FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Marta Kossaczká

Entropy numbers

Department of Mathematical Analysis

Supervisor of the bachelor thesis:	Mgr. Jan Vybíral, Ph.D.
Study programme:	Mathematics
Study branch:	General mathematics

Prague 2016

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date .... signature of the author

Title: Entropy numbers

Author: Marta Kossaczká

Department: Department of Mathematical Analysis

Supervisor: Mgr. Jan Vybíral, Ph.D., Department of Mathematical Analysis

Abstract: In this work we study entropy numbers of linear operators. We focus on entropy numbers of identities between real finite-dimensional sequence spaces and present detailed proofs of their estimates. Then we describe relation between entropy numbers of identities between real spaces and between complex spaces, which allows us to establish similar estimates for complex spaces.

Keywords: Entropy numbers, compact operators, functional analysis

I would like to thank my supervisor Mgr. Jan Vybíral, Ph.D. for helpful consultations and inspiring advice. I would also like to thank my family for their patience and support.

Contents

List of Notations

$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\lVert\cdot\rVert_V$	norm (p-norm, quasinorm) on the vector space V
B_V	the set $\{x \in V : \lVert x\rVert_V < 1\}$
$B(x, \varepsilon)$	the set $\{y \in V : \lVert x - y\rVert_V < \varepsilon\}$
$\mathcal{L}(X, Y)$	The set of all linear continuous mappings from X to Y

Introduction

Entropy numbers are closely associated with the metric entropy which was introduced by Kolmogorov in the 1930s. In this work we focus on estimates of entropy numbers of natural identity between finite-dimensional sequence spaces, which was given by Schütt [1984]. The upper estimate was proved by Edmunds and Triebel [1996], while the lower estimate was completed by Kühn [2001]. We summarize these estimates and present detailed proof.

The work is divided into 3 chapters. In the first chapter we give definitions of l_p^n-spaces and entropy numbers, and elementary properties of entropy numbers of linear operators in general. We also compute volume of the unit ball in $l_p^n(\mathbb{R})$ and prove its estimate, which is essential for estimating entropy numbers. Our aim in the second chapter is to prove Theorem 2.1, which estimates entropy numbers of identities between real finite-dimensional sequence spaces $e_k(id : l_p^n \to l_q^n)$. The first section of this chapter deals with upper estimate with $p \leq q$. We present detailed proofs. The second section deals with lower estimate with $p \leq q$, and the last section presents estimate for $p \geq q$. In the third chapter we prove that similar estimates for entropy numbers of identities between complex finite-dimensional sequence spaces.

1. Definitions and elementary properties

This chapter introduces us to the concept of entropy numbers.

1.1 Vector spaces

To define entropy number of an operator we need some kind of ”norm structure ” on its domain and range. We will be mostly concerned with the sequence spaces l_p^n. We are familiar with normed spaces, unfortunately, l_p^n is not necessarily normed vector space. Precisely, for $p \geq 1$ is l_p^n normed vector space, and for $0 < p < 1$ is it only p-normed vector space. Therefore we start with definition of p-norm.

Definition 1. *Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). The function $\|\cdot\|_V : V \to \mathbb{R}$ is called p-norm if it satisfies following conditions:*

For all $v, u \in V$ and $a \in \mathbb{R}$ (or $\mathbb{C}$),

1. $\|v\|_V = 0$ *iff v is the zero vector,*
2. $\|a \cdot v\|_V = |a| \cdot \|v\|_V$,
3. $\|u + v\|_V^p \leq \|u\|_V^p + \|v\|_V^p$ *(triangle inequality).*

If instead of 3, it holds

$$\|u + v\|_V \leq K(\|u\|_V + \|v\|_V),$$

for some fixed $K > 1$, then $\|\cdot\|_V$ is called quasinorm.

Vector space V equipped with p-norm (quasinorm) is called p-normed vector space (quasinormed vector space).

Complete p-normed (quasinormed) vector space is called p-Banach (quasi-Banach) space.

In following definition we define sequence spaces l_p^n, which are essential for our work.

Definition 2. *Let $(V, \|\cdot\|_V)$ be a normed vector space. Let $n \in \mathbb{N}$ and $p \in (0, \infty]$. We define $l_n^p(V) := (V^n, \|\cdot\|_{l_p^n(V)})$ where V^n is cartesian product of vector spaces, equipped with norm (p-norm) $\|\cdot\|_{l_p^n(V)}$ defined for all $(v_1, \ldots, v_n) \in V^n$ as follows:*

$$\|(v_1, \ldots, v_n)\|_{l_p^n(V)} = \Big(\sum_{i=1}^{n} \|v_i\|_V^p\Big)^{\frac{1}{p}}, \quad \text{if} \quad p < \infty \tag{1.1}$$

$$\|(v_1, \ldots, v_n)\|_{l_p^n(V)} = \max_{i=1\ldots n} \|v_i\|_V \quad \text{if} \quad p = \infty \tag{1.2}$$

If $1 \leq p \leq \infty$, then $l_p^n(V)$ is normed vector space with norm $\|\cdot\|_{l_p^n(V)}$.
If $0 < p < 1$, then $l_p^n(V)$ is p-normed vector space with p-norm $\|\cdot\|_{l_p^n(V)}$, satisfying triangle inequality for p-norms

$$\|v + u\|_{l_p^n(V)}^p \leq \|v\|_{l_p^n(V)}^p + \|u\|_{l_p^n(V)}^p, \tag{1.3}$$

for all u, v elements of $l_p^n(V)$.

Remark. Note that if V is complete then also $l_p^n(V)$ is complete.

Remark. Hölder's inequality gives us that for $p \leq q$ it holds

$$\|\cdot\|_{l_q^n(V)} \leq \|\cdot\|_{l_p^n(V)} \leq n^{\frac{1}{p}-\frac{1}{q}} \|\cdot\|_{l_q^n(V)}. \tag{1.4}$$

Remark. If there is no chance of misunderstanding we will write $\|\cdot\|_p$ instead of $\|\cdot\|_{l_p^n(V)}$.

Example. Let $0 < p < \infty$ and let n be a natural number. By $l_p^n(\mathbb{R})$ we denote the vector space $\mathbb{R}^n$ with standard operations $+$ and $\cdot$, equipped with norm (p-norm) defined as follows:

For all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ it holds

$$\|x\|_{l_p^n(\mathbb{R})} = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}.$$

Example. Let $0 < p < \infty$ and let n be a natural number. By $l_p^n(\mathbb{C})$ we denote the vector space $\mathbb{C}^n$ with standard operations $+$ and $\cdot$, equipped with norm (p-norm) defined as follows:

For all $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ it holds

$$\|x\|_{l_p^n(\mathbb{C})} = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}.$$

Example. Let $0 < p < \infty$ and let n be a natural number. By $l_p^n(l_2^2(\mathbb{R}))$ we denote the vector space $(\mathbb{R}^2)^n = \mathbb{R}^{2n}$ with standard operations $+$ and $\cdot$, equipped with norm (p-norm) defined as follows:

For all $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ it holds

$$\|x\|_{l_p^n(l_2^2(\mathbb{R}))} = (\sum_{i=1}^n (\sqrt{|x_{2i-1}|^2 + |x_{2i}|^2})^p)^{\frac{1}{p}}.$$

1.2 Entropy numbers

Definition 3. *Let X, Y be Banach spaces, p-Banach spaces or quasi-Banach spaces. Let $T \in \mathcal{L}(X,Y)$. We define the sequence $(e_n(T))_{n=1}^\infty$ of entropy numbers as follows*

$$e_n(T) = \inf\{\varepsilon > 0 : \exists y_1, \dots, y_{2^{n-1}} \in Y : T(B_X) \subset \bigcup_{i=1}^{2^{n-1}} (y_i + \varepsilon B_Y)\} \tag{1.5}$$

Remark. It is not necessary for T to be a continous linear operator. The concept of entropy numbers works also for any mapping between two Banach spaces, p-Banach spaces or quasi-Banach spaces.

The following theorem gives us few elementary properties entropy numbers. It can be found in [Vybíral, Thm.8] and partially in [Edmunds and Triebel, 1996, Lemma 1.].

Theorem 1.1. *Let X,Y,Z be Banach or p-Banach spaces. Let $S,T \in \mathcal{L}(X,Y)$ and $R \in \mathcal{L}(Y,Z)$. Then it holds:*

1. $\|T\| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0$ *(monotocity of entropy numbers);*
2. $e_n(T) \to 0$ *iff T is compact;*
3. *If Y is a Banach space, then* $e_1(T) = \|T\|$*;*
4. *For every* $m_1, m_2 \in \mathbb{N}$ *it holds* $e_{m_1+m_2-1}(R \circ T) \leq e_{m_1}(R)e_{m_2}(T)$*;*
5. *If Y is a p-Banach space, then for all* $m_1, m_2 \in \mathbb{N}$ *holds* $e^p_{m_1+m_2-1}(S+T) \leq e^p_{m_1}(S) + e^p_{m_2}(T)$.

Proof. 1. First inequality follows from the definition of the operator norm $\|T\| = \sup\{\|T(x)\| : x \in B_X\}$. Therefore $T(B_X) \subset B(0, \|T\|)$. The inequality $e_i(T) \geq e_{i+1}(T)$ is obvious from the definition of entropy numbers.

2. The sequence $e_n(T)$ is bounded, monotonic, and nonnegative therefore it has a limit $\lim_{n\to\infty} e_n(T) \geq 0$. Let us suppose that T is compact. Then for all $\varepsilon > 0$ there exist a natural number n_ε, and $y_1, \ldots, y_{2^{n_\varepsilon-1}} \in Y$, such that

$$\overline{T(B_X)} \subset \bigcup_{i=1}^{2^{n_\varepsilon-1}} B(y_i, \varepsilon).$$

That implies $e_{n_\varepsilon}(T) \leq \varepsilon$. Since the limit of $e_n(T)$ is nonnegative we obtain $e_n(T) \to 0$.

Now we suppose $e_n(T) \to 0$. Let M be an infinite subset of $\overline{T(B_X)}$. We will prove that M has a limit point in $\overline{T(B_X)}$. First we put $M_0 = M$. Let us suppose that we have an infinite set $M_k \subset \overline{T(B_X)}$. We define M_{k+1} and z_{k+1} as follows.

Because $e_n(T) \to 0$, we observe that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ and $y^k_1, \ldots, y^k_{2^{n_k-1}} \in Y$ such that

$$T(B_X) \subset \bigcup_{i=1}^{2^{n_k-1}} B\Big(y^k_i, \frac{1}{4k}\Big).$$

Hence $\overline{T(B_X)} \subset \bigcup_{i=1}^{2^{n_k-1}} B(y^k_i, \frac{1}{2k})$. We know that $M_k \subset \overline{T(B_X)}$ is infinite and we have only finitely many balls covering $\overline{T(B_X)}$, so there exists y^k_j such that $B(y^k_j, \frac{1}{2k}) \cap M_k$ is infinite. Now we choose arbitrary $z_k \in M_k$ and define $M_{k+1} := B(y^k_j, \frac{1}{2k}) \cap M_k \backslash z_k$. Using this induction we gain a Cauchy sequence $\{z_n\}_{n=1}^\infty$. Y is Banach (p-Banach) space therefore this sequence has a limit z which is also a limit point of M. At last z lies within $\overline{T(B_X)}$ because this set is closed.

3. Let us suppose $e_1(T) < \|T\|$. Then there exist $y \in Y$ and $0 < \varepsilon < \|T\|$, such that $T(B_X) \subset B(y, \varepsilon)$, and $x \in B_X$ such that $\|T(x)\| > \varepsilon$. Naturaly $-x \in B_X$ and $\|T(-x)\| > \varepsilon$. Hence

$$\|T(x) - T(-x)\| = \|T(x) + T(x)\| > 2\varepsilon.$$

Using the triangle inequality for the norm of a Banach space we have

$$\|T(x) - y\| + \|y - T(-x)\| > 2\varepsilon.$$

Therefore $\|T(x) - y\| > \varepsilon$ or $\|T(-x) - y\| > \varepsilon$. This is contradiction with $T(B_X) \subset B(y, \varepsilon)$ and therefore $e_1(T) \geq \|T\|$. From 1, we have $e_1(T) \leq \|T\|$, so finally we have $e_1(T) = \|T\|$.

4. Let $\varepsilon_1 > e_{m_1}(R), \varepsilon_2 > e_{m_2}(T)$. Then from (1.5) there exist $y_1, \dots, y_{2^{m_2-1}} \in Y$ and $z_1, \dots, z_{2^{m_1-1}} \in Z$ such that

$$T(B_X) \subset \bigcup_{i=1}^{2^{m_2-1}} (y_i + \varepsilon_2 B_Y) \quad \text{and} \quad R(B_Y) \subset \bigcup_{j=1}^{2^{m_1-1}} (z_j + \varepsilon_1 B_Z).$$

   Hence from linearity of R we gain

$$R(T(B_X)) \subset R(\bigcup_{i=1}^{2^{m_2-1}} (y_i + \varepsilon_2 B_Y)) = \bigcup_{i=1}^{2^{m_2-1}} (R(y_i) + \varepsilon_2 R(B_Y))$$

   and

$$\begin{aligned} R \circ T(B_X) &\subset \bigcup_{i=1}^{2^{m_2-1}} (R(y_i) + \varepsilon_2 \bigcup_{j=1}^{2^{m_1-1}} (z_j + \varepsilon_1 B_Z)) \\ &= \bigcup_{i=1}^{2^{m_2-1}} \bigcup_{j=1}^{2^{m_1-1}} ((R(y_i) + \varepsilon_2 z_j) + \varepsilon_1 \varepsilon_2 B_Z). \end{aligned}$$

   We have found $2^{m_1+m_2-2}$ balls with radius $\varepsilon_1\varepsilon_2$ that cover $(R \circ T)(B_X)$. Therefore from (1.5) it holds $e_{m_1+m_2-1}(R \circ T) \leq \varepsilon_1\varepsilon_2$. We have chosen $\varepsilon_1 > e_{m_1}(R)$ and $\varepsilon_2 > e_{m_2}(T)$ arbitrarily, therefore it also holds $e_{m_1+m_2-1}(R \circ T) \leq e_{m_1}(R) e_{m_2}(T)$.

5. Let $\varepsilon_1 > e_{m_1}, \varepsilon_2 > e_{m_2}$. Then from (1.5) there exist $y_1, \dots, y_{2^{m_2-1}} \in Y$ and $c_1, \dots, c_{2^{m_1-1}} \in Y$ such that

$$S(B_X) \subset \bigcup_{i=1}^{2^{m_1-1}} (y_i + \varepsilon_1 B_Y) \quad \text{and} \quad T(B_X) \subset \bigcup_{j=1}^{2^{m_2-1}} (c_j + \varepsilon_2 B_Y).$$

   If $x \in B_X$ then there exist $i \in 1, \dots, 2^{m_1-1}$ and $j \in 1, \dots, 2^{m_2-1}$ such that $\|S(x) - y_i\|_Y^p < \varepsilon_1^p$ and $\|T(x) - c_j\|_Y^p < \varepsilon_2^p$.
   Hence $\|S(x) + T(x) - (y_i + c_j)\|_Y^p < \varepsilon_1^p + \varepsilon_2^p$ and

$$(S + T)(B_X) \subset \bigcup_{i=1}^{2^{m_1-1}} \bigcup_{j=1}^{2^{m_2-1}} (y_i + c_j + (\varepsilon_1^p + \varepsilon_2^p)^{\frac{1}{p}} B_Y).$$

   We have found $2^{m_1+m_2-2}$ balls with radius $(\varepsilon_1^p + \varepsilon_2^p)^{\frac{1}{p}}$ that cover $(S + T)(B_X)$. Therefore from (1.5) we gain $e^p_{m_1+m_2-1}(S + T) \leq \varepsilon_1^p + \varepsilon_2^p$. We chose $\varepsilon_1 > e_{m_1}(R)$ and $\varepsilon_2 > e_{m_2}(S)$ arbitrarily, therefore it also holds $e^p_{m_1+m_2-1}(S + T) \leq e^p_{m-1}(S) + e^p_{m_2}(T)$

□

Remark. Note that to prove (4) we needed for R and T to be additive and homogenous only for real nonnegative scalars. This fact allows us to prove Theorem 3.2, in the third chapter.

1.3 Volume of the unit ball in $l_p^n(\mathbb{R})$

In this section we will compute and estimate volume of the unit ball in $l_p^n(\mathbb{R})$, which will be widely used in the second chapter. The computation was given in [Pisier, 1989, 1.17], we also present the proof of the estimate mentioned in [Pisier, 1989, 1.18].

Lemma 1.2. *Let $p \in (0, \infty)$ and $n \in \mathbb{N}$. Denote $l_p^n = l_p^n(\mathbb{R})$. Let $t > 0$. Then*

$$\operatorname{vol}\big(t \cdot B_{l_p^n}\big) = t^n \operatorname{vol}(B_{l_p^n}). \tag{1.6}$$

Proof. Denote $x = (x_1, x_2, \dots, x_n)$. Then

$$\begin{aligned}\operatorname{vol}\big(t \cdot B_{l_p^n}\big) &= \operatorname{vol}\{x \in \mathbb{R}^n : \|x\|_p < t\} \\ &= \int_{\{x \in \mathbb{R}^n : \|x\|_p < t\}} dx_1 \dots dx_n \\ &= \int_{\{x \in \mathbb{R}^n : \|\frac{x}{t}\|_p < 1\}} dx_1 \dots dx_n.\end{aligned}$$

We will use transformation of coordinates from $(x_1, x_2, \dots, x_n)$ to $(y_1, y_2, \dots, y_n)$ such that for $i = 1, \dots, n$ it holds $x_i = y_i t$. Then

$$\operatorname{vol}\big(t \cdot B_{l_p^n}\big) = \int_{\{y \in \mathbb{R}^n : \|y\|_p < 1\}} |\mathbf{J}| dy_1 \dots dy_n,$$

where $|\mathbf{J}|$ is Jacobian determinant. Let $i, j \in \{1, \dots, n\}$. Then $\frac{\partial x_i}{\partial y_i} = t$ and if $i \neq j$ then $\frac{\partial x_i}{\partial y_j} = 0$. Therefore

$$\begin{aligned}\operatorname{vol}\big(t \cdot B_{l_p^n}\big) &= \int_{\{y \in \mathbb{R}^n : \|y\|_p < 1\}} t^n dy_1 \dots dy_n \\ &= t^n \int_{\{y \in \mathbb{R}^n : \|y\|_p < 1\}} dy_1 \dots dy_n \\ &= t^n \operatorname{vol}(B_{l_p^n}).\end{aligned}$$

□

Remark. The (1.6) holds also for $p = \infty$.

Before we proceed to the computation, we recall the definition of the Gamma function.

Definition 4. *Let y be a complex number with positive real part. We define gamma function as follows:*

$$\Gamma(y) = \int_0^\infty t^{y-1} exp(-t) dt$$

Remark. We will often use that for all $a > 0$, it holds

$$a\Gamma(a) = \Gamma(a + 1), \tag{1.7}$$

which can be easily proved with integration by parts.

Theorem 1.3. *(by Pisier [1989]) Let* $0 < p < \infty$ *and let* n *be a natural number. Then it holds*

$$\operatorname{vol}(B_{l_p^n}) = \frac{(2\Gamma(1+\frac{1}{p}))^n}{\Gamma(1+\frac{n}{p})}. \tag{1.8}$$

Proof. Denote

$$I = \int_{\mathbb{R}^n} \exp(-\|x\|_p^p)dx$$

From Fubini's theorem we gain

$$I = \left(\int_{\mathbb{R}} \exp(-|t|^p)dt\right)^n = \left(2\int_0^\infty \exp(-|t|^p)dt\right)^n. \tag{1.9}$$

Using

$$\int_{\|x\|_p}^\infty \frac{d}{dt}(-\exp(-t^p))dt = \lim_{t\to\infty}\big(-\exp(-t^p)\big) - \lim_{t\to\|x\|_p}\big(-\exp(-t^p)\big) = \exp(-\|x\|_p^p),$$

we can express

$$I = \int_{\mathbb{R}^n}\int_{\|x\|_p}^\infty \frac{d}{dt}(-\exp(-t^p))dtdx = \int_{\mathbb{R}^n}\int_{\|x\|_p}^\infty pt^{p-1}\exp(-t^p)dtdx$$

We use Fubini's theorem once again and obtain

$$\begin{aligned} I &= \int_{\{(x,t);x\in\mathbb{R}^n t>\|x\|_p\}} pt^{p-1}\exp(-t^p)dtdx \\ &= \int_0^\infty pt^{p-1}\exp(-t^p)\int_{\{x\in\mathbb{R}^n;\|x\|_p<t\}} 1dxdt \\ &= \int_0^\infty pt^{p-1}\exp(-t^p)t^n\operatorname{vol}(B_{l_p^n})dt \\ &= \operatorname{vol}(B_{l_p^n})\int_0^\infty pt^{p+n-1}\exp(-t^p)dt, \end{aligned}$$

where we used also Lemma 1.2. Together with (1.9), we gain

$$vol(B_{l_p^n}) = \frac{(2\int_0^\infty \exp(-t^p)dt)^n}{\int_0^\infty pt^{p+n-1}\exp(-t^p)dt} \tag{1.10}$$

Now we proceed to the gamma function. Substituting z for t^p we gain $t = z^{\frac{1}{p}}$ and $dt = \frac{1}{p}z^{\frac{1-p}{p}}dz$, therefore

$$\int_0^\infty \exp(-t^p)dt = \int_0^\infty \frac{1}{p}z^{\frac{1-p}{p}}\exp(-z)dz = \frac{1}{p}\Gamma\bigg(\frac{1}{p}\bigg) = \Gamma\bigg(1+\frac{1}{p}\bigg),$$

where in the last equation we used (1.7). On the other hand, with same substitution z for t^p, we obtain

$$\begin{aligned} \int_0^\infty pt^{p+n-1}\exp(-t^p)dt &= \int_0^\infty \frac{1}{p}z^{\frac{1-p}{p}}pz^{\frac{p+n-1}{p}}\exp(-z) \\ &= \int_0^\infty z^{\frac{n}{p}}\exp(-z) \\ &= \Gamma\bigg(1+\frac{n}{p}\bigg) \end{aligned}$$

Combining these conclusions with (1.10) we gain

$$\mathrm{vol}(B_{l_p^n}) = \frac{(2\Gamma(1+\frac{1}{p}))^n}{\Gamma(1+\frac{n}{p})}.$$

□

Using this result, we prove the estimate of the volume of the unit ball, which is essential for many proofs in the second chapter.

Theorem 1.4. *Let $p > 0$, then there exist positive constants c and C (depending only on p) such that for all $x \in [1, \infty)$ it holds*

$$cx^{\frac{1}{p}} \le \left(\Gamma\left(1 + \frac{x}{p}\right)\right)^{\frac{1}{x}} \le Cx^{\frac{1}{p}}.$$

Proof. Denote $m = \min_{x\in[1,1+p]} \frac{(\Gamma(1+\frac{x}{p}))^{\frac{1}{x}}}{x^{\frac{1}{p}}}$. We define $c = \min\{(\frac{1}{ep})^{\frac{1}{p}}, m\}$. Since $\Gamma(a)$ is positive for all $a > 0$, it holds $m > 0$ and $c > 0$. It is obvious that

$$cx^{\frac{1}{p}} \le \left(\Gamma\left(1 + \frac{x}{p}\right)\right)^{\frac{1}{x}} \tag{1.11}$$

holds for all $x \in [1, 1 + p]$. Now we will prove that if (1.11) holds for $x = y$ then it holds also for $x = y + p$. Let us suppose that $cy^{\frac{1}{p}} \le (\Gamma(1 + \frac{y}{p}))^{\frac{1}{y}}$. We have

$$c \le \left(\frac{1}{ep}\right)^{\frac{1}{p}} \quad \text{and} \quad c^p p \le \frac{1}{e}.$$

For all positive z it holds $\frac{1}{e} \le (\frac{z}{z+1})^z$, hence

$$c^p p \le \left(\frac{y}{y+p}\right)^{\frac{y}{p}} \quad \text{and} \quad c^{y+p}p(y+p)^{\frac{y}{p}} \le c^y y^{\frac{y}{p}}.$$

Our assumption gives us

$$c^y y^{\frac{y}{p}} \le \Gamma\left(1 + \frac{y}{p}\right), \quad \text{and therefore} \quad c^{y+p}p(y+p)^{\frac{y}{p}} \le \Gamma\left(1 + \frac{y}{p}\right).$$

Hence

$$c^{y+p}(y+p)^{\frac{y+p}{p}} \le \frac{y+p}{p}\Gamma\left(1 + \frac{y}{p}\right).$$

Using (1.7), we obtain

$$c^{(y+p)}(y+p)^{\frac{y+p}{p}} \le \Gamma\left(1 + \frac{y+p}{p}\right) \quad \text{hence} \quad c(y+p)^{\frac{1}{p}} \le \left(\Gamma\left(1 + \frac{y+p}{p}\right)\right)^{\frac{1}{y+p}}.$$

We know that (1.11) holds for all $x \in [1, 1+p]$ and we proved that if it holds for $x = y$ then it holds also for $x = y + p$. Therefore it holds for all $x \in [1, \infty)$.

We will deal with second inequality in a similar way.

Let us denote $M = \max_{x\in[1,1+p]} \frac{(\Gamma(1+\frac{x}{p}))^{\frac{1}{x}}}{x^{\frac{1}{p}}}$ We define $C = \max\{(\frac{1}{p})^{\frac{1}{p}}, M\}$. Since

Γ is continous on $(0,\infty)$, it is bounded on $[1, 1+\frac{1}{p}]$ and therefore $M<\infty$ and $C<\infty$. It is obvious that

$$\left(\Gamma\left(1+\frac{x}{p}\right)\right)^{\frac{1}{x}} \le Cx^{\frac{1}{p}} \tag{1.12}$$

holds for all $x \in [1, 1+p]$. Now we prove that (1.12) holds for $x = y$ then it holds also for $x = y+p$. Let us suppose that $(\Gamma(1+\frac{y}{p}))^{\frac{1}{y}} \le Cy^{\frac{1}{p}}$. We have

$$\left(\frac{1}{p}\right)^{\frac{1}{p}} \le C \quad \text{and therefore} \quad 1 \le pC^p.$$

We know that for all positive y, p it holds $\frac{y}{y+p} \le 1$ and also $(\frac{y}{y+p})^{\frac{y}{p}} \le 1$. Hence

$$\left(\frac{y}{y+p}\right)^{\frac{y}{p}} \le pC^p \quad \text{and therefore} \quad C^y y^{\frac{y}{p}} \le pC^{y+p}(y+p)^{\frac{y}{p}}.$$

Our assumption gives us

$$\Gamma\left(1+\frac{y}{p}\right) \le C^y y^{\frac{y}{p}} \quad \text{therefore} \quad \Gamma\left(1+\frac{y}{p}\right) \le pC^{y+p}(y+p)^{\frac{y}{p}}.$$

Hence

$$\frac{y+p}{p}\Gamma\left(1+\frac{y}{p}\right) \le C^{y+p}(y+p)^{\frac{y+p}{p}}.$$

We use (1.7), and obtain

$$\Gamma\left(1+\frac{y+p}{p}\right) \le C^{y+p}(y+p)^{\frac{y+p}{p}} \quad \text{hence} \quad \left(\Gamma\left(1+\frac{y+p}{p}\right)\right)^{\frac{1}{y+p}} \le C(y+p)^{\frac{1}{p}}.$$

Now we know that (1.12) holds for all $x \in [1, 1+p]$ and if it holds for $x = y$ then it holds also for $x = y+p$. Therefore it holds for all $x \in [1,\infty)$. ☐

Theorem 1.5. *Let $0 < p \le \infty$. Then there exists positive constants c_1, c_2 (depending only on p) such that for all $n \in \mathbb{N}$ it holds*

$$c_1 n^{\frac{-1}{p}} \le (vol(B_{l_p^n}))^{\frac{1}{n}} \le c_2 n^{\frac{-1}{p}}. \tag{1.13}$$

Proof. Let $0 < p < \infty$. From Theorem 1.4, we gain positive constants c, C such that for all $n \in \mathbb{N}$ it holds

$$cn^{\frac{1}{p}} \le (\Gamma(1+\frac{n}{p}))^{\frac{1}{n}} \le Cn^{\frac{1}{p}}.$$

The Theorem 1.3 gives us

$$\mathrm{vol}(B_{l_p^n}) = \frac{(2\Gamma(1+\frac{1}{p}))^n}{\Gamma(1+\frac{n}{p})}.$$

Therefore

$$2\Gamma(1+\frac{1}{p})C^{-1}n^{-1/p} \le \mathrm{vol}(B_{l_p^n})^{\frac{1}{n}} \le 2\Gamma(1+\frac{1}{p})c^{-1}n^{-1/p}.$$

That completes the proof for $0 < p < \infty$.
If $p = \infty$ then $\mathrm{vol}\, B_{l_p^n} = 2^n$ and (1.13) obviuosly holds. ☐

2. Entropy numbers of $id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})$

This chapter focuses on the estimate of entropy numbers of natural identity between $l_p^n(\mathbb{R})$ and $l_q^n(\mathbb{R})$. Every vector space used in this chapter will be $l_p^n(\mathbb{R})$ for some $0 < p \leq \infty$ and n a natural number. To deal with triangle inequalities of norms and p-norms together, we define $\hat{p} = \min\{p, 1\}$. The triangle inequality for both $0 < p \leq 1$ and $1 \leq p$ can be rewritten as

$$\|u + v\|_p^{\hat{p}} \leq \|u\|_p^{\hat{p}} + \|v\|_p^{\hat{p}}. \tag{2.1}$$

The main result is following theorem.

Theorem 2.1. *(by Schütt [1984]) Let $0 < p \leq \infty$, $0 < q \leq \infty$ and let n be a natural number. If $0 < p \leq q \leq \infty$ then for all $k \in \mathbb{N}$ it holds*

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \sim \begin{cases} 1 & \textit{if} \quad 1 \leq k \leq \log_2 n, \\ \left(k^{-1} \log_2(nk^{-1} + 1)\right)^{\frac{1}{p} - \frac{1}{q}} & \textit{if} \quad \log_2 n \leq k \leq n, \\ 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}} & \textit{if} \quad n \leq k, \end{cases} \tag{2.2}$$

and if $0 < q \leq p \leq \infty$ then for all $k \in \mathbb{N}$ it holds

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \sim 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}}. \tag{2.3}$$

If we have $q = \infty$ (perhaps even $p = \infty$) we define $\frac{1}{q} = 0$ (or $\frac{1}{p} = 0$).

The equivalence $\sim$ from the previous theorem is defined as follows.
Let $x(n,k), y(n,k) : \mathbb{N}^2 \to \mathbb{R}$, then

$$x \sim y \quad \text{iff} \quad c \cdot y(n,k) \leq x(n,k) \leq C \cdot y(n,k),$$

where c, C are positive constants idependent of n and k. For example

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \sim 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}} \quad \text{if and only if}$$

$$c \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}} \leq e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \leq C \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}}$$

for some positive constants c, C independent of n and k, but possibly depending on p and q.

2.1 Upper estimate with $p \leq q$

The upper estimate with $p \leq q$ was proved by [Edmunds and Triebel, 1996, chap.3, 3.2.2 Proposition]. We divided the proof of this proposition into Theorems 2.2, 2.4 and 2.5.

Theorem 2.2. *Let $0 < p \leq q \leq \infty$ and let n be a natural number. Then for each natural number k it holds*

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}) \leq 1.$$

Proof. As $0 < p \le q \le \infty$, $B_X \subset B_Y$ and therefore $e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}) \le 1$ for every $k \in \mathbb{N}$. $\square$

Lemma 2.3. *Let $0 < p \le \infty$ and $0 < q \le \infty$ and let n be a natural number. We put $X = l_p^n(\mathbb{R})$ and $Y = l_q^n(\mathbb{R})$. Let $T \in \mathcal{L}(X,Y)$ and let $r > 0$. By N we denote the maximal number such that there exist $y_1, \dots, y_N \in T(B_X)$ with $\|y_i - y_j\|_q > r$ for every $i \ne j$. Then it holds*

$$N\Big(r \cdot 2^{\frac{-1}{\hat{q}}}\Big)^n \operatorname{vol} B_Y \le \operatorname{vol}(T(B_X) + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y).$$

Let k be a natural number such that $2^{k-1} \ge N$. Then $e_k(T) \le r$.

Proof. Let N and $y_1 \dots y_N \in T(B_X)$ from the statement of the lemma. Let $i, j \in \{1, \dots, N\}$ such that $i \ne j$ and let $z \in Y$. From (2.1) we have

$$r^{\hat{q}} < \|y_i - y_j\|_q^{\hat{q}} \le \|y_i - z\|_q^{\hat{q}} + \|z - y_j\|_q^{\hat{q}}.$$

Hence $\|y_i - z\|_q > r2^{\frac{-1}{\hat{q}}}$ or $\|y_j - z\|_q > r2^{\frac{-1}{\hat{q}}}$, and therefore

$$\big(y_i + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y\big) \cap \big(y_j + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y\big) = \emptyset.$$

And because for all $i = 1, \dots, N$ it holds $\big(y_i + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y\big) \subset \big(T(B_X) + r \cdot 2^{-\frac{1}{\hat{q}}} B_Y\big)$, we immediatly gain the inequality in lemma.

Now let k be a natural number such that $2^{k-1} \ge N$ and let $\varepsilon > r$. Because N is the largest number with mentioned property, for every $z \in T(B_X)$ there exist $i \in \{1, \dots, N\}$ such that $\|y_i - z\|_q \le r < \varepsilon$. Hence $e_k(T) \le \varepsilon$. We see that for all $\varepsilon > r$ it holds $e_k(T) \le \varepsilon$, and therefore $e_k(T) \le r$.

$\square$

Theorem 2.4. *Let $0 < p \le q \le \infty$ and let n be a natural number. We denote $X = l_p^n(\mathbb{R})$ and $Y = l_q^n(\mathbb{R})$. For each $k \in \mathbb{N}$ we denote $e_k(id_{p,q}) = e_k(id : X \to Y)$ Then there exist positive constant $\tilde{c} \ge 1$ depending only on p and q, such that for each natural number $\tilde{k} \ge \tilde{c}n$ it holds*

$$e_{\tilde{k}}(id_{p,q}) \le C \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}},$$

with C a positive constant depending only on p and q.

Proof. Let $k \ge n$ be a natural number. We define $r = 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q} - \frac{1}{p}}$. Let us consider $y_1, \dots, y_N$ from Lemma 2.3. Using the same lemma we gain

$$N\big(r \cdot 2^{\frac{-1}{\hat{q}}}\big)^n \operatorname{vol} B_Y \le \operatorname{vol}(B_X + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y) \tag{2.4}$$

Let $v \in B_X + r \cdot 2^{\frac{-1}{\hat{q}}} B_Y$. Then there exist $v_1 \in B_X$ and $v_2 \in r \cdot 2^{\frac{-1}{\hat{q}}} B_Y$ such that $v = v_1 + v_2$. From (1.4) we know that $\|v_2\|_p \le n^{\frac{1}{p} - \frac{1}{q}} \|v_2\|_q$,
hence $v_2 \in 2^{\frac{-(k-1)}{n} - \frac{1}{\hat{q}}} B_X \subset B_X$. Triangle inequality (2.1) gives us

$$\|v\|_p^{\hat{p}} \le \|v_1\|_p^{\hat{p}} + \|v_2\|_p^{\hat{p}} \le 1 + 1.$$

Hence $\|v\|_p \le 2^{\frac{1}{\hat{p}}}$ and $v \in 2^{\frac{1}{\hat{p}}} B_X$. Together with (2.4) we have

$$N\big(r \cdot 2^{\frac{-1}{\hat{q}}}\big)^n \operatorname{vol} B_Y \le 2^{\frac{n}{\hat{p}}} \operatorname{vol} B_X \quad \text{hence} \quad N^{\frac{1}{n}} \le 2^{\frac{1}{\hat{p}} + \frac{1}{\hat{q}}} r^{-1} \Big(\frac{\operatorname{vol} B_X}{\operatorname{vol} B_Y}\Big)^{\frac{1}{n}}.$$

From Theorem 1.5 we have

$$\left(\frac{\operatorname{vol} B_X}{\operatorname{vol} B_Y}\right)^{\frac{1}{n}} \leq c \cdot n^{\frac{1}{q}-\frac{1}{p}},$$

where c is a positive constant depending only on p and q. Therefore we gain

$$N^{\frac{1}{n}} \leq c \cdot 2^{\frac{1}{\hat{p}}+\frac{1}{\hat{q}}} 2^{\frac{k-1}{n}} \leq 2^{c'} 2^{\frac{k-1}{n}} = 2^{\frac{k-1+c'n}{n}}, \tag{2.5}$$

where c' is a positive constant depending only on p and q such that $2^{c'} \geq c \cdot 2^{\frac{2}{\hat{p}}}$.

Finaly we define $\tilde{c} = c' + 1$. Let $\tilde{k} \geq \tilde{c}n$, and k be the largest natural number such that $\tilde{k} \geq k + c'n$. Note that $k \geq n$. From (2.5) we have $N^{\frac{1}{n}} \leq 2^{\frac{k+c'n-1}{n}} \leq 2^{\tilde{k}-1}$, hence from Lemma 2.3, we gain that

$$\begin{aligned} e_{\tilde{k}}(id_{p,q}) \leq r &= 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}} \\ &= 2^{\frac{-(\tilde{k}-1)}{n}} 2^{c'} 2^{\frac{\tilde{k}-k-c'n}{n}} n^{\frac{1}{q}-\frac{1}{p}} \\ &\leq 2^{c'} 2^{\frac{1}{n}} 2^{\frac{-(\tilde{k}-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}}. \end{aligned}$$

Therefore

$$e_{\tilde{k}}(id_{p,q}) \leq C \cdot 2^{\frac{-(\tilde{k}-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}},$$

where C is a positive constant depending only on p, q.

□

Theorem 2.5. *Let $0 < p < \infty$ and let n be a natural number. For each $k \in \mathbb{N}$ we denote $e_k(id_{p,\infty}) = e_k(id : l_p^n(\mathbb{R}) \to l_\infty^n(\mathbb{R}))$. Let $\tilde{c}$ be the positive constant defined in Theorem 2.4.*

Then there exist positive constants $c_8 > 1$ and c_{10} depending only on p such that for every $1 \leq k \leq \tilde{c}c_8 n$ it holds

$$e_k(id_{p,\infty}) \leq c_{10}\Big(k^{-1}\log_2(nk^{-1}+1)\Big)^{\frac{1}{p}}. \tag{2.6}$$

Proof. Let $1 \leq k \leq \tilde{c}n$. We put $c_1 > \Big(\tilde{c}^{-1}\log_2(1+\tilde{c}^{-1})\Big)^{\frac{-1}{p}}$ and

$$t = c_1\Big(k^{-1}\log_2(nk^{-1}+1)\Big)^{\frac{1}{p}}. \tag{2.7}$$

This choice gives us

$$t > n^{-\frac{1}{p}}, \quad \text{and} \quad k \geq \tilde{c}t^{-p}. \tag{2.8}$$

If $t \geq 1$ then we from Theorem 2.2 immediatly gain (2.6). Now we will assume that $t < 1$. We denote by n_t, the largest natural number such that there exist $x \in B_{l_p^n(\mathbb{R})}$ with n_t coordinates in absolute value greater than t. Because $t < 1$ we note that $n_t \geq 1$ and

$$n_t t^p < 1. \tag{2.9}$$

From (2.8) we have $n_t < n$. Because n_t is the largest natural number with the mentioned property, and $n_t \geq 1$, it holds

$$t^p - 1 \leq n_t < t^{-p}.$$

Since $t < 1$, we gain that $t^{-p} > 1$. If $t^{-p} \geq 2$ then it follows $\frac{1}{2}t^{-p} \leq n_t$. If $1 < t^{-p} < 2$ then $n_t = 1 > \frac{1}{2}t^{-p}$. Both cases gives us

$$\frac{1}{2}t^{-p} \leq n_t < t^{-p}. \tag{2.10}$$

We define

$$e_k^{(t)} = e_k(id : l_p^{n_t}(\mathbb{R}) \to l_\infty^{n_t}(\mathbb{R})).$$

According to (2.8) and (2.9) holds $k \geq \tilde{c}t^{-p} \geq \tilde{c}n_t$. Hence, from Theorem 2.4 we gain

$$e_k^{(t)} \leq C \cdot 2^{\frac{-(k-1)}{n_t}} n_t^{\frac{-1}{p}} < c_3 n_t^{\frac{-1}{p}}, \tag{2.11}$$

where c_3 is a positive constant depending only on p. From (2.10) we have $\frac{1}{2}t^{-p} \leq n_t$ which gives us $c_4 t \geq c_3 n_t^{\frac{-1}{p}}$, where $c_4 > 1$ and depens only on p. Hence from (1.5) we know that there exis $x_1, \ldots, x_{2^{k-1}} \in l_\infty^n$ such that

$$B_{l_p^{n_t}(\mathbb{R})} \subset \bigcup_{i=1}^{2^{k-1}} (x_i + c_4 t B_{l_\infty^{n_t}(\mathbb{R})}) \tag{2.12}$$

For every $i \in \{1, \ldots, 2^{k-1}\}$ and x_i defined in (2.12) we denote $x_i = (x_i^1, \ldots x_i^{n_t})$.

Now we prove that there exist $2^{k-1}\binom{n}{n_t}$ balls in $l_\infty^n(\mathbb{R})$ with radius $c_4 t$ covering $B_{l_p^n(\mathbb{R})}$. We define $z_{i,j}$ for all $i \in \{1, \ldots, 2^{k-1}\}$ and $j \in \{1, \ldots, \binom{n}{n_t}\}$, which will be centres of those balls, as follows.

Since,there is $\binom{n}{n_t}$ ways of choosing n_t coordinates out of n, every j represent one of the possible choices of n_t coordinates. The values of $z_{i,j}$ on these n_t coordinates are $x_i^1, \ldots, x_i^{n_t}$, and the rest of coordinates are zeros.

Let $y \in B_{l_p^n(\mathbb{R})}$. From definition of n_t, we gain that y has at most n_t coordinates in in absolute value greater than t. We find $j \in \{1, \ldots, \binom{n}{n_t}\}$ representing the same choice of n_t coordinates. We know that $|0 - t| = t < c_4 t$,and therefore from (2.12) we gain that there is $i \in \{1, \ldots, 2^{k-1}\}$ such that $\|z_{i,j} - y\|_\infty < c_4 t$

Hence there exist $2^{k-1}\binom{n}{n_t}$ balls in $l_\infty^n(\mathbb{R})$ with radius $c_4 t$ covering $B_{l_p^n(\mathbb{R})}$.

Now we will prove that there exist a positive constant c_8 such that

$$2^{k-1}\binom{n}{n_t} \leq 2^{c_8 k - 1}.$$

We recall the well known inequality

$$m! \geq \left(\frac{m}{e}\right)^m,$$

for every $m \in \mathbb{N}$. This inequality gives us

$$\log_2(n_t!) \geq n_t \log_2 n_t - 2n_t.$$

We have

$$\begin{aligned} \log_2 \binom{n}{n_t} &\leq n_t \log_2 n - \log_2(n_t!) \\ &\leq n_t \log_2 n - n_t \log_2 n_t + 2n_t = n_t \log_2\left(\frac{4n}{n_t}\right) \\ &\leq 4n_t \log_2\left(\frac{n}{n_t} + 1\right). \end{aligned}$$

Using (2.10) we gain $\frac{n}{n_t} \leq 2nt^p$ and therefore

$$\log_2 \binom{n}{n_t} \leq 8n_t \log_2(nt^p + 1).$$

Hence

$$2^{k-1} \binom{n}{n_t} \leq 2^{k-1+8n_t \log_2(nt^p+1)}.$$

Finally from (2.10) we have

$$2^{k-1} \binom{n}{n_t} \leq 2^{k+c_5 t^{-p} \log_2(nt^p+1)-1} \tag{2.13}$$

By defintion of t we have

$$nt^p = c_1 nk^{-1} \log_2(nk^{-1} + 1) \leq c_1 n^2 k^{-2}.$$

Hence

$$\log_2(nt^p + 1) \leq \log_2(c_1 n^2 k^{-2} + 1) \leq c_6 \log_2(nk^{-1} + 1),$$

and from (2.13) we gain

$$2^{k-1} \binom{n}{n_t} \leq 2^{k+c_7 t^{-p} \log_2(nk^{-1}+1)-1} \tag{2.14}$$

where c_7 is a positive constant depending only on p. From (2.7) we have

$$t^{-p} = c_1^{-p} k(\log_2(nk^{-1} + 1))^{-1}.$$

This combined with (2.14) gives us

$$2^{k-1} \binom{n}{n_t} \leq 2^{k+c_7 c_1^{-p} k(\log_2^{-1}(nk^{-1}+1))^{-1} \log_2(nk^{-1}+1)-1} = 2^{k(1+c_7 c_1^{-p})-1}.$$

By c_8 we denote the smallest natural number greater than $1 + c_7 c_1^{-p}$. Hence there exist $2^{c_8 k-1}$ balls with radius $c_4 t$ covernig B_X, with c_4 and c_8 depending only on p, with $c_8 \geq 1$. Therefore from (2.7) we have

$$\begin{aligned} e_{c_8 k}(id_{p,\infty}) &\leq c_1 c_4 \big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}} \\ &\leq c_1 c_4 \big(k^{-1} c_8^{-1} c_8 \log_2(nk^{-1} c_8^{-1} c_8 + 1)\big)^{\frac{1}{p}} \\ &\leq c_1 c_4 \big(k^{-1} c_8^{-1} c_8^2 \log_2(nk^{-1} c_8^{-1} + 1)\big)^{\frac{1}{p}} \\ &\leq c_1 c_4 c_8^{\frac{2}{p}} \big(k^{-1} c_8^{-1} \log_2(nk^{-1} c_8^{-1} + 1)\big)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$e_k(id_{p,\infty}) \leq c_9 \big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}} \tag{2.15}$$

for all $1 \leq k \leq \tilde{c}n$, such that $k = c_8 k'$ for some $k' \in \{1, \dots, n\}$. The constants c_8 and c_9 are positive and depend only on p.

We will prove that (2.15) holds for all $1 \leq k \leq \tilde{c}n$. If $1 \leq k \leq c_8$ then (2.15) follows from Theorem 2.2 and

$$c_8^{-1} \log_2(c_8^{-1} + 1) \leq k^{-1} \log_2(nk^{-1} + 1).$$

Now let $c_8 \le k \le \tilde{c}n$. By k_1 we define the largest natural number less or equal to k, such that $k_1 = c_8 k'$ for some $k' \in \{1, \dots, n\}$. From monotonicity of entropy numbers and (2.15) we gain

$$\begin{aligned} e_k(id_{p,\infty}) \le e_{k_1}(id_{p,\infty}) &\le c_9 \big(k_1^{-1} \log_2(nk_1^{-1}+1)\big)^{\frac{1}{p}} \\ &\le c_9 \big(2c_8(k_1+c_8)^{-1} \log_2(n2c_8(k_1+c_8)^{-1}+1)\big)^{\frac{1}{p}} \\ &\le c_9 \big(2c_8^2(k_1+c_8)^{-1} \log_2(n(k_1+c_8)^{-1}+1)\big)^{\frac{1}{p}} \\ &\le c_9 (2c_8)^{\frac{2}{p}} \big((k_1+c_8)^{-1} \log_2(n(k_1+c_8)^{-1}+1)\big)^{\frac{1}{p}}, \end{aligned}$$

where we used that for all $k_1 \ge 1$ and $c_8 \ge 1$ it holds $2c_8 k_1 \ge k_1 + c_8$. Finally, because the function $f(x) = x^{-1}\log_2(nx^{-1}+1)$ is decreasing on $[1,\infty)$, and $k \le k_1 + c_8$ we obtain

$$e_k(id_{p,\infty}) \le c_9' \big(k^{-1} \log_2(nk^{-1}+1)\big)^{\frac{1}{p}}, \tag{2.16}$$

where c_9' is positive and depens only on p. □

Following lemma is a special case of [Edmunds and Triebel, 1996, Theorem 1.3.2(i)].

Lemma 2.6. *Let* $0 < p \le q < \infty$ *and let* n *be a natural number. We define* $X = l_p^n(\mathbb{R})$, $Y = l_q^n(\mathbb{R})$ *and* $Z = l_\infty^n(\mathbb{R})$. *Let* k_1, k_2 *be natural numbers. Then*

$$e_{k_1+k_2-1}(id : X \to Y) \le 2^{\frac{1}{p}} e_{k_1}^{\frac{p}{q}}(id : X \to X) e_{k_2}^{1-\frac{p}{q}}(id : X \to Z).$$

Proof. Let $0 < p \le q < \infty$ and let n be a natural number. We define $X = l_p^n$, $Y = l_q^n$ and $Z = l_\infty^n$. Since for all $x \ge 0$ it holds $x^q = x^p x^{q-p}$, we have that for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ it holds

$$\sum_{i=1}^n |x_i|^q \le \max_{i=1,\dots,n} |x_i|^{q-p} \sum_{i=1}^n |x_i|^p.$$

Hence for all $x \in \mathbb{R}^n$ holds

$$\|x\|_q \le \|x\|_p^{\frac{p}{q}} \cdot \|x\|_\infty^{1-\frac{p}{q}}. \tag{2.17}$$

Let k_1 and k_2 be natural numbers. Let $\varepsilon > 0$. We put $e_{1,k_1} = e_{k_1}(id : X \to X)$ and $e_{2,k_2} = e_{k_2}(id : X \to Z)$. Then there exist $x_1, \dots, x_{2^{k_1-1}} \in X$ and $z_1, \dots, z_{2^{k_2-1}} \in Z$ such that

$$B_X \subset \bigcup_{i=1}^{2^{k_1-1}} (x_i + (1+\varepsilon)e_{k_1} B_X) \quad \text{and} \quad B_X \subset \bigcup_{i=1}^{2^{k_2-1}} (z_i + (1+\varepsilon)e_{2,k_2} B_Z). \tag{2.18}$$

For all $j = 1, \dots, 2^{k_1-1}$ we put

$$A_j = B_X \cap (x_j + (1+\varepsilon)e_{k_1} B_X). \tag{2.19}$$

We prove that each of A_j can be covered with 2^{k_2-1} balls in Z
with radius $(1+\varepsilon)e_{k_2} 2^{\frac{1}{p}}$ and which centres are in A_j.

Let $j \in \{1, \dots, 2^{k_1-1}\}$. Since $A_j \subset B_X$, we gain from (2.18) that

$$A_j \subset \bigcup_{i=1}^{2^{k_2-1}} (z_i + (1+\varepsilon)e_{k_2}B_Z). \tag{2.20}$$

For every $i = 1, \dots, 2^{k_2-1}$ we define $z_{i,j} \in A_j$ as follows.
If $A_j \cap (z_i + (1+\varepsilon)e_{k_2}B_Z) = \emptyset$ then let $z_{i,j}$ be an arbitrary point in A_j. If $A_j \cap (z_i + (1+\varepsilon)e_{k_2}B_Z) \neq \emptyset$, we put $z_{i,j} \in A_j \cap (z_i + (1+\varepsilon)e_{k_2}B_Z)$.

Let us suppose that $z_{i,j} \in A_j \cap (z_i + (1+\varepsilon)e_{k_2}B_Z)$, and let $z \in (z_i + (1+\varepsilon)e_{k_2}B_Z)$. Then from triangle inequality (2.1) we have

$$\|z - z_{i,j}\|_\infty \leq \|z - z_i\|_\infty + \|z_i - z_{i,j}\|_\infty \leq 2(1+\varepsilon).$$

Hence

$$(z_i + (1+\varepsilon)e_{2,k_2}B_Z) \subset z_{i,j} + 2(1+\varepsilon)B_Z.$$

Using this, and (2.18) we obtain that for all $j = 1, \dots, 2^{k_1-1}$ holds

$$A_j \subset \bigcup_{i=1}^{2^{k_2-1}} (z_{i,j} + 2(1+\varepsilon)e_{2,k_2}2B_Z).$$

Finally this, with (2.18) and (2.19) shows us that for every $a \in B_X$ there exist $z_{i,j}$ such that

$$\|a - z_{i,j}\|_\infty \leq 2(1+\varepsilon)e_{2,k_2}.$$

From (2.1) we have

$$\|a - z_{i,j}\|_p^{\hat{p}} \leq \|a - x_j\|_p^{\hat{p}} + \|z_{i,j} - x_j\|_p^{\hat{p}} \leq 2(1+\varepsilon)^{\hat{p}} e_{1,k_1}^{\hat{p}}.$$

Hence

$$\|a - z_{i,j}\|_\infty \leq 2^{\frac{1}{\hat{p}}}(1+\varepsilon)e_{2,k_2} \quad \text{and} \quad \|a - z_{i,j}\|_p \leq 2^{\frac{1}{\hat{p}}}(1+\varepsilon)e_{1,k_1} \tag{2.21}$$

and from (2.17) we have

$$\|a - z_{i,j}\|_q \leq 2^{\frac{1}{\hat{p}}}(1+\varepsilon)e_{1,k_1}^{\frac{p}{q}} e_{2,k_2}^{1-\frac{p}{q}}.$$

Hence

$$e_{k_1+k_2-1}(id : X \to Y) \leq 2^{\frac{1}{\hat{p}}}(1+\varepsilon)e_{1,k_1}^{\frac{p}{q}} e_{2,k_2}^{1-\frac{p}{q}}.$$

Passing to the infimum over $\varepsilon > 0$ completes the proof. □

Theorem 2.7. *(by Edmunds and Triebel [1996]) Let $0 < p \leq q \leq \infty$ and let n be a natural number. For each k natural number, let us denote $e_k = e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}))$. Then*

$$e_k \leq c \cdot \begin{cases} 1 & \text{if} \quad 1 \leq k \leq \log_2 n \\ \left(k^{-1}\log_2(nk^{-1}+1)\right)^{\frac{1}{p}-\frac{1}{q}} & \text{if} \quad \log_2 n \leq k \leq n \\ 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}} & \text{if} \quad n \leq k, \end{cases} \tag{2.22}$$

where c is a positive constant depending only on p and q.

Proof. Let $0 < p \leq q \leq \infty$. The first case follows immediatly from Theorem 2.2.

If $q = \infty$ then from Theorem 2.5 we gain the second inequality, provided $k > c_8$, where c_8 is positive constant depending only on p. If $q < \infty$ then from Lemma 2.6, where we put $k_1 = 1$ and $k_2 = k$ we have

$$e_k \leq 2^{\frac{1}{p}} e_1^{\frac{p}{q}}(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) e_k^{1-\frac{p}{q}}(id : l_p^n(\mathbb{R}) \to l_\infty^n(\mathbb{R})).$$

From Theorem 2.2 and Theorem 2.5 we gain

$$e_k \leq c \cdot \big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}}, \tag{2.23}$$

where c is a positive constant depending only on p and q, and with $k \geq c_8$.

Now let $k \leq c_8$. Then for all $n \in \mathbb{N}$ holds

$$\big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}} \geq \big(c_8^{-1} \log_2(c_8^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}} = c_{11}.$$

As we can see, c_{11} is positive and depens only on p and q. Therefore from Theorem 2.2 we have

$$e_k \leq c_{11} \cdot c_{11}^{-1}.$$

Hence

$$e_k \leq c_{11}^{-1}\big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}}.$$

This together with (2.23) completes the proof of the second inequality. We also note that the second inequality holds also with $\log_2 n \leq k \leq \tilde{c}c_8 n$ and that $c_8 \geq 1$.

As for the third inequality, firstly we put $n \leq k \leq \tilde{c}n$. Then

$$2^{\frac{-k}{n}} n^{\frac{1}{q}-\frac{1}{p}} \geq \big(2^c(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}}$$

We have already proved, that

$$e_k \leq c \cdot \big(k^{-1} \log_2(nk^{-1} + 1)\big)^{\frac{1}{p}-\frac{1}{q}}.$$

Hence there is a positive constant c_{12} such that

$$e_k \leq c_{12} \cdot 2^{\frac{-k}{n}} n^{\frac{1}{q}-\frac{1}{p}},$$

which is the third inequality provided $n \leq k \leq \tilde{c}n$.

And finally Theorem 2.4 proves the third inequality provided $k \geq \tilde{c}n$. □

2.2 Lower estimate with $p \leq q$

The proof of the lower estimate for $1 \leq k \leq \log_2 n$ and $k \geq n$ is straightforward and was shown by [Triebel, 1997, Theorem 7.3]. We divided it into Theorems 2.8 and 2.10. The lower estimate for $\log \leq k \leq n$ is more difficult to prove and it was shown by Kühn [2001].

Theorem 2.8. *Let $0 < p \leq q \leq \infty$ and let n be a natural number. Then for each natural number $k \leq \log_2 n$ holds*

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \geq c,$$

where c is a positive constant depending only on p and q.

Proof. Let k be a natural number such that $k \leq \log_2 n$. For all $\varepsilon > e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}))$, there exist $y_1, \dots, y_{2^{k-1}} \in Y$ such that

$$B_X \subset \bigcup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y). \tag{2.24}$$

Let us recall, that we denote by $e_1, \dots, e_n$ the canonical vectors in $\mathbb{R}^n$. As their number n is by our assumption larger than the number of balls on the right-hand side of (2.24), we may find $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $m \in \{1, \dots, 2^{k-1}\}$ such that both $\frac{1}{2} \cdot e_i, \frac{1}{2} \cdot e_j \in y_m + \varepsilon B_Y$. From (2.1) we obtain

$$\left\| \frac{1}{2} e_i - \frac{1}{2} e_j \right\|_q^{\hat{q}} \leq \left\| \frac{1}{2} e_i - y_m \right\|_q^{\hat{q}} + \left\| y_m - \frac{1}{2} e_j \right\|_q^{\hat{q}} \leq 2\varepsilon^{\hat{q}},$$

We have

$$\left\| \frac{1}{2} e_i - \frac{1}{2} e_j \right\|_q = (2^{-q} + 2^{-q})^{\frac{1}{q}} = 2^{\frac{1}{q} - 1},$$

hence

$$2^{\frac{1}{q} - \frac{1}{\hat{q}} - 1} \leq \varepsilon.$$

As this holds for every $\varepsilon > e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}))$, the same inequality is true also for $e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}))$ and we finish the proof.

□

The idea of the following lemma is pretty simple. If some set is a subset of some union of sets then the volume of the first set has to be less or equal to the sum of volumes of the sets from that union.

Lemma 2.9. *Let $T \in \mathcal{L}(X, Y)$ and k be a natural number. We denote n dimension of Y. Then it holds*

$$e_k(T) \geq 2^{\frac{-k+1}{n}} \left(\frac{\operatorname{vol} T(B_X)}{\operatorname{vol} B_Y} \right)^{\frac{1}{n}}.$$

Proof. Let $\varepsilon > e_k(T)$. Then there exist $y_1, \dots, y_{2^{k-1}} \in Y$ such that

$$T(B_X) \subset \bigcup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y).$$

Therefore

$$2^{k-1} \operatorname{vol} \varepsilon B_Y \geq \operatorname{vol} T(B_X).$$

That gives us

$$2^{k-1} \varepsilon^n \operatorname{vol} B_Y \geq \operatorname{vol} T(B_X) \quad \text{and} \quad \varepsilon \geq 2^{\frac{-k+1}{n}} \left(\frac{\operatorname{vol} T(B_X)}{\operatorname{vol} B_Y} \right)^{\frac{1}{n}}$$

And with passing to the infimum with ε, we gain desired inequality. □

Theorem 2.10. *Let $0 < p \leq \infty$ and $0 < q \leq \infty$ and let n be a natural number. We put $X = l_p^n(\mathbb{R})$ and $Y = l_q^n(\mathbb{R})$.*
For each $k \in \mathbb{N}$ we denote $e_k = e_k(id : X \to Y)$. Then

$$e_k \geq c \cdot 2^{\frac{-k}{n}} n^{\frac{1}{q} - \frac{1}{p}},$$

where c is a positive constant depending only on p, q.

Proof. From Lemma 2.9 we have

$$e_k \geq 2^{\frac{-k+1}{n}} \Big(\frac{\operatorname{vol} B_X}{\operatorname{vol} B_Y} \Big)^{\frac{1}{n}}.$$

From Theorem 1.5 we know, that there exist positive constants c_p, c_q depending only on p and q respectively, such that

$$\operatorname{vol}(B_X)^{\frac{1}{n}} \geq c_p n^{\frac{1}{p}} \quad \text{and} \quad \operatorname{vol}(B_Y)^{\frac{1}{n}} \leq c_q n^{\frac{1}{q}}.$$

Therefore

$$e_k \geq c_p c_q^{-1} \cdot 2^{\frac{-k+1}{n}} n^{\frac{1}{q} - \frac{1}{p}} = c \cdot 2^{\frac{-k+1}{n}} n^{\frac{1}{q} - \frac{1}{p}}.$$

□

Theorem 2.11. *(by Kühn [2001]) Let $0 < p \leq q \leq \infty$ and let n be a natural number. For each $k \in \mathbb{N}$ we denote $e_k = e_k(id : l_p^n \to l_q^n)$.*

If $\log_2 n \leq k \leq n$, then

$$e_k \geq c \cdot (k^{-1} \log_2(nk^{-1} + 1))^{\frac{1}{p} - \frac{1}{q}}, \tag{2.25}$$

where c is a positive constant depending only on p and q.

Proof. Let $0 < p \leq q \leq \infty$ and let n be a natural number. Firstly we prove that (2.25) holds for any natural number k such that $\log_2 n \leq k \leq \frac{c_1 n}{2}$, where $c_1 < 1$ is a positive constant depending only on p and q. As there is no natural number between $\log_2 n$ and $\frac{c_1 n}{4}$, if $n \leq 3$, we can suppose that $4 \leq n$. For all $m \in \mathbb{N}$ such that $m \leq \frac{n}{4}$, we put

$$S_m = \Big\{ x = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n : \sum_{i=1}^{n} |x_i| = 2m \Big\}, \tag{2.26}$$

and note that S_m has cardinality

$$|S_m| = \binom{n}{2m} \cdot 2^{2m}. \tag{2.27}$$

For all $x, y \in \mathbb{R}^n$ we define Hamming distance as follows

$$h(x, y) = |\{i : x_i \neq y_i\}|.$$

Let $x \in S_m$. We put

$$H_m(x) = \{y \in S : h(x, y) \leq m\}.$$

As for the cardinality of $H_m(x)$, there is $\binom{n}{m}$ ways to choose m coordinates where x and y may differ, and 3 possible values for each of that coordinates for y. Hence

$$|H_m(x)| \leq \binom{n}{m} \cdot 3^m. \tag{2.28}$$

We put $a_m = \binom{n}{2m} \cdot \binom{n}{m}^{-1}$. Let A be any subset of S with cardinality at most a_m. Then it holds

$$|\{y \in S_m : \exists x \in A \quad \text{with} \quad h(x,y) \leq m\}| \leq \sum_{x \in A} |H_m(x)| \leq |A| \cdot \binom{n}{m} \cdot 3^m < |S_m|,$$

where we used (2.28) in the second inequality and (2.27) in the third. Therefore, for any $A \subset S_m$ with $|A| \leq a_m$ there is $x \in S_m$ such that for any $y \in A$ holds $h(x,y) > m$. Hence we can inductively find $A_m \subset S_m$ with cardinality greater than a_m, such that for all distinct $x, y \in A$ is $h(x,y) > m$.

From definition of $h(x,y)$, and $A_m \subset S_m$, we see that for all $x, y \in A_m$ such that $x \neq y$ is

$$\|x-y\|_q = \Big(\sum_{i=1}^{n} |x_i - y_i|^q\Big)^{\frac{1}{q}} > m^{\frac{1}{q}}. \tag{2.29}$$

We put

$$B_m = \{b : b = (3m)^{-\frac{1}{p}} \cdot x \quad \text{for some} \quad x \in A_m\},$$

and note $|B_m| = |A_m| > a_m$. Since $A_m \subset S_m$, for all $b \in B_m$ we have

$$\|b\|_p = \big(2m \cdot (3m)^{-1}\big)^{\frac{1}{p}} < 1,$$

hence $B \subset B_{l_p^n(\mathbb{R})}$. We can also see that for all distinct $b_1, b_2 \in B$ holds

$$\|b_1 - b_2\|_q > (3m)^{-\frac{1}{p}} m^{\frac{1}{q}}.$$

Hence, there exist a positive constat c' depending only on p and q such that for all $k \leq \log_2 a_m$ it holds

$$e_k \geq c' m^{\frac{1}{q}-\frac{1}{p}}, \tag{2.30}$$

which can be proved similarly as lemma 2.3.

We have

$$a_m = \binom{n}{2m} \cdot \binom{n}{m}^{-1} = \frac{m!(n-m)!}{(2m)!(n-2m)!} = \prod_{i=1}^{m} \frac{n-2m+i}{m+i},$$

and we know that function $f(x) = \frac{n-2m+x}{m+x}$ is decreasing in interval $(0, \infty)$. Therefore

$$\Big(\frac{n-m}{2m}\Big)^m \leq a \leq \Big(\frac{n-2m}{m}\Big)^m.$$

That implies

$$c_1 m \log_2 \frac{n}{m} \leq \log_2 a \leq m \log_2 \frac{n}{m}, \tag{2.31}$$

where c_1 is a positive constant less than 1 and independent of n and m.

Now let $k \in \mathbb{N}$, such that $c_1 \cdot m \log_2\Big(\frac{n}{m}\Big) \le k \le \frac{c_1 n}{2}$, for some $m \in \{1, \dots, \frac{n}{4}\}$. We will prove that then it holds

$$m^{-1} \ge c_2 \cdot \frac{\log_2(\frac{n}{k}+1)}{k}, \tag{2.32}$$

where c_2 is positive and depens only on p and q. The function $f(x) = x \log_2 \frac{n}{x}$ is strictly increasing on $[1, \frac{n}{4}]$ and maps this interval onto $[\log_2 n, \frac{n}{2}]$. Therefore it has on $[1, \frac{n}{4}]$ inverse function, which is also increasing. We can easily verify that if $x \in [1, \frac{n}{4}]$ then

$$x \le \frac{f(x)}{\log_2(\frac{n}{f(x)})} \tag{2.33}$$

We use (2.33) and that $f(x)$ is increasing on $[1, \frac{n}{4}]$, and gain

$$m^{-1} \ge \frac{\log_2(\frac{n}{c_1^{-1}k})}{c_1^{-1}k}. \tag{2.34}$$

Since $k \le \frac{c_1 n}{2}$ we have $\frac{n}{c_1^{-1}k} \ge 2$ therefore

$$2\log_2\Big(\frac{n}{c_1^{-1}k}\Big) \ge \log_2\Big(\frac{n}{c_1^{-1}k}+1\Big). \tag{2.35}$$

We have also

$$(c_1^{-1}+1)\log_2\Big(\frac{n}{c_1^{-1}k}+1\Big) \ge \log_2(\frac{n}{k}+1). \tag{2.36}$$

Combinig (2.35) and (2.36) with (2.34), we gain desired (2.32).

Finally, let k be a natural number such that $\log_2 n \le k \le \frac{c_1 n}{2}$. If there exist natural number $m \le \frac{n}{4}$ such that

$$c_1 m \log_2\Big(\frac{n}{m}\Big) \le k \le \log_2 a_m,$$

then (2.25) follows from (2.30) and (2.32). If there is no such m, then we find the largest natural number m such that $c_1 m \log_2\Big(\frac{n}{m}\Big) < k$.
We have $c_1 \log_2(\frac{n}{1}) \le \log_2 n$ and $c_1 \frac{n}{4} \log_2(\frac{n}{\frac{n}{4}}) = \frac{c_1 n}{2}$. Therefore we note that $1 \le m \le \frac{n}{4} - 1$, and because m is the largest number with mentioned property we gain

$$c_1 m \log_2\Big(\frac{n}{m}\Big) < k \le c_1(m+1)\log_2\Big(\frac{n}{m+1}\Big). \tag{2.37}$$

From (2.32) we gain

$$2m \le 2 \cdot \frac{k}{\log_2(\frac{n}{k})+1}$$

and because $m \ge 1$

$$m+1 \le 2 \cdot \frac{k}{\log_2(\frac{n}{k})+1}. \tag{2.38}$$

Since $k \le c_1(m+1)\log_2\Big(\frac{n}{m+1}\Big) \le \log_2 a_{m+1}$, we obtain from (2.30) that it holds

$$e_k \ge 2c'(m+1)^{\frac{1}{q}-\frac{1}{p}}$$

Combining this with (2.38) gives us that (2.25) holds for any $\log_2 n \le k \le \frac{c_1 n}{2}$. To prove (2.25) for $\frac{c_1 n}{2} \le k \le n$ we use Theorem 2.10 and monotonicity of entropy numbers.

□

Finally, Theorems 2.8, 2.10 and 2.11 give us the following Theorem.

Theorem 2.12. *(by Schütt [1984]) Let $0 < p \le q \le \infty$ and let n be a natural number. For each k natural number, let us denote $e_k = e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R}))$. Then*

$$e_k \ge c \cdot \begin{cases} 1 & \text{if } \; 1 \le k \le \log_2 n \\ \left(k^{-1}\log_2(nk^{-1}+1)\right)^{\frac{1}{p}-\frac{1}{q}} & \text{if } \; \log_2 n \le k \le n \\ 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}} & \text{if } \; n \le k, \end{cases} \tag{2.39}$$

where c is a positive constant depending only on p and q.

2.3 Estimate with $p \ge q$

The estimate with $p \ge q$ easily follows from Theorem 2.10 and Theorem 1.1(4).

Theorem 2.13. *Let $0 < q \le p \le \infty$, and let n be a natural number. Then for all $k \in \mathbb{N}$ holds*

$$c \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}} \le e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \le C \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}}, \tag{2.40}$$

where c and C are positive constants depending only on p and q.

Proof. The lower estimate follows directly from Theorem 2.10.

To prove the upper estimate, we use Theorem 1.1(4), with $R = id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})$, $T = id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})$ and $m_1 = k$ and $m_2 = 1$. This choice gives us

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \le e_k(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) \cdot e_1(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})). \tag{2.41}$$

Since $q \le p$, from (1.4) and Theorem 1.1(1) we gain

$$e_1(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) \le n^{\frac{1}{q}-\frac{1}{p}}. \tag{2.42}$$

To estimate $e_k(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R}))$ we use Theorem 2.7, where we put $p = q$. We obtain that if $k \le n$ then

$$e_k(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) \le c_1,$$

and since $k \le n$ we have

$$e_k(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) \le c_2 \cdot 2^{\frac{-(k-1)}{n}},$$

where c_2 is positive and independent of n and k. On the other hand, if $k \ge n$, Theorem 2.7 gives us

$$e_k(id : l_p^n(\mathbb{R}) \to l_p^n(\mathbb{R})) \le C \cdot 2^{\frac{-(k-1)}{n}}.$$

Combining these conclusions with (2.41) and (2.42), we gain that there exist a positive constant C depending only on p and q such that for all $k \in \mathbb{N}$ it holds

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \le C \cdot 2^{\frac{-(k-1)}{n}} n^{\frac{1}{q}-\frac{1}{p}}.$$

□

3. Entropy numbers of $id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})$

In the previous chapter we have presented several estimates for entropy numbers of $id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})$. We will show that similar estimates hold also for entropy numbers of $id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})$, which is formulated in Theorem 3.5. This relation between real and complex case was remarked by [Kühn, 2001, Remark 1.]. We present proof.

Lemma 3.1. *Let $p \in (0, \infty]$ and $n \in \mathbb{N}$. We define mapping $I_p^n : l_p^n(\mathbb{C}) \to l_p^n(l_2^2(\mathbb{R}))$ for all $(z_1, \dots, z_n) \in \mathbb{C}^n$ as follows*

$$I_p^n(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n).$$

Then I_p^n is bijection and $\|(z_1, \dots, z_n)\|_{l_p^n(\mathbb{C})} = \|I_p^n(z_1, \dots, z_n)\|_{l_p^n(l_2^2(\mathbb{R}))}$.

Proof.

$$\begin{aligned}
\|(z_1, \dots, z_n)\|_{l_p^n(\mathbb{C})} &= (\sum_{i=1}^{n} |z_i|^p)^{\frac{1}{p}} \\
&= (\sum_{i=1}^{n} ((\operatorname{Re} z_i)^2 + (\operatorname{Im} z_i)^2)^{\frac{1}{2}})^{\frac{1}{p}} \\
&= \|I_p^n(z_1, \dots, z_n)\|_{l_p^n(l_2^2(\mathbb{R}))}.
\end{aligned}$$

□

Theorem 3.2. *Let $p, q \in (0, \infty]$ and $n \in \mathbb{N}$. Denote $id_1 = id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})$ and $id_2 = id : l_p^n(l_2^2(\mathbb{R})) \to l_q^n(l_2^2(\mathbb{R}))$ Then for all k positive integers it holds*

$$e_k(id_1) = e_k(id_2).$$

Proof. Using mappings I_p^n and I_q^n defined in the previous lemma, we have

$$id_1 = (I_q^n)^{-1} \circ id_2 \circ I_p^n.$$

From Theorem 1.1 we gain

$$\begin{aligned}
e_k(id_1) &= e_{k+1-1}((I_q^n)^{-1} \circ id_2 \circ I_p^n) \\
&\le e_1((I_q^n)^{-1}) e_k(id_2 \circ I_p^n) \\
&\le e_1((I_q^n)^{-1}) e_k(id_2) e_1(I_p^n).
\end{aligned}$$

We know that both I_p^n and $(I_q^n)^{-1}$ are isometric, therefore $e_1((I_q^n)^{-1}) = 1 = e_1(I_p^n)$. Hence $e_k(id_1) \le e_k(id_2)$. On the other hand, we have

$$id_2 = I_q^n \circ id_1 \circ (I_p^n)^{-1},$$

and we can similarly prove that $e_k(id_2) \le e_k(id_1)$. □

Theorem 3.3. *Let $0 < p \leq \infty$, and let n be a natural number. Then there exist positive constants c_1 and c_2 depending only on p, such that for all $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ it holds*

$$c_1 \cdot \|x\|_{l_p^{2n}(\mathbb{R})} \leq \|x\|_{l_p^n(l_2^2(\mathbb{R}))} \leq c_2 \cdot \|x\|_{l_p^{2n}(\mathbb{R})}. \tag{3.1}$$

Proof. Let $0 \leq a$ and $0 \leq b$. Then from the inequality between the arithmetic and geometric mean we gain

$$2^{-1}(a+b) \leq \sqrt{a^2+b^2} \leq a+b. \tag{3.2}$$

Let $0 < p$. We have

$$2^p \cdot (a^p + b^p) \geq 2^p \cdot \max\{a^p, b^p\} = \max\{(2a)^p, (2b)^p\} \geq (a+b)^p.$$

Together with (3.2) it follows that

$$\left(\sqrt{a^2+b^2}\right)^p \leq 2 \cdot 2^p \cdot (a^p + b^p). \tag{3.3}$$

On the other hand, we see, that

$$2^{-p} \cdot 2^{-1} \cdot (a^p + b^p) \leq 2^{-p} \cdot \max\{a^p, b^p\} \leq 2^{-p}(a+b)^p.$$

Together with (3.2) it follows that

$$2^{-1-p}(a^p + b^p) \leq \left(\sqrt{a^2+b^2}\right)^p. \tag{3.4}$$

Using (3.3) and (3.4) we obtain, that for all $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ and $k = 1, \ldots, n$ it holds

$$2^{-1-p}(|x_{2k-1}|^p + |x_{2k}|^p) \leq \left(\sqrt{|x_{2k-1}|^2 + |x_{2k}|^2}\right)^p \leq 2 \cdot 2^p(|x_{2k-1}|^p + |x_{2k}|^p).$$

Hence

$$2^{-1-p}\sum_{k=1}^{n} |x_{2k-1}|^p + |x_{2k}|^p \leq \sum_{k=1}^{n}\left(\sqrt{|x_{2k-1}|^2 + |x_{2k}|^2}\right)^p \leq 2^{1+p}\sum_{k=1}^{n} |x_{2k-1}|^p + |x_{2k}|^p. \tag{3.5}$$

Finally, the definition of $\|\cdot\|_{l_p^n(l_2^2(\mathbb{R}))}$ and $\|\cdot\|_{l_p^{2n}(\mathbb{R})}$ implies that

$$2^{\frac{-1-p}{p}}\|\cdot\|_{l_p^{2n}(\mathbb{R})} \leq \|\cdot\|_{l_p^n(l_2^2(\mathbb{R}))} \leq 2^{\frac{1+p}{p}}\|\cdot\|_{l_p^{2n}(\mathbb{R})}.$$

□

Theorem 3.4. *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and n be a natural number. Then there exist positive constants c and C depending only on p and q such that for all $k \in \mathbb{N}$ holds*

$$c \cdot e_k(id : l_p^{2n}(\mathbb{R}) \to l_q^{2n}(\mathbb{R})) \leq e_k(id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})) \leq C \cdot e_k(id : l_p^{2n}(\mathbb{R}) \to l_q^{2n}(\mathbb{R})). \tag{3.6}$$

Proof. According Theorem 3.2

$$e_k(id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})) = e_k(id : l_p^n(l_2^2(\mathbb{R})) \to l_q^n(l_2^2(\mathbb{R}))). \tag{3.7}$$

For all $0 < r \leq \infty$, we denote $X_r = l_r^{2n}(\mathbb{R})$ and $Y_r = l_r^n(l_2^2(\mathbb{R}))$. Then it holds

$$id : (Y_p \to Y_q) = (id : Y_p \to X_p) \circ (id : X_p \to X_q) \circ (id : X_q \to Y_q)$$

Let k be a natural number. We use Theorem 1.1(4) with $R = id : Y_p \to X_p$, $T = (id : X_p \to X_q) \circ (id : X_q \to Y_q)$ and $m_1 = 1$, $m_2 = k$, and then we use it once again with $R = id : X_p \to X_q$, $T = id : X_q \to Y_q$ and $m_1 = k$, $m_2 = 1$. Alltogether we gain

$$e_k(id : Y_p \to Y_q) \leq e_1(id : Y_p \to X_p) \cdot e_k(id : X_p \to X_q) \cdot e_1(id : X_q \to Y_q) \tag{3.8}$$

Now we combine Theorem 1.1(1) with Theorem 3.3 and obtain that

$$e_1(id : Y_p \to X_p) \leq c_1^{-1} \quad \text{and} \quad e_1(id : X_q \to Y_q) \leq c_2,$$

where c_1 and c_2 are positive and depend only on p and q respectively. Combining this conclusions with (3.8) and (3.7) we gain

$$e_k(id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})) \leq c_1^{-1} c_2 e_k(id : l_p^{2n}(\mathbb{R}) \to l_q^{2n}(\mathbb{R})).$$

The second inequality can be proved similarly. □

And the corollary of the Theorem 3.4 is following theorem.

Theorem 3.5. *Let* $0, p \leq \infty$*,* $0 < q \leq \infty$ *and let* n *be a natural number. If* $0 < p \leq q \leq \infty$ *then for all* $k \in \mathbb{N}$ *it holds*

$$e_k(id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})) \sim \begin{cases} 1 & \text{if} \quad 1 \leq k \leq \log_2 2n, \\ \left(k^{-1} \log_2(2nk^{-1} + 1)\right)^{\frac{1}{p} - \frac{1}{q}} & \text{if} \quad \log_2 2n \leq k \leq 2n, \\ 2^{\frac{-(k-1)}{2n}} (2n)^{\frac{1}{q} - \frac{1}{p}} & \text{if} \quad 2n \leq k, \end{cases} \tag{3.9}$$

and if $0 < q \leq p \leq \infty$ *then for all* $k \in \mathbb{N}$ *it holds*

$$e_k(id : l_p^n(\mathbb{C}) \to l_q^n(\mathbb{C})) \sim 2^{\frac{-(k-1)}{2n}} (2n)^{\frac{1}{q} - \frac{1}{p}}. \tag{3.10}$$

If we have $q = \infty$ *(perhaps even* $p = \infty$*) we define* $\frac{1}{q} = 0$ *(or* $\frac{1}{p} = 0$*).*

The equivalence $\sim$ from the previous Theorem is defined as follows.
Let $x(n,k), y(n,k) : \mathbb{N}^2 \to \mathbb{R}$, then

$$x \sim y \quad \text{iff} \quad c \cdot y(n,k) \leq x(n,k) \leq C \cdot y(n,k),$$

where c, C are positive constants idependent of n and k. For example

$$e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \sim 2^{\frac{-(k-1)}{2n}} (2n)^{\frac{1}{q} - \frac{1}{p}} \quad \text{if and only if}$$

$$c \cdot 2^{\frac{-(k-1)}{2n}} (2n)^{\frac{1}{q} - \frac{1}{p}} \leq e_k(id : l_p^n(\mathbb{R}) \to l_q^n(\mathbb{R})) \leq C \cdot 2^{\frac{-(k-1)}{2n}} (2n)^{\frac{1}{q} - \frac{1}{p}}$$

for some positive constants c, C independent of n and k, but possibly depending on p and q.

Conclusion

The aim of this study was to introduce the concept of entropy numbers of an operator and show detailed proof of the estimates of the entropy numbers of natural identity between finite-dimensional sequence spaces.

In the first chapter we defined entropy numbers and l_p spaces. We also computed and estimated volume of the unit ball in l_p^n. In the second chapter we summarized and proved the estimates of entropy numbers of natural identity between $l_p^n(\mathbb{R})$ and $l_q^n(\mathbb{R})$. The main idea in the proofs was using the estimates of volumes of the unit balls as well as combinatorial aspects. The third chapter extends the estimates from the second chapter to the complex sequence spaces.

The relation between entropy numbers and eigenvalues of compact operators could be subject to further study.

Bibliography

D.E. Edmunds and H. Triebel. *Function Spaces, Entropy Numbers and Differential Operators.* Cambridge Tracts in Mathematics (No. 120). Cambridge Univ. Press, Cambridge, UK, 1996.

Thomas Kühn. A Lower Estimate for Entropy Numbers. *Journal of Approximation Theory*, 5:120–124, 2001. doi: 10.1006/jath.2000.3554. URL `http://dx.doi.org/10.1006/jath.2000.3554`.

G. Pisier. *The Volume of Convex Bodies and Banach spaces.* Cambridge Tracts in Mathematics (No. 94). Cambridge Univ. Press, Cambridge, UK, 1989.

Carsten Schütt. Entropy Numbers of Diagonal Operators between Symmetric Banach Spaces. *Journal of Approximation Theory*, 8:121–128, 1984. doi: 10.1016/0021-9045(84)90021-2. URL `http://dx.doi.org/10.1016/0021-9045(84)90021-2`.

H. Triebel. *Fractals and Spectra.* Birkhäuser, Basel, 1997.

Jan Vybíral. *Modern Approximation Theory.* online. URL `http://www.karlin.mff.cuni.cz/~vybiral/Vort/Vort.pdf`.