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Jan Novák

The mathematical theory of perturbations in cosmology

Ústav teoretické fyziky

Supervisor of the doctoral thesis: Mgr. Vojtěch Pravda Ph.D.

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I want to dedicate this work to Ann Donskikh.

"Life is the most miraculous thing in this Universe."

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: The mathematical theory of perturbations in cosmology

Autor: Jan Novák

Katedra: Ústav teoretické fyziky, V Holešovičkách 2, MFF UK

Vedoucí disertační práce: Mgr. Vojtěch Pravda Ph.D., Matematický ústav akademie věd, Žitná 25

Abstrakt: V práci se zabývám teorií kosmologických perturbací. V první kapitole zkoumám obecnou teorii relativity ve vyšších dimenzích. Zmiňuji se o GHP-formalizmu a představuji klasifikaci prostoročasů. Hodně prostoru věnuji spinorům, které používám pro další argument, který se týká speciálnosti prostoročasů v dimenzi 4. Také zavádím Kundtovy prostoročasy.
Druhá kapitola je věnována perturbacím FLRW prostočasů v GHP formalizmu, které plánujeme použít na kosmologickou inflaci.
Závěrečná kapitola patří skalárním perturbacím v f(R)-kosmologiích, které můžeme použít na zrychlující se expanzi v posledních 5 miliardách let. Zkoumám Vesmír na škálách do 150 Mpc, kde nemám možnost použít hydrodynamický přístup. Ale pracuji se zobecněním Landauova mechanického přístupu. Pro získání potenciálů Φ a Ψ používám kvazi-statickou aproximaci. Výsledek plánuji také použít na numerickou simulaci pohybu galaxií v těchto potenciálech.

Klíčová slova: gravitace, teorie kosmologických perturbací, f(R)-kosmologie

Title: The mathematical theory of perturbations in cosmology

Author: Jan Novák

Department: Department of Theoertical Physics, V Holešovičkách 2, Faculty of Mathematics and Physics, Charles University

Supervisor: Mgr.Vojtěch Pravda PhD., Matematický ústav akademie věd, Žitná 25

Abstract: In this thesis we deal with cosmological perturbation theory in my work. We investigate General Theory of Relativity in Higher Dimensions in the Chapter 1. I mention GHP-formalism and algebraical classification of spacetimes. I use spinors to show that spacetimes of dimension 4 are special. I discuss also Kundt spacetimes, which are interesting for perturbation theory of black holes.
I work with perturbations of FLRW ST's in GHP formalism in Chapter 2, which we want to use in Cosmological Inflation.
The final part of my thesis is connected with scalar perturbations in f(R)-cosmologies, that can be used for explaining accelerated expansion in the last 5 billion years. I investigate the Universe at the scales of 150 Mpc, where I could not use the hydrodynamical approach. Thus I work with the generalization of the Landau's mechanical approach. I need quasi-static approximation for getting the potentials Φ and Ψ, since the equations are too complicated for direct integration. I plan to use the result also for numerical simulation of motions of dwarf galaxies in these potentials.

Keywords: cosmological perturbation theory, f(R)-cosmologies, gravity

Contents

1. Introduction

1.1 General introduction

The main topic of this thesis is cosmology, in particular cosmological perturbation theory. Let me start with an introduction to this topic. At this moment it is known that there are three fundamental interactions, which are described by Standard Model: electromagnetic, weak and strong nuclear forces; The only missing knowledge is the neutrino mass. Gravity is described by the standard General Relativity (SGR), which is a theory of spacetime and matter. There is no contradiction of this theory with empirical observations at this moment. One prediction of SGR which was not directly confirmed yet is the experimental observation of gravitational waves. Some researchers are already very optimistic that the gravitational waves will be found soon.

It looks like that we should be satisfied from purely empirical point of view. However, from a mathematical point of view, the situation is still not satisfactory. While the Standard Model is based on Quantum Field Theory (QFT), the theory of gravity is purely classical.

One of the approaches to Quantum Gravity (QG) is the String Theory. According to this theory the basic objects are small vibrating strings. String theory was formulated in 10 or 11 dimensional spacetime. From one of the previous articles of our coworker, [1], we have a hint that we are not living in a higher-dimensional spacetime (ST). However, it should be also possible to formulate String Theory in the four-dimensional spacetime. Physical idea behind the String Theory is different from other theories. It is not a direct quantization of SGR or any other classical theory of gravity. It is a prototype of unified theory of all interactions. Gravity as well as other interactions, only emerges in an appropriate limit. Strings are one dimensional objects characterized by one parameter α or the string length $l_s = \sqrt{2\alpha\hbar}$. Strings forms a two dimensional surface in the spacetime, the world sheet. Closer inspection of strings leads also to other objects known as D-branes. String theory necessarily contains gravity, because the graviton - the hypothetical particle - appears as an excitation of closed strings.

There are also other approaches toward Quantum Gravity, instead of String Theory, we could mention, for example, e.g. the Loop Quantum Gravity. It still remains a challenge for us to formulate a consistent theory of QG, which could then be applied to cosmology.

1.2 Cosmology - historical background

We want to devote this part to historical background of Cosmology. In a sense this scientific discipline is like the archeology. Something happened in the past and now we uncover the remnants of these events by modern technologies. The results of modern experiments should confirm our theoretical models. The disadvantage is that we have only one Universe. However, we use accelerators for simulation of the very hot and dense state at the beginning of the Universe.

Our present understanding of the Universe is based upon the successful hot Big Bang theory, which explains its evolution from the first fraction of a second to our present age, 13.7 billion years later. This theory rests upon standard General Relativity and was experimentally verified by three observational facts: the expansion of the Universe (Edwin P. Hubble in 1930's), the relative abundance of light elements (George Gamow in 1940's) and finally by the cosmic microwave background (Arno A. Penzias and Robert W. Wilson in 1965).

1.3 FLRW

The modern cosmology is based on the, so called, cosmological principle. This means that the Universe looks the same for observers in different points. We could talk about extremal Copernican principle, [2]: Universe looks like homogeneous and isotropic on big scales (more than 100 Mpc); These assumptions lead to an essential simplification in the form of the, so called, FLRW metrics, which we then plug to Einstein's equations. The Friedmann equation gives us the evolution equation. Let us now write the FLRW metric for open, flat and close universe, which corresponds to K equal to -1, 0 and 1 respectively. The 3-dimensional spatial slices are hyperbolic surfaces with negative curvature, flat Euclidean surfaces with zero curvature or 3-spheres with positive curvature:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\big[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\big] = g_{\mu\nu}dx^\mu dx^\nu, \tag{1.1}$$

where $a(t)$ is the so called scale factor, which determines the physical size of the Universe. $\{r, \theta, \phi\}$ are co-moving coordinates, a particle initially at rest in these coordinates remains at rest.

The physical separation between freely moving particles at $(t, 0)$ and (t, r) is

$$d(t,r) = \int ds = a(t)\int_0^r \frac{\mathrm{d}s}{\sqrt{1-Ks^2}}. \tag{1.2}$$

We know already from 1920's that galaxies are receding, and therefore the Universe is expanding. The distance increases with time in an expanding Universe ($\dot{a} > 0$):

$$\dot{d} = \frac{\dot{a}}{a}d \equiv H(t)d, \tag{1.3}$$

with $H(t)$ being the Hubble parameter, sometimes called the Hubble constant. We can say that the velocity of receding of galaxies is proportional to a distance. This relation was observed on scales of 100 of MPc, but it was found on scales of 10 - 30 MPc. We know today that it holds even on scales of units of MPc. The value for the Hubble parameter in present days is

$$H_0 \approx 69 \pm 9 \text{ km/s/Mpc}. \tag{1.4}$$

We can write the metric (1.3) also in another form where we will use notation: $S(r) = \frac{\sin(\sqrt{K}r)}{\sqrt{K}}$ (where the case $K = 0$ can be obtained by limitting procedure);

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\big[\mathrm{d}r^2 + S^2(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\big] \tag{1.5}$$

The form of the FLRW metric in the $K = 0$ case can be transformed to a form similar to the Schwarzschild metric in standard coordinates with the difference that there is a function of time and radial coordinate R in front of the $\mathrm{d}T^2$ and $\mathrm{d}R^2$:

$$\mathrm{d}s^2 = \mathrm{F(T,R)dT^2} - \frac{1}{\mathrm{F(T,R)}}\mathrm{dR^2} - (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{1.6}$$

where F(T,R) is a function of T(t,r) and R(t,r).

The spatial curvature of the universe is equal to:

$$R^{(3)} = \frac{6K}{a^2(t)}. \tag{1.7}$$

As we have already written, spatially open, flat and closed universes have different geometries. Light geodesics in these universes behave differently, and thus can be in principle distinguished experimentally. We can also compute a four-dimensional spacetime curvature (see e.g., [3]) :

$$R^{(4)} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right). \tag{1.8}$$

Finally, we could also change the time coordinate to conformal time: $\mathrm{dt} = \mathrm{a}\mathrm{d}\eta$. We get from (1.5):

$$\mathrm{d}s^2 = a^2(t)\left\{\mathrm{d}\eta^2 - \left[\mathrm{dr^2} + \mathrm{S^2(r)}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]\right\}. \tag{1.9}$$

This metric is conformal to Minkowski in the case $K = 0$ and this will be the case in which we will be interested the most in this work.

The time evolution of the scale factor is governed by Einstein's equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu},$$

$$\text{curvature} = \text{matter}$$

with R the scalar curvature and $R_{\mu\nu}$ the Ricci curvature tensor respectively. We have geometrical quantities on the left-hand side and a matter content on the right-hand side. This equation means that the geometry of spacetime is influenced by the matter and that there is a reaction of motion of matter on the spacetime geometry.

A matter is represented by the energy-momentum tensor. Depending on the dynamics - and thus a matter-energy content of the universe - we will have different possible outcomes of the evolution. The universe may expand forever, re-collapse in the future or approach an asymptotic state in between. The matter fluid which is consistent with the homogeneity and isotropy is a perfect fluid, one in which an observer, co-moving with the fluid, would see the universe around it as isotropic. The energy-momentum tensor associated with such a fluid can be written as

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu}, \tag{1.10}$$

where $p(t)$ and $\rho(t)$ are pressure and energy density of the matter in given time of the expansion, and U^μ is the co-moving four-velocity satisfying $g_{\mu\nu}U^\mu U^\nu = 1$.

Let us write now the equations of motion in an expanding universe. According to SGR, these equations can be deduced from Einstein's equations (1.10), where we substitute the FLRW metric and the perfect fluid tensor (1.10). This leads to the famous Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{K}{a^2}. \tag{1.11}$$

The conservation of energy, a direct consequence of the general covariance of the theory, can be written as

$$\frac{d}{dt}(\rho a^3) + p\frac{d}{dt}(a^3) = 0. \tag{1.12}$$

We will introduce the equation of state for the perfect fluid, $p = w\rho$, where w is a constant. Then the continuity equation can be integrated to give

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \implies \rho \sim \ a^{-3(1+w)}. \tag{1.13}$$

From the two equations (1.11) and (1.12) one can derive the third, so called Raychaudhuri equation. (This is a general feature of f(R)-theories as we will mention in Chapter 4.) :

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.14}$$

From (1.11), neglecting the curvature terms, it then follows

$$a \sim \begin{cases} t^{2/(3(1+w))}, & w \neq -1 \\ e^{Ht}, & w = -1 \end{cases} \tag{1.15}$$

The matter in the universe consists of several fluids $T_{\mu\nu} = \sum_i T^{(i)}_{\mu\nu}$, with i for non-relativistic matter, radiation, or cosmological constant. If the energy exchange between them is negligible, then it follows that all fluids separately satisfy the continuity equation. We can define an equation of state for each fluid separately $p_i = w_i\rho_i$. (We can even define different temperature for different fluid, but this is another thing.)

The velocities of motion of the non-relativistic matter are much smaller than the velocity of light. If the universe is dominated by non-relativistic matter, then $w = 0$ and $\rho_m \sim \frac{1}{a^3}$. It follows from (1.15) that $a \sim t^{\frac{2}{3}}$.

Radiation includes all relativistic species, for example photons. For radiation $w_{rad} = \frac{1}{3}$ and from (1.14) we have that $\rho_{rad} \sim \frac{1}{a^4}$. If the universe is dominated by radiation, it follows from (1.15) that $a \sim \sqrt{t}$.

The least understood component of energy-momentum is vacuum energy. It remains constant with time. If it dominates a universe, then $a(t) \ \sim e^{Ht}$. We could formally add the cosmological constant on the right-hand side of Einstein's equations in the energy-momentum sector.

Define $\Omega_i = \frac{\rho_i}{3H^2}$. Then the Friedmann equation (1.11) becomes

$$\Omega = \sum_i \Omega_i = 1 + \frac{K}{aH^2}. \tag{1.16}$$

Thus Ω is smaller, equal or larger than unity for open, flat, or closed universe respectively. We find for the present values $\Omega_b \sim 0.04$ (baryons), $\Omega_{dm} \sim 0.31$ (dark matter), $\Omega_{rad} \sim 10^{-5}$ (radiation), $\Omega_\Lambda \sim 0.069$ (cosmological constant) - approximate numbers from the Planck-collaboration project. Visible matter makes up only a very small part.

Hubble's law and other observations indicate that the Universe is expanding. We even know today that the expansion is accelerating. It follows that the temperature decreases like $T \sim \frac{1}{a}$ with the expansion. The universe is initially hot and dense and it cools as it expands. The key ingredients of the Big Bang model is the primordial nucleosynthesis, matter-antimatter relation, matter-radiation equality, recombination, formation of gravitationally-bounded systems and temperature of relic radiation.

We will not discuss now the basic cosmological models, these could be found e.g. in the book [3]. We will rather say more about the cosmological constant. People began to add again the cosmological constant to Einstein's equations because of the accelerated expansion. (However, it should have a much more bigger value when we take into account QFT.)

In spite of theoretical prejudice towards $\Lambda = 0$, there are new observational arguments for a non-zero value, $\Lambda > 0$. The most important ones are recent evidence that we live in a flat Universe, together with indications of low mass density. This indicates that some kind of dark energy must make up the rest of the energy density. In addition, the disagreement between the ages of globular clusters and the expansion age of the Universe may be resolved with $\Lambda \neq 0$. Finally, it was experimentally verified that we live in an accelerating universe (the well-known experiment with supernovas, [4]).

The so called dark energy has to resist gravitational collapse, otherwise it would have been detected already as a part of the energy in the halos of the galaxies. However, if most of the energy of the Universe resists gravitational collapse, it is impossible for structure in the universe to grow. This dilemma can be resolved if the hypothetical dark energy was negligible in the past and only recently became dominant. In other words: gravitation is not attractive on big scales, but it behaves more like a Yukawa interaction; When the Universe was smaller, there was an attractive force which deccelerated the expansion. Later came a repulsion, when the distances between galaxies became bigger!

The dark energy has a negative pressure. This rules out all of the usual suspects like neutrinos, cold dark matter, radiation, etc. It is possible that the non-zero cosmological constant has something to do with limits of SGR, so that we will need other classical theory of Gravity. (Maybe a road toward the f(R)-gravities is a promising one.)

What are the shortcomings of the Big-Bang model? When we omit the basic problem of initial singularity, we still have many other troubles:

Photons travel on null geodesics with

$$\mathrm{ds}^2 = 0 \rightarrow \mathrm{dr} = \frac{\mathrm{dt}}{\mathrm{a(t)}} \tag{1.17}$$

for a radial path, where ds is an element of infinitesimal distance in spacetime.

The particle horizon is the type of horizon defined by the distance which light signal can travel between $t_0 = 0$ and t. Let's have a comoving observer which is located on (r_0, θ_0, ϕ_0). The signal starts its journey at $t = 0$. For what values of (r, θ, ϕ) it reaches the value $r = 0$? Light travels on geodesics $ds^2 = 0$ and because universe is homogeneous, we could set $r_0 = 0$. The geodesics passing through $r = 0$ are lines of constant θ and ϕ, so $d\theta = d\phi = 0$. The choice of θ_0 and ϕ_0 is irrelevant, due to the isotropy of space.

We have from the FLRW metric that

$$\int_0^t \frac{\mathrm{dt}'}{a(t')} = -\int_{r_p}^0 \frac{\mathrm{dr}'}{\sqrt{1 - Kr'^2}}. \tag{1.18}$$

So the particle horizon is equal to

$$R_p(t) = a(t) \int_0^t \frac{\mathrm{dt}'}{a(t')} = a(t) \int_0^{r_p} \frac{\mathrm{dr}'}{\sqrt{1 - Kr'^2}} = a(t) \int_0^a \frac{\mathrm{d(ln\,a)}}{aH} \sim t, \tag{1.19}$$

and it gives size to a causal region. Note that the particle horizon is set by comoving Hubble radius $(aH)^{-1}$. Physical lengths λ are stretched by the expansion-scale factor a, $\lambda \sim a$. For more information see e.g. [5].

Scales that are inside the horizon at present were outside in earlier times. Consider two photons of CMB, which were emitted at the time of last scattering t_{ls}. The distance $\lambda(t_0)$ between the two points is today smaller than the particle horizon $R_p(t_0)$. The $\lambda(t_{ls}) > R_p(t_{ls})$ was bigger than the particle horizon. According to standard physical theories causal contact was not possible. Although these photons came from two disconnected regions, they have nearly the same temperature with a very good precision. How could it be possible? This is the horizon problem, but there are other two classical problems with standard Big-Bang model. Let us now summarize these problems:

- Horizon problem: although universe was vanishingly small, the quick expansion didn't allow causal contact. The CMB (cosmic microwave background) has a perfect black body spectrum. Two photons coming from the opposite directions of the universe have nearly equal temperatures. The two elementary particles - photons, which are coming from the different parts of the sky, could not have a causal contact with each other.

- Flatness problem:

  Consider the Friedmann equation in the form $\Omega - 1 = \frac{K}{(aH)^2}$, where Ω is the density as defined before (now without index). The co-moving Hubble radius $(aH)^{-1}$ grows with time, and thus $\Omega = 1$ is a special point called unstable fixed point in the language of ODE's. Therefore the value of Ω had to be extremely balanced.

- Monopole problem:

  If the Universe can be extrapolated back in time to high temperatures (we have to remember that we only have direct evidence for the Big- Bang picture for low temperatures), the Universe should go through series of phase transitions during its evolution. The electroweak and QCD phase transitions, and also other transitions were considered at the Grand Unified

Theory scales. Topological defects - cosmic strings or monopoles - could be formed depending on the broken symmetry in the phase transition.

It depends on fundamental group, which topological defect will form. We define a theory with a Lagrangian L, which has a symmetry with respect to group G. This group G is spontaneously broken to subgroup $G^{'}$. The symmetry of the vacuum manifold is the coset group $M = G/G'$. According to the Kibble mechanism, topological defects form after a phase transition, if one of the homotopy groups π_n, $0 \leq n \leq 3$, of the vacuum manifold is nontrivial. When $\pi_0(M) \neq 1$ then domain walls form, when $\pi_1(M) \neq 1$ then cosmic strings form, when $\pi_2(M) \neq 1$ then monopoles form and when $\pi_3(M) \neq 1$ then cosmic textures form.

For example, monopoles are heavy pointlike objects, which behave as cold matter $\rho_{mp} \sim \frac{1}{a^3}$. If the monopoles had been produced in the early Universe, the energy density of monopoles would have decreased slowlier than the radiation background, and would have began to dominate the energy density in the Universe very early, and this is in conflict with observations.

Other relics which were not observed are particles which are also useful as viable dark matter candidates.

1.4 Inflation

The hot Big Bang model could not also explain the origin of structure in the Universe, the origin of matter and radiation, and the initial singularity. In particular, the questions, why is this Universe so close to the spatially flat one and why is the matter so homogeneously distributed on large scales, could be solved by the so called Cosmological Inflation.[1] This theory was invented at the beginning of 1980's by A. Guth, A. Linde and A. A. Starobinsky. The Cosmological Inflation is like a phase transition and it is mathematically characterized by $\ddot{a} > 0$. It is an epoch when the universe was exponentially expanding for a tiny moment, when it was approximately only $10^{-43} - 10^{-32}$ second old. According to this model the universe is filled with homogeneusly distributed scalar field. This field then decays. There is a similarity to the current situation in our Universe because we have also an accelerating epoch, but the difference is in the length of the accelerating period, for instance. The today's acceleration epoch lasts approximately 5 billion years. It was announced that from the result of experiment BICEP2, which was published in March 2014, there were indirectly measured primordial gravitational waves. However, this result was not confirmed. It seems that there was a contribution from "magnetized gas". (Result from September 2014, see e.g.[6])

The vacuum like period (the homogeneously distributed scalar field) that drives the Cosmological Inflation must be dynamic, it can't be real cosmological constant, because inflation must end. If we want to violate the strong energy condition and get a system with $\rho = -p$, we can use scalar fields. Let us explain the basic concept of scalar fields minimally coupled to matter, which are one of the triggers of the Cosmological Inflation. We will consider for simplicity the

[1] Details could be found, for example, in [7].

single scalar field. Let's take the following action for the scalar field, which we will call the inflaton field (see e.g. the lectures [8]):

$$S = \int \sqrt{-g}[-\frac{1}{2}R + L_{\varphi}]\ d^4x, \tag{1.20}$$

with $L_{\varphi} = \frac{1}{2}g^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi - V(\varphi)$ where $g = \det[g_{\mu\nu}] = -a^6$ (FLRW).
We can make a variation with respect to the scalar field and we arrive to Euler-Lagrange equations:

$$\frac{\partial L_{\varphi}}{\partial\varphi} - \nabla^{\mu}[\frac{\partial L_{\varphi}}{\partial(\nabla_{\mu}\varphi)}] = 0. \tag{1.21}$$

However

$$\frac{\partial L_{\varphi}}{\partial\varphi} = -V(\varphi), \quad \frac{\partial L_{\varphi}}{\partial(\nabla_{\rho}\varphi}) = \nabla_{\rho}\varphi. \tag{1.22}$$

So the standard result is

$$\Box\varphi + \frac{\partial V}{\partial\varphi} = 0. \tag{1.23}$$

With the definition

$$T_{\mu\nu} \equiv 2\frac{1}{\sqrt{-g}}\frac{\partial S_{\varphi}}{\partial g^{\mu\nu}}, \tag{1.24}$$

we get also

$$T_{\mu\nu} \equiv -\partial_{\mu}\varphi\partial_{\nu}\varphi + g_{\mu\nu}(\frac{1}{2}\partial_{\rho}\varphi\partial^{\rho}\varphi - V(\varphi)). \tag{1.25}$$

If we take the homogeneous field $\varphi(t)$, the equation (1.23) will reduce to

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0. \tag{1.26}$$

One can describe the evolution of $\varphi(t)$ and $a(t)$ together with Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\rho_{\varphi},\ \rho_{\varphi} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi),\ M_P^2 = \frac{1}{\sqrt{8\pi G_N}}. \tag{1.27}$$

We could also characterize the inflation by the following expressions

$$\rho_{\varphi} + 3p_{\varphi} < 0,\ p_{\varphi} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{1.28}$$

It is often the case that the potential energy dominates the kinetic energy during the inflation. This is one of the conditions for slow-roll approximation

$$\ddot{\varphi} << 3H\dot{\varphi} + V^{'}(\varphi),\ \dot{\varphi}^2 << V. \tag{1.29}$$

Inflation is characterized also by accelerated expansion, but very rapid one. The number

$$N(t) = \int_t^{t_e} H(t^{'})\ dt^{'}, \tag{1.30}$$

serves as a measure of the expansion rate.

One of the choices for the potential is so called chaotic inflation:

$$V = \frac{1}{2}m^2\varphi^2. \tag{1.31}$$

The scale factor then behaves as

$$a = a_0 \exp[\frac{\varphi_0^2 - \varphi^2}{4M_P^2}] \tag{1.32}$$

and the number of e-foldings in (1.30) in slow-roll approximation is

$$N = \frac{\varphi^2 - \varphi_e^2}{4M_p^2}, \tag{1.33}$$

where $\varphi_e = \varphi(t_e)$ and t_e denotes the end of inflation.

We already mentioned the primordial gravitational waves. Let me thus make a few remarks about spectrum for gravitons. The space-part of the metric can be written in the following form:

$$h_{ij} = 2\psi\delta_{ij} + 2\partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + h_{ij}^{TT}, \tag{1.34}$$

where ψ and E are scalar quantities, F_i are vector quantities and h_{ij}^{TT} are tensor quantities.

The evolution equation for tensor perturbations is

$$(\partial^2 + 2H^{'}\partial_0)h_{ij}^{TT} = 0. \tag{1.35}$$

We should apply the standard quantization procedure and then we get the spectrum of gravitational waves. Observing gravitational waves would give us another chanel for our knowledge about early Universe.

We are almost certain that a phase transition really happened at the beginning of evolution of the Universe. I included this part here because we want to study the Cosmological Inflation by the methods described in Chapter 3, of my dissertation. It is also possible to model the Cosmological Inflation together with the accelerated epoch through the f(R)-cosmologies, which I study in Chapter 4, of my thesis.

1.5 Cosmological perturbation theory

Let's formulate first the key idea of the theory of cosmological perturbations.[2] We have two objects, background spacetime and perturbed spacetime. The perturbed spacetime is a little bit deformed version of the background spacetime. It is advantageous to assume small perturbations, because we will do first order perturbation theory. We can have more coordinates in the background and perturbed spacetime and we want to perform coordinate transformations.

Let us illustrate this on the example of scalar quantity s:

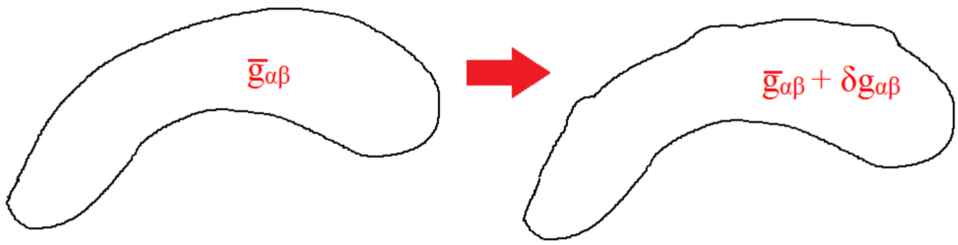
$$s = \overline{s} + \delta s. \tag{1.36}$$

We divide the scalar quantity s between background and perturbation, but the perturbation δs is gauge dependent!

Now we consider a spacetime (ST) and a perturbed ST that is close to the background ST. We have an example of the background and a perturbed ST in the figure 1.1 . We will take the following metric on the perturbed ST:

[2] We use a signature $(-,+,+,+)$ in this thesis. In Chapter 3 signature $(+,-,-,-)$ is used.

Figure 1.1: **Perturbation of background spacetime**



$$g_{\mu\nu}(t,\vec{x}) = \overline{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x}), \tag{1.37}$$

where bar means again the background and δ is a small change - perturbation - of the metric. We also assume that first and second partial derivatives are small, because we have second order PDE's. The field equations after subtraction read:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}, \tag{1.38}$$

where $\delta G_{\mu\nu}$ is a perturbation of the Einstein tensor, and $\delta T_{\mu\nu}$ is a perturbation of energy-momentum tensor.

Given a background coordinate system, we have many coordinate systems in the perturbed one, for which (1.37) holds. The choice among coordinates is called a gauge choice.

There are two possibilities how to deal with gauge transformations:

let's say that the association between background and perturbed ST is done with a coordinate system x^α ; we could have more coordinates in the perturbed spacetime, we will call it $\hat{x}^\alpha$ and $\tilde{x}^\alpha$. Now, we can perform a coordinate transformations in the perturbed spacetime. Or we can perform coordinate transformation in the background and this will induce a coordinate transformation into the perturbed spacetime. The advantage of the second choice is clear: we could respect by transformations the symmetries of the background Friedmann-Lemaitre-Robertson-Walker (FLRW) ST (homogeneous transformations of time like conformal time and transformations of space coordinates like rotations)

We drop from our equations all terms which are products of small quantities of $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\sigma}$ and $\delta g_{\mu\nu,\sigma\tau}$ in first order perturbation theory. The field equations then become the linear differential equations for $\delta g_{\mu\nu}$. We will use the conformal time in the following computations, because it is easier to work with expressions in metric.

The background ST will be FLRW ST. We will concentrate mainly on flat space (FLRW(0)). The metric in co-moving coordinates is

$$\mathrm{d}s^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{1.39}$$

where $a(\eta)$ follows from Friedmann evolution equations with cosmological constant equal to zero. We will denote again the background quantities by overbar. The Friedmann equations could be rewritten as

$$H_c^2 = \frac{8\pi\overline{\rho}\ a^2(\eta)}{3}, \tag{1.40}$$

$$H_c^{'} = \frac{-4\pi}{3}(\overline{\rho} + 3\overline{p})\ a^2(\eta), \tag{1.41}$$

where $H_c^{'} = \frac{dH_c(\eta)}{d\eta}$ is the derivative with respect to the conformal time. The energy- continuity equation is then

$$\overline{\rho}^{'} = -3H_c(\overline{\rho} + \overline{p}). \tag{1.42}$$

From (1.40) and (1.41) (with notation $w \equiv \frac{\overline{p}}{\overline{\rho}}$)

$$H_c^{'} = \frac{(-1 - 3w)}{2} H_c^2. \tag{1.43}$$

We need to define also other type of horizon, so called Hubble radius. It is defined by

$$R_H = \frac{1}{H}. \tag{1.44}$$

We could measure by Hubble radius whether particles are causally connected. It tells us the distance, how far could light travel during one expansion time. When particles are separated by a distance bigger than Hubble radius, they can not currently communicate.

These equations show that $w = \frac{-1}{3}$ corresponds to constant comoving Hubble radius, when the previous equation is zero. But for $w < \frac{-1}{3}$ the comoving Hubble radius shrinks with time (this is a typical situation for the Cosmological Inflation), whereas for $w > \frac{-1}{3}$ it grows with time.

The metric could be rewritten similarly as the scalar quantity: background and perturbation; We can write the metric of the perturbed FLRW(0) universe as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}), \tag{1.45}$$

where $h_{\mu\nu}$, as well as $h_{\mu\nu,\rho}$ and $h_{\mu\nu,\rho\sigma}$ are assumed to be small. We are doing the first order perturbation theory, so we shall drop from the equations all the terms which are of order $O(h^2)$ or higher, defining

$$h^\mu{}_\nu \equiv \eta^{\mu\rho}\eta^\sigma{}_\nu h_{\rho\sigma},\ h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}. \tag{1.46}$$

The inverse metric of the perturbed ST is in first order

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu}). \tag{1.47}$$

We will now give different names for the time and space components of the perturbed metric. We define

$$h_{\mu\nu} = \begin{pmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix},$$

where $D = -\frac{1}{6}h^i{}_i$ carries the trace of the spatial metric perturbation h_{ij}, and E_{ij} is traceless,

$$\delta^{ij}E_{ij} = 0. \tag{1.48}$$

Since indices on $h_{\mu\nu}$ are raised and lowered with $\eta_{\mu\nu}$, we immediately have

$$h^{\mu\nu} = \begin{pmatrix} -2A & B_i \\ B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix}.$$

The line element is thus

$$\mathrm{d}s^2 = a^2(\eta)\left\{-(1+2A)\mathrm{d}\eta^2 - 2\ B_i\ \mathrm{d}\eta\mathrm{d}x^i + \Big[(1-2D)\delta_{ij} + 2E_{ij}\Big]\mathrm{d}x^i\mathrm{d}x^j\right\}. \tag{1.49}$$

We will use this form of metric for the case of scalar perturbations in $f(R)$-cosmologies in the third chapter. We would fix the correspondence between background and peturbed spacetime as we said. We will perform transformations of the background FLRW(0), which respect the symmetries. We will divide the perturbations into scalar, vector and tensor ones.

We respect the homogeneity property in the background, which gives us unique slicing of the ST into homogeneous, $t = const.$, spacelike slices. This leaves us homogeneous transformations of the time coordinate, which we have as an example, when we switch from the cosmic time t to the conformal time η, (1.9). We can make transformations in the space coordinates

$$x^{i'} = X^{i'}{}_k x^k, \tag{1.50}$$

where $X^{i'}{}_k$ is independent of time. We had chosen Euclidean coordinates for the 3-metric in our background and this leaves us rotations. We have for the space-part of the metric:

$$g_{ij} = a^2\delta_{ij} \tag{1.51}$$

The transformation matrices read

$$X^{\mu'}{}_\rho = \begin{pmatrix} 1 & 0 \\ 0 & X^{i'}{}_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^{i'}{}_k \end{pmatrix},$$

and

$$X^{\mu}{}_{\rho'} = \begin{pmatrix} 1 & 0 \\ 0 & R^{i}{}_{k'} \end{pmatrix},$$

where $R^{i'}{}_k$ is a rotation matrix, with the property $R^TR = I$ or $R^{i'}{}_k R_{i'l} = \delta_{kl}$. Thus $R^T = R^{-1}$, so that $R^{i'}{}_k = R_k{}^{i'}$.

This coordinate transformation in the background induces the corresponding transformation,

$$x^{\mu'} = X^{\mu'}{}_{\rho} x^{\rho}, \tag{1.52}$$

in the perturbed ST. Here the metric is

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} -1-2A & -B_i \\ -B_i & 1-2D\delta_{ij}+2E_{ij} \end{pmatrix} \\ &= a^2\eta_{\mu\nu} + a^2 \begin{pmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij}+2E_{ij} \end{pmatrix}. \end{aligned}$$

Let's transform the metric according to

$$g_{\rho'\sigma'} = X^{\mu}{}_{\rho'} X^{\nu}{}_{\sigma'} g_{\mu\nu}. \tag{1.53}$$

After computation for the perturbations in the new coordinates, we obtain

$$\begin{aligned} A^{'} &= A, \\ D^{'} &= D, \\ B_{l'} &= R^{j}{}_{l'} B_j, \\ E_{k'l'} &= R^{i}{}_{k'} R^{j}{}_{l'} E_{ij}. \end{aligned} \tag{1.54}$$

Thus we see that A and D transform like scalars in the background spacetime coordinates, B_i like a 3-vector and E_{ij} like a tensor. We could think of them as scalar, vector and tensor fields on the 3-dimensional background spacetime. However, we can extract two more scalar quantities from B_i and E_{ij}, and a vector quantity from B_i and E_{ij}. We can write

$$\vec{B} = \vec{B}_S + \vec{B}_V, \tag{1.55}$$

$\vec{B}_S = -\nabla B$ and $\nabla.\vec{B}_V = 0$. So $\vec{B}_S$ is gradient of a scalar quantity and $\vec{B}_V$ has zero divergence. We could write similarly

$$E_{ij} = E^{S}_{ij} + E^{V}_{ij} + E^{T}_{ij}, \tag{1.56}$$

where $E^S_{ij} = E_{,ij} + \delta_{ij}\delta^{kl}E_{,kl}$ and $E^V_{ij} = -\frac{1}{2}(E_{i,j} - E_{j,i})$, where $\delta^{ij}E_{i,j} = 0$. $\delta^{jk}E^T_{ij,k} = 0$, $\delta^{ij}E^T_{ij} = 0$. Altogether: 4 scalar quantities A, B, D, E, 2 vector quantities B_i and E_i with 2 constraints and a symmetric tensor quantity E^T_{ij} with 4 constraints. We have together 10 degrees of freedom corresponding to the 10 components of the metric perturbation $h_{\mu\nu}$.

Thus the metric perturbation can be divided into scalar, vector and tensor part and these names refer to their transformation property in the background spacetime. In all textbooks it is written that scalar, vector and tensor perturbations do not couple to each other and they evolve independently.

The scalar perturbations are for us the most important. They couple to the density and pressure perturbations and exhibit gravitational instability. They are responsible for formation of structure in the Universe from small initial perturbations.

The vector perturbations tend to decay in the expanding universe and are therefore not important in cosmology. Tensor perturbations have cosmological importance, since they have an observable effect on the anisotropy of the cosmic microwave background.

We now define the following two quantities called the Bardeen potentials:

$$\boxed{\begin{array}{rcl}\Phi & \equiv & A + H_c(B - E^{'}) + (B - E^{'})^{'}, \\ \Psi & \equiv & D + \frac{1}{3}\nabla^2 E - H_c(B - E^{'}) = \psi - H_c(B - E^{'}),\end{array}}$$

where $\psi = D + \frac{1}{3}\nabla^2 E$.

The quantities are invariant under gauge transformations when we use that $\tilde{H}_c = H_c$, because H_c do not transform - the background quantity. These potentials were introduced by Bardeen and they are the simplest gauge-invariant linear combination of A, D, B and E, which span a two-dimensional space of gauge invariant variables and which can be constructed from metric-variables alone.

We can use the gauge freedom to set the scalar perturbations B and E to zero.

Doing this gauge transformation we arrive at a commonly used gauge, which has a name conformal-Newtonian gauge (other names are also used for this gauge). We will denote quantities in this gauge with the superscript N. Thus $B^N = E^N = 0$, whereas we immediately see that

$$\begin{aligned} A^N &= \Phi, \\ D^N &= \Psi. \end{aligned} \tag{1.57}$$

Thus the Bardeen potentials are equal to the two nonzero metric perturbations in the conformal-Newtonian gauge. We could use also different gauges but we are interested mainly in the conformal-Newtonian gauge.

General relativistic effects become unimportant for subhorizon scales and a Newtonian description becomes valid. In this limit, the issue of gauge choice become irrelevant as all "sensible" gauges approach each other, and the conformal-Newtonian density and velocity perturbations become those of a Newtonian description. The Bardeen potential can then be understood as a Newtonian gravitational potential due to density perturbations.

More about perturbations can be found in the lectures of Hanu-Kurki Suonio and Tomislav Prokopec ([8], [9], [10]). For general introduction to cosmology, see, for example, [3].

1.6 The plan of this thesis

The thesis is divided into 3 main chapters. We study general theory of relativity in higher dimensions in Chapter 2. Then Chapter 3, where we study FLRW spacetimes using the GHP formalism, follows. We introduce general f(R)-theories in Chapter 4. We published the original result from this chapter in EPJ C, [67].

2. Standard General Relativity in Higher Dimensions

2.1 Introduction

The studying of standard General Relativity (SGR) in higher dimensions (HD) - generally for dimension of spacetime d with non-compact dimensions - serves here as a nice preparation for our other works in the second and third part of this dissertation (Chapter 3 and Chapter 4). I begin with this topic and briefly discuss the GHP formalism in HD[1]. Then a section about classification of the Weyl tensor in HD follows. I also included a section about spinors and I mentioned Kundt spacetimes, because of their usefulness in perturbations of black holes.

Before we will begin our own work, let us mention the following inspirative ideas. Details could be found, for example, in [11], however, various authors also discuss this topic in other sources. The author of this book was looking at the matter in a curved $4d$ ST, which can be regarded as the result of the embedding in a x^4 - dependent $5d$ ST, ($x^0 = t$, x^1 , x^2 , x^3 , x^4). The nature of the 4d matter depends on the signature of the $5d$ metric. Finally, what is most important for us, the $4d$ source depends on the extrinsic curvature of the embedded $4d$ ST and the scalar field associated with the extra dimension. I would like to show that increasing the number of spacetime dimensions leads to interesting ideas with applications to our topic, cosmology.

First of all, note that the field equations of SGR in HD are - of course - more complicated and the computations more involved. Because in this thesis we want to concentrate on perturbation theory, we should mention that mathematical description of perturbations of rotating objects are more complex. For example, the perturbation theory of Schwarzchild black hole was studied, even by analytical methods, already by Chandrasekhar, [12], in 1983. We have already much more difficult problem, when we consider the Kerr black hole. The difficulty increases as we go to higher and higher dimensions. People are often using numerical simulations for studying the stability of such objects [13]. The features of event horizons are strongly dimension dependent as was pointed out already in [14]. Black hole thermodynamics is also used in this analysis, [15].

The generalization of the Kerr solution - the rotating black hole - into higher dimensions is so called Myers-Perry solution. It is a hard problem to solve the stability issues for this solution. When people try to solve these questions, they usually begin with rotations in a single plane. Natural parameters of this solution are angular momentum parameters and mass. From the formula for mass, it seems that the properties of these black holes do not differ too much from their counterparts in four dimensions, however this in not true, as we can see from the

[1] GHP formalism is a more compact version of the NP formalism; these formalisms serve for computations in perturbation theory of SGR; we project all quantities on a null tetrad, we define new differential operators and Einstein's equations transform then to a system of first order differential equations; further comments are given in Chapter 3.

formula for gravitational attraction and centrifugal repulsion

$$\frac{\Delta}{r^2} - 1 = \frac{-\mu}{r^{d-3}} + \frac{a^2}{r^2}, \tag{2.1}$$

where μ is a parameter characterizing mass and a angular momentum (details are in [13]).

Few remarks: the full solution can be analyzed using the method of the phase space; We know the term phase space from the theory of ODE's. However, here it means that we fix the mass and we define a quantity j_i for every angular momentum parameter. So we have a system of parameters j_i, where i goes to $d-3$. Only some values of these parameters - regions - are allowed. We can find the qualitative behavior of solutions in dimension d, if we know it already in dimension $d-2$, [14]. If we are able to find the region where the regular black hole exist, we could express all physical magnitudes as functions of phase space variables j_i.

People faced for many years the problems with the stability issues and some questions are not answered yet. The interesting feature occurred for the ultraspinning regime of rotating black holes. They are dynamically unstable for dimension $d \geq 6$ and they come apart into pieces, [16].

General structure of these solutions is more complex than in 4 dimensions. Black rings and black saturns also exist, [17]. Therefore it is important to find which solutions are stable in the linearized sense.

We will not talk more about standard perturbation theory of black hole ST, but we only mention that the study of the linearized perturbations have connections with isometries of black hole spacetimes, [18]. An alternative approach was made by Teukolsky. His approach was used for dealing with perturbations of the Kerr solution, [12]. As we discuss later, the Newman-Penrose scalars Ψ_i encode the information about the Weyl tensor. The perturbation of Ψ_i, $i = 0, 1, 2, 3, 4$ will be denoted by $\Psi_i^{(1)}$ and the unperturbed value is Ψ_i. We have a gauge freedom in infinitesimal coordinate transformations and infinitesimal changes of tetrad. The following interesting lemma was discussed in [18]:

$\Psi_0^{(1)}$ is gauge invariant, if l is repeated principal null direction of the background ST. (We will shed more light on this terminology in the next section about classification of the Weyl tensor.)

The linearized equations will lead to coupled equations of motion for the quantities $\Psi_i^{(1)}$ in general spacetime. In an algebraically special vacuum ST, Teukolsky showed that one can decouple these equations to obtain a single second order wave equation for $\Psi_0^{(1)}$. If the ST is of the algebraical type D, both $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ are gauge invariant. They satisfy decoupled equations of motion. (We use the GHP formalism in the second part of this thesis and the analogy with Schwarzschild ST to show that $d\Psi_4$ decouple for the case of FLRW ST's , when we make appropriate simplifications of the RHS - we ignore the RHS sources.)

There is one class of ST's which are called Kundt spacetimes. (The mathematical definition: they admit shear-free, expansion-free and twist-free geodesic null vector l;) They have nothing to do with black holes. However, we will try to show on a simple example of electromagnetism - what happens for the case of these ST's.

The highest boost weight components (boost is one of the transformations of the null tetrad, which will be discussed it in the next section) in electromagnetism - "Maxwell" scalars, which carry information about the electromagnetic field - are denoted by ϕ_j. These quantities satisfy a decoupled system of equations in 4d and in [18] it was investigated, how it works in general dimension d. We present - just for example - the master equation for the case of perturbation of some sort of Einstein spacetime and we arrive to a second order differential equation for a Maxwell test field (the spin coefficients and the symbols for differential operators, which are here generalized for HD, will be introduced in the part about GHP-formalism in Chapter 3 of the thesis) :

$$\begin{aligned}(2\text{þ}^{'}\text{þ} + \eth_j\eth_j + \rho^{'}\text{þ} - 4\tau_j\eth_j + \phi - \frac{2(d-3)}{(d-1)}\Lambda)\phi_i \\ + (-2\tau_i\eth_j + 2\tau_j\eth_i + 2\phi^S_{ij} + 4\phi^A_{ij})\phi_j = 0.\end{aligned} \tag{2.2}$$

We obtain the, so called, decoupling for the case of Kundt ST's. Now it is interesting to compare the first bracket with the equation for massive scalar field ϕ

$$(\nabla_\mu\nabla^\mu - \mu^2)\phi = 0. \tag{2.3}$$

When we write this equation in the GHP formalism, we get the following result:

$$(2\text{þ}^{'}\text{þ} + \eth_i\eth_i + \rho^{'}\text{þ} - 2\tau_i\eth_i + \rho\text{þ}^{'} - \mu^2)\phi = 0. \tag{2.4}$$

We see the clear similarity between the first bracket in (2.2) and the expression in (2.4). This is important for the second part of my thesis.

One more comment about gravitational perturbations: The basic objects are the gravitational perturbations Ω_{ij}, which are the analogues of Weyl scalar Ψ_0 - the highest boost weight projections of the Weyl tensor on the null tetrad. We have again a similar result as in the case of $4d$, [18]:

$\Omega^{(1)}_{ij}$ is gauge invariant quantity, if l is a multiple Weyl aligned null direction (WAND) of the background Einstein spacetime.

Thus we should study Einstein spacetimes for which l is a multiple WAND. In that case we have that Ω_{ij} and Ψ_{ijk} vanish in the background and that $\Omega_{ij} = \Omega^{(1)}_{ij}$ and $\Psi_{ijk} = \Psi^{(1)}_{ijk}$. So the final result is the same: we could achieve decoupling when the ST is Kundt, [18].

We are interested in black holes, also because of cosmology. (We could find some informations, for example, in works of S. Hawking and R. Penrose, [19].) Let us mention here that perhaps the most fascinating objects in this Universe are various galaxies and quasars. We suppose, and all indirect measurements are in concordance with this, that super-massive black holes are in the centers of such objects.

We will return back to Kundt ST's now. Kundt ST's have nothing to do with black holes, however when we study perturbation theory of black holes, we use so called near-horizon geometry approach, [20]. The decoupling of the basic quantities for the case of Kundt ST's is then very useful.

We thus have many reasons for presenting the concept of the algebraic classification [21].

2.2 Algebraic classification of STs in higher dimensions

We will consider the most important concepts from the algebraic classification of the Weyl tensor in higher d-dimensional pseudo-Riemanian manifolds, [22]. The present classification reduces to the classical Petrov classification in 4 dimensions. The basic idea is the following: we have the Weyl tensor, which is the traceless part of the Riemann tensor; now we consider a null frame, which consists from four vectors. We could decompose the metric with respect to these vectors, and we make different projections of the Weyl tensor. We obtain 5 complex numbers and when there exists such a null direction l in the spacetime, so that different projections - according to the notation - are zero, we obtain the notion of algebraical classification. Let us present the classification even for the dimension of spacetime d;

We shall consider a null frame l, n, $m_{(i)}$ (l and n null with $l^\mu l_\mu = n^\mu n_\mu = 0$, $l^\mu n_\mu = 1$, $m^{(i)}$ real and spacelike with $m^\nu_{(i)} m^{(j)}_\nu = \delta^{(j)}_{(i)}$; all other products vanish) in a d-dimensional Lorentzian ST with signature $(-,+,+...,+)$ (we could choose also $l^\mu n_\mu = -1$, both alternatives are possible), so that

$$g_{\mu\nu} = 2l_{\langle\mu}n_{\nu\rangle} + \delta_{(j)(k)} m^{(j)}_\mu m^{(k)}_\nu, \tag{2.5}$$

where $\langle ... \rangle$ is symmetrization, $\mu, \nu = 0, ..., d-1$ and $i, j, k = 2, ..., d-1$.

The frame is covariant relative to the group of real orthochronous (=preserves the direction of time) linear Lorentz transformations, generated by null rotations, boosts and spins. This means that we can choose different tetrads that preserve a decomposition of the metric (2.5).

Null rotations:
rotations of one of the null basis vectors about the other. A null rotation about l takes the form

$$l \mapsto l,\ n \mapsto n + z_i m_{(i)} - \frac{1}{2} z^2 l,\ m_{(i)} \mapsto m_{(i)} - z_i l, \tag{2.6}$$

where $z^2 = z_i z_i$.

Boosts:
these are rescalings of the null basis vectors that preserve the scalar product $l.n = 1$

$$l \mapsto \lambda l,\ n \mapsto \lambda^{-1} n,\ m_{(i)} \mapsto m_{(i)}, \tag{2.7}$$

where λ is an arbitrary non-zero function. We shall say that l, n and $m_{(i)}$ have boost weights 1, -1 and 0 respectively. Generally, we say that a tensor quantity $T_{i_s ... i_s}$ has a boost weight b, if it transforms as

$$T_{i_1 .... i_s} \mapsto \lambda^b T_{i_1 ... i_s} \tag{2.8}$$

under boosts.
Spins:
these are $SO(d-2)$ rotations of the spatial basis vectors

$$m_{(i)} \mapsto X_{ij} m_{(j)} \tag{2.9}$$

Any tensor T can be expanded with respect to the basis $l, n, m_{(i)}$, where we use a collective notation for all three vectors $l^{(a)}$,

$$T_{\mu\nu\ldots\sigma} = l_\mu^{(a)} l_\nu^{b)} \ldots l_\sigma^{(d)} T_{(a)(b)\ldots(d)}, \tag{2.10}$$

so, for example, (lowered) indices 0 correspond to contractions with l. (We will not use 0 and 1 with brackets in the following text.) The objects $T_{(a)(b)\ldots(d)}$ are ST scalars, but transform as tensor components under local Lorentz transformations, corresponding to changes in the choice of basis vectors. We write the covariant derivatives of the basis vectors as

$$L_{\mu\nu} = \nabla_\nu l_\mu, \;\; N_{\mu\nu} = \nabla_\nu n_\mu, \;\; M_{\mu\nu}^{(i)} = \nabla_\nu m_{(i)\mu}, \tag{2.11}$$

and then project into the basis to obtain the scalars $L_{(a)(b)}, N_{(a)(b)}$ and $M_{(a)(b)}^{(i)}$. From the orthogonality properties of the basis vectors we have the identities

$$N_{0(a)} + L_{1(a)} = 0, \;\; M_{0(a)}^{(i)} + L_{(i)(a)} = 0, \;\; M_{1(a)}^{(i)} + N_{(i)(a)} = 0, \;\; M_{(j)(a)}^{(i)} + M_{(i)(a)}^{(j)} = 0. \tag{2.12}$$

This is just rewritting of the orthogonality properties. We introduce the notation

$$T_{\{pqrs\}} \equiv \frac{1}{2}(T_{[pq][rs]} + T_{[rs][pq]}). \tag{2.13}$$

We can decompose the Weyl tensor and sort the components of the Weyl tensor by boost weight:

$$\begin{aligned}
C_{\mu\nu\rho\sigma} = {} & \overbrace{4C_{0(i)0(j)} n_{\{\mu} m_\nu^{(i)} n_\rho m_{\sigma\}}^{(j)}}^{\text{boost weight 2}} \\
& + 8\overbrace{C_{010(i)} n_{\{\mu} l_\nu n_\rho m_{\sigma\}}^{(i)} + 4C_{0(i)(j)(k)} n_{\{\mu} m_\nu^{(i)} m_\rho^{(j)} m_{\sigma\}}^{(k)}}^{\text{boost weight 1}} \\
& \left.\begin{aligned} & +4C_{0101} n_{\{\mu} l_\nu n_\rho l_{\sigma\}} + 4C_{01(i)(j)} n_{\{\mu} l_\nu m_\rho^{(i)} m_{\sigma\}}^{(j)} \\ & +8C_{0(i)1(j)} n_{\{a} m_b^{(i)} l_c m_{d\}}^{(j)} + C_{(i)(j)(k)(l)} m_{\{a}^{(i)} m_b^{(j)} m_c^{(k)} m_{d\}}^{(l)} \end{aligned}\right\}\text{boost weight 0} \\
& + \overbrace{8C_{101(i)} l_{\{a} n_b l_c m_{d\}}^{(i)} + 4C_{1(i)(j)(k)} l_{\{a} m_b^{(i)} m_c^{(j)} m_{d\}}^{(k)}}^{\text{boost weight -1}} \\
& + \overbrace{4C_{1(i)1(j)} l_{\{a} m_b^{(i)} l_c m_{d\}}^{(j)}}^{\text{boost weight -2}}
\end{aligned} \tag{2.14}$$

The Weyl tensor is generically of boost order 2. If all $C_{0(i)0(j)}$ vanish, but some $C_{010(i)}$ or $C_{0(i)(j)(k)}$ do not, then the boost order is 1, etc. The Weyl scalars also satisfy a number of additional relations, which follow from the curvature tensor symmetries and from the trace-free condition:

$$C^{(i)}{}_{0(i)0} = 0, \;\; C_{010(j)} = C^{(i)}{}_{0(i)(j)}, \;\; C_{0[(i)(j)(k)]} = 0, \tag{2.15}$$

$$C_{0101} = C^{(i)}{}_{0(i)1}, \;\; C_{(i)[(j)(k)(l)]} = 0 \;\; C_{0(i)1(j)} = -\frac{1}{2}C^{(k)}{}_{(i)(k)(j)} + \frac{1}{2}C_{01(i)(j)}, \tag{2.16}$$

$$C_{011(j)} = -C^{(i)}{}_{1(i)(j)}, \;\; C_{1[(i)(j)(k)]} = 0, \;\; C^{(i)}{}_{1(i)1} = 0. \tag{2.17}$$

A real null rotation about l fixes the leading terms of a tensor. The boosts and spins subject the leading terms to an invertible transformation. It follows that the boost order (along l) of a tensor is a function of the null direction l(only). We shall therefore denote boost order by $B(l)$. We define a null vector l to be aligned with the Weyl tensor whenever $B(l) \leq 1$ (and we shall therefore refer to l as a Weyl aligned null direction (WAND)). We call an integer $1-B(l) \in \{0,1,2,3\}$ the order of alignment. The alignment equations are $d(d-3)/2$ degree - 4 polynomial equations in $(d-2)$ variables, which are in general overdetermined and hence have no solutions for $d > 4$.
We say that the principal type of the Weyl tensor in a Lorentzian manifold is I, II, III, N according to whether there exists an aligned l of alignment order $0, 1, 2, 3$, respectively. If no aligned l exists we will say that the manifold is of general type G. If the Weyl tensor vanishes, we will say that the manifold is of type O. The algebraically special types are summarized as follows:

$$\begin{aligned}
I &: C_{0(i)0(j)} = 0 \\
II &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = 0 \\
III &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = C_{(i)(j)(k)(l)} = C_{01(i)(j)} = 0 \\
N &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = C_{(i)(j)(k)(l)} = C_{01(i)(j)} = C_{1(i)(j)(k)} = 0 \quad (2.18)
\end{aligned}$$

A 4-dimensional Weyl tensor always possesses at least one aligned direction. A Weyl tensor does not possess any aligned directions in general for HD. It was shown that if $d \geq 5$, then the set of Weyl tensors with alignment type G is a dense, open subset of the set of all d-dimensional Weyl tensors. We could find further informations about classifications, for example, in [23].

2.2.1 Shear, twist and expansion

Since

$$l_{\alpha;\beta} l^{\beta} = L_{10} l_{\alpha} + L_{(i)0} m^{(i)}_{\alpha} \quad (2.19)$$

see ([24]),
we say that

$$l \text{ is geodetic} \Leftrightarrow L_{(i)0} = 0. \quad (2.20)$$

In this case the matrix $L_{(i)(j)}$ acquires a special meaning since it is then invariant under the null rotations preserving l. It is then convenient to decompose $L_{(i)(j)}$ into its tracefree symmetric part $\sigma_{(i)(j)}$ (shear), its trace θ (expansion) and its antisymmetric part $A_{(i)(j)}$ (twist) as

$$L_{(i)(j)} = \sigma_{(i)(j)} + \theta \delta_{(i)(j)} + A_{(i)(j)}, \quad (2.21)$$

$$\sigma_{(i)(j)} \equiv L_{\langle (i)(j) \rangle} - \frac{1}{n-2} L_{(k)(k)} \delta_{(i)(j)}, \; \theta \equiv \frac{1}{n-2} L_{(k)(k)}, \; A_{(i)(j)} \equiv L_{[(i)(j)]}. \quad (2.22)$$

If l is affinely parametrized, i.e. $L_{10} = 0$, the optical scalars take the form

$$\sigma^2 = l_{\langle \alpha;\beta \rangle} l^{\langle \alpha;\beta \rangle} - \frac{1}{d-2} (l^{\alpha}{}_{;\alpha})^2, \quad (2.23)$$

$$\theta = \frac{1}{d-2} l^{\alpha}{}_{;\alpha}, \quad (2.24)$$

$$w^2 = l_{[\alpha;\beta]} l^{\alpha;\beta}. \quad (2.25)$$

The Kundt spacetimes admit a null direction l, so that the shear, expansion and twist is zero.

2.3 Spinor approach

So, we developed the algebraical classification in dimension d, but we have interest especially in dimension four. Therefore we present here the following concept, which can be generalized to HD. However classification in, so called, spinors, is not so powerful in HD.

Spinors are more simpler objects than tensors [2], intuitive comparison is that it is a square root from a tensor. Why were spinors so interesting for us in the approach for SGR ? Authors of two books [25] and [26] showed that every vector could be written in the language of spinors. We could translate every vectorial equation into this formalism. Some equations looks simpler in this formalism.

There are more consistent definitions, let's make completely elementary introduction. The basic spin-space for us is 2-dimensional complex vector space equipped with skew-symmetric bilinear form and the objects from this space are 2-components spin-vectors, which are the simplest spinors. We denote the symplectic form, which plays the role of the metric by ϵ_{AB}. We could raise and lower indices with this object and if we have two contravariant spinors ψ^A and ϕ^B, we could make an action of the bilinear form on these spinors and we get a complex number. The forms ϵ^{AB} and ϵ_{AB} provide a natural isomorphism between spin-space and its dual.

From a group theoretical methods the spinors are elements of the representation of the group $SL(2, \mathbf{C})$. And for us the mix spinor-tensor objects which make the bridge between spinors and tensors are interesting. So we have the correspondence

$$g_{\mu\nu}\sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'} \leftrightarrow \epsilon_{AB}\epsilon_{A'B'}. \tag{2.26}$$

But all this is written and is well known from the literature, for example [25], [26], [27], in various notations. But, as authors from [25] wrote in their book, the complications with formulating the physical laws were due to the tensorial approach. When we take as the basic building blocks the spinors, the difficulties disappear. As we already wrote, it is possible to build spinors in all dimensions, however the dimension of the spin-space goes like $2^{\frac{d}{2}}$, so the efficiency of the spinor formalism is very low in higher dimensions. Later we will mention the spinor classification of the Weyl tensor in higher dimensions. It was shown in the work of [28] that it is not so useful as the standard classification (and not equivalent).

I will now add another idea, why 4-dimensions should be special also from geometrical point of view. (I mentioned already in the Introduction, that we have the result about speciality of 4-dimensional ST from physical point of view.) It is possible to use spinors for algebraical classifications. The approach is quite similar to standard classifications. We will begin with mentioning of the classification of Maxwell tensor which will be a toy-model for us. Then we could use the analogous approach for the gravitational field. Well-known fact is that classification in spinors are not equivalent to standard algebraical classifications in HD.

[2] We denote them usually with capital index.

However, we stress by different argument the uniqueness of $4d$ ST in classification of gravitational field.

2.3.1 Complex three space

We will follow the notation of [29] and I repeat here these nice ideas, which are in the core of classification procedure of the Weyl tensor and spinor, and which we use for our argument of uniqeness of $4d$ ST. Let $F_{\mu\nu}$ be the Maxwell tensor and let $*F_{\mu\nu}$ be its dual. The $F_{\mu\nu}$ carries the information about the electromagnetic field. Let us also define the tensor $F^{+}_{\mu\nu}$ by

$$F^{+}_{\mu\nu} = F_{\mu\nu} + i * F_{\mu\nu}, \tag{2.27}$$

so that $*F^{+}_{\mu\nu} = -iF^{+}_{\mu\nu}$, where i is the imaginary unity. The spinor equivalent of the tensor $F_{\mu\nu}$ was found in section 8.2 of [29], whereas that of the tensor $F^{+}_{\mu\nu}$ is given by

$$F^{+}_{AB'CD'} = 2\phi_{AC}\ \epsilon_{B'D'}, \tag{2.28}$$

where ϕ_{AC} is the Maxwell spinor. Classification of the electromagnetic field can be made by classifying ϕ_{AB}. Therefore one studies eigenspinors and eigenvalues of the spinorial equation

$$\phi^{A}{}_{B}\alpha^{B} = \lambda\alpha^{A}. \tag{2.29}$$

To study this equation one introduces the basis in our spin space. Let the two spinors of the basis be denoted by l_A and n_A, satisfying the normalization condition $l_A n^A = 1$. This basis induces another basis, given by

$$\xi_{0AB} = n_A n_B, \quad \xi_{1AB} = -2l_{(A}n_{B)}, \quad \xi_{2AB} = l_A l_B, \tag{2.30}$$

in the three dimensional space, E_3 of bispinors. This means a bispinor ϕ_{AB} can be written in terms of the basis (2.30) as

$$\phi_{AB} = \sum_{m=0}^{2} \phi_m\ \xi_{mAB}, \tag{2.31}$$

where ϕ_0, ϕ_1 and ϕ_2 are called dyad components of the bispinor and corresponds to six real components of the tensor $F_{\mu\nu}$. The spin frame l_A and n_A induces other basis in E_3

$$\eta_{0AB} = \frac{1}{\sqrt{2}}i\ (l_A n_B + l_B n_A), \tag{2.32}$$

$$\eta_{1AB} = \frac{1}{\sqrt{2}}(l_A l_B + n_A n_B), \tag{2.33}$$

$$\eta_{2AB} = \frac{1}{\sqrt{2}}i\ (l_A l_B - n_A n_B). \tag{2.34}$$

This basis satisfies the orthogonality relation

$$\eta_{mAB}\eta_{n}^{AB} = \delta_{mn}. \tag{2.35}$$

In terms of this last basis ϕ_{AB} can now be written as

$$\phi_{AB} = \sum_{m=0}^{2} \chi_m \eta_{mAB}. \tag{2.36}$$

The two sets of three components χ and ϕ are then related by

$$\chi_0 = \sqrt{2} i \ \phi_1, \tag{2.37}$$

$$\chi_1 = \frac{1}{\sqrt{2}}(\phi_0 + \phi_2), \tag{2.38}$$

$$\chi_2 = \frac{1}{\sqrt{2}} i \ (\phi_0 - \phi_2). \tag{2.39}$$

We do these substitutions from the group theoretical reasons, because of symmetry.

2.3.2 Classification of Maxwell spinor

In terms of the dyad components ϕ_m, the eigenvalue equation (2.29) becomes

$$\Phi\alpha = \lambda\alpha, \tag{2.40}$$

where Φ is a matrix of rank 2, given by

$$\begin{pmatrix} \phi_1 & \phi_2 \\ -\phi_0 & -\phi_1 \end{pmatrix}$$

and α is a column matrix given by $\begin{pmatrix} \alpha^0 \\ \alpha^1 \end{pmatrix}$,
where α^a are the dyad components of α^A, i.e., $\alpha^a = \zeta^a{}_A \alpha^A$, and we have denoted ${\zeta_0}^A = l^A$ and ${\zeta_1}^A = n^A$. The two eigenvalues of equation (2.29) are $\lambda = \pm\sqrt{({\phi_1}^2 - \phi_0\phi_2)}$. One, therefore, has two cases: $\sqrt{({\phi_1}^2 - \phi_0\phi_2)} \neq 0$, in which case there are two different eigenspinors; and $\sqrt{({\phi_1}^2 - \phi_0\phi_2)} = 0$ in which case there is only one eigenspinor. This similar idea is the cornerstone of Petrov classification of the Weyl tensor (so the gravitational case) in dimension 4, therefore we are repeating it here.

2.3.3 Classification of Weyl spinor

Bivectors were discussed in the previous section. We will use now this knowledge and we will apply them to the Weyl spinor.
The Weyl tensor $C_{\alpha\beta\gamma\delta}$ has the same symmetry properties as the Riemann tensor. In addition, it satisfies

$$C^{\rho}{}_{\alpha\rho\beta} = 0. \tag{2.41}$$

These identities reduce the number of independent components to ten. We could find that the spinor equivalent of $C_{\alpha\beta\gamma\delta}$ is completely symmetric spinor of four indices, ψ_{ABCD},

$$-C_{AB'CD'EF'GH'} = \epsilon_{AC}\epsilon_{EG} \ \overline{\psi}_{B'D'F'H'} + \psi_{ACEG} \ \epsilon_{B'D'}\epsilon_{F'H'} \tag{2.42}$$

We use the Weyl spinor for the classification in the characteristic equation.

2.3.4 Complex 5-space

In order to classify the Weyl tensor we classify the Weyl spinor ψ_{ABCD} in terms of its eigenvalues and eigenspinors. The characteristic equation is now:

$$\psi_{ABCD}\phi^{CD} = \lambda\phi_{AB} \tag{2.43}$$

The basis l_A, n_A in spinorial space induces the basis

$$\zeta_{0ABCD} = n_A n_B n_C n_D, \tag{2.44}$$

$$\zeta_{1ABCD} = -4l_{(A}n_B n_C n_{D)}, \tag{2.45}$$

$$\zeta_{2ABCD} = 6l_{(A}l_B n_C n_{D)}, \tag{2.46}$$

$$\zeta_{3ABCD} = -4l_{(A}l_B l_C n_{D)}, \tag{2.47}$$

$$\zeta_{4ABCD} = l_A l_B l_C l_D. \tag{2.48}$$

We want to make a remark that l and n are different from the elements of the null tetrad now.

2.3.5 Change of frame

The key ingredient of these formulas is the following: we have 5 complex vectors ζ which correspond to 5 complex Newman-Penrose scalars; These vectors create 5 dimensional space of completely symmetric 4-spinors. These scalars transform as vectors in this space. But why we are saying all this, although these facts were well described in the literature [29]? It is not possible to generalize this concept of classification, (see e.g. [30]) to dimension 5. It is impossible to achieve this construction with vectors ζ, because there is not such a nice relation like inclusion of E_5 to $E_3 \times E_3$ (in contrast with four dimensions).

We can again write the five complex Weyl scalars:

$$\begin{aligned}
\Psi_0 &= l^A l^B l^C l^D \Psi_{ABCD},\\
\Psi_1 &= l^A l^B l^C n^D \Psi_{ABCD},\\
\Psi_2 &= l^A l^B n^C n^D \Psi_{ABCD},\\
\Psi_3 &= l^A n^B n^C n^D \Psi_{ABCD},\\
\Psi_4 &= n^A n^B n^C n^D \Psi_{ABCD},
\end{aligned} \tag{2.49}$$

where Ψ_{ABCD} is the spinor counter-part to the Weyl tensor with the following symmetries:

$$\Psi_{ABCD} = \Psi_{(AB)CD} = \Psi_{AB(CD)} = \Psi_{CDAB},\ \Psi_{ABCD} = \Psi_{(ABCD)} \tag{2.50}$$

But we can have also other "Weyl scalars" in dimension 5:

$$\begin{aligned}
\Psi_0^{**} &= l^A l^B l^C \bar{l}^D \Psi_{ABCD},\\
\Psi_1^{**} &= l^A l^B n^C \bar{l}^D \Psi_{ABCD},\\
\Psi_2^{**} &= l^A n^B n^C \bar{l}^D \Psi_{ABCD},\\
\Psi_3^{**} &= \bar{l}^A n^B n^C n^D \Psi_{ABCD},
\end{aligned} \tag{2.51}$$

and

$$
\begin{aligned}
\Psi_1^* &= l^A l^B l^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_2^* &= l^A l^B n^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_3^* &= l^A n^B n^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_4^* &= n^A n^B n^C \overline{n}^D \Psi_{ABCD},
\end{aligned}
\tag{2.52}
$$

$$
\begin{aligned}
\Psi_{01}^* &= l^A l^B \overline{l}^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_{02}^* &= l^A l^B \overline{n}^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_{12}^* &= l^A n^B \overline{n}^C \overline{n}^D \Psi_{ABCD}.
\end{aligned}
\tag{2.53}
$$

All 16 numbers were complex with one or two elements from the basic spin-space with complex conjugation. We have additional 3 real numbers:

$$
\begin{aligned}
\Psi_{00}^* &= l^A l^B \overline{l}^C \overline{l}^D \Psi_{ABCD}, \\
\Psi_{11}^* &= l^A n^B \overline{l}^C \overline{n}^D \Psi_{ABCD}, \\
\Psi_{22}^* &= n^A n^B \overline{n}^C \overline{n}^D \Psi_{ABCD}.
\end{aligned}
\tag{2.54}
$$

This corresponds together to 35 real components of Weyl tensor. We don't make a detailed calculation, but it should be clear that there are too many components for achieving the vectorial properties.

We can conclude that we found other very simple argument, why are four-dimensional ST's exceptional for spinors, as was already mentioned in the work [25]. I stress here once again the result from [1] and related works, for example [31], that HD - $d > 4$ - are not compatible in cosmology with dust-like matter. [3] This is a task for future confirmations, but we have strong hints that $4d$ physical theories should be preferable. We are studying $4d$ $f(R)$-cosmologies in part three of my thesis.

2.4 Kundt class

The motivation for studying the Kundt class is from the point of view of perturbation theory the following (for example) : Kundt class does not contain any black holes, however the studying of the so-called near horizon geometry, [20], played a role in the past and lead toward the studies of this type of ST. [4]

[3] Also in pure geometry the dimension four has nice properties.

[4] The motivation to study these STs came also from the theory of supergravity.

We could find more information about algebraic types of Kundt solutions in [32]. As in four dimensions it is characterized by having a shear-free, non-expanding, non-twisting geodesic null congruence $l = \partial_v$. HD Kundt class can be written in canonical form in the following way:

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}v + H(u,v,x^k)\mathrm{d}u + W_i(u,v,x^k)\mathrm{d}x^i] + g_{ij}(u,x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{2.55}$$

The spatial coordinates are $(x^1, ..., x^{d-2})$; g_{ij} is the Riemanian metric.
It follows from [34] that

$$C_{vijk} = \frac{1}{2(d-2)}[g_{ik}W_{j,vv} - g_{ij}W_{k,vv}] \tag{2.56}$$

and

$$C_{vivj} = 0. \tag{2.57}$$

Therefore the frame component $C_{0(i)0(j)} = 0$ for the natural tetrad:

$$l_\mu = (1, 0, 0, ...), \; n_\mu = (H, 1, W_1, W_2, ..., W_{d-2}), \tag{2.58}$$

(where the general prescription for m_μ depends on the special form of g_{ij}.) Therefore, [34]:

Lemma 2.1. *The Weyl type of Kundt metrics is I or more special.*

This follows from algebraical criteria, when we use the equation (2.57) and we used this in the introductional section to this part of thesis. All the coordinate transformations preserving the form of metric ([33], [34]) are:
• $(v', u', x'^i) = (v, u, f^i(x^k))$ and $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$H' = H, \quad W'_i = W_j (J^{-1})^j{}_i, \quad g'_{ij} = g_{kl}(J^{-1})^k{}_i (J^{-1})^l{}_j \tag{2.59}$$

• $(v', u', x'^i) = (v + h(u, x^k), u, x^i)$

$$H' = H - h_{,u}, \quad W'_i = W_i - h_{,i}, \quad g'_{ij} = g_{ij} \tag{2.60}$$

• $(v', u', x'^i) = (v/k_{,u}(u), k(u), x^i)$

$$H' = \frac{1}{k_{,u}^2}(H + v\frac{k_{,uu}}{k_{,u}}), \quad W'_i = \frac{1}{k_{,u}}W_i, \quad g'_{ij} = g_{ij} \tag{2.61}$$

• $(v', u', x'^i) = (v, u, f^i(u; x^k))$ and $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$\begin{aligned} H' &= H + g_{ij} f^i_{,u} f^j_{,u} - W_j (J^{-1})^j{}_i f^i_{,u}, \quad W'_i = W_j (J^{-1})^j{}_i - g_{ij} f^j_{,u}, \\ g'_{ij} &= g_{kl}(J^{-1})^k{}_i (J^{-1})^l{}_j \end{aligned} \tag{2.62}$$

The higher - dimensional Kundt class contains a number of interesting subclasses, which we describe in what follows. And it would be definitely interesting to continue in mathematical studies of perturbations of these subclasses.

pp-wave ST:
Higher-dimensional pp-wave spacetimes are defined as in four dimensions, as spacetimes which admit a covariantly constant null vector. The most general d-dimensional pp-wave ST is given by the Brinkmann metric (we could find reference in the section 8.3.3. of paper [23]:

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}v + H(u, x^k)\mathrm{d}u + W^i(u, x^k)\mathrm{d}x^i] + g_{ij}(u, x^k)\mathrm{d}x^i\mathrm{d}x^j \qquad (2.63)$$

For example, from the Bel-Debever criteria it follows that the Weyl type is N in four dimensions.
Now we would like to solve the question, if there exist pp-waves of Weyl type I in higher dimensions. I assert that no and we mention here this simple argument (well-known from literature): according to [34], $\omega = R_{(0)(0)} = R_{vv} = 0$ for natural frame vector $l^\mu = (0, 1, 0, ..., 0)$ and $\psi_i = R_{(0)(i)} = 0$ for natural frame vectors l^μ and $m^\mu_{(i)}$; Now we use Proposition 2 from [24]. It follows that pp-waves couldn't be of Weyl type I. In fact, pp-waves are of Weyl type II, III or N in higher dimensions.

Higher-dimensional Lorentzian ST's with vanishing scalar curvature invariants of all orders are so called VSI ST's. It was found that all such ST's belong to the Kundt class and in fact, can be written in the canonical form, [34]

$$ds^2 = 2(e^i(x^k, u)+f^i(x^k, u)v)dx^idu+2dudv+(av^2+bv+c)du^2+\delta_{ij}dx^idx^j. \qquad (2.64)$$

Such ST's have a Weyl tensor of algebraic type III, or more special. Two subclasses can be distinguished, namely the case $f^i = 0$ for all $i = 1, 2, ..., D-2$ and the case $f_1 \neq 0$ with $f_i = 0, i = 2, 3, ..., D-2$. A generalization of the VSI ST's belonging to the Kundt class, such that all polynomial scalar invariants constructed from the Riemann tensor and its derivatives are constant, are called CSI ST's.

In fact both alternative assumptions 1) and 2) in the lemma below uniquely identify the Kundt class of non-expanding, twist-free and shear-free ST, i.e. $L_{(i)(j)} = 0$ ($L_{(i)0} = 0$), [24].

Lemma 2.2. *Given a geodetic null congruence with tangent vector l in an arbitrary d-dimensional ST ($d \geq 4$), the following implications hold:*
1)$R_{00} = 0,\ \theta = 0 = \sigma_{(i)(j)} \Longrightarrow A_{(i)(j)} = 0,\ C_{0(i)0(j)} = 0$
2)$R_{00} = 0,\ \theta = 0 = A_{(i)(j)} \Longrightarrow \sigma_{(i)(j)} = 0,\ C_{0(i)0(j)} = 0$

In view of 2), we can conclude that one can not generalize the Kundt solutions by allowing for non-zero shear (as long as $R_{00} = 0$ and one insists on the non-expanding and twistfree conditions). We only note that the assumed condition $R_{00} = 0$ from previous lemma 2.2 on the matter content is satisfied in a large class of ST's such as vacuum with a possible cosmological constant, aligned pure radiation and aligned Maxwell fields.

Further, in both above cases 1) and 2), the fact that the tangent vector is necessary a WAND (because of $C_{0(i)0(j)} = 0$) implies for $d > 4$ that the considered ST is algebraically special, i.e. it can not be of type G. In addition, if we now substitute $L_{(i)0} = 0 = L_{(i)(j)}$ in one of the Ricci identities (11k) in [24] and we further assume $R_{0(i)} = 0$, we obtain $C_{0(i)(j)(k)} = 0$. Recalling the identity $C_{0101} = C_{0(j)(i)(j)}$, we find also $C_{010(i)} = 0$, so that with previous lemma we conclude [24]:

Lemma 2.3. *Under the assumption $R_{00} = 0 = R_{0(i)}$ on the matter fields, $d \geq 4$ Kundt ST's ($L_{(i)0} = 0 = L_{(i)(j)}$) are of type II (or more special).*

2.4.1 Recurrent spacetimes

Let us mention one subclass of Kundt class, because of their geometrical significance. We define a special class of Kundt ST, which we will call recurrent (RNV): there must exist a null vector l_μ in a neighborhood of every point such that

$$l_{\mu;\nu} = \alpha \; l_\mu l_\nu, \tag{2.65}$$

where α is a scalar function;
As we could see from standard literature, there exist coordinates u,v,x^i (where $i = 1, ..., d-2$), such that the metric has the canonical form

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}v + H(u, v, x^k)\mathrm{d}u + W_i(u, x^k)\mathrm{d}x^i] + g_{ij}(u, x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{2.66}$$

where $g = g_{ij}(x^k, u)\mathrm{d}x^i\mathrm{d}x^j$ is a u-dependent family of Riemanian metrics. The vector field $\partial_v = \frac{\partial}{\partial v}$ is null and recurrent, [35]. [5] We would like to stress that W_i are not functions of v. (If W_i was also a function of v, then the metric (2.66) would be a general Kundt metric.) We have the following lemma, which is clear from definition:

Lemma 2.4. *For recurrent spacetimes ($W_i = W_i(u, x^k)$) the vector $\frac{\partial}{\partial v} = v^\mu = (0, 1, 0, ..., 0)$ is the recurrent vector and $v_{0;0} = \alpha = \frac{\partial H}{\partial v}$.*

The following result is important in the classification scheme: it follows from [34] that $C_{0(i)(j)(k)} = 0$. Using Bel-Debever criteria in [36] we arrive to:

---

[5] The following remark applies only for four-dimensional ST, however we want to illustrate the geometrical significance of this subclass. One could use a similar argument in dimension d. We would like to understand more geometrically what does it mean that a spacetime is RNV. We can contract the equation (2.65) with vector l, n and m. After doing this we get:

$$Dl^\mu = 0,$$
$$\Delta l^\mu = \alpha l^\mu,$$
$$\delta l^\mu = 0,$$

where α is a function.
From the second equation we see the geometrical picture. Now we transport the vector l in direction n and the change of the vector l is again in the direction l.

Further, from the well known propagation equations

$$Dl^\mu = (\epsilon + \overline{\epsilon})l^\mu - \overline{\kappa}m^\mu - \kappa\overline{m}^\mu,$$
$$\Delta l^\mu = (\gamma + \overline{\gamma})l^\mu - \overline{\tau}m^\mu - \tau\overline{m}^\mu,$$
$$\delta l^\mu = (\overline{\alpha} + \beta)l^\mu - \overline{\rho}m^\mu - \sigma\overline{m}^\mu,$$

we get for the natural l vector the conditions for the spin coefficients:

$$\kappa = 0, \quad \sigma = 0, \quad \tau = 0, \quad \rho = 0.$$

Lemma 2.5. *The recurrent STs are of algebraical type II or more special.*

This lemma 2.5 is in correspondence with the results of T. Málek. ([37]).

2.5 Conclusion

In this first part of my dissertation, Chapter 2, I studied SGR in higher dimensions. I reviewed briefly the algebraic classification of Weyl tensor in higher dimensions after introduction, I mentioned the classification in spinors and then I studied the so called Kundt class, which admits twist-free, non-expanding and shear-free null direction l. This class played a role in the perturbation theory in studying of the, so called, near-horizon geometries, [20]. I discussed the interesting case of recurrent spacetimes, which contains the *pp*-waves as a special case. This class of exact solution has a connection to the search for the gravitational waves, which is a missing experimental consequence of SGR. It is expected to find the gravitational waves with future generation of detectors.

Finally, this interesting discipline served to us as a motivation for other works. Here we present further literature ([21], [22], [24], [39], [40]). Gravitational instability in higher dimensions was studied, for example, in [41], [42], [43], and [16].

Other sources for gravity in higher dimensions are, for example, [44], [45].

More information about recurrent spacetimes can be found in [34], [35] and [38]. Further information about general relativity are in [12], [29] and [32].

3. Perturbation of FLRW spacetimes in GHP formalism

3.1 Introduction

Our goal is to use the algebraical formalisms (GHP - formalism) in reformulation of perturbations of FLRW ST. GHP - formalism (a more compact version of NP - formalism) is a convenient formalism, because it allows us to work with partial differential equations of the first order. The scalar and tensor perturbations are for us the most interesting because of the origin of structure. I will show, how to apply the GHP-formalism for decoupling of the quantity $d\psi_4$ in Bianchi equation. It is also possible to reformulate the central equation from the article [46] in the GHP-formalism and so to reproduce the result. These calculations are done for the case of the simplified righthand side (RHS without sources).

Everything is prepared for applications when we have also sources (RHS). In the future we plan to apply this machinery to the Cosmological Inflation, which is a kind of phase transition at the beginning of the evolution of our Universe. [1]

3.2 NP-formalism

NP- and GHP-formalisms are mathematical approaches which help us, for example, in perturbation theory to simplify calculations in SGR, as we already mentioned at the beginning of the previous part [12]. We decompose the metric with respect to the null tetrad and then we project all quantities on this tetrad (in the NP-formalism). The basic quantities are spin-coefficients - projections of the derivatives of the null tetrad, then projections of the Ricci tensor and already mentioned Weyl scalars. We could then rewrite the Einstein's equations by the 18 Ricci, 8+3 Bianchi and 4 commutation equations, which are only first order PDE's, when we define new derivatives in the direction of the tetrad (D,Δ, δ, $\overline{\delta}$). Let us to introduce, for illustration, the basic quantities and equations now:

We will denote the 12 spin coefficients by S_{ijk} (standard notation is with γ), where the three indices mean, which element of the tetrad we are using (where the null tetrad is defined as in section 2.2):

$$\alpha = \frac{1}{2}(S_{214} + S_{344}),\ \beta = \frac{1}{2}(S_{213} + S_{343}),\ \gamma = \frac{1}{2}(S_{212} + S_{342}),\ \mu = S_{243},\ \rho = S_{314}$$

$$\epsilon = \frac{1}{2}(S_{211} + S_{341}),\ \tau = S_{312},\ \kappa = S_{311},\ \sigma = S_{313}, \pi = S_{241}, \lambda = S_{244}, \nu = S_{242}$$

For example:

$$\rho = m^{\mu} l_{\mu;\nu} \overline{m}^{\nu}$$

[1] The correspondence with metric perturbations was shown already in [46] (for the case of NP-formalism).

Projections of the Ricci tensor (we will omit the brackets by tetrad indices in this part of thesis):

$$\Phi_{(i)(j)},\ i,j=0,1,2,3,$$

$$\Phi_{(0)(0)}=-\frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu}$$

So, let's define the projections of the Ricci tensor by the following notation, [12]:

$$\begin{aligned}
\Phi_{00}&=-\frac{1}{2}R_{11}, & \Phi_{01}&=-\frac{1}{2}R_{13}, & \Phi_{10}&=-\frac{1}{2}R_{14},\\
\Phi_{11}&=-\frac{1}{4}(R_{12}+R_{34}), & \Phi_{21}&=-\frac{1}{2}R_{24}, & \Phi_{22}&=-\frac{1}{2}R_{22},\\
\Phi_{12}&=-\frac{1}{2}R_{23}, & \Phi_{02}&=-\frac{1}{2}R_{33}, & \Phi_{20}&=-\frac{1}{2}R_{44},\\
\Lambda&=\frac{1}{12}(R_{12}-R_{34}).
\end{aligned}\tag{3.1}$$

Weyl scalars (5 in dimension 4):

$$\Psi_i,\ i=0,1,2,3,4\tag{3.2}$$

$$\begin{aligned}
\Psi_0&=l^{\mu}m^{\nu}l^{\rho}m^{\sigma}C_{\mu\nu\rho\sigma},\\
\Psi_1&=l^{\mu}n^{\nu}l^{\rho}m^{\sigma}C_{\mu\nu\rho\sigma},\\
\Psi_2&=l^{\mu}m^{\nu}\overline{m}^{\rho}n^{\sigma}C_{\mu\nu\rho\sigma},\\
\Psi_3&=n^{\mu}l^{\nu}n^{\rho}\overline{m}^{\sigma}C_{\mu\nu\rho\sigma},\\
\Psi_4&=n^{\mu}\overline{m}^{\nu}n^{\rho}\overline{m}^{\sigma}C_{\mu\nu\rho\sigma}.
\end{aligned}\tag{3.3}$$

Now we present one Ricci, one Bianchi and one commutation relation:

Ricci identities (18 equations)

$$D\rho-\delta^{*}\kappa-\rho^{2}-\sigma\overline{\sigma}-\rho\epsilon-\rho\overline{\epsilon}+\overline{\kappa}\tau+(3\alpha+\overline{\beta}-\pi)\kappa=\Phi_{00},\tag{3.4}$$

Bianchi identities (11 equations)

$$\begin{aligned}
&-\delta^{*}\Psi_0+D\Psi_1+(4\alpha-\pi)\Psi_0-2(2\rho+\epsilon)\Psi_1+3\kappa\Psi_2=\\
&-D\Phi_{01}+\delta\Phi_{00}+2(\epsilon+\overline{\rho})\Phi_{01}+2\sigma\Phi_{10}-2\kappa\Phi_{11}-\overline{\kappa}\Phi_{02}+\\
&(\overline{\pi}-2\overline{\alpha}-2\beta)\Phi_{00},
\end{aligned}\tag{3.5}$$

Commutation relations (4 equations):

$$\Delta D - D\Delta = (\gamma + \overline{\gamma})D + (\epsilon + \overline{\epsilon})\Delta - (\overline{\tau} + \pi)\delta - (\tau + \overline{\pi})\overline{\delta}. \tag{3.6}$$

We can rotate the tetrad and we can get a transformation property of these quantities. However, there exists also a more compact version of the NP-formalism, so called GHP-formalism. One makes the following redefinitions of the derivative operators:

$$þ = D - p\epsilon - q\overline{\epsilon}, \tag{3.7}$$

$$þ^{'} = \Delta - p\gamma - q\overline{\gamma}, \tag{3.8}$$

$$ð = \delta - p\beta - q\overline{\alpha}, \tag{3.9}$$

$$ð^{'} = \overline{\delta} - p\overline{\beta} - q\alpha. \tag{3.10}$$

We have 4 different operators and two, so called, dualities in dimension 4 (star - duality, $(p,q) \to (p,-q)$, is for exchange of the vector l and m, the prime duality, $(p,q) \to (-p,-q)$, for the exchange of the l and n; we could not use both dualities in HD, because we have odd dimensions)

$$\Sigma\eta,\ \Sigma = \left\{þ,\ þ^{'}, ð, ð^{'}\right\}, \tag{3.11}$$

where Σ is an operator acting on a quantity η.

But we need the notion of the GHP scalars when we make the following transformations:

$$l^{\nu} \to \lambda^{-1} l^{\nu}, \tag{3.12}$$

$$n^{\mu} \to \lambda n^{\mu}, \tag{3.13}$$

$$m^{\rho} \mapsto e^{i\theta} m^{\rho}. \tag{3.14}$$

We say that a quantity η is a GHP-scalar of type (p,q), if it transforms like (analogical definition for the case of higher dimensions is in [39]):

$$\eta \to \lambda^{(p+q)/2} e^{i(p-q)\theta/2} \eta. \tag{3.15}$$

We will use the star and prime duality in a standard way [12]:

$$\begin{array}{lllll}\Phi'_{00}=\Phi_{22}, & \Phi'_{11}=\Phi_{11}, & \Phi'_{10}=\Phi_{12}, & \Phi'_{02}=\Phi_{20}, & \Lambda'=\Lambda,\\ \Phi^*_{00}=\Phi_{02}, & \Phi^*_{01}=-\Phi_{01}, & \Phi^*_{10}=\Phi_{12}, & \Phi^*_{11}=-\Phi_{11}, & \Phi^*_{21}=-\Phi_{21},\\ \Phi^*_{22}=\Phi_{20}, & \Phi^*_{12}=\Phi_{10}, & \Phi^*_{02}=\Phi_{00}, & \Phi^*_{20}=\Phi_{22}, & \Lambda^*=-\Lambda.\end{array} \tag{3.16}$$

The types for these quantities are the following (for illustration):

$$\begin{array}{lllll}\Phi_{00}:(2,2), & \Phi_{01}:(2,0), & \Phi_{10}:(0,2), & \Phi_{11}:(0,0), & \Phi_{20}:(-2,2),\\ \Phi_{22}:(-2,-2), & \Phi_{12}:(0,-2), & \Phi_{21}:(-2,0), & \Phi_{02}:(2,-2), & \Lambda:(0,0).\end{array} \tag{3.17}$$

3.3 Computations

Now we will apply the GHP-formalism in perturbation theory of FLRW ST. It will be done, of course, in classical manner. However, we obtain a new result with this formalism. Because we want to apply this in Cosmological Inflation, every reformulation of the perturbation of FLRW is useful.

Reference [46] will be of great importance for us, where the following fact can be found: the only non-vanishing spin coefficients for the case of FLRW are α, β, γ, μ and ρ. These are the same non-zero spin coefficients as for the case of the Schwarzschild solution. This fact can be employed in the analysis of unperturbed equations. This means that we can get rid of many terms in the resulting equations. We get rid of α, β, γ and ϵ because they are absorbed into þ and ð (þ$'$ and ð$'$). Together there are 12 spin coefficients, thus there remain yet 8 more: τ, σ, κ, μ, ρ, λ, π and ν ;

The course of action will be now the following. We will write the general Bianchi equations for the case of FLRW ST with sources. We will show our result for the special case of the delta function on the RHS and we will mention the result for scalar perturbations of FLRW ST.

We have the following 2 equations in GHP formalism for the case of FLRW ST. We have 8 equations in standard NP-formalism, but this formalism is even more efficient. (We stress once again that we have sources on the right hand side of the equations contrary to the Schwarzschild ST.) The equations read

$$þ\Psi_1-ð'\Psi_0+\tau'\Psi_0-4\rho\Psi_1+3\kappa\Psi_2=þ\Phi_{01}-ð\Phi_{00}-\overline{\pi}\Phi_{00}-2\overline{\rho}\Phi_{01}+\overline{\kappa}\Phi_{02}+2\kappa\Phi_{11}-2\sigma\Phi_{10}, \tag{3.18}$$

$$þ\Psi_2-ð'\Psi_1-\sigma'\Psi_0+2\tau'\Psi_1-3\rho\Psi_2+2\kappa\Psi_3=-ð'\Phi_{01}+þ'\Phi_{00}-\overline{\rho}'\Phi_{00}+2\overline{\tau}\Phi_{01}-2\rho\Phi_{11}-\overline{\sigma}\Phi^*_{00}+2\tau\Phi_{10}+2þ\Lambda, \tag{3.19}$$

where we defined the NP components of the Weyl tensor in the standard way.

According to the [46] – and this should be clear[2] – the Ψ_0 and Ψ_4 are connected with the tensor perturbations, Ψ_1 and Ψ_3 are connected with the vector

[2] We can use their boost weights like an argument.

perturbations and Ψ_2 is connected with the scalar perturbations. [3]

Now we will follow the approach presented in [47]. The difference, as we already mentioned, is that we have sources on the RHS. However, we can make the same steps: we will take the first equation and we will apply operator ð, we make the star duality and we add the first and this modified equation. Then we plug from the Ricci identities, we eliminate some of these combinations of spin coefficients (we make also prime and star dualities of these Ricci identities) and we arrive at the following result (the second equation could be obtained in a similar way).

$$\begin{aligned}
&\left[þ'þ - ð'ð - (4\rho' + \overline{\rho}')\, þ - \rho þ' + (4\tau' + \overline{\tau})\, ð + \tau ð' + 4\rho\rho' - 4\tau\tau' - 2\Psi_2 + 2\Lambda\right] \Psi_4 + \\
&\quad + \left[4þ\kappa' - 4ð\sigma' - 4\,(\overline{\rho} - 2\rho)\,\kappa' + 4\,(\overline{\tau} - 2\tau)\,\sigma' + 10\Psi_3\right] \Psi_3 \\
&\quad + \left[-4\sigma'þ' + 4\kappa'ð' - 12\kappa'\tau' + 12\rho'\sigma' - 3\Psi_0\right] \Psi_2 = 0.
\end{aligned} \tag{3.22}$$

This equation contains information from both (3.18) and (3.19). It is interesting that for this case of FLRW spacetimes, we have cancellations of all extra terms in front of Ψ_2 and Ψ_3. The terms in the brackets in front of Ψ_2 and Ψ_3 are exactly the same (except of one term $3\Psi_0\Psi_2$) as for the case of the Schwarzschild spacetime. This means that when we will make perturbations of these equations, the second and third term disappear. So, we obtain a decoupling of the quantity $d\Psi_4$.

This is other new information, when we compare it with [46]. Here we were interested in equations without sources, i.e. when we put just delta-function on the RHS. But in later work we could be interested in the same problem but with sources, as was already suggested in this article. It is an advantage to write all sources in one compact form.

I also present the expression J which occurs at the RHS, when we perform the computations with sources:

---

[3] In the case of non-zero sources we have also other two equations:

$$\begin{aligned}
&-[þ' - 2\overline{\tau}^* + \pi^*]\Phi_{01} + [-þ - 2\tau^* + \overline{\pi}^*]\Phi_{12} + \\
&\quad [ð - 2(\rho^* + \overline{\rho}^*]\Phi_{11} - [-ð' + \mu^* + \overline{\mu}^*]\Phi_{02} + \\
&\quad \overline{\sigma}^*\Phi^*_{02} + \sigma^*\Phi^*_{20} - \overline{\kappa}^*\Phi^*_{12} - \kappa^*\Phi^*_{21} + 3ð\Lambda = 0,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&[ð - 2\tau + 2\pi^*]\Phi_{11} - 3ð\Lambda + [-þ + 2\rho + \overline{\rho}]\Phi_{12} + \\
&\quad [-þ' - 2\overline{\mu} - \mu]\Phi_{01} + [ð' - \tau^* + \pi]\Phi_{02} \\
&\quad - \kappa\Phi_{22} + \overline{\nu}\Phi_{00} + \sigma\Phi_{21} - \overline{\lambda}\Phi_{10} = 0.
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&ðþ\Phi_{01} - ðð\Phi_{00} - ð(\overline{\pi}\Phi_{00}) - 2ð(\overline{\rho}\Phi_{01}) + ð(\overline{\kappa}\Phi^{*}_{00}) + 2ð(\kappa\Phi_{11}) - 2ð(\sigma\Phi_{10}) \\
&\quad þð\Phi_{01} - þþ\Phi_{02} + þ(\overline{\mu}\Phi_{02}) - 2þ(\overline{\tau}\Phi_{01}) - þ(\overline{\sigma}\Phi_{00}) + 2þ(\sigma\Phi_{11}) - 2þ(\kappa\Phi_{12}) \\
&\quad - \overline{\tau}'(þ\Phi_{01} - ð\Phi_{00} - \overline{\pi}\Phi_{00} - 2\overline{\rho}\Phi_{01} + \overline{\kappa}\Phi_{02} + 2\kappa\Phi_{11} - 2\sigma\Phi_{10}) + \\
&\quad \kappa(-ð'\Phi_{02} + þ'\Phi_{01} + \overline{\kappa}\Phi_{00} + 2\overline{\rho}\Phi_{01} + \overline{\tau}'\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12} - 2ð\Lambda) + \\
&\quad \overline{\rho}(-ð\Phi_{01} + þ\Phi_{02} - \overline{\mu}\Phi_{02} + 2\overline{\tau}\Phi_{01} + \overline{\sigma}\Phi_{00} - 2\sigma\Phi_{11} + 2\kappa\Phi_{12}) \\
&\quad - \sigma(þ'\Phi_{00} - ð'\Phi_{01} - \overline{\sigma}\Phi_{02} + 2\overline{\tau}\Phi_{01} - \overline{\rho}'\Phi_{00} + 2\tau\Phi_{10} + 2þ\Lambda - 2\rho\Phi_{11}) \\
&\quad - 4\tau(þ\Phi_{01} - ð\Phi_{00} + \overline{\pi}\Phi_{00} - 2\overline{\rho}\Phi_{01} + \overline{\kappa}\Phi_{02} + 2\kappa\Phi_{11} - 2\sigma\Phi_{10}) + \\
&\quad 4\rho(-ð\Phi_{01} + þ\Phi_{02} - \overline{\mu}\Phi_{02} + 2\overline{\tau}\Phi_{01} + \overline{\sigma}\Phi_{00} - 2\sigma\Phi_{11} + 2\kappa\Phi_{12}) = J. \qquad (3.23)
\end{aligned}
$$

We can collect terms now, so we will get the following equation:

$$
\begin{gathered}
[ðþ - 2ð\overline{\rho} - 2\overline{\rho}ð + þð - 2þ\overline{\tau} - 2\overline{\tau}þ - \overline{\tau}'þ + 2\overline{\rho\tau}' + \kappa þ' + 2\overline{\rho}\kappa - \\
-\overline{\rho}ð + 2\overline{\tau\rho} + \sigma ð' - 2\overline{\tau}\sigma - 4\tau þ + 8\tau\overline{\rho} - 4\rho ð + 8\overline{\tau}\rho]\Phi_{01} + \\
[-ðð - ð\overline{\pi} - \overline{\pi}ð - þ\overline{\sigma} - \overline{\sigma}þ + \overline{\pi\tau}' + \overline{\tau}'ð + \kappa\overline{\kappa} + \overline{\rho\sigma} - \\
-\sigma þ' + \sigma þ' + \sigma\overline{\rho}' + 4\tau ð - 4\tau\overline{\pi} + 4\rho\overline{\sigma}]\Phi_{00} + \\
[-2ð\sigma - 2\sigma ð + 2\overline{\tau}'\sigma + 6\sigma\tau]\,\Phi_{10} + \\
+ [2ð\kappa + 2\kappa ð + 2þ\sigma + 2\sigma þ - 2\kappa\overline{\tau}' - 2\sigma\overline{\rho} - 6\kappa\tau + 10\rho\sigma]\,\Phi_{11} - \\
[-2þ\kappa - 2\kappa þ + 6\rho\kappa + 2\kappa\overline{\rho}]\,\Phi_{12} + [-2\kappa ð - 2\sigma þ]\,\Lambda + \\
[ð\overline{\kappa} + \overline{\kappa}ð - þþ + þ\overline{\mu} + \overline{\mu}þ - \overline{\kappa\tau}' - \kappa ð' + \kappa\overline{\tau}' + \\
+\overline{\rho}þ - \overline{\mu}\rho + \sigma\overline{\sigma} - 4\overline{\kappa}\tau + 4\rho þ - 4\rho\overline{\mu}]\Phi_{02}. \qquad (3.24)
\end{gathered}
$$

We can also reproduce the result from the article of [46], when we use our GHP - approach. This means that the dependence of scalar perturbation $d\Psi_2$ goes like

$$
d\Psi_2 \sim \frac{1}{a^3 r^3} \qquad (3.25)
$$

for the case of FLRW ST and

$$
d\Psi_2 \sim \frac{1}{r^3} \qquad (3.26)
$$

for the case of Schwarzschild ST.[4] (And as we know, scalar perturbations are important for origin of structures. So we need to know the dependence of this functions on the radial coordinate.)

Conclusion

I have been studying perturbation theory of FLRW ST in GHP formalism. We obtained a new interesting observation, which could be used for other research in the realm of Cosmological Perturbation Theory. We want to apply these results to Cosmological Inflation in the future.

[4] We could find more informations about perturbations of Schwarzschild spacetime here: [42] and [43]

4. Scalar perturbations in $f(R)$-cosmology in the late universe

4.1 Introduction

It was experimentally verified that the expansion of the Universe is accelerating . The theoretical challenge is to find out why. f(R)-cosmologies - or f(R)-gravities, when we apply them in cosmology - were already long time a promising way how to model the accelerated epochs in the evolution of our Universe: Cosmological Inflation in the first second and accelerated expansion in the last 5 billion years; It was also a logical step to investigate the perturbations of these models.

We have been investigating the scalar perturbations, which are important for the origin of structures. We focused our research on the case of the cell of uniformity 150 Mpc. We could not use the hydrodynamical approach inside this cell, because the Universe is highly inhomogeneous here. Therefore we used the, so called, mechanical approach, which is a generalization of the well-known Landau's approach for the case of cosmology. We wanted to gain the scalar potentials Φ and Ψ in this first step of our work. Our result could have also applications in astrophysical simulations.

4.2 Scalar perturbations in f(R)-cosmologies in the late universe

Now we will use the results summarized in the Appendix A and we will study scalar perturbations of the metrics for non-linear $f(R)$-models, [48], which are examples of the so called scalar-tensor theories. We will consider the Universe at the late stage of its evolution and deep inside the cell of uniformity.[1] We will investigate the astrophysical approach in the case of Minkowski spacetime background and two cases in the cosmological approach: the large scalaron mass approximation and the quasistatic approximation. We get explicit expressions for scalar perturbations for both these cases. We will consider a special class of $f(R)$-models which have solutions R_{dS} of the equation (see Appendix A)

$$F(R)R - 2f(R) = 0. \tag{4.1}$$

This equation follows from (4.58) in Appendix A for the case of vacuum solutions for which Ricci scalar is constant. Such solutions are called de Sitter points. We can expand the function $f(R)$ in the vicinity of one of these points:

$$f(R) = f(R_{dS}) + F(R_{dS})(R - R_{dS}) + o(R - R_{dS}) = -f(R_{dS}) + \frac{2f(R_{dS}}{R_{dS}}R + o(R - R_{dS}), \tag{4.2}$$

[1] Some basic facts about Hubble flows inside the cell of uniformity are given in Appendix B.

where we used equation (4.1). Now we suppose that parameters of the model can be chosen in such a way that

$$\frac{2f(R_{dS})}{R_{dS}} = 1 \Rightarrow f(R_{dS}) = \frac{R_{dS}}{2}. \tag{4.3}$$

Therefore we get

$$f(R) = -2\Lambda + R + o(R - R_{dS}), \tag{4.4}$$

where $\Lambda = \frac{R_{dS}}{4}$. The stability of these points was discussed in [48] and [49]. Obviously, these models go asympotically to the de Sitter space when $R \to R_{dS} \neq 0$ with a cosmological constant $\Lambda = \frac{R_{dS}}{4}$. This happens when the matter content becomes negligible with respect to Λ as it is in the case with late Friedmann-Lemaitre-Robertson-Walker cosmology. We can also consider a zero solution $R_{dS} = 0$ of equation (4.1). Such points are called Minkowski points. Here, $\Lambda = 0$ and such models go asymptotically to the Minkowski space. In particular, three popular models, Starobinsky, Hu-Sawicky and MJWQ ([50], [51], [52]), have stable de Sitter points in the future (approximately at the redshift $z = -1$).

Basic Friedmann equations in case of $f(R)$-theories are given in the Appendix A, (4.62). They describe homogeneous background. We consider the Universe at late stages of its evolution, when galaxies and cluster of galaxies have already formed and when the Universe is highly inhomogeneous inside the cell of uniformity, which is approximately 150 Mpc in size. These inhomogeneities perturb the homogeneous background. At scales larger than the cell of the uniformity, the matter fields are well described by the hydrodynamical approach. On the smaller scales the mechanical approach is more adequate. In the mechanical approach, galaxies, dwarf galaxies, and clusters of galaxies (composed of baryonic and dark matter) can be considered as separate compact objects. Moreover, at distances much greater than their characteristic sizes they can be well described as point-like matter sources with the rest mass density

$$\rho = \frac{1}{a^3}\sum_i m_i \delta(\vec{r} - \vec{r_i}) \equiv \frac{\rho_c}{a^3}, \tag{4.5}$$

where $\vec{r_i}$ is the radius-vector of the i-th gravitating mass in the co-moving coordinates. This is the generalization of the well known astrophysical approach to the case of dynamical cosmological background, [53]. Usually, the gravitational fields of these inhomogeneities are weak and their peculiar velocities are much smaller than the speed of light. All these inhomogeneities result in scalar perturbations of the FLRW metrics. In the conformal Newtonian gauge such perturbed metrics are

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)(dx^2 + dy^2 + dz^2), \tag{4.6}$$

where scalar perturbations $\Phi, \Psi << 1$. The smallness of non-relativistic gravitational potentials Φ and Ψ and the smallness of peculiar velocities are two independent conditions. We will split the investigation of galaxy dynamics into two steps. First we neglect the peculiar velocities and we define gravitational potential Φ. Then we use this potential to determine dynamical behavior of galaxies. This enables us to take into account both the gravitational attraction between inhomogeneities and the global cosmological expansion of the Universe.

The case $f(R) = R$ was already investigated in [54].[2] This result is devoted to the first step in the program. We are going to define scalar perturbations Φ and Ψ for the $f(R)$ gravitational models.

Under our assumptions and according to [48], these perturbations satisfy the following system of equations:

$$-\frac{\Delta\Psi}{a^2} + 3H(H\Phi + \dot{\Psi}) = -\frac{1}{2F}\big[(3H^2 + 3\dot{H} + \frac{\Delta}{a^2})\delta F - 3H\dot{\delta F} + +3H\dot{F}\Phi + 3\dot{F}(H\Phi + \dot{\Psi}) + \kappa^2\delta\rho\big]\,, \tag{4.7}$$

$$H\Phi + \dot{\Psi} = \frac{1}{2F}(\dot{\delta F} - H\delta F - \dot{F}\Phi)\,, \tag{4.8}$$

$$-F(\Phi - \Psi) = \delta F\,, \tag{4.9}$$

$$3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Phi}) + 6H(H\Phi + \dot{\Psi}) + 3\dot{H}\Phi + \frac{\nabla\Phi}{a^2} = \frac{1}{2F}\big[3\ddot{\delta F} + 3H\dot{\delta F} - -6H^2\delta F - \frac{\Delta\delta F}{a^2} - 3\dot{F}\dot{\Phi} - 3\dot{F}(H\Phi + \dot{\Psi}) - (3H\dot{F} + 6\ddot{F})\Phi + \kappa^2\delta\rho\big], \tag{4.10}$$

$$\ddot{\delta F} + 3H\dot{\delta F} - \frac{\Delta\delta F}{a^2} - \frac{1}{3}R\delta F = \frac{1}{3}\kappa^2(\delta\rho - 3\delta P) + \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) + +2\ddot{F}\Phi + 3H\dot{F}\Phi - \frac{1}{3}F\delta R, \tag{4.11}$$

$$\delta F = F'\delta R, \quad \delta R = -2\big[3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) + 12H(H\Phi + \dot{\Psi}) + \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2}\big]\,. \tag{4.12}$$

In these equations, the function F, its derivative $F^{'}$ and the scalar curvature R are unperturbed background quantities. Here, Δ is a Laplacian in the comoving coordinates. As a matter source, we consider dust like matter. Therefore $\delta P = 0$ and

$$\delta\rho = \rho - \overline{\rho} = \frac{(\rho_c - \overline{\rho}_c)}{a^3}, \tag{4.13}$$

where $\overline{\rho}$ and ρ are defined in previous text.

4.2.1 Astrophysical approach

It can be easily verified that in the linear case $f(R) = R \Rightarrow F(R) = 1$, this system of equations is reduced to equations (2.18) - (2.20) in [54]. Now we will consider previous equations (4.7) - (4.12) in the astrophysical approach. This means that we neglect the time dependence of functions in these equations by

[2] Further information about related topics can be found, which we study here, in [57], [58].

setting all time derivatives equal to zero. It is also supposed that the background model is matter-free, i.e. $\overline{\rho} = 0$. There are two types of vacuum background solutions of the equation (4.58): de Sitter spacetime with

$$R_{dS} = 12H^2 = \text{const.} \neq 0 \tag{4.14}$$

and Minkowski spacetime with $R = 0$ and $H = 0$. However the system of equations was obtained for FLRW metrics, where we explicitly took into account the dependence of the scale factor a on time. Therefore if we want to get the time independent astrophysical equations directly from (4.7)-(4.12), we should also neglect the time dependence of a, the background parameter $H = 0$. This means that the background solution is the Minkowski spacetime. This background is perturbed by dust-like matter with the rest mass density, (4.5). Keeping in mind that $\overline{\rho} = 0$ we have $\delta\rho = \rho$.
In the case of Minkowski background and dropping the time derivatives, equations (4.7-4.12) in the astrophysical approach are reduced to the following system:

$$-\frac{\Delta}{a^2}\Psi = -\frac{1}{2F}\left(\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\right), \tag{4.15}$$

$$-F(\Phi - \Psi) = \delta F, \tag{4.16}$$

$$\frac{\Delta}{a^2}\Phi = \frac{1}{2F}\left(-\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\right), \tag{4.17}$$

$$-\frac{\Delta}{a^2}\delta F = \frac{1}{3}\kappa^2\delta\rho - \frac{1}{3}F\delta R, \tag{4.18}$$

$$\delta F = F'\delta R, \quad \delta R = -2\left(\frac{\Delta}{a^2}\Phi - 2\frac{\Delta}{a^2}\Psi\right). \tag{4.19}$$

From (4.15) and (4.17) we obtain respectively

$$\Psi = \frac{1}{2F}\delta F + \frac{\varphi}{a} = \frac{F'}{2F}\delta R + \frac{\varphi}{a}, \quad \Phi = -\frac{1}{2F}\delta F + \frac{\varphi}{a} = -\frac{F'}{2F}\delta R + \frac{\varphi}{a}, \tag{4.20}$$

where the function φ satisfies the equation

$$\Delta\varphi = \frac{1}{2F}\kappa^2 a^3\delta\rho = \frac{1}{2F}\kappa^2\delta\rho_c = 4\pi G_N\delta\rho_c, \quad G_N = \frac{\kappa^2}{8\pi F}. \tag{4.21}$$

Here we took into consideration that in the astrophysical approach $\delta\rho_c = \rho_c$ where ρ_c is defined by (4.5). It is worth noting that in the Poisson equation the Newtonian gravitational constant G_N is replaced by an effective one

$$G_{eff} = G_N/F. \tag{4.22}$$

Equation (4.16) follows directly from (4.20) and consequently, may be dropped, while from (4.18) we get the following Helmholtz equation with respect to the scalaron function δR :

$$\Delta\delta R + \frac{a^2}{3}\left(R - \frac{F}{F'}\right)\delta R = -\frac{a^2}{3F'}\kappa^2\delta\rho. \tag{4.23}$$

On the other hand, it can be easily seen that the substitution of equations (4.20) and (4.21) into (4.19) results in the same equation (4.23). Therefore, in the case of Minkowski background, the mass squared of the scalaron is

$$M^2 = \frac{a^2}{3}\frac{F}{F'}. \tag{4.24}$$

Now we want to take into consideration the cosmological evolution. This means that the background functions may depend on time. In this case, it is hardly possible to solve the system directly. Therefore, first we study the case of very large mass of the scalaron. It should be .also noted that we investigate the Universe filled with nonrelativistic matter with the rest mass density $\rho \sim \frac{1}{a^3}$. Hence we will drop all terms which decrease (with increasing a) faster than $\frac{1}{a^3}$. This is the accuracy of our approach. Within this approach, $\delta\rho \sim \frac{1}{a^3}$, [54].

4.2.2 Large scalaron mass

As we can see from equation (4.5), the limit of large scalaron mass corresponds to $F^{'} \to 0$. Then δF is also negligible. Therefore, equations (4.7)-(4.12) read

$$-\frac{\Delta\Psi}{a^2} + 3H(H\Phi + \dot{\Psi}) = -\frac{1}{2F}\left[3H\dot{F}\Phi + 3\dot{F}(H\Phi + \dot{\Psi})\right] , \tag{4.25}$$

$$H\Phi + \dot{\Psi} = \frac{1}{2F}(-\dot{F}\Phi) , \tag{4.26}$$

$$\Phi - \Psi = 0 , \tag{4.27}$$

$$3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) + 6H(H\Phi + \dot{\Psi}) + 3\dot{H}\Phi + \frac{\Delta\Phi}{a^2} = \frac{1}{2F}\left[-3\dot{F}\dot{\Phi} - 3\dot{F}(H\Phi + \dot{\Psi}) - (3H\dot{F} + 6\ddot{F})\Phi\right] , \tag{4.28}$$

$$0 = \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) + 2\ddot{F}\Phi + 3H\dot{F}\Phi , \tag{4.29}$$

$$0 = 3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) + 12H(H\Phi + \dot{\Psi}) + \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2} \tag{4.30}$$

From (4.26) and (4.27) we get

$$\Psi = \Phi = \frac{\varphi}{a\sqrt{F}}, \tag{4.31}$$

where the introduced function φ depends only on spatial coordinates. Substituting (4.31) into (4.25), leads to

$$\frac{1}{a^3\sqrt{F}}\Delta\varphi + \frac{3\dot{F}^2\varphi}{4aF^2\sqrt{F}} = \frac{1}{2F}\kappa^2\delta\rho \tag{4.32}$$

As we mentioned above, neglecting relativistic matter in the late Universe we have $\delta\rho \sim \frac{1}{a^3}$([54]). This approximation is getting better and better in the limit $a \to \infty$. We assume that this limit corresponds to the final stage of the Universe evolution. The similar limit with respect to the scalar curvature is $R \to R_\infty$, where the value R_∞ is just finite. Then from (4.32) we immediately come to the condition

$$F = \text{const.} + o(1), \tag{4.33}$$

where $o(1)$ is decreasing function of a. This condition holds at the considered late stage. One can naively suppose that in the late Universe $\dot{F} \approx \frac{1}{a} + o(\frac{1}{a})$.

However this is wrong. Obviously, without loss of generality, we can suppose that const. = 1. From the condition (4.33) we get

$$F = 1 + o(1) \Rightarrow f = -2\Lambda + R + o(R - R_\infty), \tag{4.34}$$

where Λ is the cosmological constant. Therefore the limit of the large scalaron mass takes place for models which possess the asymptotic form of (4.34). For example, R_∞ may correspond to the de Sitter point R_{dS} in future. All three popular models, Starobinsky, Hu-Sawicky and MJWQ [50],[51], [52] have such stable de-Sitter points in the future (approximately at the redshift $z = -1$) ([59],[60]). The condition of stability is $0 < \frac{RF'}{F} < 1$. Since $F \approx 1$, this condition reads $0 < R < \frac{1}{F'}$, which is fulfilled for the de Sitter points in the above-mentioned models. The reason is the smallness of F'.
We now return to the remaining equations (4.28) - (4.30) to show that they are satisfied within the considered accuracy. First, we study (4.28), which after the substitution of (4.31), (4.32) and some simple algebra takes the form

$$\frac{\varphi \dot{H}}{a} - \frac{\varphi}{2aF}(H\dot{F} - \ddot{F}) = 0. \tag{4.35}$$

To estimate $\dot{F}$ and $\ddot{F}$, we take into account that in the limit $R \to R_\infty$, $F \approx 1$, $H \approx const. \Rightarrow \dot{H} \approx 0$, and $F'(R_\infty)$ is some finite positive value. Then,

$$\dot{F} = F'\dot{R} \approx F'(R_\infty)\dot{R} \approx \dot{T} \approx d(1/a^3)/dt \approx H(1/a^3) \approx 1/a^3 \tag{4.36}$$

and $\ddot{F} \approx \dot{a}/a^4 \approx \frac{1}{a^3}$. Therefore, the LHS of equation (4.35) is of order $o(1/a^3)$ and we can set it to zero within the accuracy of our approach. Similarly, equations (4.29) and (4.30) are satisfied within the considered accuracy. It can be also seen that the second term on the left hand side of equation (4.32) is of order $O(1/a^7)$ and should be eliminated. Thus, in the case of the large enough scalaron mass we reproduce the linear cosmology from the nonlinear one, as it should be.

4.2.3 Quasi-static approximation

Now we do not want to assume a priori that the scalaron mass is large, i.e. F' can have arbitrary values. Hence, we will preserve the δF terms in equations (4.7) - (4.12). Moreover, we should keep the time derivatives in these equations. Such a system is very complicated for direct integration. However, we can investigate it in the quasi-static approximation. According to this approximation, the spatial derivatives give the main contribution to equations (4.7)-(4.12), ([61], [62]). Therefore, first, we should solve "astrophysical" equations (4.15)-(4.19), and then check whether the found solutions satisfy (up to the adopted accuracy) the full system of equations. In other words, in the quasi-static approximation it is naturally supposed that the gravitational potentials (the functions Φ, Ψ) are produced mainly by the spatial distribution of astrophysical/cosmological bodies. As we have seen, equations (4.15) - (4.19) result in (4.20) - (4.23). Now, we should keep in mind that we have the cosmological background. Moreover, we consider the late Universe which is not far from the de Sitter point R_{dS} in the future. This means that $\delta\rho = \rho - \overline{\rho}$ in (4.21), all background quantities are calculated roughly speaking at R_{dS} and the scalaron mass squared (4.5) reads now

$$M^2 = \frac{a^2}{3}\left(\frac{F}{F'} - R_{dS}\right). \tag{4.37}$$

Let us now consider equation (4.23) with the mass squared (4.37). Taking into account that now $\delta\rho_c = \rho_c - \overline{\rho_c}$, we can rewrite this equation as follows:

$$\Delta\widetilde{\delta R} - M^2\widetilde{\delta R} + \frac{a^2}{3F'}\frac{\kappa^2}{a^3}\sum_i m_i\delta(\vec{r} - \vec{r_i}) = 0, \tag{4.38}$$

where

$$\widetilde{\delta R} = \delta R + \frac{\kappa^2}{(F - F'R_{dS})a^3}\kappa^2\rho_c. \tag{4.39}$$

Then, the general solution for a full system is the sum over all gravitating masses. As a boundary conditions, we require for each gravitating mass the behavior $\delta R \sim \frac{1}{r}$ at small distances r and $\widetilde{R} \to 0$ for $r \to \infty$. Taking all these remarks into consideration, we obtain for the full system

$$\delta R = \frac{\kappa^2}{12\pi a F'}\sum_i \frac{m_i \exp(-M_i|\vec{r} - \vec{r_i}|)}{|\vec{r} - \vec{r_i}|} - \frac{\kappa^2\overline{\rho_c}}{(F - F'R_{dS})a^3}. \tag{4.40}$$

It is worth noting that averaging over the whole co-moving spatial volume V gives the zero value $\overline{\delta R}$. This result is reasonable because the rest mass density fluctuation $\delta\rho$, representing the source of the metric and the scalar curvature fluctuations Φ, Ψ and $\delta\rho$, has a zero average value $\overline{\delta\rho} = 0$. Consequently, all enumerated quantities should also have zero average values : $\overline{\Phi} = \overline{\Psi} = 0$ and $\overline{\delta R} = 0$, in agreement with (4.40).
From equation (4.20) we get the scalar perturbation functions Φ and Ψ in the following form:

$$\Psi = \frac{F'}{2F}\left[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r} - \vec{r_i}|)}{|\vec{r} - \vec{r_i}|} - \frac{\kappa^2}{(F - F'R_{dS})a^3}\overline{\rho_c}\right] + \frac{\varphi}{a}, \tag{4.41}$$

$$\Phi = \frac{-F'}{2F}\left[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r} - \vec{r_i}|)}{|\vec{r} - \vec{r_i}|} - \frac{\kappa^2}{(F - F'R_{dS})a^3}\overline{\rho_c}\right] + \frac{\varphi}{a}, \tag{4.42}$$

where φ satisfies equation (4.21) with $\delta\rho$ in the form (4.13) (i.e., $\overline{\rho}_c \neq 0$). Obviously when $F' \to 0$, $M \to \infty$, and we have

$$\exp(-M|\vec{r} - \vec{r_i}|)/|\vec{r} - \vec{r_i}| \to 4\pi\delta(\vec{r} - \vec{r_i})/M^2, \tag{4.43}$$

so the expression in the square brackets in (4.41) and (4.42) is equal to

$$\kappa^2\delta\rho_c/[(F - F'R_{dS})a^3]. \tag{4.44}$$

Therefore, in the considered limit $F' \to 0$ we reproduce the scalar perturbations Φ, Ψ from the previous large scalaron mass case, as it certainly should be.

Thus neglecting for a moment the influence of the cosmological background, but not neglecting the scalaron's contribution, we have found the scalar perturbations. They represent the mix of the standard potential $\frac{\varphi}{a}$ (see the linear case [54]) and the additional Yukawa term which follows from the nonlinearity.
Now we should check that these solutions satisfy the full system (4.7)-(4.12). To do it, we substitute (4.40), (4.41) and (4.42) into this system of equations. Obviously the spatial derivatives disappear. Keeping in mind this fact, the system (4.7)-(4.12) is reduced to the following equations:

$$3H\left(H\Phi+\dot{\Psi}\right)=-\frac{1}{2F}\left[\left(3H^2+3\dot{H}+\frac{\Delta}{a^2}\right)\delta F-3H\delta\dot{F}+3H\dot{F}\Phi+3\dot{F}\left(H\Phi+\dot{\Psi}\right)\right], \tag{4.45}$$

$$H\Phi+\dot{\Psi}=\frac{1}{2F}\left(\delta\dot{F}-H\delta F-\dot{F}\Phi\right), \tag{4.46}$$

$$3\left(\dot{H}\Phi+H\dot{\Phi}+\ddot{\Psi}\right)+6H\left(H\Phi+\dot{\Psi}\right)+3\dot{H}\Phi+\frac{\Delta\Phi}{a^2}=\frac{1}{2F}\left[3\delta\ddot{F}+3H\delta\dot{F}-6H^2\delta F-\frac{\Delta\delta F}{a^2}-3\dot{F}\dot{\Phi}-3\dot{F}\left(H\Phi+\dot{\Psi}\right)-\left(3H\dot{F}+6\ddot{F}\right)\Phi\right], \tag{4.47}$$

$$\delta\ddot{F}+3H\delta\dot{F}-\frac{\Delta\delta F}{a^2}=\dot{F}(3H\Phi+3\dot{\Psi}+\dot{\Phi})+2\ddot{F}\Phi+3H\dot{F}\Phi, \tag{4.48}$$

$$\delta F=F'\delta R, \tag{4.49}$$

$$\frac{F'}{F}R_{dS}\delta R=-2\left[3\left(\dot{H}\Phi+H\dot{\Phi}+\ddot{\Psi}\right)+12H\left(H\Phi+\dot{\Psi}\right)+\frac{\Delta\Phi}{a^2}+3\dot{H}\Phi-2\frac{\Delta\Psi}{a^2}\right]. \tag{4.50}$$

Here the term $\frac{F'}{F}R_{dS}\delta R$ in the left hand side of (4.50) disappear due to the redefinition of the scalaron mass squared (4.37).
It can be easily seen that all terms in (4.40), (4.41) and (4.42) depend on time, and therefore may contribute to equations (4.45)-(4.50). As we wrote above, according to our non-relativistic approach, we neglect the terms of the order $o(1/a^3)$. On the other hand, exponential functions decrease faster than any power function. Moreover, we can write the exponential term in (4.40) as follows:

$$\frac{\kappa^2}{12\pi F'}\sum_i\frac{m_i\exp(-\sqrt{\frac{1}{3}(\frac{F}{F'}-R_{dS})}|r_{ph}-r_{ph,i}|)}{|r_{ph}-r_{ph,i}|} \tag{4.51}$$

where we introduced the physical distance $r_{ph}=ar$. It is well known that astrophysical tests impose strong restrictions on the non-linearity [63, 64] (the local gravity tests impose even stronger constraints [54, 63, 64]). According to these constraints, (4.51) should be small at the astrophysical scales. Consequently, on

the cosmological scales it will be even much smaller. So we will not take into account the exponential terms in the above equations. However, in (4.40), (4.41) and (4.42), we have also $\frac{1}{a^3}$ and $\frac{1}{a}$ terms which we should examine. Before performing this, it should be recalled that we consider the late Universe which is rather close to the de Sitter point. Therefore, as we already noted in the previous subsection, $F \approx 1$, $H \approx$ const. $\rightarrow \dot{H} \approx 0$, $R_{dS} = 12H^2$ and $F^{'}(R_{dS})$ is some finite positive value. Additionaly, $\dot{F}, \ddot{F}, \dot{F}^{'} \approx \frac{1}{a^3}$. Hence, all terms of the form of $\dot{F}, \ddot{F}, \dot{F}^{'} \times \Phi, \Psi, \dot{\Phi}, \dot{\Psi}$ are of the order $o(1/a^3)$ and should be dropped. In other words, the functions F and $F^{'}$ can be considered as time independent.
First, let us consider the terms $\Psi = \Phi = \varphi/a$ in equations (4.41) and (4.42) and substitute them into equations (4.45) - (4.50). Such $1/a$ term is absent in δR. So we should put $\delta R = 0$, $\delta F = 0$. Obviously, this is the linear theory case. It can be easily seen that all equations are satisfied.
Now, we study the terms $\sim 1/a^3$, i.e.,

$$\delta R = -\frac{\kappa^2}{(F - F^{'}R_{dS})}\frac{\overline{\rho}_c}{a^3}\,\Psi = -\frac{\kappa^2 F^{'}}{2F(F - F^{'}R_{dS})}\frac{\overline{\rho}_c}{a^3}\,\Phi = \frac{\kappa^2 F^{'}}{2F(F - F^{'}R_{dS})}\frac{\overline{\rho}_c}{a^3}. \tag{4.52}$$

Let us examine, for example equation (4.45). Keeping in mind that $\delta F = F^{'}\delta R$, one can easily get

$$12H_c^2\frac{\kappa^2 F^{'}}{2F(F - F^{'}R_{dS})}\frac{\overline{\rho}_c}{a^3} = 12H_c^2\frac{\kappa^2 F^{'}}{2F(F - F^{'}R_{dS})}\frac{\overline{\rho}_c}{a^3} + o(1/a^3). \tag{4.53}$$

Therefore, the terms $\sim \frac{1}{a^3}$ exactly cancel each other, and this equation is satisfied up to the adopted accuracy $o(1/a^3)$. One can easily show that the remaining equations are fulfilled with the same accuracy.

Thus we have proved that the scalar perturbation functions Φ and Ψ in the form (4.41) and (4.42) satisfy the system of equations (4.45)-(4.50) with the required accuracy. Both of these functions contain the nonlinearity function F and the scale factor a. Therefore both effects of nonlinearity and the dynamics of the cosmological background are taken into account. The function Φ corresponds to the gravitational potential of the system of inhomogeneities. Hence we can study the dynamical behavior of the inhomogeneities including into consideration their gravitational attraction and cosmological expansion, and also taking into account the effects of nonlinearity. For example, the non-relativistic Lagrange function for a test body of the mass m in the gravitational field described by the metric (4.6) has the form ([54]):

$$L \approx -m\Phi + \frac{ma^2\vec{v}^2}{2},\ \vec{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2. \tag{4.54}$$

We can use this Lagrange function for analytical and numerical study of mutual motion of galaxies. Such investigation was performed in the case of the linear theory, e.g., in [57].

More information about related topics can be found, for example, in [59-61] and [62-64].

4.3 Conclusion

We have studied scalar perturbations of the metrics in nonlinear $f(R)$ gravity. The Universe has been considered at the late stage of its evolution and at scales much less than the cell of uniformity size which is approximately 150 Mpc. At such distances, our Universe is highly inhomogeneous, and the averaged hydrodynamic approach does not work here. We need to take into account the inhomogeneities in the form of galaxies, groups and clusters of galaxies. The peculiar velocities of these inhomogeneities are much smaller than the speed of light, and we can use the non-relativistic approximation. This means that in equations for scalar perturbations, we first neglect peculiar velocities and solve these equations with respect to scalar perturbation functions Φ and Ψ. The function Φ represents the gravitational potential of inhomogeneities. Then we use the explicit expression for Φ to describe the motion of inhomogeneities. Such mechanical approach is well known in astrophysics ([53]). We generalized it to the case of dynamical cosmological background ([54],[55]). The main objective of this work was to find explicit expressions for Φ and Ψ in the framework of nonlinear $f(R)$ models. Unfortunately, in the case of nonlinearity, the system of equations for scalar perturbations is very complicated. It is hardly possible to solve it directly. Therefore, we have considered the following approximations: the astrophysical approach; the large scalaron mass case and quasistatic approximation. We found the explicit expressions for the scalar perturbation functions Φ and Ψ in all three cases up to the required accuracy. The latter means that, because we considered non-relativistic matter with the averaged rest mass density $\rho \sim \frac{1}{a^3}$, all quantities in the cosmological approximation are also calculated up to corresponding orders of $\frac{1}{a}$. It should be noted that in the cosmological approach our consideration is valid for nonlinear models where functions of $f(R)$ have the stable de Sitter points in the future with respect to the present time, and as we go closer to R_{dS}, the approximation is more correct. All three popular models, Starobinsky, Hu-Sawicky, and Miranda, have such stable de Sitter points in the future.
The quasi-static approximation is most interesting from the point of view of the large scale structure investigations. Here, the gravitational potential Φ contains both the nonlinearity function F and the scale factor a. Hence we can study the dynamical behavior of the inhomogeneities including into consideration their gravitational attraction and the cosmological expansion, and also taking into account the effect of nonlinearity. All this make it possible to carry out the numerical [3] and analytical analysis of the large scale structure dynamics in the late Universe for $f(R)$ models as was done in [57] for the case of SGR.

4.4 Appendix A: Basic facts from $f(R)$-cosmology

We get the Einstein's equations in SGR from the variational principle. We do the variation of the following action

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\; f(R)\; d^4x + S_m, \tag{4.55}$$

[3]We suggest to use, for example, the explixit Euler method. We could use the equation (51) from [56]

where S_m is action for matter and $\kappa^2 = 8\pi G$.

We need to do the variation with respect to the action, where we have an analytical function $f(R)$ of the Ricci scalar R. We get the following field equations for the case of $f(R)$-theories:

$$\Sigma_{\mu\nu} \equiv F(R)g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Box F g_{\mu\nu} - \nabla_\mu\nabla_\nu F = \kappa^2 T^M_{\mu\nu}, \tag{4.56}$$

$T^M_{\mu\nu}$ is again the energy momentum tensor defined by the variational derivative of L_M in terms of $g^{\mu\nu}$:

$$T^M_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}}. \tag{4.57}$$

This tensor satisfies the continuity equation, as well as $\Sigma_{\mu\nu}$, $\nabla^\mu \Sigma_{\mu\nu} = 0$.
Now, Einstein gravity without the cosmological constant corresponds to $f(R) = R$ and $F(R) = 1$, so that the term $\Box F$ in

$$3\Box F - FR - 2f = \kappa^2 T \tag{4.58}$$

vanishes. In this case we have $R = -\kappa^2 T$ and hence the Ricci scalar is directly determined by the matter. The term $\Box F$ does not vanish in the case of modified gravities which means there is a propagating scalar degree of freedom, $\phi = F(R)$, so called scalaron. The trace equation determines the dynamics of the scalar field. Again,

$$G_{\mu\nu} = \kappa^2(T^M_{\mu\nu} + T^{eff}_{\mu\nu}), \tag{4.59}$$

$$\kappa^2 T^{eff}_{\mu\nu} \equiv g_{\mu\nu}\frac{(f-R)}{2} + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F + (1-F)R_{\mu\nu}. \tag{4.60}$$

Since $\nabla^\mu G_{\mu\nu} = 0$ and $T^M_{\mu\nu} = 0$, then $\nabla^\mu T^{eff}_{\mu\nu} = 0$. Hence the continuity equation holds not only for $\Sigma_{\mu\nu}$, but also for the effective $T^{eff}_{\mu\nu}$! We consider only $f(R)$ -theories which admit de Sitter points: these points corresponds to vacuum solutions at which is the Ricci scalar constant. So,

$$RF - 2f(R) = 0. \tag{4.61}$$

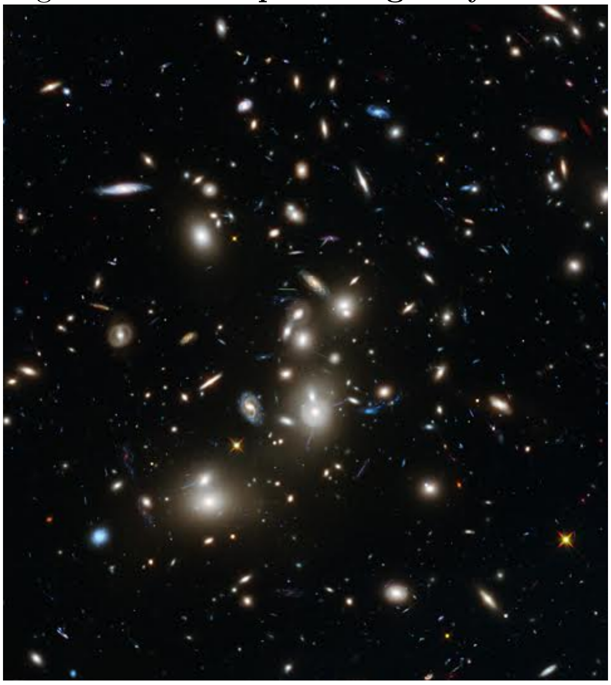
We need the de Sitter points, because we want to model inflation and accelerated expansion of our Universe.
In the case of $f(R)$-cosmologies we get again two - but more complicated - Friedmann equations:

$$3FH^2 = \frac{FR - f}{2} - 3H\dot{F} + \kappa^2\rho_M, \tag{4.62}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2(\rho_M + p_M), \tag{4.63}$$

plus the continuity equation $\dot{\rho_M} + 3H(\rho_M + p_M) = 0$. We have again that the first equation with the continuity equation imply the second equation. But the steps are different than in the case of SGR.

Figure 4.1: **Example of a galaxy cluster**



4.5 Appendix B: Hubble flows in observable Universe

The picture 4.1 represents an example of a galaxy cluster: Abell[4] 2744 galaxy cluster; Galaxies are grouped - as we all know - into larger units called clusters, superclusters and voids. The fact that the system of structures ends - according to the current knowledges - is important. This means that Universe starts to be homogeneous and isotropic on the scales bigger than 150 Mpc and is well described by the modern realization of Friedmann models, so called ΛCDM model. One of the characteristics of this model is linear velocity-distance relation between receding motion of galaxies due to the expansion of the Universe, so called Hubble flow.

We can make a rough estimate, where the gravitational attraction prevails the cosmological expansion. If we plug $v \approx 300$ km/s (peculiar velocities) and $H \approx 70$ km/s.Mpc, we get a rough estimate $3-6$ Mpc for our local group of galaxies. From this point of view it seems reasonable that Edwin Hubble observed the flow on distances 10 - 30 Mpc.

But recent observations indicate the presence of Hubble flows on distances of few Mpc from the center of our group of galaxies. There was a suggestion that the cosmological constant is responsible for this local Hubble flow, but the answer is negative! The global cosmological expansion is responsible for local cold flow, but there is a less diffusion in the presence of cosmological constant ([54]).

[4]George Ogden Abell (1927-1983) was an american astronomer. Abell's catalog is a list of approximately 4000 groups of galaxies, which have at least 30 members.

5. Conclusion

I have been studying cosmological perturbation theory in this work. I studied SGR in HD with extended extra spatial dimensions. I discussed how GHP-formalism can be used for perturbations in higher dimensions. I reviewed the algebraic classification of spacetimes in HD and I also discussed the spinor approach. I presented other argument why $4d$ ST's should be special. There was a series of results of my co-workers, from which it follows that $4d$ ST's should be preferable from the physical point of view. I applied the GHP formalism for the perturbations of FLRW ST's in the second part, Chapter 2. I obtained a new insight to perturbation theory of these ST's. We want to apply these results for studying the phase transition at the beginning of the Universe in the next works.

In Chapter 3 I studied so called $f(R)$-cosmologies, which are a promising road for modelling the accelerated expansion of the Universe. $f(R)$-gravity is a different theory than SGR. Variational procedure leads to more complicated field equations. We studied scalar perturbations because of coupling to matter. We used a generalization of the mechanical approach for the case of cosmological background. The hydrodynamical approach is not applicable for the cell of 150 Mpc, where the homogeneous Friedmann background is perturbed by inhomogeneities. Then we dealt with the so called quasi-static approximation for obtaining the scalar potentials Φ and Ψ, because these equations were complicated for direct integration. We worked in astrophysical approach first, where we neglected the time derivatives, and then the large scalaron mass approximation. This approximation leads back to the Standard General Relativity in the limit of late universe.

In our published paper, [67], we gave explicit expressions for scalar potentials Φ and Ψ for all three cases: astrophysical approach, large-scalaron mass approximation and quasi-static approximation; One term in these expressions is the Yukawa term and other part is the contribution from standard potential. I want to apply our work for models with torsion in the future. I think that it would be interesting to concentrate first on the Hu-Sawicky function, from the recently published paper, [69], and to try to overcome so the difficulties with the $f(R)$-gravities. I believe that these theories will play a role in finding, the so called, cold dark matter.

6. Literature

I use here the most important list of used publications. The first part of the thesis is mainly connected with [1], [21] and [30]. The second part is based on [46] and [47]. The third part about $f(R)$-cosmologies is based on [67].
We could find more about Quantum Cosmology in [68].

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List of Abreviations

QG ... Quantum Gravity
SGR ... Standard General Relativity
ST ... Spacetime
FLRW ... Friedman-Lemaitre-Robertson-Walker
QFT ... Quantum Field Theory
QCD ... Quantum Chromodynamics
HD ... Higher Dimensions
NP ... Newman-Penrose formalism
GHP ... Geroch-Held-Penrose formalism
ODE ... ordinary differential equations
RNV ... reccurent null vector-field
WAND ... Weyl aligned null direction
RHS ... right-hand side