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MASTER THESIS

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Microscopic sets and drops in Banach spaces

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date ....

Název práce: Mikroskopické množiny a kapky v Banachových prostorech

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Abstrakt: Nejprve definujeme mikroskopické množiny na reálné ose a zkoumáme jejich vztah k množinám Hausdorffovy a Lebesgueovy míry nula a k množinám první kategorie. V druhé části dokazujeme Bishop-Phelpsovu větu a její ekvivalenci s Ekelandovým variačním principem, větou o okvětních plátcích, Danešovou větou o kapce, Brézis-Browderovou větou a Caristi-Kirkovou větou. Přitom definujeme pojem kapky jako konvexní obal množiny a bodu. V části třetí dokazujeme, že vlastnost kapky je v jistém smyslu ekvivalentní reflexivitě. Prostor má vlastnost kapky, pokud kapku z Danešovy věty lze najít i v obecnějším případě, než zaručuje věta samotná. Dále tuto vlastnost charakterizujeme pomocí aproximativní kompaktnosti. V poslední části se zabýváme mikroskopickou vlastností kapky, která je oproti původní vlastnosti kapky méně přísná. Zjistíme však, že tyto dva pojmy jsou pro nekompaktní množiny v reflexivních prostorech ekvivalentní.

Klíčová slova: Banachovy prostory, mikroskopické množiny, Danešova věta o kapce, vlastnost kapky

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Abstract: First we define microscopic sets on the real axis and study their relation to the sets of Hausdorff and Lebesgue measure zero and the sets of first category. In the second part, we prove the Bishop-Phelps' theorem and its equivalence with the Ekeland's variational principle, the Daneš's drop theorem, the Brézis-Browder's theorem and the Caristi-Kirks's theorem. Doing so we define the notion of a drop as the convex hull of a set and a point. In the third part we prove that the drop property equals reflexivity in some sense. A space has the drop property if it is possible to find the drop from the Daneš's theorem even in a more general case than the theorem itself guarantees. Furthermore, we characterize this property using the approximative compactness. Last, we study the microscopic drop property that is more relaxed than the original drop property. We find out that those two notions are for noncompact sets in reflexive spaces equivalent.

Keywords: Banach spaces, microscopic sets, Daneš's drop theorem, drop property

Contents

1 Preliminaries

Throughout, X will always be a real Banach space. Unless otherwise stated, B is the closed unit ball in X; where confusion might arise, this ball will be denoted with B_X. The unit sphere will be labelled S_X. $B(x,r)$ denotes a closed ball with centre $x \in X$ and radius $r > 0$.

The convex hull of points $x_1, ..., x_n$ will be denoted by $\text{conv}(x_1, ..., x_n)$. The linear hull will be labelled $\text{span}(x_1, ..., x_n)$.

The expression $K(B(z,r),x)$ describes the smallest convex cone with its edge at x that includes the ball $B(z,r)$, that means,

$$K(B(z,r),x) = \{t(y-x)+x : y \in B(z,r), t > 0\}.$$

For $A \subset \mathbb{R}$, $|A|$ will denote the Lebesgue measure of A.

2 Microscopic sets

In this introductory section we define the microscopic sets on the real axis and present several reasults that show the connection of this notion to other characteristics of "smallness" of a set. We will find out that in comparison with sets of Lebesgue measure 0, Hausdorff dimension 0 and the sets of first category the microscopic sets are the "smallest" - more precisely, each microscopic set has Lebesgue measure zero, Hausdorff dimension zero and is of the first category but the converse is not true.

Unless stated otherwise, the results of this section are due to J. Appell [1].

Definition 1. *: A set $M \in \mathbb{R}$ is called* ***microscopic*** *if for any $\varepsilon > 0$ there exists a sequence of intervals I_n such that*

$$M \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } |I_n| < \varepsilon^n \text{ for every } n \in \mathbb{N}. \tag{1}$$

This property is invariant with regard to translation: if it is possible to cover a set M with intervals of the form (a_n, b_n), then the set $M_t := \{x + t : x \in M\}$ can be covered by intervals of the form $(a_n + t, b_n + t)$ for any $t \in \mathbb{R}$.

Example 2.1. *If a set M containts an interval, then it cannot be microscopic.*

Proof. Without loss of generality, let M by an interval of the length $L > 0$ - if a subset of M cannot be covered then M cannot be covered as well. Assume for contradiction that M is a microscopic set and choose $\varepsilon > 0$. The total length of the intervals covering M is bounded from definition. Therefore,

$$L = |M| \leq \sum_{n=1}^{\infty} |I_n| \leq \sum_{n=1}^{\infty} \varepsilon^n = \frac{\varepsilon}{1-\varepsilon}.$$

This fraction converges to zero for $\varepsilon \to 0$ and is thus for a sufficiently small ε smaller than L. We get $L \leq \frac{\varepsilon}{1-\varepsilon} < L$ which is a contradiction. □

As a consequence, an open set is never microscopic as it always includes a neighbourhood of its inner point, or in other words, an interval.

Example 2.2. *Any countable set is microscopic.*

Proof. We order the set in a sequence $(x_n)_{n \in \mathbf{N}}$. Choose $\varepsilon > 0$. For all natural n, set $I_n := (x_n - \frac{\varepsilon^n}{2}; x_n + \frac{\varepsilon^n}{2})$. Those intervals cover M and their length is for any natural number bounded by ε^n. □

Theorem 2.3. *Any microscopic set has Lebesgue measure 0.*

Proof. Let M be microscopic. Choose $\varepsilon > 0$ and find intervals I_n covering M according to the definition. For the Lebesgue measure of M we then have

$$|M| \leq \sum_{n=1}^{\infty} |I_n| \leq \sum_{n=1}^{\infty} \varepsilon^n = \frac{\varepsilon}{1-\varepsilon}$$

which for $\varepsilon \to 0$ tends to zero. □

Theorem 2.4. *Every microscopic set has the Hausdorff dimension 0.*

Proof. Let us recall first that the Hausdorff dimension of a set M is given by

$$\dim_H(M) := \inf\{\alpha > 0 : \mathcal{H}^\alpha(M) = 0\},$$

where

$$\mathcal{H}^\alpha(M) := \liminf_{\delta\to\infty}\{\sum_{n=1}^{\infty}(\mathrm{diam} M_n)^\alpha : M \subseteq \bigcup_{n=1}^{\infty} M_n,\ \mathrm{diam} M_n < \delta\}$$

is the Hausdorff measure with respect to $\alpha \geq 0$.

We want to show that $\mathcal{H}^\alpha(M) = 0$ for any $\alpha > 0$. Because M is microscopic, for $\delta > 0$ given there exists a sequence of intervals I_n such that M is covered by their union and $|I_n| \leq \delta^n < \delta$ (we are assuming without loss of generality that $\delta < 1$). Those intervals are considered in the computation of the Hausdorff measure and they fulfil

$$\sum_{n=1}^{\infty}(\mathrm{diam} I_n)^\alpha \leq \sum_{n=1}^{\infty}(\delta^n)^\alpha = \sum_{n=1}^{\infty}(\delta^\alpha)^n = \frac{\delta^\alpha}{1-\delta^\alpha} \to 0$$

for $\delta \to 0$, and so $\mathcal{H}^\alpha(M) = 0$. □

Theorem 2.5. *Any microscopic set is of the first Baire category.*

Proof. Suppose M is not of the first category. Then M is not nowhere dense, or in other words, the interior of M is nonempty. Then the set M contains an interval and cannot be microscopic. □

Theorem 2.6. *There exists a set that is not microscopic despite being of the first category and having Lebesgue measure 0.*

Proof. The Cantor set on the interval $[0, 1]$ has Hausdorff dimension $\frac{\ln 2}{\ln 3}$ and therefore cannot be microscopic. At the same time it is a set of the first category and Lebesgue measure zero. □

Theorem 2.7. *[3] There exists a set that is not microscopic despite having Hausdorff dimension 0.*

Proof. Our example will be a certain kind of the Cantor set. Choose a constant $c \geq 3$. We start with the interval $[0,1]$. In the first step we cut from its middle a segment of the length $1 - 2c^{-1}$. Two connected sets will be created, each of the length c, from whose centres we will in the second step cut intervals of the length $c - 2c^{-4}$. In the n-th step, we will then have 2^n intervals of the length c^{-n^2} and we will be cutting from the middle of each one of them a segment of the measure $c^{-n^2} - 2c^{-(n+1)^2}$. The set we will end up with at the end of this construction will be labelled N.

First we choose $\alpha > 0$ and compute the Hausdorff measure of the set N with the coefficient α. Because N is an intersection of a nested sequence of sets, it is for each natural n a subset of those intervals that are remaining in the appropriate step; there are 2^n such intervals and they are of length c^{-n^2}. The Hausdorff measure can thus be estimated as

$$\mathcal{H}^\alpha(N) \leq 2^n (c^{-n^2})^\alpha.$$

However, this expression decreases as n grows: Because $c > 1$ and $\alpha > 0$ is a fixed number, we have $(c^\alpha)^n \to \infty$ for $n \to \infty$. Thus there exists a natural number m such that $(c^\alpha)^m > 2$. For a sufficiently large n it is then possible to estimate

$$2^n (c^{-n^2})^\alpha = \frac{2^n}{(c^\alpha)^{n^2}} \leq (\frac{2}{c^{\alpha m}})^n \to 0.$$

The Hausdorff measure of the set N is thus zero for every positive α and so its Hausdorff dimension is zero as well.

Now we show that N is not microscopic. To this end, we choose $\varepsilon = c^{-4}$ and a sequence of intervals such that $|I_k| \leq \varepsilon^k$ for all natural numbers k. We want to find a point in N that is contained in none of the intervals I_k.

Choose a natural number n and find all k fulfilling $\frac{(n+1)^2}{4} > k \geq \frac{n^2}{4}$. After completing the $(n-1)$st step of the construction of the Cantor set, we have 2^n intervals in total, each of the length c^{-n^2}. Between them lies at least the distance of the line segment we have cut out in this step, which is $c^{-(n-1)^2} - 2c^{-n^2}$. However, for the length of the intervals I_k for the above mentioned k we have

$$|I_k| \leq \varepsilon^k \leq \varepsilon^{\frac{n^2}{4}} = c^{-n^2} < c^{-(n-1)^2} - 2c^{-n^2}. \tag{2}$$

The gap between each two intervals that are remaining before the step n is taken is therefore bigger than the length of any line segment I_k from the given set; this means that each I_k can intersect at maximum one of the 2^n intervals that are remaining before the n-th step.

Now we take into account all $k < \frac{(n+1)^2}{4}$ and denote by a_n the number of intervals that were created in the $(n-1)$st step of the construction of the Cantor set and that are intersected by one of the line segments I_k.

We know already that if k is bounded from below by the expression $k \geq \frac{n^2}{4}$, we will only have one such intersection for every interval I_k. Given the computation

$$\frac{(n+1)^2}{4} - \frac{n^2}{4} = \frac{2n+1}{4} < n,$$

the number of such intervals is bounded by n.

If on the other hand k does not fall into this category, this means if $k < \frac{n^2}{4}$, then the interval I_k intersects one of the line segments that were created in the construction of the Cantor set before the step $n-1$, and is therefore accounted for in the number a_{k-1}. By carrying out the $(n-1)$th step, this line segment breaks in two. Both can be perhaps intersected by I_k, however, no more segments can be added, because all other are subsets of sets that are not intersected by I_k. Together we get the recursive expression

$$a_n \leq a_{n-1} + n.$$

As there exists no natural number fulfilling $k < \frac{n^2}{4}$ for $n = 1$, we in addition get $a_1 = 0$. Thanks to this, we can use induction and write the expression in the form

$$a_n \leq 2^n - n.$$

Thus, for every n there exist some intervals from the construction of the Cantor set that are not covered by any segment I_k. As they are all disjoint and their number is finite, their union is a closed set; in addition, this union is a subset of all unions of the uncovered intervals that were left out in the previous steps. This means that we have obtained a sequence of nested closed sets. Because $\operatorname{diam}(I_n) \to 0$ from (2), the Cantor intersection theorem gives a nonempty intersection of those sets. This intersection is the element of the set N that is not covered by the union of our intervals I_k. □

Theorem 2.8. *There exists an uncountable microscopic set.*

Proof. We begin with the interval $[0,1]$ and order the rational numbers contained in it in a sequence $(r_n)_{n\in\mathbf{N}}$. For each pair of natural numbers j, k we then define an interval

$$I_{j,k} := [0,1] \cap (r_j - 2^{-j-k}; r_j + 2^{-j-k})$$

and set

$$M := [0,1] \backslash N \text{ where } N := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} I_{j,k}.$$

We claim that M is of first category while N is microscopic and uncountable.

We start by writing M as

$$M = [0,1] \cap N^c = [0,1] \cap (\bigcup_{k=1}^{\infty} \bigcap_{j=1}^{\infty} I_{j,k}^c).$$

Now we show that $\bigcap_{j=1}^{\infty} I_{j,k}^c$ is a nowhere dense set for any k. First, $I_{j,k}^c$ is closed as the complement of an open interval in $[0,1]$. Therefore, $\bigcap_{j=1}^{\infty} I_{j,k}^c$ is closed as well.

Suppose for contradiction there exists a point x in the interior of this set. Then, from the openness of the interior, a neighbourhood U of this point would be contained there as well:

$$U \subseteq \operatorname{int}(\bigcap_{j=1}^{\infty} I_{j,k}^c) \subset \bigcap_{j=1}^{\infty} I_{j,k}^c.$$

In each open interval, we can find a rational number. However, we have defined the sets M and N in such a way that all rational numbers fall in N, and because $N \cap U = \emptyset$, we get a contradiction. Therefore, no such point x can exist, the interior is empty and $\cap_{j=1}^{\infty} I_{j,k}^c$ is nowhere dense. Thus, M is a countable union of nowhere dense sets, that is, a set of first category.

Now we show that N is microscopic. Choose $\varepsilon > 0$ and for each $j \in \mathbb{N}$ find $k(j)$ such that $|I_{j,k(j)}| \leq \varepsilon^j$. This is possible because

$$|I_{j,k}| = |(r_j - 2^{-k-j}; r_j + 2^{-k-j})| = 2^{-k-j+1},$$

which can be made arbitrarily small for k large enough.

The intervals then cover N:

$$N = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} I_{j,k} \subset \bigcup_{j=1}^{\infty} I_{j,k(j)}.$$

It remains to be shown that N is uncountable. If N was countable, however, then it would be possible to order its elements in an increasing sequence $(y_i)_{i \in \mathbf{N}}$. Then, $M = [0,1] \backslash N$ would be the union of all intervals of the form (y_i, y_{i+1}) and could thus never be a set of first category. □

To conclude this section we will try to describe the behavior of the system of microscopic sets.

Theorem 2.9. *The set of all microscopic sets $\mathcal{M}$ is a σ-ideal.[2]*

Proof. We will verify the properties that define a σ-ideal.

(1) The empty set is surely contained in $\mathcal{M}$ as it is the subset of any interval.

(2) If $A \subset B$ and B is microscopic, then A is microscopic as well as any covering of B is at the same time a covering of A.

(3) If (A_n) is a sequence of microscopic sets, then even their union is microscopic. To see this, denote

$$\bigcup_{n=1}^{\infty} A_n =: A$$

and choose $\varepsilon > 0$. Denote $\varepsilon_k := \varepsilon^{2^k}$. Each of the sets A_k can be covered with intervals I_n^k of the lengths $\varepsilon_k^n = \varepsilon^{2^k n}$. We order them in a sequence in the following way: Each natural number can be uniquely represented as a product of a power of two and an odd number, this follows from the prime factorization. The map

$$\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N},\ \psi(n,k) := 2^{k-1}(2n-1)$$

is thus bijective, which means that for any natural number m we can find exactly one pair (n,k) such that $\psi(n,k) = m$. Set $J_m := I_{\psi(n,k)} = I_n^k$. Because ψ is a bijection, each interval I_n^k corresponds with some J_m. Thus,

$$A \subset \bigcup_n \bigcup_k I_n^k = \bigcup_m J_m.$$

In addition,

$$|J_m| = |I_n^k| \leq \varepsilon_k^n = \varepsilon^{2^{k-1} 2n} < \varepsilon^{2^{k-1}(2n-1)} = \varepsilon^m.$$

It follows that A is microscopic. □

3 Daneš's drop theorem and its equivalents

3.1 The Bishop-Phelps' theorem

Theorem 3.1 (Bishop-Phelps). *Let X be a Banach space over the real numbers. Let C be a nonempty bounded closed subset of X. Then the functionals that attain their maximum at C are norm-dense in X^*.*

As a preparation for the proof, we will, repeating the reasoning of Johnson and Lindenstrauss [4], first show the following statement:

Lemma 3.2. *Let (U, d) be a complete metric space and f a continuous bounded function on U. Let C be a closed bounded subset of U. Then, for every $\varepsilon > 0$ there exists a point $u_0 \in C$ with the property that for any other point $u \in C$,*

$$f(u_0) \geq f(u) - \varepsilon d(u, u_0).$$

Proof of the lemma. Define a partial ordering on C in the following way:

$$u \prec v \text{ if } f(u) \leq f(v) - \varepsilon d(u, v).$$

Now we claim that every chain has an upper bound. Since this is obvious for finite chains, we will now consider an infinite chain, that is, a sequence $(u_k)_{k\in\mathbb{N}}$ such that for any $k < j$ it holds $u_k \prec u_j$. Since f is bounded on every ball in U and C is as a bounded subset of U contained in some ball, we can estimate

$$f(v) - d(u, v) \leq \sup\{f(w) : w \in C\} < \infty$$

for any $u, v \in C$. The function f is bounded itself, though, and therefore the distances $d(u_k, u_{k+1})$ have to converge to zero.

This is easy to see: should they not converge, there would exist an $\alpha > 0$ such that $d(u_k, u_{k+1}) > \alpha$ for some subsequence of $(u_j)_{j\in\mathbb{N}}$; then, however, it would follow that $f(u_j) \geq f(u_{j-1}) + \alpha$ and subsequently $f(u_j) \geq f(u_1) + j\alpha$. The function f would then be unbounded on C.

This means that we have constructed a Cauchy sequence $\{u_k\}$. As C is a closed subset of the complete space U, there exists a limit point $u \in C$. From the continuity of f we then get

$$f(u) = \lim_{k\to\infty} f(u_k).$$

Because $f(u_k)$ is an increasing sequence, it holds that $f(u) \geq f(u_k)$ for any $k \in \mathbb{N}$ and thus $u \succ u_k$. In other words, we have found an upper bound for our chain.

Since our chain was arbitrary, we can use the Zorn lemma and conclude that there exists some maximal element $u_0 \in C$ such that $u_0 \succ u$ for every $u \in C$. This point then surely fulfils the inequality in the statement of our lemma. □

Proof of Bishop-Phelps. Let C be as in the statement, choose an $\varepsilon > 0$ and a functional $x^* \in X^*$. From the above lemma we know that there exists an $x_0 \in C$ such that

$$x^*(x_0) \geq x^*(x) - \varepsilon\|x - x_0\|.$$

We now consider a topological sum of X with the real numbers and define the following two sets:

$$K_1 := \{(x,t) \in X \oplus R : x \in C, x^*(x) \geq t\}$$
$$K_2 := \{(x,t) \in X \oplus R : x \in X, t \geq x^*(x_0) + \varepsilon\|x - x_0\|\}.$$

Notice that for each fixed point x in X there has to be some number t such that (x,t) lies in K_1 and another number that forces the pair to K_2. Also note that while only points from C can build pairs lying in K_1, any x is elligible to form a point in K_2.

Since K_2 has a nonempty interior whose intersection with K_1 is empty, we follow from the Hahn-Banach separation theorem that there exists some continuous linear functional g on $X \oplus R$ fulfilling

$$g((x,t)) \geq \beta \text{ for } (x,t) \in K_1$$

$$g((x,t)) \leq \beta \text{ for } (x,t) \in K_2.$$

A functional on this sum space has to be linear in both the variables separately, however. This means that g has to have the form

$$g((x,t)) = u^*(x) + \alpha t$$

for some real α and some functional $u \in X^*$. Moreover, this α has to be negative: for a fixed x and huge t, the pair (x,t) lies in K_2 and so the condition

$$u^*(x) + \alpha t \leq \beta$$

is only fulfilled when t gets substracted; on the other hand, for the same fixed x but t deep under zero, we need to satisfy

$$u^*(x) + \alpha t \geq \beta$$

which requires again that we substract this negative t. Put together and renormed, we can thus set $\alpha = -1$ and say that our functional is of the form

$$g(x,t) = u^*(x) - t.$$

Finally, we observe that the point $(x_0, x^*(x_0))$ lies in the intersection of K_1 and K_2. This gives us a way to express the number β:

$$\beta = (u^* - x^*)(x_0).$$

We now claim that the functional $x^* - u^*$ attains its maximum on C at the point x_0 and is a sufficiently good approximation of x^*.

For the first statement, we consider K_1. We know that for any pair (y,t) from this set, it holds that

$$u^*(y) - t \geq \beta \text{ or } t - u^*(y) \leq -\beta.$$

In addition, from the definition of K_1 it is clear that $(y, x^*(y))$ is its point. Therefore, we can estimate

$$(x^* - u^*)(y) \leq -\beta = (x^* - u^*)(x_0).$$

In other words, x_0 is a point in C where the maximum of $x^* - u^*$ is attained.

Now we want to show that the norm of u^* is at most ε. We notice that for each x from X, there exists some real number $t_0 = x_0 + \varepsilon\|x - x_0\|$ such that the pair (x, t_0) lies in K_2. Because for any pair (x, t) lying in K_2, we have $u^*(x) \leq \beta + t$, for our particular choice of t we get $u^*(x) \leq \beta + x^*(x_0) + \varepsilon\|x - x_0\|$.

Let us choose an arbitrary natural number n and denote S_n the sphere in X with radius n, this means, $S_n := \{x \in X : \|x\| = n\}$. Compute the norm of $\|u^*\|$ as the supremum of the functional values over this sphere:

$$\|u*\| = \sup_{x \in S_n} \frac{|u^*|}{\|x\|} \leq \sup_{x \in S_n} (\frac{\beta + x*(x_0)}{\|x\|} + \frac{\varepsilon\|x - x_0\|}{\|x\|}).$$

This expression holds without any requirements on n, however. This means that we can choose bigger and bigger spheres, letting n to infinity. This way, we can make the first fraction arbitrarily small as both the numbers in the numerator are fixed. For the second fraction, we can use the triangle inequality to estimate

$$\frac{\varepsilon\|x - x_0\|}{\|x\|} \leq \varepsilon\frac{\|x\|}{\|x\|} + \varepsilon\frac{\|x_0\|}{\|x\|}.$$

Using once more the argument that x_0 is fixed, we conclude our proof. □

3.2 The Ekeland's variational principle

The Ekeland's variational principle is best known in the simple form that is formulated below. However, we will immediatelly generalize it[5]:

Theorem 3.3 (Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ a lower-semicontinuous function which is bounded from below. Then there exists a point x_0 in X fulfilling*

$$f(x_0) < f(x) + d(x, x_0) \text{ for every } x \in X \setminus \{x_0\}. \tag{3}$$

Theorem 3.4 (Altered Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ a lower-semicontinuous function which is bounded from below. Then, for every $\varepsilon > 0$ and every $y \in X$, there exists a point x_0 fulfilling:*

1) $f(x_0) < f(x) + \varepsilon d(x, x_0)$ for every $x \in X$ which is not equal x_0

2) $f(x_0) \leq f(y) - \varepsilon d(x_0, y)$.

Before we examine how is this theorem related to other important results, let us first show that those two versions are equivalent.

The direction from the altered theorem to the basic one is obvious: just choose $\varepsilon = 1$ and y arbitrary.

For the other direction, first note that if $d(x, y)$ is a metric on X, then $\varepsilon d(x, y)$ is a metric which keeps the completeness of X - it is easy to check that $\varepsilon d(x, y)$ fulfils all requirements from the definition of a metric. Therefore, we can apply the basic theorem to the space $(X, \varepsilon d)$ and in this way we obtain the first inequality of the altered one.

Choose $y \in X$. If y fulfils the condition 1) (possibly with regard to a rescaled metric), we are ready because y surely fulfils the second condition as well. If not, then there exists x_1 such that

$$f(y) \geq f(x_1) + \varepsilon d(y, x_1).$$

In other words, x_1 then satisfies the condition 2). Again, if this x_1 satisfies (1) as well, we end right there; if not, an x_2 can be found with

$$f(x_2) \leq f(x_1) - \varepsilon d(x_1, x_2) \leq f(y) - \varepsilon d(x_1, y) - d(x_2, x_1).$$

In this way, provided the process does not end at any finite step in which case the last point chosen would be the minimizer we are searching, we construct a (not necessarily unique) sequence $(x_k)_{k\in\mathbb{N}}$ such that for each natural k,

$$f(x_k) \leq f(y) - \sum_{n=2}^{k} \varepsilon d(x_n, x_{n-1}) - \varepsilon d(x_1, y).$$

We know, however, that the function f is bounded from below. Therefore, the distances between successive points of the sequence have to shrink so fast that $f(x_k)$ stays above this bound for any k. Therefore, $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, and because we are in a complete space, there exists a unique limit point z.

Fix $\delta > 0$. Since f is lower semicontinuous, we know that on some neighbourhood of z, it holds that

$$f(z) \leq f(x) + \delta.$$

However, by choosing k sufficiently large we know that x_k lies in this neighbourhood. Therefore, we can compute:

$$\begin{aligned} f(z) \leq f(x_k) + \delta \leq f(y) - \sum_{n=1}^{k-1} \varepsilon d(x_n, x_{n+1}) - \varepsilon d(x_1, y) + \delta \\ \leq f(y) - \varepsilon d(y, x_l) + 2\delta. \end{aligned}$$

In the last step we used the triangle inequality and the fact that $\varepsilon d(z, x) < \delta$. Because this expression holds true for any δ, we conclude that

$$f(z) \leq f(y) - \varepsilon d(y, x_l).$$

To sum up, we have proved that while we cannot guarantee our z is unique or has the property $f(z) < f(x) + \varepsilon d(x, x_0)$ no matter the choice of x we do know that z minimizes the sequence from our construction.

Define a partial ordering on X in the following way:

$$z \prec x \text{ if } f(z) < f(y) + \varepsilon d(x, z).$$

From above we know that every chain has a lower bound which in addition satisfies

$$f(x_l) \leq f(y) - \varepsilon d(x_l, y).$$

Therefore, using Zorn's lemma, there exists a minimal element x_0 fulfilling

$$f(x_0) < f(x) + \varepsilon d(x_0, x)$$

for any $x \in X$. Because this x_0 is the limit of some chain, it also satisfies

$$f(x_0) \leq f(y) - \varepsilon d(x_0, y)$$

and so it is the point we are looking for in the altered Ekeland's theorem.

Theorem 3.5. *The Bishop-Phelps theorem implies the Ekeland's variational principle.*

Proof. The idea is to apply the Bishop-Phelps theorem to the epigraph of a lower semicontinuous function. The epigraph of a function f is the set

$$\{(x, \mu) : f(x) \leq \mu\}.$$

First, observe that in the space $X \oplus \mathbb{R}$, each continuous linear functional ψ of the norm 1 has the form $\psi((x,t)) = \psi'(x) + \alpha t$ for some $\psi' \in X^*$ and $\alpha \in \mathbb{R}$ such that $\|\psi'\| + |\alpha| \leq 1$.

Let f be a lower semicontiuous function on X that is bounded from below by a real number L. Denote $C := \{(x, f(x)) : x \in X\}$ the epigraph of f and define a functional $\phi \in (X \oplus \mathbb{R})^*$ as

$$\phi((x,t)) = t.$$

If ϕ happens to attain its minimum on C then we have found the desired minimal point from the Ekeland's variational principle. If not, choose $\varepsilon > 0$ and apply the Bishop-Phelps theorem to C. To be more precise, we use it on the set $C_0 := C \cap \{(x,t) \in X \oplus \mathbb{R} : t \leq K\}$ for some K that is very large in comparison to L. We will receive a functional ψ that attains its minimum somewhere on C_0. However, because $\|\phi - \psi\|_{(X \oplus \mathbb{R})^*} < \varepsilon$, ψ will be growing rapidly with growing t and the minimum will not be attained on a point (x,t) with $t = K$. Thus, the functional ψ will attain its minimum even on C.

Decompose

$$\psi(x,t) = \psi'(x) + \alpha t.$$

On C, we can rewrite this as $\psi(x, f(x)) = \psi'(x) + \alpha f(x)$. Since $\|\psi\| = 1$ and $\|\psi\| = \|\psi'\| + |\alpha|$, it follows $1 - \varepsilon < \alpha \leq 1$ and $\|\psi'\| = 1 - \alpha$.

Denote x_0 the point at which ψ attains its minimum and assume it is not the point sought in the Ekeland's variational principle, that is, assume there exists $x \in C$ such that $f(x) < f(x_0) - d(x, x_0)$. Then, using that $\|\psi'\| = 1 - \alpha$, it follows that

$$\begin{aligned}
\psi(x, f(x)) = \psi'(x) + \alpha f(x) &< \\
&< \psi'(x) + \alpha f(x_0) - \alpha d(x, x_0) \leq \\
&\leq \psi'(x_0) + (1-\alpha) d(x, x_0) + \alpha f(x_0) - \alpha d(x, x_0) = \\
&= \psi'(x_0) + \alpha f(x_0) + (1 - 2\alpha) d(x, x_0) = \\
&= \psi(x_0, f(x_0) + (1 - 2\alpha) d(x, x_0) < \\
&< \psi(x_0, f(x_0)).
\end{aligned}$$

The last inequality folows from the fact that $\alpha > 1 - \varepsilon$ which for a small enough ε means that $1 - 2\alpha < 0$.

We have obtained the inequality $\psi(x, f(x)) < \psi(x_0, f(x_0))$ which is a contradiction as $(x_0, f(x_0))$ is the minimum of ψ on C. Thus, x_0 has to be the minimizing point in the sense of the Ekeland's variational principle. □

We end the chapter by the observation that we can use Ekeland's theorem on subsets of a complete metric space as well. Formalized, the claim is as follows:

Theorem 3.6. *Let X be a complete metric space, $A \subseteq X$ its closed subset and $f : A \to \mathbb{R}$ a lower semi-continuous function which is bounded from below on A. Then for every $y \in X$ and $\varepsilon > 0$, we can find a point $x_0 \in A$ satisfying*

$$1) f(x_0) < f(x) + \varepsilon d(x_0, x) \text{ for every } x \in A \setminus \{a\} \text{ and}$$
$$2) f(x_0) \leq f(y) - \varepsilon d(x_0, y).$$

Proof. Denote

$$L := \inf\{f(z) : z \in A\}.$$

Suppose first that A is bounded. Extend the function f to a function g operating on the whole space X such that $f = g$ on A, g is lower semicontinuous and for any $z \in X \setminus A$ it holds

$$g(z) > L + \varepsilon \text{diam}(A) + 1.$$

This is possible with the following construction: any lower semi-continuous function is a limit of continuous functions. As A is closed, they can be extended to continuous functions on the whole space using the Tietze's theorem. We add to this set the continuous function

$$h(x) := L + \text{dist}(x, A)$$

and take the supremum. This operation provides a lower semicontinous function. Adding 1 to this supremum function everywhere outside A, the lower semicontinuity is not lost and so we get the function g we desire.

Using the Altered Ekeland's theorem, we find a point $x_0 \in X$ fulfilling (3). However, if we choose $z \in X \setminus A$ and $x \in A$ such that $f(x) < L + 1$, from the definition of g it follows that

$$f(z) = L + \text{diam}(A) + 1 > f(x) + \varepsilon d(x, z).$$

This means that $z \neq x_0$ and therefore, $x_0 \in A$. Because $f = g$ on A, x_0 is then the minimizer for f we seek.

For an unbounded set A, we can divide it to countably many bounded sets:

$$A = \bigcup_{i=1}^{\infty} A_i, \text{ where } A_n := A \cap B(0, n).$$

Now, if y lies in A, there exists some natural number n such that $y \in A_n \setminus A_{n-1}$. If we choose a different natural number $m > n$ and a point $x \in A_m \setminus A_{m-1}$, it holds that

$$f(x) + \varepsilon d(x, y) \geq L + \text{dist}(A_n, A_m) = L + (m - n - 1).$$

This means that for m_1 large enough, we get

$$f(x) + \varepsilon d(x, y) > f(y),$$

which makes the condition 2) from the statement of our theorem impossible to fulfil outside of a certain set A_{m_1}.

Moreover, on A_{m_1}, the function f is bounded from above by some constant M. Therefore, there exists $m_2 \in \mathbb{N}$, $m_2 > m_1$ such that for any $x \in A_{m_1}$ and $z \in A_{m_2} \setminus A_{m_2-1}$, we can compute

$$f(x) \leq M < L + \varepsilon \operatorname{dist}(A_{m_1}, A_{m_2} \setminus A_{m_2-1}) \leq f(z) + \varepsilon d(x, z).$$

These two statements together mean that if we find a minimizer that fulfils both inequalities in the Ekeland's variational principle on A_{m_2}, it will satisfy both the inequalities on any larger set as well and thus the statement of the theorem itself. We already know that such a minimizer exists, though, because A_{m_2} is a bounded set. □

3.3 The Flower petal theorem

Definition 2. *Let (X, d) be a metric space, $a, b \in X$ and $\gamma > 0$. The* ***petal*** *$P_\gamma(a, b)$ is the set*

$$P_\gamma(a, b) := \{x \in X : \gamma d(x, a) + d(x, b) \leq d(a, b)\}.$$

Theorem 3.7 (The Flower petal theorem)**.** *Denote A a complete subset of a metric space (X, d). Let $x_0 \in A$ and let*

$$b \in X \setminus A \textit{ and } 0 < r \leq \operatorname{dist}(b, A).$$

Then for every $\gamma > 0$ there exists $a \in A \cap P_\gamma(x_0, b)$ such that

$$P_\gamma(a, b) \cap A = \{a\}.$$

Theorem 3.8. *The altered Ekeland's variational principle imlies the Flower petal theorem.*

Proof. Let A be a closed subset of the complete metric space X; on A, we will use the metric induced from X. Choose x_0, γ and b as in the Flower petal theorem. Define

$$f : A \to \mathbb{R},\ f(x) := d(x, b).$$

This function is continuous and bounded from below on A. Therefore, by the Ekeland's variational principle for subsets (Theorem 3.6), there exists a point $a \in A$ satisfying

$$f(a) < f(x) + \gamma d(a, x) \text{ for any } x \in A$$

and

$$f(a) \leq f(x_0) - \gamma d(a, x_0).$$

If we translate the second equation using the definition of f, we get

$$d(a, b) \leq d(x_0, b) - \gamma d(a, x_0),$$

which by definition means that $a \in A \cap P_\gamma(x_0, b)$.

Choose now a point $x \in A \setminus \{a\}$. From the first equation, we know that

$$d(a, b) < d(x, b) + \gamma d(a, x)$$

which means that this point cannot lie in the petal $P_\gamma(a, b)$ as this contradicts its definition. Therefore, $A \cap P_\gamma(a, b) = \{a\}$. □

3.4 The Daneš's drop theorem

Definition 3. *Let X be a Banach space, $B \subset X$ its convex subset and $x \in X$ a point. The **drop** $D(B, x)$ is the convex hull of $\{x\} \cup B$:*

$$D(B, x) = \{x + t(y - x) : t \in [0, 1], y \in B\}.$$

The drop theorem again comes in two versions, a simpler one and a generalised one.

Theorem 3.9 (The Daneš's drop theorem)**.** *Let X be a Banach space, $C \subset X$ an nonnempty closed subset and $z_0 \in X \setminus C$. Choose $\rho > 0$, $r > 0$ and $R > 0$ such that $0 < r < R = \operatorname{dist}(z_0, C) < \rho$. Then there exists a point $a \in C$ fulfilling $\|a - z_0\| \leq \rho$ and $D(B(z_0, r), a) \cap C = \{a\}$.*

Theorem 3.10 (The generalized Daneš's drop theorem)**.** *Let X be a Banach space, $C \subset X$ a nonempty closed subset and $x_0 \in C$. Let $B \subset X$ be another subset of X that is nonempty, bounded, closed and convex and fulfils $\operatorname{dist}(C, B) > 0$. Then there exists a point $a \in C \cap D(B, x_0)$ such that $C \cap D(B, a) = \{a\}$.*

In essence, the generalized version works for any closed, convex and bounded set B with a positive distance from C while the ordinary version can only be used on balls. In addition, the generalized version gives a stricter requirement on the position of a - it not only sets a maximal distance from z_0 but also specifies that the point a lies in some cone.

Theorem 3.11. *The generalized Daneš's drop theorem implies the ordinary Daneš's drop theorem.*

Proof. From the assumptions of the ordinary drop theorem, we have a closed nonempty set C, a point $z_0 \in X \setminus C$ and a closed ball $B(z_0, r)$ that does not intersect C. Choose a point $x_0 \in C$ such that $\|x_0 - z_0\| \leq \rho$. Such a point exists because the distance of z_0 from C is strictly less than ρ. Our ball is a nonempty, convex, closed and bounded set, therefore, the generalized version of the drop theorem can be used and we find a point $a \in C \cap D(B, x_0)$ such that $C \cap D(B(z_0, r), a) = \{a\}$. Because a lies in the cone defined by x_0 and $B(z_0, r)$ and the distance between x_0 and z_0 is less than ρ, it is also true that $\|a - z_0\| \leq \rho$. □

Theorem 3.12. *The Daneš's ordinary drop theorem implies the generalized Daneš's drop theorem.*

Proof. Let B, C, x_0 be given as in the assumptions of the generalized drop theorem. Denote $D := D(B, x_0) \cap C$ the intersection of C and the drop defined by the point x_0 and the set B. Because B and D are closed sets with a positive distance (as the distance of B from C is assumed positive), there exists a ball B_0 such that $B \subset B_0$ and $B_0 \cap D = \emptyset$; this B_0 might intersect C somewhere but not inside the drop $D(B, x_0)$. From the ordinary drop theorem with some large ρ, we find a point $a \in D$ such that $B_0 \cap D(B_0, a) = \{a\}$. Because B is contained in B_0, the drop $D(B, a)$ is a subset of $D(B_0, a)$ and therefore, it holds that $D(B, a) \cap D = \{a\}$.

It remains to show that even $D(B, a) \cap C = \{a\}$. We know, however, that $D(B, a) \subset D(B, x_0)$. Thus, the drop $D(B, a)$ can never contain points of C that lie outside of $D(B, x_0)$. This means that $C \cap D(B, a) = D \cap D(B, a) = \{a\}$. □

Theorem 3.13. *The Flower petal theorem implies the Daneš's drop theorem.*

Proof. [5] According to the assumptions of the drop theorem, choose a nonempty closed set C and a $z_0 \in X \setminus C$ with $d := \text{dist}(z_0, C) > 0$ and denote $B := B(z_0, r)$ the ball centered at z_0 for some $r < \text{dist}(C, z_0)$. Choose an arbitrary $x_0 \in C$ and set $A := C \cap D(B, x_0)$ and $\gamma := \frac{d-r}{d+r}$. As A is a closed subset of the complete space X, we can use the Flower petal theorem to obtain a point $a \in A \cap P_\gamma(x_0, z_0)$ such that $P_\gamma(a, z_0) \cap A = \{a\}$. We claim that a is the point we seek in the drop theorem.

The fact that $a \in C$ is obvious as $a \in A = C \cap D(B, x_0)$. In addition, since our a lies in $A = C \cap D(B, x_0)$ we know that $D(B, a) \subset D(B, x_0)$.

Now we want to show that $D(B, a) \subset P_\gamma(a, z_0)$. As a preparation, for a real number t we compute

$$\begin{aligned} \frac{d-r}{d+r} &\leq \frac{t-r}{t+r} & \iff \\ (t+r)(d-r) &\leq (t-r)(d+r) & \iff \\ -tr + rd &\leq tr - rd & \iff \\ d &\leq t. \end{aligned}$$

This in particular means that

$$\gamma = \frac{d-r}{d+r} \leq \frac{t-r}{t+r}$$

if we set $t := \|a - z_0\|$. This number is surely greater or equal d because a is a point of C.

First, choose a $z \in B(z_0, r)$. From the fact that this point fulfils

$$\|x - a\| \leq \|a - z_0\| + r \text{ and}$$

$$\|x - z_0\| \leq r$$

and from the above computation, we can conclude the following estimate:

$$\gamma\|a - z\| + \|z - z_0\| \leq \frac{t-r}{t+r}\|z - a\| + \|z - z_0\| \leq \frac{t-r}{t+r}(\|a - z_0\| + r) + r = t.$$

A general point x lying in $D(B, a)$ can be expressed as a convex combination of a point of B and a:

$$x = \alpha a + (1-\alpha)z, \text{ where } z \in B(z_0, r) \text{ and } \alpha \in [0, 1].$$

Using this, we can finally compute

$$\begin{aligned}\gamma\|x-a\| + \|x-z_0\| &= \gamma\|\alpha a + (1-\alpha)z - a\| + \|\alpha a + (1-\alpha)z - z_0\| \leq \\ &\leq \alpha\|a-z_0\| + (1-\alpha)(\gamma\|z-a\| + \|z-z_0\|) \leq \\ &\leq \alpha\|a-z_0\| + (1-\alpha)\|a-z_0\| = \|a-z_0\|\end{aligned}$$

which, according to the definition, states that $x \in P_\gamma(a, z_0)$.

Finally, we can show that a is the only intersection of C and $D(B, a)$:

$$D(B, a) \cap C \subset D(B, a) \cap (C \cap D(B, x_0)) \subset P_\gamma(a, z_0) \cap A = \{a\}.$$

□

3.5 The Brézis-Browder's theorem

Theorem 3.14 (The Brézis-Browder theorem)**.** *Let C be an nonempty closed subset of a Banach space X and $z_0 \in X \setminus C$. Let $0 < r < \operatorname{dist}(z_0, C)$ and $x_0 \in C$. Denote*

$$K := \{tx : x \in B(z_0 - x_0, r), t > 0\} \tag{4}$$

the convex cone generated by the ball $B(z_0 - x_0, r)$. Then there exists a point $a \in C \cap K(B(z_0, r), x_0)$ such that

$$C \cap (a + K) \cap B(a, \delta) = \{a\}$$

for any $0 < \delta < \operatorname{dist}(z_0, C) - r$.

For the proof of this statement, we will use the following [6]:

Lemma 3.15. *Let X be a Banach space, $B \subset X$ convex nonempty, $s_0 \in X$ arbitrary and $s \in D(B, s_0)$. Find an $\alpha \in \mathbb{R}$ and a $u \in B$ such that*

$$s = \alpha s_0 + (1-\alpha)u.$$

Then the following holds true:

1) For any $t \in [0, \alpha]$ we have $s + t(D(B, s_0) - s_0) \subset D(B, s) \subset D(B, s_0)$.

2) Assume in addition that $\operatorname{dist}(s_0, B) > 0$ and set $t_0 = \frac{\operatorname{dist}(s,B)}{\operatorname{dist}(s_0,B)}$. Then $t_0 \leq \alpha$ and for any $0 \leq t_1 \leq t_2 \leq 1$ we get

$$\begin{aligned}&(s + K(B - s_0)) \;\cap\; B(s, t_1 \operatorname{dist}(s, B)) \subset \\ &s + t_2 t_0 (D(B, s_0) - s_0) \subset \\ &D(B, s) \subset \\ &D(B, s_0).\end{aligned}$$

Here, the expression $K(B - s_0)$ denotes the convex cone generated by the set $B - s_0$: $K(B - s_0) = \{tx \in X : t > 0, x \in B - s_0\}$.

Proof. 1) For a $t \in [0, \alpha]$, choose an arbitrary $x \in s + t(D(B, s_0) - s_0)$. Then for some $\beta \in [0, 1]$ and $z \in B$ we can write

$$x = s + t(bs_0 + (1-b)z - s_0) = s + t(1-b)(z - s_0).$$

If $\alpha = 0$, then x lies in $B \subset D(B, s)$ and there is nothing to show.

Assume from now on that $\alpha > 0$. Then, using $s = \alpha s_0 + (1-\alpha)u$, we can write $s_0 = \alpha^{-1}s - \alpha^{-1}(1-\alpha)u$ which then yields

$$\begin{aligned} x &= s + t(1-b)(z - \alpha^{-1}s - \alpha^{-1}(1-\alpha)u) \\ &= s + \alpha^{-1}t(1-b)(\alpha z - s - (1-\alpha)u) \\ &= (1 - \alpha^{-1}t(1-b))s + \alpha^{-1}t(1-b)(\alpha z - (1-\alpha)u) \\ &= (1-c)s + cw, \end{aligned}$$

where $c = \alpha^{-1}t(1-b)$ and $w = \alpha z + (1-\alpha)u$. Since both z and u lie in the convex set B, w is a point of B, too. Moreover, because t was chosen such that $t \le \alpha$, we know $0 \le c \le 1 - b \le 1$. This together with $s \in B$ means that x is a convex combination of s and a point lying in B and therefore, $x \in D(B, s)$. This proves the first inclusion in the lemma.

For the second one, just observe that $s \in D(B, s_0)$, $D \subset D(B, s_0)$ and the drop $D(B, s_0)$ is convex.

2) First we notice that the cone $K(B - s_0) = \{tx \in X : t \in \mathbb{R}, x \in B - s_0\}$ does not change if multiplied by a real constant. This together with $t_1 \le t_2 \le 1$ means that

$$\begin{aligned} (s + K(D - s_0)) \cap B(s, t_1 \operatorname{dist}(s, B)) &= s + t_1(K(B - s_0) \cap B(0, \operatorname{dist}(s, B))) \\ &\subset s + t_2(K(B - s_0) \cap B(0, \operatorname{dist}(s, B))) \\ &= (s + K(B - s_0)) \cap B(s, \operatorname{dist}(s, B)). \end{aligned}$$

Similarly one gets

$$s + t_2 t_0(D(B, s_0) - s_0) \subset s + t_0(D(B, s_0) - s_0).$$

Therefore, it suffices to show that

$$(s + K(B - s_0)) \cap B(s, \operatorname{dist}(s, B)) \subset s + t_0(D(B, s_0) - s_0) \subset D(B, s).$$

To this end, fix an $x \in (s + K(B - s_0)) \cap B(s, \operatorname{dist}(s, B))$. Because this x is a point of the cone $s + K(B - s_0)$, there exist $r \ge 0$ and $z \in B$ such that $x = s + r(z - s_0)$.

At the same time, x lies in the ball $B(s, \operatorname{dist}(s, B))$ and so it holds

$$\operatorname{dist}(s, B) \ge \|x - s\| = r\|z - s_0\|.$$

From this we can deduce

$$r \le \frac{\operatorname{dist}(s, B)}{\|z - s_0\|} \le \frac{\operatorname{dist}(s, B)}{\operatorname{dist}(s_0, B)} = t_0.$$

If it happens that $t_0 = 0$, it means that $\operatorname{dist}(s, B) = 0$. In such a case,

$$B(s, t\,\mathrm{dist}(s, B)) = \{s\} \subset D(B, s)$$

and the statement is true.

Assume now $t_0 > 0$ and set $b := 1 - rt_0^{-1}$. Then, because $r \leq t_0$, we know that $b \in [0, 1]$. Thus, because $bs_0 + (1 - b)z$ lies in $D(B, s_0)$ as a convex combination of s_0 and a point of B, the point $s + t_0(bs_0 + (1 - b)z - s_0)$ is contained in the set $s + t_0(D(B, s_0) - s_0)$. This point, however, is exactly x, as the following computation shows:

$$\begin{aligned} s + t_0(bs_0 + (1 - b)z - s_0) &= s + t_0((1 - rt_0^{-1})s_0 + rt_0^{-1}z - s_0) = \\ &= s + t_0(rt_0^{-1}z - rt_0^{-1}s_0) = s + r(z - s_0) = x. \end{aligned}$$

Summed up, we have shown that the statement $x \in s + t_0(D(B, s_0) - s_0)$ is true.

For the inequality $t_0 \leq \alpha$, we use the convexity of B to compute

$$\begin{aligned} \mathrm{dist}(s, B) &= \mathrm{dist}(\alpha s_0 + (1 - \alpha)u, B) \leq \\ &\leq \alpha\,\mathrm{dist}(s_0, B) + (1 - \alpha)\,\mathrm{dist}(u, B) = \alpha\,\mathrm{dist}(s_0, B). \end{aligned} \tag{5}$$

With this relation known, we can deduce from the first part of the lemma the inclusion $s + t_0(D(B, s_0) - s_0) \subset D(B, s)$. □

Theorem 3.16. *The generalized Daneš's drop theorem implies the Brézis-Browder theorem.*

Proof. [6] From assumptions of the Brézis-Browder theorem, we have a nonempty closed set $C \subset X$, points $x_0 \in C$ and $z_0 \in X \setminus C$ and a real number r so that $0 < r < \mathrm{dist}(z_0, C)$. In addition, we set $K := \{tx : t > 0, x \in B(z_0 - x_0, r)\}$.

Denote $B = B(z_0, r)$. This B is naturally nonempty, convex, closed and bounded. Furthermore, C is another nonempty closed set fulfilling $\mathrm{dist}(C, B) > 0$. Thus, the assumptions of the drop theorem are met and we obtain a point $a \in C \cap D(B, x_0)$ such that $C \cap D(B, a) = \{a\}$.

Now we want to show that

$$C \cap (a + K) \cap B(a, \delta) \subset C \cap D(B, a) = \{a\} \tag{6}$$

for any $\delta > 0$ such that $\delta < \mathrm{dist}(z_0, C) - r$. From part 2) of the above lemma, setting $t_1 = 0$, it follows

$$(a + K) \cap B(a, \mathrm{dist}(a, B)) \subset D(B, a).$$

Using the facts that $B = B(z_0, r)$ and $a \in C \setminus B$, we get

$$d(a, B(z_0, r)) = \|a - z_0\| - r \geq d(z_0, C) - r.$$

This means that $B(a, \mathrm{dist}(z_0, C) - r) \subset B(a, \mathrm{dist}(a, B))$ and from there, we can deduce

$$(a + K) \cap B(a, \mathrm{dist}(z_0, C) - r) \subset D(B, a).$$

Thus, the equation (6) holds and the Brézis-Browder theorem is proved. □

3.5.1 Brézis-Browder implies Bishop-Phelps

Finally, we are going to close the circle by proving the theorem which we have originally started with. To do this, we slightly adjust the original proof of the Bishop-Phelps theorem[7].

Lemma 3.17. *Let $f, g \in S_{X^*}$. If for some $\varepsilon > 0$ we have $g(x) < \frac{\varepsilon}{2}$ whenever $\|x\| \leq 1$ and $f(x) = 0$, then either $\|f - g\|_{X^*} \leq \varepsilon$, or $\|f + g\|_{X^*} \leq \varepsilon$.*

Proof. If we restrict g to the subspace $Y := f^{-1}(0) = \{x \in X : f(x) = 0\}$, then from assumption

$$\|g\|_{Y^*} \leq \frac{\varepsilon}{2}.$$

Using the Hahn-Banach theorem, we can extend this restriction back to the whole space X while controlling its norm. In this way we obtain a new functional $h \in X^*$ fulfilling $\|h\|_{X^*} \leq \frac{\varepsilon}{2}$. Because $h = g$ on $f^{-1}(0)$, we in addition know that $g - h = \alpha f$ for some $\alpha \in \mathbb{R}$. Thus, $\|g - \alpha f\| = \|h\| \leq \frac{\varepsilon}{2}$.

We will handle here the case $\alpha \geq 0$; the situation for a negative α would be almost the same. Suppose first that $\alpha \geq 1$. Since both f and g are of norm 1, we can compute

$$\alpha = \|\alpha f\| \leq \|g\| + \|g - \alpha f\| \leq 1 + \|g - \alpha f\|,$$

and use it to estimate

$$1 - \alpha^{-1} \leq 1 - (1 + \|g - \alpha f\|)^{-1} = \|g - \alpha f\|(1 + \|g - \alpha f\|)^{-1}.$$

Finally we obtain

$$\|g - f\| = \|(1 - \alpha^{-1})g + \alpha^{-1}(g - \alpha f)\| \leq 1 - \alpha^{-1} + \alpha^{-1}\|g - \alpha f\| \leq 2\|g - \alpha f\| \leq \varepsilon.$$

In the case $\alpha < 1$, the estimate goes as follows:

$$\|g - f\| \leq \|g - \alpha f\| + \|(1 - \alpha)f\| = \|g - \alpha f\| + 1 - \alpha = \\ \|g - \alpha f\| + \|g\| - \|\alpha f\| \leq 2\|g - \alpha f\| \leq \varepsilon.$$

This means that in case $\alpha \geq 0$, g is a good approximation of f. In the case $\alpha < 0$ we would need to approximate with $-g$. □

Before we move on, we introduce the following notation: For a functional $f \in X^*$ and a positive number α, $K(f, \alpha)$ will be a cone definited as
$K(f, \alpha) = \{x \in X : \|x\|_X \leq \alpha f(x)\}$.

This is indeed a cone as whenever $x \in K(f, \alpha)$ and $t > 0$, we can estimate $\|tx\| \leq t\alpha f(x) = \alpha f(tx)$ and so $tx \in K(f, \alpha)$. In addition, for $\alpha > 1$ the cone is nonempty: emptiness of this set would mean that $\|x\| > \alpha f(x) > f(x)$ for every $x \in X$ which is impossible for a functional of norm one.

Lemma 3.18. *Let $f, g \in S_{X^*}$ and $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{2}$ and choose an $\alpha > 2\frac{1+\delta}{\delta}$. Suppose that g is nonnegative on $K(f, \alpha)$. Then $\|f - g\|_{X^*} \leq \varepsilon$.*

Proof. We want to use the previous lemma. To this end, we choose an $y \in S_X$ such that $f(y) = 0$ and want to show that $g(y) \leq \delta$. We find an $x \in X$ fulfilling $\|x\| = \delta$ and $f(x) > \frac{\delta}{2}$; such a point has to exist as the norm of f is 1. Then,

$$\|x \pm y\| \leq \|x\| + \|y\| < \delta + 1 < \frac{\alpha\delta}{2} \leq \alpha f(x) = \alpha f(x \pm y).$$

This means that $x \pm y \in K(f, \alpha)$ and so $g(x \pm y) \geq 0$. This in turn implies that

$$|g(y)| \leq g(x) \leq \|x\| = \delta$$

and from the previous lemma it follows that either $\|f - g\| \leq \varepsilon$, or $\|f + g\| \leq \varepsilon$.

Now we rule out the case $\|f + g\| \leq \varepsilon$. Since $\|f\| = 1$ and both ε and α^{-1} are smaller than 1, we can find a $z \in X$ such that $\|z\| = 1$ and $f(z) > \max\{\alpha^{-1}, \varepsilon\}$. Thus, $f(z) \geq \alpha^{-1}\|z\|$ which means that $z \in K(f, \alpha)$ and $g(z) = 0$. This in turn implies that $\|f + g\| \geq (f + g)(z) > \varepsilon$.

Thus it cannot be true that $\|f + g\| \leq \varepsilon$ and the only remaining possibility is $\|f - g\| \leq \varepsilon$. □

Theorem 3.19. *The Brézis-Browder theorem implies the Bishop-Phelps theorem.*

Proof. Choose a closed bounded set C and suppose at first that it is in addition convex. Furthermore, choose a functional $f \in S_{X^*}$ and an $\varepsilon > 0$. We want to find a functional $g \in S_{X^*}$ that attains its norm on C such that $\|f - g\| \leq \varepsilon$.

Choose a real number $\alpha > 1 + \frac{2}{\varepsilon}$ and a point $z_1 \in C$. From the Brézis-Browder theorem, there exists a point $z_0 \in C$ such that

$$\{z_0\} = (z_0 + K(f, \alpha)) \cap C. \tag{7}$$

To be more precise, the Brézis-Browder theorem only guarantees that the sets $(z_0 + K)$ and C intersect in no other point besides z_0 on some ball around z_0. Assume therefore that there exists some point $z_2 \neq z_1$ fulfilling

$$\{z_2\} = (z_0 + K(f, \alpha)) \cap C.$$

We can apply the Brézis-Browder theorem again to the same cone but this time with regard to the point z_2. We find some point z_3, then we use the theorem on this new point z_3 and iterate this argument. If we at any step find a point $z_k := z_0$ fulfilling (7), we will be ready. If not, from the boundedness and closedness of C we will get a convergent sequence and the desired property will hold by the limit point.

From the Hahn-Banach separation theorem, there exists a functional $g \in S_{X^*}$ such that

$$\begin{aligned} g(x_0) = \sup\{g(x) : x \in C\} \leq \\ \leq \inf\{g(x) : x \in x_0 + K(f, \alpha)\} = \\ = g(x_0) + \inf\{g(x) : x \in K(f, \alpha)\}. \end{aligned}$$

This means that $0 \leq \inf\{g(x) : x \in K(f, \alpha)\}$ and so, using Lemma 3.18, we get $\|f - g\| \leq \varepsilon$.

Let now C be a general closed and bounded set. Choose a functional f that attains its supremum on $\operatorname{conv}(C)$. We want to show that it attains its norm on C itself. Denote z the point where $f(z) = \sup\{f(x) : x \in \operatorname{conv}(C)\}$. We can write

z as a convex combination of a finite number of points $z_i \in C$ and compute using the linearity of f

$$z = \sum_{i=1}^{n} \lambda_i z_i,$$

$$f(z) = \sum_{i=1}^{n} \lambda_i f(z_i), \text{ where } \sum_{i=1}^{n} \lambda_i = 1.$$

Because $f(z) \geq f(z_i)$ for all i, it follows $f(z) = f(z_i)$ for every i which means that f attains its supremum at C in all the respective points z_i.

□

3.6 The Caristi-Kirk's theorem

The theorems we have proved in the previous sections are equivalent to some more famous statements. One of them is the following:

Theorem 3.20 (Caristi-Kirk). *Let (X, d) be a complete metric space, f a real lower semicontinuous function on X that is bounded from below and $T : X \to X$ a map such that for any $x \in X$, $d(x, Tx) \leq f(x) - f(Tx)$. Then T has a fixed point.*

Theorem 3.21. *The Ekeland's variational principle implies the Caristi-Kirk theorem.*

Proof. Let T and f be as in the requirements of the Caristi-Kirk theorem. It is possible to apply the Ekeland's theorem to f; doing so, we obtain a point $x_0 \in X$ satisfying

$$f(x_0) < f(x) + d(x, x_0) \text{ for every } x \in X \setminus \{x_0\}.$$

Assume x_0 is not a fixed point of T and set $x := T(x_0)$. From the above inequality we have

$$f(x_0) - f(T(x_0)) < d(T(x_0), x_0)$$

but, at the same time, the definition of T requires

$$f(x_0) - f(T(x_0)) \geq d(T(x_0), x_0).$$

This is a contradiction and so x_0 has to be a fixed point of T. □

Theorem 3.22. *The Caristi-Kirk theorem implies the Ekeland's variational principle.*

Proof. We will use a proof by a contradiction. If the Ekeland's theorem is not true then for any $x \in X$ there exists some $Tx \in X \setminus \{x\}$ such that

$$f(x) \geq f(Tx) + d(x, Tx).$$

This means, however, that T is a map that fulfils the requirement of the Caristi-Kirk theorem and so it has to have a fixed point. As we have constructed T so that it never maps a point on itself, we have found our contradiction. □

4 The drop property

In the previous chapters we have discussed the drop theorem and several other statements in general Banach spaces. As those results are often very powerful one can ask if their statements can be generalized in some ways.

The first obvious way to do this is to extend them to more general spaces. It turns out that at least in locally convex spaces, those statements make good sense and lead to interesting results. This kind of extension goes beyond the scope of this work, however, an interested reader might consult the paper of Hamel [8], for example.

Another natural way to relax the requirements of the drop theorem is to allow the distance of the two sets in question, the unit ball and the closed set S, to be equal 0. More precisely, we can do it as follows:

Definition 4. *The norm of a Banach space X has the **drop property** if for any closed set S disjoint with the unit ball B there exists a point $x \in S$ such that $S \cap D(B, x) = \{x\}$.*

A very easy example of a space with the drop property would be $\mathbb{R}^n$ with the Eucleidian norm. Since the unit sphere has a positive distance from any closed set that does not intersect it, we know that this space has the drop property from the Daneš's drop theorem.

On the other hand, the space l^1 does not have the drop property. To show this, we will construct a countable set $S = \{f_n\}$ of points that have no convergent subsequence. This way, S will be closed as a discrete set. In addition, having constructed the first n points we will find the $(n+1)$-th one inside the drop $D(B, f_n)$, making it impossible to find a drop such as required in the definition of the drop property.

Set $f_1 := (2, 0, 0, ...)$. For $n \geq 2$, define $f_{n+1} := \frac{1}{2} f_n + \frac{1}{2} e_{n+1}$, where e_k is the standard unit vector. Because f_{n+1} is a convex combination of f_n and $e_{n+1} \in B$, it indeed lies in the drop $D(B, f_n)$.

First we compute the norm of f_n. Because f_n is zero on the only coordinate where e_{n+1} is nonzero, we have

$$\|f_n + e_{n+1}\| = \|f_n\| + \|e_{n+1}\| = \|f_n\| + 1$$

and therefore

$$\|f_{n+1}\| = \|\frac{1}{2} f_n + \frac{1}{2} e_{n+1}\| = \frac{1}{2}(\|f_n\| + 1).$$

Because $\|f_1\| = 2$, it follows that $\|f_n\| > 1$ for all $n \in \mathbb{N}$ and $S \cap B = \emptyset$. In addition, we see that $\|f_n\|$ converges to 1 which means that $\operatorname{dist}(S, B) = 0$.

To show that S has no convergent subsequence we estimate the distance between its points f_n and f_m where it is assumed without loss of generality that $m > n$. Then, however, it follows from the construction that the m-th coordinate of f_n is zero while the m-th coordinate of f_m is $\frac{1}{2}$. Thus,

$$\|f_n - f_m\| \geq \frac{1}{2} \text{ for all } n, m \in \mathbb{N},$$

which makes the convergence for any subsequence impossible. Together this means that l_1 indeed does not have the drop property.

Now when we know there are spaces both with and without the drop property we would like to have some characterizations of this notion. The first uses a notion introduced by Rolewicz [9].

Definition 5. *A sequence* $(x_n) \subset X$ *such that* $\|x_n\| > 1$ *is called a* ***stream*** *if* $x_{n+1} \in D(B, x_n)$ *for each* $n \in \mathbb{N}$.

The assumption that the norms of the points are bigger than one is important: We want the stream to be a sequence that lies outside the unit ball but is nearing its surface in some restricted area. If we abandoned this requirement, any sequence inside the unit ball would be a stream, even sequences with a very wild behavior. As a consequence, the following useful theorem would not be true.

Theorem 4.1. *X has the drop property if and only if each stream has a convergent subsequence.*

Proof. (i) Assume that X does not have the drop property. This means that there exists a closed set S with $S \cap B = \emptyset$ such that for any $x \in S$ there exists $y \in S$, $y \neq x$ such that $y \in D(B, x)$. From the Daneš's drop theorem we know that $\mathrm{dist}(S, B) = 0$.

Fix $x \in S$. We observe that we can choose an $y \in D(B, x)$ with norm as close to 1 as we like: from the definition of a drop, we can write this y as $y = \lambda x + (1 - \lambda)s$ for an $s \in S_X$ and λ an arbitrary number in $[0, 1]$. By choosing this λ very small, we keep the norm of y as close to 1 as needed.

Knowing this, we can construct a stream $(y_n) \subset S$ such that

$$\mathrm{dist}(B, y_n) < 1 + \frac{1}{n}.$$

Because each stream has a convergent subsequence and S is closed, there exists a limit point $y \in S$ with $\mathrm{dist}(B, y) = 0$ which means that $y \in B$ and therefore, $S \cap B \neq \emptyset$ which is a contradiction to the choice of S.

(ii) Choose a stream $S = \{y_n : n \in \mathbb{N}\}$ that does not contain a convergent subsequence, this means that S is closed. Assume X has the drop property. Then there exists $n \in \mathbb{N}$ such that $S \cap D(B, y_n) = \emptyset$. At the same time, however, $y_m \in D(B, y_n)$ for every $m \geq n$, which means that $y_m = y_n$ for all such m. We have found out the stream converges despite we have assumed there is no convergent subsequence; this is a contradiction and so the proof is completed. □

Now we turn our attention to the relation between the drop property and reflexivity. The finding that the drop property implies reflexivity is due to Rolewicz [9]; the opposite direction was later proved by Montesinos [10]. First, however, we have to do some preparatory work.

Definition 6. *For* $A \subset X$, *the* ***Kuratowski measure of noncompactness*** $\alpha(A)$ *is a nonnegative real number defined as*

$$\alpha(A) := \inf\{r > 0 : A \subset \bigcup A_n \textit{ for some finite family of sets } A_n \subset X \\ \textit{satisfying}\, \mathrm{diam}(A_n) \leq r\}.$$

Note that we can assume all the covering sets A_n to actually be balls as any set of diameter less than r is contained in a ball of radius $\frac{r}{2}$.

Lemma 4.2. *Suppose there is a continuous linear functional $f \in X^*$ with $\|f\| = 1$ such that for the sets*

$$G_\varepsilon := \{x \in B_X : |f(x)| \geq 1 - \varepsilon\}$$

the Kuratowski measure of noncompactness does not tend to zero as ε goes to zero, that is, $\gamma := \inf\{\alpha(G_n) : n \in \mathbb{N}\} > 0$. Then the space X does not have the drop property.

Proof. Choose a positive $\delta < \frac{\gamma}{4}$. We want to construct a sequence (x_n) such that

$$|f(x_n)| > 1 \text{ and}$$
$$\inf\{\|x_n - z\| : z \in \operatorname{span}(x_0, ..., x_{n-1})\} > \delta.$$

First we observe that for any finite-dimensional subspace $L \in X$ and any $\varepsilon > 0$, there exists a point $x \in G_\varepsilon$ such that $\operatorname{dist}(x, L) \geq \frac{\gamma}{4}$. Suppose this is not the case, this means, $\operatorname{dist}(G_\varepsilon, L) < \frac{\gamma}{4}$. Write $X = L \times T$, this means T is an infinite-dimensional subspace of X.

Define by $\mathcal{U}$ the system of all open sets $U \in X$ satisfying

$$1)\ U = G + \frac{\gamma}{4} B_T : G \text{ is open in } L$$
$$2)\ \operatorname{diam}(G) < \frac{\gamma}{2}.$$

The system $\{G : U = G + \frac{\gamma}{4} B_T \in \mathcal{U}\}$ is an open covering of B_L. Because L is finite dimensional, its unit ball is compact and so we can find a finite subsystem $\mathcal{V} \subset \mathcal{U}$ such that $G \subset \{G : U = G + \frac{\gamma}{4} B_T \in \mathcal{V}\}$. Then, however, the set G_ε is covered by $\mathcal{V}$ as any point $x \in G_\varepsilon$ can be represented in the form

$$x = g + t : g \in B_L, t \in \frac{\gamma}{4} B_T.$$

This is a contradiction to the assumption that the Kuratowski index of noncompactness of G_ε is not less than γ.

We begin the construction of our desired sequence by choosing an arbitrary $x_0 \in X$ fulfilling $f(x_0) > 1$.

We will proceed by induction. Having constructed x_i for every natural $i \leq n$, we choose an $e_n > 0$ satisfying

$$f(x_n) - 1 > \varepsilon_n$$

and find $y_{n+1} \in G_{\varepsilon_n}$ such that

$$\inf\{\|y_{n+1} - z\| : z \in \operatorname{span}\{x_0, ..., x_n\}\} > \delta.$$

This is possible thanks to the observation our proof begins with and to the choice of δ. Using the definition of y_{n+1}, we set

$$x_{n+1} := \frac{y_{n+1} + x_n}{2}.$$

This especially means that $x_{n+1} \in D(B, x_n)$ because $G_{\varepsilon_n} \subset B$. In addition, using that $y \in G_\varepsilon$ and the choice of ε_n, we get

$$f(x_{n+1}) = \frac{1}{2}(f(x_n) + f(y_{n+1}) \geq \frac{1}{2}(f_{x_n} + (1 - \varepsilon_n)) > 1$$

and

$$\inf\{\|x_{n+1}-z\| : z \in \text{span}(x_0, ..., x_n) = \frac{1}{2}\inf\{\|y_{n+1}-z\| : z \in \text{span}(x_0, ..., x_n)\} > \frac{\delta}{2}.$$

This together means that $\{x_n : n \in \mathbb{N}\}$ is a stream that has no convergent subsequence. Therefore, the norm cannot have the drop property. □

Theorem 4.3. *Any real Banach space with the drop property is reflexive.*

Proof. Choose a continuous linear functional f with $\|f\| = 1$. Define

$$G_n := \{x \in B_X : |f(x)| \geq 1 - \frac{1}{n}\}.$$

From Lemma 4.2 we know that the Kuratowski measure of noncompactness of the sets G_n tend to zero as n goes to infinity. Set

$$G := \bigcap_{n \in \mathbb{N}} G_n.$$

We want to show that G is nonempty. Because $G = \{x \in B_X : |f(x)| = 1\}$, we will then know that f attains its norm at some point of the unit ball. As f is arbitrary, we will then be able to conclude that X is reflexive from the James' characteristic.

For the proof that G is nonempty, we know that G_1 is covered by a finite number of balls with radius less than some number δ_1, let us call them $C_{1,i}$ where i is an index that goes from 1 to some finite number. Since $G_{n+1} \subset G_n$ for every $n \in \mathbb{N}$, there exists a $C_{1,i}$ that intersects every G_n.

Should this not be the case, it would mean that for every $C_{1,i}$ there exists some G_{n_i} such that $G_{n_i} \cap C_{1,i} = \emptyset$. Denote j the maximum of such n_i - this number exists as we are examining only a finite family of sets $C_{1,i}$. Because $G_j \subset G_{n_i}$ for every i, we would have $G_j \cap C_{1,i} = \emptyset$ for all i. On the other hand, from $G_1 \subset \bigcup_i C_{1,i}$ it would follow that $G_1 \cap G_j = \emptyset$ which is impossible.

Using this, we can set $K_1 := G_1 \cap \overline{C_{1,i}}$ for this $C_{1,i}$ intersecting every G_n. This is a closed set that is containted in G_1 and intersects every G_n. In addition, its diameter is smaller than δ_1.

Moving to G_2, we know that $G_2 \subset \bigcup_i C_{2,i}$ for some balls $C_{2,i}$ with radii smaller than some $\delta_2 < \frac{\delta_1}{2}$. Using similar arguments as above, there exists some $C_{2,i}$ intersecting K_1 and every G_n for $n > 2$. Set $K_2 := K_1 \cap G_2 \cap \overline{C_{2,i}}$. This is a closed set with diamater smaller than δ_2 intersecting every G_n for $n > 2$.

Iterating this argument, we get a sequence of closed sets K_n such that $K_{n+1} \subset K_n$ and $\text{diam}(K_n) < \delta_n$. Because $\delta_n \to 0$ for $n \to \infty$, it follows from the Cantor intersection theorem that there exists some point $x \in \cap K_n \subset \cap G_n = G$. □

Having proved that drop property implies reflexivity, we will now turn our attention to the converse statement. We start by defining the following notions:

Definition 7. *A norm in X is **locally uniformly rotund (LUR)** if for any sequence $(x_n)_{n=1}^{\infty}$ and any $x \in X$ such that $\|x_n\| \leq 1$, $\|x\| = 1$ and $\lim_{n\to\infty} \|x + x_n\| = 2$ it follows that $x_n \to x$ in norm.*

Definition 8. *A norm in X has the* ***Radon-Riesz property (RR)*** *if for any sequence $(x_n)_{n=1}^{\infty}$ in X with $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ we have $x_n \to x$ strongly.*

Lemma 4.4. *It is sufficient to test the Radon-Riesz property on a unit ball. More precisely: Suppose we know that for any sequence fulfilling*

$$\|x_n\| \to \|x\| = 1, \|x_n\| \leq 1, x_n \to x \text{ weakly}$$

it follows that $x_n \to x$ in norm. Then this norm already has the Radon-Riesz property.

Proof. Choose an arbitrary (x_n) converging weakly to a point $x \in X$ such that $\|x_n\| \to \|x\|$. For every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that $\frac{x_n}{\|x\|} \in (1+\varepsilon)B_X$ for every $n > n_0$. In addition, there exist real numbers q_n such that $q_n \frac{x_n}{\|x\|} \in B_X$ and $\|q_n \frac{x_n}{\|x\|}\| \to 1$. This rescaled sequence still converges to $\frac{x}{\|x\|}$ weakly: since $q_n \to 1$ and $x_n \to x$ weakly, for any functional f we can write

$$f(q_n \frac{x_n}{\|x_n\|}) = q_n f(\frac{x_n}{\|x\|}) \to f(\frac{x}{\|x\|}).$$

From our assumption it then follows that $q_n \frac{x_n}{\|x_n\|} \to \frac{x}{\|x\|}$ in norm, and since $\frac{q_n}{\|x_n\|} \to \frac{1}{\|x\|}$, we can conclude that $x_n \to x$. □

Theorem 4.5. *A locally uniformly rotund norm has the Radon-Riesz property.*

Proof. From the above lemma we know it is sufficient to work on the unit ball. Choose (x_n) that fulfils all assumptions of the Lemma 4.4. First, we can estimate

$$\lim_{n\to\infty} \|x_n + x\| \leq \lim_{n\to\infty} (\|x_n\| + \|x\|) = 2.$$

Choose $\varepsilon > 0$ and assume $\|x_n + x\| < 2 - \varepsilon$. From the dual representation of the norm, we then know that for any functional $f \in B_{X^*}$, $|f(x + x_n)| < 2 - \varepsilon$. However, from the Hahn-Banach separation theorem, there exists a functional $g \in B_{X^*}$ such that $g(x) = 1$. Then, because $x_n \to x$, it follows that

$$g(x + x_n) \to 2g(x) = 2,$$

which is a contradiction. Therefore, $\|x_n + x\| \to 2$. This means that all assumptions on the sequence from the (LUR) property are fulfilled and thus, $x_n \to x$ in norm. □

Theorem 4.6. *The drop property implies the Radon-Riesz property.*

Proof. We know from Theorem 4.2 that for any functional $f \in B_{X^*}$ the Kuratowski index of noncompactness of the sets $G_\delta := \{x \in B_X : f(x) > 1 - \delta\}$ fulfils

$$\alpha(G_\delta) \to 0 \text{ as } \delta \to 0.$$

Choose an arbitrary sequence $(x_n) \subset B_X$ such that $x_n \to x_0$ weakly for some $x_0 \in X$ with $\|x_0\| = 1$. From the Hahn-Banach theorem we can find a functional $f \in B_{X^*}$ such that $f(x_0) = 1$. We define the sets G_δ as above with respect to f and find for a given $\varepsilon > 0$ a $\delta > 0$ fulfilling $\alpha(G_\delta) < \varepsilon$.

Since y_n converges weakly to x_0, we can find $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have $y_n \in G_\delta$. Because $\alpha(G_\delta) < \varepsilon$, the set G_δ is covered by a finite number of balls of radius smaller than ε. Thus, there exists some y_n such that $\|y_n - x_0\| < \varepsilon$. From a diagonal argument we can therefore find a subsequence $(z_n) \subset (y_n)$ that converges to x_0 strongly.

Since our choice of y_n was arbirary, it follows that $x_n \to x_0$ strongly. Indeed, should this not be the case, we would be able to find a subsequence $(y_n) \subset (x_n)$ such that $\|y_n - x_0\| > \kappa$ for some $\kappa > 0$ and for any $n \in \mathbb{N}$. However, it would be impossible to choose a convergent subsequence of this (y_n).

This concludes the proof for sequences that are subsets of the unit ball. For a general sequence, use Lemma 4.4. □

Lemma 4.7. *Let $(x_1, x_2, ..., x_n)$ be the first n points of a stream, that means, $x_i \notin B$, $x_{i+1} \in D(B, x_i)$. Let $z \in \operatorname{span}\{x_1, ..., x_{n-1}\}$. Then $z \notin B$ and $x_n \in D(B, z)$.*

Proof. We will proceed by induction. For $n = 2$ the claim is obviously true as in this case, $z = x_1$. For $n = 3$, we can assume without loss of generality that $x_1 \neq x_2$, $z \neq x_i$, $i = 1, 2$. Since points x_2 and x_3 lie in certain drops, we can write the following convex combinations:

$$\begin{aligned} x_2 &= \lambda_2 x_1 + (1 - \lambda_2) b_2, \quad b_2 \in B_X, \lambda_2 \in (0, 1) \\ x_3 &= \lambda_3 x_2 + (1 - \lambda_3) b_3, \quad b_3 \in B_X, \lambda_3 \in (0, 1) \\ z &= \theta x_1 + (1 - \theta) x_2, \qquad\qquad\quad \theta \in (0, 1). \end{aligned}$$

Hence,

$$\begin{aligned} z &= \theta x_1 + (1 - \theta)[\lambda_2 x_1 + (1 - \lambda_2) b_2] \\ &= k x_1 + (1 - \theta)(1 - \lambda_2) b_2, \end{aligned}$$
$$\text{where } k = \theta + (1 - \theta)\lambda_2 \in (\lambda_2, 1).$$

Therefore,

$$x_1 = \frac{1}{k}[z - (1 - \theta)(1 - \lambda_2) b_2]$$

and, plugging the definition of x_1 and x_2 in the formula for x_3,

$$\begin{aligned} x_3 &= \lambda_3 x_2 + (1 - \lambda_3) b_3 \\ &= \lambda_3[\lambda_2 x_1 + (1 - \lambda_2) b_2] + (1 - \lambda_3) b_3 \\ &= \lambda_3 \lambda_2 x_1 + \lambda_3 (1 - \lambda_2) b_2 + (1 - \lambda_3) b_3 \\ &= \lambda_3 \lambda_2 \frac{1}{k}[z - (1 - \theta)(1 - \lambda_2) b_2] + \lambda_3 (1 - \lambda_2) b_2 + (1 - \lambda_3) b_3 \\ &= \frac{1}{k} \lambda_3 \lambda_2 z + \frac{1}{k}[k \lambda_3 (1 - \lambda_2) - (1 - \theta)(1 - \lambda_2) \lambda_3 \lambda_2] b_2 + (1 - \lambda_3) b_3. \end{aligned}$$

This means that x_3 is a convex combination of the points z, b_2 and b_3, and thus $x_3 \in D(B,z)$. Indeed,

$$\begin{aligned}
&\frac{1}{k}\lambda_3\lambda_2 + \frac{1}{k}[k\lambda_3(1-\lambda_2) - (1-\theta)(1-\lambda_2)\lambda_3\lambda_2] + (1-\lambda_3) \\
&= \lambda_3(1-\lambda_2) - \frac{1}{k}(1-\theta)\lambda_2\lambda_3(1-\lambda_2) + \frac{1}{k}\lambda_2\lambda_3 + 1 - \lambda_3 \\
&= \lambda_3 - \lambda_2\lambda_3 - [\frac{1}{k}\lambda_2\lambda_3 - \frac{1}{k}\theta\lambda_2\lambda_3 - \frac{1}{k}\lambda_2^2\lambda_3 + \frac{1}{k}\theta\lambda_2^2\lambda_3] + \frac{1}{k}\lambda_2\lambda_3 + 1 - \lambda_3 \\
&= 1 - \lambda_2\lambda_3 + \frac{1}{k}\lambda_2\lambda_3[\theta + \lambda_2 - \theta\lambda_2] \\
&= 1 - \lambda_2\lambda_3 + \lambda_2\lambda_3 = 1.
\end{aligned}$$

In addition, if the norm of z were less or equal 1, we would have $\|x_3\| \leq 1$ as x_3 would be a convex combination of points from the unit ball. Since we know that $\|x_3\| > 1$, it follows that $\|z\| > 1$. This completes the proof of the case $n = 3$.

For the general case, let us assume the lemma is true for $n \geq 3$ and choose points $x_1, ..., x_{n+1}$ and $z \in \text{conv}\{x_1, ..., x_n\}$. There exists $u \in \text{conv}(x_{n-1}, x_n)$ such that $z \in \text{conv}(x_1, ..., x_{n-2}, u)$. Since $u \in D(B, x_{n-2})$, it follows from the case $n = 3$ that $u \notin B$ and $x_{n+1} \in D(B,u)$. From the induction hypothesis we can conclude that $z \notin B$ and $x_{n+1} \in D(B,z)$. □

Theorem 4.8. *If $(X, \|.\|)$ is a reflexive Banach space with a norm that has the Radon-Riesz property, then the norm has the drop property.*

Proof. Choose a stream $(x_n)_{n=1}^{\infty}$. We want to show that this stream has a convergent subsequence. Once this is established, the claim will follow from Theorem 4.1.

If there exists some index such that the sequence is constant beyond this point then the whole sequence converges. Therefore, we can assume from now on that there exists no index such that (x_n) would be constant beyond this point.

Suppose that (x_n) has no convergent subsequence and furthermore that there is no subsequence of (x_n) of elements whose norms converge to 1. Then, $\{x_n\}$ is a closed set in X with a positive distance from the unit ball. From the Daneš's drop theorem we can find an $x_m, m \in \mathbb{N}$ such that

$$\{x_m\} = \{x_n : n \in \mathbb{N}\} \cap D(B, x_m).$$

However, from the definition of a stream it then follows that $x_n = x_m$ for every natural number $n \geq m$ which is a contradiction.

The only case remaining is the situation where we can find a subsequence of (x_n) of points whose norm tends to 1. Without loss of generality, assume that $\|x_n\| \to 1$. Because the set $\{x_n : n \in \mathbb{N}\}$ is bounded we can find a weakly convergent subsequence converging to a point $x_0 \in X$. Again, assume without loss of generality that $x_n \to x_0$ weakly. Because the norm is weakly lower semicontinuous, we know that

$$\|x_0\| \leq \liminf_{n\to\infty} \|x_n\| = 1. \tag{8}$$

On the other hand, the weak topology coincides with the strong topology on convex sets. Therefore,

$$x_0 \in \overline{\text{conv}\{x_n : n \in \mathbb{N}\}}^{w} = \overline{\text{conv}\{x_n : n \in \mathbb{N}\}}^{\|\cdot\|}.$$

From the previous lemma we know that $\|x_0\| \geq 1$. Together with (8) we deduce $\|x_n\| = 1$.

Summed up, we have found a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x_0$ weakly and $\|x_n\| \to \|x_0\|$. From the Radon-Riesz property we can conclude that $x_n \to x_0$ in the norm topology, too. This means that even in this last remaining case we have for our stream found a convergent subsequence. □

Theorem 4.9. *In any reflexive Banach space there exists an equivalent norm with the drop property.*

Proof. Any reflexive Banach space X is weakly compactly generated. From the Troyanski's renorming theorem (see [11]), there exists an equivalent norm on X which is locally uniformly rotund. We have shown in Theorem 4.5 that this norm also has the Radon-Riesz property. From Theorem 4.8 it then follows that this norm has the drop property. □

The last characteristic of a norm with the drop property uses the following definition:

Definition 9. *A set $M \subset X$ is called* ***approximatively compact*** *if for any sequence (y_n) in X such that $\|y_n - x\| \to \operatorname{dist}(x, M)$ for some $x \in X$ one can find a convergent subsequence.*

Theorem 4.10. *A norm in X has the drop property if and only if each closed, convex and bounded set in X is approximately compact.*

Proof. Choose an $M \subset X$ closed, convex and bounded. Furthermore, choose an arbitrary $x \in X$ and a sequence $(y_n) \subset M$ that minimizes the distance of x from M. We can assume without loss of generality that $x = 0 \notin M$, this means, $\|y_n\| \to \operatorname{dist}(0, M)$. Since the sequence (y_n) is bounded, there exists a weakly convergent subsequence y_{n_k} converging to some point $y \in M$. From the computation

$$\operatorname{dist}(0, M) \leq \|y\| \leq \operatorname{liminf}\|y_{n_k}\| = \operatorname{dist}(0, M)$$

we get that $\|y_{n_k}\| \to \|y\|$. Since, according to Theorem 4.6, a norm with the drop property has the Radon-Riesz property and we know that $y_{n_k} \to y$ weakly, it follows that $y_{n_k} \to y$ strongly and M is approximately compact.

For the opposite direction, assume for a contradiction that there exists a closed set $S \in X$ such that $S \cap B = \emptyset$ but for any point $x \in S$, $S \cap D(B, x) \neq \{x\}$.

Using these assumptions, we can construct a stream $(x_n) \subset S$ such that $\|x_n\| \to 1$. Define $M = \overline{\operatorname{conv}}(x_1, x_2, x_3, ...)$. This set is convex and closed and therefore approximately compact by assumption. Furthermore, the stream (x_n) minimizes the distance of M from the origin. This means that there exists a subsequence $(x_{n_k}) \subset (x_n)$ that converges strongly to some point $z \in M$; we then have $\|x_n\| \to \|z\|$ and $\|x_n\| \to 1$ which means that $z \in B$. However, because $(x_n) \subset S$ and S is closed, z lies in S as well. Therefore, $S \cap B$ is not an empty, which is a contradiction. □

5 Microscopic sets on Banach spaces

In the first chapter we have introduced the notion of a microscopic set on the real line. Here, we will generalize it and explore the connection between microscopic sets and drops.

According to Donnini and Martellotti [12], there are at least two ways how to introduce this notion:

Definition 10. *$M \subset X$ is called **microscopic** if for any $\varepsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $M \subset \bigcup_{n=1}^{\infty}(x_n + \varepsilon^n B_X)$. $M \subset X$ is called **scalarly microscopic** if for any functional $f \in X^*$ the set $f(M) \subset \mathbb{R}$ is microscopic.*

Using the properties of microscopic sets on the real line, one can easily find out the following:

Lemma 5.1. *1) Any microscopic set is scalarly microscopic.*
2) If $A \subset B$ and B is scalarly microscopic, then A is scalarly microscopic.
3) If $A = \bigcup_{n=1}^{\infty} A_n$ and A_n are all scalarly microscopic, then A is scalarly microscopic.
4) If M is scalarly microscopic, then $x + M$ and αM are scalarly microscopic for all $x \in X$ and $\alpha \in \mathbb{R}$.
5) Any countable set is scalarly microscopic.

Proof. 1) Choose $f \in X^*$ and let M be microscopic. Since $f(B_{X^*})$ is containted in the closed interval $[-\|f\|, \|f\|]$, we have

$$f(M) \subset f(\bigcup_{n=1}^{\infty}(x_n + \varepsilon^n B_X)) \subset \bigcup_{n=1}^{\infty}[f(x_n) - \varepsilon^n\|f\|, f(x_n) + \varepsilon^n\|f\|].$$

2) $f(A) \subset f(B)$ which implies that any covering of $f(B)$ covers $f(A)$.
3) $f(A) \subset \bigcup_{n=1}^{\infty} f(A_n)$. A countable union of microscopic sets is microscopic.
4) It suffices to shift and rescale the intervals covering the original set.
5) The image of a countable set is countable and each countable subset of the real axis is microscopic. □

As an example, X is not scalarly microscopic as for a functional $f \in X^* \setminus \{0\}$, $(0, \|f\|) \subset f(X)$ and intervals are not microscopic.

Define an interval in a Banach space as the convex hull of two points x and y, $x \neq y$. Such an interval is not scalarly microscopic, either: there exists a functional $f \in X^*$ such that $f(x) < f(y)$. Then, $f(\mathrm{conv}(\{x\}, \{y\})) = [f(x), f(y)]$ and again, an interval is never microscopic.

Before we move on to drops, we will need to introduce the following notion that will be useful in later proofs.

Definition 11. *Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. The **induced polygonal** $P(x_n)$ is defined as the union of the intervals connecting the adjacent points of the sequence:*

$$P(x_n) = \bigcup_{n=1}^{\infty} \mathrm{conv}(\{x_n\}, \{x_{n+1}\}).$$

*A sequence $(x_n)_{n\in\mathbb{N}}$ is called a **stream with basis** C for a set $C \subset X$ if $x_{n+1} \in D(C, x_n)$ but $x_n \notin C$ for all $n \in \mathbb{N}$.*

Lemma 5.2. *Let $(y_n)_{n\in\mathbb{N}}$ be a dyadic stream with the basis $C \subset X$, that is, a stream such that for any $n \in \mathbb{N}$, there exists an $x_n \in C$ fulfilling*

$$y_n = \frac{y_{n-1} + x_n}{2} \text{ and}$$
$$y_1 = \frac{x + x_1}{2} \text{ for some } x \in X \setminus C.$$

Suppose there exists a $\delta > 0$ such that $\operatorname{dist}(y_n, \operatorname{span}\{y_1, \dots, y_{n-1}\}) > \delta$. Then the induced polygonal $P := P(y_n)$ is closed.

Proof. See [12]. □

In the previous chapters, we have studied the drop property for a norm. Here, we will use a slightly more general version and define the drop property of a set. It will be easy to see that a norm has the drop property if the unit ball has the drop property.

Definition 12. *A set $C \subset X$ has the* ***drop property*** *if for any $F \subset X$ closed, convex and disjoint with C one can find a point $x_0 \in F$ such that*

$$D(C, x_0) \cap F = \{x_0\}.$$

A set $C \subset X$ has the ***microscopic drop property*** *if for any $F \subset X$ closed, convex and disjoint with C one can find a point $x_0 \in F$ such that $D(C, x_0) \cap F$ is scalarly microscopic.*

Obviously, the drop property implies the microscopic drop property. We will prove, however, that in reflexive spaces those two notions are actually the same. For the proof, we will need the following:

Definition 13. *Let $C \subset X$, $\delta > 0$ and $f \in X^*$ a functional that is bounded on C. Denote $M := \sup_{x\in C}\{f(x)\}$. The* ***slice*** *$S(f, C, \delta)$ is a subset of C where the functional almost attains its supremum:*

$$S(f, C, \delta) = \{x \in C : f(x) > M - \delta\}.$$

The set C has the ***property*** *(α) if for each functional $f \in X^* \setminus \{0\}$,*

$$\lim_{\delta\to 0^+} \alpha(S(f, C, \delta)) = 0,$$

where α is the Kuratowski measure of noncompatness.

Theorem 5.3. *Suppose $C \subset X$ is a convex set with the microscopic drop property and choose a stream (x_n) with basis C. Then the induced polygonal is not closed.*

Proof. Assume for contradiction that there exists a stream $(x_n)_{n\in\mathbb{N}}$ whose induced polygonal P is closed. $P \cap C = \emptyset$ as otherwise, a point of the stream itself would lie in the set C.

Choose $y \in P$. Then there exists a $k \in \mathbb{N}$ such that y lies in the interval between x_k and x_{k+1}. Then, $x_{k+1} \in D(C, y)$ because $x_{k+1} \in [x_k, c]$ for some $c \in C$ and y as a convex combination of x_k and x_{k+1} lies on the same line as x_k,

x_{k+1} and c and further from C than x_k; this means x_{k+1} can be easily written as a convex combination of y and c.

Since y and x_{k+1} are containted in $D(C, y)$, it follows from convexity that the whole interval $[y, x_{k+1}] \in D(C, y)$. However, this interval is not microscopic and so the drop cannot be microscopic as well.

This already means that C cannot have the microscopic drop property. If it was the case, because $P \cap C = \emptyset$, there would exist $y \in P$ such that $D(C, y)$ would be microscopic which is according to the previous arguments never the case. $\square$

Theorem 5.4. *If X is reflexive and a noncompact set $C \subset X$ has the microscopic drop property, then it has a nonempty interior and fulfils the property (α).*

Proof. Assume C does not have the property (α). Then there exists a functional $f \in X^*$ such that $\inf_{\varepsilon>0} \alpha(S(f, C, \varepsilon)) > 0$. We find a stream (x_n) such that $\operatorname{dist}(x_n, \operatorname{span}\{x_1, \dots, x_{n-1}\}) > \frac{\delta}{2}$, where $0 < \delta < \frac{1}{2} \inf_{\varepsilon>0} \alpha(S(f, C, \varepsilon))$ - the construction goes as in Lemma 4.2. Set $P := P(x_n)$. P is closed from Lemma 5.2 which is a contradiction to Theorem 5.3.

For the proof that the interior is not empty, see [12]. $\square$

Lemma 5.5. *For arbitrary two sets C and D, the Kuratowski measure of noncompatness fulfils $\alpha(C + D) \leq \alpha(C) + \alpha(D)$.*

Proof. For each $\gamma > \alpha(C)$ and $\delta > \alpha(D)$, there exist finitely many points c_i and d_j such that $A \subset \bigcup(c_i + \gamma B)$, $D \subset \bigcup(d_j + \delta B)$. Choose an arbitrary point $c + d \in C + D$. There exist c_i and d_j such that $\|c_i - c\| < \gamma$, $\|d_j - d\| < \delta$. Then, $\|(c + d) - (c_i + d_j)\| \leq \|c_i - c\| + \|d_j - d\| < \gamma + \delta$ which means that $\alpha(C + D) < \gamma + \delta$. $\square$

Theorem 5.6. *If X is reflexive and $C \subset X$ is a closed convex set with nonempty interior and the property (α), then C has the drop property.*

Proof. Since X is reflexive, it has an equivalent norm with the drop property as we have shown in the previous chapter. Both the drop property and the property (α) are invariant under isomorphisms, however. Consequently, we can assume without loss of generality that the norm itself already has the drop property.

Assume for contradiction that C does not have the drop property. Then there exists a closed set A, $A \cap C = \emptyset$ such that for any $x \in A$ we can find an $a \in D(C, x)$, $a \neq x$. As we have already seen, we have

$$\operatorname{dist}(a, C) < \operatorname{dist}(x, C). \tag{9}$$

Choose $x_1 \in A$ arbitrary. We construct a stream $(x_n)_{n\in\mathbb{N}} \subset A$ with basis C such that if we set

$$d_n := \operatorname{dist}(x_n, C) \text{ and}$$
$$d_n' := \inf\{\operatorname{dist}(a, C) : a \in A \cap D(C, x_n)\}$$

we will have

$$d_{n+1} < d_n' + \frac{1}{n}. \tag{10}$$

The sequence $(d_n)_{n\in\mathbb{N}}$ is convergent as it is decreasing and bounded by $\operatorname{dist}(C, A)$. Set

$$\varepsilon := \lim_{n\to\infty} d_n.$$

There are two cases that we will handle separately:

In the first case, assume $\varepsilon = 0$, which means that $\text{dist}(A, C) = 0$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - x_n\| \to 0$. Set $A_1 = \text{conv}(x_n)$. Because (x_n) is a stream and C is convex, we know that $A_1 \cap C = \emptyset$. Therefore, there exists a functional $f \in X^* \setminus \{0\}$ that separates A_1 and C:

$$M := \sup\{f(x) : x \in C\} \leq \inf\{f(x) : x \in A_1\}.$$

Because of this, $f(y_n) \to M$. From the property (α) it then follows that $\{y_n\}$ is bounded as it can be covered by a finite number of balls. Thus, because X is reflexive, $(y_n)_{n\in\mathbb{N}}$ has a subsequence that is weakly convergent to a point y. Because C is closed, it is also weakly closed and so $y \in C$. From the weak convergence, $f(y) = M$.

In addition, we show that the convergence $y_n \to y$ is actually strong. Suppose this is not the case. Then, for a sufficiently small neighbourhood U of y, we can find a subsequence $(z_k)_{k\in\mathbb{N}} \subset (y_n)_{n\in\mathbb{N}}$ such that $z_k \notin U$ for every $k \in \mathbb{N}$. Denote $Z := \text{conv}\{z_k : k \in \mathbb{N}\}$. If $y \notin Z$, we can separate y and Z by a continuous linear functional and then, y cannot be a weak limit of $(z_k)_{k\in\mathbb{N}}$, which is a contradiction as (z_k) is a subsequence of a sequence that converges weakly to y. However, the case $y \in Z$ is impossible as well, because $f(z_j) < M$ for all $j \in \mathbb{N}$ and therefore,

$$f(y) = M > f(\sum_{j=1}^{m} c_j z_j)$$

for any convex combination of points from the sequence (z_k).

Now that we know that $\|y_n - y\| \to 0$, we use the triangle inequality to find out that $\|x_n - y\| \to 0$. Since A is closed, however, it follows $y \in A \cap C$, which is a contradiction with the choice of A.

In the second case we assume $\varepsilon > 0$. Set $C_1 := C + \varepsilon B$. First we show that C_1 is weakly closed. Choose a sequence $(c_n + b_n)_{n\in\mathbb{N}} \subset C_1$ such that $c_n \in C$ and $b_n \in \varepsilon B$ for every $n \in \mathbb{N}$. Suppose that this sequence converges weakly to some $c + b$. We want to show that $c + b \in C_1$. Since εB is weakly compact, $(b_n)_{n\in\mathbb{N}} \subset \varepsilon B$ has a weakly convergent subsequence $(b_{n_k})_{k\in\mathbb{N}}$. As εB is weakly closed, its limit point lies in εB. Furthermore, because $(c_{n_k} + b_{n_k})_{k\in\mathbb{N}}$ converges to $c + b$ as a subsequence of a sequence converging to the same limit, c_{n_k} has to be convergent as well with a limit in C, and since the limit of $b_{n_k} + c_{n_k}$ is unique, $b_{n_k} + c_{n_k} \to c + b \in C + \varepsilon B$. This shows that C_1 is weakly closed and therefore it is closed in norm as well.

Now we show that C_1 has the property (α). To this end, choose a functional $f \in X^*$ such that $\sup_{C_1} f < \infty$ (which implies $\sup_C f < \infty$). Observe that $\sup_C f + \sup_{\varepsilon B} f = \sup_{C_1} f$ since for any $c \in C$ and $b \in \varepsilon B$, the value $f(c) + f(b) = f(c + b)$ is attained at the point $c + b \in C + \varepsilon B$ and the reverse inequality holds for any sum of supremums. Therefore, for any $\delta > 0$, whenever

$$f(c) < \sup_C f - \delta \text{ and } f(b) < \sup_{\varepsilon B} f - \delta$$

for a point $c + b \in C + \varepsilon B$, we have

$$f(c + b) = f(c) + f(b) < \sup_C f + \sup_{\varepsilon B} f - 2\delta = \sup_{C_1} f - 2\delta.$$

This then implies
$$S(f, C_1, \delta) \subset S(f, C, \delta) + S(f, \varepsilon B, \delta)$$
and from the previous lemma it finally follows

$$\alpha(S(f, C_1, \delta)) \leq \alpha(S(f, C, \delta)) + \alpha(S(f, \varepsilon B, \delta)).$$

We know that $x_{n+1} \in D(C, x_n) \subset D(C_1, x_n)$. In addition, since $d_n \to \varepsilon$, it also holds $\text{dist}(x_n, C_1) \to 0$. Therefore, as in the first part of this proof, the stream $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence that we will without loss of generality call $(x_n)_{n\in\mathbb{N}}$ as well. We denote its limit point x. Since A is closed, $x \in A$.

We observe that the drop $D(C, x_n)$ is closed for every natural n. This follows from the closedness of C:

Choose an arbitrary convergent sequence $p_k \in D(C, x_n)$ and denote the limit point p. From the construction of a drop, we can write $p_k = \lambda_k x_n + (1 - \lambda_k) q_k$ where $q_k \in C$. The coefficients λ_k form a sequence in the compact interval $[0, 1]$, which means that there exists a convergent subsequence - assume without loss of generality that $\lambda_k \to \lambda$. Then, the corresponding q_k converge as well:

$$\|q_k - q_l\| = \|\frac{1}{\lambda_k} p_k - \frac{1}{\lambda_l} p_l\| \leq \frac{1}{\lambda_k}\|p_k - p_l\| + |\frac{1}{\lambda_k} - \frac{1}{\lambda_l}|\|p_l\| \to 0.$$

However, C is closed, which means that there exists $q \in C$ such that $q_k \to q$. Then, $p = \lambda x_n + (1 - \lambda) q$ which means that $p \in D(C, x_n)$.

Because $x_n \in D(C, x_m)$ whenever $n \geq m$, from closedness we have $x \in D(C, x_n)$ for all $n \in \mathbb{N}$. From (9), there exists $a \in D(C, x)$, $a \neq x$ fulfilling $\text{dist}(a, C) < \text{dist}(x, C)$. Therefore, we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \text{dist}(x, C) - \text{dist}(a, C).$$

Thus,

$$d_{n+1} = \text{dist}(x_{n+1}, C) > \text{dist}(x, C) > \text{dist}(a, C) + \frac{1}{n} \geq d_n + \frac{1}{n},$$

which contradicts (10). □

Summed up, we have proved the following:

Theorem 5.7. *For X reflexive and C non-compact, the following statements are equivalent:*
1) C has the drop property.
2) C has the microscopic drop property.
3) C has nonempty interior and the property (α).

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