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MASTER THESIS

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Numerical solution of nonlinear transport problems

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Numerické řešení nelineárních transportních problémů

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Abstrakt: Práce je zaměřená na numerickou simulaci dvoufázového proudění. Je studován matematický model a numerická aproximace toku dvou nemísitelných nestlačitelných tekutin. Rozhraní mezi tekutinami je popsáno pomocí pomocí tzv. level set metody. Představena je diskretizace problému v prostoru a v čase. Metoda konečných prvků se zpětnou Eulerovou metodou je aplikována na Navierovy-Stokesovy rovnice a časoprostorová nespojitá Galerkinova metoda je použita k řešení transportního problému. Důraz je kladen na analýzu chyby nespojité Galerkinovy metody přímek a časoprostorové nespojité Galerkinovy metody pro transportní problém. Jsou prezentovány numerické výsledky.

Klíčová slova: level set funkce, Navierovy-Stokesovy rovnice, časoprostorová nespojitá Galerkinova metoda

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Abstract: This work focuses on the study of two-phase flow. Mathematical model and numerical approximation of the flow of two immiscible incompressible fluids is studied. The interface between the fluids is described with the aid of the level set method. The discretization in space and in time is introduced. The finite element method with backward Euler time discretization is applied to the Navier-Stokes equations and space-time discontinuous Galerkin method is used for solving the non-linear transport problem. The emphasis is put on the error analysis of discontinuous Galerkin method of lines and space-time discontinuous Galerkin method for the transport problem. The numerical results are shown.

Keywords: level set function, Navier-Stokes equations, space-time discontinuous Galerkin method

Contents

1 Introduction

In several areas of science and technology we meet the necessity to simulate two-phase flows. As an example we can mention chemical technology, where the motion of gas bubbles in liquids occurs. Another possibility is to consider the flow of melted metals or glass or flow of fresh concrete. Further, we can mention two-phase flow in steam turbines, water turbines, flow in nuclear reactors and groundwater flow.

The importance of flow-phase flows is manifested by a number of works dealing with this subject. We can mention, for example, [1], [4], [5], [14], [15], [16], [17], [20], [21], [22], [23], [24] and [25].

This thesis is devoted to the study of mathematical model describing flow of two immiscible incompressible fluids. We consider the incompressible Navier-Stokes system formed by the momentum equations and continuity equation, coupled with a transport equation used for the identification of the interface between the fluids.

We introduce the formulation of the coupled problem consisting of the incompressible Navier-Stokes system with variable piecewise constant density and viscosity, corresponding to different properties of both phases. The behaviour of the interface between two phases and its motion is described with the aid of the level set method (see, e.g., [19]), which is based on a hyperbolic transport equation.

The main emphasis is put on the numerical solution of the resulting nonlinear transport problem. The Navier-Stokes system is discretized in space by the finite element method using the Taylor-Hood $P2/P1$ elements. The time discretization is carried out by the backward difference formula. The resulting discrete problem is solved with the aid of Oseen iterations. The discretization of the level set transport equation is realized by the discontinuous Galerkin method (DGM) of lines and also by the space-time discontinuous Galerkin method.

The main part of the thesis is devoted to the error analysis of the discretization of the transport equation. We estimate the error in space and also in time and consider the dependence on the partition in time and space.

The use of the worked out method is demonstrated by numerical experiments showing the development of the interface between two fluid phases.

1.1 Some definitions and notation

Let us define spaces of functions we shall work with in the following text. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The vector $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \geq 0$ are integers, is called a multindex. We set $|\alpha| = \alpha_1 + \alpha_2$.

By L^p, $p \in [1, \infty)$, we denote the Lebesgue space defined as

$$L^p(\Omega) = \left\{ f \text{ messurable on } \Omega | \int\limits_\Omega |f|^p \, dx < \infty \right\},$$

with norm

$$\|f\|_{L^p} = \left(\int\limits_\Omega |f|^p \, dx \right)^{\frac{1}{p}}.$$

For $p = \infty$ we set

$$L^\infty(\Omega) = \left\{ f \text{ messurable on } \Omega | \, \exists C \in \mathbb{R}^+ \, |f(x)| \leq C \text{ a.e. on } \Omega \right\},$$

with norm

$$\|f\|_{L^\infty(\Omega)} = \inf \{ C \geq 0 : \, |f(x)| \leq C \text{ for almost every } x \}.$$

By $W^{k,p}$, $k \in \mathbb{N}$, $p \in [1, \infty]$, we denote the Sobolev spaces defined as

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega), \forall \alpha \, |\alpha| \leq k, \, D^\alpha u \in L^p(\Omega) \},$$

with norm

$$\|u\|_{k,p,\Omega} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \, p \in [1, \infty),$$

$$\|u\|_{k,\infty,\Omega} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}$$

and with seminorm

$$|u|_{k,p,\Omega} = \left(\sum_{|\alpha| = k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^\alpha} = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \right) u.$$

For the case $p = 2$ we set$H^k(\Omega) = W^{k,2}(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$.

In the following section we shall also use the Bochner spaces $C\left([0,T]\,;\,W^{k,p}\left(\Omega\right)\right)$ defined by

$$C\left([0,T]\,;\,W^{k,p}\left(\Omega\right)\right)=\left\{u:[0,T]\rightarrow W^{k,p}\left(\Omega\right),\;u\left(t\right)\;\text{ continuous}\right\}$$

with norm

$$\left\|u\right\|_{C\left([0,T];\,W^{k,p}(\Omega)\right)}=\max_{t\in[0,T]}\left\|u\left(t\right)\right\|_{k,p,\Omega}.$$

Further, if $s,q\geq 1$ are integers, we define the following Bochner spaces

$$L^{2}\left([0,T]\,;\,H^{s}\left(\Omega\right)\right)=\left\{u:(0,T)\rightarrow X;\;\int\limits_{0}^{T}\left\|u\left(t\right)\right\|_{H^{s}(\Omega)}^{2}dt<+\infty\right\}$$

with norm

$$\left\|u\right\|_{L^{2}([0,T];\,H^{s}(\Omega))}=\left(\int\limits_{0}^{T}\left\|u\left(t\right)\right\|_{H^{s}(\Omega)}^{2}dt\right)^{1/2}$$

and seminorm

$$\left|u\right|_{L^{2}([0,T];\,H^{s}(\Omega))}=\left(\int\limits_{0}^{T}\left|u\left(t\right)\right|_{H^{s}(\Omega)}^{2}dt\right)^{1/2},$$

$$\begin{aligned}H^{q+1}\left([0,T]\,;\,H^{s}\left(\Omega\right)\right)\;&=\;\Big\{u\in L^{2}\left([0,T]\,;\,H^{s}\right),\,\frac{\mathrm{d}^{j}u}{\mathrm{d}t^{j}}\in L^{2}\left([0,T]\,;\,H^{s}\right),\\&\qquad j=1,\ldots,q+1\Big\},\end{aligned}$$

where $\frac{\mathrm{d}^{j}u}{\mathrm{d}t^{j}}$ are distributional derivative. This space is equipped with the seminorm

$$\left|u\right|_{H^{q+1}([0,T];\,H^{s}(\Omega))}=\left(\int\limits_{0}^{T}\left|\frac{\mathrm{d}^{q+1}u\left(t\right)}{\mathrm{d}t^{q+1}}\right|_{H^{s}(\Omega)}dt\right)^{1/2}.$$

If X is a linear space, then we set $X^{2}=X\times X$.

2 Mathematical Model

Let $\Omega \subset \mathbb{R}^2$ be a domain such that its closure $\overline{\Omega}$, consists of two different time dependent parts $\overline{\Omega}_1(t)$ and $\overline{\Omega}_2(t)$ i.e., $\overline{\Omega} = \overline{\Omega}_1(t) \cup \overline{\Omega}_2(t)$, where $\Omega_1(t)$ and $\Omega_2(t)$ are disjoint domains. We assume that the domain $\Omega_1(t)$ is occupied by a liquid and the domain $\Omega_2(t)$ is occupied by a gas. The boundary of Ω is Lipschitz continuous and consists of several parts: $\partial\Omega(t) = \Gamma_D \cup \Gamma_S \cup \Gamma_I(t)$, where the boundary part $\Gamma_I(t)$ denotes the interface boundary between the two fluids. We assume that $\Gamma_I(t)$ is a closed curve. On Γ_D and Γ_S we prescribe different boundary conditions.

We consider the following model problem consisting of the Navier-Stokes system of momentum equations

$$\rho^{(k)}\frac{\partial v_i^{(k)}}{\partial t}+\rho^{(k)}\sum_{j=1}^{2}\frac{\partial\left(v_i^{(k)}v_j^{(k)}\right)}{\partial x_j} = -\frac{\partial p^{(k)}}{\partial x_i}+2\mu^{(k)}\sum_{j=1}^{2}\frac{\partial d_{i,j}\left(\boldsymbol{v}^{(k)}\right)}{\partial x_j}+\rho^{(k)}f_i, \quad i=1,2, \tag{2.1}$$

and the continuity equation

$$\mathrm{div}\boldsymbol{v}^{(k)} = 0. \tag{2.2}$$

Here the index $k = 1,\,2$ denotes the two fluids with pressure $p^{(k)}$, viscosity coefficient $\mu^{(k)}$, density $\rho^{(k)}$, the fluid velocity $\boldsymbol{v}^{(k)}$ with components $v_i^{(k)}$, $i = 1,\,2$, and f_i denote the components of the body forces, $i = 1,\,2$. We denote

$$d_{i,j}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i^{(k)}}{\partial x_j}+\frac{\partial v_j^{(k)}}{\partial x_i}\right), \quad i,\,j=1,2. \tag{2.3}$$

Since the velocity of the gas will be small, we shall assume that both fluids are incompressible and we consider the continuity equation in the form (2.2).

The functions $\boldsymbol{v}^{(k)}$ and $p^{(k)}$ are defined in the domain $\Omega_k(t)$ and time $t \in [0,T]$, where $T > 0$. We assume that $\rho^{(k)}$ and $\mu^{(k)}$ are positive constants. For simplicity we shall use the notation $\boldsymbol{v}$ and p defined by

$$\boldsymbol{v}|_{\Omega_k} = \boldsymbol{v}^{(k)}\text{and } p|_{\Omega_k} = p^{(k)}, \quad k=1,2. \tag{2.4}$$

Further, $\boldsymbol{\sigma}|_{\Omega_k} = \boldsymbol{\sigma}^{(k)}$, $\boldsymbol{\sigma}^{(k)} = \left(\sigma_{ij}^{(k)}\right)_{i,j=1}^{2}$, $k = 1,\,2$, denotes the total stress tensor with components

$$\sigma_{ij}^{(k)} = -p^{(k)}\delta_{ij} + \mu^{(k)} \left(\frac{\partial v_i^{(k)}}{\partial x_j} + \frac{\partial v_j^{(k)}}{\partial x_i} \right), \tag{2.5}$$

where δ_{ij} is the Kronecker symbol.

Using the notation (2.5), we can write system (2.1) in the form

$$\rho^{(k)} \left(\frac{\partial \boldsymbol{v}^{(k)}}{\partial t} + \left(\boldsymbol{v}^{(k)} \cdot \nabla \right) \boldsymbol{v}^{(k)} \right) = \rho^{(k)} \boldsymbol{f} + \mathrm{div} \boldsymbol{\sigma}^{(k)} \text{ in } \Omega^{(k)}, \quad k = 1, 2. \tag{2.6}$$

In section 3.2, we shall equip system (2.1) and (2.2) by boundary and initial conditions.

To model the movement of the free boundary $\Gamma_I(t)$ we use the level set method. We couple equations (2.1) and (2.2) with the transport problem

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla \varphi = 0, \tag{2.7}$$

$$\varphi = \varphi_D \quad \text{on } \partial\Omega^- \times (0, T), \tag{2.8}$$

$$\varphi(x, 0) = \varphi^0(x), \quad x \in \Omega, \tag{2.9}$$

where the function $\varphi > 0$ and $\varphi < 0$ in $\Omega_1(t)$ and $\Omega_2(t)$, respectively. The interface $\Gamma_I(t)$ is characterized by the condition $\varphi = 0$. In (2.8), $\partial\Omega^-$denotes the part of $\partial\Omega$, on which $\boldsymbol{v} \cdot \boldsymbol{n} < 0$, where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$. (See Section 3.)

The following two sections will be devoted to the numerical solution of the level set problem.

3 Discretization of the level set problem

3.1 Level set method

First formulate the weak solution of the level set equation (2.7).

Let us assume that the transport flow velocity function $\boldsymbol{v} : \overline{Q_T} \to \mathbb{R}^d$, is prescribed. We consider the boundary $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+$ and for all $t \in (0,T)$ we assume that

$$\begin{aligned} \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) &< 0 \text{ on } \partial\Omega^-, \\ \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) &\geq 0 \text{ on } \partial\Omega^+, \end{aligned}$$

where $\boldsymbol{n}(x)$ denotes the unit outer normal to the boundary $\partial\Omega$, the part $\partial\Omega^-$ represents the inlet part of the boundary (through which the fluid enters the domain Ω), the part of $\partial\Omega^+$, where $\boldsymbol{v} \cdot \boldsymbol{n} \geq 0$ represents the outlet part (through which the fluid leaves the domain and $\boldsymbol{v} \cdot \boldsymbol{n} > 0$) and the impermeable walls (where $\boldsymbol{v} \cdot \boldsymbol{n} = 0$).

In this section we shall be concerned with the linear initial-boundary value convection problem (2.7)-(2.9): Find $\varphi : Q_T \to \mathbb{R}$ such that

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{v} \cdot \nabla\varphi = 0 \quad \text{in } Q_T, \tag{3.1}$$

$$\varphi = \varphi_D \quad \text{on } \partial\Omega^- \times (0,T), \tag{3.2}$$

$$\varphi(x,0) = \varphi^0(x); \quad x \in \Omega. \tag{3.3}$$

Let us have the following assumptions on the data:

$$\varphi_D \text{ is the trace of some } \varphi^* \in C\left([0,T]; H^1(\Omega)\right) \cap L^\infty(Q_T) \text{ on } \partial\Omega^- \times (0,T), \tag{3.4}$$

$$\varphi_0 \in L^2(\Omega), \tag{3.5}$$

$$\boldsymbol{v} \in C\left([0,T]; W^{1,\infty}(\Omega)\right), \; |\boldsymbol{v}| \leq C_v \text{ in } \overline{Q_T}, \; |\nabla\boldsymbol{v}| \leq C_v \text{ a.e. in } Q_T. \tag{3.6}$$

Now we derive the week formulation. We multiply the equation by any

$$\psi \in V = \left\{\psi \in H^1(\Omega); \psi|_{\partial\Omega^-} = 0\right\},$$

apply the Green theorem and use the boundary and initial conditions. We obtain the following definition:

Definition 3.1. We call the function φ a weak solution to problem (3.1)-(3.3), if it satisfies the conditions

$\varphi - \varphi^* \in L^2(0,T;V)\,,\ \varphi \in L^\infty(Q_T)\,;$
$\frac{d}{dt}\int_\Omega \varphi\psi dx - \int_\Omega \varphi\nabla\cdot(\psi\boldsymbol{v})\,dx + \int_{\partial\Omega^+}(\boldsymbol{v}\cdot\boldsymbol{n})\,\varphi\psi dS = 0$
for all $\psi \in V$ in the sense of distributions on $(0,T)\,,$
$\varphi(0) = \varphi^0$ in Ω.

We assume the existence and sufficient regularity of the weak solution φ, namely,

$$\frac{\partial\varphi}{\partial t} \in L^2\left(0,T;\ H^s(\Omega)\right), \tag{3.7}$$

where $s \geq 2$ is an integer. Then $\varphi \in C\left([0,T]\,;\ H^s(\Omega)\right).$

3.1.1 Space discretization

For the space discretization we shall use the discontinuous Galerkin method (DGM).

We consider a system of triangulations $\{\mathcal{T}_h\}_{h\in(0,\overline{h})}$ with $\overline{h} > 0$ of the closure of the domain Ω into a finite number of closed triangles with disjoint interiors.

Now let us denote by $\mathcal{F}_h$ the system of all faces Γ of all elements $K \in \mathcal{T}_h$. Furthermore, we denote the set of all boundary faces by $\mathcal{F}_h^B$, the set of the "Dirichlet" boundary faces by $\mathcal{F}_h^D$, and the set of all inner faces by $\mathcal{F}_h^I$. Hence, if $\Gamma \in \mathcal{F}_h^B$, then $\Gamma \subset \partial\Omega$ and if $\Gamma \in \mathcal{F}_h^D$, then $\Gamma \subset \partial\Omega^-$ (where the Dirichlet condition is prescribed).

We introduce the assumptions on the meshes $\mathcal{T}_h$:

(A1) The triangulations $\mathcal{T}_h$, $h \in \left(0,\overline{h}\right)$, are conforming. This means that for two elements K, $K' \in \mathcal{T}_h$, $K \neq K'$, either $K \cap K' = \emptyset$ or $K \cap K'$ is a common vertex or $K \cap K'$ is a common face of K and K'.

(A2) The system $\{\mathcal{T}_h\}_{h\in(0,\overline{h})}$ of triangulations is shape-regular: there exists a positive constant C_R such that

$$\frac{h_K}{\rho_K} \leq C_R \quad \forall K \in \mathcal{T}_h \quad \forall h \in \left(0,\overline{h}\right),$$

where ρ_K denotes the radius of the largest circle inscribed into K and h_K denotes the diameter of K .

(A3) Every $\Gamma \in \mathcal{F}_h$, $h \in \left(0,\overline{h}\right)$, is associated with the quantity $h_\Gamma > 0$, which represents a “one-dimensional" size of the face Γ, and satisfies the equivalence condition with h_K, i.e., there exist constants C_T, $C_G > 0$ independent of h, K and Γ such that

$$C_T h_K \leq h_\Gamma \leq C_G h_K, \quad K \in \mathcal{T}_h, \Gamma \in \mathcal{F}_h, \Gamma \subset \partial K.$$

As an example we can define h_Γ as the length of Γ. Another possibility is to set $h_\Gamma = \frac{(h_K + h_{K'})}{2}$, where K and K' are such elements that $\Gamma \subset K \cap K'$.

For $K \in \mathcal{T}_h$ we define the inlet and outlet parts of the boundary of K as follows:

$$\begin{aligned} \partial K^- (t) &= \{x \in \partial K; \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) < 0\}, \\ \partial K^+ (t) &= \{x \in \partial K; \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) \geq 0\}, \end{aligned}$$

where $\boldsymbol{n}$ denotes the unit outer normal to ∂K.

For any $k \in \mathbb{N}$, over a triangulation $\mathcal{T}_h$ we define the broken Sobolev space

$$H^k(\Omega, \mathcal{T}_h) = \left\{\psi \in L^2(\Omega); \boldsymbol{v}|_K \in H^k(K) \ \forall K \in \mathcal{T}_h\right\}$$

with norm

$$\|\psi\|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} \|\psi\|^2_{H^k(K)}\right)^{1/2}$$

and seminorm

$$|\psi|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |\psi|^2_{H^k(K)}\right)^{1/2}.$$

Now we discretize the convective term $\boldsymbol{v} \cdot \nabla\varphi$. We follow the treatment from [9, Chapter 4]. We multiply this term by any $\psi \in H^1(\Omega, \mathcal{T}_h)$, integrate it over an element K and apply the Green theorem. (For a moment we do not emphasize the dependence on t.) We obtain

$$\begin{aligned} \int_K (\boldsymbol{v} \cdot \nabla\varphi)\, \psi dx &= -\int_K \varphi \nabla \cdot (\psi \boldsymbol{v})\, dx + \int_{\partial K} (\boldsymbol{v} \cdot \boldsymbol{n})\, \varphi\psi dS \\ &= -\int_K \varphi \nabla \cdot (\psi \boldsymbol{v})\, dx + \int_{\partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\, \varphi\psi dS + \int_{\partial K^+} (\boldsymbol{v} \cdot \boldsymbol{n})\, \varphi\psi dS. \end{aligned}$$

In what follows, on the inflow part of the boundary ∂K we shall write φ^- instead of φ. In case when $\varphi \in H^1(\Omega, \mathcal{T}_h)$ the symbol φ^- denotes the trace of φ on ∂K from the side of elements adjacent to ∂K from outside of K. Of course, if φ satisfies (3.7), then $\varphi = \varphi^-$. Hence we use only the information from outside the element K. If $x \in \partial\Omega^-$, then we put $\varphi^-(x) = \varphi_D(x)$. Using all these informations, we obtain

$$\begin{aligned}
&\int_K (\boldsymbol{v} \cdot \nabla\varphi)\,\psi dx && (3.8)\\
=\ & -\int_K \varphi\nabla \cdot (\psi\boldsymbol{v})\,dx + \int_{\partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi^-\psi dS + \int_{\partial K^+} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS\\
=\ & -\int_K \varphi\nabla \cdot (\psi\boldsymbol{v})\,dx + \int_{\partial K} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS - \int_{\partial K^+ \cup \partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS\\
& + \int_{\partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi^-\psi dS + \int_{\partial K^+} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS.
\end{aligned}$$

Equation (3.8) holds since $\int_{\partial K} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS - \int_{\partial K^+ \cup \partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\,\varphi\psi dS = 0$.
Applying the Green theorem to the first term we find that

$$\begin{aligned}
&\int_K (\boldsymbol{v} \cdot \nabla\varphi)\,\psi dx && (3.9)\\
=\ & \int_K (\boldsymbol{v} \cdot \nabla\varphi)\,\psi dx + \int_{\partial K^-} (\boldsymbol{v} \cdot \boldsymbol{n})\left(\varphi^- - \varphi\right)\psi dS\\
=\ & \int_K (\boldsymbol{v} \cdot \nabla\varphi)\,\psi dx - \int_{\partial K^- \setminus \partial\Omega} (\boldsymbol{v} \cdot \boldsymbol{n})\,[\varphi]\,\psi dS - \int_{\partial K^- \cap \partial\Omega} (\boldsymbol{v} \cdot \boldsymbol{n})\,(\varphi - \varphi_D)\,\psi dS,
\end{aligned}$$

where $[\varphi]$ is the jump of a function φ:

$$[\varphi] = \varphi - \varphi^- \text{on } \partial K^- \setminus \partial\Omega. \tag{3.10}$$

Now we are ready to derive the discrete problem. Under assumption (3.7) we multiply equation (3.1) by any $\psi \in H^1(\Omega, \mathcal{T}_h)$, integrate it over each element K and sum it over all elements $K \in \mathcal{T}_h$. We get

$$\sum_{K \in \mathcal{T}_h} \int_K \frac{\partial\varphi}{\partial t}\psi dx + \sum_{K \in \mathcal{T}_h} \int_K (\boldsymbol{v} \cdot \nabla\varphi)\,\psi dx = 0.$$

Using identity (3.9) for the convective term we obtain

$$\sum_{K\in\mathcal{T}_h}\int_K \frac{\partial\varphi}{\partial t}\psi dx + \sum_{K\in\mathcal{T}_h}\int_K (\boldsymbol{v}\cdot\nabla\varphi)\,\psi dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,[\varphi]\,\psi dS$$
$$- \sum_{K\in\mathcal{T}_h}\int_{\partial K^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,(\varphi-\varphi_D)\,\psi dS \;=\; 0.$$

This identity leads us to the definition of the forms $(\cdot,\cdot)$, b_h, l_h defined as

$$\left(\frac{\partial\varphi}{\partial t},\psi\right) = \int_\Omega \frac{\partial\varphi}{\partial t}\psi dx, \tag{3.11}$$

$$\begin{aligned} b_h(\varphi,\psi) &= \sum_{K\in\mathcal{T}_h}\int_K (\boldsymbol{v}\cdot\nabla\varphi)\,\psi dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,[\varphi]\,\psi dS \\ &\quad - \sum_{K\in\mathcal{T}_h}\int_{\partial K^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\varphi\psi dS, \end{aligned} \tag{3.12}$$

$$l_h(\psi)(t) = - \sum_{K\in\mathcal{T}_h}\int_{\partial K^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\varphi_D(t)\,\psi dS. \tag{3.13}$$

We see that the exact solution u satisfies the following identity for each $\psi \in H^1(\Omega,\mathcal{T}_h)$:

$$\left(\frac{\partial\varphi(t)}{\partial t},\psi\right) + b_h(\varphi(t),\psi) = l_h(\psi)(t). \tag{3.14}$$

We want to define an approximate solution for each $t\in(0,T)$. To this end, we introduce the finite dimensional space

$$S_{hp} = \left\{\psi\in L^2(\Omega),\ \psi|_K \in P^p(K)\,\forall K\in\mathcal{T}_h\right\}, \tag{3.15}$$

where $p \geq 1$ is an integer and $P^p(K)$ is the space of polynomials on K of degree at most p.

Definition 3.2. The DG approximate solution of the level set problem (3.1)-(3.3) is defined as a function φ_h such that

$$\begin{aligned} &\varphi_h \in C^1([0,T);\,S_{hp}) \qquad\qquad (3.16)\\ &\left(\frac{\partial\varphi_h(t)}{\partial t},\psi_h\right) + b_h(\varphi_h(t),\psi_h) = l_h(\psi_h)(t) \quad \forall\psi_h\in S_{hp} \quad \forall t\in(0,T), \\ &(\varphi_h(0),\psi_h) = \left(\varphi^0,\psi_h\right) \qquad \forall\psi_h\in S_{hp}. \end{aligned}$$

3.1.2 Space-time discretization

In the previous section we have derived the space discretization of our problem (3.1)-(3.3) using the DGM. Now we shall apply to our problem a time discretization. We use the space-time discontinuous Galerkin method, STDGM for short.

Let $M > 1$ be an integer. We shall construct a partition of the time interval $[0, T]$ as follows:

$$0 = t_0 < t_1 < \cdots < t_M = T,\, t_k = k\tau,$$

where $\tau > 0$ is a time step.

We denote

$$I_m = (t_{m-1},\, t_m)\,, \quad \overline{I_m} = [t_{m-1},\, t_m]\,.$$

So it holds

$$[0, T] = \cup_{m=1}^{M} \overline{I_m}, \quad I_m \cap I_n = \emptyset\ for\, m \neq n,\ m, n = 1, \ldots, M.$$

For a function ψ defined in $\cup_{m=1}^{M} I_m$ we introduce the notation

$$\psi_m^{\pm} = \lim_{t \to t_m \pm} \psi\,(t)\,, \quad \{\psi\}_m = \psi_m^{+} - \psi_m^{-},$$

assuming that the one-sided limits exist.

We still discretize problem (3.1)-(3.3) with the same data assumption and domain Ω. We just use different partitions and function spaces on different time intervals I_m. For each time instant t_m, $m = 0, \ldots, M$, and interval I_m, $m = 1, \ldots, M$, we consider a partition $\mathcal{T}_{h,m}$ of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles with mutually disjoint interiors. We also assume that the system of triangulations

$$\{\mathcal{T}_{h,\tau}\}_{h \in \left(0, \overline{h}\right)}\,,\, \overline{h} > 0,\, \tau_0 > 0, \quad \mathcal{T}_{h,\tau} = \{\mathcal{T}_{h,m}\}_{m=0}^{M}$$

is conforming (cf. condition (A1) from Section 3.1.1) and satisfies the shape regularity condition ((A2) from Section 3.1.1) a the equivalence condition ((A3) from Section 3.1.1):

$$\begin{aligned} \frac{h_K}{\rho_K} &\leq C_R, \qquad K \in \mathcal{T}_{h,m},\, m = 0, \ldots, M,\, h \in \left(0, \overline{h}\right), \\ C_T h_K &\leq h_\Gamma \leq C_G h_K, \quad K \in \mathcal{T}_{h,m},\, \Gamma \in \mathcal{F}_{h,m},\, \Gamma \subset \partial K,\, m = 0, \ldots, M,\, h \in \left(0, \overline{h}\right). \end{aligned}$$

Instead of the triangulation $\mathcal{T}_h$ for the DGM the triangulations $\mathcal{T}_{h,m}$ may be different for different m.

We shall use the similar notation as for the DGM, only with one more subscript m. We denote by $\mathcal{F}_{h,m}$ the system of all faces of all elements $K \in \mathcal{T}_{h,m}$. Similarly as in Section (3.1.1) we denote the set of all boundary faces by $\mathcal{F}_{h,m}^{B}$, the set of the "Dirichlet" boundary faces by $\mathcal{F}_{h,m}^{D}$ and the set of all inner faces by $\mathcal{F}_{h,m}^{I}$.

We set

$$h_K = \operatorname{diam}(K) \;\text{ for } K \in \mathcal{T}_{h,m}, \quad h_m = \max_{K \in \mathcal{T}_{h,m}} h_K, \quad h = \max_{m=1,\dots,M} h_m$$

and by ρ_K we denote the radius of the largest ball inscribed into K.

For $K \in \mathcal{T}_{h,m}$ and $t \in I_m$ we define the inlet and outlet parts of the boundary of K as follows:

$$\begin{aligned} \partial K^{-}(t) &= \{x \in \partial K;\, \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) < 0\}, \\ \partial K^{+}(t) &= \{x \in \partial K;\, \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) \geq 0\}, \end{aligned}$$

where $\boldsymbol{n}$ denotes the unit outer normal to ∂K.

For any $k \geq 1$, over a triangulation $\mathcal{T}_{h,m}$ we define the broken Sobolev space

$$H^k(\Omega,\, \mathcal{T}_{h,m}) = \left\{\psi \in L^2(\Omega);\, \boldsymbol{v}|_K \in H^k(K) \;\forall K \in \mathcal{T}_{h,m}\right\}$$

and seminorm

$$|\psi|_{H^k\left(\Omega,\mathcal{T}_{h,m}\right)} = \left(\sum_{K \in \mathcal{T}_{h,m}} |\psi|^2_{H^k(K)}\right)^{1/2}.$$

The jump of a function is defined in the same way as in Section (3.1.1).

Now we need to define the finite-dimensional space analogous to S_{hp}.

Let $p,\, q \geq 1$ be integers. For each $m = 1, \dots, M$ we define the finite-dimensional space

$$S_{h,m}^{p} = \left\{\psi \in L^2(\Omega);\, \psi|_K \in P^p \;\forall K \in \mathcal{T}_{h,m}\right\}.$$

The approximate solution will be sought in the space of piecewise polynomial functions in space and also in time. Therefore, we define a space $S_{h,\tau}^{p,q}$:

$$\begin{aligned} S_{h,\tau}^{p,q} = & \left\{\psi \in L^2(Q_T);\, \psi(x,t)\,|_{t \in I_m} = \sum_{i=0}^{q} t^i \psi_{m,i}(x)\right. \\ & \left.\text{with } \psi_{m,i} \in S_{h,m}^{p},\; i = 0, \dots, q,\; m = 1, \dots, M\right\}. \end{aligned}$$

We consider problem (3.1)-(3.3) with regular exact solution satisfying the condition

$$\frac{\partial \varphi}{\partial t} \in L^2\left(0, T;\, H^1(\Omega)\right). \tag{3.17}$$

Later, in the error analysis, we shall assume that

$$\varphi \in C\left([0, T];\, H^s(\Omega)\right) \cap H^{q+1}\left(0, T;\, H^1(\Omega)\right) \tag{3.18}$$

with integers $q \geq 1$ and $s \geq 2$.

Now we discretize the term with time derivative. Let $m \in \{1, \ldots, M\}$ be arbitrary but fixed. We multiply equation (3.1) by $\psi \in S_{h,\tau}^{p,q}$, integrate it over $K \times I_m$ and sum over all elements $K \in \mathcal{T}_{h,m}$.

We use the notation $\varphi' = \frac{\partial \varphi}{\partial t}$. By the integration of the time derivative term by parts we get

$$\int\limits_{I_m} (\varphi', \psi)\, dt = -\int\limits_{I_m} (\varphi, \psi')\, dt + \left(\varphi_m^-, \psi_m^-\right) - \left(\varphi_{m-1}^+, \psi_{m-1}^+\right). \tag{3.19}$$

It follows from (3.17) that the exact solution is continuous with respect to time, which implies that $\varphi_{m-1}^- = \varphi_{m-1}^+$. Thus,

$$\left(\varphi_{m-1}^-, \psi_{m-1}^+\right) = \left(\varphi_{m-1}^+, \psi_{m-1}^+\right). \tag{3.20}$$

Putting equations (3.19) and (3.20) together and integrating them by parts, we obtain the desired approximation

$$\begin{aligned}
\int\limits_{I_m} (\varphi', \psi)\, dt &= -\int\limits_{I_m} (\varphi, \psi')\, dt + \left(\varphi_m^-, \psi_m^-\right) - \left(\varphi_{m-1}^+, \psi_{m-1}^+\right) \\
&= \int\limits_{I_m} (\varphi', \psi)\, dt - \left(\varphi_{m-1}^-, \psi_{m-1}^+\right) + \left(\varphi_{m-1}^+, \psi_{m-1}^+\right) \\
&= \int\limits_{I_m} (\varphi', \psi)\, dt + \left(\{\varphi\}_{m-1}, \psi_{m-1}^+\right).
\end{aligned}$$

The discretization of the convective term and the right-hand side of equation (3.1) is similar as in Section 3.1.1. We can define the forms

$$
\begin{aligned}
b_{h,m}(\varphi,\psi) &= \sum_{K\in\mathcal{T}_{h,m}} \int_K (v\cdot\nabla\varphi)\,\psi dx - \sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K^-\setminus\partial\Omega} (v\cdot n)\,[\varphi]\,\psi dS \\
&\quad - \sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K^-\cap\partial\Omega} (v\cdot n)\,\varphi\psi dS, \\
l_{h,m}(\psi)(t) &= -\sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K^-\cap\partial\Omega} (v\cdot n)\,\varphi_D(t)\,\psi dS.
\end{aligned}
$$

Let us define for each element K from $\mathcal{T}_{h,m}$, $m = 1,\dots,M$, and $\varphi \in L^2(K)$ the projection as follows:

$$
\begin{aligned}
(\Pi_{h,m}\varphi)\,|_K &\in P^p(K), \\
(\Pi_{h,m}\varphi - \varphi, \psi)_{L^2(K)} &= 0 \quad \forall\psi \in P^p(K),
\end{aligned}
$$

and hence, if $\varphi \in L^2(\Omega)$, then

$$
\begin{aligned}
\Pi_{h,m}\varphi &\in S^p_{h,m}, \\
(\Pi_{h,m}\varphi - \varphi, \psi)_{L^2(K)} &= 0 \quad \forall\psi \in S^p_{h,m}.
\end{aligned}
$$

This means that $\Pi_{h,m}$ is the L^2-projection on the space $S^p_{h,m}$.

Definition 3.3. We say that the function $\Phi \in S^{p,q}_{h,\tau}$ is an approximate solution of problem (3.1)-(3.3), if it satisfies the identity

$$
\begin{aligned}
\int_{I_m} \left((\Phi',\psi) + b_{h,m}(\Phi,\psi)\right) dt + \left(\{\Phi\}_{m-1}, \psi^+_{m-1}\right) &= \int_{I_m} l_{h,m}(\psi) dt \qquad (3.21) \\
\forall\psi \in S^{p,q}_{h,\tau}, \quad m = 1,\dots,M, \quad \Phi^-_0 = \Pi_{h,0}\varphi^0, &
\end{aligned}
$$

where $\Pi_{h,0}\varphi^0$ is the $S^p_{h,0}-$interpolation and Φ' denotes the time derivative of Φ.

We can notice that this scheme is constructed in such a way that the exact solution φ satisfying (3.18) satisfies the identity

$$
\begin{aligned}
\int_{I_m} \left((\varphi',\psi) + b_{h,m}(\varphi,\psi)\right) dt + \left(\{\varphi\}_{m-1}, \psi^+_{m-1}\right) &= \int_{I_m} l_{h,m}(\psi) dt, \\
\forall\psi \in S^{p,q}_{h,\tau}, \quad m = 1,\dots,M & \qquad (3.22)
\end{aligned}
$$

3.2 Navier-Stokes equations

Let us formulate the weak solution of the momentum equations (2.1) together with the continuity equation (2.2).

As above we set $Q_T = \Omega \times (0,T)$. We want to find $v:\ Q_T \to \mathbb{R}^2$ such that

$$
\begin{aligned}
\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) &= \rho\boldsymbol{f} + \mathrm{div}\boldsymbol{\sigma} \text{ in } \Omega_1(t)\cup\Omega_2(t), &(3.23)\\
\mathrm{div}\boldsymbol{v} &= 0 \text{ in } \Omega_1(t)\cup\Omega_2(t) &(3.24)\\
\boldsymbol{v}(x,0) &= \boldsymbol{v}_0, \quad x\in\Omega, &(3.25)\\
\boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on } \Gamma_D, &(3.26)\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0, \quad \boldsymbol{t}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_S, &(3.27)\\
\boldsymbol{v}^{(1)} &= \boldsymbol{v}^{(2)} \quad \text{on } \Gamma_I, &(3.28)\\
\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n} - \boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n} &= \gamma\kappa\boldsymbol{n} \quad \text{on } \Gamma_I. &(3.29)
\end{aligned}
$$

In (3.27), $\boldsymbol{n}$ and $\boldsymbol{t}$ denote the unit normal and unit tangent to Γ_S. In (3.29) $\boldsymbol{n}$ denotes the unit normal to $\Gamma_I(t)$ pointing form $\Omega_1(t)$ into $\Omega_2(t)$. Further, γ denotes the surface tension coefficient and κ is the curvature of the interface $\Gamma_I(t)$.

We define the velocity spaces $\boldsymbol{V}$, $\boldsymbol{V}_0$ and the pressure space Q as follows:

$$\boldsymbol{V} = H^1(\Omega)^2, \quad \boldsymbol{V}_0 = \{\boldsymbol{w}\in\boldsymbol{V};\ \boldsymbol{w}|_{\Gamma_D} = 0,\ (\boldsymbol{w}\cdot\boldsymbol{n})\,|_{\Gamma_S} = 0\}, \quad Q = L^2(\Omega).$$

Let us have the following assumptions on the data:

$\boldsymbol{v}_D$ is the trace of some $\boldsymbol{v}^* \in C([0,T];\, H^1(\Omega))$ on $\Gamma_D\times(0,T)$,
$\boldsymbol{v}_0 \in L^2(\Omega)^2$,
$\boldsymbol{f}\in L^2(Q_T)^2$.

We multiply the momentum equation (3.23) by an arbitrary test function $\boldsymbol{w}\in\boldsymbol{V}_0$, integrate it over $\Omega^{(k)}$ and apply the Green theorem to the term div$\boldsymbol{\sigma}$. We obtain the relation

$$
\begin{aligned}
&\int_{\Omega_k} \rho^{(k)}\left(\frac{\partial \boldsymbol{v}^{(k)}}{\partial t} + \left(\boldsymbol{v}^{(k)}\cdot\nabla\right)\boldsymbol{v}^{(k)}\right)\boldsymbol{w}\,dx &(3.30)\\
&+\int_{\Omega_k}\boldsymbol{\sigma}^{(k)}\cdot(\nabla\boldsymbol{w})\,dx - \int_{\partial\Omega_k}\boldsymbol{\sigma}^{(k)}\cdot\boldsymbol{n}^{(k)}\boldsymbol{w}\,dS = \int_{\Omega_k}\rho^{(k)}\boldsymbol{f}\boldsymbol{w}\,dx, \quad k=1,2.
\end{aligned}
$$

Using the boundary conditions, interface conditions and properties of $\boldsymbol{w}\in\boldsymbol{V_0}$, summing the equations (3.30) for $k=1,2$, and taking into account that $\boldsymbol{n}^{(1)} = -\boldsymbol{n}^{(2)}$, we get the identity

$$\int_{\Omega} \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right) \boldsymbol{w} + \boldsymbol{\sigma} \cdot (\nabla \boldsymbol{w}) \, dx = \int_{\Gamma_I} \gamma \kappa \boldsymbol{n} \cdot \boldsymbol{w} \, dS + \int_{\Omega} \rho \boldsymbol{f} \boldsymbol{w} \, dx, \, \forall w \in V_0.$$

We also multiply the continuity equation (3.24) by an arbitrary test function $q \in Q$ and get

$$\int_{\Omega} (\nabla \cdot \boldsymbol{v}) \, q dx = 0.$$

Now we can define the weak solution of our problem:

Definition 3.4. We call a function $\boldsymbol{v}$ defined by 2.4 the weak solution to the problem (2.1)-(2.2), if it satisfies the conditions

$\boldsymbol{v} - \boldsymbol{v}^* \in L^2 \left(0, T; V\right)^2, \, \boldsymbol{v} \in L^\infty \left(0, T; L^2 \left(\Omega\right)\right)^2;$
$\int_{\Omega} \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right) \boldsymbol{w} + \sigma \cdot (\nabla \boldsymbol{w}) + (\nabla \cdot \boldsymbol{v}) \, q dx = \int_{\Gamma_I} \gamma \kappa \boldsymbol{n} \cdot \boldsymbol{w} \, dS + \int_{\Omega} \rho \boldsymbol{f} \boldsymbol{w} \, dx$
for all $\boldsymbol{w} \in \boldsymbol{V}_0$ and $q \in Q$in the sense of distributions on $(0, T)$,
$\boldsymbol{v}(0) = \boldsymbol{v}^0$ in Ω.

In what follows we shall be concerned with the term $\int_{\Gamma_I} \gamma \kappa \boldsymbol{n} \cdot \boldsymbol{w} dS$. We refer to the works [4], [7] and [1].

Let $\chi = \chi(\xi)$ denote the parametrization of Γ_I. Then we set $g = \left(\frac{\mathrm{d}\chi}{\mathrm{d}\xi} \right)^{-1}$ and for a differentiable vector function $f : \Gamma_I \to \mathbb{R}^2$ we define the tangential derivative $\nabla_\Gamma f$ by

$$\nabla_\Gamma f = \frac{\mathrm{d}(f \circ \chi)}{\mathrm{d}\xi}$$

and the Laplace-Beltrami operator

$$\triangle_\Gamma f = \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} g \nabla_\Gamma f \right)}{\partial \xi}.$$

Moreover, by [1],

$$\kappa \boldsymbol{n} = \triangle_\Gamma \boldsymbol{x},$$

where $\boldsymbol{n}$ is the unit normal to Γ_I, and the integration by parts yields

$$\int_{\Gamma_I} \kappa \boldsymbol{n} \cdot \boldsymbol{w} dS = \int_{\Gamma_I} (\triangle_\Gamma \boldsymbol{x}) \cdot \boldsymbol{w} dS = - \int_{\Gamma_I} \nabla_\Gamma \boldsymbol{x} \cdot \nabla_\Gamma \boldsymbol{w} dS.$$

Then we get the following identity:

$$\int_{\Omega} \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right) \boldsymbol{w} + \boldsymbol{\sigma} \cdot (\nabla \boldsymbol{w}) + (\nabla \cdot \boldsymbol{v}) \, q dx \tag{3.31}$$
$$= -\int_{\Gamma_I} \gamma (\nabla_\Gamma \boldsymbol{x}) \cdot (\nabla_\Gamma \boldsymbol{w}) \, dS + \int_{\Omega} \rho \boldsymbol{f} \boldsymbol{w} \, dx, \, \forall \boldsymbol{w} \in \boldsymbol{V}_0, \, \forall q \in Q.$$

3.2.1 Space discretization

For the space discretization we shall use the finite element method.

We assume that the domain Ω is polygonal and consider a system of triangulations $\{\mathcal{T}_h\}_{h \in (0, \overline{h})}$ with $\overline{h} > 0$ of the closure of the domain Ω into a finite number of closed triangles with disjoint interiors.

Now we approximate the function spaces $\boldsymbol{V}$ and Q by the finite dimensional spaces $\boldsymbol{V}_h$ and Q_h defined over the triangulation $\mathcal{T}_h$. We set

$$\begin{aligned}
\boldsymbol{V}_h &= \left\{ \boldsymbol{v_h} \in C\left(\overline{\Omega}\right)^2; \, \boldsymbol{v_h}|_K \in P^p(K)^2 \; \forall K \in \mathcal{T}_h \right\}, \\
\boldsymbol{V}_{h0} &= \left\{ \boldsymbol{w}_h \in \boldsymbol{V}_h; \, \boldsymbol{w_h}|_{\Gamma_D} = 0, \, (\boldsymbol{w_h} \cdot \boldsymbol{n})\,|_{\Gamma_S} = 0 \right\}, \\
Q_h &= \left\{ q_h \in C\left(\overline{\Omega}\right); \, q_h|_K \in P^s(K) \; \forall K \in \mathcal{T}_h \right\},
\end{aligned}$$

where p, $s \geq 1$ are integers. $P^p(K)$ and $P^s(K)$ are spaces of polynomials on K of degree at most p and s, respectively.

We introduce the discrete formulation. We proceed as in the continuous problem. We multiply the momentum equation (3.23) by an arbitrary function $\boldsymbol{w}_h \in \boldsymbol{V}_{h0}$ and the continuity equation (3.24) by an arbitrary function $q_h \in Q_h$, integrate them over an arbitrary element K and apply the Green theorem and sum over all elements K from $\{\mathcal{T}_h\}_{h \in (0, \overline{h})}$. We look for functions $\boldsymbol{v_h} : [0, T] \to \boldsymbol{V_h}$ and $p_h : [0, T] \to Q_h$ such that for all $\boldsymbol{w}_h \in \boldsymbol{V}_{h0}$, all $q_h \in Q_h$ and all $t \in (0, T)$ they satisfy the equation

$$\begin{aligned}
&\sum_{K \in \mathcal{T}_h} \int_K \Bigg(\rho \left(\frac{\partial \boldsymbol{v}_h(t)}{\partial t} + (\boldsymbol{v}_h(t) \cdot \nabla) \boldsymbol{v}_h(t) \right) \boldsymbol{w}_h - p_h \nabla \cdot \boldsymbol{w}_h \\
&\quad + 2\mu \sum_{i,j=1}^{2} d_{i,j}(\boldsymbol{v}_h(t)) \frac{\partial \boldsymbol{w}_{ki}}{\partial x_j} + (\nabla \cdot \boldsymbol{v}_h(t)) \, q_h \Bigg) dx \\
= &-\int_{\Gamma_I} \gamma (\nabla_\Gamma \boldsymbol{x}) \cdot (\nabla_\Gamma \boldsymbol{w}_h) \, dS + \sum_{K \in \mathcal{T}_h} \int_K \rho \boldsymbol{f} \boldsymbol{w}_h dx,
\end{aligned}$$

and $\boldsymbol{v}_h$ attains the values of $\boldsymbol{v}_D$ at vertices and midpoints of sides of elements

$K \in \mathcal{T}_h$ adjacent to Γ_D.

3.2.2 Space-time discretization

In this section we introduce the time discretization of the flow problem.

Let $M > 1$ be an integer. We shall construct a partition of the time interval $[0, T]$:

$$0 = t_0 < t_1 < \cdots < t_M = T, \quad t_k = k\tau,$$

where $\tau > 0$ denotes a time step.

By $\boldsymbol{v}_h^{(k)} \in \boldsymbol{V}_{h0}$ and $p_h^{(k)} \in Q_h$ we denote the approximation of the functions $\boldsymbol{v}_h(t_k)$ and $p_h(t_k)$, respectively. We expect that this notation will not lead to some confusion in comparison to notation in Section 2. The time derivative $\frac{\partial \boldsymbol{v}_h}{\partial t}$ at time t_k with $k \geq 2$ is approximated by the backward difference formula (BDF)

$$\frac{\partial \boldsymbol{v}_h(t_k)}{\partial t} \approx D\boldsymbol{v}_h^{(k)} = \frac{3\boldsymbol{v}_h^{(k)} - 4\boldsymbol{v}_h^{(k-1)} + \boldsymbol{v}_h^{(k-2)}}{2\tau}. \tag{3.32}$$

For $k = 1$ we use the approximation

$$\frac{\partial \boldsymbol{v}_h(t_1)}{\partial t} \approx D\boldsymbol{v}_h^{(1)} = \frac{\boldsymbol{v}_h^{(1)} - \boldsymbol{v}_h^{(0)}}{\tau}, \tag{3.33}$$

where $\boldsymbol{v}_h^0$ is the $\boldsymbol{V}_h$ interpolation of the function $\boldsymbol{v}^0$ from condition (3.25).

Now the full space-time approximate solution is defined as the sequences $\left\{\boldsymbol{v}_h^{(k)}\right\}_{k=0}^M$, $\left\{p_h^{(k)}\right\}_{k=1}^M$ satisfying the relations

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \int_K \Bigg(\rho \left(D\boldsymbol{v}_h^{(k)} + \left(\overline{\boldsymbol{v}}_h^{(k)} \cdot \nabla\right) \boldsymbol{v}_h^{(k)} \right) \cdot \boldsymbol{w}_h - p_h^{(k)} \nabla \cdot \boldsymbol{w}_h \\
&+2\mu \sum_{i,j=1}^{2} d_{ij}\left(\boldsymbol{v}_h^{(k)}\right) \frac{\partial w_{ki}}{\partial x_j} + \left(\nabla \cdot \boldsymbol{v}_h^{(k)}\right) q_h \Bigg) dx \\
= \; &-\int_{\Gamma_I} \gamma \left(\nabla_\Gamma \boldsymbol{x}\right) \cdot \left(\nabla_\Gamma \boldsymbol{w}_h\right) dS + \sum_{K\in\mathcal{T}_h} \int_K \rho \boldsymbol{f} \boldsymbol{w}_h dx, \\
&\forall \boldsymbol{w}_h = (w_{h1}, w_{h2}) \in \boldsymbol{V}_{h0}, \qquad \forall q_h \in Q_h, \qquad k = 1, \cdots, M.
\end{aligned} \tag{3.34}$$

Here by $\overline{\boldsymbol{v}}_h^{(k)}$ we denote the time approximation of $\boldsymbol{v}_h^{(k)}$. If we set

$$\overline{\boldsymbol{v}}_h^{(k)} = \boldsymbol{v}_h^{(k)}, \tag{3.35}$$

then the discrete problem (3.34) is nonlinear and has to be solved by a suitable iterative process. Another possibility is to use time extrapolation

$$\text{a) } \overline{\boldsymbol{v}}_h^{(k)} = \boldsymbol{v}_h^{(k-1)} \qquad \text{or} \quad \text{b) } \overline{\boldsymbol{v}}_h^{(k)} = 2\boldsymbol{v}_h^{(k-1)} - \boldsymbol{v}_h^{(k-2)}.$$

Then on each time level we solve a linear problem.

4 Error estimation of level set method

4.1 Error estimation for space discretization

In this section we shall be concerned with the estimation of the error of the scheme (3.16).

We shall consider a system $\{\mathcal{T}_h\}_{h\in(0,\overline{h})}$, $\overline{h} > 0$, of triangulations of the domain Ω and assume that satisfies assumption (A1), (A2) and (A3) form Section 3.1.1.

The error of the method is defined as a function $e = \varphi_h - \varphi$. In the derivation of the error estimate, we shall need to introduce the $S_{hp}-$interpolation Π_{hp} which will be defined in the following way.

Let us define for each element $K \in \mathcal{T}_h$ a mapping $\pi_{Kp} : L^2(K) \to P^p(K)$ such that for each $\varphi \in L^2(K)$ we have

$$\begin{aligned} \pi_{Kp}\varphi &\in P^p(K), \\ \int_K (\pi_{Kp}\varphi)\, v dx &= \int_K \varphi v dx \quad \forall v \in P^p(K). \end{aligned} \tag{4.1}$$

Then we can define the $S_{hp}-$interpolation Π_{hp} in such a way that:

$$(\Pi_{hp}\varphi)\,|_K = (\pi_{Kp}\,(\varphi|_K)) \tag{4.2}$$

and hence, if $\varphi \in L^2(\Omega)$, then

$$\begin{aligned} \Pi_{hp}\varphi &\in S_{hp}, \\ \int_K (\Pi_{hp}\varphi)\, v dx &= \int_K \varphi v dx \;\forall v \in S_{hp},\ \forall K \in \mathcal{T}_h. \end{aligned} \tag{4.3}$$

That means that Π_{hp} is the L^2-projection on the space S_{hp}.

Now we shall express the error $e_h = \varphi_h - \varphi$ in the form

$$\begin{aligned} e_h &= \xi + \eta, \\ \xi &= \varphi_h - \Pi_{hp}\varphi, \\ \eta &= \Pi_{hp}\varphi - \varphi. \end{aligned} \tag{4.4}$$

The function η represents an $S_{hp}-$interpolation error.

Let us assume that the exact solution satisfies the regularity condition

$$\frac{\partial\varphi}{\partial t} \in L^2\left([0,T]\,;\, H^s\left(\Omega\right)\right), \tag{4.5}$$

where $s \geq 2$ is an integer. Then, if the system of triangulations is shape-regular, for all $K \in \mathcal{T}_h$ and $h \in \left(0, \overline{h}\right)$ we have

$$\begin{aligned} \|\eta\|_{L^2(K)} &\leq C_A h^\mu \, |\varphi|_{H^\mu(K)} \, , && (4.6)\\ |\eta|_{H^1(K)} &\leq C_A h^{\mu-1} \, |\varphi|_{H^\mu(K)} \, , && (4.7)\\ \|\eta\|_{L^2(\Omega)} &\leq C_A h^\mu \, |\varphi|_{H^\mu(\Omega)} \, , && (4.8)\\ \left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(\Omega)} &\leq C_A h^\mu \left|\frac{\partial \varphi}{\partial t}\right|_{H^\mu(\Omega)} && (4.9) \end{aligned}$$

almost everywhere in $(0, T)$, where $\mu = \min\left(p+1, s\right)$. These estimates are the consequences of approximation properties of the interpolation operator Π_{hp}. See, e.g., [9].

Our main task will be to estimate ξ in terms of η. Then, using the η- estimates we obtain the result for the error e_h.

We proceed in such a way that we subtract (3.14) from identity (3.16), put $\psi := \xi(t)$ and obtain the following equality:

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) + b_h\left(\xi, \xi\right) = -\left(\frac{\partial \eta}{\partial t}, \xi\right) - b_h\left(\eta, \xi\right). \tag{4.10}$$

It is obvious that

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) = \frac{1}{2}\frac{d}{dt}\left\|\xi\right\|^2_{L^2(\Omega)}. \tag{4.11}$$

Now we need to estimate the term with the form b_h. First let us recall some inequalities needed in the error analysis.

Lemma 4.1. *Multiplicative trace inequality*

Let the system $\{\mathcal{T}_h\}_{h\in\left(0,\overline{h}\right)}$ be shape-regular. Then there exists a constant $C_M >$ 0 such that

$$\|v\|^2_{L^2(\partial K)} \leq C_M \left(\|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \, \|v\|_{L^2(K)}\right) \tag{4.12}$$

for all $K \in \mathcal{T}_h$, $h \in \left(0, \overline{h}\right)$ and all $v \in H^1(K)$.

For proof see for example [9, Chapter 2].

Lemma 4.2. *Inverse inequality*

Let the system $\{\mathcal{T}_h\}_{h\in\left(0,\overline{h}\right)}$ be shape-regular. Then there exists a constant $C_I >$ 0 independent of h such that

$$|v_h|_{H^1(K)} \leq C_I h_K^{-1} \, \|v_h\|_{L^2(K)} \tag{4.13}$$

for all $K \in \mathcal{T}_h$, $h \in \left(0, \overline{h}\right)$ *and all* $v_h \in P^P(K)$.

For proof see for example [9, Chapter 2].

Lemma 4.3. *Gronwall lemma*

Let y, q, z, $r \in C([0,T])$, $r \geq 0$ *and let*

$$y(t) + q(t) \leq z(t) + \int_0^t r(s)y(s)ds, \quad t \in [0,T].$$

Then

$$\begin{aligned} & y(t) + q(t) + \int_0^t r(\vartheta)q(\vartheta)\exp\left(\int_\vartheta^t r(s)ds\right)d\vartheta \\ \leq\ & z(t) + \int_0^t r(\vartheta)z(\vartheta)\exp\left(\int_\vartheta^t r(s)ds\right)d\vartheta, \quad t \in [0,T]. \end{aligned} \tag{4.14}$$

For proof we refer to [9, Chapter 1].

We shall also need the modified version of this lemma.

Lemma 4.4. *Modified Gronwall lemma*

Let for all $t \in [0,T]$

$$\chi^2(t) + R(t) \leq 2\int_0^t B(\upsilon)\chi(\upsilon)d\upsilon,$$

where $R, A, B, \chi \in C([0,T])$ *are nonnegative functions. Then for any* $t \in [0,T]$ *we obtain the inequality*

$$\sqrt{\chi^2(t) + R(t)} \leq \max_{0 \leq \upsilon \leq t} A^{1/2}(\upsilon) + \int_0^t B(\upsilon)d\upsilon. \tag{4.15}$$

See [9, Chapter 1].

Now we introduce a norm over a subset ω of either $\partial\Omega$ or ∂K:

$$\|\varphi\|_{v,\omega} = \left\|\sqrt{|v \cdot \boldsymbol{n}|}\varphi\right\|_{L^2(\omega)}, \tag{4.16}$$

where $\boldsymbol{n}$ denotes the corresponding outer unit normal to $\partial\Omega$ or ∂K.

Now we can formulate the estimation for the form b_h.

Lemma 4.5. *There exist positive constants C_b and C_b' independent of φ and h such that*

$$
\begin{aligned}
|b_h(\eta,\xi)| \;\leq\; & \frac{1}{4}\sum_{K\in T_h}\left(\|\xi\|^2_{v,\partial K^+\cap\partial\Omega}+\|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right) \\
& + C_b\sum_{K\in T_h}\|\eta\|_{L^2(K)}\|\xi\|_{L^2(K)} \\
& + C_b'\sum_{K\in T_h}\left(\|\eta\|_{L^2(K)}|\eta|_{H^1(K)}+h_K^{-1}\|\eta\|^2_{L^2(K)}\right),
\end{aligned}
\tag{4.17}
$$

where

$$
C_b' = C_vC_M, \qquad C_b \;=\; C_v(1+C_AC_I), \tag{4.18}
$$

C_v is given by the data assumption of problem (3.1)-(3.3), C_A is the approximation constant, and C_M and C_I are the constants from the multiplicative trace inequality and the inverse inequality, respectively.

Proof. Using the definition of the form b_h (3.12) and the Green theorem, we find that

$$
\begin{aligned}
b_h(\eta,\xi) \;=\; & \sum_{K\in\mathcal{T}_h}\left(\int_K(\boldsymbol{v}\cdot\nabla\eta)\,\xi dx\right. \\
& \left. -\int_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\eta-\eta^-\right)dS-\int_{\partial K^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS\right) \\
=\; & \sum_{K\in\mathcal{T}_h}\left(\int_{\partial K}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS-\int_K\eta\,(\boldsymbol{v}\cdot\nabla\xi)\,dx-\int_K\eta\xi\nabla\cdot v dx\right. \\
& \left. -\int_{\partial K^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS-\int_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\eta-\eta^-\right)dS\right),
\end{aligned}
\tag{4.19}
$$

where the superscript $^-$ denotes the values on ∂K from the outside the element

K. Hence

$$
\begin{aligned}
\left|b_h\left(\eta,\xi\right)\right| \;\; \leq \;\; & \left|\sum_{K\in\mathcal{T}_h}\int_K \eta\left(\boldsymbol{v}\cdot\nabla\xi\right)dx\right| + \left|\sum_{K\in\mathcal{T}_h}\int_K \eta\xi\nabla\cdot\boldsymbol{v}dx\right| \\
& + \left|\sum_{K\in\mathcal{T}_h}\left(\int_{\partial K}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\xi\eta dS - \int_{\partial K^-\cap\partial\Omega}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\xi\eta dS\right.\right. \\
& \left.\left. - \int_{\partial K^-\setminus\partial\Omega}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\xi\left(\eta-\eta^-\right)dS\right)\right|.
\end{aligned}
\tag{4.20}
$$

The second term on the right-hand side of (4.20) is estimated easily with the aid of the Cauchy inequality and assumption (3.6).

$$
\left|\sum_{K\in\mathcal{T}_h}\int_K \eta\xi\nabla\cdot\boldsymbol{v}dx\right| \leq C_v\sum_{K\in\mathcal{T}_h}\left\|\eta\right\|_{L^2(K)}\left\|\xi\right\|_{L^2(K)}. \tag{4.21}
$$

Since

$$
\sum_{K\in\mathcal{T}_h}\int_{\partial K^-\setminus\partial\Omega}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\xi\eta dS = -\sum_{K\in\mathcal{T}_h}\int_{\partial K^-\setminus\partial\Omega}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\xi^-\eta^- dS
$$

and $\boldsymbol{v}\cdot\boldsymbol{n}\geq 0$ on K^+, with the aid of the Young inequality, the set decomposition

$$
\partial K = \partial K^+ \cup \left(\partial K^-\cap\partial\Omega\right)\cup\left(\partial K^-\setminus\partial\Omega\right)
$$

and notation (4.16), the third term on the right-hand side of (4.20) can be rewritten and then estimated in the following way:

$$
\begin{aligned}
& \left| \sum_{K\in\mathcal{T}_h} \left(\int\limits_{\partial K^+} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS + \int\limits_{\partial K^-\setminus\partial\Omega} \left\{ (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta - (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\eta-\eta^-\right)\right\} dS \right) \right| \\
= & \left| \sum_{K\in\mathcal{T}_h} \left(\int\limits_{\partial K^+\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS + \int\limits_{\partial K^+\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta + \int\limits_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\eta^-\xi dS \right) \right| \\
= & \left| \sum_{K\in\mathcal{T}_h} \left(\int\limits_{\partial K^+\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS + \int\limits_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\eta^-\left(\xi-\xi^-\right) dS \right) \right| \\
\leq & \frac{1}{4}\sum_{K\in\mathcal{T}_h} \left(\int\limits_{\partial K^+\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS + \int\limits_{\partial K^-\setminus\partial\Omega} |\boldsymbol{v}\cdot\boldsymbol{n}|\,[\xi]^2 dS \right) \qquad (4.22) \\
& + \sum_{K\in\mathcal{T}_h} \left(\int\limits_{\partial K^+\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\eta^2 dS + \int\limits_{\partial K^-\setminus\partial\Omega} |\boldsymbol{v}\cdot\boldsymbol{n}|\left(\eta^-\right)^2 dS \right) \\
\leq & \frac{1}{4}\sum_{K\in\mathcal{T}_h} \left(\|\xi\|^2_{v,\partial K^+\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega} \right) \\
& \sum_{K\in\mathcal{T}_h} \left(\|\eta\|^2_{v,\partial K^+\cap\partial\Omega} + \left\|\eta^-\right\|^2_{v,\partial K^-\setminus\partial\Omega} \right).
\end{aligned}
$$

Using the multiplicative trace inequality, the boundedness of $\boldsymbol{v}$ and estimates (4.6) and (4.7), we get

$$
\begin{aligned}
& \sum_{K\in\mathcal{T}_h} \left(\|\eta\|^2_{v,\partial K^+\cap\partial\Omega} + \left\|\eta^-\right\|^2_{v,\partial K^-\setminus\partial\Omega} \right) \qquad (4.23) \\
\leq & C_v \sum_{K\in\mathcal{T}_h} \left(\|\eta\|^2_{L^2(\partial K^+\setminus\partial\Omega)} + \left\|\eta^-\right\|^2_{L^2(\partial K^-\cap\partial\Omega)} \right) \\
\leq & C_v \sum_{K\in\mathcal{T}_h} \|\eta\|^2_{L^2(\partial K)} \leq C_v C_M \sum_{K\in\mathcal{T}_h} \left(\|\eta\|_{L^2(K)} |\eta|_{H^1(K)} + h_K^{-1}\|\eta\|^2_{L^2(K)} \right).
\end{aligned}
$$

In virtue of the definition (4.4) of η and (4.1)-(4.3), the first term on the right-hand side of (4.20) vanishes, if the vector v is constant or piecewise linear, because $\boldsymbol{v}\cdot\nabla\xi|_K \in P^p(K)$ in this case. If this is not the case, we have to proceed in a more sophisticated way. For every $t\in[0,T)$ we introduce a function $\Pi_{h1}\boldsymbol{v}(t)$ which is a piecewise linear L^2- projection of $\boldsymbol{v}(t)$ on the space S_{hp}. Under assumption

(3.6) we have

$$\|\boldsymbol{v} - \Pi_{h1}\boldsymbol{v}\|_{L^\infty(K)} \le C_A h_K \, |\boldsymbol{v}|_{W^{1,\infty}(K)}\,, \quad K \in \mathcal{T}_h,\, h \in \left(0, \overline{h}\right). \tag{4.24}$$

as follows from [6].

The first term in (4.20) is then estimated with the aid of (4.1), (4.13), (4.24), the Cauchy inequality and assumption (3.6) in the following way:

$$\begin{aligned}
& \left| \sum_{K\in\mathcal{T}_h} \int_K \eta \, (\boldsymbol{v}\cdot\nabla\xi)\, dx \right| \qquad (4.25)\\
\le\ & \sum_{K\in\mathcal{T}_h} \left| \int_K \eta\,(\Pi_{h1}\boldsymbol{v}\cdot\nabla\xi)\, dx \right| + \sum_{K\in\mathcal{T}_h} \left| \int_K \eta\,((\boldsymbol{v}-\Pi_{h1}\boldsymbol{v})\cdot\nabla\xi)\, dx \right| \\
=\ & \sum_{K\in\mathcal{T}_h} \left| \int_K \eta\,((\boldsymbol{v}-\boldsymbol{v})\cdot\nabla\xi)\, dx \right| \le \sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}-\Pi_{h1}\boldsymbol{v}\|_{L^\infty(K)} \|\eta\|_{L^2(K)} \, |\xi|_{H^1(K)} \\
\le\ & \sum_{K\in\mathcal{T}_h} C_A h_K \, |\boldsymbol{v}|_{W^{1,\infty}(K)} \|\eta\|_{L^2(K)} \, C_I h_K^{-1} \|\xi\|_{L^2(K)} \\
\le\ & C_v C_A C_I \sum_{K\in\mathcal{T}_h} \|\eta\|_{L^2(K)} \|\xi\|_{L^2(K)}\,.
\end{aligned}$$

Using (4.21), (4.22) and (4.25) in (4.20), we obtain (4.17) with constants defined in (4.18). This finishes the proof of Lemma 4.5. □

From now on we shall use the notation

$$D(\eta) = 2C_b' \sum_{K\in\mathcal{T}_h} \left(\|\eta\|_{L^2(K)} \, |\eta|_{H^1(K)} + h_K^{-1} \|\eta\|^2_{L^2(K)} \right).$$

We consider the term $b_h(\xi,\xi)$ from equation (4.10) and rewrite it in the following way:

$$
\begin{aligned}
b_h(\xi,\xi) &= \sum_{K\in\mathcal{T}_h}\left(\int\limits_K (\boldsymbol{v}\cdot\nabla\xi)\,\xi dx\right.\\
&\quad \left. - \int\limits_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,[\xi]\,\xi dS - \int\limits_{\partial K^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right)\\
&= \sum_{K\in\mathcal{T}_h}\left(-\frac{1}{2}\int\limits_K (\nabla\cdot\boldsymbol{v})\,\xi^2 dx + \frac{1}{2}\int\limits_{\partial K}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right.\\
&\quad \left. - \int\limits_{\partial K^-\setminus\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\xi-\xi^-\right) dS - \int\limits_{\partial K^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right)
\end{aligned}
$$

Due to the decomposition $\partial K = \partial K^- \cup \partial K^+$ we have

$$
\begin{aligned}
b_h(\xi,\xi) &= \sum_{K\in\mathcal{T}_h}\frac{1}{2}\left(-\int\limits_K \xi^2\nabla\cdot\boldsymbol{v} dx - \int\limits_{\partial K^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right.\\
&\quad - \int\limits_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\left(\xi^2-2\xi\xi^-\right)dS\\
&\quad \left. + \int\limits_{\partial K^+\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS + \int\limits_{\partial K^+\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right).
\end{aligned}
$$

Using the relation

$$
\sum_{K\in\mathcal{T}_h}\int\limits_{\partial K^+\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS = -\sum_{K\in\mathcal{T}_h}\int\limits_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\left(\xi^-\right)^2 dS,
$$

we get

$$
\begin{aligned}
b_h(\xi,\xi) &= \sum_{K\in\mathcal{T}_h}\frac{1}{2}\left(-\int\limits_K \xi^2\nabla\cdot\boldsymbol{v} dx - \int\limits_{\partial K^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right. \qquad (4.26)\\
&\quad \left. - \int\limits_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\left(\xi^2-2\xi\xi^-+\left(\xi^-\right)^2\right)dS + \int\limits_{\partial K^+\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi^2 dS\right)\\
&= \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\|\xi\|^2_{v,\partial K^-\cap\partial\Omega} + \|\xi\|^2_{v,\partial K^+\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right) - \frac{1}{2}\int\limits_\Omega \xi^2\nabla\cdot\boldsymbol{v} dx.
\end{aligned}
$$

Putting relations (4.26) and (4.11) into equation (4.10), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{K\in T_h}\left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right) \\ \le\ & \left|\left(\frac{\partial\eta}{\partial t},\xi\right)\right| + |b_h(\eta,\xi)| + \int_\Omega \xi^2\nabla\cdot\boldsymbol{v}dx. \end{aligned}$$

For simplicity we shall write

$$\sigma(\xi) = \frac{1}{2}\sum_{K\in T_h}\left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right). \tag{4.27}$$

Finally we use inequality (4.17) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} + \frac{1}{2}\sigma(\xi) \\ \le\ & C_b\|\xi\|_{L^2(\Omega)}\left(\|\eta\|_{L^2(\Omega)} + \left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(\Omega)}\right) + \frac{1}{2}D(\eta) + \int_\Omega \xi^2\nabla\cdot\boldsymbol{v}dx. \end{aligned} \tag{4.28}$$

We shall integrate this inequality from 0 to t and apply the initial condition $\xi(0) = 0$. We get

$$\begin{aligned} &\|\xi\|^2_{L^2(\Omega)} + \int_0^t \sigma(\xi(\vartheta))d\vartheta \\ \le\ & 2C_b\int_0^t \|\xi(\vartheta)\|_{L^2(\Omega)}\left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\|\frac{\partial\eta}{\partial t}(\vartheta)\right\|_{L^2(\Omega)}\right)d\vartheta \\ &+ \int_0^t D(\eta)d\vartheta + 2\int_0^t\left(\int_\Omega \xi^2(x,\vartheta)\,\nabla\cdot\boldsymbol{v}(x,\vartheta)\,dx\right)d\vartheta. \end{aligned} \tag{4.29}$$

4.1.1 Incompressible flow

First, let us consider the case of incompressible flow, when div$\boldsymbol{v} = 0$. We shall use a more general case with div$\boldsymbol{v} \le 0$ in Q_T.

We shall prove the following result:

Theorem 4.6. *Let assumptions (A1), (A2), (A3) from Section 3.1.1, assumptions (3.4)-(3.6) on data and (4.5) be satisfied. Moreover, let div*$\boldsymbol{v} \le 0$ *in*

Q_T. *Then there exists a constant* $C > 0$ *independent of* h *and* T *such that*

$$\|e\,(t)\|^2_{L^2(\Omega)} \leq Ch^{2\mu-1} \qquad \forall t \in [0,T]\,,\; h \in \left(0,\overline{h}\right), \tag{4.30}$$

where $\mu = min\,(p+1, s)\,.$

Proof. If div$\boldsymbol{v} \leq 0$, then we can omit the term with divergence operator and simplify inequality (4.29) as follows:

$$\begin{aligned} & \|\xi\|^2_{L^2(\Omega)} + \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta \\ \leq\; & 2C_b \int\limits_0^t \|\xi(\vartheta)\|_{L^2(\Omega)} \left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}(\vartheta) \right\|_{L^2(\Omega)} \right) d\vartheta + \int\limits_0^t D(\eta)d\vartheta. \end{aligned} \tag{4.31}$$

In the next step we use the modified Gronwall lemma 4.4, where we have

$$\begin{aligned} \chi(t) &= \|\xi(t)\|_{L^2(\Omega)}\,, \\ R(t) &= \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta, \\ A(t) &= \int\limits_0^t D(\eta)(\vartheta)d\vartheta, \\ B(\vartheta) &= C_b \left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\,(\vartheta) \right\|_{L^2(\Omega)} \right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \left(\|\xi\,(t)\|^2_{L^2(\Omega)} + \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta \right)^{1/2} \\ \leq\; & \max_{0\leq\gamma\leq t} \left(\int\limits_0^\gamma D(\eta)(\vartheta)d\vartheta \right)^{1/2} + \int\limits_0^t C_b \left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\,(\vartheta) \right\|_{L^2(\Omega)} \right) d\vartheta. \end{aligned} \tag{4.32}$$

We have

$$D(\eta) \;=\; 2C'_b \sum_{k=K\in\mathcal{T}_h} \left(\|\eta\|_{L^2(K)}\,|\eta|_{H^1(K)} + h_K^{-1}\,\|\eta\|^2_{L^2(K)} \right) \geq 0$$

and hence, the function "$\gamma \to \int_0^\gamma D\left(\eta\left(\vartheta\right)\right) d\vartheta$" is non-decreasing. This implies that

$$\max_{0\le\gamma\le t}\left(\int_0^{\gamma} D\left(\eta\left(\vartheta\right)\right)d\vartheta\right)^{1/2}\le\left(\int_0^{t} D\left(\eta\left(\vartheta\right)\right)d\vartheta\right)^{1/2}.$$

Putting all these estimates together with inequality (4.32), we obtain

$$\begin{aligned}
&\|\xi(t)\|^2_{L^2(\Omega)}+\int_0^t\sigma(\xi(\vartheta))d\vartheta \qquad (4.33)\\
\le\;& 2\int_0^t D\left(\left(\vartheta\right)\right)d\vartheta+2C_b\left(\int_0^t\left(\|\eta(\vartheta)\|_{L^2(\Omega)}+\left\|\frac{\partial\eta}{\partial t}\left(\vartheta\right)\right\|_{L^2(\Omega)}\right)d\vartheta\right)^2\\
\le\;& 2\int_0^t D\left(\eta\left(\vartheta\right)\right)d\vartheta+4C_b\left[\left(\int_0^t\|\eta(\vartheta)\|_{L^2(\Omega)}\right)^2+\left(\int_0^t\left\|\frac{\partial\eta}{\partial t}\left(\vartheta\right)\right\|_{L^2(\Omega)}d\vartheta\right)^2\right]\\
\le\;& 2\int_0^t D\left(\eta\left(\vartheta\right)\right)d\vartheta+4C_bT\int_0^t\left(\|\eta(\vartheta)\|^2_{L^2(\Omega)}+\left\|\frac{\partial\eta}{\partial t}\left(\vartheta\right)\right\|^2_{L^2(\Omega)}\right)d\vartheta.
\end{aligned}$$

According to the approximation properties (4.6)-(4.9) we get the estimates

$$\begin{aligned}
\|\eta\|_{L^2(\Omega)} &\le C_Ah^{\mu}\left|\varphi\right|_{H^{\mu}(\Omega)}, &(4.34)\\
\left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(\Omega)} &\le C_Ah^{\mu}\left|\frac{\partial\varphi}{\partial t}\right|_{H^{\mu}(\Omega)}, &(4.35)\\
D(\eta) &= 2C_b'\sum_{K\in\mathcal{T}_h}\left(\|\eta\|_{L^2(K)}\left|\eta\right|_{H^1(K)}+h_K^{-1}\left\|\eta\right\|^2_{L^2(K)}\right) &(4.36)\\
&\le 2C_b'\sum_{K\in\mathcal{T}_h}\left(C_A^2h_K^{2\mu-1}\left|\varphi\right|^2_{H^{\mu}(K)}+C_A^2h_K^{2\mu-1}\left|\varphi\right|^2_{H^{\mu}(K)}\right)\\
&= 4C_b'C_A^2h^{2\mu-1}\left|\varphi\right|^2_{H^{\mu}(\Omega)},
\end{aligned}$$

where $\mu=\min\left(p+1,s\right)$.

Using the multiplicative trace inequality (4.12), we estimate the terms of $\sigma\left(\eta\right)$. We have

$$\begin{aligned}
\|\eta\|^2_{v,\partial K\cap\partial\Omega} &\le \int_{\partial K\cap\partial\Omega}\left|\eta\right|^2\left|\boldsymbol{v}\right|dS &(4.37)\\
&\le C_v\int_{\partial K}\left|\eta\right|^2dS\\
&\le C_vC_M\left(\|\eta\|^2_{L^2(K)}h_K^{-1}+\|\eta\|^2_{L^2(K)}\left|\eta\right|_{H^1(K)}\right),
\end{aligned}$$

$$
\begin{aligned}
\|[\eta]\|^2_{v,\partial K^-\setminus\partial\Omega} &\leq \int\limits_{\partial K^-\setminus\partial\Omega} |\boldsymbol{v}| \left|\eta_\Gamma^{(L)} - \eta_\Gamma^{(R)}\right|^2 dS \\
&\leq C_v \int\limits_{\partial K} \left|\left|\eta_\Gamma^{(L)}\right| + \left|\eta_\Gamma^{(R)}\right|\right|^2 dS \\
&\leq 2C_v \left(\int\limits_{\partial K_\Gamma^{(L)}} \left|\eta_\Gamma^{(L)}\right|^2 dS + \int\limits_{\partial K_\Gamma^{(R)}} \left|\eta_\Gamma^{(R)}\right|^2 dS \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h} \|[\eta]\|^2_{v,\partial K^-\setminus\partial\Omega} &\leq 4C_v \sum_{K\in\mathcal{T}_h} \int\limits_{\partial K} |\eta|^2\, dS \qquad (4.38) \\
&\leq 4C_v C_M \sum_{K\in\mathcal{T}_h} \left(\|\eta\|^2_{L^2(K)} h_K^{-1} + \|\eta\|^2_{L^2(K)} |\eta|_{H^1(K)} \right).
\end{aligned}
$$

The above estimates together with (4.36) imply that

$$
\begin{aligned}
2\int\limits_0^t \sigma(\eta(\vartheta))d\vartheta &= \int\limits_0^t \sum_{K\in T_h} \left(\|\eta(\vartheta)\|^2_{v,\partial K\cap\partial\Omega} + \|[\eta(\vartheta)]\|^2_{v,\partial K^-\setminus\partial\Omega} \right) d\vartheta \qquad (4.39) \\
&\leq 5C_v C_M \int\limits_0^t \sum_{K\in T_h} \left(\|\eta(\vartheta)\|^2_{L^2(K)} h_K^{-1} + \|\eta(\vartheta)\|^2_{L^2(K)} |\eta(\vartheta)|_{H^1(K)} \right) d\vartheta \\
&\leq 5C_v C_M \int\limits_0^t \sum_{K\in T_h} \left(C_A^2 h^{2\mu-1} |\varphi(\vartheta)|^2_{H^\mu(K)} + C_A^2 h^{2\mu-1} |\varphi(\vartheta)|^2_{H^\mu(K)} \right) d\vartheta \\
&\leq 10 C_v C_M C_A^2 h^{2\mu-1} \int\limits_0^t |\varphi(\vartheta)|^2_{H^\mu(\Omega)} d\vartheta.
\end{aligned}
$$

Now we return to the original equation (4.4) for the error e_h and start to estimate the term $\|e_h(t)\|^2_{L^2(\Omega)} + \int_0^t \sigma(e_h(\vartheta))d\vartheta$. From (4.27), (4.33)-(4.39) we get

$$\begin{aligned}
& \|e_h(t)\|^2_{L^2(\Omega)} + \int_0^t \sigma(e_h(\vartheta))d\vartheta \\
\leq\ & 2\left(\|\xi(t)\|^2_{L^2(\Omega)} + \int_0^t \sigma(\xi(\vartheta))d\vartheta + \|\eta(t)\|^2_{L^2(\Omega)} + \int_0^t \sigma(\eta(\vartheta))d\vartheta \right) \\
\leq\ & 4\int_0^t D\left(\eta\left(\vartheta\right)\right) d\vartheta + 8C_bT \int_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\left(\vartheta\right) \right\|^2_{L^2(\Omega)} \right) d\vartheta \\
& +2\left\|\eta(t)\right\|^2_{L^2(\Omega)} + \int_0^t \sum_{K\in T_h} \left(\|\eta(\vartheta)\|^2_{v,\partial K\cap\partial\Omega} + \|[\eta(\vartheta)]\|^2_{v,\partial K^-\setminus\partial\Omega} \right) d\vartheta \\
\leq\ & 16C_b' \int_0^t C_A^2 h^{2\mu-1} \left|\varphi\left(\vartheta\right)\right|^2_{H^\mu(\Omega)} d\vartheta \\
& +8C_bT \int_0^t \left(C_A^2 h^{2\mu} \left|\varphi\left(\vartheta\right)\right|^2_{H^\mu(\Omega)} + C_A^2 h^{2\mu} \left| \frac{\partial \varphi}{\partial t}\left(\vartheta\right) \right|^2_{H^\mu(\Omega)} \right) d\vartheta \\
& +2\left(C_A h^\mu \left|\varphi\left(t\right)\right|_{H^\mu(\Omega)} \right)^2 + 10C_vC_M \int_0^t C_A^2 h^{2\mu-1} \left|\varphi\left(\vartheta\right)\right|^2_{H^\mu(\Omega)} d\vartheta.
\end{aligned}$$

Since the term $\int_0^t \sigma(e_h(\vartheta))d\vartheta$ is non-negative, we can write

$$\begin{aligned}
\|e_h(t)\|^2_{L^2(\Omega)} \ \leq\ & C_e \Bigg(h^{2\mu-1} \int_0^t \left(\left|\varphi\left(\vartheta\right)\right|^2_{H^\mu(\Omega)} \right) d\vartheta \qquad (4.40) \\
& + h^{2\mu} \int_0^t \left(\left|\varphi\left(\vartheta\right)\right|^2_{H^\mu(\Omega)} + \left| \frac{\partial \varphi}{\partial t}\left(\vartheta\right) \right|^2_{H^\mu(\Omega)} \right) d\vartheta + h^{2\mu} \left|\varphi\left(t\right)\right|^2_{H^\mu(\Omega)} \Bigg) \\
& \leq C_e \Bigg(\left(h^{2\mu-1} + h^{2\mu} \right) \left|\varphi\right|^2_{L^2(0,T;\, H^\mu(\Omega))} \\
& + h^{2\mu} \left(\left|\varphi\left(\vartheta\right)\right|^2_{L^\infty(0,T;\, H^\mu(\Omega))} + \left| \frac{\partial \varphi}{\partial t}\left(\vartheta\right) \right|^2_{L^2(0,T;\, H^\mu(\Omega))} \right) \Bigg),\ t\in\left[0,T\right],
\end{aligned}$$

where

$$C_e = \max\left\{ C_A^2\left(16C_b' + 10C_vC_M\right);\ 8C_A^2C_bT;\ 2C_A^2 \right\}. \qquad (4.41)$$

Obviously the error of our discretization is of the order $O\left(h^{2\mu-1}\right)$. This means that we have proved estimate (4.30). □

4.1.2 General case of div$\boldsymbol{v}$

Let us consider the general case, when we have to estimate also the term with the divergence operator. We now have

$$\begin{aligned}
&\|\xi\|^2_{L^2(\Omega)} + \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta \qquad (4.42)\\
\leq\ & 2C_b \int\limits_0^t \|\xi(\vartheta)\|_{L^2(\Omega)} \left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}(\vartheta) \right\|_{L^2(\Omega)} \right) d\vartheta \\
&+ \int\limits_0^t D(\eta)d\vartheta + 2\int\limits_0^t \left(\int\limits_\Omega \xi^2\,(x,\vartheta)\,\nabla\cdot\boldsymbol{v}dx \right) d\vartheta.
\end{aligned}$$

If we use the Young inequality, we get

$$\begin{aligned}
&\int\limits_0^t \|\xi(\vartheta)\|_{L^2(\Omega)} \left(\|\eta(\vartheta)\|_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}(\vartheta) \right\|_{L^2(\Omega)} \right) d\vartheta \\
\leq\ & \frac{1}{2}\int\limits_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)} + \int\limits_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}(\vartheta) \right\|^2_{L^2(\Omega)} \right) d\vartheta.
\end{aligned}$$

The term with the divergence operator can be estimated as follows:

$$\begin{aligned}
\int\limits_0^t \left(\int\limits_\Omega \xi^2\,(x,\vartheta)\,\mathrm{div}\boldsymbol{v}dx \right) d\vartheta &\leq \int\limits_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,d\vartheta\,\|\boldsymbol{v}\|_{W^{1,\infty}(Q_T)} \qquad (4.43)\\
&\leq C_v \int\limits_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,d\vartheta
\end{aligned}$$

Thus, now we prepare the inequality for the Gronwall lemma 4.3. We shall use the following substitutions:

$$\begin{aligned}
y(t) &= \|\xi(t)\|^2_{L^2(\Omega),} \\
g(t) &= \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta, \\
z(t) &= C_2 \int\limits_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(\Omega)} + D(\eta(\vartheta)) \right) d\vartheta, \\
C_2 &= \max\left(2C_b,\, 1\right), \\
r(s) &= 2C_v.
\end{aligned}$$

In this case we know that

$$\exp\left(\int\limits_\vartheta^t r(s)ds \right) = \exp\left(2C_v\left(t-\vartheta\right)\right),$$

and applying the Gronwall lemma 4.3 we obtain

$$\begin{aligned}
&\|\xi\left(t\right)\|^2_{L^2(\Omega)} + \int\limits_0^t \sigma(\xi(\vartheta))d\vartheta \qquad (4.44) \\
\leq\ & C_2 \int\limits_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(\Omega)} + D(\eta(\vartheta)) \right) d\vartheta \\
& + C_3 \int\limits_0^t \exp\left(2C_v\left(t-\vartheta\right)\right) \left(\int\limits_0^\vartheta \left(\|\eta(s)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\left(s\right) \right\|^2_{L^2(\Omega)} + D(\eta(s)) \right) ds \right) d\vartheta,
\end{aligned}$$

for all $t \in [0, T]$, where

$$C_3 = 2C_vC_2.$$

We can simplify this estimate using the inequalities

$$\begin{aligned}
& \int\limits_0^\vartheta \left(\|\eta(s)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\left(s\right) \right\|^2_{L^2(\Omega)} + D(\eta(s)) \right) ds \\
\leq\ & \int\limits_0^T \left(\|\eta(s)\|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t}\left(s\right) \right\|^2_{L^2(\Omega)} + \int\limits_0^T D(\eta(s)) \right) ds,
\end{aligned}$$

$$\int_0^t \exp\left(2C_v\left(t-\vartheta\right)\right) d\vartheta = \tfrac{1}{2C_v} \left[-\exp\left(2C_v\left(t-\vartheta\right)\right)\right]_{\vartheta=0}^{\vartheta=t} = \frac{1}{2C_v}\left(-1+\exp\left(2C_v t\right)\right)$$
$$\le \frac{1}{2C_v}\left(\exp\left(2C_v T\right)-1\right).$$

Using all these estimates we are in the same situation as in Section (4.1.1). We have the same inequality as (4.33) except for the constants:

$$\begin{aligned}
\|\xi(\vartheta)\|^2_{L^2(\Omega)} + \int_0^t \sigma(\xi(\vartheta))d\vartheta &\le C_2 \int_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(\Omega)} + D(\eta(\vartheta)) \right) d\vartheta \\
&\quad + C_4 \int_0^t \left(\|\eta(s)\|^2_{L^2(\Omega)} + \left\|\frac{\partial \eta}{\partial t}(s)\right\|^2_{L^2(\Omega)} + D(\eta(s)) \right) ds \\
&\le C \int_0^t \left(\|\eta(\vartheta)\|^2_{L^2(\Omega)} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(\Omega)} + D(\eta(\vartheta)) \right) d\vartheta.
\end{aligned}$$

$$C_4 = C_3 T \frac{1}{2C_v}\left(\exp\left(2C_v T\right)-1\right).$$
$$C = \max\left\{C_2;\, C_4\right\}.$$

Now the same process as in Section (4.1.1) leads to estimate (4.40). However, the constant replacing C_e is of order $O\left(\exp\left(2C_v T\right)\right)$, which attains very large values for large T. The result can be formulated in the following way:

Theorem 4.7. *Let assumptions (A1), (A2), (A3) from Section 3.1.1, assumptions (3.4)-(3.6) on data and (4.5) be satisfied. Moreover, let div$\boldsymbol{v} < 0$ in Q_T. Then there exists a constant $C > 0$ of order $O\left(\exp\left(2C_v T\right)\right)$ such that*

$$\|e\left(t\right)\|^2_{L^2(\Omega)} \le C h^{2\mu-1} \qquad \forall t \in \left[0, T\right],\, h \in \left(0, \overline{h}\right), \tag{4.45}$$

where $\mu = min\left(p+1, s\right)$.

4.2 Error estimation for space-time discretization

We already have the error estimate for the space discretization (3.16). Now we shall estimate the error for the space-time discretization (3.21).

In this section we shall consider a system of triangulations

$$\left\{\mathcal{T}_{h,\tau}\right\}_{h\in\left(0,\overline{h}\right),\tau\in\left(0,\tau_0\right)},\ \overline{h} > 0,\ \tau_0 > 0, \qquad \mathcal{T}_{h,\tau} = \left\{\mathcal{T}_{h,m}\right\}_{m=0}^{M},$$

satisfying the conditions (A1) - (A3) from Section 3.1.1.

We define the error as a function $e = \Phi - \varphi$. The error depends, of course, on h and τ, but we do not emphasize it by notation. Analogously as in Section 4.1 we shall need to use the L^2-interpolation $\Pi_{h,m}$.

Now we introduce the $S_{h,\tau}^{p,q}-$interpolation defined as the space-time projection operator $\pi : C([0,T]\,;\,L^2(\Omega)) \to S_{h,\tau}^{p,q}$ as follows: if $\varphi \in C([0,T]\,;\,L^2(\Omega))$, then

$$
\begin{aligned}
&\pi\varphi \in S_{h,\tau}^{p,q},\\
&(\pi\varphi)(x,t_m-) = \Pi_{h,m}\varphi(x,t_m-) \text{ for almost all } x\in\Omega \text{ and all } m=1,\ldots,M,\\
&\int_{I_m} (\pi\varphi-\varphi,\psi)\,dt = 0 \text{ for all } \psi \in S_{h,\tau}^{p,q-1} \text{ and all } m=1,\ldots,M,\\
&(\pi\varphi)(x,0-) := \Pi_{h,0}\varphi(x,0)\,.
\end{aligned}
$$

The properties of this interpolation were proved in [12] and are also contained in [9].

Now we shall express the error $e = \Phi - \varphi$ in the form

$$
\begin{aligned}
e &= \xi + \eta, \\
\xi &= \Phi - \pi\varphi, \\
\eta &= \pi\varphi - \varphi,
\end{aligned}
\tag{4.46}
$$

where $\eta$is the $S_{h,\tau}^{p,q}-$interpolation error.

Using the fact that $\pi\varphi|_{I_m} = \pi(\Pi_{h,m}\varphi)|_{I_m}$, $m=1,\ldots,M$, proved in [6], we can express the term η in the following way:

$$
\begin{aligned}
&\eta|_{I_m} = (\pi\varphi-\varphi)|_{I_m} = \eta^{(1)} + \eta^{(2)}, \qquad m=1,\ldots,M\\
&\text{with } \eta^{(1)} = (\Pi_{h,m}\varphi-\varphi)|_{I_m},\ \eta^{(2)} = (\pi(\Pi_{h,m}\varphi)-\Pi_{h,m}\varphi)|_{I_m}.
\end{aligned}
$$

We assume that the weak solution φ of problem (3.1)-(3.3) satisfies the regularity condition

$$
\varphi \in C([0,T]\,;\,H^s(\Omega)) \cap H^{q+1}\left(0,T;\,H^1(\Omega)\right), \tag{4.47}
$$

where $s \geq 2$ is and integer. Then φ satisfies relation (3.21).

Now we can formulate estimates of η. Let us assume that p, $s \geq 1$ be integers and $\varphi \in H^s(\Omega)$.Then by (4.6)-(4.7) for all $K \in \mathcal{T}_{h,m}$, $m = 1, \ldots, M$ we have

$$\left\|\eta_0^-\right\|^2_{L^2(\Omega)} \leq C_A^2 h^{2\mu} \left|\varphi(0)\right|_{H^\mu(\Omega)}, \tag{4.48}$$

$$\left\|\eta_m^-\right\|^2_{L^2(\Omega)} \leq C_A^2 h^{2\mu} \left|\varphi(t_m)\right|_{H^\mu(\Omega)}, \tag{4.49}$$

$$\int\limits_{I_m} \left\|\eta^{(1)}\right\|^2_{L^2(K)} dt \leq C_A^2 h_K^{2\mu} \left|\varphi\right|^2_{L^2(I_m;\, H^\mu(K))}, \tag{4.50}$$

$$\int\limits_{I_m} \left|\eta^{(1)}\right|^2_{H^1(K)} dt \leq C_A^2 h_K^{2(\mu-1)} \left|\varphi\right|^2_{L^2(I_m;\, H^\mu(K))}, \tag{4.51}$$

where $\mu = \min(p+1, s)$.

Further, if moreover $q \geq 1$, then by [12] we have

$$\int\limits_{I_m} \left\|\eta^{(2)}\right\|^2_{L^2(K)} dt \leq C\tau_m^{2(q+1)} \left|\varphi\right|^2_{H^{q+1}(I_m;\, L^2(K))}, \tag{4.52}$$

$$\int\limits_{I_m} \left|\eta^{(2)}\right|^2_{H^1(K)} dt \leq C\tau_m^{2(q+1)} \left|\varphi\right|^2_{H^{q+1}(I_m;\, H^1(K))}, \tag{4.53}$$

where C_1 and C_2 are positive constants.

4.2.1 Abstract error estimate

Our first task will be to estimate ξ with respect to the interpolation error η. Then using the η- estimates we obtain the result for the error e in terms of h and τ.

Assuming our discretization from Section 3.1.2, subtracting (3.22) from discretization (3.21) and putting $\psi := \xi(t)$, we obtain following equality:

$$\begin{aligned} &\int\limits_{I_m} (\xi', \xi)\, dt + \left(\{\xi\}_{m-1}, \xi_{m-1}^+\right) + \int\limits_{I_m} b_{h,m}(\xi, \xi) \\ = &\int\limits_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi_{m-1}^+\right) - \int\limits_{I_m} b_{h,m}(\eta, \xi). \end{aligned} \tag{4.54}$$

Its easy to show that the following inequality holds:

$$\begin{aligned} &\int\limits_{I_m} (\xi', \xi)\, dt + \left(\{\xi\}_{m-1}, \xi_{m-1}^+\right) + \int\limits_{I_m} b_{h,m}(\xi, \xi) \\ \leq &\left|\int\limits_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi_{m-1}^+\right)\right| + \int\limits_{I_m} \left|b_{h,m}(\eta, \xi)\right|. \end{aligned} \tag{4.55}$$

For the form $b_{h,m}$ we can use relation (4.26), i.e.,

$$b_{h,m}(\xi,\xi) = \frac{1}{2}\sum_{K\in\mathcal{T}_{h,m}}\left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right) - \frac{1}{2}\int_\Omega \xi^2\nabla\cdot\boldsymbol{v}dx, \tag{4.56}$$

but not estimate (4.17) from Section 4.1. Here we have to proceed in the following way. We have the estimate (4.20):

$$\begin{aligned}|b_{h,m}(\eta,\xi)| \le\ & \left|\sum_{K\in\mathcal{T}_{h,m}}\int_K \eta(\boldsymbol{v}\cdot\nabla\xi)\,dx\right| + \left|\sum_{K\in\mathcal{T}_{h,m}}\int_K \eta\xi\nabla\cdot\boldsymbol{v}dx\right| \\ & + \left|\sum_{K\in\mathcal{T}_{h,m}}\left(\int_{\partial K}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS - \int_{\partial K^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS\right.\right. \\ & \left.\left. - \int_{\partial K^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\eta-\eta^-\right)dS\right)\right|.\end{aligned}$$

For the second and third term we can use estimates (4.21), (4.22) and (4.23) as in Section 4.1. We have

$$\begin{aligned}&\left|\sum_{K\in\mathcal{T}_{h,m}}\int_K \eta\xi\nabla\cdot\boldsymbol{v}dx\right| \\ \le\ & C_v\sum_{K\in\mathcal{T}_{h,m}}\|\eta\|_{L^2(K)}\|\xi\|_{L^2(K)} \\ \le\ & C_v\sum_{K\in\mathcal{T}_{h,m}}\|\eta\|^2_{L^2(K)} + \frac{1}{4}C_v\sum_{K\in\mathcal{T}_{h,m}}\|\xi\|^2_{L^2(K)}, \\ &\left|\sum_{K\in\mathcal{T}_{h,m}}\left(\int_{\partial K^+}(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta dS + \int_{\partial K^-\setminus\partial\Omega}\left\{(\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\eta - (\boldsymbol{v}\cdot\boldsymbol{n})\,\xi\left(\eta-\eta^-\right)\right\}dS\right)\right| \\ \le\ & \frac{1}{4}\sum_{K\in\mathcal{T}_{h,m}}\left(\|\xi\|^2_{v,\partial K^+\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right) \\ +\ & C_vC_M\sum_{K\in\mathcal{T}_{h,m}}\left(\|\eta\|_{L^2(K)}|\eta|_{H^1(K)} + h_K^{-1}\|\eta\|^2_{L^2(K)}\right).\end{aligned}$$

The first term $\sum_{K\in\mathcal{T}_{h,m}}\int_K \eta\left(\boldsymbol{v}\cdot\nabla\xi\right)dx$ must be estimate in another way.

$$\begin{aligned}\left|\sum_{K\in\mathcal{T}_{h,m}}\int_K \eta\left(\boldsymbol{v}\cdot\nabla\xi\right)dx\right| &\leq C_v\sum_{K\in\mathcal{T}_{h,m}}\left(\|\eta\|_{L^2(K)}\left|\xi\right|_{H^1(K)}\right)\\ &\leq C_vC_I\sum_{K\in\mathcal{T}_{h,m}}\left(h_K^{-1}\|\eta\|_{L^2(K)}\|\xi\|_{L^2(K)}\right)\\ &\leq C_vC_I^2\sum_{K\in\mathcal{T}_{h,m}}h_K^{-2}\|\eta\|^2_{L^2(K)}+\frac{1}{4}C_v\sum_{K\in\mathcal{T}_{h,m}}\|\xi\|^2_{L^2(K)}.\end{aligned}$$

Putting all these inequalities together we obtain an estimate

$$\begin{aligned}\left|b_{h,m}\left(\eta,\xi\right)\right| &\leq \frac{1}{4}\sum_{K\in\mathcal{T}_{h,m}}\left(\|\xi\|^2_{v,\partial K^+\cap\partial\Omega}+\|[\xi]\|^2_{v,\partial K^-\setminus\partial\Omega}\right)\\ &\quad+\frac{1}{2}C_v\sum_{K\in\mathcal{T}_{h,m}}\|\xi\|^2_{L^2(K)}+\omega\left(\eta\right),\end{aligned}\tag{4.57}$$

where

$$\begin{aligned}\omega(\eta) &= C_v\sum_{K\in\mathcal{T}_{h,m}}\|\eta\|^2_{L^2(K)}\\ &\quad+C_vC_I^2\sum_{K\in\mathcal{T}_{h,m}}h_K^{-2}\|\eta\|^2_{L^2(K)}\\ &\quad+C_vC_M\sum_{K\in\mathcal{T}_{h,m}}\left(\|\eta\|_{L^2(K)}\left|\eta\right|_{H^1(K)}+h_K^{-1}\|\eta\|^2_{L^2(K)}\right).\end{aligned}\tag{4.58}$$

We shall simplify the term $\int_{I_m}\left(\xi',\xi\right)dt+\left(\{\xi\}_{m-1},\xi^+_{m-1}\right)$:

$$\int_{I_m}\left(\xi',\xi\right)dt+\left(\{\xi\}_{m-1},\xi^+_{m-1}\right)=\frac{1}{2}\int_{I_m}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)}dt+\left(\xi^+_{m-1}-\xi^-_{m-1},\xi^+_{m-1}\right).\tag{4.59}$$

Using the following identities

$$\begin{aligned}\int_{I_m}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)}dt &= \left\|\xi^-_m\right\|^2_{L^2(\Omega)}-\left\|\xi^+_{m-1}\right\|^2_{L^2(\Omega)}\\ 2\left(\xi^+_{m-1}-\xi^-_{m-1},\xi^+_{m-1}\right) &= \left(\xi^+_{m-1}-\xi^-_{m-1},\xi^+_{m-1}\right)+\left(\xi^+_{m-1}-\xi^-_{m-1},\xi^+_{m-1}\right)\\ &= \left\|\xi^+_{m-1}\right\|^2_{L^2(\Omega)}-\left(\xi^-_{m-1},\xi^+_{m-1}\right)\\ &\quad+\left\|\{\xi\}_{m-1}\right\|^2_{L^2(\Omega)}+\left(\xi^+_{m-1},\xi^-_{m-1}\right)-\left\|\xi^-_{m-1}\right\|^2_{L^2(\Omega)}\\ &= \left\|\xi^+_{m-1}\right\|^2_{L^2(\Omega)}+\left\|\{\xi\}_{m-1}\right\|^2_{L^2(\Omega)}-\left\|\xi^-_{m-1}\right\|^2_{L^2(\Omega)},\end{aligned}$$

we get

$$\int_{I_m} (\xi', \xi)\, dt + \left(\{\xi\}_{m-1}, \xi^+_{m-1}\right) = \frac{1}{2}\left(\left\|\xi^-_m\right\|^2_{L^2(\Omega)} + \left\|\{\xi\}_{m-1}\right\|^2_{L^2(\Omega)} - \left\|\xi^-_{m-1}\right\|^2_{L^2(\Omega)}\right). \tag{4.60}$$

Now we shall estimate the term $\int_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right)$ from relation (4.54). We integrate it by parts and obtain

$$\begin{aligned}
\int_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) &= \int_{I_m} (\eta', \xi)\, dt + \left(\eta^+_{m-1}, \xi^+_{m-1}\right) - \left(\eta^-_{m-1}, \xi^+_{m-1}\right) \\
&= -\int_{I_m} (\eta, \xi')\, dt + \left(\eta^-_m, \xi^-_m\right) - \left(\eta^+_{m-1}, \xi^+_{m-1}\right) \\
&\quad + \left(\eta^+_{m-1}, \xi^+_{m-1}\right) - \left(\eta^-_{m-1}, \xi^+_{m-1}\right).
\end{aligned}$$

Since $\int_{I_m} (\eta, \xi')\, dt = 0$, $(\eta^-_m, \xi^-_m) = 0$ and $\left(\eta^-_{m-1}, \xi^-_{m-1}\right) = 0$ as follows from the definition of η and π, we have the identity

$$\begin{aligned}
\int_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) &= -\left(\eta^-_{m-1}, \xi^+_{m-1}\right) \qquad (4.61) \\
&= \left(\eta^-_{m-1}, \xi^-_{m-1}\right) - \left(\eta^-_{m-1}, \xi^+_{m-1}\right) \\
&= -\left(\eta^-_{m-1}, \{\xi\}_{m-1}\right).
\end{aligned}$$

Applying the Young inequality to (4.61)we obtain two relations

$$\left|\int_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right)\right| \le \frac{1}{\delta}\left\|\eta^-_{m-1}\right\|^2_{L^2(\Omega)} + \frac{\delta}{4}\left\|\{\xi\}_{m-1}\right\|^2_{L^2(\Omega)}, \tag{4.62}$$

$$\begin{aligned}
\left|\int_{I_m} (\eta', \xi)\, dt + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right)\right| &\le \left|\left(\eta^-_{m-1}, \xi^+_{m-1}\right)\right| \qquad (4.63) \\
&\le \frac{1}{\delta_1}\left\|\eta^-_{m-1}\right\|^2_{L^2(\Omega)} + \frac{\delta_1}{4}\left\|\xi^+_{m-1}\right\|^2_{L^2(\Omega)}
\end{aligned}$$

where δ, $\delta_1 > 0$.

Putting relations (4.60), (4.62), (4.57) and (4.26) into inequality (4.55), we get

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\xi_m^-\right\|_{L^2(\Omega)}^2+\left\|\{\xi\}_{m-1}\right\|_{L^2(\Omega)}^2-\left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2\right)\\
&+\frac{1}{4}\sum_{K\in T_{h,m}}\int\limits_{I_m}\left(\|\xi\|_{v,\partial K\cap\partial\Omega}^2+\|[\xi]\|_{v,\partial K^-\setminus\partial\Omega}^2\right)\\
\leq\ &\frac{1}{2}\int\limits_{I_m}\int\limits_{\Omega}\xi^2\nabla\cdot\boldsymbol{v}dxdt+\frac{1}{2}C_v\int\limits_{I_m}\sum_{K\in\mathcal{T}_{h,m}}\|\xi\|_{L^2(K)}^2\\
&+\int\limits_{I_m}\omega(\eta)\,dt+\frac{1}{\delta}\left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2+\frac{\delta}{4}\left\|\{\xi\}_{m-1}\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

If we use estimate (4.43) for the term with div$\boldsymbol{v}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\xi_m^-\right\|_{L^2(\Omega)}^2+\left\|\{\xi\}_{m-1}\right\|_{L^2(\Omega)}^2-\left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2\right)\\
&+\frac{1}{4}\sum_{K\in T_{h,m}}\int\limits_{I_m}\left(\|\xi\|_{v,\partial K\cap\partial\Omega}^2+\|[\xi]\|_{v,\partial K^-\setminus\partial\Omega}^2\right)dt\\
\leq\ &C_v\int\limits_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt+\int\limits_{I_m}\omega(\eta)\,dt+\frac{1}{\delta}\left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2+\frac{\delta}{4}\left\|\{\xi\}_{m-1}\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Now we set $\delta:=2$, multiply the above inequality by 2 and get

$$\begin{aligned}
&\left\|\xi_m^-\right\|_{L^2(\Omega)}^2-\left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2+\frac{1}{2}\sum_{K\in T_{h,m}}\int\limits_{I_m}\left(\|\xi\|_{v,\partial K\cap\partial\Omega}^2+\|[\xi]\|_{v,\partial K^-\setminus\partial\Omega}^2\right)dt\\
\leq\ &2C_v\int\limits_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt+2\int\limits_{I_m}\omega(\eta)\,dt+\left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2. \qquad (4.64)
\end{aligned}$$

As next step we need to estimate the term $\int_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt$. We modify the identity (4.59):

$$\begin{aligned}
\int\limits_{I_m}(\xi',\xi)\,dt+\left(\{\xi\}_{m-1},\xi_{m-1}^+\right) &= \frac{1}{2}\int\limits_{I_m}\frac{d}{dt}\|\xi\|^2\,dt+\left(\xi_{m-1}^+-\xi_{m-1}^-,\xi_{m-1}^+\right)\\
&= \frac{1}{2}\left(\left\|\xi_m^-\right\|_{L^2(\Omega)}^2-\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2\right)\\
&\quad+\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2-\left(\xi_{m-1}^-,\xi_{m-1}^+\right).
\end{aligned}$$

Using this identity, (4.56), (4.57) and (4.63), we have

$$\frac{1}{2}\left(\left\|\xi_m^-\right\|_{L^2(\Omega)}^2+\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2\right)+\frac{1}{4}\sum_{K\in\mathcal{T}_{h,m}}\int_{I_m}\left(\|\xi\|_{v,\partial K\cap\partial\Omega}^2+\|[\xi]\|_{v,\partial K^-\backslash\partial\Omega}^2\right)dt$$
$$\le C_v\int_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt+\int_{I_m}\omega(\eta)\,dt+\left|\left(\eta_{m-1}^-,\xi_{m-1}^+\right)\right|+\left|\left(\xi_{m-1}^-,\xi_{m-1}^+\right)\right|.$$

We multiply this inequality by 2 and apply the Young inequality to the terms $\left|\left(\xi_{m-1}^-,\xi_{m-1}^+\right)\right|$ and $\left|\left(\eta_{m-1}^-,\xi_{m-1}^+\right)\right|$. We obtain

$$\begin{aligned}&\left\|\xi_m^-\right\|_{L^2(\Omega)}^2+\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2+\frac{1}{2}\sum_{K\in\mathcal{T}_{h,m}}\int_{I_m}\left(\|\xi\|_{v,\partial K\cap\partial\Omega}^2+\|[\xi]\|_{v,\partial K^-\backslash\partial\Omega}^2\right)dt\\ \le\ & 2C_v\int_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt+2\int_{I_m}\omega(\eta)\,dt+\frac{1}{\delta_1}\left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2\\ &+\frac{1}{\delta_1}\left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2+2\delta_1\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2.\end{aligned}\tag{4.65}$$

For $l=0,\dots,q$ let us set

$$t_{m-1+l/q}=t_{m-1}+\frac{l}{q}\left(t_m-t_{m-1}\right),$$

$$\xi_{m-1+l/q}=\xi\left(t_{m-1+l/q}\right),$$

and denote by $P^q(0,1)$ and $P^q(I_m)$ the spaces of polynomials of degree $\le q$ on $(0,1)$ and I_m, respectively.

We shall prove two lemmas:

Lemma 4.8. *There exist constants L_q, $M_q>0$ dependent on q only such that*

$$\sum_{l=0}^{q}\left\|\xi_{m-1+l/q}\right\|_{L^2(\Omega)}^2\ge\frac{L_q}{\tau_m}\int_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt,\tag{4.66}$$

$$\left\|\xi_{m-1}^+\right\|_{L^2(\Omega)}^2\le\frac{M_q}{\tau_m}\int_{I_m}\|\xi\|_{L^2(\Omega)}^2\,dt.\tag{4.67}$$

Proof. Let $\Theta\in P^q(0,1)$ be an arbitrary polynomial depending on $\vartheta\in(0,1)$ of degree $\le q$ and let us defined the norms $\|\cdot\|_1$, $\|\cdot\|_2$ by

$$\|\Theta\|_1^2=\sum_{l=0}^{q}\left(\Theta(l/q)\right)^2,\qquad\|\Theta\|_2^2=\int_0^1\Theta^2\,d\vartheta.$$

Since $P^q(0,1)$ is a finite dimensional space, these norms are equivalent, and, hence, there exist constants L_q, $M_q > 0$ dependent on q only such that

$$L_q \int_0^1 \Theta^2 d\vartheta \le \sum_{l=0}^{q} (\Theta(l/q))^2 \le M_q \int_0^1 \Theta^2 d\vartheta.$$

Using the substitution theorem for $\vartheta = \frac{t-t_{m-1}}{\tau_m}$, $t \in I_m$, we obtain

$$\sum_{l=0}^{q} p^2\left(t_{m-1+l/q}\right) \ge \frac{L_q}{\tau_m} \int_{I_m} p^2 dt,$$
$$p^2(t_{m-1}) \le \frac{M_q}{\tau_m} \int_{I_m} p^2 dt,$$

for all $p \in P^q(I_m)$. The substitution $p(t) = \xi(x,t)$ for each $x \in \Omega$ yields the inequalities

$$\sum_{l=0}^{q} \xi^2\left(x,\, t_{m-1+l/q}\right) \ge \frac{L_q}{\tau_m} \int_{I_m} \xi^2(x,t)\, dt,$$
$$\xi^2(x,\, t_{m-1}) \le \frac{M_q}{\tau_m} \int_{I_m} \xi^2(x,t)\, dt, \qquad x \in \Omega.$$

Now the integration over Ω with respect to x and the Fubini theorem immediately lead to the desired inequalities (4.66) and (4.67). $\square$

Lemma 4.9. *There exist constants $C, C^* > 0$ such that*

$$\int_{I_m} \|\xi\|^2_{L^2(\Omega)} dt \le C\tau_m \left(\int_{I_m} \omega(\eta)\, dt + \left\|\eta^-_{m-1}\right\|^2_{L^2(\Omega)} + \left\|\xi^-_{m-1}\right\|^2_{L^2(\Omega)} \right), \qquad (4.68)$$

where $\omega(\eta)$ is defined in (4.58), provided

$$0 < \tau_m \le C^*. \qquad (4.69)$$

Proof. For simplicity we consider only the case $q = 1$. From the previous lemma and inequality (4.65)

$$
\begin{aligned}
&\frac{L_q}{\tau_m} \int\limits_{I_m} \|\xi\|^2_{L^2(\Omega)}\,dt + \frac{1}{2} \sum_{K\in T_{h,m}} \int\limits_{I_m} \left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\backslash\partial\Omega} \right) dt \\
\leq 2 \quad &\left(C_v + \frac{\delta_1 M_q}{\tau_m} \right) \int\limits_{I_m} \|\xi\|^2_{L^2(\Omega)}\,dt + 2 \int\limits_{I_m} \omega(\eta)\,dt \\
&+ \frac{1}{\delta_1} \left\| \eta^-_{m-1} \right\|^2_{L^2(\Omega)} + \frac{1}{\delta_1} \left\| \xi^-_{m-1} \right\|^2_{L^2(K)}.
\end{aligned}
$$

We can omit the non-negative term $\frac{1}{2}\sum_{K\in T_{h,m}} \int_{I_m} \left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\backslash\partial\Omega} \right) dt$ and if we set

$$
\delta_1 = \frac{L_q}{8M_q},
$$

then, under the condition

$$
0 < \tau_m \leq C^* := \frac{L_q}{4C_v},
$$

we get

$$
\frac{L_q}{2\tau_m} \int\limits_{I_m} \|\xi\|^2_{L^2(\Omega)}\,dt \quad \leq \quad 2 \int\limits_{I_m} \omega(\eta)\,dt + \frac{2}{\delta_1} \left\| \eta^-_{m-1} \right\|^2_{L^2(\Omega)} + \frac{2}{\delta_1} \left\| \xi^-_{m-1} \right\|^2_{L^2(K)},
$$

which implies the desired statement (4.68). □

In what follows we shall derive the abstract error estimate in the $L^2(Q_T)$ −norm. We put estimate (4.68) into inequality (4.64) to obtain

$$
\begin{aligned}
&\left\| \xi^-_m \right\|^2_{L^2(\Omega)} - \left\| \xi^-_{m-1} \right\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{K\in T_{h,m}} \int\limits_{I_m} \left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\backslash\partial\Omega} \right) dt \\
\leq \quad & 2C_v C \tau_m \left(\int\limits_{I_m} \omega(\eta)\,dt + \left\| \eta^-_{m-1} \right\|^2_{L^2(\Omega)} + \left\| \xi^-_{m-1} \right\|^2_{L^2(\Omega)} \right) \\
&+ 2 \int\limits_{I_m} \omega(\eta)\,dt + \left\| \eta^-_{m-1} \right\|^2_{L^2(\Omega)}.
\end{aligned}
$$

If we omit the expression $\frac{1}{2}\sum_{K\in T_{h,m}} \int_{I_m} \left(\|\xi\|^2_{v,\partial K\cap\partial\Omega} + \|[\xi]\|^2_{v,\partial K^-\backslash\partial\Omega} \right) dt \geq 0$ and write i instead of m, we get

$$
\begin{aligned}
& \left\|\xi_i^-\right\|_{L^2(\Omega)}^2 - (1+2C_vC\tau_i)\left\|\xi_{i-1}^-\right\|_{L^2(\Omega)}^2 \\
\leq\ & 2(C_vC\tau_i+1)\int\limits_{I_i}\omega(\eta)\,dt + (1+2C_vC\tau_i)\left\|\eta_{i-1}^-\right\|_{L^2(\Omega)}^2 .
\end{aligned} \tag{4.70}
$$

Summing inequality (4.70) over $i = 1, \ldots, m \leq M$ and taking into account that $\tau_i < T$ for all $i = 1, \ldots, m$ and $\xi_0^- = 0$, we have

$$
\begin{aligned}
& \left\|\xi_m^-\right\|_{L^2(\Omega)}^2 \leq C\sum_{i=1}^{m}\tau_i\left\|\xi_{i-1}^-\right\|_{L^2(\Omega)}^2 + C\sum_{i=1}^{m}\left(\int\limits_{I_i}\omega(\eta)\,dt + \left\|\eta_{i-1}^-\right\|_{L^2(\Omega)}^2\right) \\
& m = 0, \ldots, M
\end{aligned} \tag{4.71}
$$

where C is a positive constant independent of h, τ, M.

We shall use the following lemma:

Lemma 4.10. *(Discrete Gronwall lemma) Let x_m, a_m, b_m and c_m, where $m = 1, 2, \ldots,$ be non-negative sequences and let the sequence a_m be nondecreasing. Then, if*

$$
\begin{aligned}
x_0 + c_0 &\leq a_0, \\
x_m + c_m &\leq a_m + \sum_{i=0}^{m-1} b_i x_i \quad \text{for } m \geq 1,
\end{aligned}
$$

we have

$$
x_m + c_m \leq a_m \prod_{i=0}^{m-1}(1+b_i) \quad \text{for } m \geq 0,
$$

[9].

Applying the Discrete Gronwall lemma to (4.71) with terms

$$
\begin{aligned}
x_0 &= c_0 = a_0 = 0, \\
x_m &= \left\|\xi_m^-\right\|_{L^2(\Omega)}^2, \\
c_m &= 0, \\
a_m &= C\sum_{i=1}^{m}\left(\int\limits_{I_i}\omega(\eta)\,dt + \left\|\eta_{i-1}^-\right\|_{L^2(\Omega)}^2\right), \\
b_i &= C\tau_{i+1}, \quad i = 0, 1, \ldots, m-1,
\end{aligned}
$$

we obtain

$$\left\|\xi_m^-\right\|_{L^2(\Omega)}^2 \le C \sum_{i=1}^{m} \left(\int_{I_m} \omega(\eta)\, dt + \left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2 \right) \prod_{j=0}^{M-1} (1 + C\tau_{m+1}) .$$

Since $1 + C\tau_{i+1} \le \exp(C\tau_{i+1})$, we have

$$\prod_{i=0}^{m-1} (1 + C\tau_{i+1}) \le \exp\left(C \sum_{i=1}^{m} \tau_i \right) = \exp(Ct_m) \le \exp(CT) .$$

Hence for $m = 1, \ldots M$ we have the estimate

$$\left\|\xi_m^-\right\|_{L^2(\Omega)}^2 \le \widetilde{C} \sum_{i=1}^{m} \left(\int_{I_i} \omega(\eta)\, dt + \left\|\eta_{i-1}^-\right\|_{L^2(\Omega)}^2 \right) . \tag{4.72}$$

For the final error estimation we use the inequalities

$$\begin{aligned} \left\|e_m^-\right\|_{L^2(\Omega)}^2 &\le 2\left(\left\|\xi_m^-\right\|_{L^2(\Omega)}^2 + \left\|\eta_m^-\right\|_{L^2(\Omega)}^2 \right) , && (4.73) \\ \|e\|_{L^2(\Omega)}^2 &\le 2\left(\|\xi\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2 \right) . && (4.74) \end{aligned}$$

From (4.72) and (4.73) we obtain

$$\begin{aligned} \left\|e_m^-\right\|_{L^2(\Omega)}^2 &\le C_{AE} \sum_{i=1}^{m} \left(\int_{I_i} \omega(\eta)\, dt + \left\|\eta_{i-1}^-\right\|_{L^2(\Omega)}^2 \right) + 2\left\|\eta_m^-\right\|_{L^2(\Omega)}^2 , && (4.75) \\ & m = 1, \ldots, r, \quad h \in \left(0, \overline{h}\right) . \end{aligned}$$

Using estimate (4.68) and putting it into inequality (4.74) we get

$$\begin{aligned} \|e\|_{L^2(Q_T)}^2 &= \sum_{m=1}^{M} \int_{I_m} \|e\|_{L^2(\Omega)}^2 \\ \le\; & C \sum_{m=1}^{M} \tau_m \left(\int_{I_m} \omega(\eta)\, dt + \left\|\eta_{m-1}^-\right\|_{L^2(\Omega)}^2 + \left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2 \right) + 2 \int_0^T \|\eta\|_{L^2(\Omega)}^2\, dt. \end{aligned}$$

Now we use (4.72) for the estimate of $\left\|\xi_{m-1}^-\right\|_{L^2(\Omega)}^2$ and the relations $\xi_0^- = 0$, $\eta_0^- =$

$\Pi_{h,0}\varphi^0 - \varphi^0$, we obtain the following estimate for the error e

$$\begin{aligned}\|e\|^2_{L^2(Q_T)} \quad &\leq \quad C_{AE}\sum_{m=1}^{M}\tau_m\left(\int\limits_{I_m}\omega\left(\eta\right)dt + \left\|\eta^-_{m-1}\right\|^2_{L^2(\Omega)}\right. \\ &+ \quad \left.\sum_{i=1}^{m}\left(\int\limits_{I_i}\omega\left(\eta\right)dt + \left\|\eta^-_{i-1}\right\|^2_{L^2(\Omega)}\right)\right) + 2\left\|\eta\right\|^2_{L^2(Q_T)}, \quad h \in \left(0,\overline{h}\right).\end{aligned} \tag{4.76}$$

The above results can be summarized in the following way.

Theorem 4.11. *Under assumptions (3.4)-(3.6), (A1)-(A3) from Section 3.1.1 and (4.69) the abstract error estimates (4.75) and (4.76) hold in terms of the interpolation error η.*

4.2.2 Error estimation in terms of h and τ

Theorem 4.12. *Let assumptions of Theorem 4.11 and the regularity assumption (4.47) be satisfied. Then there exists a constant $C_E > 0$ independent of h and τ such that*

$$\|e\|^2_{L^2(Q_T)} \leq C_E\left(h^{2\mu-2} + \tau^{2q}\right), \quad h \in \left(0,\overline{h}\right), \tau \in \left(0,\overline{\tau}\right), \tag{4.77}$$

where $\mu = min\left(p+1, s\right)$.

Proof. To complete the error analysis we need to estimate the terms $\int_{I_m}\omega\left(\eta\right)dt$ and $\left\|\eta^-_{i-1}\right\|_{L^2(\Omega)}$, $i = 1,\dots,M$. We know that

$$\begin{aligned}\int\limits_{I_m}\omega(\eta)dt \quad &= \quad \int\limits_{I_m}C_v\sum_{K\in\mathcal{T}_h}\|\eta\|_{L^2(K)}\,dt \\ &+\int\limits_{I_m}C_vC_I^2\sum_{K\in\mathcal{T}_{h,m}}\left(h_K^{-2}\|\eta\|^2_{L^2(K)}\right)dt \\ &+\int\limits_{I_m}C_vC_M\sum_{K\in\mathcal{T}_{h,m}}\left(\|\eta\|_{L^2(K)}\left|\eta\right|_{H^1(K)} + h_K^{-1}\|\eta\|^2_{L^2(K)}\right)dt.\end{aligned} \tag{4.78}$$

We use estimates (4.48)-(4.53) and take in account that $\eta|_{I_m} = \left(\pi\varphi - \varphi\right)|_{I_m} = \eta^{(1)} + \eta^{(2)}$, so we have

$$\begin{aligned}\int\limits_{I_m}\sum_{K\in\mathcal{T}_{h,m}}C_v\|\eta\|^2_{L^2(K)}\,dt \quad &\leq \quad C_v^2\sum_{K\in\mathcal{T}_{h,m}}\left(C_A^2h_K^{2\mu}\left|\varphi\right|^2_{L^2(I_m;\,H^\mu(K))}\right. \\ &\left.+C\tau_m^{2(q+1)}\left|\varphi\right|^2_{H^{q+1}(I_m;\,L^2(K))}\right).\end{aligned} \tag{4.79}$$

Similarly we estimate the second term and get

$$\begin{aligned}
& C_v C_I^2 \int\limits_{I_m} \sum_{K \in \mathcal{T}_{h,m}} h_K^{-2} \|\eta\|_{L^2(K)}^2 \, dt \qquad (4.80)\\
& \leq C_v C_I^2 \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h_K^{2\mu-2} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 \\
& + C \tau_m^{2(q+1)} h_K^{-2} |\varphi|_{H^{q+1}(I_m;\, L^2(K))}^2 \Big) .
\end{aligned}$$

Further, we obtain the estimate for the third term from (4.78):

$$\begin{aligned}
& \int\limits_{I_m} C_v C_M \sum_{K \in \mathcal{T}_{h,m}} \Big(\|\eta\|_{L^2(K)} |\eta|_{H^1(K)} \Big) \, dt \qquad (4.81)\\
\leq \; & C_v C_M \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h^{2\mu-1} |\varphi|_{L^2(I_m;\, H^\mu(\Omega))}^2 + C \tau_m^{2(q+1)} |\varphi|_{H^{q+1}(I_m;\, H^1(K))}^2 \Big) , \\
& \int\limits_{I_m} C_v C_M \sum_{K \in \mathcal{T}_{h,m}} h_K^{-1} \|\eta\|_{L^2(K)}^2 \, dt \qquad (4.82)\\
\leq \; & C_v C_M \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h_K^{2\mu-1} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 + C \tau_m^{2(q+1)} h_K^{-1} |\varphi|_{H^{q+1}(I_m;\, L^2(K))}^2 \Big) .
\end{aligned}$$

Putting all these estimates (4.79)-(4.82) we have

$$\begin{aligned}
\left| \int\limits_{I_m} \omega(\eta) dt \right| \; \leq \; & C_v \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h_K^{2\mu} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 + C \tau_m^{2(q+1)} |\varphi|_{H^{q+1}(I_m;\, L^2(K))}^2 \Big) \\
+ \; & C_v C_I^2 \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h_K^{2\mu-2} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 + C \tau_m^{2(q+1)} h_K^{-2} |\varphi|_{H^{q+1}(I_m;\, L^2(K))}^2 \Big) \\
+ \; & C_v C_M \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h^{2\mu-1} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 + C \tau_m^{2(q+1)} |\varphi|_{H^{q+1}(I_m;\, H^1(K))}^2 \Big) \\
+ \; & C_v C_M \sum_{K \in \mathcal{T}_{h,m}} \Big(C_A^2 h_K^{2\mu-1} |\varphi|_{L^2(I_m;\, H^\mu(K))}^2 + C \tau_m^{2(q+1)} h_K^{-1} |\varphi|_{H^{q+1}(I_m;\, L^2(K))}^2 \Big) .
\end{aligned}$$

We have $h_K \leq h$ for all $K \in \mathcal{T}_{h,m}$, $\tau_m \leq \tau$ for all $m = 1, \ldots, M$. Let us assume that $\tau_m \leq \widetilde{C} h_K$ for all $K \in \mathcal{T}_{h,m}$, with a constant $\widetilde{C} > 0$ independent of m, h, and K. Then $h_K^{-1} \leq \frac{1}{\widetilde{C}} \tau_m^{-1}$ and we have

$$\left| \int\limits_{I_m} \omega(\eta) dt \right| \; \leq \; C \left(h^{2\mu-2} + \tau^{2q} \right), \quad h \in \left(0, \overline{h}\right), \tau \in (0, \overline{\tau}),$$

where $C > 0$ is a constant depending on norms of the solution φ. Now we use this estimate and (4.48), (4.49) in (4.76) and get

$$\begin{aligned}
\|e\|^2_{L^2(Q_T)} &\leq C_{AE}\sum_{m=1}^{M}\tau_m\Bigg(C\left(h^{2\mu-2}+\tau^{2q}\right)+C_A^2h^{2\mu}\left|\varphi\left(t_m\right)\right|^2_{H^\mu(\Omega)} \\
&\quad +\sum_{i=1}^{m}\left(C\left(h^{2\mu-2}+\tau^{2q}\right)+C_A^2h^{2\mu}\left|\varphi\left(t_{i-1}\right)\right|^2_{H^\mu(\Omega)}\right)\Bigg) \\
&\quad +2C_A^2h^{2\mu}\left|\varphi\right|^2_{L^2(I_m;\,H^\mu(Q_T))}+2C\tau_m^{2(q+1)}\left|\varphi\right|^2_{H^{q+1}(I_m;\,L^2(Q_T))} \\
&\leq C_E\left(h^{2\mu-2}+\tau^{2q}\right),\quad h\in\left(0,\overline{h}\right),\,\tau\in\left(0,\overline{\tau}\right).
\end{aligned}$$

This finishes the proof of Theorem 4.12. □

5 Numerical experiments

Now we deal with 2D numerical experiments on the rectangular domain $\Omega = (-0.5, 0.5) \times (-0.5, 1.5)$ for simulation of two-phase flow.

Like in Section 3.2.1 and 3.2.2 the Navier-Stokes equations are solved by the finite element method and for the level set problem we use the space-time discontinuous Galerkin method.

For the approximation of $\boldsymbol{v}$ and p the Taylor-Hood finite elements are used. We work with spaces $\boldsymbol{V}_h = H_h \times H_h$, where

$$H_h = \left\{ \boldsymbol{w_h} \in C\left(\overline{\Omega}\right);\, \boldsymbol{w}_h|_K \in P^2(K)\ \forall K \in \mathcal{T}_h \right\},$$

and $P^2(K)$ denotes the space of all polynomials on K of degree less or equal 2 and

$$Q_h = \left\{ q_h \in C\left(\overline{\Omega}\right);\, q_h \in P^1(K)\ \forall K \in \mathcal{T}_h \right\},$$

where $P^1(K)$ denotes the space of all linear functions.

To treat the discontinuity of the pressure due to the presence of the surface tension we apply the extended finite element method (XFEM).

The original space Q_h is enlarged using a localization by an enrichment function. We use the original basis functions of Q_h. Let $\mathcal{J} = \{1, \cdots, n\}$ denote the index set, where $n = \dim Q_h$, x_j, $j \in \mathcal{J}$ denote the mesh nodes, $q_i \in Q_h$, $i \in \mathcal{J}$, denote the basis functions that satisfy $q_i(x_j) = \delta_{ij}$. Let us denote by $\mathcal{J}'$ the subset of indices of all neighbours of the interface $\Gamma_I(t)$, $\mathcal{J}' = \{j \in \mathcal{J} :\ \mathrm{supp} q_j \cap \Gamma_I(t) \neq \emptyset\}$. We use the discontinuous enrichment function $H_\Gamma(x)$ given by the Heaviside function

$$H_\Gamma(x) = \begin{cases} 0 & \text{if } x \in \Omega_1 \quad \text{on } \Gamma, \\ 1 & \text{if } x \in \Omega_2 \quad \text{on } \Gamma, \end{cases}$$

and define the discontinuous basis functions q_j^{xfe} given by

$$q_j^{xfe} = q_j(x)\left(H_\Gamma(x) - H_\Gamma(x_j)\right).$$

The function q_j^{xfe} is equal to zero for every node x_i, $i \in \mathcal{J}$, and also the support of q_j^{xfe} is localized only to the elements containing the interface $\Gamma_I(t)$. Hence, we replace the original space Q_h by the extended space $Q_h^{xfe} = Q_h \oplus \mathrm{span}\left\{ q_j^{xfe} :\ j \in \mathcal{J}' \right\}$.

To obtain better results we can modify the XFEM method. We do not use all basis function $\left\{ q_j^{xfe} \right\}_{j \in J'}$, but basis function with very small support are not used in the computation. These basis functions cause defects in the numerical results, as we can see in Figure 5.2. We calculate the ration between the areas of supports of q_j^{xfe} and q_j. If the ratio is smaller than 2%, the basis function q_j^{xfe} is omitted.

For the approximation of the level set function φ we use the space

$$S_{hm}^{p} = \left\{ \psi_h \in L^2(\Omega);\, \psi_h|_K \in P^1\ \forall K \in \mathcal{T}_{h,m} \right\}.$$

In computing the level set function a reinitialization is needed. Let us set V_{ref} the referential volume of Ω_2. After each step of computing φ we calculate the actual volume $V_{\Omega_2}(\varphi)$ of the area Ω_2. In the case when $|V_{ref} - V_{\Omega_2}(\varphi)| > \varepsilon$, with a given constant $\varepsilon > 0$, we recalculate the level set function as follows:

1. If $|V_{ref} - V_{\Omega_2}(\varphi)| > \varepsilon,$ then compute $V_{ref} - V_{\Omega_2}(\varphi) = \triangle V$.

2. Compute $l = \int_C 1dS$, where C denotes the curve where $\varphi = 0$.

3. Set $\triangle\varphi = \triangle V / l$.

4. Set$\widetilde{\varphi} = \varphi + \triangle\varphi$.

5. Then in the next steps use $\varphi := \widetilde{\varphi}$.

5.1 Algorithm

Let us described the algorithm of our computation.

Let us prescribe the initial condition $\boldsymbol{v}^0 = \boldsymbol{v}(t)|_{t=0}$, $p^0 = p(t)|_{t=0}$, $\varphi^0 = (t)|_{t=0}$ and $\rho^{(k)}$, $\mu^{(k)}$, for $k = 1, 2$, that denotes the pressure and viscosity for the fluids 1 and 2. We set

$$\begin{aligned} \rho^0(x) : &= \rho(t,x)|_{t=0} = \rho^{(1)} + H\left(\varphi^0(x)\right)\left(\rho^{(2)} - \rho^{(1)}\right), \\ \mu^0(x) : &= \mu(t,x)|_{t=0} = \mu^{(1)} + H\left(\varphi^0(x)\right)\left(\mu^{(2)} - \mu^{(1)}\right). \end{aligned}$$

Then for $n = 0, 1, 2, 3, \ldots$ we proceed in the following way:

1. On the interval $[t_n, t_{n+1}]$ compute the level set function from (3.21) using the STDGM wiht $\boldsymbol{v} = \boldsymbol{v}^n$ and then set $\varphi^{n+1} = \Phi(t_{n+1})$. For initial condition use the φ^n from the previous time level.

2. Perform the reinitialization of φ^{n+1} defined above by 1.-5.

3. Using the approximation φ^{n+1}, determine μ^{n+1}, and ρ^{n+1} as follows

$$\begin{aligned}\rho^{n+1}(x) &= \rho^{(1)} + H\left(\varphi^{n+1}(x)\right)\left(\rho^{(2)} - \rho^{(1)}\right),\\ \mu^{n+1}(x) &= \mu^{(1)} + H\left(\varphi^{n+1}(x)\right)\left(\mu^{(2)} - \mu^{(1)}\right),\end{aligned}$$

where $H_\varepsilon(\varphi)$ denotes the Heaviside function

$$H(\varphi) = \begin{cases} 0 & \text{if } \varphi > 0, \\ 1 & \text{if } \varphi < 0. \end{cases}$$

4. Solve (3.34) using the time extrapolation $\overline{\boldsymbol{v}}^n = \boldsymbol{v^n}$ and obtain $\boldsymbol{v}^{n+1}$, p^{n+1}.

5. Set $n = n + 1$ and go to step 1.

5.2 Results

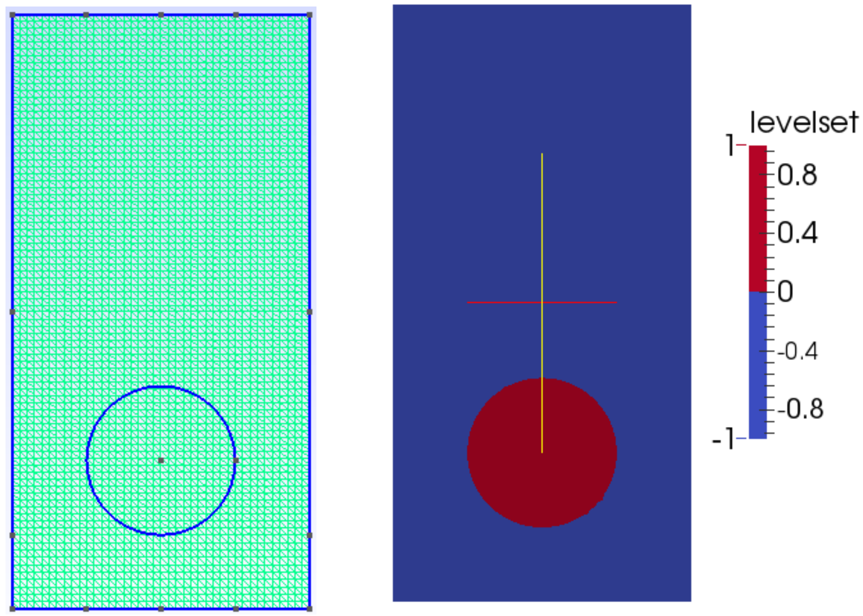
Our numerical results are obtained for the case of a rising bubble in the rectangular domain $\Omega = [-0.5, 0.5] \times [-0.5, 1.5]$. Let us set the following values for the fluids 1 and 2:

$$\begin{aligned}\rho^{(1)} &= 1000\frac{\mathrm{kg}}{\mathrm{m}^3},\ \mu^{(1)} = 10\mathrm{Pa\ s},\\ \rho^{(2)} &= 100\frac{\mathrm{kg}}{\mathrm{m}^3},\ \mu^{(1)} = 1\mathrm{Pa\ s},\\ f &= (0, -0.98)\,\frac{\mathrm{m}}{\mathrm{s}^2}.\end{aligned}$$

The fluid 2 in located in the circle of the diameter $d = 0.5$, the centre of the circle lies 0.5 up from the bottom of the domain. The bottom and top of the domain are denoted by Γ_D, the rest of the boundary is denoted by Γ_S.

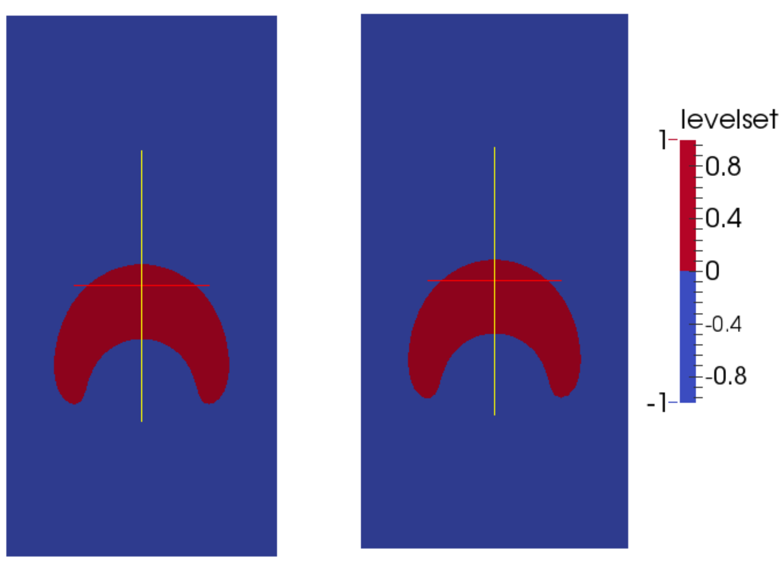
We perform the computation on a triangular mesh with uniform partition with spatial step $h = \frac{\sqrt{2}}{40}$, see Figure 5.1.

Figure 5.1: a) Mesh $h = \frac{1}{40}$. b) Initial position at $t = 0$.



For our computation we use the surface tension equal to zero. We compare the results using all XFEM basis functions $\left\{q_j^{xfe}\right\}_{j\in J}$ in computation with the results obtained after the modification of XFEM basis functions and using only system of functions $\left\{q_j^{xfe}\right\}_{j\in J'}$. In our computation we set the time step $\triangle t = 0.002$. Till time $t = 2$ we see no difference between both cases, see Figure 5.2.

Figure 5.2: a) Result at $t = 2$ with all XFEM basis functions, b) result at $t = 2$ after modification.



For $t > 2$ we can see different behaviour of the results, see Firuge 5.3. The problem by using all XFEM basis functions is with pressure that oscillates and takes large values, see Figuer 5.4.

Figure 5.3: a) Result at $t = 3$ with all XFEM basis functions, b) result at $t = 3$ after modification.

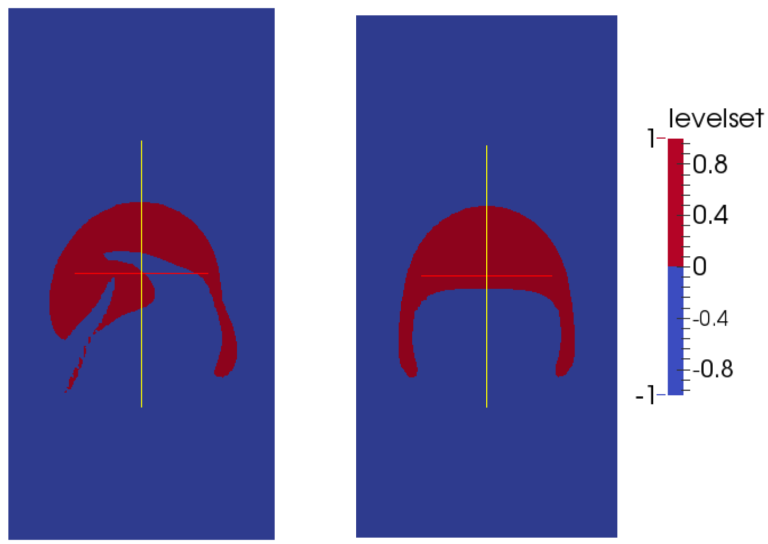
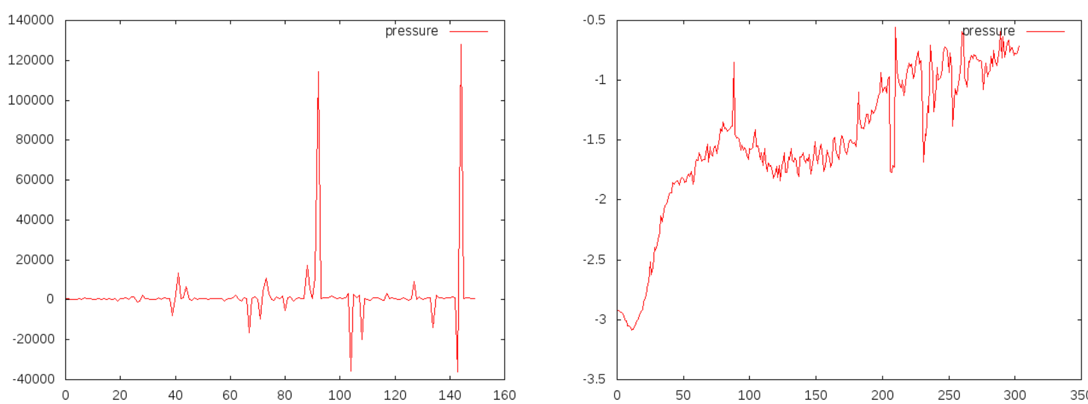
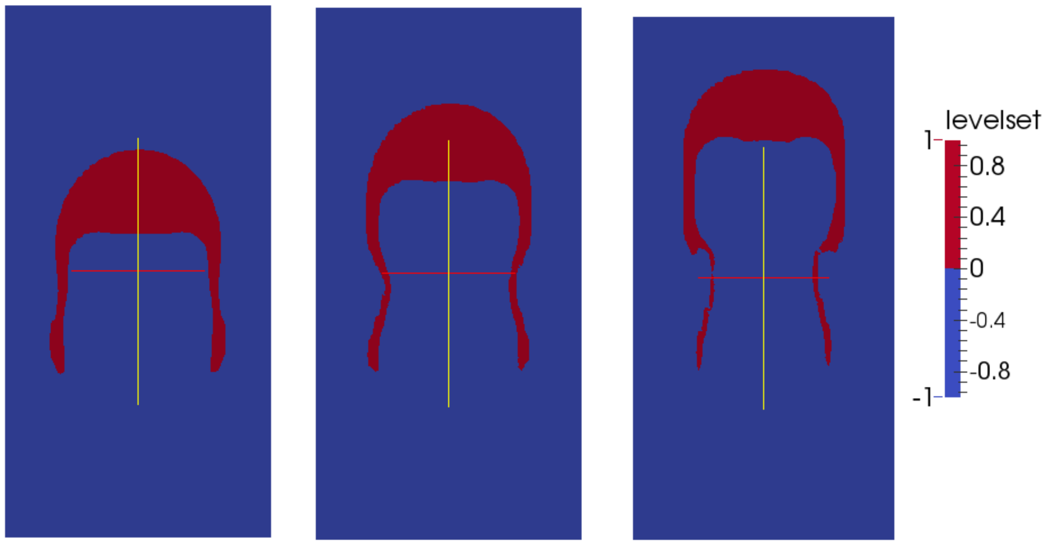


Figure 5.4: a) Pressure before the modification, b) pressure after the modification. x-axis denotes the number of time levels, y-axis presents the value of the pressure.



We can observe the deformation of the bubble that is increasing with increasing time, see Figure 5.4. The interface between both fluids is sharp, which demonstrates that the developed method is very accurate.

Figure 5.5: The modified results at time a) $t = 4$ b) $t = 5$ c) $t = 6$.



6 Conclusion

We worked out a model and method for the solution of two-phases flow. We discretized the equations in space and time. The conforming finite element method combined with backward difference formula is used for the Navier-Stokes problem and the discontinuous Galerkin method of lines and space-time Galerkin method are used for discretization of the level set equation.

We studied the error of discretization for the level set equation. The estimates for space discretization were derived for incompressible flow and also for general case. Surprisingly the results differ from each other only with a constant. Error of space discretization is of the order $O\left(h^{2\mu-1}\right)$. However, the constant in the general case is of order $O\left(\exp\left(2C_vT\right)\right)$, which attains very large values for large T. For the space-time DGM we obtained an estimate of order $O\left(h^{2\mu-2}+\tau^{2q}\right)$, under the assumption that the time step is bounded by the spatial step: $\tau_m \leq \widetilde{C}h_K$.

In the last section the numerical results are presented. An algorithm of computing is described. We use the modified XFEM method to obtain accurate results. We do not take the surface tension into account in our calculation. It will need a further investigation.

References

[1] BAKER, G. R. and D. W. MOORE. The rise and distortion of a two-dimensional gas bubble in an inviscid liquid. Phys. Fluids A 1 (9). 1989, ISBN 10.1063/1.857322.

[2] M. BALÁZSOVÁ, M. FEISTAUER, M. HADRAVA and A. KOSÍK, On the stability of the space-time discontinuous Galerkin method for the numerical solution of nonstationary nonlinear convection-diffusion problems. Journal of Numerical Mathematics, to appear.

[3] E. BÄNSCH, Finite element discretization of the Navier–Stokes equations with a free capillary surface. Numerische Mathematik. 2001, vol. 88, issue 2, s. 203-235. DOI: 10.1007/PL00005443.

[4] J. W. BARRETT, H. GARCKE and R. NÜRNBERG. Eliminating spurious velocities with a stable approximation of viscous incompressible two-phase Stokes flow. Computer Methods in Applied Mechanics and Engineering. 2013, 267: 323-359. DOI: 10.1007/bfb0086916.

[5] R. CAIDEN, R. P. FEDKIW and C. ANDERSON. A Numerical Method for Two-Phase Flow Consisting of Separate Compressible and Incompressible Regions. Journal of Computational Physics. 2001, 166(1): 1-27. DOI: 10.1006/jcph.2000.6624.

[6] P. G. CIARLET, The finite Element Method for elliptic Problems. 1st Ed. Amsterdam: North -Holland Publ. Comp., 1978, 530 s. ISBN 04-448-5028-7.

[7] S. DELLACHERIE, Numerical resolution of a potential diphasic low Mach number system. Journal of Computational Physics. 2007, vol. 223, issue 1, s. 151-187. DOI: 10.1016/j.jcp.2006.09.009.

[8] U. DIERKES, Minimal surfaces. New York: Springer-Verlag, c1992, 2 v. ISBN 038753170X2.

[9] V. DOLEJŠÍ and M. FEISTAUER, Discontinuous Galerkin Method - Analysis and Applications to Compressible Flow. Springer, to be published in 2015.

[10] M. FEISTAUER, V. KUČERA, K. NAJZAR and J. PROKOPOVÁ. Analysis of space–time discontinuous Galerkin method for nonlinear

convection–diffusion problems. Numerische Mathematik. 2010, 117(2): 251-288. DOI: 10.1007/s00211-010-0348-x.

[11] M. FEISTAUER, Mathematicel and computational methods for compressible low. 2014. vyd. Oxford: Clarendon Press, 2003, 535 s. ISBN 01-985-0588-4.

[12] M. FEISTAUER, Mathematical methods in fluid dynamics. New York: Longman, 1993, 657 s. ISBN 05-822-0988-9.

[13] S. GALLOT and D. HULIN, Riemannian geometry. 3rd ed. Berlin: Springer, 2004. ISBN 978-364-2188-558.

[14] H.-P. GITTEL, M. GÜNTHER, G. STRÖHMER and J. VESELÝ. Remarks on a nonlinear transport problem. Journal of Differential Equations. 2014, 256(3): 125-132. DOI: 10.1007/978-1-4613-4425-4_10.

[15] J. GLIMM, D. MARCHESIN and O. MCBRYAN. A numerical method for two phase flow with an unstable interface. Journal of Computational Physics. 1981, 39(1): 179-200. DOI: 10.1016/0021-9991(81)90144-3.

[16] C. W. HIRT and B.D NICHOLS. Volume of fluid (VOF) method for the dynamics of free boundaries. Journal of Computational Physics. 1981, 39(1): 201-225. DOI: 10.1016/0021-9991(81)90145-5.

[17] S. HYSING, et al. Quantitative benchmark computations of two-dimensional bubble dynamics. International Journal for Numerical Methods in Fluids. 2009, 60(11): 1259-1288. DOI: 10.1002/fld.1934.

[18] D. KUZMIN, On the design of general-purpose flux limiters for finite element schemes. I. Scalar convection. Journal of Computational Physics. 2006, vol. 219, issue 2, s. 513-531. DOI: 10.1016/j.jcp.2006.03.034.

[19] S. J. OSTER and R. P. FEDKIW. Level set methods: An overview and some recent results. J. Comput. Phys. 2001, vol.169, 463-502.

[20] Y. PENEL, Existence of global solutions to the 1D abstract bubble vibration model, Differential Integral Equations. 2013, vol. 26 (1-2), s. 59-80.

[21] Y. PENEL, S. DELLACHERIE and O. LAFITTE. Theoretical Study of an Abstract Bubble Vibration Model. Zeitschrift für Analysis und ihre Anwendungen. 2013, vol. 32, issue 1, s. 19-36. DOI: 10.4171/zaa/1472.

[22] H. B. STEWART and B. WENDROFF. Two-phase flow: Models and methods. Journal of Computational Physics. 1984, 56(3): 363-409. DOI: 10.1016/0021-9991(84)90103-7.

[23] P. SVÁČEK, Numerical simulation of free-surface flows with surface tension, Programs and Algorithms of Numerical Matematics 17. Institute of Mathematics AS CR, Prague, 2015. pp. 207-214

[24] P. SVÁČEK, On approximation of non-Newtonian fluid flow by the finite element method. Journal of Computational and Applied Mathematics. 2008, vol. 218, issue 1, s. 167-174. DOI: 10.1016/j.cam.2007.04.040.

[25] S. O. UNVERDI and G. TRYGGVASON. A front-tracking method for viscous, incompressible, multi-fluid flows. Comp. Phys. 1992, ISBN 10.1016/0021-9991(92)90294-9.