X, Y be Banach spaces and $T\colon X \to Y$ an operator. Let us set

$$\begin{aligned} \mathrm{wcc}(T) &= \sup\{\, \mathrm{dist}(w^*\text{-}\lim(Tx_n), Y) : (x_n) \text{ is a } w\text{-Cauchy sequence in } B_X\} \\ &= \sup\{\, \mathrm{wk}_Y(\{Tx_n : n \in \mathbb{N}\}) : (x_n) \text{ is a } w\text{-Cauchy sequence in } B_X\}, \\ \mathrm{wcc}_\omega(T) &= \sup\{\omega(\{Tx_n : n \in \mathbb{N}\}) : (x_n) \text{ is a } w\text{-Cauchy sequence in } B_X\}, \\ \mathrm{Rcc}(T) &= \sup\{\, \mathrm{dist}(\mu(Y^{**}, Y^*)\text{-}\lim(Tx_n), Y) : (x_n) \text{ is a } \rho\text{-Cauchy sequence in } B_X\} \\ &= \sup\{\, \mathrm{dist}(w^*\text{-}\lim(Tx_n), Y) : (x_n) \text{ is a } \rho\text{-Cauchy sequence in } B_X\} \\ &= \sup\{\, \mathrm{wk}_Y(\{Tx_n : n \in \mathbb{N}\}) : (x_n) \text{ is a } \rho\text{-Cauchy sequence in } B_X\}, \\ \mathrm{Rcc}_\omega(T) &= \sup\{\omega(\{Tx_n : n \in \mathbb{N}\}) : (x_n) \text{ is a } \rho\text{-Cauchy sequence in } B_X\}. \end{aligned}$$

The first two quantities measure (in two different ways) weak non-complete continuity of T, the last two are measures of Right non-complete continuity of T.

The following theorem contains quantitative versions of all the implications in (9).

Theorem 5.1. *Let X, Y be Banach spaces and $T\colon X \to Y$ an operator. Then*

$$\begin{array}{ccccccc}
 & & 2\omega(T^*) & & & & \\
 & & \rotatebox{90}{$\leq$} & & & & \\
 & & \mathrm{cc}_\rho(T) & \leq & \mathrm{cc}(T) & \leq 4\cdot & \chi(T) \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
 & & \mathrm{Rcc}_\omega(T) & \leq & \mathrm{wcc}_\omega(T) & \leq & \omega(T) \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\frac{1}{4}\,\mathrm{uc}(T) & \leq & \mathrm{Rcc}(T) & \leq & \mathrm{wcc}(T) & \leq & \mathrm{wk}_Y(T).
\end{array}$$

Proof. All the inequalities

$$\begin{array}{ccc}
4\chi(T) & & \\
\rotatebox{90}{$\leq$} & & \\
\mathrm{cc}(T) & \leq & \chi(T) \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\mathrm{wcc}_\omega(T) & \leq & \omega(T) \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\mathrm{wcc}(T) & \leq & \mathrm{wk}_Y(T)
\end{array}$$

has already been proved (or simply observed) in [18, § 3]. The inequality $\mathrm{cc}_\rho(T) \leq 2\omega(T^*)$ follows from [18, (2.1) and (4.1)].

The inequalities $\mathrm{cc}_\rho(T) \leq \mathrm{cc}(T)$, $\mathrm{Rcc}(T) \leq \mathrm{wcc}(T)$, and $\mathrm{Rcc}_\omega(T) \leq \mathrm{wcc}_\omega(T)$ are trivial, since every Right-Cauchy sequence is weakly Cauchy. By (4), $\mathrm{wk}_Y(A) \leq \omega(A)$ for every bounded $A \subset Y$. Therefore $\mathrm{Rcc}(T) \leq \mathrm{Rcc}_\omega(T)$ (as well as $\mathrm{wcc}(T) \leq \mathrm{wcc}_\omega(T)$, which has already been noted).

Let us show that $\mathrm{Rcc}_\omega(T) \leq \mathrm{cc}_\rho(T)$. Suppose that $\mathrm{cc}_\rho(T) < \delta$. Let (x_n) be a Right-Cauchy sequence in B_X. Then $\mathrm{ca}\left((Tx_n)\right) < \delta$, hence we can find $n_0 \in \mathbb{N}$ such that $\|Tx_n - Tx_{n_0}\| < \delta$ whenever $n > n_0$. Set $K = \{Tx_1, \ldots, Tx_{n_0}\}$. Then K is weakly compact, and $\hat{\mathrm{d}}(\{Tx_n : n \in \mathbb{N}\}, K) \leq \delta$. Therefore $\omega(\{Tx_n : n \in \mathbb{N}\}) \leq \delta$. We thus get $\mathrm{Rcc}_\omega(T) \leq \delta$, and consequently $\mathrm{Rcc}_\omega(T) \leq \mathrm{cc}_\rho(T)$.

Finally, we prove the inequality $\mathrm{uc}(T) \leq 4\,\mathrm{Rcc}(T)$. By Theorem 3.6, it is enough to show that $\mathrm{fix}_{c_0}(T) \leq 2\,\mathrm{Rcc}(T)$. If $\mathrm{fix}_{c_0}(T) = 0$, then it is obvious. Suppose that $\mathrm{fix}_{c_0}(T) > 0$ and fix $0 < \delta < \mathrm{fix}_{c_0}(T)$. By the definition of $\mathrm{fix}_{c_0}(T)$ we find a subspace X_0 of X isomorphic to c_0 and onto isomorphisms $U\colon c_0 \to X_0$, $V\colon T(X_0) \to c_0$ such that $(T \restriction_{X_0})^{-1} = U \circ V$, and $(\|U\|\|V\|)^{-1} > \delta$.

Set $f_n = \sum_{i=1}^n e_n \in c_0$, $n \in \mathbb{N}$. Then (f_n) is a weakly Cauchy sequence in c_0. Since the space c_0 enjoys the Dunford-Pettis property (see e.g. [11, p. 597]), the weak and the Right topology coincide sequentially on it by [17, Proposition 3.17]. Therefore the sequence (f_n) is Right-Cauchy. Let us define $x_n = \frac{1}{\|U\|} U f_n$, $n \in \mathbb{N}$. By the continuity of U, (x_n) is a Right-Cauchy sequence in B_X. Since T is bounded, we also have that (Tx_n) is a Right-Cauchy sequence in Y. Let y^{**} be its $\mu(Y^{**}, Y^*)$-limit in Y^{**}. We will show that $\mathrm{dist}(y^{**}, Y) > \frac{\delta}{2}$.

Let us set $Y_0 = \overline{\mathrm{span}}\{Tx_n : n \in \mathbb{N}\} = \overline{\mathrm{span}}\{(T \circ U) f_n : n \in \mathbb{N}\} = \overline{\mathrm{span}}\{(T \circ U) e_n : n \in \mathbb{N}\} = (T \circ U)(c_0)$. If $T \circ U$ is regarded as an isomorphism from c_0 onto Y_0, then

$(T \circ U)^{**}$ is an isomorphism from c_0^{**} onto Y_0^{**}, and $\|((T \circ U)^{**})^{-1}\| = \|(T \circ U)^{-1}\| = \|V\|$. Let $y_0 \in Y_0$ be arbitrary. We find $z \in c_0$ which satisfies $\frac{1}{\|U\|}(T \circ U)z = y_0$. Then

$$\begin{aligned}
\|y^{**} - y_0\| &= \left\| \mu(Y^{**}, Y^*)\text{-}\lim \left(\tfrac{1}{\|U\|}(T \circ U) f_n \right) - \tfrac{1}{\|U\|}(T \circ U)z \right\| \\
&= \frac{1}{\|U\|} \left\| (T \circ U)^{**}((\mu(Y^{**}, Y^*)\text{-}\lim f_n) - z) \right\| \\
&\geq \frac{1}{\|U\|} \left\| ((T \circ U)^{**})^{-1} \right\|^{-1} \|(\mu(Y^{**}, Y^*)\text{-}\lim f_n) - z\| \\
&= \|U\|^{-1} \|V\|^{-1} \|(w^*\text{-}\lim f_n) - z\| \geq (\|U\|\|V\|)^{-1},
\end{aligned}$$

where the last inequality follows from the fact, that $w^*\text{-}\lim f_n = (1, 1, 1, \dots) \in \ell^\infty \cong c_0^{**}$ whereas $z \in c_0$, so the distance between these two elements is at least $\lim_{n\to\infty} |1 - z(n)| = 1$. Therefore $\operatorname{dist}(y^{**}, Y_0) \geq (\|U\|\|V\|)^{-1} > \delta$. By [15, Lemma 2.2], $\operatorname{dist}(y^{**}, Y_0) \leq 2 \operatorname{dist}(y^{**}, Y)$, and hence $\operatorname{dist}(y^{**}, Y) > \frac{\delta}{2}$.

We thus have $\operatorname{Rcc}(T) > \frac{\delta}{2}$. It follows that $\operatorname{fix}_{c_0}(T) \leq 2 \operatorname{Rcc}(T)$, which completes the proof. □

Remark. The inequality $\operatorname{cc}_\rho(T) \leq 2\omega(T^*)$ from the above theorem quantifies the implication

$$T \text{ is weakly compact } \Rightarrow T \text{ is pseudo weakly compact}$$

due to the Gantmacher theorem. We cannot obtain a better quantification either with $\gamma(T)$ or with $\omega(T)$ instead of $\omega(T^*)$. The space X constructed in [18, Example 10.1(v)] forms a counterexample. Since this space enjoys the Dunford-Pettis property, the weak and the Right topology coincide sequentially on X (see [17, Proposition 3.17]), thus $\operatorname{cc}(T) = \operatorname{cc}_\rho(T)$ for each operator $T\colon X \to Y$ (Y a Banach space). But there are operators $T_n\colon X \to c_0$, $n \in \mathbb{N}$, such that $\operatorname{cc}(T_n) \geq 1$ for each $n \in \mathbb{N}$ and $\omega(T_n) = \operatorname{wk}_{c_0}(T_n) \to 0$. The measures wk_{c_0} and γ are equivalent by (3), hence there is no constant $C > 0$ such that $\operatorname{cc}_\rho(T) = \operatorname{cc}(T) \leq C\gamma(T)$ or $\operatorname{cc}_\rho(T) = \operatorname{cc}(T) \leq C\omega(T)$ for each operator $T\colon X \to c_0$.

5.2 Properties of Banach spaces related to above-defined properties of operators and a relationship between their quantitative versions

Let us recall some properties of Banach spaces, whose definitions use the above-introduced properties of operators. We follow the the notation of [17]. Let X be a Banach space. We say that

- X has the *reciprocal Dunford-Pettis property (RDP)* if for every Banach space Y every cc operator $T\colon X \to Y$ is weakly compact,

- X has the *Dieudonné property (D)* if for every Banach space Y every wcc operator $T\colon X \to Y$ is weakly compact,

- X has the *Right Dieudonné property (RD)* if for every Banach space Y every Rcc operator $T\colon X \to Y$ is weakly compact,

- X is *sequentially Right (SR)* if for every Banach space Y every pwc operator $T\colon X \to Y$ is weakly compact.

The following implications are an immediate consequence of (9):

$$
\begin{tikzcd}[column sep=small]
 & & & X \text{ is (SR)} \arrow[dr, Rightarrow] & \\
\text{(10)} \qquad X \text{ has (V)} \arrow[r, Rightarrow] & X \text{ has (RD)} \arrow[urr, Rightarrow] \arrow[drr, Rightarrow] & & & X \text{ has (RDP)} \\
 & & & X \text{ has (D)} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

All these properties have their quantitative versions, obtained in a standard way. First we define quantitative versions of the properties (SR) and (RDP) analogous to (V_q), $(\mathrm{V}_q)_\omega$, $(\mathrm{V}_q)^*_\omega$.

Definition. We say that a Banach space X has the property (RDP_q), $(\mathrm{RDP}_q)_\omega$, or $(\mathrm{RDP}_q)^*_\omega$ if there is a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\gamma(T) \le C\,\mathrm{cc}(T), \qquad \omega(T) \le C\,\mathrm{cc}(T), \qquad \text{or} \qquad \omega(T^*) \le C\,\mathrm{cc}(T),$$

respectively. Analogously we define the properties (SR_q), $(\mathrm{SR}_q)_\omega$, and $(\mathrm{SR}_q)^*_\omega$ – we just replace cc in the above inequalities by cc_ρ.

For the details about a quantification of the reciprocal Dunford-Pettis property we refer the reader to [21]. Regarding the properties (D) and (RD), there are even more possibilities of quantification. Besides the measures of weak non-compactness of T, we can also choose between two different quantities which measure weak non-complete continuity and Right non-complete continuity of T.

Definition. We say that a Banach space X has the property (D_q), $(\mathrm{D}_q)_\omega$, $(\mathrm{D}_q)^*_\omega$, (D^ω_q), $(\mathrm{D}^\omega_q)_\omega$, or $(\mathrm{D}^\omega_q)^*_\omega$ if there is a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\begin{array}{lll}
\gamma(T) \le C\,\mathrm{wcc}(T), & \omega(T) \le C\,\mathrm{wcc}(T), & \omega(T^*) \le C\,\mathrm{wcc}(T), \\
\gamma(T) \le C\,\mathrm{wcc}_\omega(T), & \omega(T) \le C\,\mathrm{wcc}_\omega(T), \quad \text{or} & \omega(T^*) \le C\,\mathrm{wcc}_\omega(T).
\end{array}$$

The properties (RD_q), $(\mathrm{RD}_q)_\omega$, $(\mathrm{RD}_q)^*_\omega$, (RD^ω_q), $(\mathrm{RD}^\omega_q)_\omega$, and $(\mathrm{RD}^\omega_q)^*_\omega$ are defined in the same way, the quantities wcc and wcc_ω are replaced by Rcc and Rcc_ω, respectively.

Clearly, if X has the property $(\mathrm{P}_q)_\omega$ or $(\mathrm{P}_q)^*_\omega$, then it also has the property (P_q) by (3), (4), and (5). Here P stands for V, RD, D, SR, or RDP. From Theorem 5.1 we obtain the following relationships between the quantitative versions of the properties defined above.

Theorem 5.2. *For a Banach space X the following implications hold:*

$$\begin{array}{ccccccc}
X \text{ has } (V_q) & \Longrightarrow & X \text{ has } (RD_q) & \Longrightarrow & X \text{ has } (RD^\omega_q) & \Longrightarrow & X \text{ has } (SR_q) \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & X \text{ has } (D_q) & \Longrightarrow & X \text{ has } (D^\omega_q) & \Longrightarrow & X \text{ has } (RDP_q),
\end{array}$$

$$\begin{array}{ccccccc}
X \text{ has } (V_q)_\omega & \Longrightarrow & X \text{ has } (RD_q)_\omega & \Longrightarrow & X \text{ has } (RD^\omega_q)_\omega & \Longrightarrow & X \text{ has } (SR_q)_\omega \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & X \text{ has } (D_q)_\omega & \Longrightarrow & X \text{ has } (D^\omega_q)_\omega & \Longrightarrow & X \text{ has } (RDP_q)_\omega,
\end{array}$$

$$\begin{array}{ccccccc}
X \text{ has } (V_q)^*_\omega & \Longrightarrow & X \text{ has } (RD_q)^*_\omega & \Longrightarrow & X \text{ has } (RD^\omega_q)^*_\omega & \Longrightarrow & X \text{ has } (SR_q)^*_\omega \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & X \text{ has } (D_q)^*_\omega & \Longrightarrow & X \text{ has } (D^\omega_q)^*_\omega & \Longrightarrow & X \text{ has } (RDP_q)^*_\omega.
\end{array}$$

Now we are ready to prove Proposition 4.3 (ii).

Proof of Proposition 4.3 (ii). If we take the space Y constructed in [21, Example 3.2], then there is a sequence (T_n) of operators from $C_0(\Omega)$ to Y which satisfies

$$\lim_{n\to\infty} \frac{\operatorname{cc}(T_n)}{\omega(T_n)} = 0.$$

Therefore $C_0(\Omega)$ does not have the property $(\mathrm{RDP}_q)_\omega$, hence not even the property $(\mathrm{V}_q)_\omega$ by Theorem 5.2. But it follows from Theorem 4.1 that $C_0(\Omega)$ enjoys the property (V_q). □

5.3 Some corollaries for $C_0(\Omega)$ spaces

Corollary 5.3. *Let Ω be a locally compact space, Y be a Banach space, and $T\colon C_0(\Omega) \to Y$ an operator. Then*

$$\begin{array}{ccccccccccc}
 & & \operatorname{wcc}(T) & & \operatorname{wcc}_\omega(T) & & \operatorname{cc}(T) & & & & \\
 & & \parallel & & \parallel & & \parallel & & & & \\
\frac{1}{4}\operatorname{uc}(T) & \le & \operatorname{Rcc}(T) & \le & \operatorname{Rcc}_\omega(T) & \le & \operatorname{cc}_\rho(T) & \le & 2\omega(T^*) & \le & 2\pi\operatorname{uc}(T) \\
 & & \rotatebox{90}{$\ge$} & & & & & & \parallel & & \\
 & & \operatorname{wk}_Y(T) & \le & \gamma(T) & \le & \gamma(T^*) & \le & 2\operatorname{wk}_Y(T^*) & & \\
 & & \rotatebox{90}{$\ge$} & & & & & & & & \\
 & & \omega(T) & & & & & & & & \\
 & & \rotatebox{90}{$\ge$} & & & & & & & & \\
 & & \chi(T). & & & & & & & &
\end{array}$$

In particular, all the quantities except for $\omega(T)$ and $\chi(T)$ are equivalent.

Proof. Since $C_0(\Omega)$ has the Dunford-Pettis property, the weak and the Right topology coincide sequentially on X by [17, Proposition 3.17]. That is why $\operatorname{Rcc}(T) = \operatorname{wcc}(T)$, $\operatorname{Rcc}_\omega(T) = \operatorname{wcc}_\omega(T)$, and $\operatorname{cc}_\rho(T) = \operatorname{cc}(T)$. The equality $\omega(T^*) = \operatorname{wk}_Y(T^*)$ follows from [18, Theorem 7.5]. By (3) and (5) we have $\operatorname{wk}_Y(T) \le \gamma(T) \le \gamma(T^*) \le 2\operatorname{wk}_Y(T^*)$. Theorem 4.1 gives $\omega(T^*) \le \pi\operatorname{uc}(T)$. The rest follows from Theorem 5.1. □

Remark. Almost the same assertion holds for every operator $T\colon X \to Y$ if X is a real L^1 predual and Y a Banach space. We only need to adjust the constant in the inequality $\omega(T^*) \le \pi\operatorname{uc}(T)$. From the proof of Theorem 4.2 we see that it is enough to replace π by 16. All the quantities except for $\omega(T)$ and $\chi(T)$ are still equivalent.

Corollary 5.4. *Let Ω be a scattered locally compact space, Y be a Banach space, and $T\colon C_0(\Omega) \to Y$ an operator. Then*

$$\begin{array}{ccccccccccc}
 & & \operatorname{wcc}(T) & & \operatorname{wcc}_\omega(T) & & \operatorname{cc}(T) & & & & \\
 & & \parallel & & \parallel & & \parallel & & & & \\
\frac{1}{4}\operatorname{uc}(T) & \le & \operatorname{Rcc}(T) & \le & \operatorname{Rcc}_\omega(T) & \le & \operatorname{cc}_\rho(T) & \le & 2\omega(T^*) & \le & 2\pi\operatorname{uc}(T) \\
 & & \rotatebox{90}{$\ge$} & & & & \rotatebox{90}{$\le$} & & & & \\
 & & \operatorname{wk}_Y(T) & \le & \omega(T) & \le & \chi(T) & & & &
\end{array}$$

Hence all the involved quantities are equivalent.

Proof. The assertion follows from Corollary 5.3 and [18, Theorem 8.2] since $C_0(\Omega)^*$ for Ω scattered is isometric to $\ell^1(\Omega)$. □

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IV. C*-algebras have a quantitative version of Pełczyński's property (V)

(Accepted to Czechoslovak Math. J. (2016), preprint available at `http://arxiv.org/abs/1605.04900`.*)*

Abstract: A Banach space X has Pełczyński's property (V) if for every Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact. H. Pfitzner proved that C^*-algebras have Pełczyński's property (V). In the preprint [8] the author explores possible quantifications of the property (V) and shows that $C(K)$ spaces for a compact Hausdorff space K enjoy a quantitative version of the property (V). In this paper we generalize this result by quantifying Pfitzner's theorem. Moreover, we prove that in dual Banach spaces a quantitative version of the property (V) implies a quantitative version of the Grothendieck property.

1 Introduction

In 1994, H. Pfitzner proved that C^*-algebras have Pełczyński's property (V) (see [10]). The aim of this paper is to prove a quantitative version of Pfitzner's result. In this way we continue the study of quantitative versions of Pełczyński's property (V) presented in the preprint [8].

Section 2 summarizes all essential definitions and basic facts contained mostly in the preprint [8]. In Section 3 we slightly improve Behrends's quantitative version of Rosenthal's ℓ^1–theorem [2, Section 3], which we use to prove the main theorem in Section 4. Section 5 is devoted to the relationship of quantitative versions of Pełczyński's property (V) and the Grothendieck property in dual Banach spaces.

2 Preliminaries

We follow the notation of [8] with one exception. Because we deal also with C^*-algebras, we write X' (instead of X^*) for a dual to a Banach space X, since the * in C^*-algebras is already reserved for the involution. All Banach spaces are considered either real or complex, unless stated otherwise. The closed unit ball of a Banach space X is denoted by B_X.

2.1 Pełczyński's property (V) and its quantification

Let us recall some essential definitions and facts (explained in more detail in [8] with many comments). A series $\sum_{n=1}^\infty x_n$ in a Banach space X is said to be

- *unconditionally convergent* if the series $\sum_{n=1}^\infty t_n x_n$ converges whenever (t_n) is a bounded sequence of scalars,
- *weakly unconditionally Cauchy* (*wuC*) if for all $x' \in X'$ the series $\sum_{n=1}^\infty |x'(x_n)|$ converges.

A bounded linear operator $T\colon X \to Y$ between Banach spaces X and Y is called *unconditionally converging* if $\sum_n Tx_n$ is an unconditionally convergent series in Y whenever $\sum_n x_n$ is a weakly unconditionally Cauchy series in X. It is not difficult to show that T is unconditionally converging if and only if for every series $\sum_n x_n$ in X with

$$\sup_{x'\in B_{X'}} \sum_{n=1}^{\infty} |x'(x_n)| < \infty$$

the series $\sum_n Tx_n$ converges. We say that a Banach space X has *Pełczyński's property (V)* if for every Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact.

To quantify the property (V) means to replace the implication

$$T \text{ is unconditionally converging } \Rightarrow\ T \text{ is weakly compact} \tag{1}$$

by an inequality

$$\begin{aligned}&\text{measure of weak non-compactness of } T\\ &\qquad\qquad \le C \cdot \text{measure of } T \text{ not being unconditionally converging,}\end{aligned}$$

where C is some positive constant depending only on X, and the two measures are positive numbers for each operator T and are equal to zero if and only if T is weakly compact or unconditionally converging, respectively. This inequality is a strengthening of the original implication (1).

For this purpose we use the following quantities. For a bounded sequence (x_n) in a Banach space X we define

$$\operatorname{ca}((x_n)) = \inf_{n\in\mathbb{N}} \sup\{\|x_k - x_l\| :\ k,l \in \mathbb{N},\ k,l \ge n\}.$$

It is a measure of non-Cauchyness of a sequence (x_n), hence in Banach spaces it measures non-convergence. Let $T\colon X \to Y$ be a bounded linear operator between Banach spaces X and Y. We set

$$\operatorname{uc}(T) = \sup\left\{\operatorname{ca}\left(\Big(\sum_{i=1}^{n} Tx_i\Big)_n\right) :\ (x_n) \subset X,\ \sup_{x'\in B_{X'}} \sum_{n=1}^{\infty} |x'(x_n)| \le 1\right\}.$$

Then $\operatorname{uc}(T)$ measures how far is the operator T from being unconditionally converging.

Let A be a bounded subset of a Banach space X. The de Blasi measure of weak non-compactness of the set A is defined by

$$\omega(A) = \inf\{\hat{\mathrm{d}}(A,K) : \emptyset \ne K \subset X \text{ is weakly compact}\},$$

where

$$\hat{\mathrm{d}}(A,K) = \sup\{\operatorname{dist}(a,K) :\ a \in A\}.$$

De Blasi has proved that $\omega(A) = 0$ if and only if A is relatively weakly compact (see [4]). Other quantities which measure relative weak non-compactness are for example

$$\begin{aligned}\gamma(A) = \sup\{|\lim_n \lim_m x'_m(x_n) - \lim_m \lim_n x'_m(x_n)| :\ &(x_n) \text{ is a sequence in } A,\\ &(x'_m) \text{ is a sequence in } B_{X'},\ \text{and the limits exist}\}\end{aligned}$$

or

$$\mathrm{wck}_X(A) = \sup\{\mathrm{dist}(\mathrm{clust}_{(X'',w^*)}(x_n), X) : (x_n) \text{ is a sequence in } A\},$$

where $\mathrm{clust}_{(X'',w^*)}(x_n)$ stands for the set of all w^*-cluster points of the sequence (x_n) in X''. The quantities $\gamma(A)$ and $\mathrm{wck}_X(A)$ are equivalent by [1, Theorem 2.3] in the following sense:

$$\mathrm{wck}_X(A) \le \gamma(A) \le 2\,\mathrm{wck}_X(A). \tag{2}$$

However, the quantity $\omega(A)$ is not equivalent to the other two (see [1, Corollary 3.4]). We have only

$$\mathrm{wck}_X(A) \le \omega(A) \tag{3}$$

by [1, Theorem 2.3].

For measuring weak non-compactness of a bounded linear operator $T\colon X \to Y$ between Banach spaces X and Y we use the quantities $\omega(T(B_X))$, $\gamma(T(B_X))$, and $\mathrm{wck}_Y(T(B_X))$, which we denote simply by $\omega(T)$, $\gamma(T)$, and $\mathrm{wk}_Y(T)$.

We say that a Banach space X has a quantitative version of Pełczyński's property (V) – we denote it by (V_q) – if there is a constant $C > 0$ such that for every Banach space Y and every operator $T\colon X \to Y$

$$\gamma(T) \le C \cdot \mathrm{uc}(T). \tag{4}$$

If it is possible to replace $\gamma(T)$ in (4) with $\omega(T)$, we say that X has the property $(\mathrm{V}_q)_\omega$. If $\gamma(T)$ in (4) is replaced by $\omega(T')$, where $T'\colon Y' \to X'$ denotes the dual operator to T, we say that X has the property $(\mathrm{V}_q)^*_\omega$.

In [8, Proposition 3.2] it is proved that a Banach space X has the property (V_q) if and only if there exists a constant $C > 0$ such that for each bounded subset K of the dual space X'

$$\gamma(K) \le C \cdot \eta(K),$$

where

$$\eta(K) = \sup\Big\{ \limsup_n \sup_{x'\in K} |x'(x_n)| : (x_n) \subset X,\ \sup_{x'\in B_{X'}} \sum_{n=1}^{\infty} |x'(x_n)| \le 1 \Big\}.$$

Using the above-described characterization we will prove in Section 4 that C^*-algebras have the property (V_q).

Note that the quantity η is translation-invariant, that is,

$$\eta(K) = \eta(K + z'), \quad K \subset X',\ z' \in X'. \tag{5}$$

This follows from the fact that (x_n) weakly null whenever $\sum x_n$ is a wuC series in X.

2.2 Measures of weak and weak* non-Cauchyness of sequences in Banach spaces

In sections 4 and 5 we will use the following standard quantities, analogous to the quantity ca, which measure how far is a sequence in a (dual) Banach space from being weakly (weak*) Cauchy.

Let X be a Banach space and let (x_n) be a bounded sequence X. We set

$$\delta(x_n) = \sup_{x' \in B_{X'}} \lim_{n\to\infty} \sup_{k,l\geq n} |x'(x'_k) - x'(x'_l)|.$$

This quantity is a measure of weak non-Cauchyness of the sequence (x_n). Furthermore, let us set

$$\tilde{\delta}(x_n) = \inf \{\delta(x_{n_k}) : (x_{n_k}) \text{ is a subsequence of } (x_n)\}.$$

It measures how close can subsequences of (x_n) be to be weakly Cauchy.

If (x'_n) is a bounded sequence in X', we set

$$\delta_{w^*}(x'_n) = \sup_{x \in B_X} \lim_{n\to\infty} \sup_{k,l\geq n} |x'_k(x) - x'_l(x)|.$$

The last quantity is a measure of weak* non-Cauchyness of the sequence (x'_n). The quantity $\delta(x_n)$ equals 0 if and only if the sequence (x_n) is weakly Cauchy. Analogously, $\delta_{w^*}(x'_n) = 0$ if and only if (x'_n) is weak* Cauchy. If $\tilde{\delta}(x_n) = 0$, it it not clear whether (x_n) admits a weakly Cauchy subsequence.

2.3 Selfadjoint elements and selfadjoint functionals

Let A be a C^*-algebra. Let us denote by A_{sa} the selfadjoint elements of A, that is $A_{\text{sa}} = \{a \in A : a = a^*\}$. Then A_{sa} is a real Banach space and $A = A_{\text{sa}} + iA_{\text{sa}}$. If f is a bounded linear functional on A, f^* is the functional defined by $f^*(x) = \overline{f(x^*)}$, $x \in A$. Let $(A')_{\text{sa}}$ denote the set $\{f \in A' : f = f^*\}$ of selfadjoint functionals on A. Then $(A')_{\text{sa}}$ is a real Banach space, and is isometrically isomorphic to $(A_{\text{sa}})'$. We will write A'_{sa} for both these spaces. Every functional $x' \in A'$ can be decomposed as $x' = f + ig$ where $f, g \in A'_{\text{sa}}$. It suffices to set $f = (x' + (x')^*)/2$, $g = (x' - (x')^*)/(2i)$.

3 A quantitative version of Rosenthal's ℓ^1–theorem

For proving the main result we need the quantitative version of Rosenthal's ℓ^1–theorem proved by E. Behrends in [2, Section 3]. In this section we revise his theorem, because it turns out that one of the estimates there can be easily improved. We will then use this improved version.

Let us remind Behrends's definition [2, 3.1].

Definition. Let (x_n) be a bounded sequence in a Banach space X and $\varepsilon > 0$. We say that (x_n) admits ε–ℓ^1–blocks if for every infinite $M \subset \mathbb{N}$ there are scalars $a_1, \ldots, a_r$ with $\sum |a_r| = 1$ and $i_1, \ldots, i_r$ in M such that $\left\|\sum a_\rho x_{i_\rho}\right\| \leq \varepsilon$.

The revised version of the quantitative Rosenthal's ℓ^1–theorem for complex Banach spaces is the following.

Theorem 3.1. *Let X be a complex Banach space X and $\varepsilon > 0$. Let (x_n) be a sequence in X which admits ε–ℓ^1–blocks. Then there is a subsequence (x_{n_k}) of (x_n) such that for every $x' \in X'$ with $\|x'\| = 1$ the diameter of the set of cluster points of the sequence $(x'(x_{n_k}))_k$ is at most $\pi\varepsilon$.*

Remark. In the original Behrends' theorem [2, Theorem 3.3] there is a larger constant $8/\sqrt{2}$ in place of π. A similar result with the better constant π has been obtained (in a different way) by I. Gasparis [5].

Sketch of the proof of Theorem 3.1. The proof is essentially the same as the original one. Suppose that the conclusion were not true. We can find $\delta > 0$ such that the number

$$\sup_{x' \in S_{X'}} \{\text{diameter of the set of accumulation points of } (x'(x_{n_k}))_k\}$$

is greater than $\pi\varepsilon + \delta$ for any subsequence (x_{n_k}) of (x_n). Fix $\tau \in (0, 1)$ such that $(2 + \sup_n \|x_n\|)\tau < \frac{\delta}{\pi}$.

Similarly to the one in the proof of [2, Theorem 3.3 (or 3.2)] we can prove the following lemma.

Lemma. *The sequence (x_n) admits a subsequence (without loss of generality still denoted by (x_n)) which satisfies the following conditions:*

(i) Whenever C and D are disjoint finite subsets of $\mathbb{N}$, there are $z_0, w_0 \in \mathbb{C}$ with $|w_0| \geq \pi\varepsilon + \delta$ and $x' \in X'$ with $\|x'\| = 1$ such that $|x'(x_n) - z_0| \leq \tau$ for $n \in C$ and $|x'(x_n) - (z_0 + w_0)| \leq \tau$ for $n \in D$.

(ii) There are $i_1 < \cdots < i_r$ in $\mathbb{N}$ and $a_1, \ldots, a_r \in \mathbb{C}$ which satisfy

$$\sum_{\rho=1}^{r} |a_\rho| = 1, \quad \left|\sum_{\rho=1}^{r} a_\rho\right| \leq \tau, \quad \left\|\sum_{\rho=1}^{r} a_\rho x_{i_\rho}\right\| \leq \varepsilon.$$

Finally, the time has come for the modification. By [11, Lemma 6.3] we find $D \subset \{1, \ldots, r\}$ such that

$$\left|\sum_{\rho \in D} a_\rho\right| \geq \frac{1}{\pi} \sum_{\rho=1}^{r} |a_\rho| = \frac{1}{\pi}.$$

Set $C = \{1, \ldots, r\} \setminus D$. For these sets C and D we find z_0, w_0, and x' from (i) of the lemma. It follows that

$$\begin{aligned}
\varepsilon &\geq \left\|\sum_{\rho=1}^{r} a_\rho x_{i_\rho}\right\| \geq \left|\sum_{\rho=1}^{r} a_\rho x'(x_{i_\rho})\right| = \left|\sum_{\rho \in C} a_\rho x'(x_{i_\rho}) + \sum_{\rho \in D} a_\rho x'(x_{i_\rho})\right| \\
&\geq \left|\sum_{\rho \in C} a_\rho z_0 + \sum_{\rho \in D} a_\rho (z_0 + w_0)\right| - \tau \sum_{\rho=1}^{r} |a_\rho| = \left|\sum_{\rho \in D} a_\rho w_0 + \sum_{\rho=1}^{r} a_\rho z_0\right| - \tau \\
&\geq |w_0| \left|\sum_{\rho \in D} a_\rho\right| - |z_0| \left|\sum_{\rho=1}^{r} a_\rho\right| - \tau \geq \frac{|w_0|}{\pi} - |z_0|\tau - \tau \geq \frac{\pi\varepsilon + \delta}{\pi} - (1 + |z_0|)\tau \\
&= \varepsilon + \frac{\delta}{\pi} - (1 + |z_0|)\tau \geq \varepsilon + \frac{\delta}{\pi} - (2 + \sup_n \|x_n\|)\tau > \varepsilon,
\end{aligned}$$

which is a contradiction. □

4 Main theorem

This section is devoted to our main result – a quantitative version of Pfitzner's theorem (Theorem 4.1 below). We also prove a "real version" of this theorem (Theorem 4.2).

Theorem 4.1. *Let A be a C^*-algebra. Then for every bounded $K \subset A'$*

$$\mathrm{wck}_{A'}(K) \leq \pi \cdot \eta(K). \tag{6}$$

Therefore A has the property (V_q).

Proof. The quantities $\gamma(K)$ and $\mathrm{wck}_{A'}(K)$ are equivalent by [1, Theorem 2.3], more specifically, the inequality (6) implies $\gamma(K) \leq 2\pi \cdot \eta(K)$. If this holds for each bounded $K \subset A'$, Proposition [8, 3.2] mentioned also in Section 2 gives that A has the property (V_q). Let us show the inequality (6).

Let $K \subset A'$ be bounded. The case $\mathrm{wck}_{A'}(K) = 0$ is trivial. Suppose that $\mathrm{wck}_{A'}(K) > 0$ and fix an arbitrary $\lambda \in (0, \mathrm{wck}_{A'}(K))$. By the definition of the quantity $\mathrm{wck}_{A'}(K)$ we find a sequence (x'_n) in K such that

$$\mathrm{dist}\,(\mathrm{clust}_{(A''',w^*)}(x'_n), A') > \lambda.$$

Since every dual of a C^*-algebra is a predual of a von Neumann algebra, we deduce from [13, Theorem III.2.14] (see also [6, Example IV.1.1(b)]) that A' is L-embedded – it means that A' is complemented in A''' by a projection P satisfying

$$\|x'''\| = \|Px'''\| + \|x''' - Px'''\|, \quad x''' \in A'''.$$

Consequently, from [7, Theorem 1] we have

$$\begin{aligned}\tilde{\delta}(x'_n) &= \inf\{\delta(x'_{n_k}) : (x'_{n_k}) \text{ is a subsequence of } (x'_k)\} \\ &\geq 2\,\mathrm{dist}\,(\mathrm{clust}_{(A''',w^*)}(x'_n), A') > 2\lambda.\end{aligned}$$

Fix an arbitrary $\varepsilon > 0$. We now prove the following claim.

Claim. There is a sequence of self-adjoint elements (x_k) in B_A satisfying $x_i x_j = 0$, $i, j \in \mathbb{N}$, $i \neq j$, and a subsequence (x'_{n_k}) of the sequence (x'_n) such that

$$\left|x'_{n_k}(x_k)\right| > (1-\varepsilon)^2 \frac{\lambda}{\pi}, \quad k \in \mathbb{N}.$$

Proof. Each x'_n is canonically decomposed in the following way: $x'_n = f_n + ig_n$, where $f_n, g_n \in A'$ are selfadjoint functionals. It suffices to find (x_k) and (x'_{n_k}) such that

$$\left|f_{n_k}(x_k)\right| > (1-\varepsilon)^2 \frac{\lambda}{\pi} \qquad \text{or} \qquad \left|g_{n_k}(x_k)\right| > (1-\varepsilon)^2 \frac{\lambda}{\pi}.$$

Indeed, since selfadjoint functionals attain real values on selfadjoint elements of A, we have

$$\left|x'_{n_k}(x_k)\right| = \left|f_{n_k}(x_k) + ig_{n_k}(x_k)\right| \geq \begin{cases} \left|\mathrm{Re}(f_{n_k}(x_k) + ig_{n_k}(x_k))\right| = |f_{n_k}(x_k)| \\ \left|\mathrm{Im}(f_{n_k}(x_k) + ig_{n_k}(x_k))\right| = |g_{n_k}(x_k)| \end{cases}.$$

We begin by proving that there is a strictly increasing sequence of indices (n_k) such that $\tilde{\delta}(f_{n_k}) > \lambda$ or $\tilde{\delta}(g_{n_k}) > \lambda$. If $\tilde{\delta}(f_n) > \lambda$, the proof is over, so suppose that $\tilde{\delta}(f_n) \leq \lambda$. Let us find $\tau > 0$ satisfying $\tilde{\delta}(x'_n) > 2\lambda + 2\tau$. By the definition of $\tilde{\delta}(f_n)$ there is a subsequence (f_{n_k}) of the sequence (f_n) with $\delta(f_{n_k}) < \lambda + \tau$. We claim that the corresponding subsequence (g_{n_k}) of (g_n) satisfies $\tilde{\delta}(g_{n_k}) > \lambda$. To obtain a contradiction, suppose that $\tilde{\delta}(g_{n_k}) \leq \lambda$. Using the definition of $\tilde{\delta}(g_{n_k})$ we find a strictly increasing sequence of indices (k_l) such that $\delta(g_{n_{k_l}}) < \lambda + \tau$. Then

$$\begin{aligned}
\delta(x'_{n_{k_l}}) &= \delta(f_{n_{k_l}} + ig_{n_{k_l}}) \\
&= \sup_{x''\in B_{A''}} \lim_{l\to\infty} \sup_{p,q\geq l} \left|x''(f_{n_{k_p}} + ig_{n_{k_p}}) - x''(f_{n_{k_q}} + ig_{n_{k_q}})\right| \\
&\leq \sup_{x''\in B_{A''}} \lim_{l\to\infty} \sup_{p,q\geq l} \left(\left|x''(f_{n_{k_p}}) - x''(f_{n_{k_q}})\right| + \left|x''(g_{n_{k_p}}) - x''(g_{n_{k_q}})\right|\right) \\
&\leq \sup_{x''\in B_{A''}} \lim_{l\to\infty} \sup_{p,q\geq l} \left|x''(f_{n_{k_p}}) - x''(f_{n_{k_q}})\right| \\
&\qquad + \sup_{x''\in B_{A''}} \lim_{l\to\infty} \sup_{p,q\geq l} \left|x''(g_{n_{k_p}}) - x''(g_{n_{k_q}})\right| \\
&= \delta(f_{n_{k_l}}) + \delta(g_{n_{k_l}}) < \lambda + \tau + \lambda + \tau = 2\lambda + 2\tau,
\end{aligned}$$

which contradicts the fact that $\tilde{\delta}(x'_n) > 2\lambda + 2\tau$.

Without loss of generality we may assume that we have found a subsequence (f_{n_k}) of the sequence (f_n) with $\tilde{\delta}(f_{n_k}) > \lambda$ and such that $(f_{n_k}) = (f_n)$. By passing to a further subsequence we can also ensure that

$$\frac{\inf_{n\in\mathbb{N}} \|f_n\|}{\sup_{n\in\mathbb{N}} \|f_n\|} > 1 - \varepsilon.$$

Indeed, the sequence (f_n) is bounded, hence we can find its subsequence (f_{n_k}) such that the $\lim_{k\to\infty} \|f_{n_k}\|$ exists. This limit is nonzero, because otherwise we would have $\tilde{\delta}(f_n) = 0$. We thus obtain the desired subsequence by omitting finitely many members of (f_{n_k}).

The inequality $\tilde{\delta}(f_n) > \lambda$ says that for every subsequence (f_{n_k}) of (f_n) there is some $x'' \in A''$ with $\|x''\| = 1$ such that the diameter of the set of accumulation points of the sequence $(x''(f_{n_k}))_k$ is greater than λ. By Theorem 3.1 the sequence (f_n) does not admit $\frac{\lambda}{\pi}$–ℓ^1–blocks, i.e. there is an infinite $M \subset \mathbb{N}$ such that whenever $a_1, \ldots, a_r \in \mathbb{C}$ satisfy $\sum_{i=1}^r |a_i| = 1$, and $n_1 < \cdots < n_r$ are indices in M, we have $\left\| \sum_{i=1}^r a_i f_{n_i} \right\| > \frac{\lambda}{\pi}$. Hence there is a subsequence (f_{n_k}) of (f_n) such that for each nonzero $(\alpha_k) \in \ell^1$ and $N \in \mathbb{N}$ large enough

$$\left\| \sum_{k=1}^N \frac{\alpha_k}{\sum_{k=1}^N |\alpha_k|} f_{n_k} \right\| > \frac{\lambda}{\pi}.$$

By letting $N \to \infty$ we obtain

$$\frac{\lambda}{\pi} \sum_{k=1}^\infty |\alpha_k| \leq \left\| \sum_{k=1}^\infty \alpha_k f_{n_k} \right\|.$$

Therefore we have for each $(\alpha_k) \in \ell^1$

$$\frac{\lambda}{\pi \sup_{k\in\mathbb{N}} \|f_{n_k}\|} \sum_{k=1}^\infty |\alpha_k| \leq \frac{\lambda}{\pi} \sum_{k=1}^\infty \frac{|\alpha_k|}{\|f_{n_k}\|} \leq \left\| \sum_{k=1}^\infty \alpha_k \frac{f_{n_k}}{\|f_{n_k}\|} \right\| \leq \sum_{k=1}^\infty |\alpha_k|.$$

Let us set

$$r = \frac{\lambda}{\pi \sup_{k\in\mathbb{N}} \|f_{n_k}\|} \qquad \text{and} \qquad \theta = (1-\varepsilon)\, r \inf_{k\in\mathbb{N}} \|f_{n_k}\|.$$

Then

$$\theta = (1-\varepsilon)\frac{\lambda}{\pi}\frac{\inf_{k\in\mathbb{N}}\|f_{n_k}\|}{\sup_{k\in\mathbb{N}}\|f_{n_k}\|} \geq (1-\varepsilon)\frac{\lambda}{\pi}\frac{\inf_{n\in\mathbb{N}}\|f_n\|}{\sup_{n\in\mathbb{N}}\|f_n\|} \geq (1-\varepsilon)^2\frac{\lambda}{\pi}.$$

Without loss of generality we can assume that $(f_{n_k}) = (f_n)$. Then $(\frac{f_n}{\|f_n\|})_n$ is a basic sequence consisting of selfadjoint elements which satisfies

$$r\sum_{k=1}^{\infty}|\alpha_k| \leq \left\|\sum_{k=1}^{\infty}\alpha_k\frac{f_k}{\|f_k\|}\right\| \leq \sum_{k=1}^{\infty}|\alpha_k|, \quad (a_k)\in\ell^1,$$

that is (36) of [10] (where $a'_k = f_k$). By Pfitzner's proof of [10, Theorem 1] we obtain a sequence (x_k) in A and a subsequence (f_{n_k}) of (f_n) for which (35) of [10] is valid (where $a'_{n_k} = f_{n_k}$), i.e. x_k are selfadjoint elements in B_A such that $x_ix_j = 0$, $i, j \in \mathbb{N}$, $i \neq j$, and $\left|f_{n_k}(x_k)\right| > \theta \geq (1-\varepsilon)^2\frac{\lambda}{\pi}$, $k\in\mathbb{N}$. This completes the proof of the claim.

Let (x_k) and (x'_{n_k}) be sequences obtained by the claim. Since $|x'_{n_k}(x_k)| > (1-\varepsilon)^2\frac{\lambda}{\pi}$, $k\in\mathbb{N}$, we have

$$\limsup_{k\to\infty}\sup_{x'\in K}|x'(x_k)| \geq (1-\varepsilon)^2\frac{\lambda}{\pi}.$$

But $\sum x_k$ is a wuC series in A satisfying $\sup_{x'\in B_{A'}}\sum|x'(x_k)| \leq 1$. Indeed, all x_k are contained in a commutative subalgebra B of A, which can be identified with the space $C_0(\Omega)$ for some Ω by the Gelfand representatiton. Then x_k, $k\in\mathbb{N}$, are real continuous functions on Ω with $\|x_k\| = \sup_{\xi\in\Omega}|x_k(\xi)| \leq 1$ and $\{x_i \neq 0\}\cap\{x_j \neq 0\} = \emptyset$, $i \neq j$. Let $x' \in A'$, and let us set $\mu = x' \restriction_B \in B' = C_0(\Omega)' = \mathcal{M}(\Omega)$. For each $N\in\mathbb{N}$ we get

$$\begin{aligned}\sum_{k=1}^{N}|x'(x_k)| = \sum_{k=1}^{N}|\mu(x_k)| = \sum_{k=1}^{N}\left|\int_\Omega x_k\,\mathrm{d}\mu\right| &\leq \sum_{k=1}^{N}\int_{\{x_k\neq 0\}}|x_k|\,\mathrm{d}|\mu| \\ &\leq \int_\Omega 1\,\mathrm{d}|\mu| = \|\mu\| \leq \|x'\|.\end{aligned}$$

Therefore $\sup_{x'\in B_{A'}}\sum_{k=1}^{\infty}|x'(x_k)| \leq 1$.

We thus obtain $\eta(K) \geq (1-\varepsilon)^2\frac{\lambda}{\pi}$. Since $\varepsilon > 0$ and $\lambda < \mathrm{wck}_{A'}(K)$ were chosen arbitrarily, it follows that $\eta(K) \geq \frac{1}{\pi}\,\mathrm{wck}_{A'}(K)$, which completes the proof. □

Remark. It is not clear whether C^*-algebras have also the property $(\mathrm{V}_q)^*_\omega$. From [8, Theorem 4.1] it follows that the answer is affirmative for commutative C^*-algebras. In fact we do not know any example of a Banach space with the property (V_q) but not $(\mathrm{V}_q)^*_\omega$. Regarding the property $(\mathrm{V}_q)_\omega$, we know from [8, Proposition 4.3] that some (commutative) C^*-algebras enjoy this property and some do not.

The following theorem is a kind of "real version" of Theorem 4.1.

Theorem 4.2. *Let A be a C^*-algebra. Then the space A_{sa} has the property (V_q), more precisely, for every bounded $K \subset A'_{\mathrm{sa}}$*

$$\mathrm{wck}_{A'}(K) \leq \eta(K). \tag{7}$$

Proof. The proof is analogous to the previous one, it suffices to use real versions of the theorems that have allowed us to prove Theorem 4.1. Let us sketch it briefly.

Consider a bounded set $K \subset A'_{\mathrm{sa}}$ with $\mathrm{wck}_{A'_{\mathrm{sa}}}(K) > 0$ and an arbitrary $\lambda \in (0, \mathrm{wck}_{A'_{\mathrm{sa}}}(K))$. We find (f_n) in K such that

$$\mathrm{dist}\left(\mathrm{clust}_{((A'_{\mathrm{sa}})'', w^*)}(f_n), A'_{\mathrm{sa}}\right) > \lambda.$$

Since A' is L-embedded, the real version of A' (let us denote it by $(A')_{\mathbb{R}}$) is also L-embedded. But $(A')_{\mathrm{sa}}$ is a 1-complemented subspace of $(A')_{\mathbb{R}}$ and is therefore L-embedded by [6, Proposition IV.1.5]. We thus get

$$\tilde{\delta}(f_n) > 2\lambda$$

from [7, Theorem 1]. Let us fix $\varepsilon > 0$. By passing to a subsequence we arrange that

$$\frac{\inf_{n \in \mathbb{N}} \|f_n\|}{\sup_{n \in \mathbb{N}} \|f_n\|} > 1 - \varepsilon.$$

By the real version of the quantitative Rosenthal's ℓ^1–theorem [2, Theorem 3.2] the sequence (f_n) admits λ–ℓ^1–blocks, which yields a subsequence (f_{n_k}) of the sequence (f_n) that for every $(\alpha_n) \in \ell^1$ satisfies

$$\frac{\lambda}{\sup_{k \in \mathbb{N}} \|f_{n_k}\|} \sum_{k=1}^{\infty} |\alpha_k| \leq \left\| \sum_{k=1}^{\infty} \alpha_k \frac{f_{n_k}}{\|f_{n_k}\|} \right\| \leq \sum_{k=1}^{\infty} |\alpha_k|.$$

Then we proceed exactly as in the proof of Theorem 4.1 to obtain the desired conclusion.

□

5 Relation to the Grothendieck property

Let us remind that a Banach space X has the Grothendieck property if every weak* convergent sequence in its dual is weakly convergent. It is well known that for dual Banach spaces the property (V) implies the Grothendieck property. In this section we prove that this implication holds even for suitable quantitative versions of these properties.

One possible quantification of the Grothendieck property has already been studied in [3] and [9]. Let us remind the definition: Let $c > 0$. A Banach space X is *c-Grothendieck* if

$$\delta(x'_n) \leq c \cdot \delta_{w^*}(x'_n) \tag{8}$$

whenever (x'_n) is a bounded sequence in X'.

A Banach space X has the Grothendieck property if and only if for every sequence (x'_n) in X' the following implication holds:

$$(x'_n) \text{ is weak}^* \text{ Cauchy} \Rightarrow (x'_n) \text{ is weakly Cauchy.}$$

The inequality (8) quantifies this implication. But we can look at the Grothendieck property also in another way: X has the Grothendieck property if and only if every sequence (x'_n) in X' satisfies the implication

$$(x'_n) \text{ is weak}^* \text{ Cauchy} \Rightarrow \{x'_n : n \in \mathbb{N}\} \text{ is a relatively weakly compact set.}$$

If we replace this implication by an inequality

$$\operatorname{wck}_{X'}(\{x'_n : n \in \mathbb{N}\}) \leq c \cdot \delta_{w^*}(x'_n)$$

where $c > 0$ is some constant not depending on (x'_n), we obtain another quantitative version of the Grothendieck property. We will prove that all dual Banach spaces with the property (V_q) have this kind of quantitative Grothendieck property (see Corollary 5.2). We do not know whether the latter quantitative Grothendieck property implies the former one (with a larger constant).

Theorem 5.1. *Let X be a Banach space. Then for every bounded sequence (x''_n) in X''*

$$\eta(\{x''_n : n \in \mathbb{N}\}) \leq \tfrac{1}{2}\delta_{w^*}(x''_n).$$

Proof. Let (x''_n) be a bounded sequence in X''. The case $\eta(\{x''_n : n \in \mathbb{N}\}) = 0$ is trivial. Suppose that $\eta(\{x''_n : n \in \mathbb{N}\}) > 0$ and fix $\delta \in (0, \eta(\{x''_n : n \in \mathbb{N}\}))$. Let us find $\varepsilon > 0$ such that $\eta(\{x''_n : n \in \mathbb{N}\}) > \delta + \varepsilon$. By the definition of the quantity η we can find a wuC series $\sum_{k=1}^{\infty} x'_k$ in X' with $\sup_{x'' \in B_{X''}} \sum_{k=1}^{\infty} |x''(x'_k)| \leq 1$ such that $\limsup_{k\to\infty} \sup_{n\in\mathbb{N}} |x''_n(x'_k)| > \delta + \varepsilon$. Since (x'_k) is a weakly null sequence, there are subsequences of (y''_n) of (x''_n) and (y'_k) of (x'_k) which for all $n \in \mathbb{N}$ satisfy $|y''_n(y'_n)| > \delta + \varepsilon$. The sequence (y'_n) is weakly null in X' and (y''_n) is a bounded sequence in X'', hence by Simons' extraction lemma [12, Theorem 1] there is a strictly increasing sequence of indices (n_k) such that for all $k \in \mathbb{N}$

$$\sum_{\substack{m\in\mathbb{N}\\ m\neq k}} |y''_{n_k}(y'_{n_m})| < \varepsilon.$$

Let us define

$$\alpha_k = \begin{cases} (-1)^k \operatorname{sgn}^{-1}(y''_{n_k}(y'_{n_k})), & y''_{n_k}(y'_{n_k}) \neq 0, \\ 0, & y''_{n_k}(y'_{n_k}) = 0, \end{cases} \qquad k \in \mathbb{N},$$

where sgn denotes the complex signum function, i.e. $\operatorname{sgn}(z) = \frac{z}{|z|}$, $z \in \mathbb{C} \setminus \{0\}$. Set

$$x' = w^*\text{-}\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k y'_{n_k} \in X'.$$

Then $x' \in B_{X'}$ because for all $x \in B_X$

$$|x'(x)| = \left|\sum_{k=1}^{\infty} \alpha_k z'_{n_k}(x)\right| \leq \sum_{k=1}^{\infty} |z'_{n_k}(x)| \leq \sum_{n=1}^{\infty} |x'_n(x)| \leq \sup_{x''\in B_{X''}} \sum_{n=1}^{\infty} |x''(x'_n)| \leq 1.$$

For each $k \in \mathbb{N}$ even

$$\begin{aligned}
\operatorname{Re} y''_{n_k}(x') &= \alpha_k y''_{n_k}(y'_{n_k}) + \operatorname{Re}\Bigg(\sum_{\substack{m\in\mathbb{N}\\ m\neq k/2}} y''_{n_k}(\alpha_{2m} y'_{n_{2m}})\Bigg) \\
&\quad - \operatorname{Re}\Bigg(\sum_{m\in\mathbb{N}} y''_{n_k}(\alpha_{2m-1} y'_{n_{2m-1}})\Bigg) \\
&\geq |y''_{n_k}(y'_{n_k})| - \sum_{\substack{m\in\mathbb{N}\\ m\neq k/2}} |y''_{n_k}(y'_{n_{2m}})| - \sum_{m\in\mathbb{N}} |y''_{n_k}(y'_{n_{2m-1}})| \\
&= |y''_{n_k}(y'_{n_k})| - \sum_{\substack{m\in\mathbb{N}\\ m\neq k}} |y''_{n_k}(y'_{n_m})| \\
&> (\delta + \varepsilon) - \varepsilon = \delta.
\end{aligned}$$

Analogously, for each $k \in \mathbb{N}$ odd

$$\begin{aligned}
\operatorname{Re} y''_{n_k}(x') &= \alpha_k y''_{n_k}(y'_{n_k}) + \operatorname{Re}\left(\sum_{m\in\mathbb{N}} y''_{n_k}(\alpha_{2m} y'_{n_{2m}})\right) \\
&\quad - \operatorname{Re}\left(\sum_{\substack{m\in\mathbb{N}\\ m\neq (k+1)/2}} y''_{n_k}(\alpha_{2m-1} y'_{n_{2m-1}})\right) \\
&\leq -|y''_{n_k}(y'_{n_k})| + \sum_{\substack{m\in\mathbb{N}\\ m\neq k}} |y''_{n_k}(y'_{n_m})| \\
&< -(\delta+\varepsilon) + \varepsilon = -\delta.
\end{aligned}$$

Therefore

$$\inf_{n\in\mathbb{N}} \sup_{k,l\geq n} |y''_{n_k}(x') - y''_{n_l}(x')| \geq \inf_{n\in\mathbb{N}} \sup_{k,l\geq n} \left|\operatorname{Re}\left(y''_{n_k}(x') - y''_{n_l}(x')\right)\right| \geq 2\delta.$$

It follows that $\delta_{w^*}(x''_n) \geq \delta_{w^*}(y''_{n_k}) \geq 2\delta$. Since $\delta < \eta(\{x''_n : n \in \mathbb{N}\})$ was chosen arbitrarily, we obtain the desired inequality. □

Corollary 5.2. *Let X be a Banach space and $C > 0$. Suppose that each bounded $K \subset X''$ satisfy*

$$\operatorname{wck}_{X''}(K) \leq C \cdot \eta(K) \tag{9}$$

(i.e. X' enjoys the property (V_q)). Then for every bounded sequence (x''_n) in X''

$$\operatorname{wck}_{X''}\left(\{x''_n : n \in \mathbb{N}\}\right) \leq \tfrac{1}{2} C \cdot \delta_{w^*}(x''_n).$$

Proof. It suffices to combine the previous theorem with the inequality (9) applied to $K = \{x''_n : n \in \mathbb{N}\}$. □

Corollary 5.3. *Let A be a von Neumann algebra. Then A has a quantitative version of the Grothendieck property – more precisely, for every bounded sequence (x'_n) in A'*

$$\operatorname{wck}_{A'}\left(\{x'_n : n \in \mathbb{N}\}\right) \leq \tfrac{1}{2}\pi\, \delta_{w^*}(x'_n).$$

Proof. Since every von Neumann algebra is a C^*-algebra and a dual Banach space, the assertion follows from Theorem 4.1 and Corollary 5.2. □

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