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Institut ekonomických studií

Mikoláš Volek

Counterparty credit risk modelling

Diplomová práce

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Autor práce: **Mikoláš Volek**

Vedoucí práce: **doc. PhDr. Petr Teplý PhD.**

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Abstrakt

Kreditní riziko protistrany je důležitým druhem finančního rizika. Tento fakt se ukázal zejména během roku 2008 ve světle pádu mnoha velkých bank. Důležitým prvkem při kalkulaci CVA je korelace tržních proměnných. Nejdůležitějším dopadem korelace je tzv. wrong-way riziko, tedy riziko ztráty v důsledku velké korelace mezi pravděpodobností defaultu a velikostí expozice. Toto riziko základní vzorce pro CVA nepostihují a mnoho aplikací se wrong-way riziku vyhýbá, neboť jeho modelování je složité. Tato diplomová práce si klade za cíl zjistit, zda a jak dobře lze wrong-way riziko aproximovat jednoduchým faktorem, který by závisel na pozorované korelaci ceny podkladového aktiva a kreditního spreadu protistrany. Přílohou práce je plně dokumentovaná implementace modelu v programu *Mathematica*.

Abstract

Counterparty credit risk is an important type of financial risk. The importance of proper counterparty risk management became most apparent in the wake of the 2008 series of failures of several large banks. Correlation of market factors is an important issue in the calculation of CVA. A notable case of correlation is wrong-way risk which occurs whenever the probability of default of the counterparty is positively correlated with exposure. The basic formulas for CVA and basic counterparty credit risk models do not account for wrong-way risk because its modeling is nontrivial. This thesis aims to answer how well can the impact of wrong-way risk on CVA be approximated with an add-on which only depends on correlation between the price of the underlying asset and the credit spread of the counterparty. The thesis is supplemented by a fully documented implementation of the model in the *Mathematica* software.

Klíčová slova

kreditní riziko protistrany, CVA, korelace, wrong-way riziko

Keywords

counterparty credit risk, CVA, correlation, wrong-way risk

Prohlášení

1. Prohlašuji, že jsem předkládanou práci zpracoval samostatně a použil jen uvedené prameny a literaturu.
2. Prohlašuji, že práce nebyla využita k získání jiného titulu.
3. Souhlasím s tím, aby práce byla zpřístupněna pro studijní a výzkumné účely.

V Praze dne 31. července 2016 Mikoláš Volek

Poděkování

I thank doc. PhDr. Petr Teplý PhD. for taking on the role of my thesis supervisor. I also wish to express thanks to my family and friends for their endless support.

Contents

Master Thesis Proposal

Institute of Economic Studies
Faculty of Social Sciences
Charles University in Prague

Author:	**Bc. Mikoláš Volek**	Supervisor:	Petra Andrlíková, MSc.
E-mail:	mikolas@volek.cz	E-mail:	andrlikova@gmail.com
Phone:	606 794 336	Phone:	728 829 436
Specialization:	Finance, FinMarkets & Banking	Defense Planned:	January 2015

Notes: The proposal should be 2-3 pages long. Save it as "yoursurname_proposal.doc" and send it to mejstrik@fsv.cuni.cz, tomas.havranek@ies-prague.org, and zuzana.irsova@ies-prague.org. Subject of the e-mail must be: "JEM001 Proposal (Yoursurname)".

Proposed Topic:

Counterparty credit risk modeling

Motivation:

Counterparty credit risk is an important type of financial risk. It is defined as the risk that the counterparty to a financial transaction defaults on its obligations prior to the final settlement of the contract. The importance of proper counterparty risk management became most apparent in the wake of the 2008 series of failures of several large banks and insurance companies.

The measurement of counterparty credit risk is a complicated issue, especially in the case of OTC ("over-the-counter") derivatives. In order to properly manage risk, market agents apply a *credit value adjustment* (CVA) when marking their portfolios to market. The term *debt value adjustment* (DVA), in turn, stands for the improvement of the value by an amount corresponding to the entity's own probability of default. Sometimes, CVA and DVA are collectively referred to as *bilateral CVA*.

A default is not the only means by which credit risk materializes into a loss. Whilst the individual positions are regularly marked to market, a deterioration of the credit quality of the counterparty results in a higher CVA and in a drop in the value of the portfolio. To correctly determine the CVA is, however, a difficult task as the CVAs can themselves be regarded as complex option-like credit derivatives.

Correlation of market factors is an important issue in the calculation of CVA. A notable case of correlation is *wrong-way risk* which occurs whenever the probability of default of the counterparty is positively correlated with exposure. The basic formulas for CVA and basic counterparty credit risk models do not account for wrong-way risk because its modeling is nontrivial.

Hypotheses:

In the thesis, I will try to test the following main hypothesis:

> "The impact of wrong-way risk on CVA can be approximated well using an add-on which only depends on correlation between the price of the underlying asset and the credit spread of the counterparty."

The following additional hypotheses will be tested subsequently:

1. In terms of CVA calculation, past correlation (as observed in realized time series) between the credit spread and the price of the underlying asset is a good predictor for future correlation.
2. Correlation between the credit spread and the price of the underlying asset is seasonal.

Methodology:

I will build a model for the calculation of CVA. The model will rely on industry-standard pricing functions for a range of financial derivatives. The calculation of future exposure and, ultimately, CVA will be done using extensive Monte-Carlo simulation.

For the main hypothesis, I will present an approximate formula for CVA that purports to include wrong-way risk. Then, wrong-way risk will be modeled using Monte-Carlo simulations and the formula will be compared to the CVA obtained by simulation.

All calculations will be done on a range of financial derivatives. They will include:

- FX Forwards,
- European FX Options,
- Non-amortized interest rate swaps,
- Amortized interest rate swaps,
- Interest rate caps and floors.

For the sub-hypotheses, I will use actual financial data and use standard statistical inference. The data will be broken into several subsamples (periods). Should the hypothesis hold, add-ons calculated on data from period *n* should be good proxies for counterparty losses due to wrong-way risk in period *n*+1 et seq. If the calculations do not confirm this then past correlation will be deemed unsatisfactory.

Expected Contribution:

The thesis will broaden the understanding of both general and specific wrong-way risk. The thesis will show that under fairly mild conditions, wrong-way risk may be incorporated into CVA via a simple "add-on method" without the need to build a complicated model.

The use of past correlation data to model future exposure will be subject to strict scrutiny in order to justify its use in the calculation of CVA.

The possible seasonal nature of the correlation between credit spreads and the underlying series may warrant a very cautious approach to estimating wrong-way risk. This thesis will aim to show whether this in fact occurs and how it may affect the calculation of CVA.

Outline:

The structure of the master's thesis will be the following:

1. **Introduction** --- this section will contain basic concepts, measures of exposure, and the effects of collateral and netting.
2. **CVA** --- this section will introduce the concept of CVA and its calculation for selected financial instruments
3. **Add-on model for wrong-way risk** -- this section will introduce the impact of correlations on CVA; the add-on formula for wrong-way risk will be introduced and its performance will be tested on various
4. **Correlation modeling** – this section will address the issues of correlation modeling, the use of past correlation in the CVA model and the seasonality of correlations.
5. **Conclusion** --- summary of findings
6. **Appendix**
7. **References**

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Author

Supervisor

1 Introduction

1.1 Motivation

The measurement of counterparty credit risk is an involved matter, especially in the case of OTC ("over-the-counter") derivatives. In order to properly manage risk, market agents apply a credit value adjustment (CVA) when marking their portfolios to market. The term debt value adjustment (DVA), in turn, stands for the improvement of the value by an amount corresponding to the entity's own probability of default. Sometimes, CVA and DVA are collectively referred to as bilateral CVA. The importance of proper counterparty risk management became most apparent in the wake of the 2008 series of failures of several large banks and insurance companies.

A default is not the only means by which credit risk materializes into a loss. Whilst the individual positions are regularly marked to market, a deterioration of the credit quality of the counterparty results in a higher CVA and in a drop in the value of the portfolio. To correctly determine the CVA is, however, a difficult task as the CVAs can themselves be regarded as complex option-like credit derivatives.

1.2 Basic notions of credit risk

1.2.1 Default

Broad definition

This section serves as an introduction to credit risk quantification and management techniques.

A "default" of an entity occurs when the entity is unable to settle its liabilities when they have become due. Default may occur as a result of liquidity shortage, or deliberately (the entity fails to pay, and possibly seeks protection from creditors under bankruptcy law).

To tell whether a counterparty has "defaulted" is not a straightforward task. Various criteria are used to assess the state of any given counterparty (or a particular receivable) with respect to whether the counterparty has already defaulted or not (i.e., whether the particular receivable should be treated as "defaulted").

Importance

Default is a central notion in the field of credit risk management. It is an essential activity of a credit institution's management to assess the probability of default, to correctly determine whether a default has occurred (or is likely to occur soon) and to implement appropriate collection process to deal with defaulted receivables. Thus the very definition of "default" has far-reaching consequences in the whole credit risk management process[1]. (Theoretical works on credit risk management should also reflect this.)

Basel II: basic criteria

In Basel II (and, subsequently, many national legislatures to which Basel rules have been transposed) the following criteria are used. (BCBS 2006)[2] A counterparty is considered to be in default if:

- the creditor deems it unlikely that the debtor will repay the debt, or
- a payment is overdue for more than 90 days.

The definition above, however easy to understand, gives little hint as to the how exactly should a creditor determine default. For instance, a wide range of possibilities may arise as to whether a repayment will be "deemed unlikely"[3].

Basel II: additional criteria

The following situations are those in which the creditor *must* deem it unlikely that the debt will be repaid:

[1]This is why banks use multiple definitions and therefore the definition used for publicly disclosed reports is different than, say, for the collection process.

[2]http://www.bis.org/publ/bcbs128.pdf, p 100 et seq.

[3]There is also room for speculation as to what is meant by 90 days (e.g., business days, calendar days, or how to treat multiple overdue payments with different due dates).

- the creditor has written of a part of the receivable, or
- the creditor has sold the receivable at a (material) loss, or
- the creditor has agreed to lower the amount receivable when refinancing (re-structuralizing) the debt, or
- the creditor has taken steps to put the debtor into bankruptcy, or
- the debtor is in bankruptcy.

Relevance for modelling

Before interpreting the results of a model it is essential to know the definition of default used for that model. This is especially important when we have at hand results from two different models. Unless the definition of default in both models is the same their results may not be at all comparable.

However, credit risk model specifications traditionally have not relied on a particular definition of default. For instance, Merton's structural model (Merton, 1973) defines "default" as the moment when equity reaches zero (i.e, the moment when market value of company's assets is less than the company's liabilities).

Reduced models, which we rely upon in this thesis, treat default as a one-off event that occur with some probability, without necessarily tying this probability to real-world economic variables.

1.2.2 Probability of default

The probability of default corresponds to the definition of probability of an event in a given probability space, i.e. $P[A] \in [0,1]$ where $A \in \Omega$ is an event and Ω is the probability space.

Default probabilities are always expressed on a "per annum" basis, i.e., they are probabilities that a given entity defaults during a period of one year, starting from now.

Probability of default is usually abbreviated "PD".

1.2.3 Exposure at default

"Exposure at default" (or "EAD") is the amount owed by the counterparty that has defaulted. This key notion is thoroughly discussed under sections 1.3.4 and 1.4.

1.2.4 Loss given default and recovery rate

"Loss given default" (or "LGD") is the loss that the creditor incurs upon the debtor's default, expressed as a percentage of EAD.

"Recovery rate" (or "RR") is the part of the debt that the debtor has repayed (or is going to repay), expressed as a percentage of EAD. Thus, it can also be defined as $RR = 1 - EAD$.

Loss given default, or the recovery rate, depends greatly on the type of the defaulted contract. It can depend

In credit risk models (and on some instances in this thesis, too), the recovery rate is usually assumed to be zero. This is done without actual loss of generality, since all cash flows can be "discounted" to reflect only the part that has not been repaid. This is rewarded by great simplification of notation, since instead of $(1 - RR_t) \times CF_t \times DF_t$ we obtain simply DCF_t, which incorporates the recovery rate and the discount factor for tenor t.

1.3 Basic notions of counterparty credit risk

1.3.1 Counterparty credit risk

Credit risk is the main risk faced by a credit institution. It is the risk that a loan will not be repaid in full.

Counterparty credit risk is the risk that a counterparty defaults prior to the final settlement of a contract. It is a type of credit risk.

The difference between "standard" credit risk (i.e., credit risk which arises in loans etc.) and counterparty credit risk can be illustrated as follows:

"Standard" credit risk:

- arises in loans, bonds, credit lines etc.,
- has predictable Exposure at Default (it is either fixed or random, but is much easily estimated and predicted),
- is one-sided (only the bank faces credit risk, not its counterparty).

Counterparty credit risk:

- arises in derivatives, repurchase agreements etc.,
- has random Exposure at Default which must be modeled on the basis of market factors,
- is bilateral (both parties of the contract face the risk).

1.3.2 Replacement cost

The value of the contract, V_t, at time t is usually modelled as *replacement cost*, i.e. the amount of money that would be needed to enter into an identical trade at time t:

$$\text{Replacement cost} = V_t = \text{Value of contract.}$$

The time value of the contract as well as conditions in the market change constantly and so does the value of the contract.

In an actual market, the replacement cost is

$$\text{Replacement cost} = V_t + \text{Transaction costs.}$$

Transaction costs will be disregarded throughout this thesis.

1.3.3 Loss from default of counterparty

Let A and B be parties to a financial contract. Let B default at time τ, which is prior to the last settlement of the contract. Then either of the following two cases will occur:

Case (1): $V_\tau > 0$, taken from the point of view of party A. In other words, B "owes" V_τ to A and as B is in default, A will suffer loss in the amount of

$$\text{Loss} = V_\tau(1 - RR).$$

Case (2): $V_\tau < 0$, taken again from the point of view of party A. A "owes" V_τ to B. B is still entitled to receive V_τ irrespective of the fact that it has defaulted[4]. A's exposure to B remains unchanged. A will suffer zero loss.

All in all, in the case of B defaulting, A suffers loss in the amount of

$$\text{Loss} = \max(V_\tau, 0)(1 - RR).$$

1.3.4 Exposure

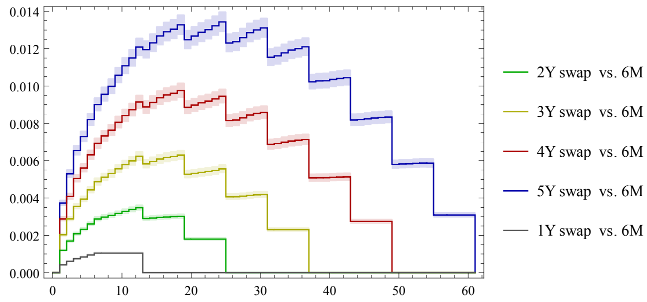
The term

$$\max(V_t, 0) = V_t^+ = E_t, \tag{1}$$

is called the *exposure* of party A towards party B at time t. Conversely, $\max(-V_t, 0)$ would be the exposure of B towards A at time t.

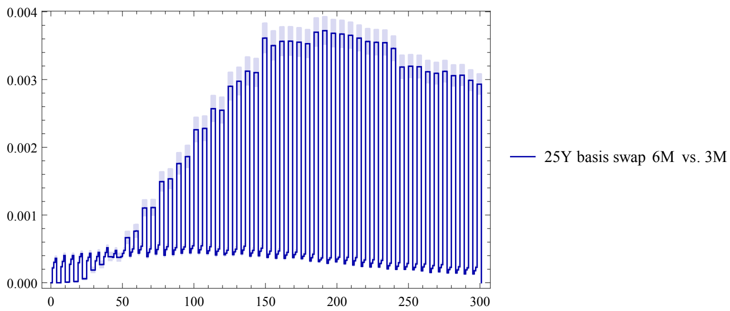
E_t is a sequence of random variables, taking values in $\mathbf{R}_0^+$ for all $t > 0$. In other words, it is a stochastic process. Examples of exposure profile are given in Figures **??** and **??**.

[4]This may depend on special features of the contract, e.g. the so-called "walkaway features".



Estimate of $\mathrm{EE}_{0,i} = B_0\, \mathbb{E}_{\mathbb{Q}}\left[B_i^{-1}\, V^{+}_{\mathrm{IRS},i} \,\middle|\, \mathcal{F}_0\right]$ with the 95% confidence interval, obtained by simulation from 5000 scenarios starting from a flat curve at 0,10%, in units of notional, for $i \in \{0, 1, \quad, 61\}$.

Fig. 1:



Estimate of $\mathrm{EE}_{0,i} = B_0\, \mathbb{E}_{\mathbb{Q}}\left[B_i^{-1}(V_{\mathrm{IRS}\,6\,M,i} - V_{\mathrm{IRS}\,3\,M,i})^{+} \,\middle|\, \mathcal{F}_0\right]$, where the $V_{\mathrm{IRS}\quad M}$ denote the value of an IRS vs the respective bases, with the 95% confidence interval, obtained by simulation from 1000 scenarios starting from the actual JPY curve as of 2 Sep 2013, in units of notional, for $i \in \{0, 1, \ldots, 301\}$.

Fig. 2:

1.3.5 Credit value adjustment

Credit value adjustment, or *CVA*, is the present value of counterparty credit risk of a given position.

The general formula for CVA is:

$$CVA_t = \mathrm{E}_{\mathbb{Q}}\left[I_{\{\tau \le T\}} DF_{t,\tau} V_\tau^+ \left(1 - RR_\tau\right)\right] \tag{2}$$

where:

- t is the current time,
- $\mathrm{E}_{\mathbb{Q}}[\ldots]$ denotes expectation, taken at time t, with respect to a risk-neutral probability measure $\mathbb{Q}$,
- τ is a random variable denoting the time of default of the counterparty (it is assumed that $\mathbb{Q}[t < \tau] = 1$),
- T is the final maturity of the contract, beyond which there is no remaining exposure to the counterparty,
- $I_{\{\tau \le T\}}$ is the indicator function, denoting whether default occurred prior to T,
- $DF_{t,\tau}$ is the discount factor for the period from t to τ,
- V_τ^+ is the exposure (see above) at time of default, and
- RR_τ is the recovery rate at time of default.

The variables τ (time of default), $DF_{t,\tau} V_\tau^+$ (present value of exposure), and RR_τ (recovery rate [5]) are random variables. Their joint probability distribution under the risk-neutral probability measure is what determines the value of CVA for a given contract.

Note that CVA_t is always ≥ 0.

[5] In most basic models for CVA, the recovery rate is assumed constant, thus non-random. Details will be elaborated on shortly.

1.3.6 Value of contract

Once we take counterparty credit risk into account, the value of the contract becomes

$$V_t^{\text{incl. CCR}} = V_t - CVA_t.$$

This formula takes into account the likelihood that the counterparty will default.

1.3.7 CVA risk

Much like V_t, the credit value adjustment is subject to movements, as time passes and the underlying factors change in a random manner.

The future value of the contract is now therefore influenced by three sources of uncertainty:

- standard market risk, which arises from the movement of market factors such as exchange rates,
- counterparty credit risk, and
- CVA risk, which arises from the movement of factors that affect CVA (see above).

1.3.8 Modelling of CVA

When modelling credit value adjustment, the following random phenomena must be properly taken care of:

- the default dynamics (the modelling of τ, RR_τ etc.),
- future exposure (E_t), and
- the interdependence between the two.

Models for CVA range from the most simple (which often take on many simplifying assumptions, e.g. independence of various random variables) to the most complicated. Monte-carlo simulation is a favorite tool for CVA modelling.

1.4 Measures of exposure

This section deals with various measures of exposure towards one counterparty. All of them depend on the notion of *exposure* defined in Equation (1).

Definitions may be found, for instance, in Canabarro and Duffie (2003).[6]

All of the following measures are defined to be "with respect to time 0", i.e., expected values are taken with respect to Ω_0 which belongs to the usual filtration $\mathscr{F}$.

1.4.1 Expected exposure

Expected exposure is simply the expected value of E_t at a given point in time:

$$EE_t = \mathrm{E}_{\mathbb{Q}}[E_t],$$

where the expectation is taken at time 0 with respect to the risk-neutral measure $\mathbb{Q}$.

1.4.2 Expected positive exposure

Expected positive exposure is the "average" expected exposure over the lifetime of the contract (from zero to T):

$$EPE = \frac{1}{T}\int_0^T EE_t \,\mathrm{d}t$$

It is of great use in simplified models where the occurrence of default is uniform on some interval $[0, T]$. Then EPE is actually "the" expected exposure at time of default.

[6]Measuring and marking counterparty risk (Canabarro and Duffie, 2003).

EPE is also the basis of the "add-on" model used in Basel II for counterparty credit risk.

1.4.3 Expected positive discounted exposure

Expected positive *discounted* exposure is the present-value variant of the preceding:

$$EPDE = \mathrm{E}_{\mathbb{Q}}\left[\frac{1}{T}\int_0^T DF_{0,t}E_t\,\mathrm{d}t\right]$$

1.4.4 Potential future exposure

Potential future exposure at significance level $\alpha \in [0,1]$, denoted $PFE_{\alpha,t}$, is defined as the α^{th} quantile of E_t:

$$\mathbb{Q}\left[E_t < PFE_{\alpha,t}\right] = \alpha,$$

where $\mathbb{Q}$ is the risk-neutral probability measure.

1.4.5 Maximum potential future exposure

Maximum potential future exposure is the "maximum" attained *PFE* over the lifetime of the contract:

$$MaxPFE_{\alpha} = \max\left\{PFE_{\alpha,t} \mid t \in [0,T]\right\}.$$

1.5 Collateral

1.5.1 Definition

Collateral is an asset that is temporarily deposited by one party with the other. It has the form of a financial instrument, usually a bond, sometimes cash. (In mortgage lending, real estate is posted as collateral.)

1.5.2 Replacement cost, loss from default and exposure

The purpose of collateral is to decrease exposure to the counterparty. The party that has received collateral may liquidate it in the event of default of the counterparty. (If no default occurs during the lifetime of the contract, collateral is returned to the counterparty.) Therefore, the replacement cost is decreased by the value of the collateral at the moment of default.

The value of collateral at time t shall henceforth be denoted C_t[7].

$$\text{Replacement cost with collateral} = V_t - C_t.$$

Consequently, loss incurred in the event of default (at time τ) is:

$$\text{Loss} = \max(V_\tau - C_\tau, 0)(1 - RR).$$

Our exposure (at time t) to the counterparty which has posted collateral is:

$$\max(V_t - C_t, 0) = (V_t - C_t)^+.$$

Conversely, $\max(C_t - V_t, 0)$ is the exposure of the counterparty towards us at time t.

Exposure for which $C_t > 0$ is said to be *collateralized*.

1.5.3 Margining

The financial contract between two parties usually stipulates:

- whether collateral is to be posted,

[7] C_t—the value at which collateral may be liquidated at time t—is a random variable. Much like V_t, it depends on the random scenario realized at time t and can also depend on whether default has occurred by time t. Throughout this text transaction costs are neglected. Therefore, liquidation cost of collateral shall be equal to its market value at time of liquidation.

- when it is to be posted (i.e., which conditions will "trigger" the posting of collateral),
- how much is to be posted,
- in which form it is to be posted.

At the beginning of the contract, an *independent amount* is posted by the counterparty.

Typically, a fixed frequency (or *remargining period*) is agreed. Every time the remargining period passes, positions and collateral holdings are marked-to-market[8]. The remargining period shall be denoted M and is usually in the range of several weeks.

Additionally, a *haircut* is applied to the value of collateral, i.e., it is, for the purpose of all additional calculations, decreased by a given percentage, H. Haircut accounts for market risk or credit risk associated with collateral itself.

If exposure (i.e., value of the position) exceeds the value of collateral:

$$V_t > (1-H)\,C_t,$$

the counterparty is asked to deliver additional collateral to fully cover exposure (i.e., a *margin call* is issued)[9].

1.5.4 Issues associated with collateral modelling

It should be obvious from the description of the practice of margining that collateralization greatly reduces counterparty credit risk. This is because several reasons:

- exposure is reduced from V_t^+ to $(V_t - C_t)^+$
- time horizon is reduced from T to M.

[8]The term "mark-to-market" (or MtM) means that a designed calculation agent, agreed on by both parties, uses a pricing model to calculate the market value of the position at a given moment.

[9]The counterparty may not be asked to deliver additional collateral unless exposure has increased by more than a certain *threshold* or it may be asked to deliver a *minimum transfer amount*.

The value of collateral, however, creates a difficulty: the calculation of *EE*, *EPE*, *PFE* and other (whereby V_t is replaced by $V_t - C_t$) may be complicated because now we also need to model C_t.

The model for C_t must give the correct value of collateral in itself but apart from that must also capture the relationship between V_t and C_t. Such relationship is governed by the assumption that C_t depends on V_u where u is the time of the last margin call prior to t.

1.6 Netting

1.6.1 Definition

Netting is the procedure whereby two parties offset exposures to each other and agree on a single exposure that incorporates exposures from all contracts.

The conditions under which netting is applied are stipulated in the contract between the two parties. They are usually termed “close-out netting stipulations”.

Typically, not all contracts between two parties are subject to netting. Instead, contracts are organized into “netting sets” and exposures are netted (i.e., offset) within each netting set. Measures of exposure are then applied to each netting set as if it were a single contract.

There is also a special type of netting—“payment netting”. Under payment netting, payments made on a given day with a single counterparty are “netted out” and thus the settlement risk is reduced.

1.6.2 Exposure in the presence of netting

Let there be n contracts within a netting set, valued $V_{1,t}$, $V_{2,t}$, ..., $V_{n,t}$. Our exposure to the counterparty arising from the netting set is:

$$E_t^{\text{netting}} = \max\left(V_{1,t} + V_{2,t} + \ldots + V_{n,t}, 0\right).$$

If we compare this to the exposure we would face absent any netting agreement, we immediately recognize why netting is a beneficial tool for counterparty risk management.

Without netting our total exposure would be:

$$E_t^{\text{no netting}} = \max\left(V_{1,t}, 0\right) + \max\left(V_{2,t}, 0\right) + \ldots + \max\left(V_{n,t}, 0\right)$$

which is in all circumstances greater or equal than E_t^{netting}.

2 CVA calculation, default modelling, and wrong-way risk

2.1 Introduction

In the most simple settings the market model (used for calculating exposure) and the default model (used for calculating the probability of default and possibly the recovery rate) are treated separately.[10] This has some virtues besides simplicity alone: namely, such models are easier to specify and calibrate and, in some cases, they allow for analytical solutions (of the CVA formula) for the most basic financial instruments. Sometimes, the simplification allows one to focus on other aspects of counterparty credit risk without complicating the model too much.[11]

The separate treatment of the market and the default mechanics is, nevertheless, not always appropriate.

The term "wrong-way risk" denotes the portion of the counterparty credit risk that arises as a consequence of the fact that the probability of default and the size of exposure are interrelated.

The (rather vague) term mainly serves to distinguish models based on whether they "do" or "do not" incorporate wrong-way risk. If they do, they must first specify what exactly constitutes "exposure" and what constitutes the "probability of default" and then specify some concrete relationship between the two variables. How this should be done is currently by far an unsettled matter.

In this section we examine in detail the problem of how to capture wrong-way risk in a CVA model.

2.2 Modelling probability of default

The distinction between risk-neutral and "actual" probability of default

[10]See, for instance, Pykhtin and Zhu (2007).
[11]See also Sorensen and Bollier (1994) and Brigo and Masetti (2006).

Before we delve into the realm of default probability models, a word of caution is in place: risk-neutral probabilities of default are *different from* (and usually *higher than*) "actual" probabilities. In other words, when we have a model, among whose outputs are probabilities of default for some counterparty, we *cannot* "back-test" these output values using real-world data (i.e., a count of defaults of that counterparty that have actually occurred over some period of time).[12]

Before we speak about *probability* we need to define a certain *probability measure*. From the theoretical standpoint, a probability measure is any function, defined on a sufficiently rich collection of sets (called events), which assigns each set (event) a number between $[0, 1]$ and which "behaves reasonably" (e.g., is σ-additive and satisfies some other conditions). The meaning of our statement "the probability of X is 0.35" ultimately depends on what the probability measure we are using is and where it came from.

The traditional way introductory texts tackle the issue of defining the probability measure by presenting the following dilemma: that there are "two" different probability measures, the "real-world" measure and the "risk-neutral" measure. The former is used for risk management purposes and the other is used for pricing.

The authors of such texts usually stop short at explaining how the "real-world" measure should be defined. The motivation behind it seems to be statistical: in the case of defaults, for instance, it is assumed that past defaults (or price gains of a stock etc.) are realisations of a random variable. By sampling, we learn about the distribution of that random variable.

Thus, the "real-world probability measure" would seem to be the measure assumed for a static model. It is inherently tied to information obtained by sampling from the past.

The risk-neutral measure, on the other hand, has nothing to do with the past. What we know about this measure is based on information we gain from *current (instantaneous)* market prices. The risk-neutral measure is tied to the possibility of hedging a claim. It yields the unique market price consistent with the condition of

[12]See also Hull (2005), p 482 et seq.

no arbitrage.[13]

This is nevertheless a simplification, as real-world markets are not perfect and mis-pricings may arise. Tthe risk-neutral probability measure is but a tool to understand the principles of financial markets, rather than something that would "exist" independently of human judgement.

2.3 Regulatory treatment: Basel II, Basel III

Basel II and the alpha multiplier

Since the introduction of Basel II, banks have been allowed to calculate their capital requirements for counterparty credit risk through with the use of their own modelled values of probability of default, exposure at default, and recovery rates.[14]

The capital requirement for counterparty credit risk revolves on one key notion, the alpha multiplier, which quantifies the influence of two key factors:

- the correlation between exposure at default and the probability of default, and
- the correlation of probabilities of default across the portfolio.

The alpha multiplier is defined thus:

$$\alpha = \frac{EC_1}{EC_2}$$

where EC_1 is the capital requirement calculated assuming wrong-way risk (i.e., calculated by joint simulation of all market factors), and EC_2 is the capital requirement calculated assuming flat exposure profile equal to the expected positive exposure (EPE, see 1.4 for definition).

The actual value of alpha may range from 1.1 for large and greatly diversified portfolios to 2.5 for small concentrated portfolios. The default regulatory value for

[13]The theory of no-arbitrage pricing is explained elsewhere, see Baxter and Rennie (1994) for an introduction.

[14]The facts presented in this section are based on Cespedes et al. (2010), pp 72–74,76–80 and BCBS (2009)

alpha is 1.4. The bank may calculate its own alpha subject to a floor of 1.2.

A sound model for the capital requirement for counterparty credit risk might therefore be of great interest to the bank if it leads to lower value of alpha (and therefore lower total capital requirement).

Basel II regulatory charge

The regulatory charge (capital requirement) for (general) credit risk against a given counterparty, as prescribed by Basel II, is the following:[15]

$$E \times (1-R) \times \left(\Phi\left(\frac{\Phi^{-1}(P) + \beta(P)\Phi^{-1}\left(\frac{999}{1000}\right)}{\sqrt{1-\left(\beta(P)\right)^2}} \right) - P \right) \times \frac{1+\left(M-\frac{5}{2}\right)b(P)}{1-\frac{3}{2}b(P)}$$

where E is exposure at default to the given counterparty, R is the recovery rate of the counterparty, P is the probability of default of the counterparty, Φ is the cumulative distribution function of the standard normal distribution,[16] and Φ^{-1} is its inverse (i.e., the quantile function). M denotes the effective maturity of the portfolio (it is essentially the average maturity of the portfolio, capped at one year; the definition of M is given in the regulation). The functions β and b are functions of the probability of default and their formulas are also given in the text of the regulation.

[15]ibid., p77, and Basel II (see references)

[16]$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t$

3 Literature review

The literature on counterparty credit risk is extremely rich and ever-growing. As the topic is both important for practitioners and interesting for academics, the works reflect different points of view and a wide range of approaches to counterparty credit risk analysis, measurement and management. For this reason, instead of detailed description of several books and articles, I present here a list of the most important works which I have used as a basis for my thesis and which I can recommend for interested readers as a starting point for their research.

3.1 Introductory

First, as an introduction to the credit risk modelling, I can recommend , de Prisco and Rosen (2005), Gregory (2015), and Brigo, Morini, *et al.* (2013). These works are an important source of basic methodology.

3.2 Wrong-way risk

Focusing on the wrong-way risk, the most important works for my research were Brigo and Chourdakis (2009), Cespedes *et al.* (2010), Rosen and Saunders (2012), Brigo, Capponi, *et al.* (2013), Ng (2013), and Černý and Witzany (2015). General wrong-way risk is treated in Pykhtin (2012). The most recent literature on wrong-way risk is El Hajjaji and Subbotin (2015), Xiao (2015), Yang *et al.* (2015), and Ghamami and Goldberg (2014).

A good introduction to wrong-way risk can be found in e.g. Redon (2006), Hull and White (2012), Carver (2013), or Ruiz *et al.* (2015).

4 The CVA model

4.1 Preliminaries

4.1.1 Definitions of probability-related terms

General definitions

For n-dimensional vectors, $n \in \mathbf{N}$, we shall use the l_2-norm, that is, $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$.

$$\mathrm{E}\left[\exp\left(\int_{t_1}^{t_2} \tfrac{1}{2} \|\boldsymbol{X}_u\|^2 \,\mathrm{d}u\right)\right] < \infty.$$

Martingales

Let $(\Omega, \{\mathscr{F}_t\}, \mathbb{P})$ be a stochastic basis. A stochastic process $\boldsymbol{X}$ is said to be a *martingale* with respect to that basis if for all t and all $u > t$:

$$\boldsymbol{X}_t = \mathrm{E}_{\mathbb{P}}[\boldsymbol{X}_u \mid \mathscr{F}_t].$$

Notation

We adopt the following notation conventions. Numbers and one-dimensional random variables are denoted thus: a, vectors and vector-valued random variables thus: $\boldsymbol{a}$, matrices thus: $\mathbf{A}$, collections of sets (algebras, σ-algebras) thus: $\mathscr{A}$, probability measures thus: $\mathbb{A}$. Other style of text might be occasionally used for some special objects.

It is in the interest of clarity that throughout the text, and especially in equations involving random variables, we omit the usual words "almost surely". Thus, all such equations and, similarly, all statements about random variables taking values in a set, are to be interpreted as being true with probability one with respect to the "prevailing" probability measure.

The link between measurability and information

A known result in probability theory is that, roughly speaking, a random variable X is $\sigma(Y)$-measurable, where $\sigma(Y)$ is the σ-algebra generated by a random variable Y, if and only if X is a deterministic function of Y.

In our model we employ the “usual” filtration $\{\mathscr{F}_i\}_{i=0}^{\infty}$ and we do so without necessarily specifying its contents and structure. Instead, we say that this filtration embodies all information that is available to market participants. We also assume the converse, that is, we take any observed (“known”) quantity to be embodied in the σ-algebras that make up the filtration.

Therefore, the statement that a random variable X is “$\mathscr{F}_i$-measurable” really means that its value (i.e., realization) can be observed by time t_i, and all subsequent times $t_{i+1}, t_{i+2}, \ldots$.

A careful consideration of which random variables are measurable with respect to which σ-algebra is essential when building a pricing model. This is exemplified in our mathematical formulation of the model in that we specify the respective σ-algebra for each expectation. This will help our intuition when handling complex formulae.

4.1.2 Basic probability theorems

The wording of the following theorems is taken from Baxter and Rennie (1996).

Theorem 1 (Cameron–Martin–Girsanov) Let $(\Omega, \mathscr{A}, \{\mathscr{F}_t\}_t, \mathbb{P})$ be a filtered probability space. Let $\boldsymbol{W}$ be an n-dimensional Wiener process, $n \in \mathbf{N}$, under the probability measure $\mathbb{P}$. Let $\boldsymbol{\gamma}$ be an n-dimensional stochastic process of class $\mathscr{E}_{[0,T]}$, adapted to $\mathscr{F}$[17] and previsible with respect to $\mathscr{F}$. Then there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$[18], such that the process

$$\left\{\boldsymbol{W}_t + \int_0^t \boldsymbol{\gamma}_u \,\mathrm{d}u\right\}_t$$

is an n-dimensional Wiener process with respect to $\mathbb{Q}$.

[17]is adaptation necessary?
[18]?: “up to time T”

Remark 2 Note: the corresponding Radon–Nikodym derivative is:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T \boldsymbol{\gamma}_t^\top \,\mathrm{d}\boldsymbol{W}_t - \frac{1}{2}\int_0^T \left\|\boldsymbol{\gamma}_t\right\|^2 \mathrm{d}t\right).$$

Theorem 3 (Martingale representation) Let $T > 0, n \in \mathbf{N}$ and let $\mathbf{S}$ be a stochastic matrix process of dimensions $n \times n$ such that $\mathbf{S}_t$ is almost surely non-singular for all $t \in [0,T]$. Let $\boldsymbol{W}$ be an n-dimensional Wiener process and let there be another n-dimensional process $\boldsymbol{X}$ defined as follows:

$$\mathrm{d}\boldsymbol{X}_t = \mathbf{S}_t^\top \,\mathrm{d}\boldsymbol{W}_t \qquad t \in [0,T].$$

Finally, let Y be a (one-dimensional) martingale process. Then there exists an n-dimensional previsible process $\boldsymbol{\phi}$ such that

$$\mathrm{d}Y_t = \boldsymbol{\phi}_t^\top \,\mathrm{d}\boldsymbol{X}_t \qquad t \in [0,T]$$

and every coordinate of $\int_0^T \left\|\mathbf{S}_t\boldsymbol{\phi}_t\right\|^2 \mathrm{d}t$ is almost surely finite. Furthermore, the process $\boldsymbol{\phi}$ with these properties is "almost unique".

4.1.3 Definitions of finance-related terms

The *numéraire* is a strictly positive process adapted to $\{\mathscr{F}_t\}$ which we use as a unit of currency. All cash flows are discounted (i.e., divided by) the value of the numeraire at the respective point in time.

The *discount T-bond* is a bond that pays exactly a unit of currency at time T. We assume that all bonds are default-free. We also assume that the market for T-bonds (for any T) satisfies the usual conditions.

4.2 Discrete Heath–Jarrow–Morton model

4.2.1 Outline of the model

General properties

Our CVA model will have a discrete time structure and a continuous state structure. The time domain is modelled by a strictly increasing (constant) sequence $\{t_i\}_{i\in\mathbf{N}}$ of real numbers. The time increment is defined as

$$\mathrm{d}t_i = t_i - t_{i-1}.$$

(Throughout the text, i, j, k, l denote integers.) The key ideas and also notation are borrowed from Baxter and Rennie (1996).

Zero-coupon bonds

A zero coupon bond that pays a unit of currency at time t_j will be called the j-bond.

We assume that any time t_i, there is a market for bonds of all maturities t_j where $j \geq i$. We assume that the *usual conditions* prevail in these markets, that is: All bonds are assumed to have unlimited liquidity[19]. All bonds are free of credit risk. We assume traders are able to sell bonds short at no cost.

The price of the j-bond at time t_i, $i \leq j$, is denoted $P_{i,j}$. The function $j \mapsto P_{i,j}$ is called the *discount factor curve* at time t_i.

Forward curve

The basic building block of our interest rate model will be the *forward curve*. The forward rate $f_{i,j}$, for $i < j$, is the rate at which funds can be borrowed or lent forward, at time t_i, for the period from t_{j-1} to t_j. The function $j \mapsto f_{i,j}$ is called the *forward curve* at time t_i. (Forward rates are expressed in continuous compounding.)

From the assumptions stated above follows that at each time t_i there is a unique discount curve and a unique forward curve in the market[20]. In order for there to be no arbitrage, the relationship between the discount factors and the

[19]The notion of unlimited liquidity means that bonds can be purchased and sold in the market at the current price at no cost. We also assume that bonds as well as the currency are infinitely divisible.
[20]explain

forward curves must be as follows:

$$P_{i,j} = \exp\left(-\sum_{m=i+1}^{j} f_{i,m}\,\mathrm{d}t_m\right)$$

for all i and all $j \geq i$. (If $i = j$ then the sum is empty and $P_{i,j} = 1$.)

Money-market account

The nature of the bond market, as stated, allows us to construct an instrument called the *money-market account*: we start with a unit of currency at t_0 and at each subsequent period t_i we invest all proceeds into the $(i+1)$-bond. The value of the money-market account at time t_i, $i > 0$, is:

$$B_i = \frac{1}{P_{0,1}P_{1,2}\ldots P_{i-1,i}} = \exp\left(\sum_{m=1}^{i} f_{m-1,m}\,\mathrm{d}t_m\right).$$

The money-market account will be used as the numéraire in our pricing model. We see that $B_{i+1} = B_i \exp\left(f_{i,i+1}\,\mathrm{d}t_{i+1}\right)$. Because both $f_{i,i+1}$ and B_i are $\mathscr{F}_i$-measurable, B_{i+1} is also $\mathscr{F}_i$-measurable and therefore B is a $\mathscr{F}$-previsible process.

4.2.2 Forward rate dynamics

Following Heath, Jarrow, and Morton (1992) we model the changes of the forward curve as follows:

$$\mathrm{d}f_{i,j} = \mu_{i,j}\,\mathrm{d}t_i + \boldsymbol{\sigma}_{i,j}^{\top}\,\mathrm{d}\boldsymbol{W}_i \tag{3}$$

where $\mathrm{d}f_{i,j} = f_{i,j} - f_{i-1,j}$, the processes μ and $\boldsymbol{\sigma}$ are $\{\mathscr{F}_t\}$-adapted and $\boldsymbol{W}$ is an n-dimensional random walk (i.e., a Wiener process, sampled at discrete time points) under the probability measure $\mathbb{P}$.

Let us denote the forward curve at time t_i by the infinite-dimensional vector $\boldsymbol{f}_i = \left(f_{i,1}, f_{i,2}, \ldots\right)$ and let us similarly define the infinite-dimensional vector $\boldsymbol{\mu}_i = \left(\mu_{i,1}, \mu_{i,2}, \ldots\right)$ and the $(\infty \times n)$-dimensional matrix $\mathbf{S}_i = \left[\boldsymbol{\sigma}_{i,1}, \boldsymbol{\sigma}_{i,2}, \ldots\right]^{\top}$. Then we may write

$$\boldsymbol{f}_i = \boldsymbol{f}_{i-1} + \mathrm{d}t_i\boldsymbol{\mu}_i + \mathbf{S}_i\,\mathrm{d}\boldsymbol{W}_i$$

and the conditional distribution of the forward rates, under $\mathbb{P}$, is joint normal:[21]

$$\boldsymbol{f}_i \mid \mathscr{F}_{i-1} \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\boldsymbol{f}_{i-1} + \mathrm{d}t_i \boldsymbol{\mu}_i, \mathrm{d}t_i \mathbf{S}_i \mathbf{S}_i^\top\right).$$

In particular, this implies:

$$\begin{aligned}
\mathrm{E}_{\mathbb{P}}\left[f_{i,j} \,\middle|\, \mathscr{F}_{i-1}\right] &= f_{i-1,j} + \mu_{i,j}\,\mathrm{d}t_i, \\
\mathrm{VAR}_{\mathbb{P}}\left[f_{i,j} \,\middle|\, \mathscr{F}_{i-1}\right] &= \left\|\boldsymbol{\sigma}_{i,j}\right\|^2 \mathrm{d}t_i, \\
\mathrm{COV}_{\mathbb{P}}\left[f_{i,j}, f_{i,k} \,\middle|\, \mathscr{F}_{i-1}\right] &= \boldsymbol{\sigma}_{i,j}^\top \boldsymbol{\sigma}_{i,k}\,\mathrm{d}t_i.
\end{aligned}$$

Let us find μ such that discounted bond prices would be martingales under the risk-neutral measure $\mathbb{Q}$. The process for the present value of the j-bond, $B_i^{-1} P_{i,j}$, is a martingale under $\mathbb{Q}$ if and only if:

$$\mathrm{E}_{\mathbb{Q}}\left[B_i^{-1} P_{i,j} \middle| \mathscr{F}_{i-1}\right] = B_{i-1}^{-1} P_{i-1,j}$$

for each i, j such that $0 < i \le j$. This is equivalent to:[22]

$$\mathrm{E}_{\mathbb{Q}}\left[\frac{B_i^{-1} P_{i,j}}{B_{i-1}^{-1} P_{i-1,j}} \middle| \mathscr{F}_{i-1}\right] = \mathrm{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{m=i+1}^{j} \mathrm{d}f_{i,m}\,\mathrm{d}t_m\right) \middle| \mathscr{F}_{i-1}\right] = 1. \tag{4}$$

Assume that there exists an $\mathscr{F}$-previsible n-dimensional process $\boldsymbol{\gamma}$[23] and define an n-dimensional "random walk with drift" $\boldsymbol{V}$ as follows:

$$\boldsymbol{V}_i = \boldsymbol{W}_i - \sum_{m=1}^{i} \boldsymbol{\gamma}_m\,\mathrm{d}t_m. \tag{5}$$

[21]From the definition of the Wiener process, $\boldsymbol{W}_i - \boldsymbol{W}_{i-1} = \mathrm{d}\boldsymbol{W}_i \sim \mathcal{N}(\mathbf{0}, \mathrm{d}t_i \mathbf{I})$.

[22]We use the fact that the random variable $B_{i-1}^{-1} P_{i-1,j}$ is $\mathscr{F}_{i-1}$-measurable and apply the following lemma: If X, Y are random variables and X is $\mathscr{A}$-measurable, then almost surely $\mathrm{E}[XY|\mathscr{A}] = X\mathrm{E}[Y|\mathscr{A}]$. The rest follows from the fact that:

$$\begin{aligned}
B_i &= B_{i-1} \exp\left(f_{i-1,i}\,\mathrm{d}t_i\right), \\
P_{i,j} &= P_{i-1,j} \exp\left(\sum_{m=i}^{j} f_{i-1,m}\,\mathrm{d}t_m - \sum_{m=i+1}^{j} f_{i,m}\,\mathrm{d}t_m\right) \\
&= P_{i-1,j} \exp\left(f_{i-1,i}\,\mathrm{d}t_i - \sum_{m=i+1}^{j} \mathrm{d}f_{i,m}\,\mathrm{d}t_m\right).
\end{aligned}$$

[23]we require $\boldsymbol{\gamma}$ to satisfy certain technical conditions—seeBaxter and Rennie (1996) for details.

By the Girsanov theorem there exists a measure $\mathbb{Q}$ such that $\boldsymbol{V}$ is a Wiener process with respect to $\mathbb{Q}$.

We are going to derive conditions for the processes $\boldsymbol{\gamma}$, μ, and $\boldsymbol{\sigma}$ under which $\mathbb{Q}$ is risk-neutral[24]. We expand the exponent in Equation (4):

$$\begin{aligned}\sum_{m=i+1}^{j} \mathrm{d}f_{i,m}\,\mathrm{d}t_m &= \sum_{m=i+1}^{j} \mu_{i,m}\,\mathrm{d}t_m\,\mathrm{d}t_i + \sum_{m=i+1}^{j} \boldsymbol{\sigma}_{i,m}^{\top}\,\mathrm{d}t_m\,\mathrm{d}\boldsymbol{W}_i \\ &= \underbrace{\sum_{m=i+1}^{j} \mu_{i,m}\,\mathrm{d}t_m\,\mathrm{d}t_i + \sum_{m=i+1}^{j} \boldsymbol{\sigma}_{i,m}^{\top}\,\mathrm{d}t_m \boldsymbol{\gamma}_i\,\mathrm{d}t_i}_{\mathscr{F}_{i-1}\text{-measurable}} + \underbrace{\sum_{m=i+1}^{j} \boldsymbol{\sigma}_{i,m}^{\top}\,\mathrm{d}t_m}_{\mathscr{F}_{i-1}\text{-measurable}}\,\mathrm{d}\boldsymbol{V}_i.\end{aligned}$$

Clearly, under $\mathbb{Q}$ and conditional on $\mathscr{F}_{i-1}$, the random variable $\sum_{m=i+1}^{j} \mathrm{d}f_{i,m}\,\mathrm{d}t_m$ has normal distribution[25] with mean $\sum_{m=i+1}^{j}\left(\mu_{i,m} + \boldsymbol{\sigma}_{i,m}^{\top}\boldsymbol{\gamma}_i\right)\mathrm{d}t_m\,\mathrm{d}t_i$ and variance $\left\|\sum_{m=i+1}^{j}\boldsymbol{\sigma}_{i,m}\,\mathrm{d}t_m\right\|^2 \mathrm{d}t_i$. This immediately means that Equation (4) is satisfied if and only if:[26]

$$\sum_{m=i+1}^{j}\left(\mu_{i,m} + \boldsymbol{\sigma}_{i,m}^{\top}\boldsymbol{\gamma}_i\right)\mathrm{d}t_m = \frac{1}{2}\left\|\sum_{m=i+1}^{j}\boldsymbol{\sigma}_{i,m}\,\mathrm{d}t_m\right\|^2 \tag{6}$$

for all i, j such that $0 < i \le j$. Because Equation (6) is satisfied for $i = j$ (both sides are zero) we may rewrite it by differentiating both sides with respect to j as follows:

$$\left(\mu_{i,j} + \boldsymbol{\sigma}_{i,j}^{\top}\boldsymbol{\gamma}_i\right)\mathrm{d}t_j = \frac{1}{2}\left(\left\|\sum_{m=i+1}^{j}\boldsymbol{\sigma}_{i,m}\,\mathrm{d}t_m\right\|^2 - \left\|\sum_{m=i+1}^{j-1}\boldsymbol{\sigma}_{i,m}\,\mathrm{d}t_m\right\|^2\right).$$

After further simplification we obtain the no-arbitrage condition for the drift of the forward rate:

$$\mu_{i,j} = \boldsymbol{\sigma}_{i,j}^{\top}\left(\sum_{m=i+1}^{j}\boldsymbol{\sigma}_{i,m}\,\mathrm{d}t_m - \tfrac{1}{2}\boldsymbol{\sigma}_{i,j}\,\mathrm{d}t_j - \boldsymbol{\gamma}_i\right). \tag{7}$$

We will assume uniform time steps $\mathrm{d}t_i = \mathrm{d}t =$ one month for all $i \in \mathbf{N}$.

From now on, we will call the whole model outlined above "the" Heath–Jarrow–Morton model, or "HJM" model.

[24]By "risk-neutral" here we mean that the market price of risk associated with the movement of interest rates is the same for all bonds and thus all tradeable instruments.

[25]We use the following lemma: If $\boldsymbol{a}$ is a constant vector and $\boldsymbol{X}$ is a vector of independent standard normal variables, then the random variable $\boldsymbol{a}^{\top}\boldsymbol{X}$ has normal distribution with zero mean and variance $\|\boldsymbol{a}\|^2$.

[26]We use the following lemma: Let X be a random variable with distribution $\mathcal{N}\left(m, s^2\right)$. Then $\mathrm{E}\left[\mathrm{e}^{-X}\right] = 1$ if and only if $m = \frac{1}{2}s^2$.

4.2.3 Forward rate volatility

The Heath–Jarrow–Morton framework allows for a rich set of possible volatility specifications. We narrow the set down significantly by assuming *time invariance*:

$$\boldsymbol{\sigma}_{i,j} = \boldsymbol{\sigma}_{i+k,j+k} \qquad \text{for all } i,j,k \in \mathbf{N},$$

and we define the symbol $\boldsymbol{s}_m = \boldsymbol{\sigma}_{i,i+m}$ for $m \in \{1,\ldots,M\}$, where $M \in \mathbf{N}$ is the maximum tenor considered in the model.[27] We also define the $(N \times M)$-matrix:

$$\mathbf{S} = [\boldsymbol{s}_1,\ldots,\boldsymbol{s}_M]^\top.$$

The time-invariance of $\mathbf{S}$ makes calibration easier because under the real-world measure the drift-adjusted forward rate movements are jointly normally distributed:

$$\begin{bmatrix} \mathrm{d}f_{i,i+1} - \mu_{i,i+1}\,\mathrm{d}t \\ \vdots \\ \mathrm{d}f_{i,i+M} - \mu_{i,i+M}\,\mathrm{d}t \end{bmatrix} = \mathbf{S}\,\mathrm{d}\boldsymbol{W}_i \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\mathbf{0}, \mathbf{S}\mathbf{S}^\top\,\mathrm{d}t\right) \qquad \text{for any } i \in \mathbf{N} \tag{8}$$

and for different i's the terms are independent. To estimate $\mathbf{S}$, we use the singular decomposition of the empirical covariance matrix.

4.2.4 Forward rate drift

Under the Heath–Jarrow–Morton framework, as defined in Equation (3), the real-world drift (i.e., drift under the real-world probability measure $\mathbb{P}$) of the forward curve is:

$$\mathrm{E}_{\mathbb{P}}\left[f_{i,i+m} - f_{i-1,i+m} \,\middle|\, \mathscr{F}_{i-1}\right] = \mu_{i,i+m}\,\mathrm{d}t.$$

From Equation (7), the drift of the m-forward rate must satisfy the no-arbitrage condition:

$$\mu_{i,i+m} = \boldsymbol{s}_m^\top \left(\boldsymbol{s}_1 + \boldsymbol{s}_2 + \cdots + \boldsymbol{s}_{m-1} + \tfrac{1}{2}\boldsymbol{s}_m\right)\mathrm{d}t - \boldsymbol{s}_m^\top \boldsymbol{\gamma}_i.$$

[27]Note that the invariance of $\boldsymbol{\sigma}$ does not imply the invariance of μ, as the latter depends on the process $\boldsymbol{\gamma}$ for which we do not assume invariance. In fact, as we only assume previsibility, $\boldsymbol{\gamma}_i$ will generally depend on market variables observed up to time t_{i-1}, which will prove useful later.

For the model to be complete, we must specify the process $\boldsymbol{\gamma}$. In fact we are free to pick *any* $\mathscr{F}$-previsible process in the place of $\boldsymbol{\gamma}$: the Girsanov theorem will ensure the existence of a probability measure $\mathbb{Q}$ and the no-arbitrage condition (7) will ensure that, under $\mathbb{Q}$, bond prices discounted with the money-market account B are martingales. We exploit this flexibility of the Heath–Jarrow–Morton framework to introduce some real-world behaviour of interest rates into the model, in particular, the observation that interest rates tend to be mean-reverting and that yield curves tend to retain their upward-sloping shapes.[28] In an N-factor model, $N \in \mathbf{N}$, we do this as follows: we fix tenors $m_1, \dots, m_N \in \{1, \dots, M\}$ and for each i we specify particular values for the expected forward rates[29] $e_{i,i+m_1}, \dots, e_{i,i+m_N}$. Then, $\boldsymbol{\gamma}_i$ is recovered from the linear system:[30]

$$\begin{bmatrix} e_{i,i+m_1} - f_{i-1,i+m_1} \\ \vdots \\ e_{i,i+m_N} - f_{i-1,i+m_N} \end{bmatrix} \frac{1}{\mathrm{d}t} = \begin{bmatrix} \boldsymbol{s}_{m_1}^\top \left(\boldsymbol{s}_1 + \boldsymbol{s}_2 + \cdots + \boldsymbol{s}_{m_1-1} + \frac{1}{2}\boldsymbol{s}_{m_1}\right) \\ \vdots \\ \boldsymbol{s}_{m_N}^\top \left(\boldsymbol{s}_1 + \boldsymbol{s}_2 + \cdots + \boldsymbol{s}_{m_N-1} + \frac{1}{2}\boldsymbol{s}_{m_N}\right) \end{bmatrix} \mathrm{d}t - \begin{bmatrix} \boldsymbol{s}_{m_1}^\top \\ \vdots \\ \boldsymbol{s}_{m_N}^\top \end{bmatrix} \boldsymbol{\gamma}_i .$$

We assume that the m_k-forward rates, where $k \in K$, $K \subset \{1, \dots, N\}$, $K \neq \emptyset$, are mean-reverting according to the formula:

$$e_{i,i+m_k} = 2^{-\frac{\mathrm{d}t}{H_k}} f_{i-1,i+m_k-1} + \left(1 - 2^{-\frac{\mathrm{d}t}{H_k}}\right) b_{m_k}$$

where $b_{m_k} \in \mathbf{R}$ is the long-term mean m_k-forward rate and $H_k \in \mathbf{T}$ drives the speed of mean reversion.[31] The other forward rates, i.e., m_l-forward rates where $l \in K'$, $K' = \{1, \dots, N\} \setminus K$, are driven, respectively, by a mean-reverting m_{k_l}-forward rate, $k_l \in K$, and by the shape-preserving property of the yield curve according to the

[28]Compare this observation to the typical behaviour of some other interest rate models, e.g., the Vašíček model where the entire yield curve reverts to a flat shape. Note also that we only expect our model to show the desired behaviour under the real-world measure $\mathbb{P}$ (not the risk-neutral measure $\mathbb{Q}$), so that we are able to extract some information about forward rate volatility, as shown in Section 4.2.3.

[29]We define a special symbol e for the one-period-ahead expectation of the forward rate:

$$e_{i,j} = \mathrm{E}_{\mathbb{P}}\left[f_{i,j} \middle| \mathscr{F}_{i-1}\right], \quad i,j \in \mathbf{N}, i < j.$$

The $e_{i,i+m_1}, \dots$ are $\mathscr{F}$-previsible processes by definition, therefore in order to be able to calibrate the model, our formulas for $e_{i,i+m_1}, \dots$ must depend only on interest rates up to time t_{i-1}.

[30]We require that for $i, j \in \{1, \dots, N\}$, $i \neq j$, the vectors $\boldsymbol{s}_{m_i}$, $\boldsymbol{s}_{m_j}$ be linearly independent. Under this assumption the system has a unique solution.

[31]H_k should be seen as the "half-life" of the difference between the current m_k-forward rate and the long-term mean. The two extreme cases are $H_k = 0$, meaning immediate mean reversion ($2^{-\frac{\mathrm{d}t}{0}} = 0$), and $H_k = \infty$, meaning no mean reversion at all ($2^{-\frac{\mathrm{d}t}{\infty}} = 1$).

formula:

$$e_{i,i+m_l} = e_{i,i+m_{k_l}} + 2^{-\frac{\mathrm{d}t}{H_l}} \left(f_{i-1,i+m_l-1} - f_{i-1,i+m_{k_l}-1} \right) + \left(1 - 2^{-\frac{\mathrm{d}t}{H_l}} \right) d_l$$

where d_l is the long-term mean slope of the forward curve between the two tenors m_{k_l}, m_l and $H_l \in \mathbf{T}$ drives the speed of mean reversion.

4.3 Pricing instruments in the discrete model

4.3.1 General pricing formula

Let there be an instrument with a single payoff X which occurs at time t_j. (Note that both X and j might be random variables.) The price of such instrument at time t_i is:

$$B_i \mathrm{E}_{\mathbb{Q}} \left[1_{j \geq i} B_j^{-1} X \middle| \mathscr{F}_i \right] \tag{9}$$

where B is the money-market account process defined in Section 4.2.1, and $1_{j \geq i}$ is an indicator that the cash flow has not occurred yet (the value of past cash flows is zero).

Equation (9) is a general pricing formula. All other pricing formulas in this section are derived from it.

Instruments with more payoffs, say, $X_1, X_2, \ldots$ can be decomposed into individual payoffs. By the linear property of the expectation operator, the sum of value of the individual payoffs makes up the value of the original instrument. (note that this statement holds for however complex relationships there might be between $X_1, X_2, \ldots$).

We will now show the pricing formulas for several financial instruments, starting with the simplest one—the zero-coupon bond. Note that unless otherwise noted, all bonds and derivatives considered here have unit principal value or notional principal value, respectively.

4.3.2 Bond (zero-coupon)

A zero-coupon bond pays a unit of currency at a single known point in time. (We shall occasionally use the term "T-bond" for a zero-coupon bond maturing at time T.)

Proposition 4 The price at time t_i of a t_j-bond is:

$$B_i \mathrm{E}_{\mathbb{Q}}\left[B_j^{-1} \middle| \mathscr{F}_i \right]$$

for $j \geq i$, or zero for $j < i$. (The proof is trivial.)

The following proposition says that this is really the same price as the price of the zero-coupon bond defined in Section 4.2.1.

Proposition 5

$$B_i \mathrm{E}_{\mathbb{Q}}\left[B_j^{-1} \middle| \mathscr{F}_i \right] = P_{i,j} \qquad \text{for } i \leq j.$$

Proof. We use the fact that if X is a random variable and $\mathscr{A}$, $\mathscr{B}$ are σ-algebras, $\mathscr{A} \subset \mathscr{B}$, then almost surely $\mathrm{E}[\mathrm{E}[X|\mathscr{B}]|\mathscr{A}] = \mathrm{E}[X|\mathscr{A}]$. From the martingale property (4) of discounted bond prices we know that:

$$P_{i,j} = B_i \mathrm{E}_{\mathbb{Q}}\left[B_{i+1}^{-1} P_{i+1,j} \middle| \mathscr{F}_i \right]$$

(we have replaced $i-1$ in the original equation with i). By repeated substitution for $P_{i+1,j}, P_{i+2,j}, \ldots$ we obtain:

$$P_{i,j} = B_i \mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[B_{i+2}^{-1} P_{i+2,j} \middle| \mathscr{F}_{i+1} \right] \middle| \mathscr{F}_i \right] = B_i \mathrm{E}_{\mathbb{Q}}\left[B_{i+2}^{-1} P_{i+2,j} \middle| \mathscr{F}_i \right] =$$
$$= B_i \mathrm{E}_{\mathbb{Q}}\left[B_{i+3}^{-1} P_{i+3,j} \middle| \mathscr{F}_i \right] = \ldots = B_i \mathrm{E}_{\mathbb{Q}}\left[B_j^{-1} P_{j,j} \middle| \mathscr{F}_i \right]$$

because $\mathscr{F}_i \subset \mathscr{F}_{i+1} \subset \ldots \subset \mathscr{F}_j$. But $P_{j,j} = 1$ which proves the proposition.

4.3.3 Bond (fixed-coupon)

A fixed-coupon bonds pays a unit of currency at a single known point in time. In addition to that, it makes periodic interest payments (known in advance) until maturity.

Note: to accommodate the various day count conventions, as well as linear and continuous compounding, we assume that interest accrues according to a function $I(c,t_a,t_b)$ where c is the coupon rate, and (t_a,t_b) is the interest period.

Thus, for continuously compounded interest, we set $I(c,t_a,t_b) = \mathrm{e}^{c(t_b-t_a)} - 1$. For linearly compounded interest, we set $I(c,t_a,t_b) = c(t_b - t_a)\xi$ where ξ is the year fraction factor (e.g., for the Act/360 convention, $\xi \approx \frac{365.25}{360}$).

Proposition 6 Let us assume a fixed-coupon bond with unit principal value, coupon rate c, day count convention I, issue date t_{p_0}, and coupon payments at times t_{p_1}, ..., t_{p_L}, respectively. (All interest payments are in arrears.) The value of such bond at time t_i is:

$$B_i \sum_{s=1}^{L} \mathrm{E}_{\mathbb{Q}}\left[1_{p_s \geq i} B_{p_s}^{-1} I\left(c, t_{p_{s-1}}, t_{p_s}\right) \middle| \mathscr{F}_i \right] + P_{i,p_L}. \tag{10}$$

Proof. Equation (10) is a trivial application of Equation (9) and Proposition 5.

Par rate, par curve, and bootstrap

If a fixed-coupon bond with maturity T trades at par (i.e., its price is equal to the principal value) then the fixed rate is called the "par rate" for T. Par rates for various maturities make up a "par curve".

The par curve can be used to determine zero rates (or, equivalently, the prices of zero-coupon bonds), as the following proposition shows. The procedure is known as "bootstrapping the par curve".

Proposition 7 (Bootstrapping the par curve) Let $L \in \mathbf{N}$, $p_0 < \ldots < p_L$. Let there be a bond, trading at par at time t_{p_0}, with a fixed coupon rate c and payments at times $t_{p_1}, \ldots, t_{p_L}$. Let the prices of the t_{p_1}-bond, t_{p_2}-bond, ..., and the $t_{p_{L-1}}$-bond be known. Then the price of the t_{p_L}-bond is:

$$P_{p_0,p_L} = \frac{1 - \sum_{s=1}^{L-1} I\left(c, t_{p_{s-1}}, t_{p_s}\right) P_{p_0,p_s}}{1 + I\left(c, t_{p_{L-1}}, t_{p_L}\right)}.$$

Proof. For a fixed-coupon bond trading at par we set (10) = 1, substitute $i = p_0$, apply Proposition 5 and rearrange to obtain:

$$\sum_{s=1}^{L} I\left(c, t_{p_{s-1}}, t_{p_s}\right) P_{p_0,p_s} + P_{p_0,p_L} = 1.$$

from where P_{p_0,p_L} can be isolated directly, Q.E.D.

Note that the corresponding spot rate (under continuous compounding) can be calculated as:

$$r_L = \frac{1}{t_{p_L} - t_{p_0}} \log \frac{1 + I\left(c, t_{p_{L-1}}, t_{p_L}\right)}{1 - \sum_{s=1}^{L-1} I\left(c, t_{p_{s-1}}, t_{p_s}\right) \mathrm{e}^{-r_s\left(t_{p_s} - t_{p_0}\right)}}.$$

4.3.4 Forward rate agreement

A forward rate agreement (FRA) pays the difference between the interest accrued under a fixed rate and a floating rate, respectively.[32]

Remark 8 We follow the convention that a "long" position is the one with positive delta, hence the cash flow of the floating leg is positive and the cash flow of the fixed leg is negative.

Remark 9 We assume $P_{i,j} = 0$ for $i > j$, which is just a notational convention.

Proposition 10 Let us assume an FRA for the period (t_a, t_b), $a < b$, with a fixed rate k and interest convention I. (The floating rate is fixed at t_a and the cash flows are exchanged at t_b.) The value, at time t_i, of the FRA is:

$$V_{\mathrm{FRA},i} = P_{i,b} \left(\frac{P_{*,a}}{P_{*,b}} - 1 - I_{k,t_a,t_b} \right) \tag{11}$$

where $* = \min(i, a)$.

Proof. The pricing of an FRA is trivial—prior to t_a, i.e., before the floating rate is known, $V_{\mathrm{FRA},i}$ is forced by the arbitrage opportunity involving the t_a-bond and the t_b-bond. After t_a, all cash flows are known and $V_{\mathrm{FRA},i}$ is forced by these and the price of the t_b-bond.

[32]Hull (2012), pp 86-89

With this in mind, we note that the floating rate is the yield of the t_b-bond at time t_a, therefore, irrespective of the convention, the interest received must be equal to $P_{a,b}^{-1} - 1$. From the pricing formula (9) we get

$$V_{\mathrm{FRA},i} = 1_{i \le b} B_i \mathrm{E}_{\mathbb{Q}} \left[B_b^{-1} \left(P_{a,b}^{-1} - 1 - I_{k,t_a,t_b} \right) \middle| \mathscr{F}_i \right].$$

Now, assuming $i \le a$, we have $\mathscr{F}_i \subset \mathscr{F}_a$, so we can write:

$$\mathrm{E}_{\mathbb{Q}} \left[B_b^{-1} P_{a,b}^{-1} \middle| \mathscr{F}_i \right] = \mathrm{E}_{\mathbb{Q}} \left[\mathrm{E}_{\mathbb{Q}} \left[B_b^{-1} P_{a,b}^{-1} \middle| \mathscr{F}_a \right] \middle| \mathscr{F}_i \right] = \mathrm{E}_{\mathbb{Q}} \left[P_{a,b}^{-1} \mathrm{E}_{\mathbb{Q}} \left[B_b^{-1} \middle| \mathscr{F}_a \right] \middle| \mathscr{F}_i \right] =$$
$$= \mathrm{E}_{\mathbb{Q}} \left[P_{a,b}^{-1} B_a^{-1} P_{a,b} \middle| \mathscr{F}_i \right] = \mathrm{E}_{\mathbb{Q}} \left[B_a^{-1} \middle| \mathscr{F}_i \right] = B_i^{-1} P_{i,a}.$$

On the other hand, for $i \ge a$, everything in the expectation except for B_b^{-1} is $\mathscr{F}_i$-measurable, so, after some rearranging, we obtain the following:

$$V_{\mathrm{FRA},i} = \begin{cases} P_{i,a} + P_{i,b} \left(-1 - I_{k,t_a,t_b} \right), & i \le a, \\ P_{i,b} \left(P_{a,b}^{-1} - 1 - I_{k,t_a,t_b} \right), & a \le i \le b, \\ 0, & b < i, \end{cases}$$

which is equivalent to (11), Q.E.D.

Remark 11 Assume the same as in Proposition 10. Let t_o be the inception date of the FRA, $o \in \mathbf{Z}$, $o < a < b$. Then, assuming that $V_{\mathrm{FRA},o} = 0$, we have:

$$I_{k,t_a,t_b} = \frac{P_{o,a}}{P_{o,b}} - 1 \qquad \text{and} \qquad V_{\mathrm{FRA},i} = P_{i,b} \left(\frac{P_{*,a}}{P_{*,b}} - \frac{P_{o,a}}{P_{o,b}} \right). \tag{12}$$

Let t_z be the time of inception of the FRA, $z < a$. In practice, the fixed rate of the FRA is set so that at t_z the value of the contract is zero.[33] This market-implied rate is called the “FRA rate”. The following proposition shows that there is really no distinction between FRA rates and forward rates derived from the prices of zero-coupon bonds.

Proposition 12 The FRA rate is equal to the forward rate implied by the prices of zero-coupon bonds.

[33] ibid

Proof. The FRA rate k can be found by setting $i = z$ and $(11) = 0$ and rearranging:

$$k = \frac{1}{t_b - t_a} \log \frac{\mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} P_{a,b}^{-1} \middle| \mathscr{F}_z \right]}{\mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} \middle| \mathscr{F}_z \right]}. \tag{13}$$

Let us calculate the conditional expectations. With the use of Proposition 5, we obtain:

$$\begin{aligned} \mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} P_{a,b}^{-1} \middle| \mathscr{F}_z \right] = \mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} P_{a,b}^{-1} \middle| \mathscr{F}_a \right] \middle| \mathscr{F}_z \right] = \mathrm{E}_{\mathbb{Q}}\left[P_{a,b}^{-1} \mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} \middle| \mathscr{F}_a \right] \middle| \mathscr{F}_z \right] = \\ = \mathrm{E}_{\mathbb{Q}}\left[P_{a,b}^{-1} B_a^{-1} P_{a,b} \middle| \mathscr{F}_z \right] = \mathrm{E}_{\mathbb{Q}}\left[B_a^{-1} \middle| \mathscr{F}_z \right] = B_z^{-1} P_{z,a} \end{aligned}$$

and we already know that $\mathrm{E}_{\mathbb{Q}}\left[B_b^{-1} \middle| \mathscr{F}_z \right] = B_z^{-1} P_{z,b}$. After substituting into Equation (13) and rearranging we obtain:

$$k = \frac{1}{t_b - t_a} \log \frac{P_{z,a}}{P_{z,b}},$$

which is the forward rate, at t_z, for the period (t_a, t_b), implied by zero-coupon bonds, Q.E.D.

4.3.5 Interest rate swap

An interest rate swap pays, periodically, the difference between the interest accrued under two different regimes over several subsequent periods. A swap is “fixed-for-floating” if the interest accrues thus:[34]

- in the “fixed leg”: under a rate that is fixed for the whole life of the swap,
- in the “floating leg”: under a rate that is periodically reset to the current market spot rate for a fixed tenor (e.g., six months).

Proposition 13 Let $L \in \mathbf{N}$, $p_0 < \ldots < p_L$. Let there be a fixed-for-floating interest rate swap with L payments in total. The floating rate is set at times $t_{p_0}, \ldots, t_{p_{L-1}}$. The payments take place at times $t_{p_1}, \ldots, t_{p_L}$ (all payments are in arrears). The interest convention is I. The fixed swap rate is k. Then the value, at time t_i, of such

[34]ibid

swap is:

$$B_i \mathrm{E}_{\mathbb{Q}}\left[\sum_{s=1}^{L} B_{p_s}^{-1}\left(I\left(x_s, t_{p_{s-1}}, t_{p_s}\right) - I\left(k, t_{p_{s-1}}, t_{p_s}\right)\right) 1_{p_s \geq i} \middle| \mathscr{F}_i\right] \tag{14}$$

where x_s is the floating rate for the s^{th} swaplet, $s \in \{1, \ldots, L\}$. The floating rate is the yield of the zero-coupon bond, at time $t_{p_{s-1}}$, maturing at time t_{p_s}. Under continuous compounding, the floating rate can be expressed as:

$$x_s = \frac{1}{t_{p_s} - t_{p_{s-1}}} \log \frac{1}{P_{p_{s-1}, p_s}}.$$

Proof. The proof follows trivially from the fact that the swap is effectively a series of FRAs and from Proposition 10.

Remark 14 Assume the same as in Proposition 13. Let t_{p_0} be the inception date of the IRS and assume that $V_{\text{IRS}, p_0} = 0$. Further assume that the reset dates are equally spaced, i.e., $t_{p_s} = t_{p_{s-1}} + D$ for all $s \in \{1, \ldots, L\}$ and some $D \in \mathbf{T}$.[35] Finally, assume linear accruals, i.e., $I_{k, t_a, t_b} = (t_a - t_b) k$. Then:

$$k = \frac{1 - P_{p_0, p_L}}{D \times \sum_{s=1}^{L} P_{p_0, p_s}} \quad \text{and} \quad V_{\text{IRS}, i} = \frac{P_{i, p_S}}{P_{p_{S-1}, p_S}} - P_{i, p_L} - \left(1 - P_{p_0, p_L}\right) \frac{\sum_{s=S}^{L} P_{i, p_s}}{\sum_{s=1}^{L} P_{p_0, p_s}}. \tag{15}$$

Let us define the "residual swap rate" k_i as the fixed rate k that would make the residual value of the swap, $V_{\text{IRS}, i}$, equal to zero. An illustration of the "residual swap rate" is shown in Figure 3. We obtain:

$$k_i = \frac{\frac{P_{i, p_S}}{P_{p_{S-1}, p_S}} - P_{i, p_L}}{D \times \sum_{s=S}^{L} P_{p_i, p_s}} \qquad \text{and} \qquad V_{\text{IRS}, i} = D \times (k_i - k) \times \sum_{s=S}^{L} P_{p_i, p_s}.$$

Similarly to FRAs, the fixed rate for a swap is set on the issue date so that the value of the swap is zero. The fixed rate with this property is called the "swap rate". Swap rates for different maturities make up the "swap curve".

The following proposition shows that swap rates are essentially equivalent to par rates. This means that we can "bootstrap" swap rates to derive the prices of zero-coupon bonds.

Proposition 15 The swap rate is equal to the par rate (assuming that the bond and the swap have the same payment schedule).

[35]For instance, $D = \frac{6}{12}$ years for an IRS vs a 6M LIBOR.

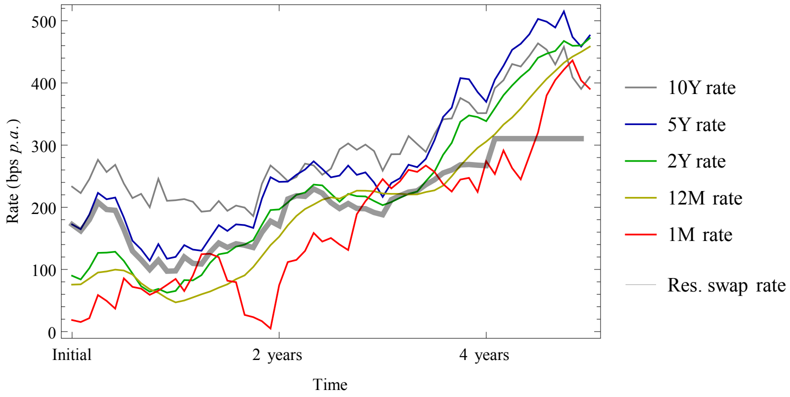


Fig. 3: A sample $\mathbb{Q}$-scenario of risk-free zero rates. The residual swap rate for a 5Y IRS v 12M LIBOR is shown in gray.

Proof. The value of the interest rate swap can be expressed as the sum of the values of the two legs: $V_{\text{float}} + V_{\text{fix}}$. The value of the floating leg of the swap at time t_{p_0} is:

$$V_{\text{float}} = B_{p_0} \mathrm{E}_{\mathbb{Q}} \left[\sum_{s=1}^{L} B_{p_s}^{-1} I\left(x_s, t_{p_{s-1}}, t_{p_s}\right) \middle| \mathscr{F}_{p_0} \right]. \tag{16}$$

Similarly to FRAs, the floating rate x_s is the spot rate for the period $\left(t_{p_{s-1}}, t_{p_s}\right)$, or the yield of the t_{p_s}-bond at time $t_{p_{s-1}}$. Therefore $I\left(x_s, t_{p_{s-1}}, t_{p_s}\right) = P_{p_{s-1},p_s}^{-1} - 1$. We also know from the proof of Proposition 12 that $\mathrm{E}_{\mathbb{Q}}\left[B_{p_s}^{-1} P_{p_{s-1},p_s}^{-1} \middle| \mathscr{F}_{p_0}\right] = B_{p_0}^{-1} P_{p_0,p_{s-1}}$. Therefore the value of the floating leg (16) is:

$$V_{\text{float}} = B_{p_0} \sum_{s=1}^{L} \mathrm{E}_{\mathbb{Q}}\left[B_{p_s}^{-1}\left(P_{p_{s-1},p_s}^{-1} - 1\right) \middle| \mathscr{F}_{p_0}\right] = \sum_{s=1}^{L} \left(P_{p_0,p_{s-1}} - P_{p_0,p_s}\right) = 1 - P_{p_0,p_L}.$$

The value of the fixed leg at time t_{p_0} is:

$$V_{\text{fix}} = -B_{p_0} \mathrm{E}_{\mathbb{Q}} \left[\sum_{s=1}^{L} B_{p_s}^{-1} I\left(k, t_{p_{s-1}}, t_{p_s}\right) \middle| \mathscr{F}_{p_0} \right] = -V_{\text{bond}} + P_{p_0,p_L},$$

where V_{bond} is the value of a fixed-coupon bond with the same payment schedule as the swap and with fixed rate k (see Proposition 6). Because k is the par rate, the bond must trade at par ($V_{\text{bond}} = 1$), therefore $V_{\text{float}} = -V_{\text{fix}}$ and the value of the swap is zero, which makes k the swap rate, Q.E.D.

4.3.6 Interest rate cap, floor, and other options

European options in general

The value of a European option at time t_i is usually of the form:

$$B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i} B_j^{-1} X^+ \middle| \mathscr{F}_i\right] \qquad \text{or} \qquad B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i} B_j^{-1} X^- \middle| \mathscr{F}_i\right]$$

where X is some payoff which occurs at time t_j.[36] In practice, X could be the value of a stock minus the strike price (for a stock option), of a quantity of some commodity minus the strike price (for a commodity option), or of a foreign-exchange forward contract (for an FX option). The + and − signs differentiate between call and put options.

Other option styles

The formula given above is for European options, where the right to exercise is constrained to just one moment. Valuation of American options, which can be exercised any time during their life, is more involved. This is not discussed in this thesis.

Interest rate cap and floor

An *interest rate cap* is an option contract whereby one party (the *short*) agrees to pay to the other party (the *long*) the *positive* difference between the interest accrued under a reference rate and the interest accrued under a fixed rate. Note that the payment is made if the reference rate is *above* the fixed rate.

Usually, the payments are made multiple times over a series of periods, with the reference rate reset for each period (the individual payments are called *caplets*).

An *interest rate floor* is similar to a cap but pays the negative difference, i.e., a payment is made only if the reference rate is below the fixed rate. The individual payments are then called *floorlets*.

[36]We use the following notation: $x^+ = \max\{0, x\}$, and $x^- = \max\{0, -x\}$.

Interest rate floors and caps are similar to fixed-for-floating interest rate swaps in the following sense: in a swap, payments are made for every period, both the negative and the positive ones. In a cap or a floor, only the positive (or negative) payments are made.

Interest rate caps and floors are purchased at a premium, paid either at the inception as a lump-sum payment, or periodically.

Proposition 16 Let $L \in \mathbf{N}$, $p_0 < \ldots < p_L$. Let there be an interest rate cap with L payments in total. The floating rate is set at times $t_{p_0}, \ldots, t_{p_{L-1}}$. The payments take place at times $t_{p_1}, \ldots, t_{p_L}$ (all payments are in arrears). The interest convention is I. The fixed cap rate is k. Then the value, at time t_i, of the cap is:

$$B_i \mathrm{E}_{\mathbb{Q}} \left[\sum_{s=1}^{L} B_{p_s}^{-1} \left(I\left(x_s, t_{p_{s-1}}, t_{p_s}\right) - I\left(k, t_{p_{s-1}}, t_{p_s}\right) \right)^{+} 1_{p_s \geq i} \middle| \mathscr{F}_i \right]$$

where x_s is the floating rate for the s^{th} caplet, $s \in \{1, \ldots, L\}$. The floating rate is the yield of the zero-coupon bond, at time $t_{p_{s-1}}$, maturing at time t_{p_s}. (See the expression for the floating rate in Proposition 13.)

4.4 Calibration of the Heath–Jarrow–Morton model

4.4.1 Source data, bootstrapping, and forward rate interpolation

Source data

Our source data consist of historical money-market rates and interest-rate swap rates for the Japanese yen (JPY).[37] The data cover the history from January 1985 to June 2015 and are sampled with daily frequency (closing mid rates were used). Figure 4 shows the history of yen money-market and swap rates (for selected tenors only).

Figure 5 shows the availability of the yen rates for various tenors over the selected period. Since not all tenors were quoted throughout the period considered, some missing data had to be filled in.

[37]The source of the data is the *Bloomberg Professional Service*.

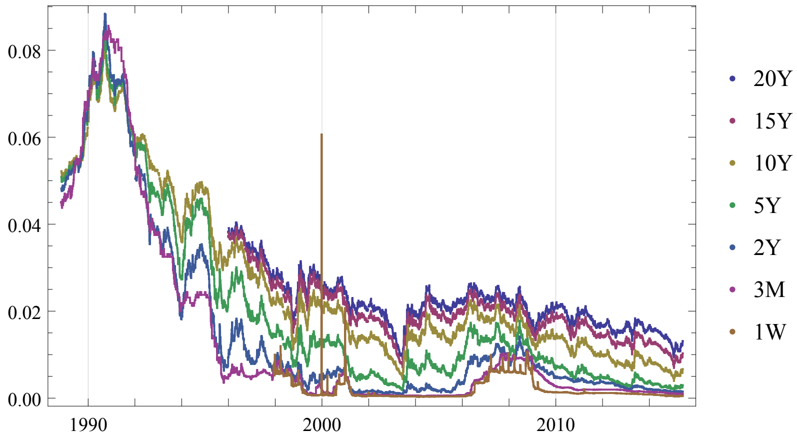


Fig. 4: Daily yen money-market and swap rates (selected tenors).

1. The 30Y rate has been extrapolated, over the period from 1 Jan 1985 until 3 Sep 1999, using a linear model based on the 10Y rate.[38]

2. Once the 30Y rates have been obtained, any missing rates for all the tenors between ON and 30Y (both included) were filled in based the on available rates for neighboring tenors for the particular date.[39]

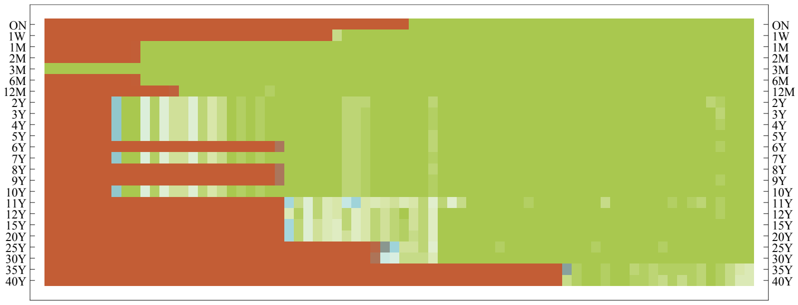


Fig. 5: The availability of daily yen money-market and swap rates, shown in 100-day baskets (the time axis runs from left to right). Green color indicates full availability, red color indicates no data, other shades indicate occasional missing data.

Bootstrapping of the yen swap curve

[38]The model was of the form $r_{30\mathrm{Y},t} = \alpha + \beta r_{10\mathrm{Y},t} + \varepsilon_t$.

[39]The algorithm works as follows: first, the nearest earlier and the nearest later dates for which a value is available for the given tenor are found. Second, the missing values are modelled as a linear interpolation between the rates the nearest existing upper and nearest existing lower tenor.

This strategy for filling in the missing data was motivated by the observation that rates for different tenors tend to move together, i.e. there is valuable information carried by the daily movement of rates for other tenors. Thus, by interpolating between adjacent tenors while keeping the date fixed we obtain a better predictor of the actual value than if we ignored other tenors and simply interpolated between the nearest earlier and nearest later available values.

For details, see the comments in the *PatchData[...]* function in *JPYrates.nb*.

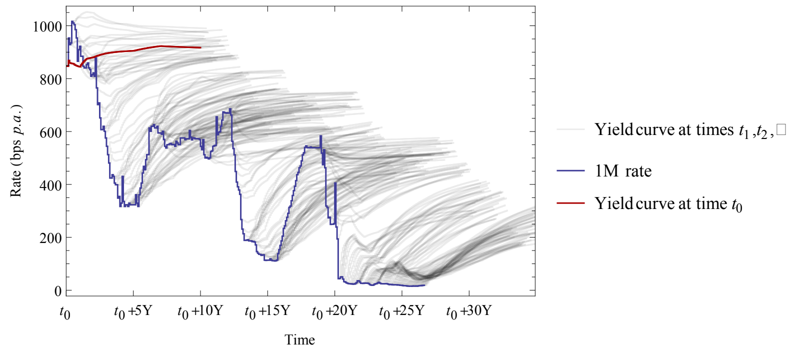


Fig. 6: A composite view of the history of the USD LIBOR+swap curve from 1984 to 2016.

Once all missing data had been filled in, the forward rates were calculated by bootstrapping the swap rates.[40] The bootstrapping procedure is described in Proposition 7.

To obtain a dataset of reasonable size and variability, forward rates were calculated in monthly steps (i.e., dt = 1 month).

Forward rate interpolation

Market swap rates are typically not available for every tenor. For those points on the yield curve that do not directly correspond to a traded instrument, the rate must be determined using other methods. This is known as “interpolation of the yield curve”.

In contemporary financial literature, interpolation of the yield curve is an open problem. Hagan and West (2006) present an overview of prevailing methods which vary in their complexity, their theoretical underpinning, and performance. Known interpolation methods range from simple, and rather technical, tools to sophisticated algorithms, sometimes even relying on no-arbitrage conditions for some particular market model. (See, for instance, Laurini and Hotta, 2010). The advantage of the former type is of course simplicity, the advantage of the latter type is in more precise results and therefore less unpleasant “surprises” when the method is used to price an instrument.

[40]See the commented code of the *BootstrapParRates[...]* and *ToForwardRates[...]* functions in *JPYrates.nb*. The “short” part of the curve (i.e., money market rates up to 12M) was not bootstrapped but only converted to continuously-compounded rates.

Our aim is *indicative* pricing, i.e., we are not interested in developing a pricing mechanism that would always yield precise and market-consistent prices. This allows us to act rather liberally in choosing an interpolation algorithm.

The interpolation method we choose ensures *piecewise constant forward rates* between quoted tenors. The method is as follows. We start with a set of spot rates $r_1,\ldots,r_n$ for tenors $t_1,\ldots,t_n$. We then calculate the numbers $r_1t_1, \ldots, r_nt_n$, and use linear interpolation on these values to produce a piecewise linear function $t \mapsto r_t t$. Because for the forward rate f_t we have $f_t = \frac{\partial}{\partial t}(tr_t)$ (where the derivative exists), the forward rate is constant between the quoted dates.

The observed behaviour of this method makes it preferable to other methods. The most compelling reason is that it is less sensitive to irregularities in the source yield curve and therefore produces less volatile forward rates.

The result is, for our yen data, a 308×360 matrix of forward rates (308 is the length of the historical period in months and 360 is the maximum tenor in months). The data are shown in Figure 7. All forward rates are expressed in continuous compounding.

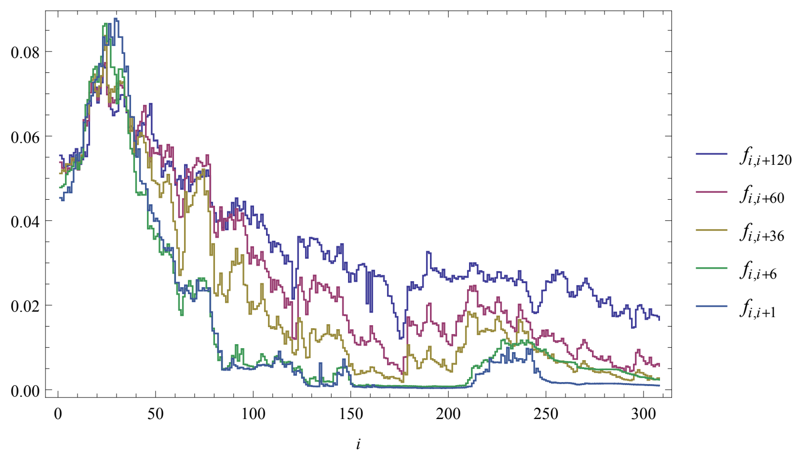


Fig. 7: Historical monthly implied yen T-forward rates $f_{i,i+T}$ for selected values of T (see Section 4.2.1 for the definition). The index $i \in \{1,\ldots,308\}$ marks monthly steps with t_1 corresponding to 1 Nov 1988 and t_{308} corresponding to 2 Jun 2014. T denotes the time to maturity in months.

4.4.2 Motivation and particular features of the model

Motivation

The forward-rate model outlined in Section 4.2.2 is a rather general description of the behaviour of the forward rate: the forward rate differential $\mathrm{d}f_{i,j}$ depends on the random variables $\mu_{i,j}$, $\boldsymbol{\sigma}_{i,j}$, and $\mathrm{d}\boldsymbol{W}_i$ where the former two are $\mathscr{F}_{i-1}$-measurable, hence "known" at time t_{i-1}, and the latter is a random variable unknown at time t_{i-1} but with a known distribution. Other than that, few restrictions are imposed on $\mu_{i,j}$, $\boldsymbol{\sigma}_{i,j}$.

This is, nevertheless, enough to formulate the martingale property for discounted bond prices under a risk-neutral measure (Equation (4) on page 35) and to derive the no-arbitrage condition for the drift (Equation (7) on page 36).

However, to price financial instruments using the Heath–Jarrow–Morton model, we need more assumptions about the dynamics of the forward rate. We know from Section 4.3, Equation (9), that the price, at time t_i, of a payoff X at time t_j is calculated from an expectation of the random variable:

$$1_{j\geq i}B_j^{-1}X,$$

conditional on the σ-algebra $\mathscr{F}_i$. As we have seen in further in Section 4.3, for common financial derivatives the random variable X often depends in non-trivial ways on $f_{m,n}$ where $m \in \{i, i+1, i+2, \ldots\}, n > m$. For instance, the last payoff of an interest rate swap depends on the floating rate for the period $\left(t_{p_{L-1}}, t_{p_L}\right)$, which is a function of the rates $f_{p_{L-1},p_{L-1}+1}, f_{p_{L-1},p_{L-1}+2}, \ldots, f_{p_{L-1},p_L}$, see Proposition 13 (page 44).

With the Heath–Jarrow–Morton framework alone, we are generally unable to price such instruments.[41]

The pricing of financial derivatives therefore warrants some prior knowledge of the joint distribution of forward rates beyond the single time step $t_i - t_{i-1}$. In other words, we must impose some assumptions on distribution of $f_{i,j}$ conditional on not just $\mathscr{F}_{i-1}$ but also on $\mathscr{F}_{i-2}, \mathscr{F}_{i-3}$, and so on.

Homogeneity

[41]Save for a handful of special cases where the instrument can be statically replicated with zero-coupon bonds, as illustrated in Section 4.3.

The Heath–Jarrow–Morton framework allows for a rich set of possible volatility specifications. By assuming *time homogeneity*, we narrow the set down significantly: We shall assume that for all $i, j, k \in \mathbf{N}$:

$$\boldsymbol{\sigma}_{i,j} = \boldsymbol{\sigma}_{i+k,j+k}.$$

This particular property makes it easier to specify and calibrate the model, because instead of $\boldsymbol{\sigma}_{i,j}$ we can simply write $\boldsymbol{\sigma}_{j-i}$.

By assuming time homogeneity (informally: "future will be the same as the past") we are able to extract useful information from historical market data.

(Note that the time homogeneity of $\boldsymbol{\sigma}$ does not imply time homogeneity of μ, as the latter depends on the process $\boldsymbol{\gamma}$ for which we do not assume homogeneity. In fact, as we only assume previsibility, $\boldsymbol{\gamma}_i$ will generally depend on market variables observed up to time $t_i - 1$, which will prove useful later.)

Number of factors

We shall assume a two-factor model ($n = 2$).

This has one key reason: one-factor models have limited scope as to the development of the forward curve (all rates move more or less simultaneously). With two-factor and richer models, several independent random shocks influence the forward curve, and this allows for some shocks to influence, for instance, the "short" end differently from the "long" end.

This is especially important for treating interest rate derivatives, whose value depends, among other things, on the shape of the curve.

At the same time, two factors seem to suffice for most applications, since we are able to model the most frequent shapes of the forward (or yield) curve.[42]

Mean reversion

[42]cite: Modely úrokových sazeb - teorie a praxe (Myška), p1

Our optimization goal is to find $\beta_i \in [0,1]$ such that for $M \in \{1,\dots,n-i\}$:

$$\mathrm{E}\left[f_{i,i+M}\right] = \beta_i f_{0,M} + \left(1-\beta_i\right) f_{i-1,i+M-1}$$

where $\beta_i = 1 - 2^{-\frac{\mathrm{d}t_i}{H}}$ is the speed of reversal (H is the half-life). But $\mathrm{E}\left[f_{i,i+M}\right] = f_{i-1,i+M} + \mu_{i,i+M}\,\mathrm{d}t_i$, therefore we obtain:

$$\mu_{i,i+M} = \frac{1}{\mathrm{d}t_i}\left(\beta_i f_{0,M} + \left(1-\beta_i\right) f_{i-1,i+M-1} - f_{i-1,i+M}\right)$$

Our optimization goal is to find $\beta_i \in [0,1]$ such that for $M \in \{1,\dots,n-i\}$:

$$\mathrm{E}\left[f_{i,i+M}\right] = \beta_i f_{0,M} + \left(1-\beta_i\right) f_{i-1,i+M-1}$$

where $\beta_i = 1 - 2^{-\frac{\mathrm{d}t_i}{H}}$ is the speed of reversal (H is the half-life). But $\mathrm{E}\left[f_{i,i+M}\right] = f_{i-1,i+M} + \mu_{i,i+M}\,\mathrm{d}t_i$, therefore we obtain:

$$\mu_{i,i+M} = \frac{1}{\mathrm{d}t_i}\left(\beta_i f_{0,M} + \left(1-\beta_i\right) f_{i-1,i+M-1} - f_{i-1,i+M}\right)$$

Or, in more detail:

$$\begin{bmatrix} \mu_{i,i+1} \\ \vdots \\ \mu_{i,n} \end{bmatrix} = \frac{1}{\mathrm{d}t_i}\left(\beta_i \begin{bmatrix} f_{0,1} \\ \vdots \\ f_{0,n-i} \end{bmatrix} + \left(1-\beta_i\right)\begin{bmatrix} f_{i-1,i} \\ \vdots \\ f_{i-1,n-1} \end{bmatrix} - \begin{bmatrix} f_{i-1,i+1} \\ \vdots \\ f_{i-1,n} \end{bmatrix}\right).$$

4.4.3 Volatility

Analysis of historical volatility

Our calibration of the Heath–Jarrow–Morton will be based on historical market data.[43]

Figure 8 shows the realized volatility of the yen forward rates in the past.

[43]If our aim was to hedge counterparty exposure in the market, then the risk-neutral valuation based on the market price of interest rate caps, floors, and swaptions, would be the right choice. If we are examining the "real-world" properties of counterparty credit risk we may as well use "real-world" realizations of the random variables in question.

The values suggest that the forward rates might be modelled with a single common volatility, σ.

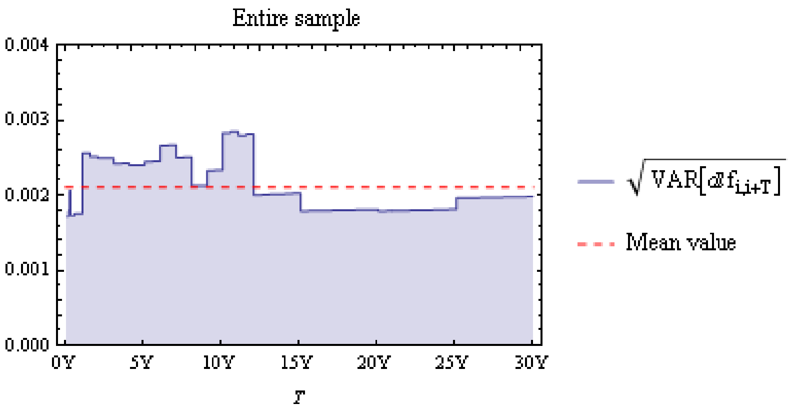


Fig. 8: Realized volatility of the monthly changes of the T-forward yen interest rate $f_{i,i+T}$ over the entire historical period. The values are *p.a.* The red dashed line depicts the average volatility over all tenors.

Analysis of historical correlation

Having analyzed the overall volatility of the yen forward curve, we now turn to the interdependence between the individual forward rates. Figure 9 shows the realized *correlation* of the monthly changes of yen forward rates in the past.

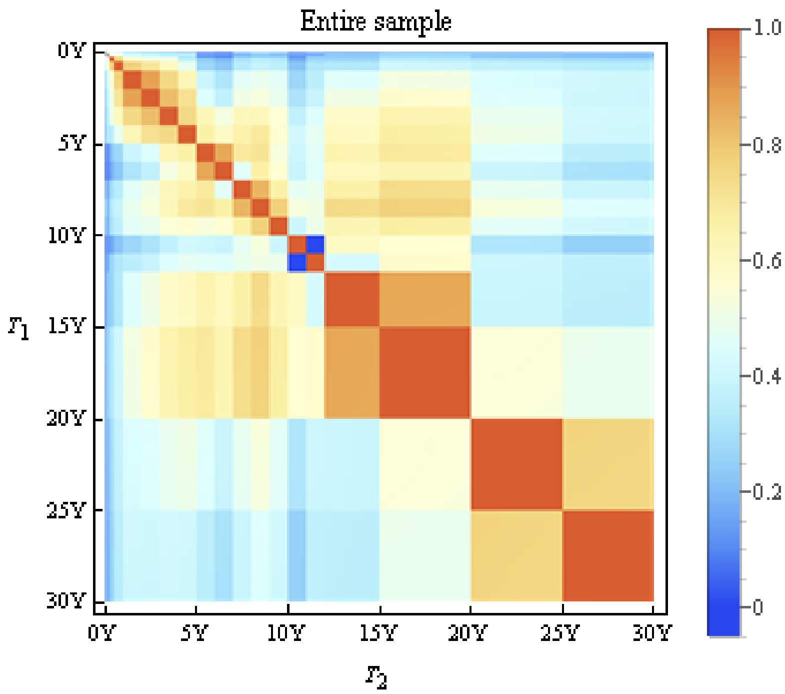


Fig. 9: Realized correlation of the monthly changes of the T_1-forward and the monthly changes of the T_2-forward yen interest rate, for $T_1, T_2 \in \{1M, \ldots, 30Y\}$, over the entire available history.

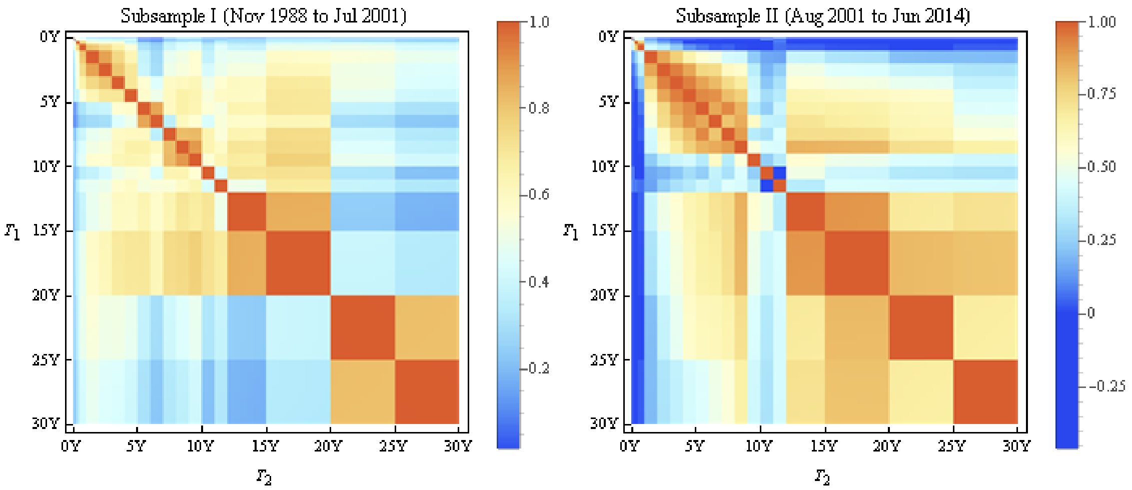


Fig. 10: Realized correlation of the monthly changes of the T_1-forward and the T_2-forward yen interest rate, for $T_1, T_2 \in \{1\text{M}, \ldots, 30\text{Y}\}$, over the two respective sub-periods.

4.5 Credit risk model

4.5.1 Credit risk and default

Default

In this section we introduce credit risk into the model. The central notion is the *default*, which we assume is a one-off event that may occur in a finite time or never. Formally, we assume a random variable:

$$d : \Omega \mapsto \mathbf{Z} \cup \{\infty\}$$

defined in the common probability space. If the default occurs in the time interval $(t_{i-1}, t_i]$, $i \in \mathbf{N}$, then $d = i$. If the default never occurs, then $d = \infty$. We assume that d is a stopping time with respect to $\{\mathscr{F}_i\}_{i \in \mathbf{N}}$, which is a formal way of saying that it is "known" at any time whether the default has occurred or not.

Definition A random variable d is a *stopping time* with respect to the filtration $\{\mathscr{F}_i\}_{i \in \mathbf{N}}$ if and only if for every $i \in \mathbf{N}$ the event $d \le i$ is $\mathscr{F}_i$-measurable.

There are two basic approaches to credit risk (default) modelling: *reduced* and *structural* models. Structural models assume some economic structure and treat default as a result of real-world phenomena (i.e., a firm's equity becoming negative).

Reduced models, on the other hand, including the one presented here, require no such structure. Default in such models is usually triggered by an exogenous “shock”, and emphasis is on the modelling of the stochastic processes on which the default depends. Brigo and Mercurio (2006, pp. 697 sqq.) offer a good overview of credit models with emphasis on interest rate modelling.

Defaultable claim

We now proceed with some basic definitions. A payoff Y occurring at time t_j[44] is said to be *defaultable* if it can be expressed as:

$$Y = X\left(1-(1-\mathscr{R})1_{j\geq d}\right) \tag{17}$$

where X is a payoff, $\mathscr{R}$ is the (possibly random) recovery rate, $\mathscr{R}[0,1)$, and d denotes the moment of default. Recalling Equation (9), the value, at time t_i, of a generic defaultable claim Y occurring at time t_j is:

$$B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i}B_j^{-1}Y\middle|\mathscr{F}_i\right]$$

this may be expanded, using (17), as:

$$\begin{aligned}
\ldots &= B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i}B_j^{-1}X\left(1-(1-R)1_{j\geq d}\right)\middle|\mathscr{F}_i\right] \\
&= B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i}B_j^{-1}X - 1_{j\geq i}B_j^{-1}X(1-R)1_{j\geq d}\middle|\mathscr{F}_i\right] \\
&= B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq i}B_j^{-1}X\middle|\mathscr{F}_i\right] - B_i \mathrm{E}_{\mathbb{Q}}\left[1_{j\geq \max\{i,d\}}B_j^{-1}X(1-R)\middle|\mathscr{F}_i\right]
\end{aligned}$$

If the time of default, t_d, the exposure X, and the recovery rate, R, are mutually independent, then the expectation in the preceding formula can be further decomposed into a product of expectations. The individual expectations (i.e., the probability of default, expected exposure at default, and the expected recovery rate, respectively) can then be calculated individually using the respective models. This is the case of “no wrong-way risk”.

Throughout the rest of the text, we assume $\mathscr{R} = 0$.[45]

[44]Note, again, that both Y and j might be random.

[45]This is not entirely justified from the practitioner’s point of view, as default events with nonzero recovery is market standard, and the rules for the calculation of regulatory capital reflect this (see, for

Defaultable bond

The simplest defaultable instrument is the defaultable zero-coupon bond with zero recovery rate, or the *defaultable t_j-bond*, $j \in \mathbf{N}$.

Proposition 17 Under the simplifying assumption of no wrong-way risk, the value of the defaultable t_j-bond at time t_i is:

$$(1 - \mathbb{Q}[d \le j | \mathscr{F}_i]) P_{i,j}.$$

Proof. The assumption of no wrong-way risk means that the time of default of the defaultable t_j-bond, d, is independent of B_j. Let us further assume that $i \le j$. Then, using Equation (17), the value of the defaultable bond is:

$$B_i \mathrm{E}_{\mathbb{Q}} \left[1_{i \le j < d} B_j^{-1} \middle| \mathscr{F}_i \right],$$

which decomposes into:

$$B_i \mathrm{E}_{\mathbb{Q}} \left[1_{j<d} \middle| \mathscr{F}_i \right] \mathrm{E}_{\mathbb{Q}} \left[B_j^{-1} \middle| \mathscr{F}_i \right].$$

But $\mathrm{E}_{\mathbb{Q}} \left[1_{i \le j < d} \middle| \mathscr{F}_i \right] = 1 - \mathbb{Q}[d \le j | \mathscr{F}_i]$ where $\mathbb{Q}[d \le j | \mathscr{F}_i]$ is the risk-neutral probability that the debtor will default prior to, or at maturity. The rest of the equation is the price of the (risk-free) zero-coupon bond.

Proposition 18 Let $i, j \in \mathbf{N}$. Then the price of the credit risk of the defaultable t_j-bond at time t_i is:

$$P_{i,j} - R_{i,j} = 1_{d>i} B_i \mathrm{E}_{\mathbb{Q}} \left[B_d^{-1} P_{d,j} \middle| \mathscr{F}_i \right]$$

Credit default swap

The credit default swap (CDS) is a financial instrument that offers one party protection from credit risk. One party (the "long" one, or protection *buyer*) makes

instance, the percentage LGD for collateralized exposures as set out in Basel Committee on Banking Supervision (2006, p. 69)). This is not a problem from the theoretical standpoint, however, as non-zero recovery may be added to the model on an individual basis as a contract-specific feature. Furthermore, the assumption of zero recovery simplifies formulas considerably.

periodic payments to the other party (the “short” one, or protection *seller*). The payments take place until (a) a default of the reference entity, or (b) the maturity of the CDS, whichever comes first. If a default of the reference entity occurs prior to the maturity of the swap, the short party pays to the long party the notional amount. This one-off payment is made immediately.[46] For simplicity we assume that the periodic payments are interest accruals under a fixed interest rate (the “CDS spread”). The fixed rate, or spread, k, is set so that at inception date the value of the swap is zero. This is similar to FRAs and interest rate swaps, as discussed in previous sections (Hull 2012, pp. 548 sqq.).[47]

Let us assume a credit default swap on a unit notional value with maturity t_j, fixed rate (i.e., CDS spread) k, with starting date t_{p_0} and with periodic payments at times $t_{p_1},\dots,t_{p_L}$ where $L \in \mathbf{N}$ and $p_0 < p_1 < p_2 < \cdots < p_L$ (all $\in \mathbf{N}$). The contract pays 1 at time t_d if a default occurs during the life of the contract ($d \le t_L$), or zero otherwise ($d > t_L$). The periodic payments stop at expiration date or upon default, whichever comes earlier.

Proposition 19 The value of the CDS is:

$$B_i \mathrm{E}_{\mathbb{Q}} \left[B_d^{-1} 1_{i \le d \le p_L} - \sum_{l=1}^{L} B_{p_l}^{-1} I\left(k, t_{p_{l-1}}, t_{p_l}\right) 1_{i \le p_l < d} \middle| \mathscr{F}_i \right].$$

4.5.2 Defaultable numéraire and the credit-risk-neutral measure

Simple models rely on the assumption of independence of market factors and the time of default, thus treating the “market model” and the “default model” as two separate tools. Our goal, to the contrary, is to build a model that would allow for a very broad range of possible default-interest rate dependencies. At the same time, we aim to keep the model mathematically tractable. The credit-risk part of our model is based on the notion of *defaultable numéraires*, which we borrow from Schonbucher (2004).[48]

[46]In practice, the actual amount delivered to the protection buyer depends, albeit indirectly, on the recovery and is determined only after the default takes place, using market prices of tradeable bonds issued by the reference entity. We ignore such details regarding the settlement of CDS. Instead, we assume no recovery whatsoever.

[47]This is another simplification, as it can also be a floating rate, such as LIBOR, plus a predefined spread.

[48]The literature on defaultable numéraires is otherwise scarce—perhaps of interest is the working paper by Travis *et al.* (2015) which contains a rigorous theoretical treatment of pricing measures

The key building block of the model is the defaultable zero-coupon bond. (Recall that we assume zero recovery.) One such bond with maturity t_j pays 1 if $j < d$ (the debtor defaults later than t_j or never, i.e. $d = \infty$), The price of such bond at time t_i is denoted $R_{i,j}$. The price satisfies:

$$R_{i,j} = B_i \mathrm{E}_{\mathbb{Q}} \left[1_{i \le j < d} B_j^{-1} \middle| \mathscr{F}_i \right]. \tag{18}$$

Analogically to the default-free money-market account, B_i, let us define C_i, the *defaultable numéraire* or *defaultable money-market account*: starting with a unit of currency at t_0 and at each time t_i reinvesting all proceeds into the defaultable t_{i+1}-bond, the value of the defaultable money-market account at time t_i, $i > 0$, is:

$$C_i = \frac{1_{d>i}}{R_{0,1} R_{1,2} \ldots R_{i-1,i}}. \tag{19}$$

The following definition and lemma provide the theoretical underpinning for the use of C_i as a basis for pricing.

Definition Let $(\Omega, \mathscr{A}, \mu)$ be a measurable space. Let ν be another measure on $\mathscr{A}$. We say that ν is *absolutely continuous* with respect to μ if $\nu(M) = 0$ for all $\mathscr{A}$-measurable events M for which $\mu(M) = 0$ (Lachout 2004, p. 106).

Lemma 20 (Radon–Nikodym theorem.) Let $(\Omega, \mathscr{A}, \mu)$ be a measurable space. Let ν be another measure on $\mathscr{A}$. Then ν is absolutely continuous with respect to μ if and only if there exists a μ-measurable function X such that $\nu(M) = \int_M X \, \mathrm{d}\mu$ for all $\mathscr{A}$-measurable events M (ibid., p. 106).

This allows us to formulate the following:

Proposition 21 There exists an equivalent-martingale measure, $\mathbb{R}$, under which the discounted price process of each defaultable bond is a martingale, where the discounting is done with respect to the defaultable money-market account:

$$R_{i,j} = C_i \mathrm{E}_{\mathbb{R}} \left[1_{i \le j} C_j^{-1} \middle| \mathscr{F}_i \right].$$

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defined on the basis of defaultable numéraires.

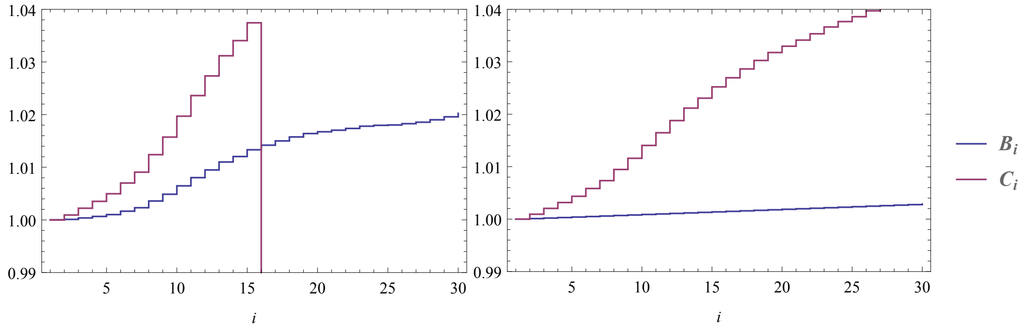

Fig. 11: An illustration of the typical development of B_i and C_i over time. On the left is a sample from $\mathbb{Q}$ where default occurred at t_{16}. On the right is a sample from $\mathbb{R}$.

Figure 11 shows the "typical" development of the two numéraires, B_i and C_i, over time. Unlike traditional numéraires, C_i may be zero which renders C_i^{-1} undefined. However, as Schonbucher (2004, p. 80) notes, this does not prevent $\mathbb{R}$ and its corresponding numéraire from being well-defined. To prevent ambiguities, we shall define $C_i^{-1} = 0$ whenever $C_i = 0$, which follows the spirit of the definition given in (19). The following proposition gives us a tool to price defaultable claims of the form $1_{d>j}X$ where X is a payoff that is known at time t_j but is paid only if a default has not occurred by that time.

Proposition 22 Let $i, j \in \mathbf{Z}, i \leq j$ and let X be $\mathscr{F}_j$-measurable. Then the price of X conditional on the counterparty not defaulting up to t_j is:

$$B_i \mathrm{E}_{\mathbb{Q}}\left[1_{d>j} B_j^{-1} X \middle| \mathscr{F}_i\right] = C_i \mathrm{E}_{\mathbb{R}}\left[C_j^{-1} X \middle| \mathscr{F}_i\right]$$

Credit-risk-neutral property of $\mathbb{R}$

The important property of $\mathbb{R}$ is that under it, the default almost surely never occurs, i.e., it assigns zero probability to any events that involve default in a finite time:

Proposition 23 ($\mathbb{R}$ is credit-risk-neutral.)

$$\mathbb{R}[d = \infty] = 1.$$

Proof. Let us fix some $i, j \in \mathbf{Z}, i < j$. Then, with the use of Proposition 22,

$$\mathbb{R}[d = j | \mathscr{F}_i] = \mathrm{E}_{\mathbb{R}}\left[1_{d=j} \middle| \mathscr{F}_i\right] = C_i^{-1} B_i \mathrm{E}_{\mathbb{R}}\left[1_{d>j} C_j B_j^{-1} 1_{d=j} \middle| \mathscr{F}_i\right] = 0,$$

because $1_{d>j}1_{d=j}$ is always zero. Now,[49]

$$\mathbb{R}[d<\infty] = \mathrm{E}_{\mathbb{R}}\left[\sum_{j\in\mathbf{Z}}\mathbb{R}\left[d=j|\mathscr{F}_{j-1}\right]\right] = \mathrm{E}_{\mathbb{R}}\left[\sum_{j\in\mathbf{Z}}0\right] = 0, \qquad \text{Q.E.D.}$$

4.5.3 Defaultable bond yield dynamics

In this section we are going to define the credit spread and introduce the dynamics of yields of defaultable bonds. Finally, we will relate the dynamics to the risk-free forward rate dynamics introduced in preceding sections.

Motivation

In the preceding sections we found a probability measure, $\mathbb{R}$, than can be used to price any claim that is conditional upon default of the reference entity.[50] To price such a claim, however, we need to know the conditional distributions of the respective variables under $\mathbb{R}$. We can recover those distributions by generating a random sample of scenarios from $\mathbb{R}$. In order to do this, we need to introduce some assumptions about the dynamics of defaultable bond yields.

The credit spread in our model is defined as the difference between the return on the defaultable and risk-free variants of the same bond. The credit spread is denoted by an indexed letter s.

Definitions

The following definitions introduce the defaultable forward rate and the forward credit spread. The relationship between the (spot) yield and the forward yield is analogical to the relationship between the (spot) rate and the forward rate.

Definition Let $i, m \in \mathbf{Z}, i < m$. The *defaultable forward rate* for the period (t_{m-1}, t_m),

[49]We use the following lemma: for any random variable X and any σ-algebra $\mathscr{A}$: $\mathrm{E}[\mathrm{E}[X|\mathscr{A}]] = \mathrm{E}[X]$.

[50]We have so far tacitly assumed that the counterparty and the issued of the bond are the same entity and that the claims from the derivative contracts are of the same seniority as the bonds. This should not pose a problem, however, as the term “defaultable bond” is really a shortcut for a theoretical object that captures the credit risk of the counterparty.

at time t_i, is an $\mathscr{F}_i$-measurable random variable $g_{i,m}$ such that:

$$R_{i,j} = 1_{d>i} \exp\left(- \sum_{m=i+1}^{j} g_{i,m} \, \mathrm{d}t_m \right) \qquad \text{for all } j \in \mathbf{Z}, j > i.$$

Definition Let $i, m \in \mathbf{Z}, i < m$. The random variable $s_{i,m} = g_{i,m} - f_{i,m}$ is called the *forward credit spread* for the period (t_{m-1}, t_m) at time t_i.

Defaultable bond yield dynamics

We will model the dynamics of the forward defaultable rates under the (historical) measure $\mathbb{P}$ using the same powerful model as for the forward risk-free rates. Borrowing from Section 4.2.2, where the Heath–Jarrow–Morton framework was introduced, we assume:

$$\mathrm{d}g_{i,j} = \lambda_{i,j} \, \mathrm{d}t_i + \boldsymbol{\rho}_{i,j}^{\top} \mathrm{d}\boldsymbol{W}_i \tag{20}$$

where $\mathrm{d}g_{i,j} = g_{i,j} - g_{i-1,j} = \mathrm{d}f_{i,j} + \mathrm{d}s_{i,j}$, and the processes λ and $\boldsymbol{\rho}$ are $\{\mathscr{F}_i\}$-adapted. Let us recall that $\boldsymbol{W}$ is an n-dimensional Wiener process under $\mathbb{P}$, sampled at discrete time points.

Note that the source of "randomness" in forward defaultable rates, the n-dimensional process $\boldsymbol{W}$ in Equation (20), is the same as for risk-free rates. It is then up to our specification of the volatility processes $\boldsymbol{\sigma}, \boldsymbol{\rho}$ to determine the relationship between the prices of risk-free and defaultable bonds. In other words, the processes $\boldsymbol{\sigma}, \boldsymbol{\rho}$ determine the dynamics of probabilities of default and also wrong-way risk.

No-arbitrage condition for defaultable bond yields

Definition Analogically to Section 4.2.2, let us define the process $\{\boldsymbol{U}_i\}_{i \in \mathbf{Z}}$ as follows:

$$\mathrm{d}\boldsymbol{U}_i = \mathrm{d}\boldsymbol{W}_i - \boldsymbol{\beta}_i \, \mathrm{d}t_i$$

for some $\mathscr{F}$-previsible n-dimensional process $\{\boldsymbol{\beta}_i\}_{i \in \mathbf{Z}}$ satisfying the same technical conditions as $\{\gamma_i\}_{i \in \mathbf{Z}}$.

Proposition 24 There exists a measure $\mathbb{R}$, absolutely continuous with respect to $\mathbb{P}$ (and thus $\mathbb{Q}$), such that $\boldsymbol{U}$ is a Wiener process under $\mathbb{R}$. (Proof follows from the fact that $\boldsymbol{W}$ is a Wiener process under $\mathbb{P}$, from the fact that $\mathbb{P}$ is equivalent to $\mathbb{Q}$, and from the Girsanov theorem.)[51]

The use of the symbol $\mathbb{R}$ in Proposition 24 is correct, as we are going to derive conditions under which $\mathbb{R}$ is indeed the credit-risk neutral measure we have defined.

Let us find the necessary and sufficient condition for $\mathbb{R}$ being the measure under which, for any $j \in \mathbf{N}$, the process for the discounted price of the defaultable t_j-bond, that is, $\{C_i^{-1} R_{i,j}\}_{i \in \mathbf{Z}}$, is a martingale. By applying similar reasoning as in Section 4.2.2, we obtain the condition:

$$\mathrm{E}_{\mathbb{R}}\left[\left.\frac{C_i^{-1} R_{i,j}}{C_{i-1}^{-1} R_{i-1,j}}\right|\mathscr{F}_{i-1}\right] = \mathrm{E}_{\mathbb{R}}\left[\left. 1_{d>i} \exp\left(-\sum_{m=i+1}^{j} \mathrm{d}g_{i,m}\,\mathrm{d}t_m\right)\right|\mathscr{F}_{i-1}\right] = 1.$$

First, note that $1_{d>i}$ is almost surely 1 under $\mathbb{R}$ (see Proposition 23), so we can leave the term out of the expression without changing the expectation and continue analogically to Section 4.2.2. The details are not reproduced here as the logic is the same as for forward risk-free rates. It follows that the no-arbitrage condition for the drift of the forward defaultable rate is:

$$\lambda_{i,j} = \boldsymbol{\rho}_{i,j}^{\top}\left(\sum_{m=i+1}^{j} \boldsymbol{\rho}_{i,m}\,\mathrm{d}t_m - \tfrac{1}{2}\boldsymbol{\rho}_{i,j}\,\mathrm{d}t_j - \boldsymbol{\beta}_i\right). \tag{21}$$

Similarly to $\boldsymbol{\sigma}$, we assume time-invariance of $\boldsymbol{\rho}$:

$$\boldsymbol{\rho}_{i,j} = \boldsymbol{\rho}_{i+k,j+k} \qquad \text{for all } i,j,k \in \mathbf{N},$$

and we define the symbol $\boldsymbol{r}_m = \boldsymbol{\rho}_{i,i+m}$ for $m \in \{1,\ldots,M\}$.

Generating credit-risk-neutral scenarios

To generate $\mathbb{R}$-random yield scenarios, we need a formula for $\mathrm{d}g_{i,j}$ in terms of $\mathrm{d}\boldsymbol{U}_i$, which we know is conditionally normally distributed under $\mathbb{R}$. If we combine Equations (20) and (21) and the definition of $\boldsymbol{U}$ we obtain. The following is the

[51]The Girsanov theorem holds for the Wiener process, which is a stochastic process defined indexed by $\mathbf{R}$, not $\mathbf{N}$. Nevertheless, the results carry over to the discrete case as we may imagine the discrete processes as being sampled at predefined time points from continuous processes. This also explains our slight abuse of notation, for instance we treat $\boldsymbol{W}$ either as a continuous process and a discrete one, depending on the circumstances.

overview of the formulas that are used to sample random scenarios of the forward risk-free and defaultable bond yields, respectively with respect to each probability measure:

$$
\begin{aligned}
&\text{To generate } \mathbb{P}\text{-scenarios:} && \mathrm{d}f_{i,j} = u_{j-i}\,\mathrm{d}t - \boldsymbol{s}_{j-i}^{\top}\boldsymbol{\gamma}_i\,\mathrm{d}t + \boldsymbol{s}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{W}_i \\
& && \mathrm{d}g_{i,j} = y_{i,j}\,\mathrm{d}t - \boldsymbol{r}_{j-i}^{\top}\boldsymbol{\beta}_i\,\mathrm{d}t + \boldsymbol{r}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{W}_i \\
&\text{To generate } \mathbb{Q}\text{-scenarios:} && \mathrm{d}f_{i,j} = u_{j-i}\,\mathrm{d}t + \boldsymbol{s}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{V}_i && (22) \\
& && \mathrm{d}g_{i,j} = y_{j-i}\,\mathrm{d}t + \boldsymbol{r}_{j-i}^{\top}\left(\boldsymbol{\gamma}_i - \boldsymbol{\beta}_i\right)\mathrm{d}t + \boldsymbol{r}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{V}_i \\
&\text{To generate } \mathbb{R}\text{-scenarios:} && \mathrm{d}f_{i,j} = u_{j-i}\,\mathrm{d}t + \boldsymbol{s}_{j-i}^{\top}\left(\boldsymbol{\beta}_i - \boldsymbol{\gamma}_i\right)\mathrm{d}t + \boldsymbol{s}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{U}_i && (23) \\
& && \mathrm{d}g_{i,j} = y_{j-i}\,\mathrm{d}t + \boldsymbol{r}_{j-i}^{\top}\,\mathrm{d}\boldsymbol{U}_i && (24)
\end{aligned}
$$

An example of a scenario from $\mathbb{R}$ is shown in Figure 12.

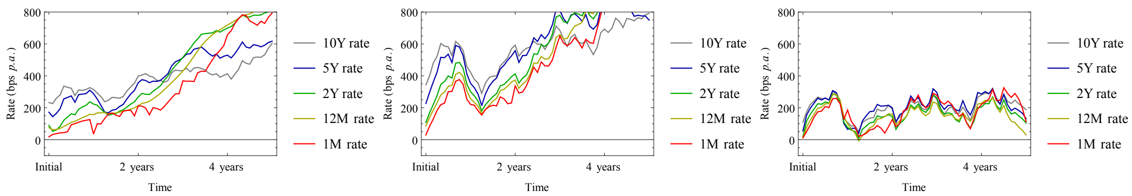

Fig. 12: An example of a random $\mathbb{R}$ scenario, shown from left to right: forward risk-free rates, forward defaultable bond yields, forward credit spreads.

4.6 Generating market-risk-neutral scenarios

Definition By a *forward rate scenario* we shall denote any realization of the random matrix:

$$
\begin{bmatrix}
f_{0,1} & f_{0,2} & \cdots & f_{0,M} \\
f_{1,2} & f_{1,3} & \cdots & f_{1,M+1} \\
\vdots & \vdots & \ddots & \vdots \\
f_{T,T+1} & f_{T,T+2} & \cdots & f_{T,T+M}
\end{bmatrix},
$$

where M is the maximum tenor considered and T is the *horizon* of the scenario, both given in units of $\mathrm{d}t$. The rates $f_{0,j}$ are the *starting rates* of the scenario (given as input to the simulation routine).[52]

[52]The starting time of the scenario is customarily set to be t_0, which is without generality as the time indices do not have any special assumptions attached to them.

Our analysis relies on random sampling from the empirical distributions of the variables in question, e.g., the loss from a counterparty's default. Because the distribution depends on the probability measure with respect to which the random sample was taken, we shall prepend the symbol for the measure ("$\mathbb{P}$-scenario") whenever ambiguity might arise.

Let us set $\boldsymbol{\gamma}_i = \mathbf{0}$ in (7) and denote the resulting expression by the indexed symbol u:

$$u_m = \boldsymbol{s}_m^\top \left(\boldsymbol{s}_1 + \boldsymbol{s}_2 + \cdots + \boldsymbol{s}_{m-1} + \tfrac{1}{2}\boldsymbol{s}_m\right) \mathrm{d}t.$$

Recall from (5) the definition of the process $\boldsymbol{V}$. We know that $\boldsymbol{V}$ is a Wiener process with respect to $\mathbb{Q}$. The following formulas may be used to generate forward rate scenarios:

$$\text{To generate } \mathbb{P}\text{-scenarios:} \qquad \mathrm{d}f_{i,j} = \mu_{i,j}\,\mathrm{d}t + \boldsymbol{s}_{j-i}^\top\,\mathrm{d}\boldsymbol{W}_i.$$
$$\text{To generate } \mathbb{Q}\text{-scenarios:} \qquad \mathrm{d}f_{i,j} = u_{j-i}\,\mathrm{d}t + \boldsymbol{s}_{j-i}^\top\,\mathrm{d}\boldsymbol{V}_i. \tag{25}$$

We now derive analytically the joint distribution of all future forward rates $f_{i,j}$, $i, j \in \mathbf{N}$ for all tenors conditional on the starting forward rates $f_{0,j}$. We do this by vectorizing, or "stacking" the random part of the forward rate scenario in a single vector:[53],[54]

$$\vec{\boldsymbol{f}} = \left[f_{1,1}, f_{1,2}, \cdots, f_{1,T+M}, f_{2,1}, \cdots, f_{2,T+M}, \cdots, f_{T+M,1}, \cdots, f_{T+M,T+M}\right]^\top. \tag{26}$$

Clearly, for $i, j \in \{1, \ldots, T+M\}$, the $((T+M)(i-1)+j)^{\text{th}}$ (or "representative") term of $\vec{\boldsymbol{f}}$ is $f_{i,j}$. To derive the distribution of $\vec{\boldsymbol{f}}$, we expand each term as:

$$f_{i,j} = f_{0,j} + \mathrm{d}f_{1,j} + \mathrm{d}f_{2,j} + \cdots + \mathrm{d}f_{i,j}. \tag{27}$$

We denote the vector of starting rates by $\boldsymbol{f}_0 = \left[f_{0,1}, f_{0,2}, \cdots, f_{0,T+M}\right]^\top$. Analogically to (26), we define the vector $\mathrm{d}\vec{\boldsymbol{f}}$ whose representative term is $\mathrm{d}f_{i,j}$.

[53]The expressions $f_{i,j}$, $\mathrm{d}f_{i,j}$, etc. are undefined for $i \geq j$ (and so are $\boldsymbol{s}_m$, u_m for $m \leq 0$), which renders undefined approx. half of the entries in the vectors and matrices below. This is in fact taken care of in the *Mathematica* code, but the undefined terms are kept in the formulas in this text for the sake of clarity and ease of presentation—allowing the elements to be indexed simply from 1 to $T+M$ etc.

[54]The "stacked" matrices and related objects are marked thus: $\vec{\boldsymbol{f}}$, $\vec{\mathbf{S}}$, etc., so that there is no confusion with symbols used elsewhere.

The next step requires some auxiliary definitions. For $n \in \mathbf{N}$ let $\boldsymbol{e}_n$ be the vector of n ones, let $\mathbf{E}_n = \boldsymbol{e}_n \boldsymbol{e}_n^\top$ be the $n \times n$-matrix of ones, let $\mathbf{I}_n$ be the $n \times n$ identity matrix, and let $\mathrm{Low}(\mathbf{X})$ be the matrix $\mathbf{X}$ with all elements *above*, but not on, the diagonal replaced by zeros. Let $\otimes$ denote the Kronecker product. We define the "accumulating matrix" as:

$$\mathbf{A} = \mathrm{Low}(\mathbf{E}_{T+M}) \otimes \mathbf{I}_{T+M}.$$

We obtain the "stacked" version of (27):

$$\vec{\boldsymbol{f}} = \boldsymbol{e}_{T+M} \otimes \boldsymbol{f}_0 + \mathbf{A}\,\mathrm{d}\vec{\boldsymbol{f}}. \tag{28}$$

We now expand the $\mathrm{d}f_{i,j}$ terms using (25). Let $\vec{\mathbf{S}}$ be the following "volatility" matrix of dimensions $(T+M)^2 \times N(T+M)$:

$$\vec{\mathbf{S}} = \begin{bmatrix} \boldsymbol{s}_0^\top & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{s}_1^\top & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{s}_{T+M-1}^\top & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{s}_{-1}^\top & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{s}_0^\top & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \boldsymbol{s}_{T+M-2}^\top & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{s}_0^\top \end{bmatrix}$$

and let $\mathrm{d}\vec{\boldsymbol{V}}$ be the vector of independent normal random variables:

$$\mathrm{d}\vec{\boldsymbol{V}} \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{N\times(T+M)}\,\mathrm{d}t\right). \tag{29}$$

The matrix $\vec{\mathbf{S}}$ and the vector $\mathrm{d}\vec{\boldsymbol{V}}$ are defined so that the representative term of the vector $\vec{\mathbf{S}}\,\mathrm{d}\vec{\boldsymbol{V}}$ is the random variable $\boldsymbol{s}_{j-i}^\top\,\mathrm{d}\boldsymbol{V}_i$. Finally, let $\vec{\boldsymbol{u}}$ be defined analogically to (26) but with the representative term u_{j-i}. We obtain the "stacked" version of (25):

$$\mathrm{d}\vec{\boldsymbol{f}} = \vec{\boldsymbol{u}}\,\mathrm{d}t + \vec{\mathbf{S}}\,\mathrm{d}\vec{\boldsymbol{V}}. \tag{30}$$

From (28) and (30) we obtain the formula that can be used directly to generate random $\mathbb{Q}$-scenarios of forward rates:

$$\vec{\boldsymbol{f}} = \boldsymbol{e}_{T+M} \otimes \boldsymbol{f}_0 + \mathbf{A}\vec{\boldsymbol{u}}\,\mathrm{d}t + \mathbf{A}\vec{\mathbf{S}}\,\mathrm{d}\vec{\boldsymbol{V}}. \tag{31}$$

From (29), (31) and the properties of the multivariate normal distribution follows that the joint distribution of the "stacked" forward rates is:

$$\vec{\boldsymbol{f}} \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\boldsymbol{e}_{T+M} \otimes \boldsymbol{f}_0 + \mathbf{A}\vec{\boldsymbol{u}}\,\mathrm{d}t,\ \mathbf{A}\vec{\mathbf{S}}\vec{\mathbf{S}}^\top \mathbf{A}^\top\,\mathrm{d}t\right).$$

5 Empirical part

5.1 CVA formula

Remark 25 Although we allow for $d = \infty$, the time domain does not include ∞ and none of the variables $f_{i,j}, B_i, V_i$, etc. are defined for $i = \infty$. To prevent ambiguity, if V is any process that gives the price of an instrument, $V_\infty = 0$. We also assume $\infty \times 0 = 0$.

Definition Let the residual value[55] of an instrument at time t_i be V_i for all $i \in \mathbf{Z}$. Then the *credit value adjustment*, or *CVA*, for such instrument at time t_i is:

$$CVA_i = B_i \mathrm{E}_{\mathbb{Q}}\left[B_d^{-1} V_d^+ \middle| \mathscr{F}_i\right] 1_{i>d}$$

The CVA of an instrument is the price of the credit risk associated with positive exposure to our counterparty. V_d^+ is the value that we lose at the time of default, t_d. The loss is zero if the default never occurs.

Definition For an instrument whose residual value at time t_i is V_i, define W_i to be the premium, at time t_i, of a call option on the residual value of the instrument one period ahead (i.e., at t_{i+1}) with zero strike:[56]

$$W_i = B_i \mathrm{E}_{\mathbb{Q}}\left[B_{i+1}^{-1} V_{i+1}^+ \middle| \mathscr{F}_i\right].$$

The following proposition gives us a formula to price CVA using the defaultable numéraire and the credit-risk-neutral pricing measure $\mathbb{R}$. It shows that CVA can be calculated using the premia of the one-period-ahead call options, discounted by the defaultable money-market account:

Proposition 26

$$CVA_i = 1_{d>i} V_i^+ + C_i \mathrm{E}_{\mathbb{R}}\left[\sum_{j=i}^{\infty} C_j^{-1}\left(W_j - V_j^+\right) \middle| \mathscr{F}_i\right] \tag{32}$$

[55]The *residual value* is the value of all *future* cash flows/claims associated with the instrument.

[56]The option pays the positive value of the instrument, V_{i+1}^+, at time t_{i+1}. Note that W_i might be *less* than V_i^+ if the instrument pays out cash at time t_i.

Proof. From the definitions of CVA_i and W_i and the properties of $\mathbb{R}$:

$$\begin{aligned} CVA_i &= \sum_{j=i+1}^{\infty} B_i \mathrm{E}_{\mathbb{Q}}\left[(1_{d>j-1} - 1_{d>j}) B_j^{-1} V_j^+ \middle| \mathscr{F}_i \right] \\ &= \sum_{j=i+1}^{\infty} \Big(B_i \mathrm{E}_{\mathbb{Q}}\left[1_{d>j-1} B_{j-1}^{-1} B_{j-1} \mathrm{E}_{\mathbb{Q}}\left[B_j^{-1} V_j^+ \middle| \mathscr{F}_{j-1} \right] \middle| \mathscr{F}_i \right] - \\ &\qquad\qquad - B_i \mathrm{E}_{\mathbb{Q}}\left[1_{d>j} B_j^{-1} V_j^+ \middle| \mathscr{F}_i \right] \Big) \\ &= \sum_{j=i+1}^{\infty} \Big(C_i \mathrm{E}_{\mathbb{R}}\left[C_{j-1}^{-1} W_{j-1} \middle| \mathscr{F}_i \right] - C_i \mathrm{E}_{\mathbb{R}}\left[C_j^{-1} V_j^+ \middle| \mathscr{F}_i \right] \Big), \end{aligned}$$

from which (32) follows after some further rearranging, Q.E.D.

5.2 Wrong-way risk

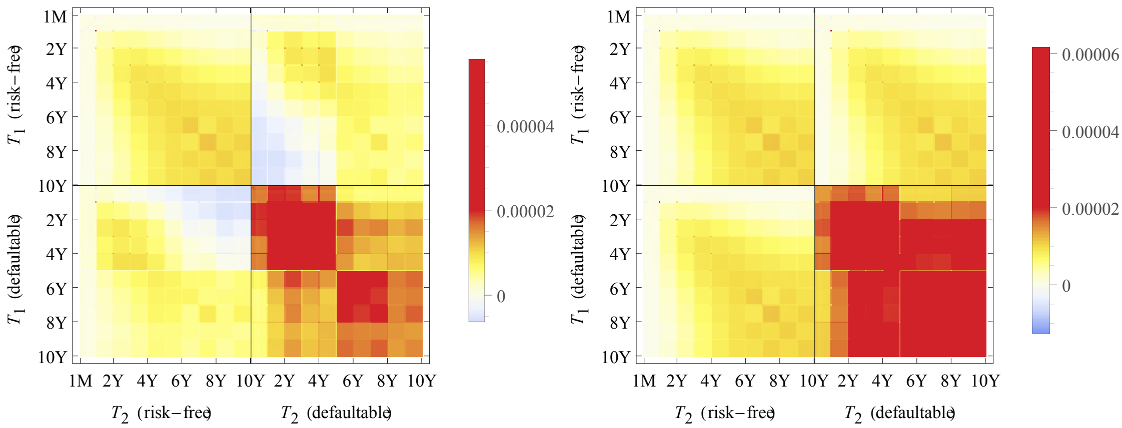

Fig. 13: The covariance matrices used in the final specification. On the left: *historical* covariance matrix, on the right: the *zero* covariance matrix.

To quantify the impact of wrong-way risk, two covariance matrices (or, equivalently, two volatility matrices **S**) were considered. The matrices are named *historical* and *zero*. The *historical* covariance matrix was directly estimated from historical risk-free interest rates and credit spreads.

For the *zero* specification, the aim was to “set the correlations” between risk-free interest rates and credit spreads were to zero while covariances among either set were kept unchanged. See the `ApplyToWrongWayRisk` function for details on how this is done. The variance matrix **S** was obtained using the four most significant columns of the singular value decomposition of the “target” matrix.

5.3 Results

The CVA was calculated by evaluating Equation (32) using the Monte–Carlo method. For each CVA estimate, Between 200–500 scenarios was generated, each involving the following steps:

- A joint scenario of risk-free and defaultable bond yields was sampled from $\mathbb{R}$ using Equations (24) and (23). The starting rates at are shown in Figure 14.
- For each month into the scenario, V_i^+, B_i etc. were computed directly using their respective definitions.
- For each month into the scenario, W_i was estimated by generating 100 risk-free rate $\mathbb{Q}$-scenarios using Equation (25).

The starting rates at are shown in Figure 14.

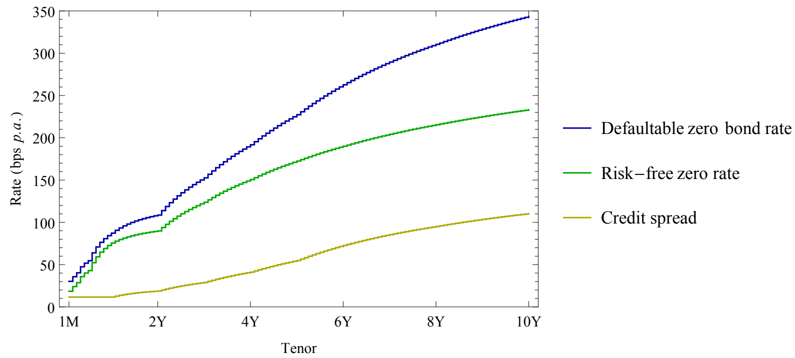

Fig. 14: The starting rates for the calculation of CVA using the Monte–Carlo method. The risk-free rate is the bootstrapped USD LIBOR and swap curve on 1 June 2015, the spread is derived from CDS spreads on AIG senior unsecured debt quoted on 29 May 2015.

Table 1: CVA at inception under two wrong-way risk specifications (*historical* and *zero*, see the main text), in bps times notional value over the lifetime of the IRS. The results are given in the form $x \pm y$, where x is the estimate of the CVA and y is the half-width of the bootstrapped confidence interval at the $\alpha = 0.01$ level.

Contract	CVA_0	
	historical	*zero*
2Y IRS v 6M LIBOR	-3 ± 2	2 ± 2
5Y IRS v 6M LIBOR	106 ± 25	137 ± 24

We refer to Table 1 for the results of the numerical study. The model correctly assigns lower credit risk to the shorter-term contract. However, the *zero* specification (i.e., no wrong-way risk, or credit spreads uncorrelated with risk-free rates) results in *higher* CVA than the baseline, which is clearly unwarranted: the counterparty to the IRS is assumed to pay the floating rate which, under the *historical* alternative, is higher when the counterparty is more likely to default, as opposed to the *zero* alternative. Hence, the opposite result was expected from the model. Another lurking problem might be the negative estimate of CVA (significant even at the 0.01 level), sharply in contradiction with the trivial fact that any unilateral price of credit risk such as CVA must be non-negative. This is likely the result of the assumption of normality of forward rate jumps. That, on the other hand, is one of the central components of the Heath–Jarrow–Morton framework and therefore will be difficult to remedy.

6 Conclusion

In my thesis I have built and implemented a comprehensive no-arbitrage model for the term structure of interest rates that is based on the discrete Heath–Jarrow–Morton framework. The model captures the real-world and risk-neutral behaviour of risk-free interest rates as well as yields of defaultable bonds. I have successfully calibrated the model to historical volatilities of interest rates and credit spread, including the correlations between the two types of financial indicators. In order to price credit risk, I have applied the concept of defaultable numéraires and derived a formula for the calculation of credit value adjustment for a generic financial derivative. The fact that the model is capable of matching quite precisely the real-world interdependence between interest rates and credit spreads makes it suitable for modelling wrong-way risk and quantifying its impact on CVA. I have conducted a numerical study in the attempt to isolate the effect of wrong-way risk on the CVA of a vanilla IRS. The results are mixed, which motivates further research on the specification of the model. The implementation of the entire model, written in *Mathematica*, is available as a supplement to this thesis. The code is fully documented and is meant to be further developed by fellow researchers.

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7 Appendix

Help on functions

The following is the overview of most user-defined functions, variables and symbols in the Mathematica implementation of the wrong-way risk model.

Data storage and manipulation

Data storage

Variables used to store financial data.

CDSAIG1Dates are the dates of the individual observations in *CDSAIG1Rates*.

CDSAIG1DatesMonthlyPositions gives the positions of
the first available observation in each respective calendar month in *CDSAIG1Dates*.

CDSAIG1ForwardsMonthly are the implied forward AIG senior–debt CDS spreads. The individual observations correspond to the first available day of each calendar month. Each observation is a list $\{f_1,f_2,\ldots\}$ of the forward spreads implied from CDS spreads quoted on the particular date, where f_k denotes the forward spread for the one–month period starting in $(k-1)$ months and ending in k months. The forward spreads compound continuously.

CDSAIG1Rates is the matrix of spreads of CDS on AIG senior debt (*p.a.* in Act/360 convention, USD, 40% recovery, Bloomberg ticker: AIG CDS USD SR *tenor* D14 Curncy), indexed by *CDSAIG1Dates* and *CDSAIG1Tenors* (see the respective descriptions). Any missing data are denoted by *Missing*[*NotAvailable*]. *CDSAIG1Rates* are supposed to be altered by data–patching routines (see the respective sections).

CDSAIG1RawData contains the raw data (AIG CDS spreads on senior debt, Act/360 convention, USD, 40% recovery, Bloomberg ticker: AIG CDS USD SR *tenor* D14 Curncy) imported from Excel.

CDSAIG1Tenors is the list of the respective tenors (given as text) in *CDSAIG1Rates*.

CDSAIG1TenorsN is the list of the respective tenors (given in years) in *CDSAIG1Rates*.

CDSAIG1Times are the dates (given in years since the first observation) of the individual observations in *CDSAIG1Rates*.

JPYDates are the dates of the individual observations in *JPYRates*.

JPYDatesMonthlyPositions gives the positions of
the first available observation in each respective calendar month in *JPYDates*.

JPYForwardsMonthly are the JPY implied forward interest rates. The individual observations correspond to the first available day of each calendar month. Each observation is a list $\{f_1,f_2,\ldots\}$ of the forward rates implied from money–market fixings and closing swap rates for the particular date, where f_k denotes the rate rate for the one–month period starting in $(k-1)$ months and ending in k months. The forward rates compound continuously.

JPYRates is the matrix of JPY interest rates *p.a.*, indexed by *JPYDates* and *JPYTenors* (see the respective descriptions). Any missing data are denoted by *Missing*[*NotAvailable*]. *JPYRates* are supposed to be altered by data–patching routines (see the respective sections).

JPYRatesOriginal gives the original JPY interest rates *p.a.*, as parsed from *JPYRawData*. (See *JPYRates* for details.)

JPYRatesPatchStartingDate[*tenor*] returns the index *i* such that any missing values of *JPYRates*⟦*j*,*tenor*⟧ for $j<i$ are to be kept (i.e., not patched).

JPYRawData contains the raw data (JPY interest rates) imported from Excel.

JPYTenors is the list of the respective tenors (given as text) in *JPYRates*.

JPYTenorsN is the list of the respective tenors (given in years) in *JPYRates*.

JPYTimes are the dates (given in years since the first observation) of the individual observations in *JPYRates*.

USDCalibrationData
is a data matrix created by stacking 60 corresponding observations from *USDForwardsMonthly* and defaultable bond yields (created synthetically as *USDForwardsMonthly* + *CDSAIG1ForwardsMonthly*). The columns up to *M* are risk–free rates and the remaining columns are defaultable bond yields.

USDDates are the dates of the individual observations in *USDRates*.

USDDatesMonthlyPositions gives the positions of the first available observation in each respective calendar month in *USDDates*.

USDForwardsMonthly are the USD implied forward interest rates. The individual observations correspond to the first available day of each calendar month. Each observation is a list $\{f_1, f_2, \ldots\}$ of the forward rates implied from money–market fixings and closing swap rates for the particular date, where f_k denotes the rate rate for the one–month period starting in $(k-1)$ months and ending in k months.

USDRates is the matrix of USD interest rates *p.a.*, indexed by *USDDates* and *USDTenors* (see the respective descriptions). Any missing data are denoted by *Missing*[*NotAvailable*].

USDRatesOriginal gives the original USD interest rates *p.a.*, as parsed from *USDRawData*. (See *USDRates* for details.)

USDRawData contains the raw data (USD interest rates) imported from Excel.

USDTenors is the list of the respective tenors (given as text) in *USDRates*.

USDTenorsN is the list of the respective tenors (given in years) in *USDRates*.

USDTimes are the dates (given in years since the first observation) of the individual observations in *USDRates*.

Data manipulation

Functions used to work with imported financial data.

BootstrapParRates[{r_1,…,r_m}, *k*, *T*, *ξ*]
returns the vector of zero rates corresponding to maturities {*T*,2*T*,…,*mT*}, calculated by bootstrapping the par rates {r_{k+1},…,r_m}, with initial money–market rates {r_1,…,r_k}.
For the money–market, linear accruals and a single (bullet) interest installment are assumed (for $1 \le i < n$, the interest payment is $T \times \xi \times i \times r_i$).
For the par bond, linear accruals and regular interest payments are assumed (for $n \le i \le m$, interest is paid at times {*T*,2*T*,…,*iT*}, each equal to $T \times \xi \times r_i$).
The returned rates are expressed in continuous compounding.

PatchData[…] fills missing values with estimates based on neighbouring tenors (see code for details).

ToContinuousCompounding[{r_1,…,r_n}, {t_1,…,t_n}, *ξ*]
returns the vector of continuously–compounded zero rates corresponding to maturities {t_1,…,t_n} calculated from the money–market rates {r_1,…,r_n} under the assumption that the money–market loans have linear accruals, year fraction factor *ξ* and a single (bullet) interest installment.

ToForwardRates[*tenorsN*, *rates*, *times*, *k*, *T*, *ξ*]
yields the forward rates {f_1,f_2,…} where f_j is the forward rate for the period ($times_{j-1}$,$times_j$). The forward rates are derived from *rates*, which is a mix of money–market and swap rates (or par rates), corresponding to *tenorsN* with linear accuals and year fraction factor *ξ*. It is assumed that the *rates* for tenors up to and including $k \times T$ are money–market rates (i.e., with bullet interest), and the rest are rates correposonding to swaps (or par bonds) with interest payments at times {*T*, 2*T*, …}. The *tenorsN*, *times*, and *T* are assumed to be given in years. The *rates* are assumed to compound linearly. The output forward rates compound continuously. The *tenorsN* and *rates* must be of the same length. Any missing rates (should be indicated by Missing[…]) are ignored.

Model calibration and scenario generation

Calibration to historical data

Functions and variables related to calibration of the wrong-way risk model.

ApplyToOffDiagonalBlocks[*f*, *X*]
applies *f* to the upper right and lower left blocks of *X*. *X* must be a square matrix of even size.

ApplyToWrongWayRisk[*f*, *Σ*]
applies *f* to the "wrong–way risk component" of the covariance matrix *Σ*. It is assumed that *Σ* is the covariance matrix of the "stacked" vector {$f_{i,i+1}$, …, $g_{i,i+1}$, …} of risk–free and defaultable bond yields. The function transforms *Σ* to extract its "wrong–way risk component", then applies *f* to it and finally applies all transformations in reverse. The "wrong–way risk component" is really the matrix of correlations between the risk–free rates and credit spreads. The function preserves as much as possible the internal correlations among either set as well as the variances of each risk–free rate or credit spread.

DriftCalibrationWeights[{l_1,…,l_N}]

returns the vector {w_1,…,w_{l_N}} of weights used to steer the one–period–ahead drift of the forward curve to the desired levels. (The calculated results are cached.)

EstimatedParameterConfidenceInterval[*param*, *data*, *n*, α]

estimates the confidence interval for *param*[*data*] at level α (.01 by default) by bootstrapping the *data* *n* times.

ExpectedMeanRevertingSpreadDrifts[{$s_{i-1,i}$, …, $s_{i-1,M-1}$}]

gives the "desired" drift of the credit spread $dg_{i,i+m} - df_{i,i+m}$ under ℝ conditional on the current credit spreads $s_{i-1,i}$, …. The "desired" drifts are calculated so that the spreads are mean–reverting. An adjustment is made to prevent negative credit spreads with 99% probability (see code for details).

FindFitOwn[*data*, *expr*, *pars*]

returns the values of *pars* (in the form of replace rules) that make *expr* closest to *data*. The following options may be given:

Weights → {w_1,w_2,…} for weighted sum of squares,

LossFunction → *f* to use a loss function *f* (default is least squares),

and all options taken by *NMinimize*.

FindVolatility[Σ, *N*]

returns the matrix **S** = {$\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …} obtained from the *N* most significant columns of the singular value decomposition of the covariance matrix Σ.

Nfac

is the number of factor of the model.

RestAppendSpreadDrift[{$f_{i-1,i}$, …, $f_{i-1,i+M-1}$, $g_{i-1,i}$, …, $g_{i-1,i+M-1}$}]

returns the vector {$f_{i-1,i+1} + s_1^{\mathsf{T}} \beta_i\, dt$, …, $f_{i-1,i+M} + s_M^{\mathsf{T}} \beta_i\, dt$, $g_{i-1,i+1}$, …, $g_{i-1,i+M}$}. It is the "variable" part of the computation of the composite curve {$f_{i,i+1}$, …, $f_{i,i+M}$, $g_{i,i+1}$, …, $g_{i,i+M}$} under the credit–risk–neutral measure ℝ and the assumption of constant γ_i. The vector β_i = {$\beta_{i,1}$, …, $\beta_{i,N}$} is chosen so that it makes the drift of the spread, $dg_{i,i+m} - df_{i,i+m}$, as close as possible to the target value (see *ExpectedMeanRevertingSpreadDrifts*) for all $m \in$ {1, …, *M*}. The function uses equally weighted least squares. Note: after every recalibration, *UpdateRestAppendSpreadDrift*[] must be run to update the definition of *RestAppendSpreadDrift*.

UpdateModelParameters[*S*]

recalculates the vectors $\boldsymbol{\mu}_0$, (symbolic) $\boldsymbol{\mu}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, etc., to reflect the new volatilities S.

UpdateRestAppendSpreadDrift[]

updates the definition of *RestAppendSpreadDrift*[…] to reflect the current model parameters.

$JointModel

is a global switch between two computational modes: *False* for a model of a single set of rates (e.g., risk–free rates), *True* for a joint model of two sets of rates (e.g., risk–free bond and defaultable bond yields), termed as the "1st" and "2nd" sets, respectively.

Model parameters

Variables and constants of the wrong-way risk model.

dt

is the constant time step dt_1=dt_2=… of the HJM model.

M

is the maximum tenor, in months, considered for the modelling of the forward curve. In a joint model (see *$JointModel*), *M* applies to each curve.

S

is the matrix {$\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_M$}. In a joint model (see *$JointModel*), it is the matrix {$\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_M$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, …, $\boldsymbol{r}_M$}.

***β*Symbolic**

serves as the symbolic expression of {$\beta_{i,1}$, …, $\beta_{i,N}$}.

***γ*Secular**

is the constant vector {γ_1, …} estimated on historical data based on the assumption that at each point in time the forward curve is expected to remain constant (for upward–sloping forward curves this typically means a slight downward drift as the forward rates are indexed by their expiry date, as opposed to time to maturity). Applies to the 1st set in the joint model.

***γ*Symbolic**

serves as the symbolic expression of {$\gamma_{i,1}$, …, $\gamma_{i,N}$}.

***λ*0**

is the drift vector {$\lambda_{i,1}$, …, $\lambda_{i,M}$} with β_i = {0, …, 0}. The drift vector is determined by the no–arbitrage condition:

$$\lambda_{i,j} = \rho_{i,j}^{\mathsf{T}} \left(\rho_{i,i+1}\, dt_{i+1} + \ldots + \rho_{i,j-1}\, dt_{j-1} + \frac{1}{2} \rho_{i,j}\, dt_j - \beta_i\right).$$

***λ*Symbolic**

is the vector {$\lambda_{i,1}$, …, $\lambda_{i,M}$} with the components of β_i given as symbols. The drift vector is determined by the no–arbitrage condition:

$$\lambda_{i,j} = \rho_{i,j}^{\mathsf{T}} \left(\rho_{i,i+1}\, dt_{i+1} + \ldots + \rho_{i,j-1}\, dt_{j-1} + \frac{1}{2} \rho_{i,j}\, dt_j - \beta_i\right).$$

***μ*0**

is the drift vector {$\mu_{i,1}$, …, $\mu_{i,M}$} with γ_i = {0, …, 0}. The drift vector is determined by the no–arbitrage condition:

$$\mu_{i,j} = \sigma_{i,j}^{\mathsf{T}} \left(\sigma_{i,i+1}\, dt_{i+1} + \ldots + \sigma_{i,j-1}\, dt_{j-1} + \frac{1}{2} \sigma_{i,j}\, dt_j - \gamma_i\right).$$

***μ*1**

is the vector {$u_1 - s_1^{\mathsf{T}} \gamma_i$), …, $u_M - s_M^{\mathsf{T}} \gamma_i$, y_1, …, y_M}. It is the "fixed" part of the drift of the composite curve {$f_{i,i+1}$, …, $f_{i,i+M}$, $g_{i,i+1}$, …, $g_{i,i+M}$} under the credit–risk–neutral measure $\mathbb{R}$ and the assumption of constant γ_i.

μSecular

is an estimate of the "secular drift", i.e., the constant drift of the risk–free forward curve given the assumption that the curve tends to preserve its shape.

μSymbolic

is the vector $\{\mu_{i,1}, \ldots, \mu_{i,M}\}$ with the components of γ_i given as symbols. The drift vector is determined by the no–arbitrage condition:

$$\mu_{i,j} = \sigma_{i,j}^{\intercal}\left(\sigma_{i,i+1}\, dt_{i+1} + \ldots + \sigma_{i,j-1}\, dt_{j-1} + \frac{1}{2}\sigma_{i,j}\, dt_j - \gamma_i\right).$$

Monte–Carlo simulation

Functions used for Monte-Carlo simulations.

Generation of scenarios and calculation of prices

IRSResidualOptionScenario[f, b, m]

returns, for the given zero bond price (discount factor) scenario *P*, the vector $\{B_0^{-1} W_{IRS,0}, B_1^{-1} W_{IRS,1}, \ldots, B_{n-2}^{-1} W_{IRS,n-2}\}$ where $W_{IRS,i} = B_i\, E_{\mathbb{Q}}[\, B_{i+1}^{-1}\, V_{IRS,i+1}^{+} \mid \mathcal{F}_i\,]$ where $V_{IRS,i+1}^{+}$ is the exposure (positive part of the value), at time t_{i+1}, of a fixed–for–floating IRS on unit notional value with inception date t_0, zero value at inception date, reset times $t_0, t_b, t_{2b}\ldots, t_{(L-1)\times b}$ and cash flows at times $t_b, t_{2b}\ldots, t_{L\times b}$, $L = \lfloor M/b \rfloor$. The B_i^{-1} are discount factors to time t_0. (For dt_i = 1 month, *b* is the repricing period in months and *M* is the tenor of the swap in months.) Note that *f* is assumed to be a risk–free forward scenario in the tenor convention, i.e., it must be of the form $\{\{f_{0,1}, f_{0,2}, \ldots\}, \{f_{1,2}, f_{1,3}, \ldots\}, \{f_{2,3}, \ldots\}, \ldots\}$. The scenario starts from time t_0. Note: it is assumed that $P_{i,j} = 0$ for i>j.

IRSResidualRateScenario[*P*, *b*, *M*]

returns, for the given zero bond price (discount factor) scenario *P*, the vector $\{k_0, k_1, \ldots, k_n\}$ where k_i is the fixed rate that would make the residual value of a particular IRS zero at time t_i. The contract is a fixed–for–floating IRS with reset times $t_0, t_b, t_{2b}\ldots, t_{(L-1)\times b}$ and cash flows at times $t_b, t_{2b}\ldots, t_{L\times b}$, $L = \lfloor M/b \rfloor$. (For dt_i = 1 month, *b* is the repricing period in months and *M* is the tenor of the swap in months.) Note that *P* must be of the form : $\begin{pmatrix} P_{0,0} & \cdots & P_{0,n} \\ \vdots & \ddots & \vdots \\ P_{n,0} & \cdots & P_{n,n} \end{pmatrix}$, i.e. the scenario starts from time t_0. Elements of *P* may be symbolic. Note: it is assumed that $P_{i,j} = 0$ for i>j.

IRSResidualValueScenario[*P*, *b*, *M*]

returns, for the given zero bond price (discount factor) scenario *P*, the vector $\{B_0^{-1} V_{IRS,0}, B_1^{-1} V_{IRS,1}, \ldots, B_n^{-1} V_{IRS,n}\}$ where $V_{IRS,i}$ is the value, at time t_i, of a fixed–for–floating IRS on unit notional value with reset times $t_0, t_b, t_{2b}\ldots, t_{(L-1)\times b}$ and cash flows at times $t_b, t_{2b}\ldots, t_{L\times b}$, $L = \lfloor M/b \rfloor$. The B_i^{-1} are discount factors to time t_0. (For dt_i = 1 month, *b* is the repricing period in months and *M* is the tenor of the swap in months.) Note that *P* must be of the form : $\begin{pmatrix} P_{0,0} & \cdots & P_{0,n} \\ \vdots & \ddots & \vdots \\ P_{n,0} & \cdots & P_{n,n} \end{pmatrix}$, i.e. the scenario starts from time t_0. Elements of *P* may be symbolic. Note: it is assumed that $P_{i,j} = 0$ for i>j.

RandomForwardRateScenario[{$f_{0,1}$,…,$f_{0,M}$}] returns a random (T+1)×M matrix of forward rates {{$f_{0,1}$,$f_{0,2}$,…,$f_{0,M}$},{$f_{1,2}$,$f_{1,3}$,…,$f_{1,1+M}$},…,{$f_{T,T+1}$,$f_{T,T+2}$,…,$f_{T,T+M}$}} obtained by simulating the discrete HJM model with starting rates $f_{0,1}$,… (only tenors up to M are taken into account).

The scenario is indexed according to the following convention (the two formulas are equivalent):

RandomForwardRateScenario[…]⟦R,C⟧=$f_{R-1,\,R+C-1}$

RandomForwardRateScenario[…]⟦i+1,j−i⟧=$f_{i,\,j}$.

In a joint model (see *$JointModel* for details), the starting rates must be of the form {$f_{0,1}$,…,$f_{0,M}$,$g_{0,1}$,…,$g_{0,M}$} and a (T+1)×2M matrix is returned with each row of the form {$f_{i,i+1}$,$f_{i,i+2}$,…,$f_{i,i+M}$,$g_{i,i+1}$,$g_{i,i+2}$,…,$g_{i,i+M}$}.

Note: rates $f_{i,j}$ where j>M should be considered invalid! They should not be used in any further computations.

The following options may be given:

Length (M by default) for the length of the scenario (T),

ProbabilityMeasure → ℙ for historical (real–world) scenarios,

ProbabilityMeasure → ℚ (default option) for market–risk–neutral scenarios,

ProbabilityMeasure → ℝ for credit–risk–neutral scenarios.

(Note: the ℙ, ℚ options are available for a single–set mode only and the ℝ option is available for the joint–model mode only.)

RandomSeed → n (*None* by default) may be used to reset the seed of the random generator to n.

SampleScenarioNotes

gives additional information on each batch of random scenario samples.

SampleScenarioSpecifications

is a list of the form {1 → Hold[f_1], 2 → Hold[f_2], …} that gives for each batch number k the function f_k used to generate each scenario in that batch.

Structural transformations of scenarios

SanitizeScenario[f]

replaces with *Undefined* those elements $f_{i,j}$ for which j>M of width M, where M is the witdth of the single forward rate scenario f.

SanitizeScenario[f, x]

uses x as replacement.

SplitScenario[f]

splits the joint scenario f into two separate scenarios.

Qualitative transformations of scenarios

ToDiscountFactors[*F*, *dt*]

converts the forward rates $F_{i,j}$ to discount factors (zero bond prices), assuming that the forward rates correspond to terms equally spaced by *dt* (if omitted, the global value of *dt* is assumed). *F* may be a vector or a matrix.

ToMaturityConvention[*X*]

converts the matrix *X* of e.g. forward rates or discount factors from the 'tenor' convention to the 'maturity' convention, i.e. *X*⟦R,C⟧ becomes *Y*⟦R,R+C⟧. The trimmed vectors are padded with zeros.

ToMaturityConvention[*X*, *padding*]

uses the given padding.

ToZeroRates[{f_1,f_2,…,f_n}]

converts the forward rates f_k to zero rates r_k=(f_1+…+f_k)/*k*, i.e., assuming uniform time steps.

ToZeroRates[*M*]

where *M* is a matrix converts each row of *M* to zero rates.

ToZeroRatesFromDF[*P*, *dt*]

converts discount factors (zero bond prices) $P_{i,j}$ to continuously−compounded zero rates $r_{i,j}$ where *i* is an index and *j* is the tenor in units of *dt* (if omitted, the global value of *dt* is assumed). *P* may be a vector or a matrix.

Graphical presentation

Functions for graphical presentation of results.

CompositeScenarioPlot[*data*]

gives a visual representation of *data*, assumed to be a zero− or forward−rate scenario.
It shows the development of the 1M rate in time with the entire curve, drawn for each date,
stemming from the corresponding points on the 1M line. The following options may be given:

Horizon → *n* to specify the range of the horizontal axis as *n* steps from the initial state — *n* may be any expression, in which any occurrences of *Automatic* are replaced by the length of data,

CurveLabel → "Yield curve"

Unit → (1/10000),

UnitName → "bps *p.a.*",

and any options taken by *ListLinePlot*.

CovariancePlot[*data*]

gives a visualisation of *data* (assumed to be a covariance matrix of forward rate differences). The settings such as the color scale, image size, etc. are fixed to allow comparison among multiple plots. The following options may be given (with defaults given after "→"):

TickStep → 12 for spaces between ticks (in units of dt),

Split → *$JointModel* to distinguish between the single−rate/joint model setting,

Names → {"risk−free", "defaultable"} to give names to the two sets of rates (only available for *Split* → True),

ColorFunctionRange → {−.00002,.00002} to specify the range of values over which the color palette should stretch (*All* to automatically span the entire range of values),

and any options for *ArrayPlot*.

CurveHistoryPlot[*data*, *dates*]

gives a visual representation of *data*, assumed to be a historical forward or zero rate scenario, where individual observations correspond to *dates*. The following options may be given:

Take → Automatic to pick approx. five representative dates, incl. the first and the last, distributed evenly (default setting),

Take → *n* to pick approx. *n* representative dates,

Take → All to pick all dates,

Take → {d_1,d_2,…} to pick dates d_1, d_2, etc.,

DateString for the date string specification of the plot legend,

and any options taken by *CurveScenarioPlot* or *ListLinePlot*.

CurveScenarioPlot[*data*]

gives a visual representation of *data*, assumed to be a forward or zero rate scenario. In that case, *CurveScenarioPlot* gives the plot of curves corresponding to selected times. The following options are available:

MaxTenor to specify the maximum tenor shown,

Horizons to specify which time points to display (it is assumed that *data*⟦*i*⟧ contains the curve (*i*–1) months from now),

Unit to specify the units in which to express data,

UnitName to specify the name of the unit,

and any options for *ListLinePlot*.

RateScenarioPlot[*data*]

gives a visual representation of *data*, assumed to be a forward or zero rate scenario. In that case, *RateScenarioPlot* gives the plot of rates corresponding to selected tenors over time. The following options are available:

Tenors to specify which tenors to display,

Unit to specify the units in which to express *data*,

UnitName to specify the name of the unit,

and any options for *ListLinePlot*.

TenorToStringLong[*T*]

gives the tenor *T*, in months, expressed in long textual form, such as 1 month, 6 months, 2 years, 5 years 3 months, etc. *T* may be a list of tenors.

TenorToStringShort[*T*]

gives the tenor *T*, in months, expressed in short textual form, such as 1M, 6M, 2Y, 5Y3M, etc. *T* may be a list of tenors.

VariancePlot[*data*, *legend*, *step*]

gives a visualisation of *data* (assumed to be a list of standard deviations of forward rate differences). Any options for *ListLinePlot* may be given. Only for the single–rate model (see *$JointModel* for details).

Other functions

Other functions.

NearestGreater[*list*, *x*]

gives the smallest element of *list* greater than or equal to *x*, or ∞ if no such element exists.

NearestLower[*list*, *x*]

gives the greatest element of *list* less than or equal to *x*, or −∞ if no such element exists.

NotebookOpenOwn[*name*]

opens the notebook *name*.nb in the same directory as this notebook. Use the option *FrontEndExecute* → {$token_1$, $token_2$, …} to execute the $token_i$ in turn on that notebook. The function returns the corresponding notebook object.

RecreateCell[*expr*, *caption*, *tag*, *options*]

prints an evaluatable cell containing *expr* (with given *options*, if any, and tagged by *tag*) with *caption* printed to a new text cell above it.

Any previously created cells with the *tag* are removed.

RestAppend[{x_1, x_2, x_3, …, x_{n-1}, x_n}]

returns {x_2, x_3, …, x_{n-1},x_n, x_n}.

SetImmediateInterruption[*True*]

makes *Mathematica* interrupt evaluation as soon as a message is generated. This helps prevent errors from accumulating. (Note: calibration requires this feature to be turned off due to the behaviour of the *NMinimize* function.).

SetImmediateInterruption[*False*]

resets the standard behaviour.

ShiftedDifferences[*X*]

gives, for a matrix *X*, the matrix of differences where from each row of *X* we subtract the row above it shifted one step to the left. (Compare to *Differences*[*X*] which works the same way save for the shift.). To obtain matrix of the same width as *X*, the last column is duplicated. Use the option *Split* → True to effectively treat *X* as two separate blocks of equal size, *Split* → *False* to treat it as a single block. The default value of *Split* is taken to be the current value of *$JointModel*.

TimingPrint[*expr*]

evaluates *expr*, and returns the result. It prints the time in seconds used in a separate cell.

ToCorrelationMatrix[*Σ*]

converts the covariance matrix *Σ* to the corresponding correlation matrix.

$ImmediateInterruption

tells whether immediate interruption has been turned on (see SetImmediateInterruption for details). If *False* or undefined then the feature has not been turned on.