Charles University in Prague
Faculty of Mathematics and Physics

**DOCTORAL THESIS**

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# Designs and their algebraic theory

Department of Algebra

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 25 May 2015 Andrew Kozlík

I would like to thank Professor Terry S. Griggs and Professor Aleš Drápal for introducing me to the beautiful world of combinatorial designs, for their kind support and for their patient guidance throughout my graduate studies. I would also like to thank Professor William W. McCune for his model builder Mace4. This program has been an invaluable tool for me in much of my work.

*To my father, Antonín Kozlík, who always encouraged me in my pursuit of science.*

Název práce: Designy a jejich algebraická teorie

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Abstrakt: Je dobře známo, že pro každý Steinerův systém trojic (STS) lze definovat binární operaci $\cdot$ na jeho nosné množině tak, že předepíšeme $x \cdot x = x$ pro všechna $x$ a $x \cdot y = z$, kde $z$ je třetí bod v bloku obsahujícím dvojici $\{x, y\}$. Totéž lze udělat i s Mendelsohnovým systémem trojic (MTS), usměrněným systémem trojic (DTS) jakož i s hybridní systémem trojic (HTS), kde dvojici $(x, y)$ chápeme jako uspořádanou. V případě STS a MTS dostáváme kvazigrupovou operaci, ale v případě DTS a HTS tomu tak být nemusí. DTS nebo HTS, který indukuje kvazigrupu nazýváme *Latinský*. Kvazigrupy asociované s STS nebo MTS splňují flexibilní zákon $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, ale v případě Latinských DTS a Latinských HTS tomu tak být nemusí. Říkáme, že DTS nebo HTS je čistý, jestliže jakožto dvojitý systém trojic neobsahuje opakující se bloky. Tato práce je věnována studiu Latinských DTS and Latinských HTS, zejména zkoumání flexibility, čistoty a dalších souvisejících vlastností v těchto systémech. Dále se zabývá Latinskými DTS a Latinskými HTS, které mají cyklický nebo rotační automorfismus. V práci jsou mimo jiné dokázány existenční spektra těchto systémů a prezentovány enumerační výsledky. Menší část práce je pak věnována studiu velikosti centra Steinerovy lupy a spojitosti s maxi-Pasch problémem v STS.

Klíčová slova: kvazigrupa, Steinerův systém trojic, usměrněný systém trojic, Mendelsohnův systém trojic, hybridní systém trojic

Title: Designs and their algebraic theory

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Abstract: It is well known that for any Steiner triple system (STS) one can define a binary operation $\cdot$ upon its base set by assigning $x \cdot x = x$ for all $x$ and $x \cdot y = z$, where $z$ is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems (MTS), directed triple systems (DTS) as well as hybrid triple systems (HTS), where $(x, y)$ is considered to be ordered. In the case of STSs and MTSs the operation yields a quasigroup, however this is not necessarily the case for DTSs and HTSs. A DTS or an HTS which induces a quasigroup is said to be *Latin*. The quasigroups associated with STSs and MTSs satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ but those associated with Latin DTSs and Latin HTSs need not. A DTS or an HTS is said to be pure if when considered as a twofold triple system it contains no repeated blocks. This thesis focuses on the study of Latin DTSs and Latin HTSs, in particular it aims to examine flexibility, purity and other related properties in these systems. Latin DTSs and Latin HTSs which admit a cyclic or a rotational automorphism are also studied. The existence spectra of these systems are proved and enumeration results are presented. A smaller part of the thesis is then devoted to examining the size of the centre of a Steiner loop and the connection to the maxi-Pasch problem in STSs.

Keywords: quasigroup, Steiner triple system, directed triple system, Mendelsohn triple system, hybrid triple system

# Contents

# Introduction

This thesis consists of the following papers, which are reproduced here as submitted to the individual journals:

(1) Drápal, A., A. Kozlik, and T.S. Griggs. Latin directed triple systems. *Discrete Math.* 2012, **312**, 597–607.

(2) Drápal, A., T.S. Griggs, and A.R. Kozlik. Basics of DTS quasigroups: Algebra, geometry and enumeration. *J. Algebra Appl.* 2015, **14**, 1550089.

(3) Drápal, A., T.S. Griggs, and A.R. Kozlik. Triple systems and binary operations. *Discrete Math.* 2014, **325**, 1–11.

(4) Kozlik, A.R. Cyclic and rotational Latin hybrid triple systems. Submitted to *Math. Slovaca.*

(5) Drápal, A., T.S. Griggs, and A.R. Kozlik. Flexible Latin directed triple systems. *Utilitas Math.* (to appear).

(6) Drápal, A., T.S. Griggs, and A.R. Kozlik. Pure Latin directed triple systems. *Australas. J. Combin.* 2015, **62**, 59–75.

(7) Kozlik, A.R. Antiflexible Latin directed triple systems. *Comment. Math. Univ. Carolin.* (to appear).

(8) Kozlik, A.R. The centre of a Steiner loop and the maxi-Pasch problem. Submitted to *J. Combin. Des.*

My contribution to the papers with coauthors is as follows: In (1), the examples. In (2), Example 2.7, Section 4 and the Appendix. In (3), Sections 1, 4 and 6, parts of Section 5, most of Section 7 and the Appendix. In (5), the results in Table 1, half of Section 4 and the Appendix. In (6), Section 2, half of Sections 3 and 4, and the Appendix. Shortly put, my contributions to (2), (3), (5) and (6) account for roughly 30–60 % of each paper. My contribution to paper (1) is somewhat smaller. This paper is included mainly because of its fundamental role in the context.

## 1. Triple systems and their connections to algebra

A *Steiner triple system* of order $n$, STS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points taken from $V$ such that every pair of distinct points from $V$ appears in precisely one triple.

The most well known examples of Steiner triple systems come from finite geometry. Let $V = \mathbb{F}_2^n \setminus \{\mathbf{0}\}$ and let $\mathcal{B}$ be the collection of all $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ are pairwise distinct and $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. Then $(V, \mathcal{B})$ is a *projective* STS$(2^n - 1)$. The points and blocks of a projective STS$(2^n - 1)$ are the projective points and projective lines of projective geometry PG$(n-1, 2)$. For $n = 3$ we get an STS(7) also known as the Fano plane. This is the only Steiner triple system of order 7 up to isomorphism.

If we let $V = \mathbb{F}_3^n$ and $\mathcal{B}$ be the collection of all $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ are pairwise distinct and $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$, then $(V, \mathcal{B})$ is an *affine* STS$(3^n)$. The points and blocks of an affine STS$(3^n)$ are the points and lines of the affine geometry AG$(n, 3)$. For $n = 3$ we get an STS(9), which is also unique up to isomorphism.

It was Plücker who in 1835 [15] encountered the STS(9) in a study of algebraic curves and was the first to generalize this type of system and ask the question for which $n$ does an STS$(n)$ exist. In 1839 [16] he gave the necessary condition $n \equiv 1$ or $3 \pmod 6$, which can be derived by a simple counting argument. The question of the existence of Steiner triple systems was settled in 1847 by Kirkman [12]. An STS$(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$. The name comes from the fact that in 1853 Steiner [17], unaware of Kirkman's result, asked a series of questions, the first of which was the existence of what came to be known as Steiner triple systems.

To date, the number of pairwise non-isomorphic STS$(n)$s has been determined for all $n \leq 19$. The uniqueness of the STS(7) and the STS(9) was apparently well known. In 1897 Zulauf [30] showed that all known STS(13)s fall into two isomorphism classes and two years later De Pasquale [8] showed that only two isomorphism classes are possible. In the 1910s White, Cole and Cummings [7, 29] determined that there are 80 non-isomorphic STS(15)s. Almost a century later, in 2004, Kaski and Östergård [19] used digital computers to find that the total number of pairwise non-isomorphic STS(19)s is 11 084 874 829. A complete enumeration and classification of STS(21)s does not look promising in the foreseeable future [19].

A *quasigroup* is an ordered pair $(Q, \cdot)$, where $Q$ is a set and $\cdot$ is a binary operation on $Q$ such that for all $a$, $b \in Q$ there exist unique elements $x$, $y \in Q$ satisfying $a \cdot x = b$ and $y \cdot a = b$. The unique solutions to these equations are written $x = a \backslash b$ and $y = b/a$. The operations $\backslash$ and $/$ are called *left division* and *right division*, respectively. For $Q$ finite, the *order* of the quasigroup $(Q, \cdot)$ is $|Q|$. It is easy to see that the multiplication table of a finite quasigroup defines a *Latin square*, i.e. a $|Q| \times |Q|$ array such that every element of $Q$ appears exactly once in each row and exactly once in each column. A *loop* is a quasigroup with an identity element, i.e. an element $e \in Q$ such that $x \cdot e = x = e \cdot x$ for all $x \in Q$. The binary operation will sometimes be replaced with juxtaposition, for example $x \cdot yz$ meaning $x \cdot (y \cdot z)$.

Given an STS $(V, \mathcal{B})$ one can define a binary operation $\cdot$ on the set $V$ by assigning $x \cdot x = x$ for all $x \in V$ and $x \cdot y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The induced operation satisfies the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x$$

for all $x$ and $y$ in $V$. Any binary operation satisfying these three identities is called an *idempotent totally symmetric quasigroup*. The process described above is reversible. Given an idempotent totally symmetric quasigroup one can obtain an STS by assigning $\{x, y, x \cdot y\} \in \mathcal{B}$ for all $x,\ y \in V,\ x \neq y$. Thus there is a one-to-one correspondence between Steiner triple systems and idempotent totally symmetric quasigroups or *Steiner quasigroups*, as they are commonly known. All Steiner quasigroups satisfy the *flexible law* $y \cdot (x \cdot y) = (y \cdot x) \cdot y$. It is also possible to define a loop operation for any Steiner triple system by adjoining an identity element $e$ to $V$. Let $L = V \cup \{e\}$ and assign $x \cdot x = e$ for all $x \in V$ and $x \cdot y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The resulting loop is called a *Steiner loop*. Again the process is reversible. This correspondence is well known in both the combinatorial and the algebraic communities, see for example [6, page 24] and [14, page 124].

If we consider oriented triples, then there are two possibilities. A *cyclic triple* $(x, y, z)$ contains the ordered pairs $(x, y)$, $(y, z)$ and $(z, x)$. A *transitive triple* $\langle x, y, z\rangle$ contains the ordered pairs $(x, y)$, $(y, z)$ and $(x, z)$; we sometimes refer to these ordered pairs as the *initial*, the *terminal* and the *long* edge, respectively.

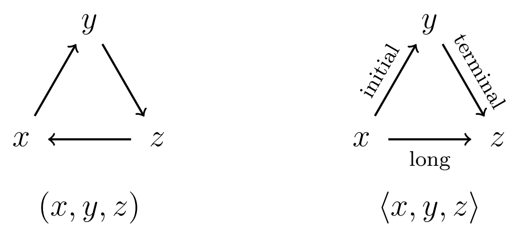


Figure 1: A cyclic triple $(x, y, z)$ and a transitive triple $\langle x, y, z\rangle$.

A *hybrid triple system* of order $n$, HTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclic and transitive triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. An HTS($n$) can also be thought of as a decomposition of the complete digraph on $n$ vertices into oriented triples which are either transitive triples or cyclic triples. The term hybrid triple system was first used in [18] but the concept appeared earlier under the name *ordered triple system* [21] and later under the name *oriented triple system* [24]. An HTS($n$) which contains only cyclic triples is known as a *Mendelsohn triple system*, MTS($n$). Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [13]. An HTS($n$) which contains only transitive triples is known as a *directed triple system*, DTS($n$). Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$ [11].

Every HTS induces a binary operation $\cdot$ upon its point set $V$. For a cyclic triple $(x, y, z)$ set $x \cdot y = z$, $y \cdot z = x$ and $z \cdot x = y$. For a transitive triple $\langle x, y, z\rangle$ set $x \cdot y = z$, $y \cdot z = x$ and $x \cdot z = y$. The *induced operation* $\cdot$ is assumed to be idempotent, i.e. $x \cdot x = x$ holds for every $x \in V$.

In the case of MTSs, the induced operation yields a *semisymmetric* quasigroup, i.e. it satisfies $x \cdot (y \cdot x) = y$ for all $x$ and $y$ in $V$. All semisymmetric quasigroups satisfy the flexible law. It is well known [1, Remark 2.12] that there is a one-to-one correspondence between MTSs and idempotent semisymmetric quasigroups or *Mendelsohn quasigroups*, as they are also known. For DTSs and HTSs, however, the induced operation may or may not yield a quasigroup. If for example $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$, then $u \cdot x = y = v \cdot x$, but $u \neq v$. Nevertheless there do exist DTSs and HTSs that yield quasigroups. If a DTS or an

HTS induces a quasigroup, then it is said to be *Latin*, to signify that the operation table forms a Latin square. The induced binary operation is then called a *DTS-quasigroup* or an *HTS-quasigroup*, respectively.

Latin directed triple systems (LDTS) were introduced in (1), where it was shown that an LDTS($n$) exists if and only if $n \equiv 0$ or 1 (mod 3) and $n \neq 4$, 6 or 10. The algebraic and geometric aspects of LDTSs were studied in (2). Together these two papers also give enumeration results for all orders less than or equal to 13.

Latin hybrid triple systems (LHTS) were introduced in (3), where it was shown that an LHTS($n$) exists if and only if $n \equiv 0$ or 1 (mod 3) and $n \neq 6$. If in addition $n \geq 9$, then there exists a proper LHTS($n$). An LHTS is said to be *proper* if the induced quasigroup is neither a Mendelsohn quasigroup nor a DTS-quasigroup. Similarly, a DTS is said to be *proper* if the induced quasigroup is not a Steiner quasigroup.

## 2. Basic properties

The correspondence between HTSs or DTSs and the induced binary operations is not one-to-one, since if the system contains a pair of triples with the same point set, say $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$, then replacing these with the pair of triples $\langle y, z, x\rangle$ and $\langle x, z, y\rangle$ gives a system which yields the same DTS-quasigroup as the first and yet the two LDTS($n$)s may be non-isomorphic, see Example 2.4 in (1). Call a triple occurring in an HTS *bidirectional* if there exists another triple in the system with the same point set, otherwise call it *unidirectional*. The point set of a bidirectional triple is called a *Steiner triple*. When presenting examples we replace any pair of bidirectional triples, say $(x, y, z)$ and $(z, y, x)$, with their underlying Steiner triple $\{x, y, z\}$. The block set of an HTS then consists of three types of triples: Steiner triples, unidirectional cyclic triples and unidirectional transitive triples. This view of HTSs results in a one-to-one correspondence between HTSs and the induced binary operations and it allows for a more precise study of these systems, namely when dealing with isomorphisms and automorphisms of HTSs.

A DTS is proper if and only if it contains at least one unidirectional triple. An HTS is proper if and only if it contains at least one unidirectional cyclic triple and at least one unidirectional transitive triple.

The following theorem proven in (3) gives a combinatorial characterisation of LDTSs and LHTSs.

**Theorem 1.** *Let $(Q, \cdot)$ be induced by an HTS (or a DTS) $(V, \mathcal{B})$. Then $Q$ is a quasigroup if and only if for each $\langle x, y, z\rangle \in \mathcal{B}$ there exist elements $x'$, $y'$, $z' \in V$ such that $\langle z', y, x\rangle$, $\langle z, y', x\rangle$ and $\langle z, y, x'\rangle$ belong to $\mathcal{B}$ as well.*

Figure 2 shows the triples discussed in the previous theorem. Note that in an LHTS, if $(x, y)$ is the initial edge of a transitive triple, then $(y, x)$ is a terminal edge in a transitive triple and vice versa. Thus initial edges match with terminal edges and similarly long edges match with long edges.

**Example 1.** Let $V = \{0, \ldots, 6\}$ and $\mathcal{B} = \big\{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \langle 1, 5, 3\rangle, \langle 3, 5, 2\rangle, \langle 2, 5, 4\rangle, \langle 4, 5, 1\rangle, \langle 1, 6, 4\rangle, \langle 4, 6, 2\rangle, \langle 2, 6, 3\rangle, \langle 3, 6, 1\rangle\big\}$. Then $(V, \mathcal{B})$ is a proper LDTS(7).

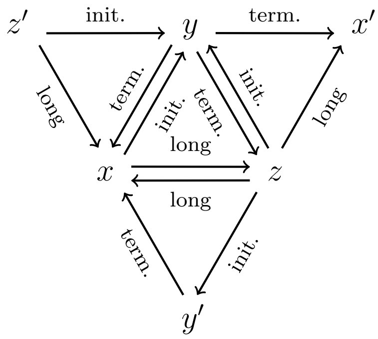


Figure 2: An illustration of the triples in Theorem 1 and their matching edges.

Let $\cdot$ be determined by an HTS $(V, \mathcal{B})$. Denote by $\mathcal{F}$ the set of all $\{x, y, z\}$ such that $\{x, y, z\}$ is the point set of a unidirectional triple of $\mathcal{B}$. Consider now $\mathcal{F}$ as a set of triangular faces. Each edge $\{u, v\}$ is incident to two faces, hence the faces can be sewn together along common edges to form a generalised pseudosurface. By splitting pinch points we obtain a generalised surface, which can be partitioned into connected components. Call such a surface component *uniform* if all its triples are either cyclic, or transitive. From Theorem 1 we see that all components are uniform if $\cdot$ yields a quasigroup. Figure 3 shows the components of the surface formed by the proper LDTS(7) from Example 1.

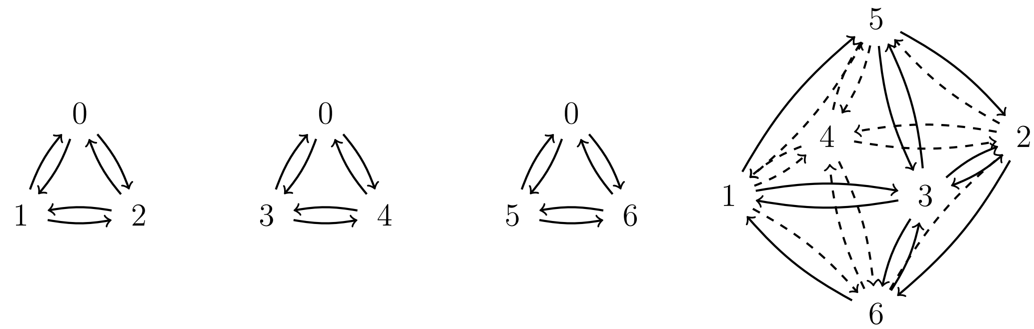


Figure 3: The components of the surface formed by the proper LDTS(7) from Example 1.

Given an LHTS $(V, \mathcal{B})$, every transitive triple $B = \langle x, y, z\rangle$ can be replaced by a cyclic triple $\overline{B} = (x, y, z)$. This yields an MTS since $\langle z', y, x\rangle$ is turned into $(z', y, x)$, $\langle z, y', x\rangle$ into $(z, y', x)$ and $\langle z, y, x'\rangle$ into $(z, y, x')$. We shall call this the *underlying* MTS of $\mathcal{B}$.

Notice that an LHTS yields the same surface as its underlying MTS. Clearly, the surface obtained from an MTS is orientable, hence any LHTS yields an orientable surface as well. This is generally not the case for HTSs.

# 3. Pure, flexible and anti-flexible Latin directed triple systems

An HTS is said to be *pure* if it does not contain a bidirectional triple. The quasigroups obtained from pure LHTSs are anti-commutative, i.e. they satisfy

$xy = yx \Rightarrow x = y$ for all $x$ and $y$. The absence of bidirectional triples implies that there is a one-to-one correspondence between pure LHTSs and anti-commutative HTS-quasigroups.

The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ but those associated with Latin directed triple systems need not. An LHTS or an LDTS is said to be *flexible* if the induced operation satisfies the flexible law. For a flexible LDTS the components of the corresponding generalized surface are all spheres. The following theorem gives a more specific description.

**Theorem 2** (5)**.** *A pure flexible LDTS(n) exists if and only if the complete graph $K_n$ can be decomposed into triangles and $k$-gonal bipyramid graphs $O_k$, $k \geq 3$, i.e. graphs of $k+2$ vertices with a cycle of length $k$, and two further points joined to every point of the cycle.*

Using a variety of recursive constructions and constructions based on group divisible designs of block size 3 we obtain the existence spectra for various types of LDTSs.

**Theorem 3** (1)**.** *The existence spectrum of non-flexible LDTS(n) is $n \equiv 0$ or $1 \pmod 3$, $n \neq 3, 4, 6, 7, 10$.*

**Theorem 4** (5)**.** *The existence spectrum of flexible LDTS(n)s is $n \equiv 0$ or $1 \pmod 3$, $n \neq 4, 6, 10, 12$.*

**Theorem 5** (6)**.** *The existence spectrum of pure non-flexible LDTS(n)s is $n \equiv 0$ or $1 \pmod 3$, $n \geq 13$.*

**Theorem 6** (6)**.** *A pure flexible LDTS(n) exists for all $n \equiv 0$ or $1 \pmod 3$, $n \geq 16$ and $n \neq 18$, possibly except $n = 24, 30, 42, 78, 114$ and $150$.*

In any LDTS, $(V, \mathcal{B})$, the following condition holds for all $x, y \in V$

$$\langle x, y, x \cdot y \rangle \in \mathcal{B} \quad \Rightarrow \quad y \cdot (x \cdot y) = (y \cdot x) \cdot y.$$

Thus in every LDTS there exist pairs of distinct points $(x, y) \in V \times V$ for which the flexible identity $(y \cdot x) \cdot y = y \cdot (x \cdot y)$ holds. In (7) we study the LDTSs whose binary operation satisfies the reverse of the condition above, i.e. for all $x, y \in V$, $x \neq y$,

$$y \cdot (x \cdot y) = (y \cdot x) \cdot y \quad \Rightarrow \quad \langle x, y, x \cdot y \rangle \in \mathcal{B}.$$

An LDTS satisfying this condition is called *antiflexible*. In other words an antiflexible DTS-quasigroup is one where the flexible identity holds for the least possible number of ordered pairs $(x, y) \in V \times V$. Thus, in a sense, antiflexible LDTSs are the LDTSs which are as distant from STSs as possible. Indeed, every antiflexible LDTS is pure. The antiflexible property can also be characterised geometrically in terms of the degrees of the vertices of the corresponding generalised surface, see Theorem 2.1 in (7).

At first glance, antiflexible LDTSs may appear to be a rather artificial construct. However, there exists a surprisingly simple cyclic construction which naturally produces antiflexible LDTSs. Let $k \geq 2$ and $n = 6k + 1$. A cyclic antiflexible LDTS($n$), $(V, \mathcal{B})$, can be obtained as follows. Set $V = \mathbb{Z}_n$ and define the set of starter triples

$$\mathcal{S} = \left\{ \langle r, k+2r, 0\rangle,\ \langle 0, 3k-r+1, r\rangle : r = 1, 2, \ldots, k \right\}.$$

If $k \equiv 1 \pmod 3$, then replace the starter triples

$$\langle \tfrac{2k+1}{3},\ k + 2\tfrac{2k+1}{3},\ 0\rangle, \quad \langle 0,\ 3k - \tfrac{2k+1}{3} + 1,\ \tfrac{2k+1}{3}\rangle \quad \text{and} \quad \langle k,\ 3k,\ 0\rangle$$

in $\mathcal{S}$ with the starter triples

$$\langle 4k+1,\ 0,\ \tfrac{1}{3}(5k+1)\rangle, \quad \langle \tfrac{1}{3}(5k+1),\ 0,\ \tfrac{1}{3}(2k+1)\rangle \quad \text{and} \quad \langle \tfrac{1}{3}(2k+1),\ 0,\ 3k+1\rangle.$$

The set of blocks $\mathcal{B}$ is then obtained by developing the set of starter blocks $\mathcal{S}$ under the action of the mapping $\alpha : i \mapsto i+1$. This construction yields cyclic antiflexible LDTSs of all admissible orders, thus we have:

**Theorem 7** (7). *A cyclic antiflexible LDTS(n) exists if and only if $n \equiv 1 \pmod 6$ and $n \geq 13$.*

Using recursive constructions and constructions based on group divisible designs with blocks of sizes 3 and 4 we obtain the complete existence spectrum of antiflexible LDTSs:

**Theorem 8** (7). *An antiflexible LDTS(n) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \geq 13$.*

# 4. Enumeration of Latin directed triple systems

To enumerate DTS-quasigroups we use the model builder Mace4 [23] which is part of the package Prover9, an automated theorem prover for first-order and equational logic. The program is provided with an algebraic description of a DTS-quasigroup given in the theorem below. Isomorphic quasigroups can then be removed using the GAP [9] package LOOPS [25].

**Theorem 9** (2). *Let $Q$ be an idempotent quasigroup. Then $Q$ is a DTS-quasigroup if and only if all $x, y \in Q$ satisfy*

*(i)* $x \cdot xy = y = yx \cdot x$ *or* $xy \cdot x = y = x \cdot yx$, *and*

*(ii)* $xy \cdot x = y$ *implies* $xy \cdot y = x$.

The enumeration of DTS-quasigroups of order up to 12 can be achieved in a matter of minutes with Mace4. However when we reach order 13, the combinatorial explosion takes over and, as reported in (2), the enumeration problem has to be split into more manageable tasks.

When dealing with HTS-quasigroups, the program is provided with the following algebraic description.

**Theorem 10** (3). *Let $Q$ be an idempotent binary operation. Then $Q$ is an HTS-quasigroup if and only if all $x, y \in Q$ satisfy*

$$y = x \cdot (x \cdot y) = (y \cdot x) \cdot x \quad \textit{or} \quad y = (x \cdot y) \cdot x = x \cdot (y \cdot x).$$

For HTS-quasigroups the combinatorial explosion takes over at order 12. Table 1 shows an overview of the number of non-isomorphic HTS quasigroups induced by different types of triple systems. The results for MTSs come from [20]. The results for LDTSs and LHTSs were obtained in (2) and (3). These two papers also give details of their automorphism groups and genera of their surface components including examples of the most interesting systems.

| | Order | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Triple system | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| STS | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 |
| Proper MTS | 0 | 1 | 0 | 3 | 19 | 241 | 9 801 188 | 13 710 290 114 |
| Proper LDTS | 0 | 0 | 0 | 1 | 3 | 0 | 2 | 1 206 967 |
| flexible | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 922 |
| pure | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 444 |
| Proper LHTS | 0 | 0 | 0 | 0 | 7 | 14 | ? | ? |
| flexible | 0 | 0 | 0 | 0 | 3 | 4 | ? | ? |
| pure | 0 | 0 | 0 | 0 | 3 | 4 | ? | ? |

Table 1: Number of non-isomorphic HTS quasigroups induced by different types of triple systems.

# 5. Cyclic and rotational systems

An HTS($n$) is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length $n$ and it is said to be *rotational* if it admits an automorphism consisting of a cycle of length $n-1$ and one fixed point. Table 2 summarises known results and new results obtained in (4) about the existence spectra of various cyclic and rotational triple systems. The main goal of (4) is to prove the existence spectra of pure and proper, cyclic and rotational, LDTSs and LHTSs. In [10] Gardner et al. went so far as to prove the existence spectrum of cyclic and rotational HTS($n$)s containing exactly $c$ cyclic triples for all admissible values of $c$. In (4) we state the admissible values of $c$ for LHTS($n$)s, but we do not prove existence for each of these values. Existence is proven only for the minimum non-zero values of $c$.

**Proposition 11** (4). *Let c be the number of cyclic triples in a cyclic LHTS(n). Then*

*(i)* if $n \equiv 0 \pmod 3$, then $c \equiv \frac{2}{3}n \pmod{2n}$;

*(ii)* if $n \equiv 1 \pmod 3$, then $c \equiv 0 \pmod{2n}$.

**Proposition 12** (4). *Let c be the number of cyclic triples in a rotational LHTS(n). Then*

*(i)* if $n \equiv 1 \pmod 6$, then $c \equiv \frac{1}{3}(n-1) \pmod{2(n-1)}$ and $c \neq \frac{1}{3}(n-1)$;

*(ii)* if $n \equiv 3 \pmod 6$, then $c \equiv 0 \pmod{n-1}$;

*(iii)* if $n \equiv 4 \pmod 6$, then $c \equiv \frac{4}{3}(n-1) \pmod{2(n-1)}$.

| Triple system | Conditions | Ref. |
|---|---|---|
| Cyclic STS | $n \equiv 1$ or $3 \pmod 6$ and $n \neq 9$ | [26] |
| Cyclic MTS | $n \equiv 1$ or $3 \pmod 6$ and $n \neq 9$ | [5] |
| Pure cyclic MTS | $n \equiv 1 \pmod 6$ | (4) |
| Cyclic DTS | $n \equiv 1, 4$ or $7 \pmod{12}$ | [4] |
| Proper cyclic LDTS | $n \equiv 1$ or $3 \pmod 6$ and $n \geq 13$ | (4) |
| Pure cyclic LDTS | $n \equiv 1 \pmod 6$ and $n \geq 13$ | (4) |
| Cyclic HTS | $n \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$ and $n \neq 9$ | [24] |
| Proper cyclic LHTS | $n \equiv 1$ or $3 \pmod 6$ and $n \geq 19$ | (4) |
| Proper pure cyclic LHTS | $n \equiv 1 \pmod 6$ and $n \geq 19$ | (4) |
| Rotational STS | $n \equiv 3$ or $9 \pmod{24}$ | [27] |
| Rotational MTS | $n \equiv 1, 3$ or $4 \pmod 6$ and $n \neq 10$ | [2] |
| Pure rotational MTS | $n \equiv 1 \pmod 3$ and $n \neq 10$ | (4) |
| Rotational DTS | $n \equiv 0 \pmod 3$ | [3] |
| Proper rotational LDTS | $n \equiv 3 \pmod 6$ and $n \geq 15$ | (4) |
| Pure rotational LDTS | does not exist | (4) |
| Rotational HTS | $n \equiv 0$ or $1 \pmod 3$ | [24] |
| Proper rotational LHTS | $n \equiv 1, 3$ or $4 \pmod 6$, $n \geq 16$ and $n \neq 19$ | (4) |
| Proper pure rotational LHTS | $n \equiv 1 \pmod 3$, $n \geq 16$ and $n \neq 19$ | (4) |

Table 2: The necessary and sufficient conditions for the existence of various cyclic and rotational triple systems.

# 6. The centre of a Steiner loop and the maxi-Pasch problem

The *left nucleus* $N_\lambda$, *middle nucleus* $N_\mu$ and *right nucleus* $N_\rho$ of a loop $L$ are defined as

$$\begin{aligned} N_\lambda(L) &= \{\, x \in L : x(yz) = (xy)z \text{ for all } y, z \in L \,\}, \\ N_\mu(L) &= \{\, y \in L : x(yz) = (xy)z \text{ for all } x, z \in L \,\}, \\ N_\rho(L) &= \{\, z \in L : x(yz) = (xy)z \text{ for all } x, y \in L \,\}. \end{aligned}$$

The *nucleus* $N(L) = N_\lambda(L) \cap N_\mu(L) \cap N_\rho(L)$ of $L$ is a subgroup of $L$. The *centre* of a loop $L$ is defined as

$$Z(L) = N(L) \cap \{\, x \in L : xy = yx \text{ for all } y \in L \,\}.$$

In (8) we derive the necessary and sufficient conditions for the existence of a Steiner loop of order $n$ with centre of order $m$.

**Theorem 13** (8)**.** *Let $n$ be a positive integer and let $k$ be the largest integer such that $2^k$ divides $n$. A nontrivial Steiner loop of order $n$ with centre of order $m$ exists if and only if $n \equiv 2$ or $4 \pmod 6$, and*

1. *$n \neq 2^k$ and $m = 2^i$, where $i \in \{0, 1, \ldots, k-1\}$, or*

2. *$n = 2^k$, $(n, m) \neq (8, 1)$ and $m = 2^i$, where $i \in \{0, 1, \ldots, k-3\} \cup \{k\}$.*

In a Steiner triple system, a collection of four triples on six points is called a *Pasch configuration* or *quadrilateral*. It is easily seen that this structure necessarily has the form $\{a, b, c\}$, $\{a, d, e\}$, $\{b, e, f\}$, $\{c, d, f\}$. Denote the number of Pasch configurations in an STS($v$), $S$, by $P(S)$. Define

$$P(v) = \max\{\, P(S) : S \text{ is an STS}(v)\,\}.$$

An STS($v$), $S$, is said to be *maxi-Pasch* if $P(S) = P(v)$.

An elementary counting argument yields $P(v) \leq v(v-1)(v-3)/24$. In [28] Stinson and Wei show that $P(v) = v(v-1)(v-3)/24$ if and only if $v = 2^n - 1$ for some $n$. The only known values of $P(v)$ when $v \neq 2^n - 1$ are $P(9) = 0$, $P(13) = 13$ and $P(19) = 84$. The Steiner loops of all known maxi-Pasch STSs have centre of maximum possible order. The results in (8) also show that for some values of $v$ the maximum known lower bound is attained by a Steiner triple system whose corresponding Steiner loop has centre of maximum possible order.

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# LATIN DIRECTED TRIPLE SYSTEMS

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Abstract. It is well known that given a Steiner triple system then a quasigroup can be formed by defining an operation $\cdot$ by the identities $x \cdot x = x$ and $x \cdot y = z$ where $z$ is the third point in the block containing the pair $\{x, y\}$. The same is true for a Mendelsohn triple system where the pair $(x, y)$ is considered to be ordered. But it is not true in general for directed triple systems. However directed triple systems which form quasigroups under this operation do exist. We call these Latin directed triple systems and in this paper begin the study of their existence and properties.

## 1. Introduction

The equivalence between Steiner triple systems, on the one hand, and Steiner quasigroups and Steiner loops, on the other hand, is well know in both the combinatorial and the algebraic communities, see for example [6, page 24] and [16, page 124]. Recall the definitions. A *Steiner triple system* of order $n$, STS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points, also called blocks, taken from $V$ such that every pair of distinct points from $V$ appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or 3 (mod 6) [11]. A *Steiner quasigroup* or *squag* is a pair $(Q, \cdot)$ where $Q$ is a set and $\cdot$ is an operation on $Q$ satisfying the identities

$$x \cdot x = x, \;\; y \cdot (x \cdot y) = x, \;\; x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS($n$), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if $Q$ is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups [16, Theorem V.1.11]. A Steiner quasigroup is also known as an *idempotent totally symmetric quasigroup* [1, Remark 2.12]. A *Steiner loop* or *sloop* is a pair $(L, \cdot)$ where $L$ is a set containing an identity element, say $e$, and $\cdot$ is an operation on $L$ satisfying the identities

$$e \cdot x = x, \;\; x \cdot x = e, \;\; y \cdot (x \cdot y) = x, \;\; x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS($n$), then a Steiner loop $(L, \cdot)$ is obtained by letting $L = V \cup \{e\}$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. Again the process is reversible.

Less well known is the following correspondence. A *Mendelsohn triple system* of order $n$, MTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclically ordered triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple.

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Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [14]. Quasigroups and loops can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for Steiner triple systems. Note that the law $y \cdot (x \cdot y) = x$ is usually called semi-symmetric. So the quasigroups are known as *idempotent semisymmetric quasigroups* [1, Remark 2.12]. However the algebraic structures might also appropriately be called *Mendelsohn quasigroups* and *Mendelsohn loops*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems, Mendelsohn quasigroups and Mendelsohn loops.

A *directed triple system* of order $n$, DTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of transitively ordered triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$ [10]. Given a DTS($n$), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$ where $z$ is the third element in the transitive triple containing the ordered pair $(x, y)$. However the structure obtained need not necessarily be a quasigroup. If $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$ then $u \cdot x = v \cdot x = y$. But as we will see, some DTS($n$)s do yield quasigroups. Such a DTS($n$) will be called a *Latin directed triple system*, and denoted by LDTS($n$), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*. In an analogous way to that described above for Steiner triple systems we may also construct a loop from a LDTS($n$); called a *DTS-loop*.

## 2. Properties

First we derive a necessary and sufficient condition for a directed triple system to be Latin.

**Proposition 2.1.** *Let $D = (V, \mathcal{B})$ be a DTS(n). Denote by $S_{a,b}$ the set of ordered pairs $(x, y)$ in positions $a$ and $b$ respectively of the triples of $\mathcal{B}$. Then $D$ is a LDTS(n) if and only if $S_{1,2} = S_{3,2}$, $S_{2,3} = S_{2,1}$, and $S_{1,3} = S_{3,1}$.*

*Proof.* Let $D$ be a LDTS($n$) and suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then $y \cdot z = x$. Now there exists $w$ such that precisely one of $\langle y, x, w\rangle$, $\langle y, w, x\rangle$, or $\langle w, y, x\rangle \in \mathcal{B}$. In the first two cases $y \cdot w = x$ and so $w = z$ which is impossible. Therefore $\langle w, y, x\rangle \in \mathcal{B}$ and $S_{1,2} \subset S_{3,2}$. Further $x \cdot y = z$. Similarly one of $\langle w, z, y\rangle$, $\langle z, w, y\rangle$, or $\langle z, y, w\rangle \in \mathcal{B}$. Again in the first two cases $w \cdot y = z$ and so $w = x$, which is also impossible. Therefore $\langle z, y, w\rangle \in \mathcal{B}$ and so $S_{3,2} \subset S_{1,2}$. Therefore $S_{1,2} = S_{3,2}$. It further follows that $S_{2,3} = S_{2,1}$. Finally since $S_{1,2} \cup S_{1,3} \cup S_{2,3} = S_{3,2} \cup S_{3,1} \cup S_{2,1}$ and all of the sets $S_{a,b}$ are disjoint, it follows that $S_{1,3} = S_{3,1}$.

Conversely suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then $x \cdot y = z$. For $D$ to be a LDTS($n$) we require that the equations $\alpha \cdot y = z$, $x \cdot \beta = z$, and $x \cdot y = \gamma$ have unique solutions, namely $x$, $y$, and $z$ respectively for $\alpha$, $\beta$, and $\gamma$. Clearly $z$ is the unique solution for $\gamma$ by definition. If $x \cdot \beta = z$ then precisely one of $\langle x, z, \beta\rangle$, $\langle x, \beta, z\rangle$, or $\langle z, x, \beta\rangle \in \mathcal{B}$. In the first case no such block exists, in the second case $\beta = y$, and in the third case no such block exists because $S_{1,3} = S_{3,1}$. If $\alpha \cdot y = z$ then precisely one of $\langle \alpha, y, z\rangle$, $\langle \alpha, z, y\rangle$, or $\langle z, \alpha, y\rangle \in \mathcal{B}$. In the first case $\alpha = x$ and in the other two cases no such block exists because $S_{2,3} = S_{2,1}$. Further if $\langle x, y, z\rangle \in \mathcal{B}$

then $x \cdot z = y$ and $y \cdot z = x$ and we need to show that for each equation, given any two of the parameters, the third is uniquely determined. The proof is similar to the case for the equation $x \cdot y = z$. □

The conditions for a LDTS($n$) given in the above proposition can be simplified but we have chosen to present them in this form because they are reminiscent of those ($S_{1,2} = S_{2,1}$, $S_{2,3} = S_{3,2}$, and $S_{1,3} = S_{3,1}$) for another class of directed triple systems, so called *Mendelsohn directed triple systems*, the existence of which was discussed in [9]. A more succinct necessary and sufficient condition is given in the next theorem

**Theorem 2.2.** *Let $D = (V, \mathcal{B})$ be a DTS(n). Then $D$ is a LDTS(n) if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle w, y, x\rangle \in \mathcal{B}$ for some $w \in V$.*

*Proof.* In the notation of Proposition 2.1, the condition in this theorem is $S_{1,2} \subset S_{3,2}$ which is trivially implied by the conditions in the proposition. We need to show that the reverse is also true. Since the cardinalities of the sets $S_{1,2}$ and $S_{3,2}$ are equal it follows that $S_{1,2} = S_{3,2}$ which, as observed in the proof of the proposition, implies the other two conditions. □

Before discussing existence and enumeration results for DTS-quasigroups and DTS-loops, it is important to point out two fundamental differences between these and their Steiner and Mendelsohn counterparts. The first concerns flexibility. The *flexible law* states that $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. As is easily verified, both Steiner quasigroups and loops and Mendelsohn quasigroups and loops all satisfy this law. But this is not the case for DTS-quasigroups and loops. Next we state and prove a necessary and sufficient condition for a DTS-quasigroup or loop to satisfy the flexible law.

**Theorem 2.3.** *A DTS-quasigroup or DTS-loop obtained from a LDTS(n), $D = (V, \mathcal{B})$ satisfies the flexible law if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$.*

*Proof.* Suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then there exists $\alpha, \beta, \gamma \in V$ such that $\langle z, y, \alpha\rangle$, $\langle z, \beta, x\rangle$, $\langle \gamma, y, x\rangle \in \mathcal{B}$. Here we allow any of the equalities $\alpha = x$, $\beta = y$, $\gamma = z$ to be satisfied in which case all three are. Consider the six possibilities.
(a) $x \cdot (y \cdot x) = x \cdot \gamma$; $(x \cdot y) \cdot x = z \cdot x = \beta$; hence we require $x \cdot \gamma = \beta$.
(b) $y \cdot (x \cdot y) = y \cdot z = x$; $(y \cdot x) \cdot y = \gamma \cdot y = x$.
(c) $y \cdot (z \cdot y) = y \cdot \alpha = z$; $(y \cdot z) \cdot y = x \cdot y = z$.
(d) $z \cdot (y \cdot z) = z \cdot x = \beta$; $(z \cdot y) \cdot z = \alpha \cdot z$; hence we require $\alpha \cdot z = \beta$.
(e) $z \cdot (x \cdot z) = z \cdot y = \alpha$; $(z \cdot x) \cdot z = \beta \cdot z$; hence we require $\beta \cdot z = \alpha$.
(f) $x \cdot (z \cdot x) = x \cdot \beta$; $(x \cdot z) \cdot x = y \cdot x = \gamma$; hence we require $x \cdot \beta = \gamma$.
Thus the flexible law is satisfied if and only if (i) $\langle x, \beta, \gamma\rangle = \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$ and (ii) $\langle \alpha, \beta, z\rangle = \langle z \cdot y, z \cdot x, z\rangle \in \mathcal{B}$. To complete the proof we need to show that the second condition can be derived from the first. We have that $\langle z, y, \alpha\rangle \in \mathcal{B}$ and the first condition implies that $\langle z, \alpha \cdot z, y \cdot z\rangle = \langle z, \alpha \cdot z, x\rangle \in \mathcal{B}$ so that $\alpha \cdot z = \beta$, i.e. $\langle \alpha, \beta, z\rangle = \langle z \cdot y, z \cdot x, z\rangle \in \mathcal{B}$. □

By analogy we will say that a LDTS($n$) is flexible if the DTS-quasigroup and DTS-loop obtained from it satisfies the flexible law. Later, we will also use partial LDTS($n$). We define these as partial DTS($n$) which satisfy the conditions of Proposition 2.1 (not Theorem 2.2). These are not the same for partial systems; the set of directed triples $\langle x, a, y\rangle$, $\langle y, a, z\rangle$, $\langle z, a, x\rangle$ which are a partial DTS(4) satisfy the condition of Theorem 2.2 but not the conditions of Proposition 2.1 and so are not a partial LDTS(4). If they are augmented by directed triples $\langle y, b, x\rangle$, $\langle z, b, y\rangle$, $\langle x, b, z\rangle$ then we have a partial LDTS(5). Partial LDTS($n$) will be called flexible or non-flexible depending on whether they satisfy the condition of Theorem 2.3.

The second difference between Latin directed triple systems and Steiner or Mendelsohn triple systems is that with the former there is not a one-one correspondence between the triple systems and the associated quasigroups or loops. Suppose that we are given the operation table of a DTS-quasigroup or DTS-loop. We wish to recover the LDTS($n$), $(V, \mathcal{B})$, from which it came. Choose $x, y, z$, $x \neq y \neq z \neq x$ with $x \cdot y = z$. Then $\langle x, y, z\rangle$ or $\langle x, z, y\rangle$ or $\langle z, x, y\rangle \in \mathcal{B}$. In order to identify which of these three possibilities is the correct one perform a number of tests:

- if $x \cdot z \neq y$, then $\langle z, x, y\rangle \in \mathcal{B}$.
- if $z \cdot y \neq x$, then $\langle x, y, z\rangle \in \mathcal{B}$.
- if $y \cdot z \neq x$ and $z \cdot x \neq y$, then $\langle x, z, y\rangle \in \mathcal{B}$.

Otherwise, $x \cdot z = y$, $z \cdot y = x$, and either $y \cdot z = x$ or $z \cdot x = y$. The only inference that can be made is that the set $\mathcal{B}$ contains one of the six directed triples formed by ordering the three points $x, y, z$, together with its reverse.

In a DTS($n$), $(V, \mathcal{B})$, any directed triple $\langle x, y, z\rangle \in \mathcal{B}$ for which also $\langle z, y, x\rangle \in \mathcal{B}$ will be called *bidirectional*. The set $\{x, y, z\}$ will be called a *Steiner triple*. Other directed triples will be called *unidirectional*. From the above discussion, if a LDTS($n$) contains a pair of bidirectional directed triples, then these can be replaced by a different pair of bidirectional triples to form a potentially non-isomorphic LDTS($n$) yet both will generate the same quasigroup and loop. This is illustrated in the following example. Here and in other places throughout the rest of this paper, where there is no danger of confusion, for simplicity we omit set brackets and commas from directed triples.

**Example 2.4.** Let $V = \{0, 1, 2, 3, 4, 5, 6\}$.
Define $\mathcal{B} = \{102, 201, 304, 403, 506, 605, 315, 416, 514, 613, 326, 425, 523, 624\}$,
and $\mathcal{B}' = \{012, 210, 034, 430, 056, 650, 315, 416, 514, 613, 326, 425, 523, 624\}$.
Both $(V, \mathcal{B})$ and $(V, \mathcal{B}')$ are LDTS(7)s but are clearly non-isomorphic as consideration of the distribution of points in the middle position of the directed triples shows. However both give the same DTS-quasigroup.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 4 | 3 | 6 | 5 |
| 1 | 2 | 1 | 0 | 6 | 5 | 3 | 4 |
| 2 | 1 | 0 | 2 | 5 | 6 | 4 | 3 |
| 3 | 4 | 5 | 6 | 3 | 0 | 1 | 2 |
| 4 | 3 | 6 | 5 | 0 | 4 | 2 | 1 |
| 5 | 6 | 4 | 3 | 2 | 1 | 5 | 0 |
| 6 | 5 | 3 | 4 | 1 | 2 | 0 | 6 |

The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_4$ of order 8 generated by the permutations (3 5 4 6) and (1 2)(5 6). Note however that this is not necessarily the automorphism group of the LDTS(7)s. The same group is the automorphism group of $(V, \mathcal{B})$ but not of $(V, \mathcal{B}')$ which has only the identity automorphism.

In view of the above, for purposes of enumeration it makes more sense to count DTS-quasigroups (or DTS-loops; these are in one-one correspondence) rather than the Latin directed triple systems from which they come. Where there are bidirectional triples, the block set $\mathcal{B}$ of a LDTS($n$) will be expressed as the union of a set of Steiner triples, $\mathcal{T}$, and a set of unidirectional directed triples, $\mathcal{D}$. Denote the cardinality of $\mathcal{T}$ by $t$, (so that the number of bidirectional triples is $2t$), and the cardinality of $\mathcal{D}$ by $d$.

A directed triple system, $(V, \mathcal{B})$, is said to be *pure* if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle z, y, x\rangle \notin \mathcal{B}$. Pure LDTS($n$) give anti-commutative DTS-quasigroups and, because there are no Steiner triples, there does exist a one-one correspondence between these. At the other extreme, commutative DTS-quasigroups correspond to the situation where every directed triple is bidirectional, i.e. where the LDTS($n$) consists of the blocks of a Steiner triple system, each in some order, together with their reverse. In short, commutative DTS-quasigroups and Steiner quasigroups are the same.

In the next section we present some enumeration results for DTS-quasigroups of small order. Then in the rest of the paper we discuss existence results. A necessary condition for the existence of a LDTS($n$) is $n \equiv 0,\ 1 \pmod 3$ and the number of directed triples is $n(n-1)/3$. For $n \equiv 1,\ 3 \pmod 6$, there exist Steiner quasigroups of these orders and, except for $n = 3$ or $9$, by choosing a Steiner triple system containing a Pasch configuration $\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}$ and replacing these Steiner triples by directed triples $\langle a, b, c\rangle, \langle a, y, z\rangle, \langle x, b, z\rangle, \langle x, y, c\rangle, \langle z, y, x\rangle, \langle c, b, x\rangle, \langle c, y, a\rangle, \langle z, b, a\rangle$ a DTS-quasigroup which is non-commutative is obtained. Replacing a Pasch configuration by the above set of directed triples is an important technique which will be used extensively in the next two sections. Note that the set of directed triples is a partial LDTS(6) and is flexible. We will denote it by $\mathcal{P}$.

But replacing a single Pasch configuration means that most of the triples will still be bidirectional. It would be of more interest to construct pure LDTS($n$) or at least ones with relatively few bidirectional triples. In Section 4 we construct flexible LDTS($n$) for $n \equiv 1,\ 3 \pmod 6$ in which the number of unidirectional triples is asymptotic to $n^2/3$. Then in Section 5 we turn our attention to non-flexible systems and determine the complete spectrum for the existence of such LDTS($n$). Again in the systems that we construct the number of unidirectional triples is asymptotic to $n^2/3$. We leave existence results for flexible LDTS($n$) of even order and pure LDTS($n$) to a future paper.

## 3. Enumeration

We present the enumeration results for DTS-quasigroups of small order in the following theorem.

**Theorem 3.1.** *The numbers of non-isomorphic DTS-quasigroups of order* $n = 3, 4, 6, 7, 9, 10, 12$ *are* $1, 0, 0, 2, 4, 0, 2$ *respectively.*

We consider each order in turn.
$n = 3$. Trivially the only DTS-quasigroup of order 3 is the Steiner quasigroup of this order.

$n = 4$. Let $V = \{0, 1, 2, 3\}$. Without loss of generality there exists a directed triple $\langle 0, 1, 2\rangle$. Therefore there also exists a directed triple $\langle 2, 1, 0\rangle$ or directed triples $\langle 2, 1, \cdot\rangle$, $\langle 2, \cdot, 0\rangle$, $\langle \cdot, 1, 0\rangle$ where the dots, both here and in other places later, represent yet to be assigned points. Neither of these two possibilities can be completed to form a LDTS(4).

$n = 6$. Let $V = \{0, 1, 2, 3, 4, 5\}$. There will be 10 directed triples in any LDTS(6). So without loss of generality there are directed triples $\langle 0, 1, 2\rangle$, $\langle 0, 3, 4\rangle$, $\langle \cdot, 0, 5\rangle$. But now the unassigned first element in the last block must also be 5.

$n = 7$. Let $V = \{0, 1, 2, 3, 4, 5, 6\}$. Given any directed triple system DTS($n$), if the ordering of the points in the blocks is suppressed then a twofold triple system TTS($n$) is obtained. There exist 4 non-isomorphic TTS(7)s which are listed in [6, page 61]. It is a straightforward exercise to take each of these in turn and try to construct LDTS(7)s by ordering the blocks. Perhaps it is appropriate to note here that there are 2368 non-isomorphic DTS(7)s, [7], but the extra constraint on Latin directed triple systems makes the exercise considerably easier. However the enumeration can be shortened as follows. In a LDTS($n$), $(V, \mathcal{B})$, for $x \in V$, denote by $f(x), m(x), l(x)$, the number of occurrences of the point $x$ in the first, middle, and last positions respectively in unidirectional triples of $\mathcal{B}$. Obviously $f(x) = l(x)$ for all $x$. Also $\Sigma_{x\in V} f(x) = \Sigma_{x\in V} m(x) = n(n-1)/3 - 2t$, where $t$ is the number of Steiner triples.

Now consider the 4 non-isomorphic TTS(7)s from [6] in turn. It will be convenient to do so in reverse order. System #4 has $t = 0$. So for each point $x$, $(f(x), m(x)) = (3, 0), (2, 2), (1, 4)$ or $(0, 6)$. But neither $m(x) = 2$ nor $f(x) = 1$ as this would imply that the directed triples come from Steiner triples. So $f(x) = 3$ or 0. But the number of unidirectional triples, 14, is not divisible by 3 and so there is no LDTS(7) from this possibility.

System #3 has one Steiner triple $\{0, 1, 2\}$. So for the three points 0, 1, 2 we have $(f(x), m(x)) = (2, 0)$ or $(0, 4)$ and for the other four points $(f(x), m(x)) = (3, 0)$ or $(0, 6)$. There are two possibilities. The first is that 0, 1, 2 have $(f(x), m(x)) = (2, 0)$, 3, 4 have $(f(x), m(x)) = (3, 0)$, and 5, 6 have $(f(x), m(x)) = (0, 6)$. But then the ordered pairs $(5, 6)$ and $(6, 5)$ cannot occur. The second possibility is that 0, 1, 2 have $(f(x), m(x)) = (0, 4)$ and 3, 4, 5, 6 have $(f(x), m(x)) = (3, 0)$. But this cannot be completed without introducing further Steiner triples. (The problem is equivalent to decomposing the complete directed graph on 4 vertices into three directed 4-cycles which is not possible.)

System #2 has three Steiner triples $\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}$. The six points other than 0 have $(f(x), m(x)) = (2, 0)$ or $(0, 4)$. So there are four points of the first type and two points, say 1 and 2, of the latter type. Without loss of generality the unidirectional triples are $\langle 3, 1, 5\rangle$, $\langle 4, 1, 6\rangle$, $\langle 5, 1, 4\rangle$, $\langle 6, 1, 3\rangle$, $\langle 3, 2, 6\rangle$, $\langle 4, 2, 5\rangle$, $\langle 5, 2, 3\rangle$, $\langle 6, 2, 4\rangle$ and the DTS-quasigroup is the one given in the example in the previous section. It is flexible.

Finally system #1 has seven Steiner triples, i.e. it is two copies of identical STS(7)s and gives the Steiner quasigroup of order 7.

$n = 9$. It is possible, but extremely tedious and time-consuming, to enumerate DTS-quasigroups of order 9 by hand. Perhaps a better approach is to adopt the same technique as for order 7 and use a computer. There exist 36 non-isomorphic TTS(9)s, [15], [13]. These are listed in [6, page 63]. It is a straightforward procedure to take each of them in turn and attempt to order the blocks in order to construct a LDTS(9). We find that there are in fact four DTS-quasigroups of order 9, including the Steiner quasigroup of this order. Details of the other three are given below, referenced as examples.

**Example 3.2.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 8\}, \{2, 5, 8\}, \{3, 6, 8\}, \{4, 7, 8\}, \{2, 4, 6\}, \{3, 5, 7\}\}$ and
$\mathcal{D} = \{207, 706, 605, 504, 403, 302, 213, 314, 415, 516, 617, 712\}$.
Then $(V, \mathcal{B})$ is a flexible LDTS(9) with $d = 12$ and $2t = 12$.
The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_6$ of order 12 generated by the permutations (2 3 4 5 6 7) and (0 1)(2 3)(4 7)(5 6).

**Example 3.3.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 8\}, \{2, 3, 4\}, \{2, 7, 8\}, \{3, 6, 8\}, \{4, 5, 8\}, \{5, 6, 7\}\}$ and
$\mathcal{D} = \{026, 035, 047, 125, 137, 146, 520, 531, 621, 640, 730, 741\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(9) with $d = 12$ and $2t = 12$.
For example $(0 \cdot 2) \cdot 0 = 6 \cdot 0 = 4$, whilst $0 \cdot (2 \cdot 0) = 0 \cdot 5 = 3$.
The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_3$ of order 6 generated by the permutations (2 3 4)(5 7 6) and (0 1)(3 4)(5 6).

**Example 3.4.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 2\}, \{3, 5, 7\}, \{4, 6, 8\}\}$ and
$\mathcal{D} = \{308, 316, 324, 403, 415, 427, 504, 518, 526, 605, 617, 623, 706, 714, 728, 807, 813, 825\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(9) with $d = 18$ and $2t = 6$.
For example $(3 \cdot 4) \cdot 3 = 2 \cdot 3 = 6$, whilst $3 \cdot (4 \cdot 3) = 3 \cdot 0 = 8$.
The automorphism group of the DTS-quasigroup is the group $\mathcal{D}_3 \times \mathcal{C}_3$ of order 18 generated by the permutations (1 2)(3 4 5 6 7 8) and (0 1 2)(3 5 7).

$n = 10$. Since $n$ is even, $m(x)$ is odd and at least 3. The number of directed triples is 30 and so it follows that for each point $x$, $(f(x), m(x)) = (3, 3)$ and there are no Steiner triples. The directed triples containing each point $x$ have the format $\langle a, x, b\rangle$, $\langle b, x, c\rangle$, $\langle c, x, a\rangle$. From these form oriented triangles $(a, b, c)$. Collectively, these triangles have the property that they contain a directed edge $(\alpha, \beta)$ iff they also contain the directed edge $(\beta, \alpha)$. Hence they can be sewn together along common edges to form an orientable surface. It will be a surface rather than a pseudosurface because $f(x) = l(x) = 3$, i.e. each vertex has valency 3. Now the Euler characteristic, #vertices + #faces − #edges = 10 + 10 − 15 which is odd; a contradiciton. Hence there is no LDTS(10).

$n = 12$. We first present a construction of LDTS(12)s based on a tetrahedron. Let the vertex set be $\{0, 1, 2, 3\}$ and choose a consistent orientation of the faces, say (0 1 2), (0 3 1), (0 2 3), (1 3 2). Each of the four 3-cycles will be regarded as a permutation $\phi_i \in \mathcal{S}_4$, with $\phi_i(i) = i$.
For every $x \in \{0, 1, 2\}$ define sets of directed triples:

$$D_x^+ = \{\langle (x, j), (x + 1, j'), (x, \phi_{j'}(j))\rangle : j, j' \in \{0, 1, 2, 3\}, j \neq j'\}$$

$$D_x^- = \{\langle (x, j), (x + 1, j'), (x, \phi_{j'}^{-1}(j))\rangle : j, j' \in \{0, 1, 2, 3\}, j \neq j'\}$$

For every $x \in \{0, 1, 2\}$ choose $D_x \in \{D_x^+, D_x^-\}$ and regard $\mathcal{D} = D_0 \cup D_1 \cup D_2$ as a set of unidirectional triples. These triples cover every pair $((x, j), (x', j'))$ from the set $\{0, 1, 2\} \times \{0, 1, 2, 3\}$ for which $j \neq j'$. By adjoining Steiner triples $\{(0, j), (1, j), (2, j)\}$ we obtain a LDTS(12).

For each $x \in \{0,1,2\}$ there are two choices for $D_x$ corresponding to the chosen orientation. However for isomorphism what is important is whether, for given $x$ and $x'$, these are the same or opposite. There must always be two that are the same so without loss of generality let $D_0 = D_0^+$ and $D_1 = D_1^+$. There are thus two isomorphism types depending on the choice of $D_2$. In the example below we explicitly list the triples of these two systems, constructed as described, where the ordered pair $(x, j)$ is represented as the integer $4x + j$ with 10 written as $T$ and 11 as $E$.

**Example 3.5.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E\}$.
Define $\mathcal{T} = \{\{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, T\}, \{3, 7, E\}\}$,
$D_0^+ = \{052, 063, 071, 160, 172, 143, 270, 241, 253, 342, 350, 361\}$,
$D_1^+ = \{496, 4T7, 4E5, 5T4, 5E6, 587, 6E4, 685, 697, 786, 794, 7T5\}$,
$D_2^+ = \{81T, 82E, 839, 928, 93T, 90E, T38, T09, T1E, E0T, E18, E29\}$, and
$D_2^- = \{81E, 829, 83T, 92E, 938, 90T, T39, T0E, T18, E09, E1T, E28\}$.
Let $\mathcal{D}^+ = D_0^+ \cup D_1^+ \cup D_2^+$ and $\mathcal{D}^- = D_0^+ \cup D_1^+ \cup D_2^-$.
Then $(V, \mathcal{T} \cup \mathcal{D}^+)$ and $(V, \mathcal{T} \cup \mathcal{D}^-)$ are both non-flexible LDTS(12)s with $d = 36$ and $2t = 8$.
For example in both systems $(0 \cdot 1) \cdot 0 = 7 \cdot 0 = 2$, whilst $0 \cdot (1 \cdot 0) = 0 \cdot 6 = 3$.
The permutations $(1\ 2\ 3)(5\ 6\ 7)(9\ T\ E)$ and $(0\ 1)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(T\ E)$, which together generate the alternating group $\mathcal{A}_4$ of order 12, stabilize each of the sets $\mathcal{T}, D_0^+, D_1^+, D_2^+$ and $D_2^-$ and give the full automorphism group of the DTS-quasigroup of the LDTS(12), $(V, \mathcal{T} \cup \mathcal{D}^-)$. The other DTS-quasigroup has an additional permutation automorphism $(0\ 4\ 8)(1\ 6\ E)(2\ 7\ 9)(3\ 5\ T)$ to give the full automorphism group of order 36.

In fact the two systems are the only two DTS-quasigroups of this order. We state this formally as a proposition.

**Proposition 3.6.** *Every DTS-quasigroup of order 12 is isomorphic to one of the two quasigroups given in Example 3.5.*

*Proof.* The proof was obtained by computer with the help of the model builder **Mace4**, which is part of the package **Prover9** [12]. The procedure can easily be repeated by giving an algebraic description of DTS-quasigroups, generating all models of order 12, and using the isomorphism filter. □

$n \geq 13$. At $n = 13$, the combinatorial explosion takes over. The smallest anti-commutative DTS-quasigroups are of this order. There are 8444 non-isomorphic such systems and an example is given below. However none of them are flexible.

**Example 3.7.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
Define $\mathcal{B} = \mathcal{D} = \{103, 142, 201, 247, 2E3, 2W5, 302, 341, 3E6, 3W7, 406, 4T5, 4E9, 4W8, 504, 518, 539, 5T7, 5E2, 5W6, 605, 619, 628, 6T4, 6E7, 6W3, 709, 715, 743, 7T6, 7E8, 7W2, 807, 816, 82T, 835, 8E4, 8W9, 908, 917, 926, 93T, 9E5, 9W4, T0W, T1E, T29, T38, E0T, E1W, W0E, W1T\}$.
Then $(V, \mathcal{B})$ is a pure non-flexible LDTS(13).
For example $(2 \cdot 3) \cdot 2 = E \cdot 2 = 5$, whilst $2 \cdot (3 \cdot 2) = 2 \cdot 0 = 1$.

In addition there are 1,197,601 non-flexible and 924 flexible (including the 2 Steiner quasigroups) DTS-quasigroups which are not anti-commutative.

It remains to identify the smallest anti-commutative, flexible DTS-quasigroups. The next order to consider is $n = 15$ but first we develop some structural theory of anti-commutative, flexible DTS-quasigroups. Let $D = (V, \mathcal{B})$ be a pure flexible LDTS($n$). Suppose that $\langle x, u, y\rangle \in \mathcal{B}$. Then there exists $z$, $v$ such that $\langle y, u, z\rangle$, $\langle y, v, x\rangle \in \mathcal{B}$ where $z \neq x$, $v \neq u$. So $(y \cdot x) \cdot y = v \cdot y$ and $y \cdot (x \cdot y) = y \cdot u = z$. Therefore $v \cdot y = z$, i.e. $\langle z, v, y\rangle \in \mathcal{B}$. It follows that $\mathcal{B}$ partitions into subsets $\{\langle x_1, u, x_2\rangle, \langle x_2, u, x_3\rangle, \ldots, \langle x_{n-1}, u, x_n\rangle, \langle x_n, u, x_1\rangle, \langle x_2, v, x_1\rangle, \langle x_3, v, x_2\rangle, \ldots,$ $\langle x_n, v, x_{n-1}\rangle, \langle x_1, v, x_n\rangle\}, n \geq 3$, which we will call *components*, with each point $u, v, x_1, x_2, \ldots, x_n$ distinct. These components can be thought of as spheres with $u$ and $v$ at the poles, both joined to $x_1, x_2, \ldots, x_n$ around the equator. In the notation used above for the case $n = 7$, for each point $x$ of a LDTS($n$), $m(x) \neq 1$, and further, if it is pure $m(x) \neq 2$. Also $n - 1 - m(x)$ is divisible by 2 and the above argument shows that if it is also pure and flexible $n - 1 - m(x)$ is divisible by 4. We now have the following result.

**Proposition 3.8.** *There is no anti-commutative, flexible DTS-quasigroup of order 15.*

*Proof.* The constraints that $14 - m(x)$ is divisible by 4 and $m(x) \neq 2$ implies that $m(x) = 14, 10$ or 6. Suppose that there are $\lambda, \mu$ and $\nu$ points with each of these three counts, respectively. Then

$$14\lambda + 10\mu + 6\nu = 70 \text{ and } \lambda + \mu + \nu = 15.$$

Hence $8\lambda + 4\mu = -20$ which is a contradiction because the coefficients cannot be negative. □

However for $n = 16$, there does exist an anti-commutative flexible DTS-quasigroup. It was found by computer using the package **Paradox** [4].

**Example 3.9.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$.
Define $\mathcal{B} = \mathcal{D} = \{801, 107, 70E, E05, 50F, F0B, B08, 198, 791, E97, 59E, F95, B9F, 89B, B36, 638, 83D, D37, 73F, F3E, E3B, 64B, 846, D48, 74D, F47, E4F, B4E, 1E6, 6EA, AE8, 8E2, 2ED, DEC, CE1, 6F1, AF6, 8FA, 2F8, DF2, CFD, 1FC, 03C, C39, 93A, A30, C40, 94C, A49, 04A, 312, 214, 413, 253, 452, 354, 026, 629, 920, 6D0, 9D6, 0D9, 56C, C67, 765, C85, 78C, 587, 27A, A7B, B72, AC2, BCA, 2CB, 1AF, FAD, DA1, FB1, DBF, 1BD\}$.
Then $(V, \mathcal{B})$ is a pure, flexible LDTS(16). It has only the identity automorphism.

The next order to consider is $n = 18$ and again we can use the theory developed above to prove that there is no pure, flexible LDTS($n$) of this order.

**Proposition 3.10.** *There is no anti-commutative, flexible DTS-quasigroup of order 18.*

*Proof.* Since 4 divides $17 - m(x)$ and $m(x) \neq 1$ then $m(x) = 17, 13, 9$ or 5. Suppose that there are $\lambda, \mu$, $\nu$ and $\rho$ points with each of these four counts, respectively. Then

$$17\lambda + 13\mu + 9\nu + 5\rho = 102 \text{ and } \lambda + \mu + \nu + \rho = 18.$$

Further $\lambda = 0$ or 1.
If $\lambda = 1$ then

$$13\mu + 9\nu + 5\rho = 85 \text{ and } \mu + \nu + \rho = 17.$$

Hence $8\mu + 4\nu = 0$ and the only solution is $(\lambda, \mu, \nu, \rho) = (1, 0, 0, 17)$. With this distribution, it is not possible to construct a pure, flexible LDTS(18) composed of components as required.
If $\lambda = 0$ then

$$13\mu + 9\nu + 5\rho = 102 \text{ and } \mu + \nu + \rho = 18.$$

Hence $8\mu + 4\nu = 12$ so $(\lambda, \mu, \nu, \rho) = (0, 0, 3, 15)$ or $(0, 1, 1, 16)$. Again it is not possible to construct a pure, flexible LDTS(18) composed of components. □

For $n = 19$, the equations lead to a unique distribution. We have that 4 divides $18 - m(x)$ and since $m(x) \neq 2$ it follows that $m(x) = 18, 14, 10$ or 6. Proceeding as before let there be $\lambda$, $\mu$, $\nu$ and $\rho$ points with each of these four counts, respectively. Then

$$18\lambda + 14\mu + 10\nu + 6\rho = 114 \text{ and } \lambda + \mu + \nu + \rho = 19$$

with again $\lambda = 0$ or 1.
If $\lambda = 1$ then

$$14\mu + 10\nu + 6\rho = 96 \text{ and } \mu + \nu + \rho = 18.$$

Hence $8\mu + 4\nu = -12$ and there is no solution.
If $\lambda = 0$ then

$$14\mu + 10\nu + 6\rho = 114 \text{ and } \mu + \nu + \rho = 19.$$

Hence $8\mu + 4\nu = 0$ and the only solution is $(\lambda, \mu, \nu, \rho) = (0, 0, 0, 19)$. This leaves open the possibility of an anti-commutative, flexible DTS-quasigroup with a cyclic automorphism and indeed such a system does exist.

**Example 3.11.** Let $V = \mathcal{Z}_{19}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
The starter blocks for $\mathcal{B} = \mathcal{D}$ are $\langle 0, 1, 6\rangle$, $\langle 6, 1, 9\rangle$, $\langle 9, 1, 0\rangle$, $\langle 6, 2, 0\rangle$, $\langle 0, 2, 9\rangle$, $\langle 9, 2, 6\rangle$. Then $(V, \mathcal{B})$ is a pure, flexible LDTS(19).

## 4. Flexible LDTS

Our constructions of flexible LDTS($n$) are of two types. The first of these uses the well-known so-called "doubling" construction for Steiner triple systems and is particularly simple. It deals with the residue classes 3, 7 (mod 12). The details are given in the proof of the following proposition.

**Proposition 4.1.** *There exists a flexible LDTS(n) for all $n \equiv 3, 7 \pmod{12}$.*

*Proof.* Put $m = (n - 1)/2$ and choose an STS($m$), $(V, \mathcal{B})$. Let $V' = \{x' : x \in V\}$ and $W = V \cup V' \cup \{\infty\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\{x, y, z\} \in \mathcal{B}$, assign $\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in \mathcal{B}'$. Further let $\{x, x', \infty\} \in \mathcal{B}'$ for all $x \in V$. Then $(W, \mathcal{B}')$ is an STS($n$). In order to obtain a LDTS($n$) replace each Pasch configuration as above by the set $\mathcal{P}$ of directed triples and retain the sets containing the point $\infty$ as Steiner triples. Because the LDTS($n$) is constructed of flexible components, i.e. just the flexible partial LDTS(6), $\mathcal{P}$, and the trivial squag on 3 points, it is also flexible. The number of unidirectional triples, $d = (n - 1)(n - 3)/3$ and the number of bidirectional triples, $2t = n - 1$. □

The second construction of LDTS($n$) uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall that a 3-GDD of type $g^u$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where $V$ is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of $V$ into $u$ subsets of cardinality $g$ called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. We will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set $V$ being of cardinality $v = gu + m$ and the partition $G$ being into $u$ subsets of cardinality $g$ and one set of cardinality $m$. Necessary and sufficient conditions for 3-GDDs of type $g^u$ were determined in [3] and for 3-GDDs of type $g^u m^1$ in [5]; a convenient reference is [8] where the existence of all the GDDs that are used can be verified.

We will also need the following system.

**Example 4.2.** Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
Define $\mathcal{T}$=$\{\{0, 4, 5\}, \{1, 7, 9\}, \{1, T, W\}, \{3, 5, 8\}, \{3, 7, W\}, \{5, 9, T\}\}$ and
$\mathcal{D} = \{103, 142, 156, 18E, 201, 243, 257, 28W, 302, 341, 60E, 629, 63T, 647, 65W, 681, 706, 74T, 75E, 782, 80T, 849, 908, 92E, 936, 94W, T07, T26, T3E, T48, E0W, E2T, E39, E46, E51, E87, W09, W4E, W52, W86\}$.
Then $(V, \mathcal{B})$ is a flexible LDTS(13) with $d = 40$ and $2t = 12$.

We can now prove the following proposition.

**Proposition 4.3.** *There exists a flexible LDTS(n) for all* $n \equiv$ *1, 9 (mod 12).*

*Proof.* The proof is divided into different residue classes.
(a) $n \equiv 1 \pmod{12}$. Take a 3-GDD of type $6^s, s \geq 3$. Inflate each point by a factor 2 and adjoin an extra point $\infty$. On each inflated group, together with the point $\infty$, place a flexible LDTS(13) given in Example 4.2. On each inflated block place the set $\mathcal{P}$ of directed triples $\langle a, b, c\rangle, \langle a, y, z\rangle, \langle x, b, z\rangle, \langle x, y, c\rangle, \langle z, y, x\rangle, \langle c, b, x\rangle, \langle c, y, a\rangle, \langle z, b, a\rangle$, with the three sets of points $\{a, x\}, \{b, y\}, \{c, z\}$ as the inflated points in the three groups. We will use $\mathcal{P}$ in this manner throughout. This simple construction gives a flexible LDTS($12s + 1$), $s \geq 3$. A count shows that $d = (n - 1)(n - 3)/3$ and $2t = n - 1$.

(b) $n \equiv 9 \pmod{24}$. Take a 3-GDD of type $4^{3s+1}, s \geq 1$. Inflate each point by a factor 2 and adjoin an extra point $\infty$. On each inflated group, together with the point $\infty$, place a flexible LDTS(9) given in Example 3.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS($24s + 9$), $s \geq 1$ with $d = (n - 1)(2n - 9)/6$ and $2t = 3(n - 1)/2$.

(c) $n \equiv 21 \pmod{24}$. Take a 3-GDD of type $4^{3s+1}6^1, s \geq 1$. Inflate each point by a factor 2 and adjoin an extra point $\infty$. On each inflated group of cardinality 8, together with the point $\infty$, place a flexible LDTS(9) given in Example 3.2 and on the inflated group of cardinality 12, together with the point $\infty$, place a flexible LDTS(13) given in Example 4.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS($24s+21$), $s \geq 1$ with $d = (2n^2 - 11n + 45)/6$ and $2t = 3(n - 5)/2$.

(d) The above constructions complete the proof of the proposition except for the two values $n = 21$ in (c) and $n = 25$ in (a). These too can be constructed by GDD techniques. For $n = 21$ take a 3-GDD of type $3^3$. Inflate each point by a factor 2 and adjoin three extra points $\infty_1, \infty_2, \infty_3$. On each inflated group, together with the three extra points, place a flexible LDTS(9) given in Example 3.2 in such a way that the triple $\{\infty_1, \infty_2, \infty_3\}$ is identified with the same Steiner triple in each LDTS(9). On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS(21) with $d = 108$ and $2t = 32$. For $n = 25$ take a 3-GDD of type $4^3$. Inflate each point by a factor 2 and adjoin an extra point $\infty$. On each inflated group, together with the point $\infty$, place a flexible LDTS(9) given in Example 3.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS(25) with $d = 164$ and $2t = 36$. □

Combining the results of the above two propositions we have proved the following result.

**Theorem 4.4.** *There exists a flexible LDTS(n) for all $n \equiv 1, 3 \pmod 6$.*

## 5. Non-flexible LDTS

Our constructions of non-flexible LDTS($n$) use a variety of techniques and divide into different residue classes. The first proposition deals with the case where $n$ is divisible by 3 and is a modification of the well-known Bose construction. First we recall some basic definitions.

Two Latin squares $L$ and $M$ are said to be mutually orthogonal if $L(x, y) = L(x', y')$ and $M(x, y) = M(x', y')$ implies that $x = x'$ and $y = y'$. A Latin square $L$ is said to be self-orthogonal if it is mutually orthogonal to its transpose $L'$. The diagonal of a self-orthogonal Latin square is a *transversal*, i.e. it contains every element precisely once; thus by relabelling the elements, a self-orthogonal Latin square can be made *idempotent*, i.e. $L(i, i) = i$.

**Proposition 5.1.** *There exists a non-flexible LDTS(n) for all $n \equiv 0 \pmod 3$, except $n = 3, 6$.*

*Proof.* Let $m = n/3$ and $L$ be a self-orthogonal Latin square of side $m$, with the rows, columns, and entries in $\mathbb{Z}_m$ and labelled in such a way as to be idempotent. Such a square exists for all $m \neq 2, 3, 6$, [2]. Denote the entry in row $x$, column $y$ by $x \star y$.
Let $V = \mathbb{Z}_m \times \mathbb{Z}_3$. Let $\mathcal{D}$, the set of unidirectional triples, be

$$\langle (x, i), (x \star y, i + 1), (y, i) \rangle, \; x, y \in \mathcal{Z}_m, \; x \neq y, \; i \in \mathcal{Z}_3$$

and $\mathcal{T}$, the set of Steiner triples, be

$$\{(x, 0), (x, 1), (x, 2)\}, \; x \in \mathcal{Z}_m.$$

Then $(V, \mathcal{B}) = (V, \mathcal{D} \cup \mathcal{T})$ is a LDTS($n$). For $m = 1$ it produces the squag of order 3. We show that for $m \neq 1$ it is not flexible. Choose any $x, y \in \mathcal{Z}_m$, $x \neq y$.
Now $[(x, i) \cdot (y, i)] \cdot (x, i) = (x \star y, i + 1) \cdot (x, i) = (z, i)$ where $z \star x = x \star y$.
Also $(x, i) \cdot [(y, i) \cdot (x, i)] = (x, i) \cdot (y \star x, i + 1) = (w, i)$ where $x \star w = y \star x$.
If $w = z$ then $(x \star y, y \star x) = (z \star x, x \star z)$ which violates $L$ being self-orthogonal. Hence $w \neq z$ and the LDTS($n$) is non-flexible. The number of unidirectional triples, $d = 3m(m - 1) = n(n - 3)/3$ and the number of bidirectional triples, $2t = 2m = 2n/3$.

It remains to consider the three values of $m$ for which there does not exist a self-orthogonal Latin square. By Theorem 3.1, for $m = 2$, there is no LDTS(6). For $m = 3$, non-flexible LDTS(9)s are given in Examples 3.3 and 3.4. For $m = 6$, we remark that the full force of self-orthogonality is not required in the above construction. Using the idempotent anti-symmetric Latin square below will produce a LDTS(18) which is non-flexible.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 5 | 4 | 1 | 3 | 2 |
| 1 | 4 | 1 | 5 | 0 | 2 | 3 |
| 2 | 3 | 0 | 2 | 5 | 1 | 4 |
| 3 | 2 | 4 | 1 | 3 | 5 | 0 |
| 4 | 5 | 3 | 0 | 2 | 4 | 1 |
| 5 | 1 | 2 | 3 | 4 | 0 | 5 |

$[(0,0)\cdot(1,0)]\cdot(0,0) = (5,1)\cdot(0,0) = (4,0)$, whilst $(0,0)\cdot[(1,0)\cdot(0,0)] = (0,0)\cdot(4,1) = (2,0)$. □

Next we deal with the case where $n \equiv 1 \pmod 6$. The following example is a non-flexible LDTS(13).

**Example 5.2.** Let $V = \{0,1,2,3,4,5,6,7,8,9,T,E,W\}$.
Define $\mathcal{T}$=$\{\{0,7,8\}$, $\{1,8,T\}$, $\{3,8,9\}$, $\{6,8,W\}$, $\{4,7,W\}$, $\{4,9,T\}\}$ and
$\mathcal{D} = \{012$, 046, 053, $0E9$, $0WT$, 145, $1WE$, 213, 240, 256, $2T7$, $2E8$, $2W9$, 310, $34E$, 357, $3T6$, $3W2$, 548, $5ET$, $5W1$, 619, 643, 650, $6T2$, $6E7$, 716, 759, $7T3$, $7E2$, 842, $8E5$, 917, 952, $9E6$, $9W0$, $TE0$, $TW5$, $E41$, $EW3\}$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(13) with $d = 40$ and $2t = 12$.
For example $(0\cdot 1)\cdot 0 = 2\cdot 0 = 4$, whilst $0\cdot(1\cdot 0) = 0\cdot 3 = 5$.

**Proposition 5.3.** *There exists a non-flexible LDTS(n) for all $n \equiv 1 \pmod 6$, except $n = 7$.*

*Proof.* We have already noted that there is no non-flexible LDTS(7) and a non-flexible LDTS(13) is given in the above example. Let $m \geq 3$ and put $n = 6m+1$. Let $(V,\mathcal{B}) = (V,\mathcal{D}\cup\mathcal{T})$ be a non-flexible LDTS($3m$), constructed as in the proof of the previous proposition. We form a LDTS($6m+1$) as follows. Let $V' = \{x' : x \in V\}$ and $W = V\cup V'\cup\{\infty\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\langle x,y,z\rangle \in \mathcal{D}$, assign $\langle x,y,z\rangle, \langle x,y',z'\rangle, \langle x',y,z'\rangle, \langle x',y',z\rangle \in \mathcal{D}'$. In addition for all $\{x,y,z\} \in \mathcal{T}$ assign $\langle x,y,z\rangle, \langle x,y',z'\rangle, \langle x',y,z'\rangle, \langle x',y',z\rangle, \langle z',y',x'\rangle, \langle z,y,x'\rangle, \langle z,y',x\rangle, \langle z',y,x\rangle \in \mathcal{D}'$. Further let $\{x,x',\infty\} \in \mathcal{T}'$, the set of Steiner triples in the LDTS($6m+1$), for all $x \in V$. Let $\mathcal{B}' = \mathcal{D}'\cup\mathcal{T}'$. Then $(W,\mathcal{B}')$ is a non-flexible LDTS($n$) with $d = (n-1)(n-3)/3$ and $2t = n-1$. □

Next we deal with the case where $n \equiv 4 \pmod{12}$. First we give three examples for the cases $n = 16, 28, 40$. The first of these is used in the proposition below, the proof of which again uses GDD techniques. The other two examples give the values which the method misses.

**Example 5.4.** Let $V = \{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$.
Define $\mathcal{T}$=$\{\{0,1,2\}$, $\{0,3,4\}\}$ and
$\mathcal{D} = \{135$, $14A$, $1C6$, 236, $25C$, 297, $2D8$, $2F4$, 468, $47B$, $4D2$, $4FC$, 506, 537, 541, $5AF$, $5B9$, $5ED$, $60F$, 631, 692, $6A7$, $6CD$, $6E5$, 705, $71F$, 732, $78E$, 796, $7AD$, $80C$, 819, $83A$, 852, $86B$, $8D4$, $90D$, $91E$, 938, 945, $9BF$, $9CA$, $A08$, $A2B$, $A39$, $A4E$, $AC1$, $B0A$, $B18$, $B2E$, $B3D$, $B64$, $B7C$, $C0E$, $C3B$, $C58$, $C74$, $CF2$, $D07$, $D1B$, $D3F$, $DA5$, $DC9$, $DE6$, $E0B$, $E17$, $E2A$, $E3C$, $E49$, $E8F$, $F09$, $F1D$, $F3E$, $F87$, $FA6$, $FB5\}$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(16) with $d = 76$ and $2t = 4$.
For example $(1\cdot 3)\cdot 1 = 5\cdot 1 = 4$, whilst $1\cdot(3\cdot 1) = 1\cdot 6 = C$.

**Example 5.5.** Let $V = \mathcal{Z}_{14} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j) \mapsto (i+1,j)$.
The starter blocks for $\mathcal{T}$ are $\{(0,0),(1,0),(3,0)\}$ and $\{(0,0),(4,0),(0,1)\}$ and for $\mathcal{D}$ are $\langle(0,0),(9,0),(12,1)\rangle$, $\langle(0,0),(1,1),(7,0)\rangle$, $\langle(0,0),(6,1),(11,1)\rangle$, $\langle(0,0),(7,1),(5,1)\rangle$, $\langle(0,0),(8,1),(4,1)\rangle$, $\langle(0,0),(9,1),(8,0)\rangle$, $\langle(0,0),(13,1),(6,0)\rangle$, $\langle(0,1),(11,0),(13,1)\rangle$, $\langle(0,1),(12,0),(3,0)\rangle$, $\langle(0,1),(2,1),(10,0)\rangle$, $\langle(0,1),(4,1),(7,1)\rangle$, $\langle(0,1),(8,1),(9,0)\rangle$, $\langle(0,1),(9,1),(1,1)\rangle$, $\langle(0,1),(11,1),(2,0)\rangle$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(28) with $d = 196$ and $2t = 56$.
For example $[(0,0)\cdot(1,1)]\cdot(0,0) = (7,0)\cdot(0,0) = (8,1)$, whilst $(0,0)\cdot[(1,1)\cdot(0,0)] = (0,0)\cdot(6,0) = (13,1)$.

**Example 5.6.** Let $V = \mathcal{Z}_{20} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j) \mapsto (i+1,j)$.
The starter blocks for $\mathcal{T}$ are $\{(0,0),(1,0),(3,0)\}$, $\{(0,0),(4,0),(9,0)\}$, $\{(0,0),(8,0),(0,1)\}$ and for $\mathcal{D}$ are $\langle(0,0),(1,1),(10,0)\rangle$, $\langle(0,0),(2,1),(7,0)\rangle$, $\langle(0,0),(3,1),(6,0)\rangle$, $\langle(0,0),(4,1),(19,1)\rangle$, $\langle(0,0),(9,1),(10,1)\rangle$, $\langle(0,0),(11,1),(7,1)\rangle$, $\langle(0,0),(14,1),(13,0)\rangle$, $\langle(0,0),(15,1),(13,1)\rangle$, $\langle(0,0),(16,1),(14,0)\rangle$, $\langle(0,0),(17,1),(6,1)\rangle$, $\langle(0,1),(2,0),(7,1)\rangle$, $\langle(0,1),(12,0),(10,1)\rangle$, $\langle(0,1),(15,0),(3,1)\rangle$, $\langle(0,1),(2,1),(13,0)\rangle$, $\langle(0,1),(4,1),(1,0)\rangle$, $\langle(0,1),(5,1),(17,1)\rangle$, $\langle(0,1),(6,1),(10,0)\rangle$, $\langle(0,1),(8,1),(14,0)\rangle$, $\langle(0,1),(11,1),(7,0)\rangle$, $\langle(0,1),(19,1),(13,1)\rangle$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(40) with $d = 400$ and $2t = 120$.
For example $[(0,0)\cdot(1,1)]\cdot(0,0) = (10,0)\cdot(0,0) = (11,1)$, whilst $(0,0)\cdot[(1,1)\cdot(0,0)] = (0,0)\cdot(7,0) = (2,1)$.

**Proposition 5.7.** *There exists a non-flexible LDTS(n) for all $n \equiv 4 \pmod{12}$, except $n = 4$.*

*Proof.* We have already noted that there is no LDTS(4) and non-flexible LDTS($n$) for $n = 16, 28, 40$ are given above. Take a 3-GDD of type $6^s 8^1$, $s \geq 3$. Inflate each point by a factor 2. On each inflated group of cardinality 12 place a non-flexible LDTS(12) constructed as in the proof of Proposition 5.1 and on the inflated group of cardinality 16 place a non-flexible LDTS(16) given in Example 5.4. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a non-flexible LDTS($12s + 16$), $s \geq 3$ with $d = (n^2 - 3n + 20)/3$ and $2t = 2(n - 10)/3$. □

Now we come to the final case where $n \equiv 10 \pmod{12}$. This in turn divides into three different residue classes, for one of which we will need the following example of a non-flexible LDTS(22).

**Example 5.8.** Let $V = \mathcal{Z}_{11} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j) \mapsto (i+1,j)$.
The starter blocks for $\mathcal{T}$ are $\{(0,0),(1,0),(3,0)\}$ and $\{(0,0),(4,0),(0,1)\}$ and for $\mathcal{D}$ are $\langle(0,0),(5,1),(8,1)\rangle$, $\langle(5,0),(0,0),(3,1)\rangle$, $\langle(1,1),(0,0),(10,1)\rangle$, $\langle(2,1),(0,0),(5,0)\rangle$, $\langle(3,1),(0,0),(2,1)\rangle$, $\langle(3,1),(0,1),(4,1)\rangle$, $\langle(4,1),(0,0),(6,1)\rangle$, $\langle(6,1),(0,0),(1,1)\rangle$, $\langle(9,1),(5,1),(0,0)\rangle$, $\langle(10,1),(0,0),(4,1)\rangle$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(22) with $d = 110$ and $2t = 44$.
For example $[(1,1)\cdot(0,0)]\cdot(1,1) = (10,1)\cdot(1,1) = (6,0)$, whilst $(1,1)\cdot[(0,0)\cdot(1,1)] = (1,1)\cdot(6,1) = (2,0)$.

**Proposition 5.9.** *There exists a non-flexible LDTS(n) for all $n \equiv 10 \pmod{12}$ except $n = 10$ and possibly except $n = 58$.*

*Proof.* We deal with the different residue classes in turn.
(a) $n \equiv 34 \pmod{36}$. Take three copies of a non-flexible LDTS($12s + 12$), $s \geq 0$, constructed as in the proof of Proposition 5.1 on point sets $\{\infty, (i, 0) : 0 \leq i \leq 12s + 10\}$, $\{\infty, (i, 1) : 0 \leq i \leq 12s + 10\}$, $\{\infty, (i, 2) : 0 \leq i \leq 12s + 10\}$ respectively. Now take an idempotent, antisymmetric Latin square of side $12s+11$, for example a self-orthogonal Latin square. Adjoin the Steiner triples $\{(x, 0), (x, 1), (x, 2)\}, x \in \mathbb{Z}_{12s+11}$ and unidirectional triples $\langle(x, 0), (y, 1), (x \star y, 2)\rangle$ and $\langle(y \star x, 2), (y, 1), (x, 0)\rangle$, $x, y \in \mathbb{Z}_{12s+11}$, $x \neq y$. This gives a non-flexible LDTS($36s + 34$), $s \geq 0$, with $d = (n^2 - 5n - 2)/3$ and $2t = 2(2n + 1)/3$.

(b) $n \equiv 10 \pmod{36}$. This case is similar to the previous one but starting with three copies of a non-flexible LDTS($12s + 4$), $s \geq 1$, constructed as in the proof of Proposition 5.7. This gives a non-flexible LDTS($36s + 10$), $s \geq 1$, with $d = (n^2 - 5n + 58)/3$ and $2t = 2(2n - 29)/3$, $n \geq 154$.

(c) $n \equiv 22 \pmod{36}$. The method used in the previous two cases is inapplicable here because of the non-existence of a LDTS($12s+8$). We revert to a GDD technique. Take a 3-GDD of type $9^{2s}11^1$, $s \geq 2$. Inflate each point by a factor 2. On each inflated group of cardinality 18 place a non-flexible LDTS(18) constructed as in the proof of Proposition 5.1 and on the inflated group of cardinality 22 place a non-flexible LDTS(22) given in Example 5.8. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a non-flexible LDTS($36s + 22$), $s \geq 2$, with $d = (n + 8)(n - 11)/3$ and $2t = 2(n + 44)/3$ and just leaves the value $n = 58$ undecided. □

It remains only to consider $n = 58$. We first need the following example which is of a non-flexible LDTS(24) which contains a LDTS(7) as a subsystem. In fact the LDTS(24) contains three disjoint LDTS(7)s but we will not need this additional property.

**Example 5.10.** Let $V = \{\mathcal{Z}_7 \times \mathcal{Z}_3\} \cup \{\infty_1, \infty_2, \infty_3\}$.
The three disjoint LDTS(7)s are defined by the triples obtained from the following starter blocks under the action of the mapping $(i, j) \mapsto (i, j+1)$ with $\infty_1, \infty_2, \infty_3$ as fixed points.
The starter blocks for the Steiner triples $\mathcal{T}_1$ are $\{(0, 0), (4, 0), (6, 0)\}$, $\{(1, 0), (5, 0), (6, 0)\}$, $\{(2, 0), (3, 0), (6, 0)\}$ and for the unidirectional triples $\mathcal{D}_1$ are $\langle(1, 0), (0, 0), (3, 0)\rangle$, $\langle(1, 0), (4, 0), (2, 0)\rangle$, $\langle(2, 0), (0, 0), (1, 0)\rangle$, $\langle(2, 0), (4, 0), (5, 0)\rangle$, $\langle(3, 0), (0, 0), (5, 0)\rangle$, $\langle(3, 0), (4, 0), (1, 0)\rangle$, $\langle(5, 0), (0, 0), (2, 0)\rangle$, $\langle(5, 0), (4, 0), (3, 0)\rangle$.
The starter blocks for the remaining Steiner triples $\mathcal{T}_2$ are $\{(0, 0), (3, 1), (3, 2)\}$, $\{(3, 0), (4, 1), (6, 2)\}$, $\{(2, 0), (6, 1), (4, 2)\}$, $\{(0, 0), (0, 1), (0, 2)\}$, $\{\infty_1, \infty_2, \infty_3\}$ and for the unidirectional triples $\mathcal{D}_2$ are $\langle(1, 0), (0, 1), (1, 2)\rangle$, $\langle(1, 0), (2, 1), (5, 2)\rangle$, $\langle(1, 0), (0, 2), (6, 1)\rangle$, $\langle(1, 0), (2, 2), (1, 1)\rangle$, $\langle(1, 0), (4, 2), (5, 1)\rangle$, $\langle(2, 0), (0, 1), (2, 2)\rangle$, $\langle(3, 0), (1, 1), (4, 2)\rangle$, $\langle(3, 0), (2, 2), (6, 1)\rangle$, $\langle(4, 0), (0, 1), (4, 2)\rangle$, $\langle(4, 0), (0, 2), (5, 1)\rangle$, $\langle(4, 0), (2, 2), (3, 1)\rangle$, $\langle(5, 0), (0, 1), (5, 2)\rangle$, $\langle(5, 0), (2, 1), (4, 2)\rangle$, $\langle(5, 0), (6, 1), (1, 2)\rangle$, $\langle(5, 0), (2, 2), (5, 1)\rangle$, $\langle(5, 0), (3, 2), (1, 1)\rangle$, $\langle(6, 0), (0, 1), (6, 2)\rangle$, $\langle(6, 0), (2, 1), (1, 2)\rangle$, $\langle(6, 0), (5, 1), (3, 2)\rangle$, $\langle(1, 0), (6, 2), \infty_1\rangle$, $\langle(3, 0), (2, 1), \infty_1\rangle$, $\langle(5, 0), (0, 2), \infty_1\rangle$, $\langle(1, 0), (3, 1), \infty_2\rangle$, $\langle(2, 0), (0, 2), \infty_2\rangle$, $\langle(5, 0), (4, 1), \infty_2\rangle$, $\langle(3, 0), (5, 2), \infty_3\rangle$, $\langle(4, 0), (1, 2), \infty_3\rangle$, $\langle(6, 0), (0, 2), \infty_3\rangle$, $\langle\infty_1, (0, 0), (1, 2)\rangle$, $\langle\infty_1, (2, 0), (3, 1)\rangle$,

$\langle\infty_1,(6,0),(5,2)\rangle$, $\langle\infty_2,(0,0),(2,2)\rangle$, $\langle\infty_2,(3,0),(5,1)\rangle$, $\langle\infty_2,(4,0),(1,1)\rangle$, $\langle\infty_3,(0,0),(4,2)\rangle$, $\langle\infty_3,(1,1),(3,2)\rangle$, $\langle\infty_3,(5,0),(6,2)\rangle$, $\langle(4,0),\infty_1,(4,1)\rangle$, $\langle(6,0),\infty_2,(6,1)\rangle$, $\langle(2,0),\infty_3,(2,1)\rangle$.
Putting $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2$ and $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2$, then $(V,\mathcal{B})=(V,\mathcal{D}\cup\mathcal{T})$ is a non-flexible LDTS(24) containg three disjoint LDTS(7)s subsystems and with $d=144$ and $2t=40$. For example $[(1,0)\cdot(0,1)]\cdot(1,0)=(1,2)\cdot(1,0)=(2,1)$, whilst $(1,0)\cdot[(0,1)\cdot(1,0)]=(1,0)\cdot\infty_1=(6,2)$.

**Proposition 5.11.** *There exists a non-flexible LDTS(58).*

*Proof.* Define sets $\mathcal{N}=\{(\infty,j):0\le j\le 6\}$, $\mathcal{M}_k=\{(i,k):0\le i\le 16\}$, $k=0,1,2$. Take three copies of a non-flexible LDTS(24) containing a LDTS(7) as a subsystem, constructed as in Example 5.10 on point sets $\mathcal{N}\cup\mathcal{M}_0$, $\mathcal{N}\cup\mathcal{M}_1$, $\mathcal{N}\cup\mathcal{M}_2$ respectively with in each case the LDTS(7) on the set $\mathcal{N}$. Now take an idempotent, antisymmetric Latin square of side 17, for example a self-orthogonal Latin square. Adjoin the Steiner triples $\{(x,0),(x,1),(x,2)\},x\in\mathbb{Z}_{17}$ and unidirectional triples $\langle(x,0),(y,1),(x\star y,2)\rangle$ and $\langle(y\star x,2),(y,1),(x,0)\rangle$, $x,y\in\mathbb{Z}_{17}$, $x\neq y$. This gives a non-flexible LDTS(58), with $d=960$ and $2t=142$. □

Collecting together all the results in this section gives the following theorem.

**Theorem 5.12.** *The existence spectrum of non-flexible LDTS(n) is $n\equiv$ 0, 1 (mod 3), $n\neq 3,4,6,7,10$.*

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# BASICS OF DTS QUASIGROUPS: ALGEBRA, GEOMETRY AND ENUMERATION

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Abstract. A directed triple system can be defined as a decomposition of a complete digraph to directed triples $\langle x, y, z\rangle$. By setting $xy = z$, $yz = x$, $xz = y$ and $uu = u$ we get a binary operation that can be a quasigroup. We give an algebraic description of such quasigroups, explain how they can be associated with triangulated pseudosurfaces and report enumeration results.

The notion of a DTS quasigroup is defined in Section 1. In Theorem 1.6 we give an algebraic characterization that is an important tool in classification and enumeration of DTS quasigroups. In this respect our main result is the classification of DTS quasigroups of order 13, where we found 1 206 969 isomorphism types. Some examples which may be of particular interest are given in the Appendix. The method of enumeration is reported in Section 4.

DTS loops are those loops that can be obtained from the (idempotent) DTS quasigroups by prolongation. Section 2 explains why there is little hope that any proper DTS loop will turn out to be of an algebraic significance. In Section 3 we show that DTS quasigroups possess a rich geometrical structure. This structure offers various invariants, some of which are exploited in the classification result.

The first paper in which DTS quasigroups were defined is [3]. The connection to this paper is explained below in Section 1.

## 1. Directed triple systems and binary operations

Consider a complete directed graph on a set $X$. If $X$ is finite of size $n$, then it contains $n(n-1)$ directed edges (arrows). The set of edges can be decomposed into $n(n-1)/3$ triples if and only if $n \neq 2 \bmod 3$. In such a decomposition each triple has a vertex set of 3, 4, 5 or 6 elements. We shall be considering only the first alternative. There are four possibilities:

(1) $\{(x, y),(y, z),(z, x)\}$ that will be recorded as $(x, y, z)$ and called a *cyclic* triple,
(2) $\{(x, y),(y, z),(x, z)\}$ that will be recorded as $\langle x, y, z\rangle$ and called a *directed* triple,
(3) $\{(x, y),(y, x),(x, z)\}$ that will be denoted by $\mathbf{i}(x, y, z)$, and
(4) $\{(x, y),(y, x),(z, x)\}$ that will be denoted by $\mathbf{o}(x, y, z)$.

Triples of types (3) and (4) will be used only in these introductory passages. Given a decomposition $\mathcal{D}$ of the complete directed graph on $X$ to triples of types (1), (2), (3) and (4) define upon $X$ an operation $\cdot$ by setting $a \cdot b = c$ whenever $\{a, b, c\}$ is the vertex set of the triple containing the directed edge $(a, b)$. To define the binary operation $\cdot$ completely put $a \cdot a = a$ for every $a \in Q$ (the operation is idempotent).

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Note that $(x, y, z) = (y, z, x) = (z, x, y)$, while $\langle x, y, z \rangle$, $\langle y, z, x \rangle$ and $\langle z, x, y \rangle$ are pairwise different. Call $(z, y, x)$ the *opposite* (opposite triple) to $(x, y, z)$, $\langle z, y, x \rangle$ the opposite to $\langle x, y, z \rangle$, $\mathbf{i}(z, y, x)$ the opposite to $\mathbf{i}(x, y, z)$ and $\mathbf{o}(z, y, x)$ the opposite to $\mathbf{o}(x, y, z)$.

Call a triple from $\mathcal{D}$ a *Steiner* triple if the opposite triple is contained in $\mathcal{D}$ as well. The Steiner triples will be denoted by $\{x, y, z\}$ as we shall pay no attention to the way how $\{x, y, z\}$ is decomposed into the two opposite triples. The reason is that we shall be interested in $X(\cdot)$ rather than $\mathcal{D}$. A Steiner triple induces upon $\{x, y, z\}$ the structure of the (only) idempotent quasigroup upon this set, and so the decomposition to the opposites bears no impact upon the definition of $\cdot$. It is easy to see that the rest of $\mathcal{D}$ (i.e. the non-Steiner triples) can be derived from the knowledge of the binary operation uniquely.

From here on assume that $\mathcal{D}$ contains only cyclic and directed triples.

**Lemma 1.1.** *Suppose that $x, y \in X$ and $x \neq y$. Then $x \cdot y = y \cdot x$ if and only if $\{x, y, x \cdot y\}$ is a Steiner triple of $\mathcal{D}$.*

*Proof.* Let $z$ be the third vertex of the triple that contains $(x, y)$. Then $(y, x)$ determines a triple with the vertex set $\{x, y, z\}$ if and only if $z = y \cdot x$. □

The binary operation will be sometimes replaced by juxtaposition, with, say, $x \cdot yz$ meaning $x \cdot (y \cdot z)$. A cyclic triple $(x, y, xy)$ fulfils both $y \cdot xy = x$ and $xy \cdot x = y$. The latter laws are called *semisymmetric*. If they hold universally, then they yield a structure of a quasigroup in which $y \backslash x = xy = y/x$ (it is well known and easy to see that each of the semisymmetric laws implies the other law).

The pair $(X, \mathcal{D})$ is called a *Mendelsohn triple system* (MTS) if all elements of $\mathcal{D}$ are cyclic. It is clear that $\mathcal{D}$ is MTS if and only if the operation $\cdot$ is semisymmetric (and idempotent, by the definition). Idempotent semisymmetric quasigroups are thus rightly known as *Mendelsohn quasigroups*. An MTS is called *pure* if it contains no Steiner triple.

Commutative semisymmetric quasigroups are called *totally symmetric* because all their parastrophes (i.e. the conjugates) coincide. Idempotent totally symmetric quasigroups are also known as *Steiner* quasigroups and they are in a 1-to-1 correspondence to *Steiner triple systems* (STS). An MTS that is not an STS is called *proper*.

In this paper we shall investigate the situation when all elements of $\mathcal{D}$ are directed triples. Then $(X, \mathcal{D})$ forms a *directed triple system* (DTS). It is called *pure* if it contains no Steiner triple. If all triples in $\mathcal{D}$ are Steiner, then we get again a Steiner quasigroup, and that happens, by Lemma 1.1, if and only if the operation $\cdot$ is commutative. Call a DTS *proper* if it is not an STS.

The purpose of this paper is to study those DTS for which $X(\cdot)$ is an (idempotent) quasigroup. Note the difference to MTS, where the semisymmetric law guarantees that we get a quasigroup structure in all cases.

From here on we shall assume that $\mathcal{D}$ is a DTS upon $X$. Our first goal will be to investigate the conditions under which $X(\cdot)$ is a quasigroup. Such systems will be called *Latin directed triple systems* (LDTS). Here and elsewhere there will be a nontrivial intersection with paper [3] where we determined the existence spectrum of (proper) LDTS. In this paper our approach is somewhat different. While [3] respects the style of exposition typical for design theory, here we concentrate on algebraic and geometrical connections that are complemented by a report on enumerations of LDTS of orders up to 13 (the enumeration strategy depends heavily upon the algebraic model).

**Lemma 1.2.** *The binary system $X(\cdot)$ is a quasigroup if and only if it is* divisible *(i.e. for all $x, y \in X$ there exist $u, v \in X$ such that $xu = y$ and $vx = y$).*

*Proof.* Assume $x \neq y$ and consider the triples that carry $(x, y)$ and $(y, x)$. Each of the two triples induces three different ordered triples $(a_1, a_2, a_3) \in X^3$ such that $a_3 = a_1 a_2$. There are thus at most six such triples for which there exist $i, j \in \{1, 2, 3\}$ with $a_i = x$ and $a_j = y$. The divisibility condition with respect to $x$ and $y$ means that such a triple exists for any choice of $i$ and $j$, $i \neq j$. However, if that is true, then the triple is determined uniquely since there are exactly six choices for $(i, j)$. The divisibility hence implies the uniqueness of divisions. □

Put $Q = X(\cdot)$ and denote by $Q^{\mathrm{op}}$ the binary system with operation $x * y = yx$. Of course, $Q^{\mathrm{op}}$ is a quasigroup if and only if $Q$ is a quasigroup, and is induced by the directed triple system $\mathcal{D}^{\mathrm{op}} = \{\langle z, y, x\rangle;\ \langle x, y, z\rangle \in \mathcal{D}\}$.

**Theorem 1.3.** *Let $\mathcal{D}$ be a directed triple system upon a set $X$. Define a binary operation $\cdot$ on $X$ in such a way that $xy = z$, $yz = x$ and $xz = y$ whenever $\langle x, y, z\rangle \in \mathcal{D}$, and that $xx = x$ for all $x \in X$. Then $X(\cdot)$ is a quasigroup if and only if for all $\langle x, y, z\rangle \in \mathcal{D}$ there exist $x', y', z' \in X$ such that*

$$\langle z', y, x\rangle, \langle z, y', x\rangle, \langle z, y, x'\rangle \in \mathcal{D}.$$

*In such a case $z' = yx$, $y' = zx$ and $x' = zy$.*

*Proof.* Consider $x, y, z \in X$ such that $\langle x, y, z\rangle \in \mathcal{D}$. Suppose first that $\langle y, z', x\rangle \in \mathcal{D}$ for some $z' \in X$. Then $z' \neq z$ since $(y, z)$ cannot be covered twice, and so $yz' = x$ implies that $X(\cdot)$ is not a quasigroup. Similarly, we cannot get a quasigroup if $\langle y, x, z'\rangle \in \mathcal{D}$ since then $xz' = y$. We have thus shown that if $X(\cdot)$ is a quasigroup, then there exists $z' \in X$ with $\langle z', y, x\rangle \in \mathcal{D}$. In such a case

$$z'y = x, \quad z'x = y \quad \text{and} \quad yx = z'.$$

By taking into account that $xy = z$, $yz = x$ and $xz = y$, we see that the divisibility condition is satisfied with respect to $x$ and $y$.

By turning to $\mathcal{D}^{\mathrm{op}}$ we get that if $X(\cdot)$ is a quasigroup, then there exists $x' \in X$ such that $\langle z, y, x'\rangle \in \mathcal{D}$. Then

$$zy = x', \quad yx' = z \quad \text{and} \quad zx' = y,$$

and $z$ and $y$ satisfy the divisibility condition.

If there exists $y' \in X$ with $\langle z, x, y'\rangle \in \mathcal{D}$, then $X(\cdot)$ is not a quasigroup by $xy' = z$. We also do not get a quasigroup if $\langle y', z, x\rangle \in \mathcal{D}$ since then $y'z = x$. Hence there exists $y' \in X$ with $\langle z, y', x\rangle \in \mathcal{D}$ if $X(\cdot)$ is a quasigroup, and then

$$zy' = x, \quad y'x = z \quad \text{and} \quad zx = y',$$

which supplies the divisibility for $x$ and $z$.

We have seen that the existence of $x', y', z' \in Q$ that satisfy the condition of the theorem is necessary if $X(\cdot)$ is a quasigroup. We have also observed that if such $x'$, $y'$ and $z'$ exist, then the operation $\cdot$ is divisible. That makes $X(\cdot)$ a quasigroup by Lemma 1.2. □

Theorem 1.3 thus yields a characterization of LDTS. Our next aim is to characterize quasigroups $X(\cdot)$ in terms of the binary operation. Such a quasigroup clearly satisfies condition (i) of Lemma 1.4. The lemma is included to show how the characterization of Theorem 1.6 was discovered.

**Lemma 1.4.** *Let $Q$ be an idempotent quasigroup. The following properties are equivalent:*

(i) *If $x, y \in Q$, and $a, b \in \{x, y, xy\}$, then $\{ab, ba\} \cap \{x, y, xy\} \neq \emptyset$.*
(ii) *If $x, y \in Q$, then $y \in \{x \cdot xy, xy \cdot x\}$ and $x \in \{xy \cdot y, y \cdot xy\}$.*
(iii) *If $x, y \in Q$, then both of the following are true*
  (a) *$y = x \cdot xy$ or $y = xy \cdot x$, and*
  (b) *$y = yx \cdot x$ or $y = x \cdot yx$.*

*Proof.* Since $Q$ is idempotent, we can consider only the case $x \neq y$. Then $x$, $y$ and $xy$ are pairwise distinct. Condition (i) needs a verification only for $\{a, b\} = \{x, xy\}$ and for $\{a, b\} = \{y, xy\}$, and that is exactly the claim of condition (ii). The first part of (ii) can be expressed by (a), and the second part is (b) with $x$ and $y$ exchanged. □

**Lemma 1.5.** *Let $Q$ be a quasigroup such that for all $x, y \in Q$ there holds at least one of the equalities $x \cdot xy = y = yx \cdot x$ and $x \cdot yx = y = xy \cdot x$. Then*

$$x \cdot xy = y \Leftrightarrow y = yx \cdot x \quad \text{and} \quad x \cdot yx = y \Leftrightarrow xy \cdot x = y.$$

*All four equalities are true if and only if $xy = yx$. If $Q$ is idempotent, then $xy = yx$ if and only if $\{x, y, xy\}$ is a subquasigroup.*

*Proof.* We shall argue by contradiction. There are four possible violations of our claim. It will suffice to consider just two of them since the other two follow by a mirror argument.

First, let $x \cdot xy = y$ and $yx \cdot x \neq y$. Then $x \cdot yx = xy \cdot x = y$. Thus $x \cdot xy = y = x \cdot yx$, and hence $xy = yx$. That yields $yx \cdot x = xy \cdot x = y$, a contradiction.

Second, let $x \cdot yx = y$ and $xy \cdot x \neq y$. Then $x \cdot xy = yx \cdot x = y$. Thus $x \cdot yx = x \cdot xy$, $xy = yx$, and $xy \cdot x = yx \cdot x = y$, a contradiction. □

Call $Q$ a *DTS quasigroup* if $Q$ can be obtained from an LDTS $\mathcal{D}$.

**Theorem 1.6.** *Let $Q$ be an idempotent quasigroup. Then $Q$ is a DTS quasigroup if and only if all $x, y \in Q$ satisfy*

(i) *$x \cdot xy = y = yx \cdot x$ or $xy \cdot x = y = x \cdot yx$, and*
(ii) *$xy \cdot x = y$ implies $xy \cdot y = x$.*

*Proof.* Let $\mathcal{D}$ be an LDTS on $X$ such that $X(\cdot)$ is a quasigroup. Assume that $\langle a, b, c\rangle \in \mathcal{D}$. By Theorem 1.3 there exist $a', b', c' \in X$ with $\langle c, b, a'\rangle$, $\langle c, b', a\rangle$, $\langle c', b, a\rangle \in \mathcal{D}$. We get the following table:

| $x$ | $y$ | $x \cdot xy$ | $yx \cdot x$ | $xy \cdot x$ | $x \cdot yx$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $b$ | $b$ | $b'$ | $ac'$ |
| $a$ | $c$ | $c$ | $c$ | $c'$ | $ab'$ |
| $b$ | $c$ | $c'$ | $a'b$ | $c$ | $c$ |

We see that (i) is obviously true and that (ii) holds if $(x, y) = (b, c)$. For the other cases of (ii) note that $\{a, b, c\}$ is a Steiner triple if $b' = b$ or $c' = c$ and that $xy \cdot x = y$ in every Steiner quasigroup. Hence (i) and (ii) hold in every quasigroup that is induced by an LDTS.

Suppose now that (i) and (ii) are true. Define $\mathcal{D}$ so that $\{x, y, xy\}$ is a Steiner triple if $x \neq y$ are elements of $X$ such that $xy = yx$. If $xy \neq yx$ let $(x, y)$ determine the following element of $\mathcal{D}$:

(1) $\langle x, y, xy\rangle$ if $x \cdot xy = yx \cdot x = y$ and $y \cdot xy = yx \cdot y = x$;

(2) $\langle x, xy, y\rangle$ if $x \cdot xy = yx \cdot x = y$ and $xy \cdot y = y \cdot yx = x$; and
(3) $\langle xy, x, y\rangle$ if $xy \cdot x = x \cdot yx = y$ and $xy \cdot y = y \cdot yx = x$.

Every pair $(x, y)$ is covered by a triple from $\mathcal{D}$. That follows from our assumption and from Lemma 1.5. The question is whether two triples have to agree if they agree in one of the directed edges. First we shall observe that none of the directed edges that is carried by a triple determined by (1–3) can appear in a Steiner triple. For that it is enough to show that any of $x \cdot xy = xy \cdot x$ and $y \cdot xy = xy \cdot y$ implies $xy = yx$. That follows from Lemma 1.5.

Now we shall show that each of conditions (1), (2) and (3) determines the same set of triples. Assume that $(x, y)$ satisfies (1). In the next paragraph we shall observe that then (a) $(x', y') = (x, xy)$ satisfies (2), (b) $(x'', y'') = (y, xy)$ satisfies (3), and that in both cases we obtain the triple $\langle x, y, xy\rangle$ again. It follows that a triple determined by (1) can be determined by (2) and (3) as well. We shall then make a similar argument starting from (2), and from (3).

By Lemma 1.5 each of conditions (1–3) contains twice more equalities than needed. When verifying (a) or (b) we shall prove only one equality for each pair. For (a) note that $x'y' = x \cdot xy = y$, $x(x \cdot xy) = xy$ and $(x \cdot xy) \cdot xy = y \cdot xy = x$. For (b) observe that $x''y'' = y \cdot xy = x$, $(y \cdot xy)y = xy$ and $(y \cdot xy) \cdot xy = x \cdot xy = y$.

Assume now (2). We shall show that (a) $(x', y') = (x, xy)$ satisfies (1), (b) $(x'', y'') = (xy, y)$ satisfies (3), and that both (a) and (b) yield $\langle x, xy, y\rangle$. We have (a) $x'y' = x \cdot xy = y$, $x(x \cdot xy) = xy$ and $xy \cdot (x \cdot xy) = xy \cdot y = x$. Furthermore, (b) $x''y'' = xy \cdot y = x$, $(xy \cdot y) \cdot xy = x \cdot xy = y$ and $(xy \cdot y)y = xy$.

Finally assume (3). We need to show that (a) $(x', y') = (xy, x)$ satisfies (1), (b) $(x'', y'') = (xy, y)$ satisfies (2), and that in both cases we obtain $(xy, x, y)$. Now, (a) $x'y' = xy \cdot x = y$, $xy \cdot (xy \cdot x) = xy \cdot y = x$ and $x \cdot (xy \cdot x) = xy$, while (b) $x''y'' = xy \cdot y = x$, $xy \cdot (xy \cdot y) = xy \cdot x = y$, and $(xy \cdot y)y = xy$.

Suppose now that a directed edge $(x, y)$ is covered in two ways. We have proved that if in one case a Steiner triple is involved, then it is involved in the other case as well. Since $x$ and $y$ cannot appear in two different Steiner triples, we can assume that none of them appears in a Steiner triple. Thus $xy \neq yx$.

Since each of (1–3) determines the same set of directed triples we need to consider only the case when for the given $(x, y)$ there are true two of conditions (1-3). However, that easily gives $xy = yx$, a contradiction. □

Laws $x \cdot xy = y$ and $yx \cdot x = y$ are known as the *left* and *right key laws*, respectively. Theorem 1.6 can be thus rephrased by saying that DTS quasigroups are those idempotent quasigroups in which (i) every pair $(x, y)$ is a *key* pair or a *semisymmetric* pair, and (ii) if $(x, y)$ is semisymmetric, then $(y, x)$ is key. One can ask what happens when condition (ii) is removed. Then we obtain quasigroups that can be induced by *hybrid* triple systems [2], i.e. triple systems which may contain both cyclic and directed triples. This will be described in detail in a future paper.

**Proposition 1.7.** *Let $Q$ be an idempotent quasigroup. Then $Q$ is a DTS quasigroup if and only if $xy = z$ implies*
*(a) $xz = y$ and $yz = x$,*
*or*
*(b) $xz = y$ and $zy = x$,*
*or*
*(c) $zx = y$ and $zy = x$,*
*for all $x, y \in Q$.*

*Proof.* Note that (a) can be rewritten as $x \cdot xy = y$ and $y \cdot xy = x$. By expressing (b) and (c) in a similar way we see that the condition of the statement follows from Theorem 1.6 immediately. Of course, it is also easy to verify it directly from the assumption that $Q$ is determined by a DTS $\mathcal{D}$.

To prove the converse we shall start by showing that $x$, $y$ and $z = xy$ form a (commutative) idempotent subquasigroup if at least two of (a), (b) and (c) can be used for a given pair $(x, y)$.

Suppose first that (a) and (c) apply. Thus $xz = y = zx$ and $yz = zy = x$. Put $u = yx$ and consider conditions (a–c) with respect to the pair $(y, x)$. Then (a,b) give $yu = x$ and (c) gives $uy = x$. We see that both cases imply $u = z$.

Assume now (a) and (b). Then $xz = y$ and $yz = x = zy$. Put $u = zx$. It suffices to show that $u = y$ since then the previous case can be used. Consider $(z, x)$. Then (a,b) $zu = x$ and (c) $uz = x$. Thus $y = u$.

Finally, let (b) and (c) be true. Then $xz = y = zx$ and $zy = x$. It suffices to show that $yz = x$. Put $u = yz$ and consider $(y, z)$. Then (a) $zu = y$ and (b,c) $uz = y$.

Let us now define $\mathcal{D}$. Assume $x \neq y$ and put $z = xy$. If $\{x, y, z\}$ forms a subquasigroup, take it as a Steiner triple. If not, include (a) $\langle x, y, z\rangle$, (b) $\langle x, z, y\rangle$, or (c) $\langle z, x, y\rangle$. We have proved that only one of these cases applies. It is now clear that each directed edge is covered by a triple of $\mathcal{D}$.

Assume (a) $xz = y$ and $yz = x$. Then $(x, z)$ fulfils (b) since $xy = z$ and $yz = x$, and $(y, z)$ fulfils (c) since $xy = z$ and $xz = y$.

Assume (b) $xz = y$ and $zy = x$. Then $(x, z)$ fulfils (a) since $xz = y$ and $zy = x$, and $(z, y)$ fulfils (c) since $xz = y$ and $xy = z$.

Assume (c) $zx = y$ and $zy = x$. Then $(z, x)$ fulfils (a) since $zx = y$ and $xy = z$, and the same equalities imply that $(z, y)$ fulfils (b).

Therefore any of the three directed edges of a triple from $\mathcal{D}$ can be used to induce the triple. Hence a directed edge $(x, y)$ might induce two different triples of $\mathcal{D}$ only if at least two of the alternatives (a–c) apply to $(x, y)$. Above we have proved that then $\{x, y, xy\}$ forms a Steiner triple. Each directed edge thus induces only one triple of $\mathcal{D}$. □

It is true that a shorter proof could be obtained by uniting Theorem 1.6 and Proposition 1.7 into one statement. We did not do so for the purpose of future references since we expect that the characterization of Theorem 1.6 will be mentioned in the future much more often than the condition of Proposition 1.7.

**Proposition 1.8.** *The class of DTS quasigroups is closed under subquasigroups and under homomorphic images. If both $Q$ and $Q \times Q$ are DTS quasigroups, then $Q$ is a Steiner quasigroup.*

*Proof.* If $Q$ fulfils the condition of Proposition 1.7, then the condition is clearly fulfilled both by subquasigroups and by homomorphic images.

Suppose now that $Q$ is a proper DTS quasigroup derived from $\mathcal{D}$. Consider $\langle x, y, z\rangle \in \mathcal{D}$. Then $(x, y)(y, z) \cdot (y, z) = (z, x)(y, z) = (zy, y)$ equals $(x, y)$ only if $\{x, y, z\}$ is a Steiner triple. However, that is also true if $(y, z) \cdot (x, y)(y, z) = (y, z)(z, x) = (x, zx)$ equals $(x, y)$. □

**Proposition 1.9.** *Let $Q$ be a DTS quasigroup. If $Q$ satisfies any of the laws $x \cdot xy = y$, $yx \cdot x = y$, $xy \cdot x = y$ or $x \cdot yx = y$, then $Q$ is a Steiner quasigroup.*

*Proof.* Let $Q$ be determined by a set of triples $\mathcal{D}$. Consider $\langle x, y, xy\rangle \in \mathcal{D}$. By Lemma 1.1 we have to show that $xy = yx$. We have $y \cdot xy = x$ and so $y \cdot yx = x$

yields $xy = yx$. We also have $x \cdot xy = y$, and so $xy = yx$ follows from $x \cdot yx = y$. For the other cases use a mirror argument (or consider $Q^{\mathrm{op}}$). $\square$

There are thus no proper semisymmetric or key DTS quasigroups. However, there exist many proper flexible DTS quasigroups. Here we refer to the *flexible* law $x \cdot yx = xy \cdot x$.

**Lemma 1.10.** *Let $Q$ be a DTS quasigroup determined by $\mathcal{D}$. Then $Q$ is flexible if $x \cdot yx = xy \cdot x$ for every $\langle x, xy, y\rangle \in \mathcal{D}$.*

*Proof.* We need to show that the restricted assumption of flexibility implies that $a \cdot ba = ab \cdot a$ for any pair $(a, b)$, where $a$ and $b$ are distinct elements of $Q$. For that it clearly suffices to consider the cases $(x, xy)$ and $(xy, y)$, where $\langle x, xy, y\rangle \in \mathcal{D}$. The latter case is immediate since $\langle y, xy, y \cdot xy\rangle \in \mathcal{D}$ by Theorem 1.3, and hence $(xy)(y \cdot xy) = y = x \cdot xy = (xy \cdot y)(xy)$. For the former case note that $(x \cdot xy)x = yx$ and that $x(xy \cdot x) = x(x \cdot yx)$ is equal to $yx$ since by Theorem 1.3 we have $\langle y, yx, x\rangle \in \mathcal{D}$ and $\langle x, yx, x \cdot yx\rangle \in \mathcal{D}$. $\square$

The above lemma can be seen as a variation of [3, Theorem 2.3]. Note that [3] assumes that the set $X$ is finite, while here we do not exclude the infinite sets. The next statement corresponds to [3, Theorem 2.2]. It weakens the condition of Theorem 1.3, but only for finite sets. Hence we include it without a proof.

**Lemma 1.11.** *Let $\mathcal{D}$ be a DTS upon a finite set $X$. Then $X(\cdot)$ is a quasigroup if and only if for every $\langle x, y, z\rangle \in \mathcal{D}$ there exists $z' \in X$ such that $\langle z', y, x\rangle \in \mathcal{D}$.*

Let us finish this section by a remark, that an LDTS $\mathcal{D}$ is pure if and only if the corresponding quasigroup is *anticommutative* (i.e. $xy = yx$ implies $x = y$). This follows, say, from Lemma 1.1.

## 2. From quasigroups to loops

A standard way how to prolong an idempotent quasigroup $Q$ into a loop $Q_1$ consists of adding a (new) neutral element 1 and setting $x^2 = 1$ for all $x \in Q$ (the loop $Q_1$ is *involutory*).

A loop will be called a *DTS loop* if it can be obtained as a prolongation of a DTS quasigroup. (Similarly we define *Steiner* and *Mendelsohn* loops.)

If $x, y \in Q$ are such that $x \cdot xy = y$ (or $yx \cdot x = x$, or $x \cdot yx = y$ or $xy \cdot x = y$), then the respective identity holds in $Q_1$ as well, and vice versa. Hence Mendelsohn loops coincide with semisymmetric loops, and Theorem 1.6 can be alternatively expressed as:

**Theorem 2.1.** *A loop $Q_1$ is a DTS loop if and only if for all $x, y \in Q_1$*

(i) *$x \cdot xy = y = yx \cdot x$ or $xy \cdot x = y = x \cdot yx$, and*
(ii) *$xy \cdot x = y$ implies $xy \cdot y = x$.*

**Proposition 2.2.** *A loop $Q_1$ is a DTS loop if and only if $xy = z$ implies*

*(a) $xz = y$ and $yz = x$,*
*or*
*(b) $xz = y$ and $zy = x$,*
*or*
*(c) $zx = y$ and $zy = x$,*

*for all $x, y \in Q_1$.*

*Proof.* Suppose first that $Q_1$ is a prolongation of a DTS quasigroup $Q$. If $xy = z$ in $Q_1$ and if none of $x$, $y$ and $z$ is equal to 1, then the implication holds in $Q_1$ because it holds in $Q$. It is easy to see that it holds as well when $1 \in \{x, y, z\}$. On the other hand if $Q_1$ fulfils the implication for all $x, y \in Q_1$, then $xy = 1$ implies $x = y$. That means that $Q_1$ is involutory and can be obtained by a prolongation of an idempotent quasigroup $Q$. If $xy = z$ in $Q$, then either $x = y = z$ or $xy = z$ in $Q_1$. Hence the implication holds in $Q$ as well and Proposition 1.7 can be used. □

Arguments used in the proof of Proposition 1.8 apply to DTS loops as well, and so we have:

**Proposition 2.3.** *The class of DTS loops is closed under subloops and under homomorphic images. If both $Q_1$ and $Q_1 \times Q_1$ are DTS loops, then $Q_1$ is a Steiner loop.*

A loop that satisfies the law $x \cdot xy = x^2y$ is called *left alternative*. The mirror law is the *right alternative* law.

A prolongation $Q_1$ of an idempotent quasigroup $Q$ is left alternative if and only if $Q$ satisfies the left key law $x \cdot xy = y$. The prolongation is semisymmetric if and only if $Q$ is semisymmetric.

**Proposition 2.4.** *Let $Q_1$ be a DTS loop. If $Q_1$ is commutative or left alternative or right alternative or semisymmetric, then it is a Steiner loop.*

*Proof.* If $Q_1$ is commutative, then it is semisymmetric (and hence also alternative), by Theorem 2.1. The rest follows from Proposition 1.9. □

**Lemma 2.5.** *Let $Q_1$ be a DTS loop. Suppose that $x, y \in Q_1$ generate a subgroup, that $1 \notin \{x, y\}$ and that $x \neq y$. Then the subgroup consists of 1, $x$, $y$ and $xy$. This takes place if and only if $\{x, y, xy\}$ forms a Steiner triple, and that is true if and only if $xy = yx$.*

*Proof.* Use Lemma 1.1 if $xy = yx$. If $\{x, y, xy\}$ forms a Steiner triple, then we clearly get a subgroup. For the converse it may be assumed that $Q_1$ is a group, by Proposition 2.3. The claim follows from Proposition 2.4 since the involutory groups are commutative. □

**Proposition 2.6.** *Let $Q_1$ be a proper DTS loop. Then it cannot be a (left or right) Bol loop, or an LC or RC loop, or a Buchsteiner loop or a left or right conjugacy closed loop.*

*Proof.* Left Bol loops and LC loops are left alternative. Right Bol loops and RC loops are right alternative. By Lemma 2.4 we hence need only to prove that $Q_1$ is commutative if it is a Buchsteiner loop or, say, a left conjugacy closed (LCC) loop.

LCC loops fulfil the identity $((xy)/x)z = x(y(x\backslash z))$. Setting $z = 1$ we get $xy = (x \cdot yx)x$ since $Q_1$ is involutory. Assume that the latter identity holds. Consider the associated LDTS $\mathcal{D}$ and assume that $\langle y, x, z\rangle \in \mathcal{D}$. Then $(x \cdot yx)x = xz \cdot x = yx = z = xy$. That makes $\{x, y, z\}$ a Steiner triple, by Lemma 1.1, and we see that $Q_1$ is commutative, as required.

In every involutory loop the Buchsteiner law $x\backslash(xy \cdot z) = (y \cdot zx)/x$ yields $x\backslash(xy \cdot x) = y/x$. Assume $\langle z, x, y\rangle \in \mathcal{D}$. Then $z = y/x$ and $xy \cdot x = zx$. Therefore $zx = xz$ and so we get the commutativity again. □

Proper DTS loops thus never belong to one of the standardly studied equational classes of loops.

Let $Q_1$ be a loop. The *left nucleus* $N_\lambda$ is formed by elements $a \in Q_1$ with $a(xy) = (ax)y$ for all $x, y \in Q_1$. By shifting $a$ to the right we get the *middle nucleus* $N_\mu$ and the right nucleus $N_\rho$. The *centre* $Z(Q_1)$ consists of all $a \in N_\lambda \cap N_\rho \cap N_\mu$ with $ax = xa$ for every $x \in Q_1$.

Set $C(Q_1) = \{a \in Q_1;\ ax = xa \text{ for all } x \in Q_1\}$. By Lemma 2.5, if $Q_1$ is a DTS loop, then its element $a \neq 1$ belongs to $C(Q_1)$ if and only if $\{a, x, ax\}$ is a Steiner triple for any $x \in Q_1 \setminus \{1, a\}$. Note that $C(Q_1)$ does not have to be a subloop—below is a counterexample of the smallest order. For simplicity, we omit commas from the triples.

**Example 2.7.** Let $X = \{2, 3, 4, 5, 6, 7, 8, 9, \mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}\}$ and let $Q_1$ be the DTS loop determined by the triples $\{\mathtt{234}\}$, $\{\mathtt{256}\}$, $\{\mathtt{278}\}$, $\{\mathtt{29A}\}$, $\{\mathtt{2BC}\}$, $\{\mathtt{2DE}\}$, $\{\mathtt{357}\}$, $\{\mathtt{36C}\}$, $\{\mathtt{38A}\}$, $\{\mathtt{39E}\}$, $\{\mathtt{3BD}\}$, $\langle\mathtt{458}\rangle$, $\langle\mathtt{469}\rangle$, $\langle\mathtt{74E}\rangle$, $\langle\mathtt{76B}\rangle$, $\langle\mathtt{85D}\rangle$, $\langle\mathtt{864}\rangle$, $\langle\mathtt{954}\rangle$, $\langle\mathtt{96D}\rangle$, $\langle\mathtt{97C}\rangle$, $\langle\mathtt{98B}\rangle$, $\langle\mathtt{A4C}\rangle$, $\langle\mathtt{A5B}\rangle$, $\langle\mathtt{A6E}\rangle$, $\langle\mathtt{A7D}\rangle$, $\langle\mathtt{B47}\rangle$, $\langle\mathtt{B59}\rangle$, $\langle\mathtt{B6A}\rangle$, $\langle\mathtt{B8E}\rangle$, $\langle\mathtt{C4D}\rangle$, $\langle\mathtt{C5E}\rangle$, $\langle\mathtt{C7A}\rangle$, $\langle\mathtt{C89}\rangle$, $\langle\mathtt{D4A}\rangle$, $\langle\mathtt{D5C}\rangle$, $\langle\mathtt{D68}\rangle$, $\langle\mathtt{D79}\rangle$, $\langle\mathtt{E4B}\rangle$, $\langle\mathtt{E5A}\rangle$, $\langle\mathtt{E67}\rangle$, $\langle\mathtt{E8C}\rangle$. Then $C(Q_1) = \{1, 2, 3\}$, but $\mathtt{2} \cdot \mathtt{3} = \mathtt{4} \notin C(Q_1)$.

**Lemma 2.8.** *Let $Q_1$ be a DTS loop. Then $N_\lambda \cup N_\rho \cup N_\mu \subseteq C(Q_1)$.*

*Proof.* Suppose that $a, x, y \in Q_1$ are such that $ax = y$. Let $a$ be first an element of $N_\lambda$. Then $ay = aa \cdot x = x$ and $a \cdot xy = a(x \cdot ax) = ax \cdot ax = 1 = ay \cdot ay = a(y \cdot ay) = a \cdot yx$. Thus $xy = yx$ and $\{1, a, x, y\}$ is a commutative subgroup of $Q_1$, by Lemma 2.5. Hence $ax = xa$.

Let $a$ be now an element of $N_\mu$. Then $ay = aa \cdot x = x$ and $yy = 1 = xx = x \cdot ay = xa \cdot y$. Thus $xa = y = ax$. □

While the existence spectrum of DTS loops is known, there seem to be no results that would specify possible sizes of nuclei.

## 3. Directed triples and surface triangulations

By a *combinatorial triangulated 2-pseudomanifold* (shortly *triangulated pseudomanifold*) we shall understand a finite family $\mathcal{F}$ of *faces* such that every face is a three-element set $\{x, y, z\}$ and there exist unique $x' \neq x$, $y' \neq y$ and $z' \neq z$ with $\{x', y, z\}, \{x, y', z\}, \{x, y, z'\} \in \mathcal{F}$. In other words, every edge of $\mathcal{F}$ is incident to exactly two faces. Each face determines three edges and three points. The edges and points yield the *graph* of $\mathcal{F}$. The pseudomanifold is said to be *connected* if the graph is connected. The pseudomanifold is *strongly connected* if for any two points $x$ and $y$ there exists a sequences of faces $F_0, \dots, F_k$ such that $F_{i-1}$ and $F_i$ share an edge, $1 \leq i \leq k$, $x$ is incident to $F_0$ and $y$ is incident to $F_k$. Note that many authors require (triangulated) pseudomanifolds to be strongly connected.

The main notion we need is that of the triangulated pseudomanifold as defined above. A more general notion of *combinatorial 2-pseudomanifolds* (shortly, *pseudomanifolds*) is defined similarly, but the faces can be $k$-gons, $k \geq 3$. Taken formally, the *face* is then a pair $\{(y_1, \dots, y_k), (y_k, \dots, y_1)\}$, where $y_1, \dots, y_k$ are pairwise distinct points and $(y_1, \dots, y_k)$ is regarded as a cyclic sequence. By choosing one element of the pair we choose an orientation of the face. A combinatorial 2-pseudomanifold is *orientable* if the orientation can be fixed in such a way that two different faces that share an edge induce upon the edge opposite orientations. If such a coherent orientation is given, we speak about an *oriented pseudomanifold*. An oriented pseudomanifold can be considered as a family of *oriented faces* $(y_1, \ldots, y_k)$. Orientable pseudomanifolds will be called here *(combinatorial) pseudosurfaces.*

Let $\mathcal{D}$ be a finite DTS. Elements $\langle x, y, z \rangle \in \mathcal{D}$ that do not yield a Steiner triple will be called *unidirectional.* Denote by $\mathcal{F}$ the set of all $\{x, y, z\}$, where $\langle x, y, z \rangle$ runs through all unidirectional triples of $\mathcal{D}$. Consider now $\mathcal{F}$ as a set of faces. Each edge $\{x, y\}$ is incident to two faces, and so we get a pseudomanifold. In general, the pseudomanifold does not have to be orientable.

Suppose now that $\mathcal{D}$ is a finite LDTS. Orient $\{x, y, z\} \in \mathcal{F}$ as $(x, y, z)$ if $\langle x, y, z \rangle \in \mathcal{D}$. It follows from Theorem 1.3 that this defines a coherent orientation. Hence $\mathcal{F}$ is a pseudosurface. We shall call it the pseudosurface of $\mathcal{D}$ (or of $Q$ if $Q$ is the DTS quasigroup that determines $\mathcal{D}$).

Consider $\langle y_0, x, y_1 \rangle \in \mathcal{D}$. There exist $k \geq 2$ and points $y_0, y_1, y_2, \ldots, y_k$ that are pairwise distinct such that $\langle y_1, x, y_2 \rangle, \ldots, \langle y_k, x, y_0 \rangle \in \mathcal{D}$. Call $(y_0, y_1, \ldots, y_k)$ an (oriented) *residual face.* The triangular faces $\{y_0, x, y_1\}, \ldots, \{y_k, x, y_0\}$ form its *cap.* The oriented residual face $(y_0, \ldots, y_k)$ is said to be *singular* if $(y_k, \ldots, y_0)$ is an oriented residual face as well. To see how singular residual faces relate to flexibility we need the following lemma. It analyzes the situation when two residual faces share an edge.

**Lemma 3.1.** *Suppose that* $\mathcal{D}$ *contains* $\langle y_0, x_1, y_1 \rangle$, $\langle y_1, x_2, y_0 \rangle$, $\langle y_1, x_1, y_2 \rangle$ *and* $\langle y_2', x_2, y_1 \rangle$. *Then* $y_2' = y_2$ *if and only if* $y_1 \cdot y_0 y_1 = y_1 y_0 \cdot y_1$.

*Proof.* By our assumptions $x_1 = y_0 y_1$, $x_2 = y_1 y_0$, $y_2 = y_1 x_1 = y_1 \cdot y_0 y_1$ and $y_2' = x_2 y_1 = y_1 y_0 \cdot y_1$. □

**Corollary 3.2.** *A finite DTS quasigroup* $Q$ *is flexible if and only if all residual faces of* $Q$ *are singular.*

*Proof.* Combine Lemma 3.1 with Lemma 1.10. □

Denote by $O_k$ a $k$-gonal bipyramid, i.e. a graph of $k+2$ vertices with a cycle of length $k \geq 3$, in which the remaining two vertices are connected to the elements of the cycle (the graph contains $3k$ edges). Corollary 3.2 immediately yields:

**Theorem 3.3.** *A flexible DTS quasigroup of order* $n$ *exists if and only if the complete graph* $K_n$ *can be decomposed to triangles and graphs* $O_k$, $k \geq 3$.

Note that the number of nonisomorphic flexible quasigroups of order $n$ can be much bigger than the number of nonisomorphic decompositions of $K_n$, as each $O_k$ can be oriented in two ways (if $k = 4$ then there are, in addition, three ways how to choose the non-oriented residual face).

The existence spectrum of odd order flexible DTS quasigroups was determined in [3, Theorem 4.4]. The even case is being investigated.

When we put aside the singular residual faces we get a set of oriented faces that yields an oriented pseudosurface. We call it the *residual pseudosurface.* It is obtained from the pseudosurface of $Q$ by cutting away the caps.

The proof that there are no DTS quasigroups of order 10 [3, Theorem 3.3] is based upon showing that the parameters of a potential residual pseudosurface induce a surface with parameters that would violate the parity of the Euler characteristic.

The notion of the strong connectivity can be used to partition the pseudosurface of a DTS quasigroup $Q$ into *components.* Each component possesses a genus, and the list of genera can be considered as an invariant of $Q$. The components induced by a singular residual face are called *flexible*. Their graph is isomorphic to $O_k$ for some $k \geq 3$.

Note however that a component may still be a proper pseudosurface, i.e. it does not have to be a (combinatorial) surface (a formal definition of a surface can be found below). This fact seems to make the geometrical approach a less potent tool than might be expected when proving the existence or non-existence of DTS quasigroups of orders greater than 10.

Nevertheless, the gained geometrical insight naturally leads to a construction that uses latin bitrades to diminish the number of Steiner triples in a DTS quasigroup (in particular, to build a proper DTS quasigroup from a Steiner quasigroup).

By a *latin bitrade* $T$ we shall understand a pair $T = (L, R)$ where $L$ and $R$ are two disjoint sets consisting of ordered triples such that if $1 \leq i < j \leq 3$, and $a = (a_1, a_2, a_3) \in L$, then $\{a_i, a_j\}$ determines the triple $a$ uniquely, $a_i \neq a_j$, and there exists $b = (b_1, b_2, b_3) \in R$ with $(a_i, a_j) = (b_i, b_j)$. The meaning of the *mates* $L$ and $R$ is interchangeable, and thus for $R$ there apply symmetric conditions.

Our definition of latin bitrades is tailored to present needs. Instead of requiring that $a_i \neq a_j$ and that $\{a_i, a_j\}$ determines the triple $a$ it is usual to require only that $(a_i, a_j)$ determines $a$. Another, a more restrictive definition, includes a condition that $a_i \neq a'_j$ for all $(a'_1, a'_2, a'_3) \in L$, $1 \leq i < j \leq 3$. These variations have no structural impact and can be solved by renaming of elements.

Note that by considering the family of all $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where $(a_1, a_2, a_3) \in L$ and $(b_1, b_2, b_3) \in R$, we get a pseudomanifold. By choosing reverse orientations for elements of $L$ and $R$ we see that the pseudomanifold is orientable (it is a pseudosurface).

**Proposition 3.4.** *Let $(L, R)$ be a Latin bitrade and let $\mathcal{D}$ be an LDTS such that $\{a_1, a_2, a_3\}$ is a Steiner triple in $\mathcal{D}$ for every $(a_1, a_2, a_3) \in L$. Change $\mathcal{D}$ into $\mathcal{D}'$ in such a way that these Steiner triples are replaced by directed triples $\langle a_1, a_2, a_3\rangle$ and $\langle b_3, b_2, b_1\rangle$, where $(a_1, a_2, a_3) \in L$ and $(b_1, b_2, b_3) \in R$. Then $\mathcal{D}'$ is an LDTS as well.*

*Proof.* Suppose that $(b_1, b_2, b_3) \in R$ is chosen in such a way that $b_1 = a_1$ and $b_3 = a_3$ where $(a_1, a_2, a_3) \in L$. Then $\langle a_1, a_2, a_3\rangle$ covers $(a_1, a_3)$ and $\langle b_3, b_2, b_1\rangle$ covers $(a_3, a_1)$. By treating cases $(b_1, b_2) = (a_1, a_2)$ and $(b_2, b_3) = (a_2, a_3)$ in a similar way we see that Theorem 1.3 can be used. $\square$

If $Q'$ is the quasigroup determined by $\mathcal{D}'$, and $Q$ is determined by $\mathcal{D}$, then we shall say that *$Q'$ is derived from $Q$ by means of a latin bitrade* $(L, R)$.

By a *surface* we understand here a strongly connected pseudosurface in which all faces incident to a point rotate around the point. To turn a strongly connected pseudosurface into a surface it suffices to divide a point into several new points (let us call them *vertices*) so that each vertex corresponds to a cycle of faces around the point. If the pseudosurface is triangulated, then such a cycle around a point $x$ takes form $\{y_0, x, y_1\}, \{y_1, x, y_2\}, \ldots, \{y_k, x, y_0\}$. A pseudosurface is thus a surface if and only if for each point $x$ there is only one such cycle.

A DTS quasigroup $Q$ yields components that are pseudosurfaces, and each such pseudosurface yields a surface by the procedure we have just described. We shall speak about a *surface constituent* of $Q$. If a vertex corresponds to the cap of a residual face, it will be referred to as a *middle* vertex, otherwise it will be referred to as a *residual* vertex.

**Proposition 3.5.** *A DTS quasigroup $Q$ can be derived by means of a latin bitrade from a Steiner quasigroup if and only if each surface constituent of $Q$ is vertex 3-colourable.*

*Proof.* In a vertex 3-colourable triangulated surface with a chosen coherent orientation the faces can be divided into two classes according to the cyclic ordering of the vertex classes that is induced by the orientation of the face. The surface is hence face 2-colourable. For each face colour consider the set of ordered triples $(a_1, a_2, a_3)$ such that $\{a_1, a_2, a_3\}$ is a face of the given colour and $a_i$ is a vertex of colour $i$. It is clear that the obtained sets are mates of a latin bitrade.

Assume that all surface constituents of $Q$ are vertex 3-colourable. Each constituent thus defines a latin bitrade. The identifications of vertices that are needed to turn the surface constituent into the corresponding (pseudosurface) component can be carried out in the bitrade structure without violating the definition of the latin bitrade. Furthermore, the obtained latin bitrades can be aggregated into one bitrade, and this bitrade determines a Steiner quasigroup from which $Q$ can be derived.

If $Q$ was derived from a Steiner quasigroup, then the used latin bitrade can be interpreted as a pseudosurface. The obtained pseudosurface coincides with the pseudosurface of $Q$. Each constituent of $Q$ can be thus interpreted as a latin bitrade in which the projections along the 1st, 2nd and 3rd coordinate yield three sets that are pairwise disjoint. These sets yield the three colours of vertices. □

Each nonflexible component of a DTS quasigroup $Q$ yields in an obvious way a *residual component* and a *residual constituent*. Note that a surface constituent is vertex 3-colourable if and only if the graph of its residual constituent is bipartite.

It is well known that triangulated surfaces of genus 0 (the spherical surfaces) are vertex 3-colourable if and only if they are Eulerian (i.e. if each vertex is of an even degree). Using Theorem 3.3 we see that a flexible DTS quasigroup can be derived from a Steiner quasigroup by means of latin bitrades if and only if each component corresponds to $O_k$ for an even $k = 2m$. The trades involved in such derivation of flexible DTS quasigroups possess a transparent structure. They are sometimes called *bicyclic* and can be represented by $L = \{(x_1, y, x_2),$ $(x_2, z, x_3), \ldots, (x_{2m-1}, y, x_{2m}), (x_{2m}, z, x_1)\}$ and by $R$ that is obtained from $L$ by exchanging all occurrences of $y$ and $z$. Note that by permuting, say, the first and second coordinate we get a latin bitrade that can be used to build a DTS quasigroup as well. However, the resulting quasigroup will not be flexible if $m \geq 3$.

If $m = 2$, then the STS of the initial Steiner quasigroup contains $\{x_1, y, x_2\}$, $\{x_3, y, x_4\}$, $\{x_1, z, x_3\}$ and $\{x_2, z, x_4\}$. This is known as a *Pasch configuration.* Its transformation via the corresponding latin bitrade is used in [3] several times (e.g. in Proposition 4.1).

## 4. Enumeration and classification

To enumerate DTS quasigroups we use the program Mace4 [6] which is part of the package Prover9, an automated theorem prover for first-order and equational logic. While Prover9 searches for a proof, Mace4 is generally used to search for finite counterexamples, however it can also be used to enumerate all structures of some finite order that satisfy a given set of equations. For example, in order to generate all proper DTS quasigroups of order 7 we provide Mace4 with the following input

```
1  assign(max_models, -1).
2  assign(domain_size, 7).
3  formulas(sos).
4      x * y = x * z  ->  y = z.
5      y * x = z * x  ->  y = z.
6      x * x = x.
7      (x * (x * y) = y  &  (y * x) * x = y) |
8          ((x * y) * x = y  &  x * (y * x) = y).
9      (x * y) * x = y  ->  (x * y) * y = x.
10     0 * 1 != 1 * 0.
11 end_of_list.
```

The equations on lines 7, 8 and 9 correspond to the characterisation of DTS quasigroups given in Theorem 1.6. Mace4 tends to generate the results faster using this characterisation than if the characterisation from Proposition 1.7 is used. When enumerating proper DTS quasigroups of order 12 it runs approximately 20 times faster. On the right side of the implication on line 9 either one of the key laws or a conjunction of the key laws can be used. Similarly the left side of the implication can be replaced with `x * (y * x) = y` or with a disjunction of the two expressions. As one might expect, using the disjunction on the left gives the worst running time of all. The remaining six possible combinations all do equally well.

Mace4 can instantly enumerate the DTS quasigroups of orders up to 9 and determine that none exist for orders 4, 6 or 10. The enumeration of DTS quasigroups of order 12 can be achieved in a matter of minutes.

The smallest proper DTS quasigroup is of order 7. It is unique up to isomorphism and yields a single surface constituent which is isomorphic to $O_4$.

For proper DTS quasigroups of order 9 there exist three isomorphism types. The first two types each yield a single surface constituent isomorphic to $O_6$, however one of these is flexible while the other is not, i.e. their residual constituents are non-isomorphic. The third type yields a surface constituent of genus 1 consisting of 3 residual faces.

For proper DTS quasigroups of order 12 there exist two isomorphism types. Their pseudosurfaces differ only in orientation. Each type yields three residual surface constituents, all isomorphic to a tetrahedron.

All DTS quasigroups of order up to 12 are explicitly described in [3].

At order 13 the combinatorial explosion takes over. If we attempt to generate the DTS quasigroups of order 13 using the above input, Mace4 soon runs out of memory. In comparison for Steiner triple systems the combinatorial explosion takes place at order 19 [5].

We split the task of enumerating DTS quasigroups of order 13 into more manageable tasks by placing restrictions on the degrees of middle vertices (cf. Section 3). We first focused on generating the DTS quasigroups with middle vertices of degree at most 6, then we focused on generating those that contain at least one middle vertex of degree greater than 6. Thus the task was split into generating proper DTS quasigroups of order 13 such that

1. all middle vertices have degree 3;
2. all middle vertices have degree at most 4 and there exists a middle vertex of degree 4;
3. all middle vertices have degree at most 6, there exists a middle vertex of degree 5 and there may or may not exist a vertex of degree 6;

| Task | Generated | Isomorphism types | Time to generate |
|---:|---:|---:|---:|
| 1 | 12 | 1 | 2 minutes |
| 2 | 217 292 | 8 004 | 24.5 hours |
| 3 | 831 487 | 106 446 | 4.0 hours |
| 4 | 1 337 912 | 87 019 | 14.2 hours |
| 5 | 1 960 056 | 258 251 | 2.0 hours |
| 6.1 | 3 368 344 | 353 637 | 3.2 hours |
| 6.2 | 1 090 528 | 34 079 | 2.4 hours |
| 6.3 (a) | 1 327 664 | 91 738 | 1.3 hours |
| 6.3 (b) | 686 064 | 299 641 | 0.6 hours |
| 7 | 325 644 | 36 184 | 0.9 hours |
| 8 | 4 779 308 | 401 683 | 3.7 hours |
| 9 | 758 160 | 63 180 | 2.2 hours |
| Total | 16 682 471 | 1 206 967 | 59.2 hours |

TABLE 1. The number of proper DTS quasigroups of order 13 generated by Mace4 in each task of the enumeration.

4. all middle vertices have degree at most 6 and there exists a middle vertex of degree 6 but no vertex of degree 5;
5. there exists a middle vertex of degree 7;
6. there exists a middle vertex of degree 8;
7. there exists a middle vertex of degree 9;
8. there exists a middle vertex of degree 10;
9. there exists a middle vertex of degree 12.

Mace4 generated a total of 16 682 471 quasigroups in 59.2 hours on a computer equipped with an Intel Xeon E5620 2.40 GHz CPU with 12 MB of cache. This does not include the time needed to remove the isomorphic quasigroups. Details are given in Table 1.

When dealing with the DTS quasigroups that have a middle vertex of degree 8, Mace4 ran out of memory. The task was split further as follows. Denote the point corresponding to the middle vertex of degree 8 as $\mathtt{0}$, the corresponding residual face as $(\mathtt{1}, \mathtt{2}, \dots, \mathtt{8})$ and the remaining points as $\mathtt{9}$, $\mathtt{T}$, $\mathtt{E}$ and $\mathtt{W}$. We split the task based on how these four remaining points relate to the point $\mathtt{0}$. There are three possibilities, one of which had to be split further because Mace4 ran out of memory.

6.1 The remaining points form two Steiner triangles with the point $\mathtt{0}$, e.g. $\{\mathtt{0}, \mathtt{9}, \mathtt{T}\}$ and $\{\mathtt{0}, \mathtt{E}, \mathtt{W}\}$;

6.2 there exists another middle vertex corresponding to the point $\mathtt{0}$ and the remaining four points correspond to vertices which form a cycle around this middle vertex, e.g. the LDTS contains the directed triples $\langle \mathtt{9}, \mathtt{0}, \mathtt{T} \rangle$, $\langle \mathtt{T}, \mathtt{0}, \mathtt{E} \rangle$, $\langle \mathtt{E}, \mathtt{0}, \mathtt{W} \rangle$ and $\langle \mathtt{W}, \mathtt{0}, \mathtt{9} \rangle$; or

6.3 there exists a residual vertex corresponding to the point $\mathtt{0}$ and the remaining four points correspond to vertices which form a cycle around this residual vertex, e.g. the LDTS contains the directed triples $\langle \mathtt{0}, \mathtt{9}, \mathtt{T} \rangle$, $\langle \mathtt{T}, \mathtt{E}, \mathtt{0} \rangle$, $\langle \mathtt{0}, \mathtt{E}, \mathtt{W} \rangle$ and $\langle \mathtt{W}, \mathtt{9}, \mathtt{0} \rangle$, and further
   (a) $\mathtt{9} \cdot \mathtt{W} = \mathtt{T}$ or
   (b) $\mathtt{9} \cdot \mathtt{W}$ is one of the points $\mathtt{1}, \dots, \mathtt{8}$.

When dealing with the case of the two Steiner triangles $\{\mathtt{0}, \mathtt{9}, \mathtt{T}\}$ and $\{\mathtt{0}, \mathtt{E}, \mathtt{W}\}$ above, $\mathtt{T} \cdot \mathtt{W}$ must be one of the points $\mathtt{1}, \dots, \mathtt{8}$. Assigning $\mathtt{T} \cdot \mathtt{W} = \mathtt{1}$ reduces the number of isomorphic models generated and was necessary to prevent Mace4 from running out of memory. Similarly in 6.3 (b) we assign $\texttt{9} \cdot \texttt{W} = \texttt{1}$.

After putting all the results together we found 1 206 969 isomorphism types of DTS quasigroups of order 13. Out of these 8 444 are pure and 924 are flexible (including the 2 Steiner quasigroups). There do not exist any pure flexible DTS quasigroups of order 13.

To remove the isomorphic models, the results were first split into smaller classes according to an invariant which is derived from how each point of the pseudosurface splits into vertices of the surface, taking into account the degree of each vertex and whether it is a middle vertex or a residual vertex. Isomorphic models were then removed from each class using a custom program which exploits the geometric structure of DTS quasigroups to find possible isomorphisms. Afterwards, each of these classes was checked using the GAP [4] package LOOPS [8] to confirm that its contents are indeed pairwise non-isomorphic.

The isomorphism types were then classified according to the genera of their surface constituents and according to their automorphism group, see Tables 2 and 3. Table 2 also gives the number of non-isomorphic pseudosurfaces yielded by the DTS quasigroups in each class. For example the last line in Table 2 indicates that there exist exactly 6 non-isomorphic DTS quasigroups of order 13 that consist of 2 surface constituents of genus 1 (see Example A.2). These 6 quasigroups yield only 2 non-isomorphic pseudosurfaces. The number of non-isomorphic pseudosurfaces in each class was determined using `shortg` from the package **nauty** [7]. The automorphism groups in Table 3 were determined using GAP. We refer to the dihedral group of order $2n$ as $D_{2n}$.

Using the sizes of the automorphism groups from Table 3, we can easily compute the total number of DTS quasigroups of order 13 by taking the sum of $13!/|\mathrm{Aut}(Q)|$ over all isomorphism types $Q$, which comes out to 7 502 250 290 008 320.

If we attempt to generate the DTS quasigroups of orders 15, 19 or 21, Mace4 instantly produces plenty of models and soon runs out of memory. For the remaining orders, the program tends to produce fewer results. Using the above input, we were not able to obtain DTS quasigroups of even orders greater than 18 in a reasonable amount of time, but we did obtain ones of orders 25, 27, 31 and 37.

To determine the existence spectrum of LDTS in [3] we needed to obtain LDTS of certain orders, which were as high as 40. We did this by prescribing a suitable automorphism as part of the input to Mace4. Generally Mace4 can then produce a model within a few seconds, but the time varies greatly. To date, the largest model that we have been able to obtain this way is a pure DTS quasigroup of order 58 with an automorphism of type $29^2$. However this technique is not always successful. For example, we were not able to generate a pure flexible DTS quasigroup of order 16, instead it was generated using the program Paradox [1] which found an automorphism-free model.

## Appendix. Examples of DTS quasigroups of order 13

It is clearly impossible to list all DTS quasigroups of order 13 but below are given some which may be of particular interest. These are the unique proper system with all middle vertices of degree 3, all six systems with two surface constituents of genus 1, all systems having an automorphism group of order greater than or equal to 4, and at least one example of a system having just one surface constituent of genus 0, 1, 2, 3 or 4, respectively.

| Number of surface constituents of genus $g$ | | | | | Number of non-isomorphic quasigroups | Number of non-isomorphic pseudosurfaces |
|---|---|---|---|---|---|---|
| $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | | |
| 0 | 1 | 0 | 0 | 0 | 392 685 | 189 280 |
| 1 | 0 | 0 | 0 | 0 | 391 805 | 166 149 |
| 2 | 0 | 0 | 0 | 0 | 152 818 | 26 227 |
| 0 | 0 | 1 | 0 | 0 | 117 368 | 58 588 |
| 1 | 1 | 0 | 0 | 0 | 80 875 | 16 100 |
| 3 | 0 | 0 | 0 | 0 | 32 100 | 2 098 |
| 1 | 0 | 1 | 0 | 0 | 14 019 | 3 162 |
| 0 | 0 | 0 | 1 | 0 | 10 636 | 5 374 |
| 4 | 0 | 0 | 0 | 0 | 6 000 | 267 |
| 2 | 1 | 0 | 0 | 0 | 5 896 | 505 |
| 5 | 0 | 0 | 0 | 0 | 955 | 28 |
| 1 | 0 | 0 | 1 | 0 | 769 | 189 |
| 3 | 1 | 0 | 0 | 0 | 533 | 36 |
| 0 | 0 | 0 | 0 | 1 | 246 | 131 |
| 2 | 0 | 1 | 0 | 0 | 178 | 18 |
| 4 | 1 | 0 | 0 | 0 | 40 | 3 |
| 6 | 0 | 0 | 0 | 0 | 24 | 3 |
| 1 | 0 | 0 | 0 | 1 | 14 | 4 |
| 0 | 2 | 0 | 0 | 0 | 6 | 2 |
| | | | | Total | 1 206 967 | 468 164 |

Table 2. Classification of the isomorphism types of proper DTS quasigroups of order 13 according to the genera of their surface constituents.

| Aut($Q$) | Number of types | Pure | Flexible |
|---|---|---|---|
| $C_1$ | 1 202 669 | 8 406 | 864 |
| $C_2$ | 4 163 | 36 | 43 |
| $C_3$ | 92 | 0 | 8 |
| $C_2 \times C_2$ | 17 | 0 | 0 |
| $C_5$ | 8 | 0 | 0 |
| $S_3$ | 7 | 0 | 1 |
| $C_6$ | 5 | 0 | 4 |
| $C_{10}$ | 2 | 0 | 2 |
| $D_{10}$ | 1 | 0 | 1 |
| $D_{12}$ | 2 | 0 | 0 |
| $C_{13}$ | 2 | 2 | 0 |
| $C_{13} \rtimes C_3$ | 1 | 0 | 1 |
| Total | 1 206 969 | 8 444 | 924 |

Table 3. Classification of the isomorphism types of DTS quasigroups of order 13 according to their automorphism group.

In the following examples let $X = \{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9},\mathtt{T},\mathtt{E},\mathtt{W}\}$. For simplicity, we omit commas from the triples.

**Example A.1.** Define $\mathcal{T} = \{$\{018\}, \{09E\}, \{0TW\}, \{19T\}, \{1EW\}, \{259\}, \{268\}, \{27W\}, \{2TE\}, \{35T\}, \{36E\}, \{379\}, \{38W\}, \{45W\}, \{46T\}, \{47E\}, \{489\}, \{58E\}, \{69W\}, \{78T\}$\}$,
$\mathcal{C}_1 = \{$⟨203⟩, ⟨304⟩, ⟨402⟩, ⟨214⟩, ⟨413⟩, ⟨312⟩$\}$ and
$\mathcal{C}_2 = \{$⟨506⟩, ⟨607⟩, ⟨705⟩, ⟨517⟩, ⟨716⟩, ⟨615⟩$\}$.
Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 0, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ is the unique proper LDTS(13) such that all middle vertices are of degree 3. The system is automorphism-free and flexible.

**Example A.2.** The 6 systems with two surface constituents of genus 1 are defined as follows.

(1) Define $\mathcal{T} = \{$\{09E\}, \{38W\}, \{48E\}$\}$,
$\mathcal{C}_1 = \{$⟨102⟩, ⟨203⟩, ⟨304⟩, ⟨405⟩, ⟨501⟩, ⟨164⟩, ⟨463⟩, ⟨365⟩, ⟨56W⟩, ⟨W62⟩, ⟨261⟩, ⟨19W⟩, ⟨W95⟩, ⟨594⟩, ⟨491⟩, ⟨1E5⟩, ⟨5E3⟩, ⟨3E2⟩, ⟨2EW⟩, ⟨WE1⟩$\}$ and
$\mathcal{C}_2 = \{$⟨607⟩, ⟨70W⟩, ⟨W0T⟩, ⟨T08⟩, ⟨806⟩, ⟨317⟩, ⟨718⟩, ⟨81T⟩, ⟨T13⟩, ⟨24T⟩, ⟨T4W⟩, ⟨W47⟩, ⟨742⟩, ⟨258⟩, ⟨857⟩, ⟨75T⟩, ⟨T52⟩, ⟨297⟩, ⟨793⟩, ⟨39T⟩, ⟨T96⟩, ⟨698⟩, ⟨892⟩, ⟨6ET⟩, ⟨TE7⟩, ⟨7E6⟩$\}$.
Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 1, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$, $(X, \mathcal{T} \cup \mathcal{C}_1^{\mathrm{op}} \cup \mathcal{C}_2)$, $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2^{\mathrm{op}})$ and $(X, \mathcal{T} \cup \mathcal{C}_1^{\mathrm{op}} \cup \mathcal{C}_2^{\mathrm{op}})$ are non-flexible, automorphism-free LDTS(13)s.

(2) Define $\mathcal{T} = \{$\{09E\}, \{137\}, \{679\}$\}$,
$\mathcal{C}_1 = \{$⟨102⟩, ⟨203⟩, ⟨304⟩, ⟨405⟩, ⟨501⟩, ⟨16W⟩, ⟨W64⟩, ⟨463⟩, ⟨365⟩, ⟨562⟩, ⟨261⟩, ⟨295⟩, ⟨594⟩, ⟨49W⟩, ⟨W92⟩, ⟨1E5⟩, ⟨5E3⟩, ⟨3E2⟩, ⟨2EW⟩, ⟨WE1⟩$\}$ and
$\mathcal{C}_2 = \{$⟨60T⟩, ⟨T07⟩, ⟨70W⟩, ⟨W08⟩, ⟨806⟩, ⟨418⟩, ⟨819⟩, ⟨91T⟩, ⟨T14⟩, ⟨427⟩, ⟨72T⟩, ⟨T28⟩, ⟨824⟩, ⟨83W⟩, ⟨W3T⟩, ⟨T39⟩, ⟨938⟩, ⟨758⟩, ⟨85T⟩, ⟨T5W⟩, ⟨W57⟩, ⟨4ET⟩, ⟨TE6⟩, ⟨6E8⟩, ⟨8E7⟩, ⟨7E4⟩$\}$.
Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 1, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2^{\mathrm{op}})$ are non-flexible, automorphism-free LDTS(13)s.

**Example A.3.** The DTS quasigroup that has automorphism group of order 39 is the Steiner quasigroup which comes from the cyclic STS(13) obtained from the starter blocks \{014\}, \{027\} under the action of the permutation
$(\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9},\mathtt{T},\mathtt{E},\mathtt{W})$.

**Example A.4.** The 2 DTS quasigroups that have automorphism group $C_{13}$ are defined by the triples obtained from the following starter blocks under the action of the permutation $(\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9},\mathtt{T},\mathtt{E},\mathtt{W})$. The starter blocks for $\mathcal{C}$ are ⟨105⟩, ⟨507⟩, ⟨703⟩, ⟨301⟩. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{C})$ and $(X, \mathcal{C}^{\mathrm{op}})$ are pure, non-flexible LDTS(13)s.

**Example A.5.** The 2 DTS quasigroups that have automorphism group $D_{12}$ of order 12 are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5})(\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9},\mathtt{T},\mathtt{E})$ and $(\mathtt{0},\mathtt{5})(\mathtt{1},\mathtt{4})(\mathtt{2},\mathtt{3})(\mathtt{6},\mathtt{8})(\mathtt{9},\mathtt{E})$. The starter blocks for $\mathcal{T}$ are \{018\}, \{024\}, \{03W\}, \{69W\}, and for $\mathcal{C}$ are ⟨60E⟩, ⟨E09⟩. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\mathrm{op}})$ are non-flexible LDTS(13)s.

**Example A.6.** The unique DTS quasigroup that has automorphism group $D_{10}$ of order 10 is defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4})(\mathtt{5},\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9})$

and $(1,4)(2,3)(6,9)(7,8)(T,E)$. The starter block for $\mathcal{C}_1$ is $\langle 0T4\rangle$, for $\mathcal{C}_2$ is $\langle 5T8\rangle$, and for $\mathcal{T}$ are $\{026\}$, $\{05W\}$, $\{078\}$, $\{TEW\}$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 0, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ is a flexible LDTS(13).

**Example A.7.** The 2 DTS quasigroups that have automorphism group $C_{10}$ can both be obtained from the starter blocks $\{017\}$, $\{05W\}$, $\{TEW\}$, $\langle 0T2\rangle$, $\langle 2E0\rangle$. The first LDTS is defined by the triples obtained from the starter blocks under the action of the permutation $(0,1,2,3,4,5,6,7,8,9)(T,E)$. The second LDTS is defined by the triples obtained from the starter blocks under the action of the permutation $(0,1,2,3,4,5,6,7,8,9)$. Both LDTS(13)s are flexible and each consists of 2 surface constituents of genus 0.

**Example A.8.** The 5 DTS quasigroups that have automorphism group $C_6$ are defined by the triples obtained from the following starter blocks under the action of the permutation $(0,1,2,3,4,5)(6,7,8,9,T,E)$.

(1) The starter blocks for $\mathcal{D}$ are $\{06W\}$, $\{68T\}$, $\langle 104\rangle$, $\langle 407\rangle$, $\langle 708\rangle$, $\langle 80E\rangle$, $\langle E0T\rangle$, $\langle T01\rangle$. Then $(X, \mathcal{D})$ and $(X, \mathcal{D}^{op})$ are flexible LDTS(13)s, each consisting of 3 surface constituents of genus 0.
(2) The starter blocks for $\mathcal{D}$ are $\{06W\}$, $\{09T\}$, $\{68T\}$, $\langle 104\rangle$, $\langle 40E\rangle$, $\langle E08\rangle$, $\langle 801\rangle$. Then $(X, \mathcal{D})$ and $(X, \mathcal{D}^{op})$ are flexible LDTS(13)s, each consisting of 3 surface constituents of genus 0.
(3) The starter blocks for $\mathcal{D}$ are $\{03W\}$, $\{68T\}$, $\{69W\}$, $\langle 106\rangle$, $\langle 607\rangle$, $\langle 702\rangle$, $\langle 20T\rangle$, $\langle T09\rangle$, $\langle 901\rangle$. Then $(X, \mathcal{D})$ is a non-flexible LDTS(13) consisting of a single surface constituent of genus 1.

**Example A.9.** The 7 DTS quasigroups that have automorphism group $S_3$ are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(0,1,2)(3,4,5)(6,7,8)(9,T,E)$ and $(0,3)(1,5)(2,4)(7,8)(T,E)$.

(1) The starter blocks for $\mathcal{T}$ are $\{678\}$, $\{69W\}$, $\{9TE\}$, for $\mathcal{C}_1$ are $\langle 061\rangle$, $\langle 163\rangle$, $\langle 0W2\rangle$, and for $\mathcal{C}_2$ are $\langle 094\rangle$, $\langle 491\rangle$, $\langle 198\rangle$, $\langle 893\rangle$. Then $\mathcal{C}_1$ is a surface constituent of genus 0, $\mathcal{C}_2$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ and $(X, \mathcal{T} \cup \mathcal{C}_1^{op} \cup \mathcal{C}_2)$ are non-flexible LDTS(13)s.
(2) The starter blocks for $\mathcal{T}$ are $\{05W\}$, $\{678\}$, $\{69W\}$, $\{9TE\}$, and for $\mathcal{C}$ are $\langle 061\rangle$, $\langle 164\rangle$, $\langle 46E\rangle$, $\langle E60\rangle$, $\langle 092\rangle$, $\langle 293\rangle$. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{op})$ are non-flexible LDTS(13)s.
(3) The starter blocks for $\mathcal{T}$ are $\{016\}$, $\{05W\}$, $\{678\}$, $\{69W\}$, $\{9TE\}$, and for $\mathcal{C}$ are $\langle 094\rangle$, $\langle 491\rangle$, $\langle 198\rangle$, $\langle 893\rangle$. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C})$ is a non-flexible LDTS(13).
(4) The starter blocks for $\mathcal{T}$ are $\{039\}$, $\{04E\}$, $\{057\}$, $\{678\}$, $\{69W\}$, $\{9TE\}$, and for $\mathcal{C}$ are $\langle 061\rangle$, $\langle 16E\rangle$, $\langle E63\rangle$, $\langle 0W2\rangle$. Then $\mathcal{C}$ is a surface constituent of genus 0, and $(X, \mathcal{T} \cup \mathcal{C})$ is a non-flexible LDTS(13).
(5) The starter blocks for $\mathcal{T}$ are $\{017\}$, $\{039\}$, $\{04E\}$, $\{05W\}$, $\{08T\}$, $\{678\}$, $\{69W\}$, $\{9TE\}$. Then $(X, \mathcal{T})$ is the non-cyclic STS(13).

**Example A.10.** The 8 DTS quasigroups that have automorphism group $C_5$ are defined by the triples obtained from the following starter blocks under the action of the permutation $(0,1,2,3,4)(5,6,7,8,9)$. The starter blocks for $\mathcal{C}_0$ are $\langle 0T1\rangle$, $\langle 1E0\rangle$, for $\mathcal{C}_1$ are $\langle 706\rangle$, $\langle 609\rangle$, $\langle 90W\rangle$, $\langle W07\rangle$, $\langle 5T7\rangle$, $\langle 5E6\rangle$, for $\mathcal{C}_2$ are $\langle 706\rangle$, $\langle 608\rangle$, $\langle 803\rangle$, $\langle 307\rangle$, $\langle 0W5\rangle$, $\langle 5W1\rangle$, $\langle 5T6\rangle$, $\langle 5E8\rangle$, for $\mathcal{T}_1$ are $\{025\}$, $\{TEW\}$, and $\mathcal{T}_2 = \{\{TEW\}\}$. Then $\mathcal{C}_0$ is a surface constituent of genus 0, $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 2, and $(X, \mathcal{T}_1 \cup \mathcal{C}_0 \cup \mathcal{C}_1)$, $(X, \mathcal{T}_1 \cup \mathcal{C}_0 \cup \mathcal{C}_1^{op})$, $(X, \mathcal{T}_1 \cup \mathcal{C}_0^{op} \cup \mathcal{C}_1)$,

$(X, \mathcal{T}_1 \cup \mathcal{C}_0^{\rm op} \cup \mathcal{C}_1^{\rm op})$, $(X, \mathcal{T}_2 \cup \mathcal{C}_0 \cup \mathcal{C}_2)$, $(X, \mathcal{T}_2 \cup \mathcal{C}_0 \cup \mathcal{C}_2^{\rm op})$, $(X, \mathcal{T}_2 \cup \mathcal{C}_0^{\rm op} \cup \mathcal{C}_2)$ and $(X, \mathcal{T}_2 \cup \mathcal{C}_0^{\rm op} \cup \mathcal{C}_2^{\rm op})$ are non-flexible LDTS(13)s.

**Example A.11.** The 17 DTS quasigroups that have automorphism group $C_2 \times C_2$ are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(\texttt{0},\texttt{1})(\texttt{2},\texttt{3})(\texttt{4},\texttt{5})(\texttt{6},\texttt{7})(\texttt{8},\texttt{9})$ and $(\texttt{0},\texttt{9})(\texttt{1},\texttt{8})(\texttt{2},\texttt{7})(\texttt{3},\texttt{6})(\texttt{4},\texttt{5})(\texttt{T},\texttt{E})$.

(1) The starter blocks for $\mathcal{T}$ are {048}, {45W}, {TEW}, for $\mathcal{D}_0$ are ⟨0T5⟩, ⟨5T8⟩, ⟨8T6⟩, ⟨6T2⟩, ⟨2T0⟩, for $\mathcal{C}_1$ are ⟨03W⟩, ⟨W38⟩, ⟨839⟩, ⟨930⟩, and for $\mathcal{C}_2$ are ⟨243⟩, ⟨346⟩. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 0, $\mathcal{D}_0$ consists of 2 surface constituents of genus 0, and $(X, \mathcal{T} \cup \mathcal{D}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2)$, $(X, \mathcal{T} \cup \mathcal{D}_0 \cup \mathcal{C}_1^{\rm op} \cup \mathcal{C}_2)$, $(X, \mathcal{T} \cup \mathcal{D}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2^{\rm op})$ and $(X, \mathcal{T} \cup \mathcal{D}_0 \cup \mathcal{C}_1^{\rm op} \cup \mathcal{C}_2^{\rm op})$ are non-flexible LDTS(13)s.

(2) The starter blocks for $\mathcal{T}$ are {01W}, {048}, {26W}, {45W}, {TEW}, for $\mathcal{C}_1$ are ⟨305⟩, ⟨507⟩, ⟨706⟩, ⟨603⟩, and for $\mathcal{C}_2$ are ⟨0T2⟩, ⟨2T5⟩, ⟨5T6⟩, ⟨6T8⟩, ⟨8T1⟩. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are surface constituents of genus 0, and $(X, \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ and $(X, \mathcal{T} \cup \mathcal{C}_1^{\rm op} \cup \mathcal{C}_2)$ are non-flexible LDTS(13)s.

(3) The starter blocks for $\mathcal{T}$ are {01W}, {048}, {27W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨305⟩, ⟨506⟩, ⟨607⟩, ⟨703⟩, ⟨0T2⟩, ⟨2T5⟩, ⟨5T7⟩, ⟨7T9⟩, ⟨9T0⟩. Then $\mathcal{C}$ is a surface constituent of genus 0, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\rm op})$ are non-flexible LDTS(13)s.

(4) The starter blocks for $\mathcal{T}$ are {01E}, {048}, {07T}, {09W}, {23W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨206⟩, ⟨603⟩, ⟨305⟩, ⟨502⟩, ⟨2T5⟩, ⟨5T3⟩. Then $\mathcal{C}$ is a surface constituent of genus 0, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\rm op})$ are non-flexible LDTS(13)s.

(5) The starter blocks for $\mathcal{T}$ are {01T}, {02E}, {048}, {09W}, {25T}, {26W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨305⟩, ⟨507⟩, ⟨706⟩, ⟨603⟩. Then $\mathcal{C}$ is a surface constituent of genus 0, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\rm op})$ are non-flexible LDTS(13)s.

(6) The starter blocks for $\mathcal{T}$ are {01W}, {048}, {23W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨206⟩, ⟨603⟩, ⟨305⟩, ⟨502⟩, ⟨0T9⟩, ⟨9T2⟩, ⟨2T5⟩, ⟨5T7⟩, ⟨7T0⟩. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\rm op})$ are non-flexible LDTS(13)s.

(7) The starter blocks for $\mathcal{T}$ are {01T}, {048}, {09W}, {25T}, {27W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨20E⟩, ⟨E03⟩, ⟨305⟩, ⟨507⟩, ⟨706⟩, ⟨602⟩. Then $\mathcal{C}$ is a surface constituent of genus 1, and $(X, \mathcal{T} \cup \mathcal{C})$ is a non-flexible LDTS(13).

(8) The starter blocks for $\mathcal{T}$ are {01E}, {048}, {09W}, {26W}, {45W}, {TEW}, and for $\mathcal{C}$ are ⟨203⟩, ⟨305⟩, ⟨506⟩, ⟨60T⟩, ⟨T07⟩, ⟨702⟩, ⟨2T5⟩, ⟨5T3⟩. Then $\mathcal{C}$ is a surface constituent of genus 2, and $(X, \mathcal{T} \cup \mathcal{C})$ and $(X, \mathcal{T} \cup \mathcal{C}^{\rm op})$ are non-flexible LDTS(13)s.

**Example A.12.** The system is defined by the triples obtained from the following starter blocks under the action of the permutation $(\texttt{0},\texttt{1},\texttt{2})(\texttt{3},\texttt{4},\texttt{5})(\texttt{6},\texttt{7},\texttt{8})(\texttt{9},\texttt{T},\texttt{E})$. The starter blocks for $\mathcal{C}$ are ⟨16E⟩, ⟨E63⟩, ⟨368⟩, ⟨862⟩, ⟨265⟩, ⟨561⟩, ⟨096⟩, ⟨69W⟩, ⟨W98⟩, ⟨893⟩, ⟨391⟩, ⟨195⟩, ⟨59T⟩, ⟨T90⟩, ⟨3W0⟩, ⟨0W5⟩, and $\mathcal{T} = \{$ {012}, {345} $\}$. Then $\mathcal{C}$ is a surface constituent of genus 3, and $(X, \mathcal{T} \cup \mathcal{C})$ is a non-flexible LDTS(13). The automorphism group of the DTS quasigroup is $C_3$.

**Example A.13.** The system is defined by the triples obtained from the following starter blocks under the action of the permutation $(\texttt{0},\texttt{1})(\texttt{2},\texttt{3})(\texttt{4},\texttt{5})(\texttt{6},\texttt{7})(\texttt{8},\texttt{9})(\texttt{T},\texttt{E})$. The starter blocks for $\mathcal{C}$ are ⟨065⟩, ⟨162⟩, ⟨260⟩, ⟨561⟩, ⟨084⟩, ⟨287⟩, ⟨382⟩, ⟨483⟩, ⟨580⟩, ⟨785⟩, ⟨0T9⟩, ⟨1T0⟩, ⟨2T1⟩, ⟨3T4⟩, ⟨4T5⟩, ⟨5T3⟩, ⟨6T7⟩, ⟨7T2⟩, ⟨8T6⟩, ⟨9T8⟩,

$\langle \texttt{0W2} \rangle$, $\langle \texttt{2W4} \rangle$, $\langle \texttt{4W6} \rangle$, $\langle \texttt{6W8} \rangle$, $\langle \texttt{8W1} \rangle$, and $\mathcal{T} = \{\{\texttt{TEW}\}\}$. Then $\mathcal{C}$ is a surface constituent of genus 4, and $(X, \mathcal{T} \cup \mathcal{C})$ is a non-flexible LDTS(13). The automorphism group of the DTS quasigroup is $C_2$.

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# TRIPLE SYSTEMS AND BINARY OPERATIONS

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Abstract. It is well known that given a Steiner triple system (STS) one can define a binary operation $*$ upon its base set by assigning $x * x = x$ for all $x$ and $x*y = z$, where $z$ is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems (MTS) as well as hybrid triple systems (HTS), where $(x, y)$ is considered to be ordered. In the case of STSs and MTSs, the operation is a quasigroup, however this is not necessarily the case for HTSs. In this paper we study the binary operation induced by HTSs. It turns out that each such operation $*$ satisfies

$$y \in \{x * (x * y), (x * y) * x\} \quad \text{and} \quad y \in \{(y * x) * x, x * (y * x)\}$$

for all $x$ and $y$ from the base set. We call every binary operation that fulfils this condition *hybridly symmetric*.

Not all idempotent hybridly symmetric operations can be obtained from HTSs. We show that these operations correspond to decompositions of a complete digraph into certain digraphs on three vertices. However, an idempotent hybridly symmetric quasigroup always comes from an HTS. The corresponding HTS is then called a *latin HTS* (LHTS). The core of this paper is the characterization of LHTSs and the description of their existence spectrum.

## 1. Introduction

Consider an ordered pair $(X, \mathcal{B})$, where $X$ is a set of points and $\mathcal{B}$ is a decomposition of the complete digraph on $X$ into *cyclic triples* $(a, b, c)$ and *transitive triples* $\langle a, b, c\rangle$. The cyclic triple $(a, b, c)$ consists of arrows (i.e. directed edges) $(a, b)$, $(b, c)$ and $(c, a)$, while a transitive triple $\langle a, b, c\rangle$ carries $(a, b)$, $(b, c)$ and $(a, c)$. If all triples in $\mathcal{B}$ are cyclic, then $(X, \mathcal{B})$ is called a *Mendelsohn triple system* (MTS). If all triples in $\mathcal{B}$ are transitive, then it is called a *directed triple system* (DTS). If we allow both cyclic and transitive triples to occur in $\mathcal{B}$, then the term *hybrid triple system* (HTS) is used, following Colbourn, Pulleyblank and Rosa [5]. However, the concept of an HTS seems to have appeared earlier (under a different name) in an article [19] of Lindner and Street, and later, independently, in [23].

Each HTS induces a binary operation, say $*$, upon its base set. For a cyclic triple $(a, b, c)$ set $a * b = c$, $b * c = a$ and $c * a = b$. For a transitive triple $\langle a, b, c\rangle$ set $a * b = c$, $b * c = a$ and $a * c = b$. The *induced operation* $*$ is assumed to be idempotent, i.e. $a * a = a$ holds for every $a$.

It is easy to see that the binary operation $*$ is induced by an MTS if and only if it is semisymmetric (i.e. $x * (y * x) = y$ for all $x$ and $y$). If a binary operation satisfies the semisymmetric law, then it is a quasigroup, and, as is well known [1, Remark 2.12], there is a one-to-one correspondence between MTS($n$)s and idempotent semisymmetric quasigroups of order $n$. In [9] and [8] we were concerned with a combinatorial and an algebraic description of those DTSs that yield a quasigroup. In this paper we give a similar description for HTSs that yield

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a quasigroup (Theorems 5.3 and 5.4). We call any such HTS a *latin hybrid triple system* (LHTS). The corresponding quasigroups can be described as idempotent hybridly symmetric quasigroups, i.e. idempotent quasigroups that fulfil

$$y \in \{x * (x * y), (x * y) * x\} \quad \text{and} \quad y \in \{(y * x) * x, x * (y * x)\}$$

for all $x$ and $y$ of the base set.

In fact, every binary operation $*$ that is induced by an HTS (regardless of whether it is a quasigroup or not) is hybridly symmetric. However, not all idempotent hybridly symmetric operations can be obtained from HTSs. We shall see that such an operation corresponds to an HTS if and only if it satisfies an additional condition (Proposition 3.5). It is then natural to ask if there is a combinatorial interpretation for idempotent hybridly symmetric operations. The answer is positive and points to the decompositions of $K_n^*$ into digraphs on three vertices as studied by Hartman and Mendelsohn [15]. There are exactly seven digraphs on three vertices which have the property that for any two distinct vertices $a$ and $b$ at least one of the arrows $(a, b)$ and $(b, a)$ is an edge of the digraph. It is possible to define $*$ for these digraphs similarly as above and, as we prove in Section 3, decompositions of $K_n^*$ into these seven digraphs correspond to idempotent hybridly symmetric operations (Theorem 3.3). The correspondence is not one-to-one, since if the decomposition contains a pair of triples with the same vertex set, say $\langle a, b, c\rangle$ and $\langle c, b, a\rangle$, then these can be replaced by a different pair of triples, say $\langle b, a, c\rangle$ and $\langle c, a, b\rangle$, however both systems induce the same binary operation. To get a one-to-one correspondence the notion of a coarse decomposition is needed (cf. Section 3).

The algebraic descriptions that are introduced in this paper have, admittedly, certain disadvantages: they cannot be used for decompositions into $\lambda$-fold digraphs if $\lambda > 1$, and they cannot be used to study mixed triple systems [11]. But there are also advantages. One of them is the ease with which examples can be produced by standard tools that generate first-order models. We exploit this feature in Section 4 where we give the number of isomorphism types for uniform systems that are based upon each of the seven digraphs on three elements, for a base set of 10 and less elements (with one exception). Also, in Section 6, we count the number of isomorphism types for proper LHTSs, again up to a base set of 10 elements (see also the Appendix). In doing so, we have used the algebraic description of LHTSs given in Theorem 5.4, which states that LHTSs correspond to idempotent binary operations $*$ satisfying

$$y = x * (x * y) = (y * x) * x \quad \text{or} \quad y = (x * y) * x = x * (y * x)$$

for all $x$ and $y$ of the base set.

In Section 7 we prove that a proper LHTS of order $n$ exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \geq 9$. The proof is constructive and yields LHTSs which are balanced in the sense that asymptotically half of the triples are transitive and the other half are cyclic.

## 2. Symmetric operations (totally, left, right, middle, hybridly)

The fact that Steiner triple systems (STS) are equivalent to totally symmetric idempotent quasigroups is often mentioned without realizing where the notion of total symmetricity comes from.

A quasigroup operation $*$ is said to be *totally symmetric* if $x * y = y * x = x/y = y/x = x\backslash y = y\backslash x$ for all $x$ and $y$. Here $\backslash$ and $/$ stand for the left and right division in the quasigroup. It is immediately clear that this class of quasigroups is determined already by the laws $x*y = y*x = y/x$. Instead of $y*x = y/x$ one can write $(y*x)*x = y$, and hence totally symmetric quasigroups can be fully described by

$$(y*x)*x = y \quad \text{and} \quad x*y = y*x.$$

A binary operation fulfilling these two laws is necessarily a quasigroup, and so there is no difference between totally symmetric (binary) operations and totally symmetric quasigroups. If the operation is idempotent (i.e. $x*x = x$ for every element $x$), then the sets $\{x, y, x*y\}$, $x \neq y$, determine an STS. The converse holds as well. The correspondence between STSs and totally symmetric loops (quasigroups with a unit) is also widely known.

The notion of totally symmetric quasigroups is natural. It is not really about a symmetry, but about the coincidence of all six parastrophic operations. The equational description of totally symmetric quasigroup seems to have been the main reason why the law $(y*x)*x = y$ has been called *right symmetric* and the law $x*(x*y) = y$ *left symmetric*. Admittedly, these terms are little illuminating in themselves. Nevertheless, they are widely accepted and in such circumstances it seems reasonable to call the law $x*(y*x) = y$ *middle symmetric*. However, this law is nearly always called semisymmetric since it represents a half of the characterization of totally symmetric quasigroups by $x*(y*x) = y$ and $x*y = y*x$.

**Lemma 2.1.** *Let $*$ be a binary operation upon a set $X$. If $x*(y*x) = y$ for all $x, y \in X$, then $(x*y)*x = y$ for all $x, y \in X$ as well.*

*Proof.* We have $(x*y)*x = (x*y)*(y*(x*y)) = y$. □

The term *hybridly symmetric* introduced above thus expresses the fact that for any $(x, y)$ we get an instance of left or middle symmetric law, and an instance of right or middle symmetric law.

From the lemma above we can also see that any semisymmetric binary operation is necessarily a quasigroup in which $y\backslash x = x*y = y/x$. If the operation is idempotent, then the sets $(x, y, x*y)$, $x \neq y$, determine an MTS. Thus there is a one-to-one correspondence between MTSs and idempotent semisymmetric binary operations.

## 3. Idempotent hybridly symmetric operations

We have started this paper by interpreting cyclic and transitive triples as digraphs upon three elements. In this way we shall also interpret a 3-element set $\{a, b, c\}$ and identify it with the complete digraph (arrows $(a, b)$, $(b, a)$, $(a, c)$, $(c, a)$, $(b, c)$ and $(c, b)$). There are four other digraphs on three elements we shall use:

- $\mathbf{d}(a, b, c)$ consists of $(a, b)$, $(b, a)$, $(b, c)$, $(c, b)$ and $(a, c)$;
- $\mathbf{t}(a, b, c)$ consists of $(a, b)$, $(b, c)$, $(a, c)$ and $(c, a)$;
- $\mathbf{i}(a, b, c)$ consists of $(a, b)$, $(c, b)$, $(a, c)$ and $(c, a)$; and
- $\mathbf{o}(a, b, c)$ consists of $(b, a)$, $(b, c)$, $(a, c)$ and $(c, a)$.

The following easy fact will be useful:

**Lemma 3.1.** *Suppose that $a$, $b$ and $c$ are three different elements. Then*

(i) $\mathbf{d}(a, b, c)$ *contains* $(a, c, b)$, $\langle a, c, b\rangle$, $\langle b, a, c\rangle$, *and* $\langle a, b, c\rangle$*;*
(ii) $\mathbf{t}(a, b, c)$ *contains* $(a, b, c)$ *and* $\langle a, b, c\rangle$*;*
(iii) $\mathbf{i}(a, b, c)$ *contains* $\langle a, c, b\rangle$ *and* $\langle c, a, b\rangle$*; and*

(iv) $\mathbf{o}(a,b,c)$ *contains* $\langle b,a,c\rangle$ *and* $\langle b,c,a\rangle$.

Let $X$ be a set and let $\mathcal{B}$ be a decomposition of the complete digraph upon $X$ into digraphs each of which is of the form $(a,b,c)$ or $\langle a,b,c\rangle$ or $\{a,b,c\}$ or $\mathbf{d}(a,b,c)$ or $\mathbf{t}(a,b,c)$ or $\mathbf{i}(a,b,c)$ or $\mathbf{o}(a,b,c)$, where $a$, $b$ and $c$ are three different elements of $X$. Such a decomposition will be called a 3-*decomposition* of $X$.

Given a 3-decomposition $\mathcal{B}$ define an operation $*$ so that $a*a=a$ for every $a\in X$, and

(1) if $(a,b,c)\in\mathcal{B}$, then $a*b=c$, $b*c=a$ and $c*a=b$;
(2) if $\langle a,b,c\rangle\in\mathcal{B}$, then $a*b=c$, $b*c=a$ and $a*c=b$;
(3) if $\{a,b,c\}\in\mathcal{B}$, then $a*b=b*a=c$, $b*c=c*b=a$ and $a*c=c*a=b$;
(4) if $\mathbf{d}(a,b,c)\in\mathcal{B}$, then $a*b=b*a=c$, $b*c=c*b=a$ and $a*c=b$;
(5) if $\mathbf{t}(a,b,c)\in\mathcal{B}$, then $a*b=c$, $b*c=a$ and $a*c=c*a=b$;
(6) if $\mathbf{i}(a,b,c)\in\mathcal{B}$, then $a*b=c$, $c*b=a$ and $a*c=c*a=b$; and
(7) if $\mathbf{o}(a,b,c)\in\mathcal{B}$, then $b*a=c$, $b*c=a$ and $a*c=c*a=b$.

The definition of $*$ is done explicitly for the purpose of reference. It obeys a general principle: if a triple in $\mathcal{B}$ contains the directed edge $(x,y)$, then $x*y$ is equal to the third element of the triple.

We see clearly that $*$ is well defined. We shall call it the binary operation *induced* by the 3-decomposition $\mathcal{B}$. Let us recall that an operation is hybridly symmetric if it satisfies $y\in\{x*(x*y),(x*y)*x\}$ and $y\in\{(y*x)*x,x*(y*x)\}$ for all $x$ and $y$ from the base set.

**Lemma 3.2.** *Every binary operation induced by a 3-decomposition is hybridly symmetric.*

*Proof.* We have to verify that $y\in\{x*u,u*x\}$ and $y\in\{v*x,x*v\}$, where $u=x*y$ and $v=y*x$. This is clear if $x=y$. Assume $x\neq y$. Then there exists an element of $\mathcal{B}$ with vertices $x$, $y$ and $u$, and an element of $\mathcal{B}$ with vertices $x$, $y$ and $v$. If the former element contains $(x,u)$, then $y=x*u$. If it contains $(u,x)$, then $y=u*x$. If the latter element contains $(v,x)$, then $y=v*x$. If it contains $(x,v)$, then $y=x*v$. □

**Theorem 3.3.** *An idempotent binary operation can be induced by a 3-decomposition if and only if it is hybridly symmetric.*

*Proof.* Let $*$ be an idempotent hybridly symmetric operation upon $X$. For each $a,b\in X$, $a\neq b$, there exists a triple $T(a,b)$ such that $c=a*b$ and

$T(a,b)=(a,b,c)$, $c*a=b$ and $b*c=a$;
$T(a,b)=\langle c,a,b\rangle$, $c*a=b$ and $c*b=a$;
$T(a,b)=\langle a,b,c\rangle$, $a*c=b$ and $b*c=a$; or
$T(a,b)=\langle a,c,b\rangle$, $a*c=b$ and $c*b=a$.

The existence of $T(a,b)$ is a direct consequence of the definition of a hybridly symmetric operation. However, the choice of $T(a,b)$ need not be unique. Let us call a triple *compatible* with $*$ if it can be obtained as $T(a,b)$ for some $a,b\in X$. It is easy to see that if $\langle a,b,c\rangle$ is a compatible triple, then it can serve not only as $T(a,b)$, but also as $T(a,c)$ and $T(b,c)$. Similarly, $(a,b,c)$ can be obtained as $T(a,b)$, $T(b,c)$ and $T(c,a)$.

Let now $\mathcal{B}'$ be the collection of all 3-element sets that carry a compatible triple. For each $C\in\mathcal{B}'$ consider the digraph obtained by unifying the digraphs which represent all compatible triples that yield $C$. Denote the collection of such digraphs by $\mathcal{B}$. Then $\mathcal{B}$ is a 3-decomposition of $X$ and this decomposition induces $*$. □

The 3-decomposition $\mathcal{B}$ defined in the proof of Theorem 3.3 has the property, that no two digraphs have the same vertex set. Such decompositions will be called *coarse*.

**Corollary 3.4.** *There is a one-to-one correspondence between idempotent hybridly symmetric binary operations upon a set $X$ and coarse 3-decompositions of the set $X$.*

**Proposition 3.5.** *Let $*$ be an idempotent hybridly symmetric binary operation that is induced by a coarse 3-decomposition $\mathcal{B}$ of a set $X$. The following are equivalent:*

(i) *if $x * y = y * x$, then $x * (x * y) = (x * y) * x$, for all $x, y \in X$;*
(ii) *the set $\mathcal{B}$ contains no element of the form $\mathbf{d}(a, b, c)$, $\mathbf{t}(a, b, c)$, $\mathbf{i}(a, b, c)$ or $\mathbf{o}(a, b, c)$; and*
(iii) *the operation $*$ is induced by an HTS.*

*Proof.* First we show that the presence of a triple of type $\mathbf{d}$, $\mathbf{t}$, $\mathbf{i}$ or $\mathbf{o}$ violates (i). Suppose that $\mathbf{d}(a, b, c) \in \mathcal{B}$. Then $a * b = b * a = c$, $a * c = b$ and $c * a \neq b$. For cases $\mathbf{t}$, $\mathbf{i}$ and $\mathbf{o}$ start from $a * c = c * a = b$. Then $a * b = c \neq b * a$ in $\mathbf{t}(a, b, c)$ and $\mathbf{i}(a, b, c)$, while $b * a = c \neq a * b$ in the case $\mathbf{o}(a, b, c)$. Thus (i) implies (ii).

To see (ii) $\Rightarrow$ (iii) is trivial: decompose every $\{a, b, c\}$ to $(a, b, c)$ and $(c, b, a)$ (alternatively to $\langle a, b, c\rangle$ and $\langle c, b, a\rangle$).

Assume that $*$ is induced by an HTS. If $x * y = y * x$ and $x \neq y$, then there exist $a, b, c \in X$ such that $\{x, y\} \subseteq \{a, b, c\}$ and the HTS contains either both $(a, b, c)$ and $(c, b, a)$, or both $\langle a, b, c\rangle$ and $\langle c, b, a\rangle$. The restriction of $*$ to $\{a, b, c\}$ yields a totally symmetric idempotent quasigroup on three elements, and hence (iii) implies (i). □

## 4. Uniform systems

Consider a 3-decomposition $\mathcal{B}$. Call the decomposition *uniform* if only one of the seven graphs is used. If all graphs are of the form $\{a, b, c\}$, then $\mathcal{B}$ is an STS. The form $(a, b, c)$ corresponds to MTS and $\langle a, b, c\rangle$ to DTS; coarse systems of both of these types are known as *pure*.

Suppose that the base set has $n$ elements. If all graphs are of the form $\mathbf{d}(a, b, c)$, then by amalgamating digraphs $\mathbf{d}(a, b, c)$ and $\mathbf{d}(c, d, a)$ we obtain a decomposition of $K_n$ into graphs $K_4 \setminus e$ (one edge removed from $K_4$). Such decompositions exist for all $n \equiv 0$ or $1 \pmod 5$, $n \neq 5$. Conversely a $K_4 \setminus e$ design may give rise to many uniform 3-decompositions of $K_n^*$ because the choice of orientation brings an additional degree of freedom.

If all graphs are of the form $\mathbf{t}(a, b, c)$, then these can be obtained from decompositions of $K_n$ into wheel graphs $W_r$, $r \geq 3$. A *wheel graph* $W_r$ is a graph with $r + 1$ vertices formed by connecting a single vertex to all vertices of an $r$-cycle.

The necessary and sufficient conditions for the existence of a coarse uniform system of order $n$ are given in Table 1 together with references. The table also gives the number of isomorphism types for coarse uniform systems of order up to 10 (with one exception). These results were obtained using the program Mace4 [21] which is part of the package Prover9, an automated theorem prover for first-order and equational logic. Given an algebraic description of a system, Mace4 can enumerate all structures of some finite order that satisfy the given set of formulas. Note that the case $\mathbf{o}(a, b, c)$ is not listed because it is dual to $\mathbf{i}(a, b, c)$.

| Form | Conditions | Ref. | Number of isomorphism types of order 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\{a,b,c\}$ | $n \equiv 1$ or $3 \pmod 6$ | [18] | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| $\langle a,b,c\rangle$ | $n \equiv 0$ or $1 \pmod 3$, $n \neq 3$ | [6] | 0 | 3 | 0 | 32 | 1016 | 0 | ? | ? |
| $(a,b,c)$ | $n \equiv 0$ or $1 \pmod 3$, $n \neq 3$ or $6$ | [2] | 0 | 1 | 0 | 0 | 2 | 0 | 7 | 60 |
| $\mathbf{d}(a,b,c)$ | $n \equiv 0$ or $1 \pmod 5$, $n \neq 5$ | [3] | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 92 |
| $\mathbf{t}(a,b,c)$ | $n \equiv 0$ or $1 \pmod 4$, $n \neq 5$ or $8$ | [15] | 0 | 1 | 0 | 0 | 0 | 0 | 16 | 0 |
| $\mathbf{i}(a,b,c)$ | $n \equiv 1 \pmod 4$ | [15] | 0 | 0 | 1 | 0 | 0 | 0 | 3 | 0 |

TABLE 1. Conditions for the existence of a coarse uniform system of order $n$ and the number of isomorphism types of order up to 10.

## 5. QUASIGROUPS

Let $*$ be an idempotent hybridly symmetric binary operation that is induced by a coarse 3-decomposition $\mathcal{B}$ of a set $X$. Each cyclic or transitive triple induces a partial binary operation and we call such a triple *compatible* if the partial binary operation agrees with $*$. The definition of compatible triples is thus the same as in the proof of Theorem 3.3.

For a transitive triple $\langle a,b,c\rangle$ call the directed edge $(a,b)$ the *initial* edge, $(b,c)$ the *terminal* edge and $(a,c)$ the *long* edge.

Consider now two triples that share two vertices, say $a$ and $b$. These triples are said to be in a *matching position* if

(1) both of them are cyclic; or
(2) both of them are transitive, one of the directed edges $(a,b)$ and $(b,a)$ is initial and the other edge is terminal; or
(3) both of them are transitive and both directed edges $(a,b)$ and $(b,a)$ are long.

The two triples *mismatch* if they are not in a matching position.

**Lemma 5.1.** *If there exist two compatible triples which share exactly two vertices and which mismatch, then $*$ is not a quasigroup operation.*

*Proof.* We shall enumerate all possible mismatching positions and explicitly show a violation of the quasigroup properties in each of them.

(a) $(a,b,c)$ and $\langle b,a,d\rangle$: $b*d = a = b*c$;
(b) $(a,b,c)$ and $\langle d,b,a\rangle$: $d*a = b = c*a$;
(c) $(a,b,c)$ and $\langle b,d,a\rangle$: $d*a = b = c*a$;
(d) $\langle a,b,c\rangle$ and $\langle b,a,d\rangle$: $b*c = a = b*d$;
(e) $\langle a,b,c\rangle$ and $\langle b,d,a\rangle$: $b*c = a = b*d$;
(f) $\langle c,a,b\rangle$ and $\langle d,b,a\rangle$: $c*a = b = d*a$; and
(g) $\langle c,a,b\rangle$ and $\langle b,d,a\rangle$: $c*a = b = d*a$.

□

**Lemma 5.2.** *If $\mathcal{B}$ contains an element in one of the forms $\mathbf{d}(a,b,c)$, $\mathbf{t}(a,b,c)$, $\mathbf{i}(a,b,c)$ or $\mathbf{o}(a,b,c)$, then the induced operation $*$ is not a quasigroup operation.*

*Proof.* In cases $\mathbf{d}$ and $\mathbf{t}$ there exists upon $\{a,b,c\}$ both a transitive triple compatible with $*$ and a cyclic triple compatible with $*$. Any compatible triple that has exactly two vertices from $\{a,b,c\}$ has to mismatch one of these two triples. Thus by Lemma 5.1, $*$ is not a quasigroup operation. To finish the proof it suffices to consider the case **i** since case **o** occurs in the mirror operation. If $\mathbf{i}(a,b,c) \in \mathcal{B}$, then both $\langle a,c,b\rangle$ and $\langle c,a,b\rangle$ are compatible triples. Suppose that $*$ is a quasigroup operation. Then Lemma 5.1 implies that $(b,a)$ should occur both as an initial edge and as a long edge in some transitive triple. That is not possible, of course. □

Hence all idempotent hybridly symmetric quasigroup operations $*$ can be obtained from HTSs, and so we will refer to them as *HTS quasigroups* for short. Quasigroups that can be obtained from DTSs have been called *DTS quasigroups*.

Call a triple occurring in an HTS *bidirectional* if there exists another triple in the system with the same vertex set, otherwise call it *unidirectional*. Let $*$ be determined by an HTS $(X,\mathcal{B})$. Denote by $\mathcal{F}$ the set of all $\{a,b,c\}$ such that $\{a,b,c\}$ is the vertex set of a unidirectional triple of $\mathcal{B}$. Consider now $\mathcal{F}$ as a set of faces. Each edge $\{a,b\}$ is incident to two faces, hence the faces can be sewn together along common edges to form a pseudosurface. By separating pinch points we obtain a surface, which can be partitioned into connected components. Call such a surface component *uniform* if all its triples are either cyclic, or transitive. From Lemma 5.1 we see that all components are uniform if $*$ yields a quasigroup.

**Theorem 5.3.** *Let $*$ be determined by an HTS $(X,\mathcal{B})$. Denote by $S_{p,q}$ the set of ordered pairs $(a,b)$ in positions $p$ and $q$ respectively of the transitive triples of $\mathcal{B}$. The following are equivalent:*

(i) *$*$ is a quasigroup operation;*
(ii) *whenever two triples in $\mathcal{B}$ share two vertices, they are in a matching position;*
(iii) *$S_{1,2} = S_{3,2}$, $S_{2,3} = S_{2,1}$ and $S_{1,3} = S_{3,1}$;*
(iv) *for each $\langle a,b,c\rangle \in \mathcal{B}$ there exist elements $a'$, $b'$, $c' \in X$ such that $\langle c',b,a\rangle$, $\langle c,b',a\rangle$, $\langle c,b,a'\rangle \in \mathcal{B}$.*

*Proof.* Assume that (i) holds. Clearly, whenever two triples in an HTS share the same vertex set, they are in a matching position. If they share exactly 2 vertices, then (ii) follows from Lemma 5.1.

Assume that (ii) holds. The set $S_{1,2}$ is a collection of all initial edges, thus $(a,b) \in S_{1,2}$ if and only if $(b,a)$ is a terminal edge in some transitive triple, i.e. $(a,b) \in S_{3,2}$. Analogously for the remaining equalities. Thus (ii) implies (iii).

Now assume that (iii) holds and let $\langle a,b,c\rangle \in \mathcal{B}$. Since $(a,b) \in S_{1,2} = S_{3,2}$, there exists $c'$ such that $\langle c',b,a\rangle \in \mathcal{B}$. Analogously for $b'$ and $a'$.

Finally assume that (iv) holds. For $*$ to be a quasigroup operation we need to show that for any $a,b \in X$, $a \neq b$, there exist $x$ and $y$ such that $b*x = a$ and $y*b = a$. If $(a,b)$ occurs in a cyclic triple, then $(b,a)$ also occurs in a cyclic triple, thus $(a,b,x)$, $(b,a,y) \in \mathcal{B}$, i.e. we can set $x = a*b$ and $y = b*a$. If $(a,b)$ is an initial edge, then use $\langle a,b,x\rangle$, $\langle y,b,a\rangle \in \mathcal{B}$. If $(a,b)$ is a terminal edge, then use $\langle y,a,b\rangle$, $\langle b,a,x\rangle \in \mathcal{B}$. If $(a,b)$ is a long edge, then use $\langle a,y,b\rangle$, $\langle b,x,a\rangle \in \mathcal{B}$. □

Note that the conditions given in the above theorem are reminiscent of those ($S_{1,2} = S_{2,1}$, $S_{2,3} = S_{3,2}$, and $S_{1,3} = S_{3,1}$) for a class of directed triple systems, so called *Mendelsohn directed triple systems*, the existence of which was discussed in [13].

Recall that an HTS that induces a quasigroup operation is called a latin hybrid triple system (LHTS). Similarly a DTS which induces a quasigroup operation will be called a *latin directed triple system* (LDTS). A *partial* LHTS is defined as a partial HTS which satisfies condition (iii) of Theorem 5.3. Partial LDTSs are defined analogously. An important example of such a system is the partial LDTS(6) consisting of the transitive triples $\{\langle x, y, z\rangle, \langle x, y', z'\rangle, \langle x', y, z'\rangle, \langle x', y', z\rangle, \langle z', y', x'\rangle, \langle z, y, x'\rangle, \langle z, y', x\rangle, \langle z', y, x\rangle\}$. We will denote this set by $\mathcal{P}$.

Given a partial LHTS $\mathcal{B}$, every transitive triple $\langle a, b, c\rangle$ can be replaced by a cyclic triple $(a, b, c)$. This yields a partial MTS since $\langle c, b', a\rangle$ is turned into $(c, b', a)$, $\langle c', b, a\rangle$ into $(c', b, a)$ and $\langle c, b, a'\rangle$ into $(c, b, a')$. We shall call this the *underlying* (partial) MTS of $\mathcal{B}$ and denote it by $\overline{\mathcal{B}}$.

Notice that an LHTS yields the same surface as its underlying MTS. Clearly, the surface obtained from an MTS is orientable, hence any LHTS yields an orientable surface as well. This is generally not the case for HTSs.

Given a set of triples $\mathcal{B}$, if every cyclic triple $(a, b, c)$ is replaced by $(c, b, a)$ and every transitive triple $\langle a, b, c\rangle$ is replaced by $\langle c, b, a\rangle$, then the resulting set is called the *converse* of $\mathcal{B}$ and is denoted $\mathcal{B}^{\mathcal{R}}$. Clearly the converse of a (partial) LHTS is also a (partial) LHTS.

**Theorem 5.4.** *Let $*$ be an idempotent binary operation upon a non-empty set $X$. The following is equivalent:*

(i) *the operation $*$ is a hybridly symmetric quasigroup operation;*
(ii) *the operation $*$ is induced by an LHTS;*
(iii) $y = x * (x * y) = (y * x) * x$ *or* $y = (x * y) * x = x * (y * x)$ *for all* $x, y \in X$.

*Proof.* The implication (i) ⇒ (ii) follows from Lemma 5.2. Let $\mathcal{B}$ be an LHTS. If $(a, b, c) \in \mathcal{B}$, then by Theorem 5.3 (ii) there exists an element $c'$ such that $(b, a, c') \in \mathcal{B}$. Then $(a * b) * a = b = a * (b * a)$. Now assume $\langle a, b, c\rangle \in \mathcal{B}$ and let $a'$, $b'$ and $c'$ be as in Theorem 5.3. Then $a * (a * b) = b = (b * a) * a$, $a * (a * c) = c = (c * a) * a$ and $(b * c) * b = c = b * (c * b)$. We see that (ii) implies (iii).

Assume that condition (iii) holds. This condition simply states that for any $(a, b)$ we get an instance of left and right symmetric law or an instance of middle symmetric law. Thus the operation is hybridly symmetric. Furthermore, the condition states that for any $x, y \in X$ there exist $u, v \in X$ such that $x * u = y$ and $v * x = y$, where $u = x * y$ and $v = y * x$, or $u = y * x$ and $v = x * y$. Thus $*$ is a quasigroup operation. □

This paper is mainly about connections of triple systems and idempotent binary operations. In Theorem 5.4 we have hence proved the equivalence of points (i) and (iii) in this context. The equivalence holds for all quasigroups, without the assumption of idempotency:

**Proposition 5.5.** *Let $Q(*)$ be a quasigroup. The following conditions are equivalent:*

(i) $y \in \{x * (x * y), (x * y) * x\}$ *and* $y \in \{(y * x) * x, x * (y * x)\}$ *for all* $x, y \in Q$, *and*
(ii) $y = x * (x * y) = (y * x) * x$ *or* $y = (x * y) * x = x * (y * x)$ *for all* $x, y \in Q$.

*Proof.* The implication (ii) ⇒ (i) is clear. Assume that (i) holds and fix $x, y \in Q$. The situations that need treatment are $y = x * (x * y) = x * (y * x)$ and $y = (x * y) * x = (y * x) * x$. Both of them yield $x * y = y * x$. So let us assume that $x * y = y * x$ and let $z = x * y$. Then $y = x * z$ or $y = z * x$ and we need to show that in fact $y = x * z = z * x$. If $y = x * z$, then $x * (x * z) = x * y = z = y * x = (x * z) * x$. From (i) we also know that $z = (z * x) * x$ or $z = x * (z * x)$. Either one of these cases implies that $z * x = x * z$. Similarly, if $y = z * x$, then $x * (z * x) = z = (z * x) * x$. Since $z = x * (x * z)$ or $z = (x * z) * x$, we again get $z * x = x * z$. □

As we discussed in Section 2, the quasigroups obtained from STSs and MTSs are always semisymmetric. It can be seen from Lemma 2.1 that they also satisfy $x*(y*x) = (x*y)*x$ for all $x$ and $y$. This is known as the *flexible* law. Generally, HTS quasigroups need not be flexible. Flexibility in DTS quasigroups has been previously studied in [9], where the existence spectrum of non-flexible LDTS($n$)s was determined to be $n \equiv 0$ or $1 \pmod 3$, $n \neq 3, 4, 6, 7, 10$. We now derive the necessary and sufficient condition for an LHTS to yield a flexible quasigroup.

**Lemma 5.6.** *Let $Q(*)$ be a quasigroup induced by an HTS $\mathcal{B}$. If $x*(y*x) = (x*y)*x$ for every $\langle x, x*y, y\rangle \in \mathcal{B}$, then $Q(*)$ is flexible.*

*Proof.* For the purposes of clarity, we shall write simply $xy$ instead of $x*y$. We need to show that the restricted assumption of flexibility implies that $a(ba) = (ab)a$ for any pair $(a, b)$, where $a$ and $b$ are distinct elements of $Q$.

If the pair $(a, b)$ lies in a cyclic triple, then by Theorem 5.3 the pair $(b, a)$ also lies in a cyclic triple. Thus $(a, b, ab), (b, a, ba) \in \mathcal{B}$ and $a(ba) = b = (ab)a$.

For transitive triples, it suffices to consider the cases $(x, xy)$ and $(xy, y)$, where $\langle x, xy, y\rangle \in \mathcal{B}$. The latter case is immediate since by Theorem 5.3 we have $\langle y, xy, y(xy)\rangle \in \mathcal{B}$, hence $(xy)(y(xy)) = y = x(xy) = ((xy)y)(xy)$. For the former case note that $(x(xy))x = yx$. By Theorem 5.3 we have $\langle y, yx, x\rangle \in \mathcal{B}$ and $\langle x, yx, x(yx)\rangle \in \mathcal{B}$, hence $x(x(yx)) = yx$. Now $(x(xy))x = yx = x(x(yx)) = x((xy)x)$. □

If an LHTS yields a flexible quasigroup, we call it a *flexible* LHTS. Denote by $O_k$ the set $\{\langle x_i, u, x_{i+1}\rangle : i \in \mathbb{Z}_k\} \cup \{\langle x_{i+1}, v, x_i\rangle : i \in \mathbb{Z}_k\}$, where $k \geq 3$ and the points $u$, $v$, $x_0$, …, $x_{k-1}$ are pairwise distinct. The set $O_k$ is a partial LHTS consisting of a single surface component. This component has the form of a $k$-gonal bipyramid, i.e. a graph with $k+2$ vertices formed by connecting the vertex $u$ and the vertex $v$ to all vertices of a $k$-cycle. Note that the set of transitive triples $\mathcal{P}$ mentioned above is a set of the form $O_4$.

**Theorem 5.7.** *An LHTS $(X, \mathcal{B})$ is flexible if and only if the unidirectional transitive triples in $\mathcal{B}$ can be partitioned into subsets of the form $O_k$, where $k \geq 3$.*

*Proof.* Consider a unidirectional triple $\langle x_0, u, x_1\rangle \in \mathcal{B}$. Assuming $X$ is finite, by Theorem 5.3 there exist pairwise distinct points $v$, $x_0$, $x_1$, …, $x_{k-1}$, $k \geq 3$, such that $\langle x_1, v, x_0\rangle \in \mathcal{B}$ and $\langle x_i, u, x_{i+1}\rangle \in \mathcal{B}$ for all $i \in \mathbb{Z}_k$. If $(X, \mathcal{B})$ is flexible and $\langle x_{i+1}, v, x_i\rangle \in \mathcal{B}$ for some $i \in \mathbb{Z}_k$, then by Theorem 5.3 the triple $\langle v*x_{i+1}, v, x_{i+1}\rangle$ belongs to $\mathcal{B}$ as well and $v*x_{i+1} = (x_{i+1}*x_i)*x_{i+1} = x_{i+1}*(x_i*x_{i+1}) = x_{i+1}*u = x_{i+2}$. Thus by induction we get a set of the form $O_k$.

If the unidirectional transitive triples in $\mathcal{B}$ can be partitioned into subsets of the form $O_k$, then for any triple of the form $\langle x_i, u, x_{i+1}\rangle \in \mathcal{B}$, $i \in \mathbb{Z}_k$, we have $x_i*(x_{i+1}*x_i) = x_{i-1} = (x_i*x_{i+1})*x_i$, and similarly for $\langle x_{i+1}, v, x_i\rangle$ we have $x_{i+1}*(x_i*x_{i+1}) = x_{i+2} = (x_{i+1}*x_i)*x_{i+1}$. Thus the condition given in Lemma 5.6 is satisfied. □

## 6. Enumeration of LHTSs

If an LHTS contains a pair of bidirectional triples, then these can be replaced by a different pair of bidirectional triples to form a potentially non-isomorphic LHTS, yet both systems will generate the same quasigroup; see [9, Example 2.4]. For purposes of enumeration, it therefore makes more sense to count HTS quasigroups rather than the LHTSs from which they come.

Call a set $\{a, b, c\}$ a *Steiner triple* if it is the vertex set of a bidirectional triple. Given an LHTS $(X, \mathcal{B})$, replace every pair of bidirectional triples with a Steiner triple to obtain a coarse 3-decomposition of $X$. The block set $\mathcal{B}$ can then be expressed as a union of a set of Steiner triples $\mathcal{T}$, a set of unidirectional transitive triples $\mathcal{D}$ and a set of unidirectional cyclic triples $\mathcal{M}$. This representation gives a one-to-one correspondence between LHTSs and HTS quasigroups.

For each $x \in X$ denote by $f(x)$, $m(x)$ and $l(x)$ the number of unidirectional transitive triples in $\mathcal{B}$ such that $x$ appears in the first, middle and last position, respectively. Note that $f(x) = l(x)$ for any $x \in X$. By $c(x)$ denote the number of unidirectional cyclic triples in $\mathcal{B}$ containing $x$. By $t(x)$ denote the number of Steiner triples in $\mathcal{B}$ containing $x$. Clearly $\sum_{x \in X} t(x) = 3|\mathcal{T}|$. The outdegree of any $x \in X$ can be expressed

$$n - 1 = 2f(x) + m(x) + c(x) + 2t(x). \tag{1}$$

**Lemma 6.1.** *Let $(X, \mathcal{B})$ be an LHTS, then $f(x) \neq 1$, $m(x) \notin \{1, 2\}$, $c(x) \notin \{1, 2\}$, for any $x \in X$.*

*Proof.* Suppose that $m(x) \in \{1, 2\}$ and $\langle y, x, z\rangle \in \mathcal{B}$ is unidirectional. By Theorem 5.3 there exist unidirectional triples $\langle z', x, y\rangle, \langle z, x, y'\rangle \in \mathcal{B}$. Clearly $m(x) \neq 1$, so $m(x) = 2$. Then $\langle z', x, y\rangle = \langle z, x, y'\rangle$, thus $z = z'$ and $\langle y, x, z\rangle$ is bidirectional, which is a contradiction. Analogously for $c(x) \in \{1, 2\}$.

Suppose that $f(x) = 1$ and $\langle x, y, z\rangle \in \mathcal{B}$ is unidirectional. By Theorem 5.3 there exist unidirectional triples $\langle z', y, x\rangle, \langle z, y', x\rangle \in \mathcal{B}$. Since $l(x) = f(x) = 1$, the two triples coincide, thus $z = z'$ and $\langle x, y, z\rangle$ is bidirectional, which is a contradiction. □

It is well known [22] that an MTS($n$) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$. The non-existence of an underlying MTS(6) implies the non-existence of an LHTS(6). Since any MTS($n$) is also an LHTS($n$), it follows that:

**Proposition 6.2.** *An LHTS(n) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$.*

It would, however, be of more interest to study LHTSs containing both unidirectional cyclic triples, as well as unidirectional transitive triples. We call such systems *proper* LHTSs. Similarly an MTS or an LDTS containing at least one unidirectional triple will be called *proper*. Clearly there exists no proper LHTS(3). There exists no proper LHTS(4), since the underlying MTS(4), which is unique up to isomorphism, has only one component. Any proper LHTS must have at least two components, one consisting of transitive triples and one consisting of cyclic triples.

**Proposition 6.3.** *There exists no proper LHTS(7).*

*Proof.* Suppose that $(\mathbb{Z}_7, \mathcal{B})$ is a proper LHTS(7). By definition, we require that $\mathcal{B}$ contains some unidirectional cyclic triple $(x, y, z)$. Then there exist points $x', y', z' \in X \setminus \{x, y, z\}$ such that $(z', y, x)$, $(z, y', x)$, $(z, y, x') \in \mathcal{B}$. There are therefore at least four pairwise distinct points in $X$ that appear in a cyclic triple.

From Equation (1) and Lemma 6.1 we see that the possible values for $(f(x), m(x), c(x), t(x))$ are $(0, 0, 0, 3)$, $(0, 0, 4, 1)$, $(0, 0, 6, 0)$, $(0, 3, 3, 0)$, $(0, 4, 0, 1)$, $(0, 6, 0, 0)$, $(2, 0, 0, 1)$ or $(3, 0, 0, 0)$. In a proper LHTS there exists a point $x$ such that $f(x) > 0$. Let us examine the two possible cases:

If there exists a point of type $(3, 0, 0, 0)$, say 0, then without loss of generality $\langle 0, 1, 2\rangle$, $\langle 0, 3, 4\rangle$, $\langle 0, 5, 6\rangle \in \mathcal{B}$. Points 2, 4 and 6 have $f(x) = l(x) > 0$, i.e. they are of type $(2,0,0,1)$ or $(3,0,0,0)$. At least four points must have $c(x) > 0$, but this is not possible.

If there exists a point of type $(2,0,0,1)$, say 0, then without loss of generality $\langle 0,1,2\rangle$, $\langle 0,3,4\rangle$, $\{0,5,6\} \in \mathcal{B}$. Points 2 and 4 have $f(x) = l(x) > 0$, i.e. they are of type $(2,0,0,1)$. At least four points must have $c(x) > 0$, thus points 5 and 6 are of type $(0,0,4,1)$ and points 1 and 3 are of type $(0,3,3,0)$. But then 3 does not divide $\sum_{x\in X} t(x) = 5$, which is a contradiction. □

Table 2 shows the number of non-isomorphic HTS quasigroups of each kind for some small orders. The proper LHTSs are listed only up to order 10, because at order 12 the combinatorial explosion takes over. These quasigroups were found by the program Mace4 [21]. Using the algebraic description of an HTS quasigroup in Theorem 5.4 (iii), Mace4 can enumerate all structures of some finite order that satisfy the given formulas. Isomorphic quasigroups were removed using the GAP [10] package LOOPS [24].

| Order | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|
| STS | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 |
| Proper MTS | 0 | 1 | 0 | 3 | 19 | 241 | 9 801 188 | 13 710 290 114 |
| Proper LDTS | 0 | 0 | 0 | 1 | 3 | 0 | 2 | 1 206 967 |
| Proper LHTS | 0 | 0 | 0 | 0 | 7 | 14 | ? | ? |

Table 2. Number of non-isomorphic HTS quasigroups induced by the four types of triple systems.

A system that is isomorphic to its converse is called *self-converse*. Two systems are said to be *equivalent* if they are isomorphic or if one is isomorphic to the converse of the other. The number of inequivalent, rather than non-isomorphic, MTS($n$)s has already been tabulated in literature for all $n \leq 12$, see [22, 20, 7]. The number of non-isomorphic MTS($n$)s for all $n \leq 12$, as well as both inequivalent and non-isomorphic MTS(13)s are given in [17].

The LDTS($n$)s of order $n \leq 12$ can be found in [9]. The smallest proper LDTS is of order 7. This system, which is unique up to isomorphism, is given in the example below. In [8] the LDTS(13)s are classified according to the genera of their surface components and according to their automorphism group. Some examples of LDTS(13)s which may be of particular interest are given in [8] as well; namely all systems having an automorphism group of order greater than or equal to 4.

**Example 6.4.** Proper flexible LDTS(7). For simplicity, we omit commas from the triples.
$X = \{0,1,2,3,4,5,6\}$.
$\mathcal{T} = \big\{\{012\}, \{034\}, \{056\}\big\}$ and
$\mathcal{D} = \big\{\langle 315\rangle, \langle 514\rangle, \langle 416\rangle, \langle 613\rangle, \langle 326\rangle, \langle 624\rangle, \langle 425\rangle, \langle 523\rangle\big\}$.

There exist exactly 7 proper LHTS(9)s and 14 proper LHTS(10)s that are pairwise non-isomorphic. If we count only inequivalent systems then we have 6 proper LHTS(9)s and 10 proper LHTS(10)s. All of these systems are given in the appendix. Table 3 gives the classification according to their automorphism group.

| Order | Aut($Q$) | Number of types | Flexible | Self-converse |
|---|---|---|---|---|
| 9 | $C_3$ | 2 | 0 | 2 |
| 9 | $D_{10}$ | 3 | 3 | 1 |
| 9 | $A_4$ | 2 | 0 | 2 |
| 10 | $C_1$ | 3 | 1 | 1 |
| 10 | $C_3$ | 5 | 1 | 1 |
| 10 | $C_6$ | 2 | 2 | 0 |
| 10 | $C_3 \times C_3$ | 4 | 0 | 4 |

TABLE 3. Classification of the isomorphism types of proper HTS quasigroups of orders 9 and 10 according to their automorphism group.

## 7. Existence of proper LHTSs

Proper LHTS($n$)s of order $n \equiv 1 \pmod 3$, $n \geq 40$, can be easily constructed using pairwise balanced designs. A *pairwise balanced design* PBD($n, K$) of order $n$ with block sizes from $K \subseteq \mathbb{N}$ is a pair $(X, \mathcal{B})$, where $X$ is a finite set of cardinality $n$ and $\mathcal{B}$ is a family of subsets of $X$ called *blocks*, such that (1) if $B \in \mathcal{B}$ then $|B| \in K$ and (2) every pair of distinct elements from $X$ occurs in exactly one block of $\mathcal{B}$. We will need the following result.

**Theorem 7.1** (Rees and Stinson [25])**.** *Let $n \geq 3k + 1$ and*

(1) *$n \equiv 1$ or $4 \pmod{12}$, $k \equiv 1$ or $4 \pmod{12}$ and $k \neq 4$, or*
(2) *$n \equiv 7$ or $10 \pmod{12}$ and $k \equiv 7$ or $10 \pmod{12}$.*

*Then there exists a PBD($n, \{4, k\}$) containing exactly one block of size $k$.*

To construct a proper LHTS($n$) of order $n \equiv 1 \pmod 3$, proceed as follows. For $n \equiv 1$ or $4 \pmod{12}$, $n \geq 40$, take a PBD($n, \{4, 13\}$) containing one block of size 13. On each 4-block place the triples of an MTS(4) and on the 13-block place the proper LHTS(13) given in Example A.6. Similarly, for $n \equiv 7$ or $10 \pmod{12}$, $n \geq 22$, take a PBD($n, \{4, 7\}$) containing one block of size 7, and this time place the proper LDTS(7) given in Example 6.4 on the 7-block.

Replacing a single 7-block or a single 13-block means that most of the triples will be cyclic. It would be of more interest to construct LHTS($n$)s having both many unidirectional cyclic triples as well as many unidirectional transitive triples. The constructions of proper LDTS($n$)s given in [9] yield systems that have asymptotically $\frac{1}{3}n^2$ unidirectional triples. It is easy to see that using Theorem 7.1 we can use these to obtain a proper LHTS($n$) with asymptotically as many as $\frac{1}{27}n^2$ unidirectional transitive triples for any $n \equiv 1 \pmod 3$, $n \geq 40$.

We can obtain LHTSs with even more transitive triples using the following result.

**Theorem 7.2** (Hoffman and Lindner [16])**.** *Let $n, m \equiv 0$ or $1 \pmod 3$, $n \geq 2m + 1$ and $n, m \neq 6$. Then there exists an MTS(n) having an MTS(m) as a subsystem.*

Let $n \equiv 0$ or $1 \pmod 3$, $n \geq 25$. Then for $n$ odd we can take an MTS($n$) having an MTS($\frac{1}{2}(n-1)$) as a subsystem and for $n$ even we can take an MTS($n$) having an MTS($\frac{1}{2}(n-4)$) subsystem and replace the subsystem with a proper LDTS. This yields an LHTS($n$) containing asymptotically $\frac{1}{12}n^2$ unidirectional transitive triples. However, the statement of Theorem 7.2 does not tell us anything about the number of unidirectional cyclic triples in the resulting LHTS. Upon examining the proof of Lemma 2.6 in [16] it can be seen that the resulting LHTS will contain asymptotically $\frac{1}{4}n^2$ unidirectional cyclic triples.

In the remainder of this section we construct proper LHTS($n$)s in which the number of triples of each type is asymptotic to $\frac{1}{6}n^2$, i.e. the ratio is one to one. This method also has the advantage that it can be utilized to create other asymptotic ratios. The construction techniques are of two types. Both produce LHTSs which are flexible. The first of these uses the so-called "doubling" construction for Steiner triple systems. It deals with the residue classes 3 and 7 (mod 12). The details are given in the proof of the following proposition.

**Proposition 7.3.** *There exists a proper flexible LHTS(n) for all* $n \equiv 3$ *or* 7 (mod 12), $n \geq 15$.

*Proof.* Let $s = \frac{1}{4}(n-3)$, $s \equiv 0$ or 1 (mod 3), $s \geq 3$. Take an STS($2s+1$), $(X, \mathcal{B})$. Let $X' = \{x' : x \in X\}$, $Y = X \cup X' \cup \{\infty\}$ and construct a collection of triples $\mathcal{B}'$ as follows. For each of the $\frac{1}{3}s(2s+1)$ blocks $\{x, y, z\} \in \mathcal{B}$, assign $\{x, y, z\}$, $\{x, y', z'\}$, $\{x', y, z'\}$, $\{x', y', z\} \in \mathcal{B}'$. Further let $\{x, x', \infty\} \in \mathcal{B}'$ for all $x \in X$. Then $(Y, \mathcal{B}')$ is an STS($n$). In order to obtain a proper LHTS($n$) take $\frac{1}{3}s(s-1)$ of the Pasch configurations of the form $\{\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\}\}$ given above and replace each one with the set of transitive triples $\mathcal{P} = \{\langle x, y, z\rangle, \langle x, y', z'\rangle, \langle x', y, z'\rangle, \langle x', y', z\rangle, \langle z', y', x'\rangle, \langle z, y, x'\rangle, \langle z, y', x\rangle, \langle z', y, x\rangle\}$, then take the remaining $\frac{1}{3}s(s+2)$ Pasch configurations and replace each one with the set of cyclic triples $\overline{\mathcal{P}} = \{(x, y, z), (x, y', z'), (x', y, z'), (x', y', z), (z', y', x'), (z, y, x'), (z, y', x), (z', y, x)\}$.

This yields a LHTS($4s+3$) with $\frac{1}{6}(n-3)(n+5)$ unidirectional cyclic triples, $\frac{1}{6}(n-3)(n-7)$ unidirectional transitive triples and $\frac{1}{2}(n-1)$ Steiner triples. All unidirectional transitive triples are obtained from the set $\mathcal{P}$, which is a set of the form $O_4$. Therefore, by Theorem 5.7, the LHTS is flexible. □

The second construction of LHTS($n$)s uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall that a 3-GDD of type $g^u$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where $V$ is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of $V$ into $u$ subsets of cardinality $g$ called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. We will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set $V$ being of cardinality $v = gu + m$ and the partition $G$ being into $u$ subsets of cardinality $g$ and one set of cardinality $m$. Necessary and sufficient conditions for 3-GDDs of type $g^u$ were determined in [14] and for 3-GDDs of type $g^u m^1$ in [4]; a convenient reference is [12] where the existence of all the GDDs that are used can be verified.

**Proposition 7.4.** *There exists a proper flexible LHTS(n) for all* $n \equiv 0$ *or* 4 (mod 12), $n \geq 12$.

*Proof.* Let $u = \frac{1}{4}n$, $u \equiv 0$ or 1 (mod 3), $u \geq 3$. Take a 3-GDD of type $2^u$ and inflate each point by a factor of 2. On each inflated group place an MTS(4). Take half of the $\frac{2}{3}u(u-1)$ inflated blocks and on each one place the set of transitive triples $\mathcal{P}$ with the three sets of points $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$ as the inflated points in the three groups. On each of the remaining inflated blocks place the set of cyclic triples $\overline{\mathcal{P}}$. This yields an LHTS($4u$) with $\frac{1}{6}n(n+2)$ unidirectional cyclic triples and $\frac{1}{6}n(n-4)$ unidirectional transitive triples. Once again, all unidirectional transitive triples are obtained from the set $\mathcal{P}$, thus the LHTS is flexible. □

The technique described in the proof of the previous proposition is used to construct all of the remaining infinite classes of LHTSs. In each case we start with a 3-GDD of type $g^u$ or $g^u m^1$, where the group sizes $g$ and $m$ are fixed and the number of groups $u$ is $(n-c)/(2g)$ for some fixed $c \in \mathbb{Z}$. Sets of eight transitive triples $\mathcal{P}$ are placed on one half of the inflated blocks and sets of eight cyclic triples $\overline{\mathcal{P}}$ are placed on the remaining half of the inflated blocks. The total number of inflated blocks is at least $\frac{1}{6}g^2u(u-1)$. Thus the constructions given below yield systems where the number of unidirectional triples of each type is asymptotically $\frac{1}{6}n^2$.

The transitive triples in all of the remaining constructions come from the set $\mathcal{P}$ or from some flexible LHTS. Thus, by Theorem 5.7, the resulting LHTSs are always flexible.

**Proposition 7.5.** *There exists a proper flexible LHTS(n) for all* $n \equiv 1 \pmod{12}$, $n \neq 1$.

*Proof.* A proper flexible LHTS(13) is given in Example A.6.

For $n = 25$, take a 3-GDD of type $4^3$, inflate each point by a factor of 2 and adjoin an extra point $\infty$. On each inflated group together with the point $\infty$ place an MTS(9). On each inflated block place the set of transitive triples $\mathcal{P}$ or the set of cyclic triples $\overline{\mathcal{P}}$.

For $n \geq 37$, take a 3-GDD of type $6^u$, $u \geq 3$, inflate each point by a factor of 2 and adjoin an extra point $\infty$. On each inflated group together with the point $\infty$ place an MTS(13). On each inflated block place the set of transitive triples $\mathcal{P}$ or the set of cyclic triples $\overline{\mathcal{P}}$. □

**Proposition 7.6.** *There exists a proper flexible LHTS(n) for all* $n \equiv 9 \pmod{12}$.

*Proof.* A proper flexible LHTS(9) is given in Example A.2.

For $n > 9$ take a 3-GDD of type $3^{2s+1}$, $s \geq 1$, inflate each point by a factor of 2 and adjoin three extra points $\infty_1$, $\infty_2$, $\infty_3$. On each inflated group together with the three extra points place an STS(9) in such a way that the points $\infty_1$, $\infty_2$, $\infty_3$ are identified with a Steiner triple in each STS(9). On each inflated block place the set of transitive triples $\mathcal{P}$ or the set of cyclic triples $\overline{\mathcal{P}}$. □

**Proposition 7.7.** *There exists a proper flexible LHTS(n) for all* $n \equiv 10 \pmod{12}$.

*Proof.* A proper flexible LHTS(10) is given in Example A.4.

For $n > 10$ take a 3-GDD of type $3^{2s+1}$, $s \geq 1$, inflate each point by a factor of 2 and adjoin four extra points $\infty_1$, $\infty_2$, $\infty_3$, $\infty_4$. Take one of the flexible LHTS(10)s given in Example A.4 containing an MTS(4) as a subsystem. On each inflated group together with the four extra points place a copy of the LHTS(10) in such a way that the points $\infty_1$, $\infty_2$, $\infty_3$, $\infty_4$ are identified with the MTS(4) subsystem in each LHTS(10). On each inflated block place the set of transitive triples $\mathcal{P}$ or the set of cyclic triples $\overline{\mathcal{P}}$. □

**Proposition 7.8.** *There exists a proper flexible LHTS(n) for all* $n \equiv 6 \pmod{12}$, $n \neq 6$.

*Proof.* A proper flexible LHTS(18) is given in Example A.7.

For $n = 30$, take a 3-GDD of type $5^3$, inflate each point by a factor of 2. On each inflated group place an MTS(10). On each inflated block place the set $\mathcal{P}$ or $\overline{\mathcal{P}}$.

For $n > 30$ take a 3-GDD of type $3^{2s}7^1$, $s \geq 2$, inflate each point by a factor of 2 and adjoin four extra points $\infty_1$, $\infty_2$, $\infty_3$, $\infty_4$. Take one of the flexible LHTS(10)s given in Example A.4 and the flexible LHTS(18) given in Example A.7. Both of these systems contain an MTS(4) as a subsystem. On each inflated group of cardinality 6 together with the four extra points place a copy of the LHTS(10) and on the inflated group of cardinality 14 together with the four extra points place the LHTS(18) in such a way that the points $\infty_1$, $\infty_2$, $\infty_3$, $\infty_4$ are identified with the MTS(4) subsystem in each LHTS(10) as well as in the LHTS(18). On each inflated block place the set of transitive triples $\mathcal{P}$ or the set of cyclic triples $\overline{\mathcal{P}}$. □

Collecting together the existence results in this section gives the following:

**Theorem 7.9.** *A proper LHTS(n) exists if and only if* $n \equiv 0$ *or* $1 \pmod 3$ *and* $n \geq 9$.

## Appendix. Examples of HTS quasigroups

The following examples give all proper LHTSs of orders 9 and 10 up to isomorphism, as well as the proper LHTSs of orders 13 and 18 that are needed to complete the existence spectrum. For simplicity we omit commas from the triples.

In the following three examples let $X = \{0, 1, \ldots, 8\}$.

**Example A.1.** The 2 proper HTS quasigroups of order 9 that have automorphism group $C_3$ are defined by the triples obtained from the following starter blocks under the action of the permutation $(0,1,2)(3,4,5)$. The starter blocks for $\mathcal{M}$ are (012), (026), and for $\mathcal{D}$ are $\langle 405\rangle$, $\langle 508\rangle$, $\langle 803\rangle$, $\langle 307\rangle$, $\langle 704\rangle$, $\langle 365\rangle$, and $\mathcal{T} = \{\{678\}\}$. Then $\mathcal{M}$ is a surface component of genus 0, $\mathcal{D}$ is a surface component of genus 1, and $(X, \mathcal{T} \cup \mathcal{M} \cup \mathcal{D})$ and $(X, \mathcal{T} \cup \mathcal{M}^{\mathcal{R}} \cup \mathcal{D})$ are non-flexible LHTS(9)s. Both of these systems are self-converse.

**Example A.2.** The 3 proper HTS quasigroups of order 9 that have automorphism group $D_{10}$ of order 10 are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(0,1,2,3,4,5)$ and $(0,1)(2,4)(5,6)(7,8)$. The starter block for $\mathcal{D}_1$ is $\langle 051\rangle$, for $\mathcal{D}_2$ is $\langle 072\rangle$, and for $\mathcal{M}$ are (567), (578). Then $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{M}$ are surface components of genus 0, and $(X, \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{M})$, $(X, \overline{\mathcal{D}_1} \cup \mathcal{D}_2 \cup \mathcal{M})$ and $(X, \mathcal{D}_1 \cup \overline{\mathcal{D}_2} \cup \mathcal{M})$ are flexible LHTS(9)s. The first of these systems is self-converse, while the other two are not.

**Example A.3.** The 2 proper HTS quasigroups of order 9 that have automorphism group $A_4$ are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(0,1,2)(3,4,5)$ and $(0,1)(2,6)(3,4)(5,7)$. The starter block for $\mathcal{T}$ is $\{678\}$, for $\mathcal{M}$ is (012), and for $\mathcal{D}$ is $\langle 405\rangle$. Then $\mathcal{M}$ and $\mathcal{D}$ are surface components of genus 0, and $(X, \mathcal{T} \cup \mathcal{M} \cup \mathcal{D})$ and $(X, \mathcal{T} \cup \mathcal{M}^{\mathcal{R}} \cup \mathcal{D})$ are non-flexible LHTS(9)s. Both of these systems are self-converse.

In the following two examples let $X = \{0, 1, \ldots, 9\}$.

**Example A.4.** Let $\mathcal{T}_1 = \{\{038\}, \{148\}, \{258\}\}$, $\mathcal{T}_2 = \{\{038\}\}$,
$\mathcal{D}_1 = \{\langle 061\rangle, \langle 162\rangle, \langle 263\rangle, \langle 364\rangle, \langle 465\rangle, \langle 560\rangle, \langle 075\rangle, \langle 574\rangle, \langle 473\rangle, \langle 372\rangle, \langle 271\rangle, \langle 170\rangle\}$,
$\mathcal{D}_2 = \{\langle 106\rangle, \langle 605\rangle, \langle 507\rangle, \langle 701\rangle, \langle 127\rangle, \langle 723\rangle, \langle 326\rangle, \langle 621\rangle, \langle 347\rangle, \langle 745\rangle, \langle 546\rangle, \langle 643\rangle\}$,

$\mathcal{M}_0 = \{(678), (689), (697), (798)\}$,
$\mathcal{M}_1 = \{(024), (092), (294), (490)\}$, $\mathcal{M}_2 = \{(135), (193), (395), (591)\}$,
$\mathcal{M}_3 = \{(029), (042), (094), (248), (259), (285), (135), (149), (158), (184), (193), (395)\}$. Then $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are surface components of genus 0, and $(X, \mathcal{T}_1 \cup \mathcal{D}_1 \cup \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2)$ and $(X, \mathcal{T}_1 \cup \mathcal{D}_1 \cup \mathcal{M}_0^{\mathcal{R}} \cup \mathcal{M}_1 \cup \mathcal{M}_2)$ are the LHTS(10)s with automorphism group $C_6$ generated by the permutation $(0, 1, 2, 3, 4, 5)$. Furthermore $(X, \mathcal{T}_1 \cup \mathcal{D}_1 \cup \mathcal{M}_0 \cup \mathcal{M}_1^{\mathcal{R}} \cup \mathcal{M}_2)$, $(X, \mathcal{T}_1 \cup \mathcal{D}_2 \cup \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2)$, $(X, \mathcal{T}_1 \cup \mathcal{D}_2 \cup \mathcal{M}_0^{\mathcal{R}} \cup \mathcal{M}_1 \cup \mathcal{M}_2)$, $(X, \mathcal{T}_1 \cup \mathcal{D}_2 \cup \mathcal{M}_0 \cup \mathcal{M}_1^{\mathcal{R}} \cup \mathcal{M}_2)$ and $(X, \mathcal{T}_1 \cup \mathcal{D}_2 \cup \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2^{\mathcal{R}})$ are the LHTS(10)s with automorphism group $C_3$ generated by the permutation $(0, 2, 4)(1, 3, 5)$; and $(X, \mathcal{T}_2 \cup \mathcal{D}_1 \cup \mathcal{M}_0 \cup \mathcal{M}_3)$, $(X, \mathcal{T}_2 \cup \mathcal{D}_2 \cup \mathcal{M}_0 \cup \mathcal{M}_3)$ and $(X, \mathcal{T}_2 \cup \mathcal{D}_2 \cup \mathcal{M}_0^{\mathcal{R}} \cup \mathcal{M}_3)$ are the automorphism-free LHTS(10)s. All systems containing $\mathcal{D}_1$ are flexible, while those containing $\mathcal{D}_2$ are non-flexible. The two flexible systems with automorphism group $C_3$ and $C_1$ are self-converse, while the other eight are not.

**Example A.5.** The 4 proper HTS quasigroups of order 10 that have automorphism group $C_3 \times C_3$ are defined by the triples obtained from the following starter blocks under the action of the group generated by the permutations $(0, 1, 2)(3, 4, 5)(6, 7, 8)$ and $(0, 1, 2)(3, 5, 4)$. The starter blocks for $\mathcal{M}_1$ are (012), (091), for $\mathcal{M}_2$ are (345), (394), for $\mathcal{M}_3$ are (678), (697), and for $\mathcal{D}$ are $\langle 307 \rangle$, $\langle 705 \rangle$. Then $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are surface components of genus 0, $\mathcal{D}$ is a surface component of genus 1, and $(X, \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{D})$, $(X, \mathcal{M}_1^{\mathcal{R}} \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{D})$, $(X, \mathcal{M}_1 \cup \mathcal{M}_2^{\mathcal{R}} \cup \mathcal{M}_3 \cup \mathcal{D})$ and $(X, \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3^{\mathcal{R}} \cup \mathcal{D})$ are non-flexible LHTS(10)s. All four systems are self-converse.

**Example A.6.** Let $X = \mathbb{Z}_{10} \cup \{\infty_1, \infty_2, \infty_3\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$, with $\infty_1$, $\infty_2$, $\infty_3$ as fixed points.
The starter blocks for $\mathcal{T}$ are $\{0\,5\,\infty_3\}$, $\{\infty_1\,\infty_2\,\infty_3\}$, for $\mathcal{M}$ are $(0\,1\,3)$, $(0\,3\,2)$, and for $\mathcal{D}$ are $\langle 0\,\infty_1\,4 \rangle$, $\langle 0\,\infty_2\,6 \rangle$.
Then $(X, \mathcal{T} \cup \mathcal{M} \cup \mathcal{D})$ is a proper flexible LHTS(13), with $|\mathcal{M}| = 20$, $|\mathcal{D}| = 20$ and $|\mathcal{T}| = 6$.

**Example A.7.** Let $X = \mathbb{Z}_{14} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$, with $\infty_1$, $\infty_2$, $\infty_3$, $\infty_4$ as fixed points.
The starter blocks for $\mathcal{M}_0$ are $(\infty_1\,\infty_2\,\infty_3)$, $(\infty_1\,\infty_3\,\infty_4)$, $(\infty_1\,\infty_4\,\infty_2)$, $(\infty_2\,\infty_4\,\infty_3)$, for $\mathcal{M}_1$ are $(0\,1\,3)$, $(0\,3\,\infty_4)$, $(0\,4\,\infty_2)$, $(0\,5\,1)$, $(0\,7\,5)$, and for $\mathcal{D}$ are $\langle 0\,\infty_1\,6 \rangle$, $\langle 0\,\infty_3\,8 \rangle$.
Then $(X, \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{D})$ is a proper flexible LHTS(18), with $|\mathcal{M}_0 \cup \mathcal{M}_1| = 74$ and $|\mathcal{D}| = 28$.

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# CYCLIC AND ROTATIONAL LATIN HYBRID TRIPLE SYSTEMS

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Abstract. It is well known that given a Steiner triple system (STS) one can define a binary operation $*$ upon its base set by assigning $x * x = x$ for all $x$ and $x * y = z$, where $z$ is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems (MTS), directed triple systems (DTS) as well as hybrid triple systems (HTS), where $(x, y)$ is considered to be ordered. In the case of STSs and MTSs the operation yields a quasigroup, however this is not necessarily the case for DTSs and HTSs. A DTS or an HTS which induces a quasigroup is said to be Latin. In this paper we study Latin DTSs and Latin HTSs which admit a cyclic or a rotational automorphism. We prove the existence spectra for these systems as well as the existence spectra for their pure variants. As a side result we also obtain the existence spectra of pure cyclic and pure rotational MTSs.

## 1. Introduction

A *Steiner triple system* of order $n$, STS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points taken from $V$ such that every pair of distinct points from $V$ appears in precisely one triple. Given an STS $(V, \mathcal{B})$ one can define a binary operation $*$ on the set $V$ by assigning $x * x = x$ for all $x \in V$ and $x * y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The induced operation satisfies the identities

$$x * x = x, \quad y * (x * y) = x, \quad x * y = y * x$$

for all $x$ and $y$ in $V$. Any binary operation satisfying these three identities is called an *idempotent totally symmetric quasigroup*. The process described above is reversible. Given an idempotent totally symmetric quasigroup one can obtain an STS by assigning $\{x, y, x * y\} \in \mathcal{B}$ for all $x$, $y \in V$, $x \neq y$. Thus there is a one-to-one correspondence between Steiner triple systems and idempotent totally symmetric quasigroups or *Steiner quasigroups*, as they are commonly known [1, Remark 2.12].

If we consider oriented triples then there are two possibilities. A *cyclic triple* $(x, y, z)$ contains the directed edges $(x, y)$, $(y, z)$ and $(z, x)$. A *transitive triple* $\langle x, y, z\rangle$ contains the directed edges $(x, y)$, $(y, z)$ and $(x, z)$; we assign these edges the colours red, green and blue, respectively.

A *hybrid triple system* of order $n$, HTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclic and transitive triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. If every ordered pair of points from $V$ appears in *at most* one triple, then $(V, \mathcal{B})$ is called a *partial* HTS. An HTS($n$) can also be thought of as a decomposition of the complete digraph on $n$ vertices into oriented triples which are either transitive triples or cyclic triples. The term hybrid triple system was

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first used in [6] but the concept appeared earlier under the name *ordered triple system* [11] and later under the name *oriented triple system* [12]. The notation $c$-HTS($n$) is often used to denote an HTS($n$) containing $c$ cyclic triples. An HTS($n$) which contains only cyclic triples, i.e. $\frac{n(n-1)}{3}$-HTS($n$), is known as a *Mendelsohn triple system*, MTS($n$). An HTS($n$) which contains only transitive triples, i.e. 0-HTS($n$), is known as a *directed triple system*, DTS($n$).

Every HTS induces a binary operation $*$ upon its point set $V$. For a cyclic triple $(x, y, z)$ set $x * y = z$, $y * z = x$ and $z * x = y$. For a transitive triple $\langle x, y, z\rangle$ set $x * y = z$, $y * z = x$ and $x * z = y$. The *induced operation* $*$ is assumed to be idempotent, i.e. $x * x = x$ holds for every $x \in V$.

In the case of MTSs, the induced operation yields a *semisymmetric* quasigroup, i.e. it satisfies $x * (y * x) = y$ for all $x$ and $y$ in $V$. It is well known [1, Remark 2.12] that there is a one-to-one correspondence between MTSs and idempotent semisymmetric quasigroups or *Mendelsohn quasigroups*, as they are also known. For DTSs and HTSs, however, the induced operation may or may not yield a quasigroup. If a DTS or an HTS induces a quasigroup, then it is said to be *Latin*, to signify that the operation table forms a Latin square, and $(V, *)$ is then called a *DTS-quasigroup* or an *HTS-quasigroup*, respectively.

Latin directed triple systems (LDTS) were introduced in [9], where it was shown that an LDTS($n$) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 4$, 6 or 10. The algebraic and geometric aspects of LDTSs were studied in [7]. Together these two papers also give enumeration results for all orders less than or equal to 13.

Latin hybrid triple systems (LHTS) were introduced in [8]. An LHTS($n$) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$. If in addition $n \geq 9$, then there exists a proper LHTS($n$). An LHTS is said to be *proper* if the induced quasigroup is neither a Mendelsohn quasigroup nor a DTS-quasigroup. Similarly, a DTS is said to be *proper* if the induced quasigroup is not a Steiner quasigroup.

If we ignore the ordering of the triples in an HTS then we obtain a twofold triple system. An HTS is said to be *pure* if its underlying twofold triple system contains no repeated blocks. An HTS($n$) is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length $n$ and it is said to be *rotational* if it admits an automorphism consisting of a cycle of length $n-1$ and one fixed point. Table 1 gives an overview of the existence spectra for various cyclic and rotational triple systems. The main goal of this paper is to prove the existence spectra of pure and proper, cyclic and rotational, LDTSs and LHTSs. In [10] Gardner et al. determined the existence spectrum of cyclic and rotational $c$-HTS($n$)s for all admissible values of $c$. In Propositions 3.9 and 4.14 we state the admissible values of $c$ for LHTS($n$)s, but we do not go so far as to prove existence for each of these values. Existence is proven only for the minimum non-zero values of $c$.

It is easy to see that any STS can be turned into an LHTS by replacing every unordered triple $\{x, y, z\}$ with a pair of cyclic triples $(x, y, z)$ and $(z, y, x)$ or a pair of transitive triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$. The resulting LHTS then induces a Steiner quasigroup, and so this construction is of little interest. Instead, the existence proofs in this paper aim at constructing pure systems. If a pure system does not exist for a particular order, then the construction aims for the next best thing, which is minimising the number of repeated blocks in the underlying twofold triple system.

The following section introduces some additional properties and terminology surrounding LHTSs. Sections 3 and 4 deal with the structure of cyclic and rotational LHTSs and LDTSs, and with the necessary conditions for their existence.

| Triple system | Conditions | References |
|---|---|---|
| Cyclic STS | $n \equiv 1$ or $3 \pmod 6$ and $n \neq 9$ | [13] |
| Cyclic MTS | $n \equiv 1$ or $3 \pmod 6$ and $n \neq 9$ | [5] |
| Cyclic DTS | $n \equiv 1, 4$ or $7 \pmod{12}$ | [4] |
| Cyclic HTS | $n \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$ and $n \neq 9$ | [12] |
| Rotational STS | $n \equiv 3$ or $9 \pmod{24}$ | [15] |
| Rotational MTS | $n \equiv 1, 3$ or $4 \pmod 6$ and $n \neq 10$ | [2] |
| Rotational DTS | $n \equiv 0 \pmod 3$ | [3] |
| Rotational HTS | $n \equiv 0$ or $1 \pmod 3$ | [12] |

Table 1. The necessary and sufficient conditions for the existence of various triple systems.

Finally, Section 5 gives a number of constructions proving the existence of these systems.

## 2. Preliminaries

The correspondence between HTSs or DTSs and the induced binary operations is not one-to-one, since if the system contains a pair of triples with the same point set, say $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$, then these can be replaced by a different pair of triples, say $\langle y, x, z\rangle$ and $\langle z, x, y\rangle$, however both systems induce the same binary operation. Call a triple occurring in an HTS *bidirectional* if there exists another triple in the system with the same point set, otherwise call it *unidirectional*. The point set of a bidirectional triple will be called a *Steiner triple*. In the following, we will always replace any pair of bidirectional triples, say $(x, y, z)$ and $(z, y, x)$, with their underlying Steiner triple $\{x, y, z\}$. The block set of an HTS then consists of three types of triples: Steiner triples, unidirectional cyclic triples and unidirectional transitive triples. This view of HTSs allows for a more precise study of these systems and results in a one-to-one correspondence between HTSs and the induced binary operations.

An HTS is said to be *pure* if it contains no Steiner triples. A DTS is proper if and only if it contains at least one unidirectional triple. An HTS is proper if and only if it contains at least one unidirectional cyclic triple and at least one unidirectional transitive triple.

The following theorem proven in [8] gives a combinatorial characterisation of LDTSs and LHTSs.

**Theorem 2.1.** *Let $*$ be induced by an HTS (or a DTS) $(V, \mathcal{B})$. The following are equivalent:*

(i) *$*$ is a quasigroup operation;*
(ii) *for each $\langle x, y, z\rangle \in \mathcal{B}$ there exist elements $x'$, $y'$, $z' \in V$ such that $\langle z', y, x\rangle$, $\langle z, y', x\rangle$ and $\langle z, y, x'\rangle$ belong to $\mathcal{B}$ as well.*

Let $*$ be determined by an HTS $(V, \mathcal{B})$. Denote by $\mathcal{F}$ the set of all $\{x, y, z\}$ such that $\{x, y, z\}$ is the point set of a unidirectional triple of $\mathcal{B}$. Consider now $\mathcal{F}$ as a set of triangular faces. Each edge $\{u, v\}$ is incident to two faces, hence the faces can be sewn together along common edges to form a pseudosurface. By splitting pinch points we obtain a surface, which can be partitioned into connected components. For a more detailed description of this process see [7]. Call a surface component *uniform* if all its triples are either cyclic, or transitive. From Theorem 2.1 we see that all components are uniform if $*$ yields a quasigroup.

When referring to an element of $V$ the terms *point* or *element* will be used interchangeably, whilst the term *vertex* will be used when referring to vertices of the induced surface. Thus as a result of splitting pinch points, one point of an HTS may correspond to multiple vertices of the induced surface.

A *partial* LHTS is defined as a partial HTS which satisfies condition (ii) in Theorem 2.1. Given a partial LHTS $\mathcal{B}$, every transitive triple $B = \langle x, y, z\rangle$ can be replaced by a cyclic triple $\overline{B} = (x, y, z)$. This yields a partial MTS since $\langle z, y', x\rangle$ is turned into $(z, y', x)$, $\langle z', y, x\rangle$ into $(z', y, x)$ and $\langle z, y, x'\rangle$ into $(z, y, x')$. We shall call this the *underlying* (partial) MTS of $\mathcal{B}$.

Notice that an LHTS yields the same surface as its underlying MTS. Clearly, the surface obtained from an MTS is orientable, hence any LHTS yields an orientable surface as well. This is generally not the case for HTSs.

The automorphism group of an LHTS will appear as a subgroup of the automorphism group of its underlying MTS. The automorphism groups of the two systems need not be the same. For example the unique proper LDTS(7) has an automorphism group of order 8, whilst its underlying MTS has an automorphism group of order 24.

Let $(V, \mathcal{B})$ be an LHTS($n$). For each $x \in V$ denote by $f(x)$ and $m(x)$ the number of transitive triples in $\mathcal{B}$ such that $x$ appears in the first and middle position, respectively. By $c(x)$ denote the number of cyclic triples in $\mathcal{B}$ containing $x$. By $s(x)$ denote the number of Steiner triples in $\mathcal{B}$ containing $x$. The number of unidirectional transitive triples can then be expressed as $\sum_{x\in V} f(x)$ or as $\sum_{x\in V} m(x)$. The number of unidirectional cyclic triples can be expressed as $\frac{1}{3}\sum_{x\in V} c(x)$. The outdegree of any $x \in V$ can be expressed

$$n - 1 = 2f(x) + m(x) + c(x) + 2s(x). \tag{1}$$

## 3. Cyclic LHTSs

Recall that an LHTS, $(V, \mathcal{B})$, is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length $n$. Let $V = \mathbb{Z}_n$ and $\alpha = (0, 1, \ldots, n-1)$ be an automorphism of $\mathcal{B}$. Since $\alpha$ acts transitively on $V$, we have $nf(0) = \sum_{x\in V} f(x) = \sum_{x\in V} m(x) = nm(0)$. Thus $f(x) = m(x)$ for every $x \in V$ and from (1) we get

$$n - 1 = 3f(x) + c(x) + 2s(x). \tag{2}$$

A cyclic MTS($n$) exists if and only if $n \equiv 1$ or $3 \pmod 6$ and $n \neq 9$ [5]. The non-existence of an underlying cyclic MTS($n$) for some $n$ implies the non-existence of a cyclic LHTS($n$). Since any cyclic MTS($n$) is also a cyclic LHTS($n$), it follows that:

**Lemma 3.1.** *A cyclic LHTS(n) exists if and only if* $n \equiv 1$ *or* $3 \pmod 6$ *and* $n \neq 9$.

We can associate any cyclic triple $(x, y, z)$ or transitive triple $\langle x, y, z\rangle$ with a *difference triple* $[y-x, z-y, x-z]$, and any Steiner triple $\{x, y, z\}$ with a pair of distinct difference triples $[y-x, z-y, x-z]$ and $[x-y, z-x, y-z]$. Note that any transitive triple is associated with the same difference triple as its underlying cyclic triple. We shall consider the difference triples $[a, b, c]$, $[b, c, a]$ and $[c, a, b]$ to be equivalent. Any difference triple $[a, b, c]$ satisfies $a + b + c \equiv 0 \pmod n$, we say that the triple is *balanced.*

For a cyclic LHTS, an *orbit* of a triple $B \in \mathcal{B}$ can be defined as the set $\{\, \alpha^i(B) \mid i = 0, 1, \ldots, n-1 \,\}$, or equivalently as the set of all blocks that have the same difference triple as $B$ or the same pair of difference triples as $B$ if $B$ is a Steiner triple. Thus difference triples correspond to orbits of blocks and the differences correspond to orbits of edges. Notice that the orbit of an edge is single coloured, thus we can assign a colour to each difference. By Theorem 2.1 the difference $a$ is red if and only if $-a$ is green, $a$ is blue if and only if $-a$ is blue and $a$ is colourless if and only if $-a$ is colourless. The colours of the differences in the difference triple $[a, b, c]$ are either red, green and blue, respectively, or an even permutation of these colours, or all three differences are colourless.

The *length* of an orbit is its cardinality. An orbit of length $n$ is said to be *full*, otherwise *short*. The orbit of any transitive triple is full. It is easy to check that if a block $B$ of a cyclic LHTS is associated with the difference triple $[a, a, b]$, $a, b \in \mathbb{Z}_n$, then $b = a$ and the orbit of $B$ is short. A cyclic triple lies in a short orbit if and only if its difference triple is $[\frac{n}{3}, \frac{n}{3}, \frac{n}{3}]$ or $[\frac{2n}{3}, \frac{2n}{3}, \frac{2n}{3}]$.

**Proposition 3.2.** *Let $(V, \mathcal{B})$ be an LHTS(n) and let $D$ be the set of its difference triples.*

(1) *If $[a, b, c]$, $[-a, -c, -b] \in D$, then these two difference triples are associated with the orbit of a Steiner triple.*
(2) *If $[a, b, c]$, $[-a, -b, -c] \in D$, $b \neq c$, then these two difference triples are associated with orbits consisting of cyclic triples, which together form a set of surface components such that each component has genus 1 and every vertex has degree 6.*

*Proof.* (1) The two difference triples are associated with the orbits of the triples $B_1$ and $B_2$ respectively, such that $\overline{B}_1 = (0, b, -a)$ and $\overline{B}_2 = (0, -a, b)$. These are bidirectional, therefore $B_1$ and $B_2$ are bidirectional as well.

(2) The two difference triples are associated with the orbits of the triples $B_1$ and $B_2$ respectively, such that $\overline{B}_1 = (0, b, -a)$ and $\overline{B}_2 = (0, -a, c)$, which are unidirectional as long as $b \neq c$.

Assume that $[a, b, c]$ is associated with the orbit of a transitive triple and without loss of generality assume that $c$ is blue. Then $-c$ is blue and both $b$ and $-b$ are green, which is a contradiction. Both difference triples are therefore associated with orbits of cyclic triples.

The cyclic triples form a closed surface $S$. For each $x \in V$ there are exactly six triples in $S$ that contain the point $x$, namely $(x-a, x, x+b)$, $(x+b, x, x-c)$, $(x-c, x, x+a)$, $(x+a, x, x-b)$, $(x-b, x, x+c)$ and $(x+c, x, x-a)$. These six triples form a hexagon with $x$ at its center. The oriented surface formed by the two orbits therefore has no pinch points. Thus there are $n$ vertices, $2n$ faces and $3n$ edges. The sum of the genera of the surface components is equal to the number of surface components. Since the surface contains exactly one vertex for each point in $V$ and $\alpha$ acts transitively on $V$, all components of $S$ have the same genus. Therefore each component has genus 1.

□

If the set of difference triples of an LHTS($n$) contains both $[\frac{n}{3}, \frac{n}{3}, \frac{n}{3}]$ and $[\frac{2n}{3}, \frac{2n}{3}, \frac{2n}{3}]$, then these are associated with a short orbit of a Steiner triple, namely the Steiner triple $\{0, \frac{n}{3}, \frac{2n}{3}\}$.

We now see, that a cyclic LHTS($n$) induces a partitioning of $\{1, \ldots, n-1\}$ into balanced triples and at most two singletons ($\{\frac{n}{3}\}$ and $\{\frac{2n}{3}\}$). If $n \equiv 0 \pmod 3$, then both singletons must be present in the partition, and so $\{0, \frac{n}{3}, \frac{2n}{3}\} \in \mathcal{B}$. If $n \equiv 1 \pmod 3$, then no singleton may be present in the partition, and so all orbits are full.

Since the order $n$ of an LHTS must satisfy $n \equiv 1$ or $3 \pmod 6$, we have

**Lemma 3.3.** *If a pure cyclic LHTS(n) exists, then $n \equiv 1 \pmod 6$.*

Proposition 3.2 serves as a guide on how to "purify" cyclic LHTSs and in particular cyclic MTSs. Given a cyclic LHTS, one can replace any long orbit of Steiner triples with two orbits of unidirectional cyclic triples. For example the orbit of the Steiner triple $\{x, y, z\}$ may be replaced with the orbits of $(x, y, z)$ and $(y, x, x+y-z)$. This means that any cyclic LHTS($n$) or any cyclic MTS($n$), where $n \equiv 1 \pmod 6$, can be made pure. Since the existence spectrum of cyclic MTSs is $n \equiv 1$ or $3 \pmod 6$, $n \neq 9$, we have the following result

**Theorem 3.4.** *A pure cyclic MTS(n) exists if and only if $n \equiv 1 \pmod 6$.*

**Lemma 3.5.** *Let $(V, \mathcal{B})$ be a cyclic LHTS(n). Then $f(x) \equiv 0 \pmod 2$, $f(x) \neq 2$, $c(x) \equiv 0 \pmod 6$ and $s(x) \equiv 2n+1 \pmod 3$ for any $x \in V$.*

*Proof.* The number of unidirectional transitive triples is even, because every transitive triple $\langle x, y, z\rangle$ can be paired up with another transitive triple $\langle z, y', x\rangle$, for some $y' \in V$. This implies that $f(0)$ must be even as well, because the number of unidirectional transitive triples can be expressed as $nf(0)$ and $n$ is odd. Furthermore $f(0) = m(0) \neq 2$. From Equation (2) we then see that $c(0)$ is also even.

Each point $x \in V$ appears in exactly three triples of any full orbit and in exactly one triple of any short orbit. If $n \equiv 0 \pmod 3$, then there is one short orbit consisting of Steiner triples, thus $s(x) \equiv 1 \pmod 3$. If $n \equiv 1 \pmod 3$, then there are no short orbits, thus $s(x) \equiv 0 \pmod 3$. In both cases every cyclic triple has a full orbit. □

For any proper cyclic LDTS($n$) we have $f(x) \geq 4$ for all $x \in V \setminus \{\infty\}$. For any proper cyclic LHTS($n$) we have $f(x) \geq 4$ and $c(x) \geq 6$ for all $x \in V \setminus \{\infty\}$. Substituting into Equation (2) yields the following two results.

**Proposition 3.6.** *If a proper cyclic LDTS(n) exists, then $n \geq 13$. Every proper cyclic LDTS(13) is pure.*

**Proposition 3.7.** *If a proper cyclic LHTS(n) exists, then $n \geq 19$. Every proper cyclic LHTS(19) is pure.*

The next proposition follows from Lemma 3.5.

**Proposition 3.8.** *Let $s$, $c$ and $t$ be the numbers of Steiner triples, unidirectional cyclic triples and unidirectional transitive triples in a cyclic LHTS(n). Then $c \equiv 0 \pmod{2n}$, $t \equiv 0 \pmod{2n}$ and $t \neq 2n$. Furthermore*

(i) *if $n \equiv 0 \pmod 3$, then $s \equiv \frac{1}{3}n \pmod n$;*
(ii) *if $n \equiv 1 \pmod 3$, then $s \equiv 0 \pmod n$.*

If we wish to work with the conventional definition of an HTS, which does not involve Steiner triples, then the following applies.

**Proposition 3.9.** *Let $c$ be the number of cyclic triples in a cyclic LHTS(n), where all Steiner triples are replaced with bidirectional triples. Then*

(i) *if $n \equiv 0 \pmod 3$, then $c \equiv \frac{2}{3}n \pmod{2n}$;*

(ii) *if* $n \equiv 1 \pmod 3$, *then* $c \equiv 0 \pmod{2n}$.

*Proof.* If $n \equiv 1 \pmod 3$ then every orbit of Steiner triples is long and may thus be replaced with $2n$ cyclic triples or $2n$ transitive triples.

If $n \equiv 0 \pmod 3$ then one orbit of Steiner triples is short. This orbit cannot be replaced with two orbits of transitive triples, since an orbit of transitive triples must be long. Thus it must be replaced with two orbits of cyclic triples, each of length $\frac{1}{3}n$. □

## 4. Rotational LHTSs

Recall that an LHTS, $(V, \mathcal{B})$, is said to be *rotational* if it admits an automorphism consisting of a cycle of length $n-1$ and one fixed point. Let $V = \mathbb{Z}_{n-1} \cup \{\infty\}$ and $\alpha = (0, 1, \dots, n-2)(\infty)$ be an automorphism of $\mathcal{B}$.

A rotational MTS($n$) exists if and only if $n \equiv 1, 3$ or $4 \pmod 6$ and $n \neq 10$ [2]. As noted above, the same condition will apply to LHTS.

**Lemma 4.1.** *A rotational LHTS(n) exists if and only if* $n \equiv 1$, $3$ *or* $4 \pmod 6$ *and* $n \neq 10$.

**Lemma 4.2.** *All triples that contain the point* $\infty$ *lie in a single orbit. If* $\infty$ *lies in a Steiner triple, then the orbit has length* $\frac{n-1}{2}$; *if it lies in a transitive triple or in a cyclic triple, then the orbit is full.*

*Proof.* Its easy to see, that the point $\infty$ cannot lie in the first or last position of any transitive triple: If $B = \langle \infty, x, y \rangle \in \mathcal{B}$, then $\alpha^{-x}(B) = \langle \infty, 0, y-x \rangle \in \mathcal{B}$ and $\alpha^{-y}(B) = \langle \infty, x-y, 0 \rangle \in \mathcal{B}$, which is a contradiction. Similarly for $\langle x, y, \infty \rangle$.

If the point $\infty$ lies in the middle position of some transitive triple $B = \langle x, \infty, y \rangle \in \mathcal{B}$, then the orbit of $B$ under the action of the automorphism $\alpha$ is $\{\, \langle x+i, \infty, y+i \rangle \mid i \in \mathbb{Z}_{n-1} \,\}$. This orbit contains all directed edges of the form $(i, \infty)$ and $(\infty, i)$, where $i \in \mathbb{Z}_{n-1}$. Similarly if $\infty$ lies in a cyclic triple or in a Steiner triple. Thus all triples that contain the point $\infty$ lie in a single orbit.

When $\infty$ lies in a transitive triple or in a cyclic triple, the orbit is clearly full. When it lies in a Steiner triple, the orbit of this triple can be thought of as a partitioning of the set $\mathbb{Z}_{n-1}$ into pairs, so its length is $\frac{n-1}{2}$. □

We shall associate any cyclic triple $(x, \infty, y)$ and any transitive triple $\langle x, \infty, y \rangle$ with the difference $\delta_\infty = x - y$, and any Steiner triple $\{x, \infty, y\}$ with a pair of differences $\delta_\infty = x - y$ and $-\delta_\infty = y - x$. Note that the transitive triple containing the point $\infty$ is associated with the same difference as its underlying cyclic triple.

**Lemma 4.3.** *The point* $\infty$ *lies in a Steiner triple if and only if* $\delta_\infty = -\delta_\infty = \frac{n-1}{2}$.

*Proof.* If $B = \{x, y, \infty\} \in \mathcal{B}$, then $\alpha^{-x}(B) = \{0, y-x, \infty\} \in \mathcal{B}$ and $\alpha^{-y}(B) = \{x-y, 0, \infty\} \in \mathcal{B}$, therefore $\delta_\infty = x - y = \frac{n-1}{2}$.

If $B = \langle 0, \infty, \frac{n-1}{2} \rangle \in \mathcal{B}$ then $\alpha^{\frac{n-1}{2}}(B) = \langle \frac{n-1}{2}, \infty, 0 \rangle \in \mathcal{B}$, i.e. $B$ is a bidirectional triple, which is a contradiction. Similarly $(0, \infty, \frac{n-1}{2}) \in \mathcal{B}$ leads to the same contradiction. □

For any $x, y, z \in \mathbb{Z}_{n-1}$ we associate the cyclic triple $(x, y, z)$ and the transitive triple $\langle x, y, z \rangle$ with the difference triple $[y-x, z-y, x-z]$ and the Steiner triple $\{x, y, z\}$ with the pair of difference triples $[y-x, z-y, x-z]$ and $[x-y, z-x, y-z]$. The results that we obtained for difference triples of cyclic LHTS apply analogously:

Any difference triple $[a, b, c]$ satisfies $a+b+c \equiv 0 \pmod{n-1}$, we say that the triple is balanced. A cyclic triple lies in a short orbit if and only if its difference triple is $[\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-1}{3}]$ or $[\frac{2(n-1)}{3}, \frac{2(n-1)}{3}, \frac{2(n-1)}{3}]$. If the set of difference triples of an LHTS($n$) contains both $[\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-1}{3}]$ and $[\frac{2(n-1)}{3}, \frac{2(n-1)}{3}, \frac{2(n-1)}{3}]$, then these must be associated with a short orbit of a Steiner triple, namely the Steiner triple $\{0, \frac{n-1}{3}, \frac{2(n-1)}{3}\}$.

We now see that a rotational LHTS($n$) induces a partitioning of the set $\{1, \ldots, n-2\} \setminus \{\delta_\infty\}$ into balanced triples and up to two singletons ($\{\frac{n-1}{3}\}$ and $\{\frac{2n-1}{3}\}$).

**Lemma 4.4.** *Let* $(V, \mathcal{B})$ *be a rotational LHTS(n). If* $n \equiv 3 \pmod 6$*, then* $\{0, \frac{n-1}{2}, \infty\} \in \mathcal{B}$ *and the length of its orbit is* $\frac{n-1}{2}$*, while all other orbits are full.*

*Proof.* If $n \equiv 3 \pmod 6$ then 3 divides $n-3$ and so no singleton may be present in the partition. Since the sum over each difference triple is divisible by $n-1$, the sum over all differences $\{1, \ldots, n-2\} \setminus \{\delta_\infty\}$ must also be divisible by $n-1$, thus $\frac{(n-1)(n-2)}{2} - \delta_\infty \equiv 0 \pmod{n-1}$. This gives $\delta_\infty = \frac{n-1}{2}$ and by Lemma 4.3 this implies that $\{0, \frac{n-1}{2}, \infty\} \in \mathcal{B}$. □

**Lemma 4.5.** *If* $n \equiv 1 \pmod 3$*, then the element* $\infty$ *lies in a cyclic triple and there exists an orbit of length* $\frac{n-1}{3}$ *consisting of cyclic triples, all other orbits are full. Furthermore if* $n \equiv 1 \pmod 6$*, then vertices corresponding to the point* $\infty$ *have degree 6 and the differences* $\pm\frac{n-1}{3}$*,* $\pm\frac{n-1}{6}$ *and* $\frac{n-1}{2}$ *are associated with edges of cyclic triples; and if* $n \equiv 4 \pmod 6$*, then* $\{(0, \infty, k), (k, \infty, 2k), (2k, \infty, 0), (0, k, 2k)\} \subseteq \mathcal{B}$*, where* $k \in \{\pm\frac{n-1}{3}\}$*.*

*Proof.* If $n \equiv 1 \pmod 3$, then exactly one of the singletons $\{k\}$, $k \in \{\pm\frac{n-1}{3}\}$, must be present in the partition. We now have

$$\frac{(n-1)(n-2)}{2} - k - \delta_\infty \equiv 0 \pmod{n-1}. \tag{3}$$

If $n \equiv 4 \pmod 6$ then Equation (3) gives $\delta_\infty \equiv -k \equiv \mp\frac{n-1}{3} \pmod{n-1}$. We see that the underlying MTS contains the triples $(0, \infty, k)$, $(k, \infty, 2k)$ and $(2k, \infty, 0)$. Each of these three triples is adjacent to the triple $(0, k, 2k)$ which necessarily underlies a cyclic triple. This implies that the three triples containing the point $\infty$ also underlie cyclic triples.

If $n \equiv 1 \pmod 6$ then Equation (3) gives $\delta_\infty \equiv \frac{n-1}{2} - k \equiv \pm\frac{n-1}{6} \pmod{n-1}$. We see that vertices corresponding to the point $\infty$ have degree 6. Pick a surface component containing the element $\infty$. Take the set of all differences associated with the triples of this component and remove $\delta_\infty$ and $k$ (if present). Denote this set by $\Delta$. The sum of the elements of $\Delta$ is divisible by $n-1$. Furthermore, for each $x \in \Delta \setminus \{-\delta_\infty, -k\}$ we have $-x \in \Delta$, so the sum of the elements of $\Delta \setminus \{-\delta_\infty, -k, \frac{n-1}{2}\}$ is also divisible by $n-1$. This implies that the sum of the elements of $\Delta \cap \{-\delta_\infty, -k, \frac{n-1}{2}\}$ is divisible by $n-1$. The only way that this can happen is if $-k$ and $\frac{n-1}{2}$ both lie in $\Delta$. The fact that $-k \in \Delta$ means that one of the triples in the component is associated with the difference triple $[k, k, k]$ and must therefore be a cyclic triple. Since the component is uniform (cf. Section 2), the element $\infty$ lies in a cyclic triple as well. Finally, the component also contains an edge associated with the difference $\frac{n-1}{2}$, which implies that any edge associated with the difference $\frac{n-1}{2}$ lies in a cyclic triple. □

The two lemmas above account for all admissible values of $n$, yielding the following two results.

**Corollary 4.6.** *If there exists a rotational LDTS(n), then it is non-pure and* $n \equiv 3 \pmod 6$.

**Corollary 4.7.** *If there exists a pure rotational LHTS(n), then* $n \equiv 1 \pmod 3$.

Since for $n \equiv 1 \pmod 3$ any orbit of a Steiner triple is full, it is possible to replace each one with two orbits of unidirectional cyclic triples in exactly the same manner as described in Section 3. Thus any rotational LHTS($n$), $n \equiv 1 \pmod 3$, can be made pure. Since the existence spectrum of rotational MTS is $n \equiv 1, 3$ or $4 \pmod 6$, $n \neq 10$, we have the following result

**Theorem 4.8.** *A pure rotational MTS(n) exists if and only if* $n \equiv 1 \pmod 3$ *and* $n \neq 10$.

Lemmas 4.4 and 4.5 imply that the point $\infty$ does not lie in a transitive triple, i.e. $f(\infty) = 0$ and $m(\infty) = 0$. Since $\alpha$ acts transitively on $V \setminus \{\infty\}$, we have $(n-1)f(0) + f(\infty) = \sum_{x \in V} f(x) = \sum_{x \in V} m(x) = (n-1)m(0) + m(\infty)$, and again $f(x) = m(x)$ for every $x \in V$. Equation (2) therefore applies to rotational LHTSs as well.

**Lemma 4.9.** *Let* $(V, \mathcal{B})$ *be a rotational LHTS(n). Then* $f(x) \equiv 0 \pmod 2$, $f(x) \neq 2$, $c(x) \equiv 3n+3 \pmod 6$ *and* $s(x) \equiv 2n+1 \pmod 3$ *for any* $x \in V \setminus \{\infty\}$.

*Proof.* By Theorem 2.1 each transitive triple $B = \langle x, y, z \rangle \in \mathcal{B}$ can be paired up with a transitive triple $B' = \langle z, y', x \rangle \in \mathcal{B}$ for some $y' \in V$. Notice that the images of $B$ and $B'$ under $\alpha$ pair up in this manner as well. This means that we can also pair up the orbits of the two triples. However if $B$ and $B'$ lie in the same orbit, then this orbit pairs with itself. This happens if and only if $x - z \equiv z - x \pmod{n-1}$, i.e. if and only if $x - z = \frac{n-1}{2}$. By Lemma 4.4 and 4.5 if $n \equiv 3 \pmod 6$ then the difference $\frac{n-1}{2}$ is associated with the edge of a Steiner triple, if $n \equiv 1 \pmod 6$ then it is associated with the edge of a cyclic triple, otherwise it does not exist. The difference $\frac{n-1}{2}$ is never associated with the edge of a transitive triple, so $B$ and $B'$ never lie in the same orbit. The number of orbits consisting of transitive triples is therefore even.

Let $x \in V \setminus \{\infty\}$. In any orbit consisting of transitive triples there is exactly one transitive triple such that $x$ appears in the first position. The value of $f(x)$ is therefore equal to the number of these orbits which is even. From Equation (2) we then see that $c(x) \equiv n - 1 \pmod 2$.

If $n \equiv 0 \pmod 3$, then there is one short orbit consisting of Steiner triples and all other orbits are full, thus $s(x) \equiv 1 \pmod 3$ and $c(x) \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then there is one short orbit consisting of cyclic triples and one full orbit consisting of cyclic triples, which contain the point $\infty$. All other orbits are full, thus $s(x) \equiv 0 \pmod 3$. The short orbit contributes to the value $c(x)$ by 1, the orbit where $\infty$ appears contributes by 2 and every other orbit of cyclic triples contributes by 3, thus $c(x) \equiv 0 \pmod 3$. The result now follows from $c(x) \equiv n - 1 \pmod 2$. □

For any proper rotational LDTS($n$) we have $f(x) \geq 4$ for all $x \in V \setminus \{\infty\}$ and $n \equiv 3 \pmod 6$. For any proper rotational LHTS($n$) we have $f(x) \geq 4$ and $c(x) \geq 3$ for all $x \in V \setminus \{\infty\}$. Substituting into Equation (2) yields the following two results.

**Proposition 4.10.** *If a proper rotational LDTS(n) exists, then* $n \geq 15$.

**Proposition 4.11.** *If a proper rotational LHTS(n) exists, then* $n \geq 16$. *Every proper rotational LHTS(*$16$*) is pure.*

**Proposition 4.12.** *There exists no proper rotational LHTS of order 19.*

*Proof.* Assume that there exists a proper rotational LHTS(19), $(V, \mathcal{B})$. Lemma 4.9 and Equation (2) imply that $f(x) = 4$, $c(x) = 6$ and $s(x) = 0$ for any $x \in V \setminus \{\infty\}$. As noted in the proof of Lemma 4.9, for any $x \in V \setminus \{\infty\}$ the value of $f(x)$ is equal to the number of orbits consisting of transitive triples. Therefore exactly twelve of the differences in the set $\{1, 2, \ldots, 17\}$ are associated with the edges of transitive triples. By Lemma 4.5, the differences 3, 6, 9, 12 and 15 are associated with edges of cyclic triples. Thus if a proper rotational LHTS(19) exists then the set $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}$ can be partitioned into balanced triples. It is easy to check that the difference 2 can only appear in the triple $\{2, 5, 11\}$ and the difference 8 can only appear in the triple $\{8, 11, 17\}$. But these two triples cannot both be present in the partition. Such a partition therefore does not exist. □

The next proposition follows from the lemmas above.

**Proposition 4.13.** *Let* $s$, $c$ *and* $t$ *be the numbers of Steiner triples, unidirectional cyclic triples and unidirectional transitive triples in a rotational LHTS(n). Then* $t \equiv 0 \pmod{2(n-1)}$ *and* $t \neq 2(n-1)$. *Furthermore*

(i) *if* $n \equiv 1 \pmod 6$, *then* $c \neq \frac{1}{3}(n-1)$,

$$s \equiv 0 \pmod{n-1} \quad \textit{and} \quad c \equiv \tfrac{1}{3}(n-1) \pmod{2(n-1)};$$

(ii) *if* $n \equiv 3 \pmod 6$, *then*

$$s \equiv \tfrac{1}{2}(n-1) \pmod{n-1} \quad \textit{and} \quad c \equiv 0 \pmod{2(n-1)};$$

(iii) *if* $n \equiv 4 \pmod 6$, *then*

$$s \equiv 0 \pmod{n-1} \quad \textit{and} \quad c \equiv \tfrac{4}{3}(n-1) \pmod{2(n-1)}.$$

If we wish to work with the conventional definition of an HTS, which does not involve Steiner triples, then the following applies.

**Proposition 4.14.** *Let* $c$ *be the number of cyclic triples in a rotational LHTS(n), where all Steiner triples are replaced with bidirectional triples. Then*

(i) *if* $n \equiv 1 \pmod 6$, *then* $c \equiv \frac{1}{3}(n-1) \pmod{2(n-1)}$ *and* $c \neq \frac{1}{3}(n-1)$;
(ii) *if* $n \equiv 3 \pmod 6$, *then* $c \equiv 0 \pmod{n-1}$;
(iii) *if* $n \equiv 4 \pmod 6$, *then* $c \equiv \frac{4}{3}(n-1) \pmod{2(n-1)}$.

## 5. Existence of cyclic and rotational LHTS

Let us start by showing that any partition of the set $\mathbb{Z}_n \setminus \{0\}$ into balanced triples gives rise to some cyclic LDTS($n$). To prove this, we first need to define a structure, which bears a certain similarity to the notion of a current graph.

Let $\Delta \subseteq \mathbb{Z}_n \setminus \{0\}$ and let $P$ be a partition of the set $\Delta$ into balanced triples. Define the quiver $\Gamma_P = (P, \Delta)$, where the triples of $P$ are viewed as points and the differences in $\Delta$ are viewed as arrows. The arrow $\delta \in \Delta$ starts at the triple containing $\delta$ and terminates at the triple containing $-\delta \bmod n$. Figure 1 shows an example of such a quiver for $P = \{\{1, 6, 12\}, \{2, 3, 14\}, \{4, 7, 8\}, \{5, 16, 17\}, \{9, 11, 18\}, \{10, 13, 15\}\}$ over $\mathbb{Z}_{19}$.

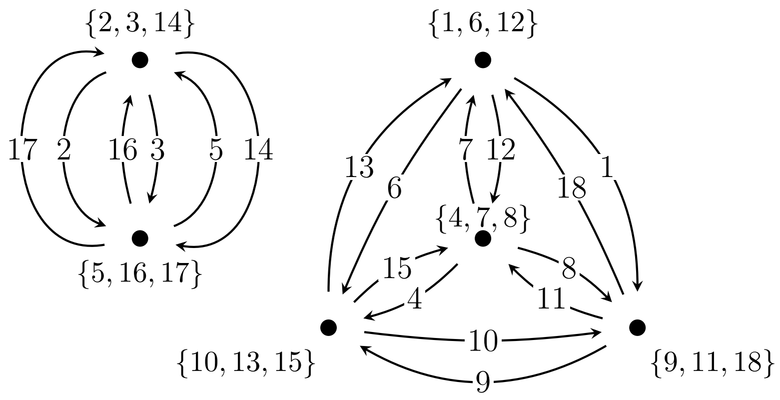


FIGURE 1. Quiver $\Gamma_P$, for $P = \{\{1,6,12\}, \{2,3,14\}, \{4,7,8\}, \{5,16,17\}, \{9,11,18\}, \{10,13,15\}\}$ over $\mathbb{Z}_{19}$.

**Lemma 5.1.** *Let $\Delta \subseteq \mathbb{Z}_n \setminus \{0\}$ such that $\Delta = -\Delta$ and let $P$ be a partition of the set $\Delta$ into balanced triples. Then there exists a partial cyclic LDTS(n) corresponding to the partition $P$.*

*Proof.* We need to show that there exists an arrow colouring of $\Gamma_P$ such that

(1) for every $a \in \Delta$, the opposite arrows $a$ and $-a$ are either both blue or one of them is red while the other is green; and
(2) every vertex has a red, a green and a blue outgoing arrow.

First let us show that there are no loops in $\Gamma_P$ and that every vertex has one or three neighbours. A vertex having a loop would be of the form $\{0, a, -a\}$, which is a contradiction with $0 \notin \Delta$. Assume that a vertex $\{a, b, c\}$ has exactly two neighbours, then without loss of generality one of the neighbours is of the form $\{-a, -b, d\}$. Since the triples are balanced, we get $d = a + b = -c$, which implies that its only neighbour is $\{-a, -b, -c\}$.

When a vertex $\{a, b, c\}$ has only one neighbour the two vertices form a component of connectivity that we shall call *degenerate*. A degenerate component can be assigned colours as follows: $a$ and $-b$ red, $b$ and $-a$ green, $c$ and $-c$ blue. This colouring corresponds to the orbit of a Steiner triple.

Denote by $G_P$ the underlying undirected graph obtained by replacing all arrows of $\Gamma_P$ with undirected edges. Every non-degenerate connected component $C$ of $G_P$ is 3-regular. Furthermore, non-degenerate components of $G_P$ can be shown to be 2-edge connected. Assume to the contrary, that there exists an edge $(p, q)$ whose removal splits $C$ into two components. In the quiver locate the corresponding pair of arrows $a$ and $-a$ between the vertices $p$ and $q$ and remove the two arrows. Denote this new quiver $\Gamma'_P$. Since the sum of every pair of opposite arrows is zero, so is the sum of all arrows in any component of the new quiver $\Gamma'_P$. The sum of the outgoing arrows from the vertices $p$ and $q$ in $\Gamma'_P$ is $a$ and $-a$, for any other vertex the sum of the outgoing arrows is zero. If the vertices $p$ and $q$ lie in separate components of $\Gamma'_P$ then the sum of the arrows in each of these two components is non-zero, which is a contradiction.

By a well known theorem of Petersen [14] every 3-regular 2-edge connected graph has a 1-factor. Thus the edge set of any non-degenerate component $C$ of $G_P$ can be partitioned into a 1-factor $M$ and a 2-factor $N$, i.e. into a perfect matching and a collection of cycles that spans all vertices of the component. For each edge in $M$ locate the corresponding pair of arrows in the quiver and colour them blue. For each cycle in $N$ locate the corresponding set of arrows in the quiver. These arrows can be viewed as a pair oriented cycles having a common vertex set but opposite orientation. Colour one of the oriented cycles red and the other green. This colouring satisfies the required conditions. $\square$

**Lemma 5.2.** *If* $n \equiv 1 \pmod 6$ *and* $n \geq 13$*, then there exists a pure cyclic LDTS(n).*

*Proof.* Let $n = 6k+1$ and $k \geq 2$, then

$$P = \{\, \{r, r+k, -2r-k\}, \{-r, -r-3k, 2r+3k\} : r = 1, \dots, k \,\}$$

is a partition of the set $\mathbb{Z}_n \setminus \{0\}$ into balanced triples. By Lemma 5.1 there exists a cyclic LDTS($n$) corresponding to the partition $P$. The system is non-pure if and only if $\{r, r+k, -2r-k\} = \{r, r+3k, -2r-3k\}$ for some $r \in \{1, \dots, k\}$. If $k \equiv 1 \pmod 3$ then this equality is satisfied only for $r = \frac{2k+1}{3}$, otherwise it is not satisfied for any $r \in \{1, \dots, k\}$. Thus for $k \equiv 0$ or $2 \pmod 3$ the partition $P$ corresponds to some pure cyclic LDTS($n$).

For $k \equiv 1 \pmod 3$ replace the triples $\{\frac{2k+1}{3},\ \frac{2k+1}{3}+k,\ -2\frac{2k+1}{3}-k\}$, $\{-\frac{2k+1}{3},\ -\frac{2k+1}{3}-3k,\ 2\frac{2k+1}{3}+3k\}$ and $\{k,\ 2k,\ -3k\}$ in $P$ with the balanced triples $T_1 = \{\frac{2k+1}{3},\ 2\frac{2k+1}{3}+3k,\ k\}$, $T_2 = \{\frac{2k+1}{3}+k,\ -\frac{2k+1}{3}-3k,\ 2k\}$ and $T_3 = \{-2\frac{2k+1}{3}-k,\ -\frac{2k+1}{3},\ -3k\}$ and apply Lemma 5.1. As long as $k > 1$ this replacement produces a partition of the set $\mathbb{Z}_n \setminus \{0\}$ into balanced triples. It remains to be shown that by replacing these triples no other pairs of opposite triples are created. Any new pair of opposite triples would have to involve $T_1$, $T_2$ or $T_3$. However the intersections $T_1 \cap -T_2$, $T_2 \cap -T_3$ and $T_3 \cap -T_1$ are all non-empty. Thus for $k \equiv 1 \pmod 3$ this partition corresponds to some pure cyclic LDTS($n$). $\square$

**Lemma 5.3.** *If* $n \equiv 1 \pmod 6$ *and* $n \geq 19$*, then there exists a proper pure cyclic LHTS(n) with* $2n$ *cyclic triples.*

*Proof.* A cyclic LHTS(19) can be obtained from the following starter blocks: $(0,2,5)$, $(0,5,3)$, $\langle 1,0,7\rangle$, $\langle 7,0,15\rangle$, $\langle 15,0,9\rangle$ and $\langle 9,0,1\rangle$.

Let $n = 6k+1$, $k \geq 4$, and consider the partition $P$ used in the proof of Lemma 5.2.

If $k$ is even, then remove the triples $\{1, k+1, -k-2\}$, $\{\frac{k}{2}, \frac{k}{2}+k, -2k\}$, $\{k, 2k, -3k\}$, $\{-\frac{k}{2}, -\frac{k}{2}-3k, 4k\}$, $\{-\frac{k+2}{2}, -\frac{k+2}{2}-3k, 4k+2\}$ and $\{-k, -4k, 5k\}$ and replace them with $\{1, \frac{k}{2}, -\frac{k+2}{2}\}$, $\{\frac{k}{2}+k, 2k, -\frac{k}{2}-3k\}$, $\{-\frac{k+2}{2}-3k, -2k, -\frac{k}{2}\}$ and $\{-3k, 4k+2, -k-2\}$.

If $k$ is odd, then remove the triples $\{1, k+1, -k-2\}$, $\{\frac{k-1}{2}, \frac{k-1}{2}+k, -2k+1\}$, $\{\frac{k+1}{2}, \frac{k+1}{2}+k, -2k-1\}$, $\{k, 2k, -3k\}$, $\{-\frac{k+1}{2}, -\frac{k+1}{2}-3k, 4k+1\}$ and $\{-k, -4k, 5k\}$ and replace them with $\{\frac{k-1}{2}, -\frac{k+1}{2}, 1\}$, $\{\frac{k+1}{2}, \frac{k-1}{2}+k, 4k+1\}$, $\{\frac{k+1}{2}+k, -\frac{k+1}{2}-3k, 2k\}$ and $\{-3k, -2k+1, -k-2\}$.

This yields a partition of the set $\mathbb{Z}_n \setminus \{0, k, k+1, 2k+1, 4k, 5k, 5k+1\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n$). Augmenting this system with the orbits of the cyclic triples $(0, k, 2k+1)$ and $(0, 5k, k)$ gives a proper cyclic LHTS($n$). If $k \equiv 1 \pmod 3$ and $k \neq 4$, then the system contains $n$ Steiner triples, otherwise it contains no Steiner triples.

In order to obtain a pure LHTS for $k \equiv 1 \pmod 3$ consider the partition $P$ used in the proof of Lemma 5.2 and remove the triples $\{\frac{2k+1}{3}, \frac{2k+1}{3}+k, -2\frac{2k+1}{3}-k\}$ and $\{-\frac{2k+1}{3}, -\frac{2k+1}{3}-3k, 2\frac{2k+1}{3}+3k\}$. Apply Lemma 5.1 and augment the resulting partial system with the orbits of the cyclic triples $(0, \frac{2k+1}{3}, \frac{7k+2}{3})$ and $(0, \frac{7k+2}{3}, \frac{5k+1}{3})$. ☐

**Lemma 5.4.** *If $n \equiv 3 \pmod{12}$ and $n \neq 3$, then there exists a proper cyclic LDTS(n) with $\frac{1}{3}n$ Steiner triples, and if in addition $n \geq 27$, then there exists one with $\frac{4}{3}n$ Steiner triples.*

*Proof.* Let $n = 12k + 3$, $k \geq 1$ and

$$\begin{aligned}
P_1 &= \big\{ \{r, r+2k+1, -2r-2k-1\} : r = 1, 2, \ldots, k-1 \big\}, \\
P_2 &= \big\{ \{r, r+2k-1, -2r-2k+1\} : r = k+2, k+3, \ldots, 2k+1 \big\}, \\
P_3 &= \big\{ \{-r, -r+6k+2, 2r-6k-2\} : r = 1, 2, \ldots, 2k \big\}, \\
P_4 &= \big\{\{k, k+1, -2k-1\}\big\},
\end{aligned}$$

then $P = P_1 \cup P_2 \cup P_3 \cup P_4$ is a partition of the set $\mathbb{Z}_n \setminus \{0, \frac{n}{3}, \frac{2n}{3}\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n$) corresponding to the partition $P$. Augmenting this system with the orbit of the Steiner triple $\{0, \frac{n}{3}, \frac{2n}{3}\}$ gives a cyclic LDTS($n$). The augmented orbit has length $\frac{n}{3}$.

Clearly the triple in $P_4$ does not correspond to an orbit of Steiner triples, since the opposite triple $\{-k, -k-1, 2k+1\}$ does not lie in $P$. Generally any pair of opposite triples will necessarily involve a triple from $P_3$. Thus the system contains additional Steiner triples if and only if

$$\{r, r+2k+1, -2r-2k-1\} = \{r, r-6k-2, -2r+6k+2\}$$

for some $r = 1, 2, \ldots, k-1$, or

$$\{r, r+2k-1, -2r-2k+1\} = \{r, r-6k-2, -2r+6k+2\}$$

for some $r = k+2, k+3, \ldots, 2k$. If $k \equiv 0 \pmod 3$ then this condition is satisfied only for $r = \frac{4k+3}{3}$, otherwise it is not satisfied for any $r \in \{1, \ldots, 2k\}$. Thus if $k \equiv 0 \pmod 3$, then $P$ yields one full orbit of Steiner triples, otherwise none.

In order to minimize the number of Steiner triples for $k \equiv 0 \pmod 3$, use the following partition instead:

$$\begin{aligned}
P' = &\big\{ \{r, r+2k-1, -2r-2k+1\} : r = 1, \ldots, k \big\} \\
&\cup \big\{ \{r, r+2k+1, -2r-2k-1\} : r = k+1, k+2, \ldots, 2k-1 \big\} \\
&\cup \big\{ \{-r, -r+6k+2, 2r-6k-2\} : r = 1, \ldots, 2k \big\} \\
&\cup \big\{\{3k, 3k+1, -6k-1\}\big\}.
\end{aligned}$$

If $k \equiv 2 \pmod 3$, then this partition yields one full orbit of Steiner triples, otherwise none.

Finally we need a partition which yields one full orbit of Steiner triples if $k \equiv 1 \pmod 3$, $k \geq 4$. Such a partition can be obtained from $P$ by removing the triples $\{k-1, 3k, 8k+4\}$, $\{2k+1, 4k, 6k+2\}$, $\{-k+1, 5k+3, 8k-1\}$, $\{-k, 5k+2, 8k+1\}$ and $\{-k-1, 5k+1, 8k+3\}$ and replacing them with $\{k-1, 5k+2, 6k+2\}$, $\{-k-1, -k, 2k+1\}$, $\{3k, 4k, 5k+3\}$, $\{5k+1, 8k+1, -k+1\}$ and $\{8k-1, 8k+3, 8k+4\}$. ☐

**Lemma 5.5.** *If $n \equiv 9 \pmod{12}$ and $n \neq 9$, then there exist proper cyclic LDTS(n)s with $\frac{1}{3}n$ Steiner triples and $\frac{4}{3}n$ Steiner triples.*

*Proof.* Let $n = 12k + 9$, $k \geq 1$ and

$$P_1 = \big\{ \{r, r + 2k + 1, -2r - 2k - 1\} : r = 1, \ldots, 2k + 1 \text{ and } r \neq k + 1 \big\},$$
$$P_2 = \big\{ \{-r, -r + 6k + 4, 2r - 6k - 4\} : r = 1, \ldots, 2k \text{ and } r \neq k \big\},$$
$$P_3 = \big\{\{k + 1, 5k + 4, 6k + 4\}, \{3k + 2, 10k + 7, 11k + 9\}, \{6k + 5, 8k + 5, 10k + 8\}\big\},$$

then $P = P_1 \cup P_2 \cup P_3$ is a partition of the set $\mathbb{Z}_n \setminus \{0, \frac{n}{3}, \frac{2n}{3}\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n$) corresponding to the partition $P$. Augmenting this system with the orbit of the Steiner triple $\{0, \frac{n}{3}, \frac{2n}{3}\}$ gives a cyclic LDTS($n$). The augmented orbit has length $\frac{n}{3}$.

If $k = 1$, then the triples $\{1, 4, -5\} \in P_1$ and $\{5, 17, 20\} \in P_3$ correspond to a full orbit of Steiner triples, otherwise the triples in $P_3$ do not correspond to orbits of Steiner triples. Thus for $k > 1$ the system contains additional Steiner triples if and only if

$$\{r, r + 2k + 1, -2r - 2k - 1\} = \{r, r - 6k - 4, -2r + 6k + 4\}$$

for some $r = 1, 2, \ldots, 2k + 1$. If $k \equiv 0 \pmod 3$ then this condition is satisfied only for $r = \frac{4k+3}{3}$, otherwise it is not satisfied for any $r \in \{1, \ldots, 2k + 1\}$.

To maximize the number of unidirectional triples for $k = 1$ use the partition $P_1 \cup \big\{\{2, 8, 11\}, \{5, 18, 19\}, \{9, 13, 20\}, \{10, 15, 17\}\big\}$.

To maximize the number of unidirectional triples for $k \equiv 0 \pmod 3$, we can proceed as in the proof of Lemma 5.2, replacing the triples $\{\frac{4k+3}{3}, \frac{4k+3}{3} + 2k + 1, -2\frac{4k+3}{3} - 2k - 1\}$, $\{-\frac{4k+3}{3}, -\frac{4k+3}{3} + 6k + 4, 2\frac{4k+3}{3} - 6k - 4\}$ and $\{6k + 5, 8k + 5, 10k + 8\}$ in $P$ with the balanced triples $\{\frac{4k+3}{3}, -\frac{4k+3}{3} + 6k + 4, 6k + 5\}$, $\{\frac{4k+3}{3} + 2k + 1, -\frac{4k+3}{3}, 10k + 8\}$ and $\{-2\frac{4k+3}{3} - 2k - 1, 2\frac{4k+3}{3} - 6k - 4, 8k + 5\}$.

Finally we need a partition which yields one full orbit of Steiner triples if $k \equiv 1$ or $2 \pmod 3$ and $k \geq 2$. This can be achieved by replacing the triples $\{k+3, 3k+4, -4k-7\}$, $\{2k, 4k+1, -6k-1\}$, $\{-2k, 4k+4, -2k-4\}$, $\{k+1, 5k+4, 6k+4\}$ and $\{6k+5, 8k+5, 10k+8\}$ in $P$ with the triples $\{2k, 4k+4, 6k+5\}$, $\{6k + 4, 8k + 5, -2k\}$, $\{k + 1, k + 3, -2k - 4\}$, $\{3k + 4, 4k + 1, 5k + 4\}$ and $\{-6k - 1, -4k - 7, 10k + 8\}$. □

**Lemma 5.6.** *If $n \equiv 3 \pmod 6$ and $n \geq 21$, then there exists a proper cyclic LHTS(n) with $\frac{1}{3}n$ Steiner triples and $2n$ cyclic triples.*

*Proof.* We shall utilise the principle derived from Proposition 3.2. By Lemmas 5.4 and 5.5 if $n \equiv 3 \pmod 6$ and $n \geq 21$, then there exists a proper cyclic LDTS($n$) with $\frac{4}{3}n$ Steiner triples. The Steiner triples form two orbits, one long and one short. Let $\{x, y, z\}$ be one of the Steiner triples in the long orbit. Replace this orbit with two orbits of the unidirectional cyclic triples $(x, y, z)$ and $(y, x, x+y-z)$. This yields a cyclic LHTS($n$) with $\frac{1}{3}n$ Steiner triples and $2n$ cyclic triples. □

**Lemma 5.7.** *If $n \equiv 3 \pmod 6$ and $n \geq 15$, then there exists a proper rotational LDTS(n) with $\frac{1}{2}(n - 1)$ Steiner triples.*

*Proof.* Let $n = 6k + 3$ and $\delta_\infty = 3k + 1$, then

$$P = \big\{ \{r, r + k, -2r - k\}, \{-r, -r + 3k + 1, 2r - 3k - 1\} : r = 1, \ldots, k \big\}$$

is a partition of the set $\mathbb{Z}_{n-1} \setminus \{0, \delta_\infty\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n-1$) corresponding to the partition $P$. Augmenting this system with the orbit of the Steiner triple $\{0, \frac{n-1}{2}, \infty\}$ gives a rotational

LDTS($n$). By Lemma 4.4 the augmented orbit has length $\frac{1}{2}(n-1)$. The system contains more than $\frac{1}{2}(n-1)$ Steiner triples if and only if $\{r, r+k, -2r-k\} = \{r, r-3k-1, 3k+1-2r\}$ for some $r \in \{1,\dots,k\}$. If $k \equiv 1 \pmod 3$ then this equality is satisfied only for $r = \frac{2k+1}{3}$, otherwise it is not satisfied for any $r \in \{1,\dots,k\}$. Thus for $k \equiv 0$ or $2 \pmod 3$ the system contains exactly $\frac{1}{2}(n-1)$ Steiner triples, whilst for $n \equiv 1 \pmod 3$ it contains an additional orbit consisting of $n-1$ Steiner triples.

To minimize the number of Steiner triples for $k \equiv 1 \pmod 3$, $k \geq 4$, replace the triples $\{k, 2k, -3k\}$, $\{-\frac{k-1}{3}, 3k+1-\frac{k-1}{3}, 2\frac{k-1}{3}-3k-1\}$ and $\{-\frac{2k+1}{3}, 3k+1-\frac{2k+1}{3}, 2\frac{2k+1}{3}-3k-1\}$ in $P$ with the triples $\{k, 3k+1-\frac{2k+1}{3}, 3k+1-\frac{k-1}{3}\}$, $\{2k, 2\frac{2k+1}{3}-3k-1, -\frac{k-1}{3}\}$ and $\{-3k, 2\frac{k-1}{3}-3k-1, -\frac{2k+1}{3}\}$. □

**Lemma 5.8.** *If $n \equiv 3 \pmod 6$ and $n \geq 21$, then there exists a proper rotational LHTS(n) with $\frac{1}{2}(n-1)$ Steiner triples and $2(n-1)$ cyclic triples.*

*Proof.* A rotational LHTS(21) can be obtained from the following starter blocks: $\{0, 10, \infty\}$, $(0, 3, 11)$, $(0, 11, 8)$, $\langle 13, 0, 15\rangle$, $\langle 15, 0, 16\rangle$, $\langle 16, 0, 14\rangle$ and $\langle 14, 0, 13\rangle$.

Let $n = 6k+3$, $k \geq 4$, and consider the partition $P$ used in the proof of Lemma 5.7.

If $k$ is even, then remove the triples $\{k, 2k, -3k\}$, $\{\frac{k}{2}, \frac{k}{2}+k, -2k\}$, $\{\frac{k+2}{2}, \frac{k+2}{2}+k, -2k-2\}$, $\{-1, 3k, 1-3k\}$, $\{-\frac{k}{2}, 3k+1-\frac{k}{2}, -2k-1\}$ and $\{-k, 2k+1, -k-1\}$ and replace them with $\{\frac{k}{2}, \frac{k+2}{2}+k, -2k-1\}$, $\{-1, -\frac{k}{2}, \frac{k+2}{2}\}$, $\{\frac{k}{2}+k, 2k+1, 3k+1-\frac{k}{2}\}$ and $\{-k-1, -2k-2, -3k+1\}$.

If $k$ is odd, then remove the triples $\{k, 2k, -3k\}$, $\{\frac{k+1}{2}, \frac{k+1}{2}+k, -2k-1\}$, $\{-1, 3k, 1-3k\}$, $\{-\frac{k-1}{2}, 3k+1-\frac{k-1}{2}, -2k-2\}$, $\{-\frac{k+1}{2}, 3k+1-\frac{k+1}{2}, -2k\}$ and $\{-k, 2k+1, -k-1\}$ and replace them with $\{\frac{k+1}{2}, -\frac{k-1}{2}, -1\}$, $\{\frac{k+1}{2}+k, 3k+1-\frac{k+1}{2}, 2k+1\}$, $\{-\frac{k+1}{2}, 3k+1-\frac{k-1}{2}, -2k-1\}$ and $\{-k-1, -2k-2, 1-3k\}$.

This yields a partition of the set $\mathbb{Z}_n \setminus \{0, k, 2k, 3k, 3k+1, 3k+2, 4k+2, 5k+2\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n-1$). Augmenting this system with the orbits of the Steiner triple $\{0, \frac{n-1}{2}, \infty\}$ and the cyclic triples $(0, k, 3k)$ and $(0, 3k, 2k)$ gives a proper rotational LHTS($n$). If $k \equiv 1 \pmod 3$ and $k \neq 4$, then the system contains $\frac{3}{2}(n-1)$ Steiner triples, otherwise it contains $\frac{1}{2}(n-1)$ Steiner triples.

To obtain an LHTS with the minimum possible number of Steiner triples for $k \equiv 1 \pmod 3$ consider the partition $P$ used in the proof of Lemma 5.7 and remove the triples $\{\frac{2k+1}{3}, \frac{2k+1}{3}+k, -2\frac{2k+1}{3}-k\}$ and $\{-\frac{2k+1}{3}, 3k+1-\frac{2k+1}{3}, 2\frac{2k+1}{3}-3k-1\}$. Apply Lemma 5.1 and augment the resulting partial system with the orbits of the Steiner triple $\{0, \frac{n-1}{2}, \infty\}$ and of the cyclic triples $(0, \frac{2k+1}{3}, \frac{7k+2}{3})$ and $(0, \frac{7k+2}{3}, \frac{5k+1}{3})$. □

**Lemma 5.9.** *If $n \equiv 4 \pmod 6$ and $n \geq 16$, then there exists a proper pure rotational LHTS(n) with $\frac{4}{3}(n-1)$ cyclic triples.*

*Proof.* Let $n \equiv 4 \pmod 6$ and $n \geq 16$. By Lemmas 5.4 and 5.5 there exists a cyclic LDTS($n-1$), $(V, \mathcal{B}_1 \cup \mathcal{B}_2)$, such that $\mathcal{B}_2$ is the orbit of the Steiner triple $\{0, \frac{n-1}{3}, \frac{2(n-1)}{3}\}$ whilst $\mathcal{B}_1$ contains no Steiner triples. Let $\mathcal{B}_2'$ be the union of the orbits of the cyclic triples $(0, \infty, \frac{n-1}{3})$ and $(0, \frac{n-1}{3}, \frac{2(n-1)}{3})$. Then $(V \cup \{\infty\}, \mathcal{B}_1 \cup \mathcal{B}_2')$ is a pure rotational LHTS($n$). □

**Lemma 5.10.** *If $n \equiv 1 \pmod{12}$ and $n \geq 25$, then there exists a proper pure rotational LHTS(n) with $\frac{7}{3}(n-1)$ cyclic triples.*

*Proof.* Let $n = 12k + 1$, $k \geq 2$ and consider the partition

$$P = \big\{ \{r, r + 2k, -2r - 2k\} : r = 1, \ldots, 2k - 1 \text{ and } r \neq k \text{ or } k + 1 \big\}$$
$$\cup \big\{ \{-r, -r + 6k - 1, 2r - 6k + 1\} : r = 1, \ldots, 2k - 2 \big\}$$
$$\cup \big\{ \{k, k + 1, -2k - 1\}, \{3k, 3k + 1, 6k - 1\}, \{6k + 1, 8k - 2, 10k + 1\} \big\}$$

of the set $\mathbb{Z}_{n-1} \setminus \{0, \pm\frac{n-1}{3}, \pm\frac{n-1}{6}, \frac{n-1}{2}\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n - 1$) corresponding to the partition $P$. Augmenting this system with the orbits of the cyclic triples $(0, \infty, 2k)$, $(0, 8k, 4k)$ and $(0, 2k, 8k)$ gives a proper rotational LHTS($n$). If $k \equiv 1 \pmod 3$ and $k \geq 7$, then the system contains $n - 1$ Steiner triples, otherwise it contains no Steiner triples.

To obtain a pure LHTS for $k \equiv 1 \pmod 3$ where $k \geq 7$, replace the triples $\{\frac{2k-2}{3}, \frac{2k-2}{3} + 2k, -2\frac{2k-2}{3} - 2k\}$, $\{\frac{4k-1}{3}, \frac{4k-1}{3} + 2k, -2\frac{4k-1}{3} - 2k\}$ and $\{6k + 1, 8k - 2, 10k + 1\}$ in $P$ with the triples $\{\frac{2k-2}{3}, \frac{4k-1}{3}, 10k + 1\}$, $\{\frac{2k-2}{3} + 2k, \frac{4k-1}{3} + 2k, 6k + 1\}$ and $\{-2\frac{2k-2}{3} - 2k, -2\frac{4k-1}{3} - 2k, 8k - 2\}$. □

**Lemma 5.11.** *If* $n \equiv 7 \pmod{12}$ *and* $n \geq 31$*, then there exists a proper pure rotational LHTS(n) with* $\frac{7}{3}(n - 1)$ *cyclic triples.*

*Proof.* Let $n = 12k + 7$, $k \geq 2$, and consider the partition

$$P = \big\{ \{r, r + 2k, -2r - 2k\} : r = 2, 3, \ldots, 2k \text{ and } r \neq k + 1 \big\}$$
$$\cup \big\{ \{-r, -r + 6k + 3, 2r - 6k - 3\} : r = 1, \ldots, 2k \text{ and } r \neq k + 2 \text{ or } 2k - 1 \big\}$$
$$\cup \big\{ \{4k + 1, 10k + 4, -2k + 1\} \big\}$$
$$\cup \big\{ \{1, k + 1, -k - 2\}, \{3k + 1, 4k + 4, 5k + 1\}, \{-4k + 1, -2k - 5, 6k + 4\} \big\} \quad (\text{if } k \neq 3)$$
$$\cup \big\{ \{1, 10, 31\}, \{4, 16, 22\} \big\} \text{ (if } k = 3)$$

of the set $\mathbb{Z}_{n-1} \setminus \{0, \pm\frac{n-1}{3}, \pm\frac{n-1}{6}, \frac{n-1}{2}\}$ into balanced triples. By Lemma 5.1 there exists a partial cyclic LDTS($n-1$) corresponding to the partition $P$. Augmenting this system with the orbits of the cyclic triples $(0, \infty, 2k + 1)$, $(0, 8k + 4, 4k + 2)$ and $(0, 2k + 1, 8k + 4)$ gives a proper rotational LHTS($n$). If $k \equiv 0 \pmod 3$ and $k \geq 6$, then the system contains $n - 1$ Steiner triples, otherwise the system is pure.

To obtain a pure LHTS for $k \equiv 0 \pmod 3$ where $k \geq 6$, replace the triples $\{-\frac{4k+3}{3}, -\frac{4k+3}{3} + 6k + 3, 2\frac{4k+3}{3} - 6k - 3\}$, $\{-\frac{2k-3}{3}, -\frac{2k-3}{3} + 6k + 3, 2\frac{2k-3}{3} - 6k - 3\}$ and $\{2k, 4k, -6k\}$ in $P$ with the triples $\{2k, -\frac{4k+3}{3} + 6k + 3, -\frac{2k-3}{3} + 6k + 3\}$, $\{4k, 2\frac{4k+3}{3} - 6k - 3, -\frac{2k-3}{3}\}$ and $\{-6k, 2\frac{2k-3}{3} - 6k - 3, -\frac{4k+3}{3}\}$. □

Putting together the results in this paper yields the following theorems:

**Theorem 5.12.** *A pure cyclic LDTS(n) exists if and only if* $n \equiv 1 \pmod 6$ *and* $n \geq 13$.

**Theorem 5.13.** *A proper cyclic LDTS(n) exists if and only if* $n \equiv 1$ *or* $3 \pmod 6$ *and* $n \geq 13$.

**Theorem 5.14.** *A proper pure cyclic LHTS(n) exists if and only if* $n \equiv 1 \pmod 6$ *and* $n \geq 19$.

**Theorem 5.15.** *A proper cyclic LHTS(n) exists if and only if* $n \equiv 1$ *or* $3 \pmod 6$ *and* $n \geq 19$.

**Theorem 5.16.** *A proper rotational LDTS(n) exists if and only if* $n \equiv 3 \pmod 6$ *and* $n \geq 15$*. There exists no pure rotational LDTS.*

**Theorem 5.17.** *A proper pure rotational LHTS(n) exists if and only if* $n \equiv 1 \pmod 3$*,* $n \geq 16$ *and* $n \neq 19$*.*

**Theorem 5.18.** *A proper rotational LHTS(n) exists if and only if* $n \equiv 1$*,* $3$ *or* $4 \pmod 6$*,* $n \geq 16$ *and* $n \neq 19$*.*

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# FLEXIBLE LATIN DIRECTED TRIPLE SYSTEMS

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Abstract. It is well known that, given a Steiner triple system, a quasigroup can be formed by defining an operation $\cdot$ by the identities $x \cdot x = x$ and $x \cdot y = z$ where $z$ is the third point in the block containing the pair $\{x, y\}$. The same is true for a Mendelsohn triple system where the pair $(x, y)$ is considered to be ordered. But it is not true in general for directed triple systems. However directed triple systems which form quasigroups under this operation do exist and we call these Latin directed triple systems. The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ but those associated with Latin directed triple systems need not. In a previous paper, [Discrete Mathematics 312 (2012), 597–607], we studied non-flexible Latin directed triple systems. In this paper we turn our attention to flexible Latin directed triple systems.

## 1. Introduction

This paper is a sequel to [4]. There, we introduced the concepts of a Latin directed triple system and a DTS-quasigroup, developed some of the basic theory and determined the existence spectrum of Latin directed triple systems whose associated quasigroups do not satisfy the flexible law. Here we turn our attention to flexible Latin directed triple systems.

First we recall the basic definitions and results which are appropriate for our purposes. A *Steiner triple system* of order $n$, STS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points, also called blocks, taken from $V$ such that every pair of distinct points from $V$ appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [8]. A *Steiner quasigroup* or *squag* or *idempotent totally symmetric quasigroup* is a pair $(Q, \cdot)$ where $Q$ is a set and $\cdot$ is an operation on $Q$ satisfying the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS($n$), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if $Q$ is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups [12, Theorem V.1.11]. This is all well-known.

Next consider ordered triples. There are two possibilities. A *cyclically ordered triple*, denoted by $(x, y, z)$, contains the ordered pairs $(x, y)$, $(y, z)$, $(z, x)$ and a *transitively ordered triple*, denoted by $\langle x, y, z\rangle$ contains the ordered pairs $(x, y)$, $(y, z)$, $(x, z)$.

A *Mendelsohn triple system* of order $n$, MTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclically ordered triples of distinct points

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taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or 1 (mod 3), $n \neq 6$ [11]. Quasigroups can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for Steiner triple systems. Note that the law $y \cdot (x \cdot y) = x$ is usually called semi-symmetric. So the quasigroups are known as *idempotent semisymmetric quasigroups* [1, Remark 2.12] or *Mendelsohn quasigroups*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems and Mendelsohn quasigroups.

A *directed triple system* of order $n$, DTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of transitively ordered triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or 1 (mod 3) [7]. Given a DTS($n$), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$ where $z$ is the third element in the transitive triple containing the ordered pair $(x, y)$. However the structure obtained is not necessarily a quasigroup. If $\langle u, x, y \rangle$ and $\langle y, v, x \rangle \in \mathcal{B}$ then $u \cdot x = v \cdot x = y$. But some DTS($n$)s do yield quasigroups. Such a DTS($n$) will be called a *Latin directed triple system*, and denoted by LDTS($n$), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*. In [4] an easy necessary and sufficient condition for a directed triple system to be Latin was proved.

**Theorem 1.1.** *Let $D = (V, \mathcal{B})$ be a DTS(n). Then $D$ is an LDTS(n) if and only if $\langle x, y, z \rangle \in \mathcal{B} \Rightarrow \langle w, y, x \rangle \in \mathcal{B}$ for some $w \in V$.*

Before proceeding further, it is important to point out two fundamental differences between DTS-quasigroups and Steiner or Mendelsohn quasigroups which motivates the study of these structures. First, DTS-quasigroups are *not* in one-one correspondence with Latin directed triple systems. Non-isomorphic LDTS($n$)s can yield identical DTS-quasigroups. In view of this, for purposes of enumeration, it makes more sense to count non-isomorphic DTS-quasigroups rather than non-isomorphic LDTS($n$)s. Secondly all Steiner and Mendelsohn quasigroups satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. DTS-quasigroups need not. In [4], the two following results were proved.

**Theorem 1.2.** *The number of non-isomorphic DTS-quasigroups of order* $n = 3$, 4, 6, 7, 9, 10, 12 *are* 1, 0, 0, 2, 4, 0, 2 *respectively.*

**Theorem 1.3.** *The existence spectrum of non-flexible LDTS(n)s is* $n \equiv 0, 1$ (mod 3), $n \neq 3$, 4, 6, 7, 10.

For flexible DTS-quasigroups, again there is an easy necessary and sufficient condition.

**Theorem 1.4.** *A DTS-quasigroup obtained from an LDTS(n), $D = (V, \mathcal{B})$, satisfies the flexible law if and only if $\langle x, y, z \rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x \rangle \in \mathcal{B}$.*

Note that trivially a Steiner quasigroup is a DTS-quasigroup. Such a DTS-quasigroup will be called *improper*; all others are *proper*. From [4], there exist only two non-isomorphic proper flexible DTS-quasigroups of order less than 13; one of order 7 and one of order 9. They are given in the two examples below, and in the same format as in [4], as Latin directed triple systems. For simplicity commas are omitted from the triples. The set $\mathcal{T}$ is the set of unordered triples or *Steiner triples*. Each triple $\{x, y, z\}$ represents a pair of transitively ordered triples in one of three ways, (i) $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$; or (ii) $\langle y, z, x\rangle$ and $\langle x, z, y\rangle$; or (iii) $\langle z, x, y\rangle$ and $\langle y, x, z\rangle$. Thus these triples are *bidirectional.* The set $\mathcal{D}$ is a set of transitively ordered triples or *unidirectional* triples. Replacing a pair of bidirectional triples in an LDTS($n$), say $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$, with a different pair of bidirectional triples, say $\langle y, z, x\rangle$ and $\langle x, z, y\rangle$, gives a system which yields the same DTS-quasigroup as the first and yet the two LDTS($n$)s may be non-isomorphic, [4, Example 2.4].

**Example 1.5.** Flexible LDTS(7).
$V = \{0, 1, 2, 3, 4, 5, 6\}$.
$\mathcal{T} = \big\{\{012\}, \{034\}, \{056\}\big\}$ and
$\mathcal{D} = \big\{\langle 315\rangle, \langle 514\rangle, \langle 416\rangle, \langle 613\rangle, \langle 326\rangle, \langle 624\rangle, \langle 425\rangle, \langle 523\rangle\big\}$.

**Example 1.6.** Flexible LDTS(9).
$V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
$\mathcal{T} = \big\{\{018\}, \{258\}, \{368\}, \{478\}, \{246\}, \{357\}\big\}$ and
$\mathcal{D} = \big\{\langle 207\rangle, \langle 706\rangle, \langle 605\rangle, \langle 504\rangle, \langle 403\rangle, \langle 302\rangle, \langle 213\rangle, \langle 314\rangle, \langle 415\rangle, \langle 516\rangle, \langle 617\rangle, \langle 712\rangle\big\}$.

## 2. Structure of flexible DTS-quasigroups

In a further paper [3], flexible DTS-quasigroups were shown to have a certain structure in terms of their topology. Let $D = (V, \mathcal{B})$ be an LDTS($n$). Denote by $F$, the set of all unordered triples $\{x, y, z\}$, where $\langle x, y, z\rangle$ runs through all unidirectional triples of $D$. Now consider $F$ as a set of faces. Each edge $\{x, y\}$ is incident to two faces and hence we get a generalized pseudosurface. By separating pinch points we obtain a set of one or more components which are an invariant of the LDTS($n$) and are very useful in determining whether two DTS-quasigroups are isomorphic.

Consider a unidirectional triple $\langle z_1, x, z_0\rangle \in \mathcal{B}$. Then, using Theorem 1.1, there exists $k \geq 3$ and points $z_0, z_1, z_2, \ldots, z_{k-1}$ such that

$$\langle z_1, x, z_0\rangle, \langle z_2, x, z_1\rangle, \ldots, \langle z_{k-1}, x, z_{k-2}\rangle, \langle z_0, x, z_{k-1}\rangle \in \mathcal{B}.$$

If $D$ is also flexible, using Theorem 1.4,

$$\langle z_1, y, z_2\rangle, \langle z_2, y, z_3\rangle, \ldots, \langle z_{k-1}, y, z_0\rangle, \langle z_0, y, z_1\rangle \in \mathcal{B}$$

where $y = z_0 \cdot z_1 = z_1 \cdot z_2 = \cdots = z_{k-2} \cdot z_{k-1} = z_{k-1} \cdot z_0$. These $2k$ transitive triples define a $k$-gonal bipyramid; denoted by $O_k$, i.e. a graph of $k+2$ vertices with a cycle of length $k$, the points of which can be thought of as situated around the equator of a sphere, and two middle vertex points which are connected to all points of the cycle and which can be thought of as situated at the poles of the sphere. Thus we have the following important result.

**Theorem 2.1.** *A flexible DTS-quasigroup of order $n$ exists if and only if the complete graph $K_n$ can be decomposed into triangles and graphs $O_k$, $k \geq 3$. The components of the generalized pseudosurface of the quasigroup are all spheres.*

It is worth remarking that when at least one of the graphs $O_k$ has $k \geq 6$ and even, the decomposition of $K_n$ as described in the theorem may also be used to obtain a non-flexible system. Replace triples

$$\langle z_{2i+1}, x, z_{2i}\rangle, \langle z_{2i+2}, x, z_{2i+1}\rangle, \langle z_{2i}, y, z_{2i+1}\rangle, \langle z_{2i+1}, y, z_{2i+2}\rangle$$

in the flexible system by triples

$$\langle z_{2i+1}, z_{2i}, x\rangle, \langle x, z_{2i+2}, z_{2i+1}\rangle, \langle y, z_{2i}, z_{2i+1}\rangle, \langle z_{2i+1}, z_{2i+2}, y\rangle,$$

$i = 0, 1, \ldots, (k-2)/2$, subscript arithmetic modulo $k$. As illustration from Example 1.6, the following is a non-flexible LDTS(9).

**Example 2.2.** Non-flexible LDTS(9).
$V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
$\mathcal{T} = \big\{\{018\}, \{258\}, \{368\}, \{478\}, \{246\}, \{357\}\big\}$ and
$\mathcal{D} = \big\{\langle 270\rangle, \langle 076\rangle, \langle 650\rangle, \langle 054\rangle, \langle 430\rangle, \langle 032\rangle, \langle 231\rangle, \langle 134\rangle, \langle 451\rangle, \langle 156\rangle, \langle 671\rangle, \langle 172\rangle\big\}$.

The number of equator cycles of each length is clearly an invariant of a flexible DTS-quasigroup. Another invariant can be calculated as follows. The *type* of a vertex is the list of valencies which it has as a middle vertex or pole of a bipyramid. The number of vertices of each type is then also an invariant.

Thus, for the DTS-quasigroup given in Example 1.5, there is an equator cycle of length 4 $(3, 5, 4, 6)$ and two points (1 and 2) of type 4. For the DTS-quasigroup given in Example 1.6 there is an equator cycle of length 6 $(2, 7, 6, 5, 4, 3)$ and two points (0 and 1) of type 6. A more instructive example however of order 13 is given below.

**Example 2.3.** Flexible LDTS(13).
$V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
$\mathcal{T}=\big\{\{045\}, \{179\}, \{1TW\}, \{358\}, \{37W\}, \{59T\}\big\}$ and
$\mathcal{D} = \big\{\langle 103\rangle, \langle 302\rangle, \langle 201\rangle, \langle 142\rangle, \langle 243\rangle, \langle 341\rangle, \langle 629\rangle, \langle 92E\rangle, \langle E2T\rangle, \langle T26\rangle, \langle 63T\rangle, \langle T3E\rangle, \langle E39\rangle, \langle 936\rangle, \langle 156\rangle, \langle 65W\rangle, \langle W52\rangle, \langle 257\rangle, \langle 75E\rangle, \langle E51\rangle, \langle 18E\rangle, \langle E87\rangle, \langle 782\rangle, \langle 28W\rangle, \langle W86\rangle, \langle 681\rangle, \langle 60E\rangle, \langle E0W\rangle, \langle W09\rangle, \langle 908\rangle, \langle 80T\rangle, \langle T07\rangle, \langle 706\rangle, \langle 647\rangle, \langle 74T\rangle, \langle T48\rangle, \langle 849\rangle, \langle 94W\rangle, \langle W4E\rangle, \langle E46\rangle\big\}$.

For this system there are equator cycles of length 3, 4, 6 and 7, two points (2 and 3) of type 4, two points (5 and 8) of type 6 and two points (0 and 4) of type $3, 7$.

At $n = 13$, the combinatorial explosion takes over and, as reported in [3], there are 1 206 969 non-isomorphic DTS-quasigroups of order 13. Details of their automorphism groups and genera of their separated surface components are also given in that paper. However, only 924 of these quasigroups are flexible including the two Steiner quasigroups of this order. Table 1 shows the classification in terms of numbers of Steiner triples ($t$), lengths of equator cycles, and types of middle valency points.

## 3. Recursive constructions

In this section we present, in the form of theorems, some recursive constructions for flexible Latin directed triple systems. Using Theorem 2.1, we express these in terms of decompositions of the complete graph $K_n$ into triangles and $k$-gonal bipyramids $O_k$. We represent the latter by the notation $[N : E_1, E_2, \ldots, E_k : S]$ where $N$ and $S$ are the poles and $(E_1, E_2, \ldots, E_k)$ is the equator cycle. The first two constructions are "doubling" and "trebling" constructions respectively which often apply for combinatorial designs.

**Theorem 3.1.** *If there exists a flexible LDTS($n$) based on a decomposition of the complete graph $K_n$ into triangles and graphs $O_k$, $k \geq 4$ and even, then there exists a flexible LDTS($2n + 1$).*

| | Equator Cycles | | | | | | | | Middle Valency Points | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | **3** | **4** | **5** | **6** | **7** | **8** | **10** | $t$ | **4** | **6** | **3, 3** | **8** | **5, 3** | **4, 4** | **10** | **7, 3** | **6, 4** | **5, 5** | **4, 3, 3** | **4, 4, 4** |
| 76 | | 2 | | 1 | | | | 12 | 4 | 2 | | | | | | | | | | |
| 69 | | 1 | | | | | 1 | 12 | 2 | | | | | | 2 | | | | | |
| 56 | | 2 | | | | | 1 | 8 | 4 | | | | | | 2 | | | | | |
| 46 | | 1 | | 1 | | | | 16 | 2 | 2 | | | | | | | | | | |
| 41 | | 1 | | 2 | | | | 10 | 2 | 4 | | | | | | | | | | |
| 36 | | 3 | | | | | | 14 | 6 | | | | | | | | | | | |
| 36 | | 3 | | | | | | 14 | 4 | | | | | 1 | | | | | | |
| 36 | | 2 | | 1 | | | | 12 | 2 | | | | | | | | 2 | | | |
| 32 | 1 | 2 | | | 1 | | | 8 | 4 | | | | | | | 2 | | | | |
| 32 | | 2 | | 1 | | | | 12 | 2 | 2 | | | | 1 | | | | | | |
| 28 | | 3 | | 1 | | | | 8 | 6 | 2 | | | | | | | | | | |
| 28 | | 3 | | 1 | | | | 8 | 4 | 2 | | | | 1 | | | | | | |
| 28 | | 3 | | 1 | | | | 8 | 4 | | | | | | | | 2 | | | |
| 28 | | 2 | | | | | | 18 | 4 | | | | | | | | | | | |
| 28 | | | | 1 | | | 1 | 10 | | 2 | | | | | 2 | | | | | |
| 24 | | 2 | | 2 | | | | 6 | 2 | 2 | | | | | | | 2 | | | |
| 24 | 1 | 1 | | | 1 | | | 12 | 2 | | | | | | | 2 | | | | |
| 22 | | 1 | | 2 | | | | 10 | | 2 | | | | | | | 2 | | | |
| 18 | | | | 2 | | | | 14 | | 4 | | | | | | | | | | |
| 16 | | 2 | | | | | 1 | 8 | 2 | | | | | 1 | 2 | | | | | |
| 13 | | | | | | | 1 | 16 | | | | | | | 2 | | | | | |
| 12 | 2 | 1 | | | | 1 | | 8 | 2 | | 2 | 2 | | | | | | | | |
| 12 | 1 | 1 | | 1 | 1 | | | 6 | 2 | 2 | | | | | | 2 | | | | |
| 12 | | 3 | | 1 | | | | 8 | 2 | 2 | | | | 2 | | | | | | |
| 12 | 1 | | | 1 | 1 | | | 10 | | 2 | | | | | | 2 | | | | |
| 12 | | 1 | | 1 | | | 1 | 6 | 2 | 2 | | | | | 2 | | | | | |
| 10 | 1 | | | | 1 | 1 | | 8 | | | | 2 | | | | 2 | | | | |
| 9 | | | | 3 | | | | 8 | | 6 | | | | | | | | | | |
| 8 | | 3 | | 1 | | | | 8 | 2 | | | | | 1 | | | 2 | | | |
| 8 | | 2 | | 2 | | | | 6 | 2 | 4 | | | | 1 | | | | | | |
| 8 | 2 | | | | | | 1 | 10 | | | 2 | | | | 2 | | | | | |
| 8 | | 2 | | | | | | 18 | 2 | | | | | 1 | | | | | | |
| 6 | 3 | | | | 1 | | | 10 | | | 2 | | | | | 2 | | | | |
| 6 | 2 | 1 | | | | 1 | | 8 | | | | 2 | | | | | | | 2 | |
| 6 | | 1 | | 3 | | | | 4 | | 4 | | | | | | | 2 | | | |
| 6 | 2 | | | | | 1 | | 12 | | | 2 | 2 | | | | | | | | |
| 6 | | 1 | 2 | | | | | 12 | 2 | | | | | | | | | 2 | | |
| 6 | 1 | | | | 1 | | | 16 | | | | | | | | 2 | | | | |
| 6 | | 1 | | 1 | | | | 16 | | | | | | | | | 2 | | | |
| 5 | | 1 | | | | | | 22 | 2 | | | | | | | | | | | |
| 5 | | | | 1 | | | | 20 | | 2 | | | | | | | | | | |
| 4 | 2 | | 1 | | 1 | | | 8 | | | | | 2 | | | 2 | | | | |
| 4 | 1 | 2 | | | 1 | | | 8 | | | | | | 2 | | 2 | | | | |
| 4 | | 3 | | 1 | | | | 8 | 3 | 2 | | | | | | | | | | 1 |
| 4 | | 2 | | 2 | | | | 6 | 4 | 4 | | | | | | | | | | |
| 4 | | 3 | | | | | | 14 | 2 | | | | | 2 | | | | | | |
| 4 | | | 2 | | | | | 16 | | | | | | | | | | 2 | | |
| 2 | 2 | 1 | | 2 | | | | 4 | | 2 | 2 | | | | | | 2 | | | |
| 2 | 2 | 1 | | 1 | | | | 10 | 2 | 2 | 2 | | | | | | | | | |
| 2 | 2 | 1 | | 1 | | | | 10 | | 2 | | | | | | | | | 2 | |
| 2 | 2 | 1 | | 1 | | | | 10 | | | 2 | | | | | | 2 | | | |
| 2 | 2 | 1 | | | | | | 16 | 2 | | 2 | | | | | | | | | |
| 2 | 2 | | | 1 | | | | 14 | | 2 | 2 | | | | | | | | | |
| 2 | | 3 | | | | | | 14 | 3 | | | | | | | | | | | 1 |
| 2 | | | | | | | | 26 | | | | | | | | | | | | |
| 1 | 2 | 1 | | 2 | | | | 4 | | 4 | | | | | | | | | 2 | |
| 1 | 2 | | | 2 | | | | 8 | | 4 | 2 | | | | | | | | | |
| 1 | 2 | 1 | | | | | | 16 | | | | | | | | | | | 2 | |
| 1 | 2 | | | | | | | 20 | | | 2 | | | | | | | | | |

TABLE 1. Classification of flexible LDTS(13)s.

*Proof.* Let $(V, \mathcal{B})$ be a decomposition of the complete graph $K_n$ into triangles and graphs $O_k$, $k \geq 4$ and even, as stated in the statement of the theorem where $V = \{0, 1, \ldots, n-1\}$. Let $V' = \{\, x' : x \in V \,\}$ and $W = V \cup V' \cup \{\infty\}$. Construct a decomposition of the complete graph $K_{2n+1}$ on the set $W$ as follows. For all $[N : E_1, E_2, E_3, \ldots, E_{2l} : S] \in \mathcal{B}$, assign

$$[N : E_1, E_2, E_3, \ldots, E_{2l} : S], \quad [N : E_1', E_2', E_3', \ldots, E_{2l}' : S],$$
$$[N' : E_1, E_2', E_3, \ldots, E_{2l}' : S'], \quad [N' : E_1', E_2, E_3', \ldots, E_{2l} : S'] \quad \in \quad \mathcal{B}'.$$

For all $\{x, y, z\} \in \mathcal{B}$, assign $[x : y, z, y', z' : x'] \in \mathcal{B}'$. Further let $\{x, x', \infty\} \in \mathcal{B}'$ for all $x \in V$. Then $(W, \mathcal{B}')$ is a decomposition of the complete graph $K_{2n+1}$ into triangles and $k$-gonal bipyramids. □

**Theorem 3.2.** *If there exists a flexible LDTS(n) based on a decomposition of the complete graph $K_n$ into triangles and graphs $O_k$, $k \geq 4$ and even, then there exists a flexible LDTS(3n).*

*Proof.* Let $(V, \mathcal{B})$ be a decomposition of the complete graph $K_n$ into triangles and graphs $O_k$, $k \geq 4$ and even, as stated in the statement of the theorem where $V = \{0, 1, \ldots, n-1\}$. Let $V' = \{\, x' : x \in V \,\}$, $V'' = \{\, x'' : x \in V \,\}$ and $W = V \cup V' \cup V''$. Construct a decomposition of the complete graph $K_{3n}$ on the set $W$ as follows. For all $[N : E_1, E_2, E_3, \ldots, E_{2l} : S] \in \mathcal{B}$, assign

$$[N : E_1, E_2, E_3, \ldots, E_{2l} : S], \quad [N : E_1', E_2'', E_3', \ldots, E_{2l}'' : S],$$
$$[N : E_1'', E_2', E_3'', \ldots, E_{2l}' : S], \quad [N' : E_1', E_2', E_3', \ldots, E_{2l}' : S'],$$
$$[N' : E_1, E_2'', E_3, \ldots, E_{2l}'' : S'], \quad [N' : E_1'', E_2, E_3'', \ldots, E_{2l} : S'],$$
$$[N'' : E_1'', E_2'', E_3'', \ldots, E_{2l}'' : S''], \quad [N'' : E_1, E_2', E_3, \ldots, E_{2l}' : S''],$$
$$[N'' : E_1', E_2, E_3', \ldots, E_{2l} : S''] \quad \in \quad \mathcal{B}'.$$

For all $\{x, y, z\} \in \mathcal{B}$, assign $\{x, y, z'\}$, $\{x, y', z''\}$, $\{x, y'', z\}$, $[x' : y, z, y', z', y'', z'' : x''] \in \mathcal{B}'$. Further let $\{x, x', x''\} \in \mathcal{B}'$ for all $x \in V$. Note that for all $\{x, y, z\} \in \mathcal{B}$, the construction yields a flexible LDTS(9) on the point set $\{x, x', x'', y, y', y'', z, z', z''\}$. Then $(W, \mathcal{B}')$ is a decomposition of the complete graph $K_{3n}$ into triangles and $k$-gonal bipyramids. □

The next construction is also a "doubling" construction and employs a Hamiltonian decomposition.

**Theorem 3.3.** *If there exists a flexible LDTS(n), then there exists a flexible LDTS($2n+1$).*

*Proof.* Let $(V, \mathcal{B})$ be a flexible LDTS($n$) where $V = \{x_1, x_2, \ldots, x_n\}$ disjoint from the set $\mathcal{Z}_{n+1}$ and $K_{n+1}$ be the complete graph on $\mathcal{Z}_{n+1}$.

Suppose that $n$ is even. Take a decomposition of $K_{n+1}$ into $n/2$ disjoint Hamiltonian cycles $H_i$, $1 \leq i \leq n/2$. For each $i$, construct a $(n+1)$-gonal bipyramid $[x_{2i-1} : H_i : x_{2i}]$ and let $\mathcal{B}'$ be the set of unidirectional triples obtained from these bipyramids. Then $(V \cup \mathcal{Z}_{n+1}, \mathcal{B} \cup \mathcal{B}')$ is a flexible LDTS($2n+1$).

Now suppose that $n$ is odd. Remove a one-factor $F$ from $K_{n+1}$ and proceed as in the even case using a decomposition of the graph $K_{n+1} \setminus F$ into $(n-1)/2$ disjoint Hamiltonian cycles $H_i$, $1 \leq i \leq (n-1)/2$. Further let $\mathcal{T}$ be the set of Steiner triples $\{a, b, x_n\}$ where the edge $\{a, b\} \in F$. Then $(V \cup \mathcal{Z}_{n+1}, \mathcal{B} \cup \mathcal{B}' \cup \mathcal{T})$ is a flexible LDTS($2n+1$). □

We remark that in the proof of the theorem we may replace the Hamiltonian decomposition by any 2-factorization of the relevant graph. In this respect, a particularly elegant and easy way of implementing the construction is for each $d \in \mathcal{Z}_{n+1} \setminus \{0\}$ and $i \in \mathcal{Z}_{n+1}$, assign $\langle i, x_d, i+d \rangle \in \mathcal{B}'$.

A directed triple system, $(V, \mathcal{B})$, is said to be *pure* if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle z, y, x\rangle \notin \mathcal{B}$. In the construction described in the above theorem, if $n$ is even and the LDTS($n$) is pure then so is the LDTS($2n+1$). The DTS-quasigroups obtained from pure Latin directed triple systems are anti-commutative.

The final construction is in a similar vein to the previous construction. We need some further definitions. In a Steiner triple system, STS($n$), a *parallel class* is a set of blocks which collectively contain every point of the STS($n$) precisely once. A *Kirkman triple system* of order $n$, KTS($n$), is a triple $(V, \mathcal{B}, \mathcal{R})$ where $(V, \mathcal{B})$ is an STS($n$) and $\mathcal{R}$ is a *partition* or *resolution* of the set of blocks $\mathcal{B}$ into parallel classes. Such systems exist if and only if $n \equiv 3 \pmod 6$, [9], [13].

**Theorem 3.4.** *If there exists a flexible LDTS($2n$), then there exists a flexible LDTS($6s+3+2n$) for all $s \geq (n-1)/3$.*

*Proof.* Let $(V, \mathcal{B})$ be a flexible LDTS($2n$) where $V = \{1, 2, \ldots, 2n\}$ and $(W, \mathcal{S}, \mathcal{R})$ be a KTS($6s+3$) where the set $W$ is disjoint from the set $V$. The partition $\mathcal{R}$ consists of $3s+1$ parallel classes $\Pi_i$, $1 \leq i \leq 3s+1$. For the first $n$ parallel classes $\Pi_i$, $1 \leq i \leq n$, construct trigonal bipyramids $[2i-1 : x, y, z : 2i]$ where $\{x, y, z\} \in \Pi_i$, and then decompose these into unidirectional triples

$$\langle x, 2i-1, y\rangle, \langle y, 2i-1, z\rangle, \langle z, 2i-1, x\rangle, \langle y, 2i, x\rangle, \langle z, 2i, y\rangle, \langle x, 2i, z\rangle.$$

Denote this set of unidirectional triples by $\mathcal{B}'$. The remaining parallel classes together form the set of unordered or Steiner triples $\mathcal{T} = \bigcup_{i=n+1}^{3s+1} \Pi_i$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}' \cup \mathcal{T})$ is a flexible LDTS($6s+3+2n$). □

## 4. Existence of flexible Latin directed triple systems

In this section we determine the existence spectrum of flexible LDTS($n$). For $n$ odd, this has previously been done in [4] but the proof is short, so we include it for completeness. We will need a definition. In a Steiner triple system, a collection of four triples on six points is called a *Pasch configuration*. It is easily seen that this structure necessarily has the form $\{a, b, c\}$, $\{a, y, z\}$, $\{x, b, z\}$, $\{x, y, c\}$. Given such a Pasch configuration, we will replace it by transitive triples $\langle a, b, c\rangle$, $\langle a, y, z\rangle$, $\langle x, b, z\rangle$, $\langle x, y, c\rangle$, $\langle z, y, x\rangle$, $\langle c, b, x\rangle$, $\langle c, y, a\rangle$, $\langle z, b, a\rangle$. These can be thought of as a partial LDTS(6) but a crucial point is that they satisfy the flexible law. We will denote this collection of eight transitive triples by $\mathcal{P}$. Part of the proof also uses a standard technique, known as Wilson's fundamental construction, for which we need the concept of a *group divisible design* (GDD). A 3-GDD of type $g^u$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where $V$ is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of $V$ into $u$ subsets of cardinality $g$ called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. In the proof for $n$ even, we will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set $V$ being of cardinality $v = gu + m$ and the partition $G$ being into $u$ subsets of cardinality $g$ and one set of cardinality $m$. Necessary and sufficient conditions for 3-GDDs of type $g^u$ were determined in [6] and for 3-GDDs of type $g^u m^1$ in [2]; a convenient reference is [5] where the existence of all the GDDs that are used can be verified.

**Proposition 4.1.** *There exists a proper flexible LDTS(n) for all* $n \equiv 1, 3 \pmod 6$.

*Proof.*

(a) $n \equiv 3, 7 \pmod{12}$. Put $m = (n-1)/2$ and choose an STS$(m)$, $(V, \mathcal{B})$. Let $V' = \{\, x' : x \in V \,\}$ and $W = V \cup V' \cup \{\infty\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\{x, y, z\} \in \mathcal{B}$, assign $\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}$, $\{x', y', z\} \in \mathcal{B}'$. Further let $\{x, x', \infty\} \in \mathcal{B}'$ for all $x \in V$. Then $(W, \mathcal{B}')$ is an STS$(n)$. In order to obtain a LDTS$(n)$ replace each Pasch configuration as above by the set $\mathcal{P}$ of transitive triples, and retain the sets containing the point $\infty$ as Steiner triples. Because the LDTS$(n)$ is constructed of flexible components, i.e. just the flexible partial LDTS(6), $\mathcal{P}$, and the trivial Steiner quasigroup on 3 points, it is also flexible.

(b) $n \equiv 9 \pmod{12}$. Put $m = (n-3)/2$ and choose an STS$(m)$, $(V, \mathcal{B})$ which contains a parallel class. Denote this parallel class by $\Pi$. Let $V' = \{\, x' : x \in V \,\}$ and $W = V \cup V' \cup \{\infty_1, \infty_2, \infty_3\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\{x, y, z\} \in \Pi$, assign $\{x, y, z\}$, $\{x', y', z'\}$, $\{x, x', \infty_1\}$, $\{y, y', \infty_1\}$, $\{z, z', \infty_1\}$, $\{x, y', \infty_2\}$, $\{y, z', \infty_2\}$, $\{z, x', \infty_2\}$, $\{x, z', \infty_3\}$, $\{y, x', \infty_3\}$, $\{z, y', \infty_3\} \in \mathcal{B}'$ and for all $\{x, y, z\} \in \mathcal{B} \setminus \Pi$, assign $\{x, y, z\}$, $\{x, y', z'\}$, $\{x', y, z'\}$, $\{x', y', z\} \in \mathcal{B}'$. Finally let $\{\infty_1, \infty_2, \infty_3\} \in \mathcal{B}'$. Then $(W, \mathcal{B}')$ is an STS$(n)$. In order to obtain an LDTS$(n)$, replace each Pasch configuration by the set $\mathcal{P}$ of transitive triples in the same way as in (a). Further replace each collection of eleven triples corresponding to each block of the parallel class, together with the set $\{\infty_1, \infty_2, \infty_3\}$, by the flexible LDTS(9) from Example 1.6, ensuring that the triple $\{\infty_1, \infty_2, \infty_3\}$ corresponds to a Steiner triple for each collection.

(c) $n \equiv 1 \pmod{12}$. Take a 3-GDD of type $6^s, s \geq 3$. Inflate each point by a factor 2 and adjoin an extra point $\infty$. On each inflated group, together with the point $\infty$, place the flexible LDTS(13) given in Example 2.3. On each inflated block place the set $\mathcal{P}$ of transitive triples $\langle a, b, c\rangle$, $\langle a, y, z\rangle$, $\langle x, b, z\rangle$, $\langle x, y, c\rangle$, $\langle z, y, x\rangle$, $\langle c, b, x\rangle$, $\langle c, y, a\rangle$, $\langle z, b, a\rangle$, with the three sets of points $\{a, x\}, \{b, y\}, \{c, z\}$ as the inflated points in the three groups. We will use $\mathcal{P}$ in this manner throughout. This misses the value $n = 25$ but this can also be constructed in a similar manner by taking a 3-GDD of type $4^3$, inflating each point by a factor 2 and adjoining an extra point $\infty$. On each inflated group, together with the point $\infty$, place the flexible LDTS(9) from Example 1.6 and on each inflated block, place the set of transitive triples $\mathcal{P}$.

□

The determination of the spectrum of flexible LDTS$(n)$ for $n$ even is more intricate, mainly because there exist no LDTS$(n)$ for $n = 4$, 6, and 10, and the only two LDTS(12)s are not flexible, [4]. The smallest even order flexible system is LDTS(16). Again we will use Wilson's fundamental construction, but we will need a variety of 3-GDDs and initial systems. Flexible LDTS$(n)$ for $n = 16$, 18, 22, 24, 28, 30, 34, 36, and 40 are given as Examples A.1 to A.9 in the Appendix and were all found by computer search.

We can now prove a series of propositions

**Proposition 4.2.** *There exists a flexible LDTS(n) for all* $n \equiv 0, 16 \pmod{24}$.

*Proof.*

(a) $n \equiv 0 \pmod{48}$. Take a 3-GDD of type $8^{3s}$, $s \geq 1$. Inflate each point by a factor 2. On each inflated group place a flexible LDTS(16) and on each inflated block, place the set of transitive triples $\mathcal{P}$.

(b) $n \equiv 16 \pmod{48}$. Proceed as in (a) starting with a 3-GDD of type $8^{3s+1}$, $s \geq 1$.

(c) $n \equiv 24 \pmod{48}$. Again proceed as in (a) starting with a 3-GDD of type $8^{3s}12^1$, $s \geq 1$, and in addition on the inflated group of cardinality 12 place a flexible LDTS(24).

(d) $n \equiv 40 \pmod{48}$. Proceed as in (c) starting with a 3-GDD of type $8^{3s+1}12^1$, $s \geq 1$. This misses the value $n = 40$, but a flexible LDTS(40) is Example A.9 in the Appendix.

□

**Proposition 4.3.** *There exists a flexible LDTS(n) for all* $n \equiv 4, 12 \pmod{24}$, $n \geq 28$ *except* $n = 52$, *60, 76, 84.*

*Proof.*

(a) $n \equiv 4 \pmod{24}$. Take a 3-GDD of type $12^s 14^1$, $s \geq 3$. Inflate each point by a factor 2. On each inflated group place a flexible LDTS(24) or LDTS(28) as appropriate and on each inflated block, place the set of transitive triples $\mathcal{P}$. This misses the values $n = 52$ and 76.

(b) $n \equiv 12 \pmod{24}$. Proceed as in (a) starting with a 3-GDD of type $12^s 18^1$, $s \geq 3$ and using a flexible LDTS(36) instead of an LDTS(28). This misses the values $n = 60$ and 84.

□

Before dealing with the next two residue classes we will need two further flexible systems of orders 42 and 46. These can be obtained using the following elementary construction techniques.

**Proposition 4.4.**

(i) *If there exists a flexible LDTS(n), then there exists a flexible LDTS(*$3n - 2$*).*

(ii) *If there exists a flexible LDTS(n) containing a Steiner triple, then there exists a flexible LDTS(*$3n - 6$*), also containing a Steiner triple.*

*Proof.*

(i) Take three copies of the LDTS($n$) on point sets $\{\infty, 0_i, 1_i, \ldots, (n-2)_i\}$, $i \in \{0, 1, 2\}$ respectively. Then take a Latin square $L(i, j)$ of order $n - 1$ on the set $\{0, 1, \ldots, n-2\}$ and adjoin all Steiner triples $\{i_0, j_1, L(i, j)_2\}$, $0 \leq i \leq n-2$, $0 \leq j \leq n-2$.

(ii) Take three copies of the LDTS($n$) on point sets $\{\infty_1, \infty_2, \infty_3, 0_i, 1_i, \ldots, (n-4)_i\}$, $i \in \{0, 1, 2\}$ respectively, where $\{\infty_1, \infty_2, \infty_3\}$ is a Steiner triple in all three systems. Then take a Latin square $L(i, j)$ of order $n - 3$ on the set $\{0, 1, \ldots, n-4\}$ and adjoin all Steiner triples $\{i_0, j_1, L(i, j)_2\}$, $0 \leq i \leq n-4$, $0 \leq j \leq n-4$.

□

We now have the required LDTS(42) and LDTS(46) using the flexible LDTS(16) from Example A.1.

**Proposition 4.5.** *There exists a flexible LDTS(n) for all* $n \equiv 6, 10 \pmod{12}$, $n \geq 18$ *except* $n = 58$, *66, 70, 78, 82.*

*Proof.*

(a) $n \equiv 18 \pmod{36}$. Take a 3-GDD of type $9^{2s+1}$, $s \geq 1$. Inflate each point by a factor 2. On each inflated group place a flexible LDTS(18) and on each inflated block, place the set of transitive triples $\mathcal{P}$.

(b) $n \equiv 22, 30, 34, 42, 46 \pmod{36}$. Proceed as in (a) starting with a 3-GDD of type $9^{2s}m^1$, $s \geq 2$ where $m \in \{11, 15, 17, 21, 23\}$ and in addition on the single larger inflated group, place a flexible LDTS($2m$).

□

It remains to deal with the nine exceptional values.

**Proposition 4.6.** *There exist flexible LDTS(n) for* $n \in \{52$, *58, 60, 66, 70, 76, 78, 82,* $84\}$.

*Proof.*

(a) The values $n = 52$, 70, 82 can be obtained from the construction of Proposition 4.4 (i) using examples on 18, 24, 28 points respectively given in the Appendix.

(b) The values $n = 60$, 66, 78, 84 can be obtained from the construction of Proposition 4.4 (ii) using examples on 22, 24, 28, 30 points respectively again given in the Appendix.

(c) For $n = 76$, take a 3-GDD of type $15^5$ and on each group together with a further point $\infty$, place the flexible LDTS(16) in Example A.1. Each block is a Steiner triple.

(d) The value $n = 58$ is the most difficult. We shall use the same approach as for the non-flexible case. Define sets $\mathcal{N} = \{\, \infty_j : 0 \leq j \leq 6 \,\}$, $\mathcal{M}_k = \{\, i_k : 0 \leq i \leq 16 \,\}$, $k = 0, 1, 2$. Take three copies of the flexible LDTS(24) containing an LDTS(7) as a subsystem, constructed as in Example A.4 on point sets $\mathcal{N} \cup \mathcal{M}_0$, $\mathcal{N} \cup \mathcal{M}_1$, $\mathcal{N} \cup \mathcal{M}_2$ respectively, in each case with the LDTS(7) on the set $\mathcal{N}$. Then take a Latin square $L(i, j)$ of side 17 on the set $\{0, 1, \ldots, 16\}$ and adjoin all Steiner triples $\{i_0, j_1, L(i, j)_2\}$.

□

Collecting together all the results in this section gives the following theorem.

**Theorem 4.7.** *The existence spectrum of flexible LDTS(n)s is* $n \equiv 0, 1 \pmod{3}$, $n \neq 4$, *6, 10, 12.*

## Appendix. Examples of flexible LDTSs

The following examples were obtained by computer with the help of the model builder Mace4 [10] using an algebraic description of a DTS-quasigroup, see [3]. We denote the elements $(i, j) \in \mathcal{Z}_m \times \mathcal{Z}_n$ as $i_j$. For simplicity, we omit commas from the triples.

**Example A.1.** Flexible LDTS(16).
$V = (\mathcal{Z}_3 \times \mathcal{Z}_5) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
The starter blocks for $\mathcal{T}$ are $\{0_0\, 1_0\, 2_0\}$, $\{0_0\, 1_3\, 1_4\}$, $\{0_1\, 2_2\, 0_4\}$, $\{0_1\, 0_3\, 2_3\}$, $\{0_2\, 2_3\, \infty\}$, and for $\mathcal{D}$ are $\langle 1_2\, 0_0\, 0_4 \rangle$, $\langle 0_4\, 0_0\, 2_3 \rangle$, $\langle 2_3\, 0_0\, 1_2 \rangle$, $\langle 1_2\, 1_4\, 2_3 \rangle$, $\langle 2_3\, 1_4\, 0_4 \rangle$, $\langle 0_4\, 1_4\, 1_2 \rangle$,

$\langle 0_0\,0_1\,1_1\rangle$, $\langle 1_1\,0_1\,2_4\rangle$, $\langle 2_4\,0_1\,0_0\rangle$, $\langle 0_0\,\infty\,2_4\rangle$, $\langle 2_4\,\infty\,1_1\rangle$, $\langle 1_1\,\infty\,0_0\rangle$, $\langle 1_0\,0_2\,0_1\rangle$, $\langle 0_1\,0_2\,1_2\rangle$, $\langle 1_2\,0_2\,1_0\rangle$, $\langle 1_0\,1_3\,1_2\rangle$, $\langle 1_2\,1_3\,0_1\rangle$, $\langle 0_1\,1_3\,1_0\rangle$.

**Example A.2.** Flexible LDTS(18).
$V = \mathcal{Z}_3 \times \mathcal{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,2_0\}$, $\{0_1\,0_4\,1_5\}$, $\{0_2\,1_2\,2_2\}$, $\{0_4\,1_4\,2_4\}$, and for $\mathcal{D}$ are $\langle 1_3\,0_0\,0_5\rangle$, $\langle 0_5\,0_0\,0_4\rangle$, $\langle 0_4\,0_0\,1_3\rangle$, $\langle 1_3\,2_3\,0_4\rangle$, $\langle 0_4\,2_3\,0_5\rangle$, $\langle 0_5\,2_3\,1_3\rangle$, $\langle 2_0\,0_1\,2_1\rangle$, $\langle 2_1\,0_1\,1_3\rangle$, $\langle 1_3\,0_1\,2_0\rangle$, $\langle 2_0\,1_4\,1_3\rangle$, $\langle 1_3\,1_4\,2_1\rangle$, $\langle 2_1\,1_4\,2_0\rangle$, $\langle 1_0\,0_2\,0_1\rangle$, $\langle 0_1\,0_2\,2_5\rangle$, $\langle 2_5\,0_2\,1_0\rangle$, $\langle 1_0\,0_5\,2_5\rangle$, $\langle 2_5\,0_5\,0_1\rangle$, $\langle 0_1\,0_5\,1_0\rangle$, $\langle 0_0\,0_3\,0_2\rangle$, $\langle 0_2\,0_3\,0_5\rangle$, $\langle 0_5\,0_3\,2_2\rangle$, $\langle 2_2\,0_3\,0_1\rangle$, $\langle 0_1\,0_3\,1_2\rangle$, $\langle 1_2\,0_3\,0_0\rangle$, $\langle 0_0\,1_4\,1_2\rangle$, $\langle 1_2\,1_4\,0_1\rangle$, $\langle 0_1\,1_4\,2_2\rangle$, $\langle 2_2\,1_4\,0_5\rangle$, $\langle 0_5\,1_4\,0_2\rangle$, $\langle 0_2\,1_4\,0_0\rangle$.

**Example A.3.** Flexible LDTS(22).
$V = \mathcal{Z}_{11} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,3_0\}$, $\{0_0\,5_1\,10_1\}$, and for $\mathcal{D}$ are $\langle 4_0\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,9_1\rangle$, $\langle 9_1\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,4_0\rangle$, $\langle 4_0\,8_1\,0_1\rangle$, $\langle 0_1\,8_1\,9_1\rangle$, $\langle 9_1\,8_1\,6_0\rangle$, $\langle 6_0\,8_1\,1_1\rangle$, $\langle 1_1\,8_1\,4_0\rangle$.

**Example A.4.** Flexible LDTS(24) containing an LDTS(7) as a subsystem.
$V = \mathcal{Z}_8 \times \mathcal{Z}_3$.
This system in fact contains three disjoint LDTS(7)s on point sets $\{0_i, 1_i, 2_i, 3_i, 4_i, 5_i, 6_i\}$, $i \in \{0, 1, 2\}$, respectively. The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto i_{j+1}$.
The starter blocks for the Steiner triples $\mathcal{T}_1$ are $\{0_0\,1_0\,2_0\}$, $\{0_0\,3_0\,4_0\}$, $\{0_0\,5_0\,6_0\}$, and for the unidirectional triples $\mathcal{D}_1$ are $\langle 3_0\,1_0\,5_0\rangle$, $\langle 5_0\,1_0\,4_0\rangle$, $\langle 4_0\,1_0\,6_0\rangle$, $\langle 6_0\,1_0\,3_0\rangle$ $\langle 3_0\,2_0\,6_0\rangle$, $\langle 6_0\,2_0\,4_0\rangle$, $\langle 4_0\,2_0\,5_0\rangle$, $\langle 5_0\,2_0\,3_0\rangle$.
The starter blocks for the remaining Steiner triples $\mathcal{T}_2$ are $\{0_0\,6_1\,2_2\}$, $\{1_0\,1_1\,1_2\}$, $\{1_0\,5_1\,5_2\}$, $\{1_0\,6_1\,6_2\}$, $\{2_0\,3_1\,4_2\}$, $\{2_0\,5_1\,3_2\}$, $\{2_0\,5_2\,7_1\}$, $\{3_0\,5_1\,4_2\}$, $\{4_0\,4_1\,4_2\}$, $\{4_0\,5_1\,6_2\}$, $\{5_0\,6_2\,7_1\}$, and for the unidirectional triples $\mathcal{D}_2$ are $\langle 1_1\,0_0\,7_0\rangle$, $\langle 7_0\,0_0\,4_2\rangle$, $\langle 4_2\,0_0\,1_1\rangle$, $\langle 1_1\,7_2\,4_2\rangle$, $\langle 4_2\,7_2\,7_0\rangle$, $\langle 7_0\,7_2\,1_1\rangle$, $\langle 3_1\,1_0\,7_0\rangle$, $\langle 7_0\,1_0\,3_2\rangle$, $\langle 3_2\,1_0\,3_1\rangle$, $\langle 3_1\,6_0\,3_2\rangle$, $\langle 3_2\,6_0\,7_0\rangle$, $\langle 7_0\,6_0\,3_1\rangle$, $\langle 0_1\,3_0\,7_0\rangle$, $\langle 7_0\,3_0\,0_2\rangle$, $\langle 0_2\,3_0\,0_1\rangle$, $\langle 0_1\,5_0\,0_2\rangle$, $\langle 0_2\,5_0\,7_0\rangle$, $\langle 7_0\,5_0\,0_1\rangle$, $\langle 1_1\,2_0\,0_2\rangle$, $\langle 0_2\,2_0\,6_1\rangle$, $\langle 6_1\,2_0\,7_2\rangle$, $\langle 7_2\,2_0\,2_2\rangle$, $\langle 2_2\,2_0\,1_1\rangle$, $\langle 1_1\,4_0\,2_2\rangle$, $\langle 2_2\,4_0\,7_2\rangle$, $\langle 7_2\,4_0\,6_1\rangle$, $\langle 6_1\,4_0\,0_2\rangle$, $\langle 0_2\,4_0\,1_1\rangle$. Put $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

**Example A.5.** Flexible LDTS(28).
$V = \mathcal{Z}_{14} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter block for $\mathcal{T}$ is $\{0_0\,1_0\,3_0\}$, and for $\mathcal{D}$ are
$\langle 4_0\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,9_0\rangle$, $\langle 9_0\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,4_0\rangle$, $\langle 4_0\,11_1\,0_1\rangle$, $\langle 0_1\,11_1\,1_1\rangle$, $\langle 1_1\,11_1\,9_0\rangle$, $\langle 9_0\,11_1\,3_1\rangle$, $\langle 3_1\,11_1\,4_0\rangle$, $\langle 2_0\,0_1\,10_0\rangle$, $\langle 10_0\,0_1\,5_1\rangle$, $\langle 5_1\,0_1\,12_1\rangle$, $\langle 12_1\,0_1\,3_0\rangle$, $\langle 3_0\,0_1\,9_0\rangle$, $\langle 9_0\,0_1\,2_0\rangle$.

**Example A.6.** Flexible LDTS(30).
$V = \mathcal{Z}_{15} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,3_0\}$, $\{0_0\,5_0\,10_0\}$, $\{0_0\,9_1\,13_1\}$, $\{0_1\,5_1\,10_1\}$, and for $\mathcal{D}$ are

$\langle 0_0\,6_0\,5_1\rangle$, $\langle 5_1\,6_0\,8_1\rangle$, $\langle 8_1\,6_0\,0_0\rangle$, $\langle 0_0\,6_1\,8_1\rangle$, $\langle 8_1\,6_1\,5_1\rangle$, $\langle 5_1\,6_1\,0_0\rangle$, $\langle 9_0\,2_0\,5_1\rangle$, $\langle 5_1\,2_0\,12_1\rangle$, $\langle 12_1\,2_0\,6_1\rangle$, $\langle 6_1\,2_0\,9_0\rangle$, $\langle 9_0\,5_0\,6_1\rangle$, $\langle 6_1\,5_0\,12_1\rangle$, $\langle 12_1\,5_0\,5_1\rangle$, $\langle 5_1\,5_0\,9_0\rangle$.

**Example A.7.** Flexible LDTS(34).
$V = \mathcal{Z}_{17} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,3_0\}$, $\{0_0\,4_0\,9_0\}$, $\{0_0\,6_0\,0_1\}$, and for $\mathcal{D}$ are
$\langle 7_0\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,7_0\rangle$, $\langle 7_0\,5_1\,2_1\rangle$, $\langle 2_1\,5_1\,6_1\rangle$, $\langle 6_1\,5_1\,7_0\rangle$, $\langle 0_1\,9_0\,6_1\rangle$, $\langle 6_1\,9_0\,16_1\rangle$, $\langle 16_1\,9_0\,14_1\rangle$, $\langle 14_1\,9_0\,5_1\rangle$, $\langle 5_1\,9_0\,0_1\rangle$, $\langle 0_1\,13_0\,5_1\rangle$, $\langle 5_1\,13_0\,14_1\rangle$, $\langle 14_1\,13_0\,16_1\rangle$, $\langle 16_1\,13_0\,6_1\rangle$, $\langle 6_1\,13_0\,0_1\rangle$.

**Example A.8.** Flexible LDTS(36).
$V = \mathcal{Z}_{18} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,3_0\}$, $\{0_0\,6_0\,12_0\}$, $\{0_1\,6_1\,12_1\}$, and for $\mathcal{D}$ are
$\langle 15_0\,4_0\,16_1\rangle$, $\langle 16_1\,4_0\,17_1\rangle$, $\langle 17_1\,4_0\,15_0\rangle$, $\langle 15_0\,6_1\,17_1\rangle$, $\langle 17_1\,6_1\,16_1\rangle$, $\langle 16_1\,6_1\,15_0\rangle$, $\langle 5_0\,0_0\,5_1\rangle$, $\langle 5_1\,0_0\,14_1\rangle$, $\langle 14_1\,0_0\,14_0\rangle$, $\langle 14_0\,0_0\,5_0\rangle$, $\langle 2_0\,8_1\,6_1\rangle$, $\langle 6_1\,8_1\,16_0\rangle$, $\langle 16_0\,8_1\,13_1\rangle$, $\langle 13_1\,8_1\,10_0\rangle$, $\langle 10_0\,8_1\,2_0\rangle$, $\langle 2_0\,9_1\,10_0\rangle$, $\langle 10_0\,9_1\,13_1\rangle$, $\langle 13_1\,9_1\,16_0\rangle$, $\langle 16_0\,9_1\,6_1\rangle$, $\langle 6_1\,9_1\,2_0\rangle$.

**Example A.9.** Flexible LDTS(40).
$V = \mathcal{Z}_{20} \times \mathcal{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
The starter blocks for $\mathcal{T}$ are $\{0_0\,1_0\,3_0\}$, $\{0_0\,4_0\,9_0\}$, $\{0_0\,8_0\,0_1\}$, and for $\mathcal{D}$ are
$\langle 0_0\,5_1\,9_1\rangle$, $\langle 9_1\,5_1\,6_0\rangle$, $\langle 6_0\,5_1\,0_0\rangle$, $\langle 0_0\,14_1\,6_0\rangle$, $\langle 6_0\,14_1\,9_1\rangle$, $\langle 9_1\,14_1\,0_0\rangle$, $\langle 13_0\,0_0\,15_1\rangle$, $\langle 15_1\,0_0\,17_1\rangle$, $\langle 17_1\,0_0\,13_0\rangle$, $\langle 13_0\,3_1\,17_1\rangle$, $\langle 17_1\,3_1\,15_1\rangle$, $\langle 15_1\,3_1\,13_0\rangle$, $\langle 1_0\,7_1\,8_1\rangle$, $\langle 8_1\,7_1\,18_1\rangle$, $\langle 18_1\,7_1\,11_0\rangle$, $\langle 11_0\,7_1\,4_1\rangle$, $\langle 4_1\,7_1\,6_0\rangle$, $\langle 6_0\,7_1\,16_0\rangle$, $\langle 16_0\,7_1\,14_1\rangle$, $\langle 14_1\,7_1\,1_0\rangle$.

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# PURE LATIN DIRECTED TRIPLE SYSTEMS

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Abstract. It is well known that, given a Steiner triple system, a quasigroup can be formed by defining an operation $\cdot$ by the identities $x \cdot x = x$ and $x \cdot y = z$ where $z$ is the third point in the block containing the pair $\{x, y\}$. The same is true for a Mendelsohn triple system where the pair $(x, y)$ is considered to be ordered. But it is not true in general for directed triple systems. However directed triple systems which form quasigroups under this operation do exist and we call these Latin directed triple systems. The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ but those associated with Latin directed triple systems need not. A directed triple system is said to be pure if when considered as a twofold triple system it contains no repeated blocks. In a previous paper, [Discrete Mathematics 312 (2012), 597–607], we studied non-pure Latin directed triple systems. In this paper we turn our attention to pure non-flexible and pure flexible Latin directed triple systems.

## 1. Introduction

In [7], the present authors introduced the concepts of a Latin directed triple system and a DTS-quasigroup and determined their existence spectrum. The latter, an algebraic structure, may be obtained from the former, a combinatorial structure, by a standard procedure explained below. A DTS-quasigroup does not necessarily satisfy the *flexible law*, i.e. $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, and a necessary and sufficient condition for it to do so was also given in [7]. The existence spectrum of flexible DTS-quasigroups was determined in [8]. These systems also possess a certain structure in terms of their topology and this is discussed in [5] and [6]. However in both [7] and [8] the Latin directed triple systems constructed are not *pure*, i.e. when considered as a twofold triple system they contain repeated blocks. Equivalently the DTS-quasigroups are not anti-commutative, i.e. they do not satisfy $x \cdot y = y \cdot x \Rightarrow x = y$. The construction of pure Latin directed triple systems is more challenging than for non-pure systems and the purpose of this paper is to present such constructions both non-flexible and flexible. We are able to adapt some of the methods used for non-pure systems, though greater care must be taken. However most of the approach in this paper uses different techniques. For pure, non-flexible Latin directed triple systems, we are able to determine the existence spectrum completely. For pure, flexible Latin directed triple systems we leave six orders unresolved. These seem to be difficult even with the aid of a computer.

First we recall some definitions. A *Steiner triple system* of order $n$, STS$(n)$, is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points, also called blocks, taken from $V$ such that every pair of distinct points from $V$ appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [14].

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A *Steiner quasigroup* or *squag* or *idempotent totally symmetric quasigroup* is a pair $(Q, \cdot)$ where $Q$ is a set and $\cdot$ is an operation on $Q$ satisfying the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS($n$), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if $Q$ is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups [19, Theorem V.1.11]. All Steiner quasigroups satisfy the flexible law.

Next consider ordered triples. There are two possibilities. A *cyclically ordered triple*, denoted by $(x, y, z)$, contains the ordered pairs $(x, y)$, $(y, z)$, $(z, x)$ and a *transitively ordered triple*, denoted by $\langle x, y, z\rangle$ contains the ordered pairs $(x, y)$, $(y, z)$, $(x, z)$.

A *Mendelsohn triple system* of order $n$, MTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclically ordered triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [18]. Quasigroups can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for Steiner triple systems. Note that the law $y \cdot (x \cdot y) = x$ is usually called semi-symmetric. So the quasigroups are known as *idempotent semisymmetric quasigroups* [2, Remark 2.12] or *Mendelsohn quasigroups*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems and Mendelsohn quasigroups. Again, all Mendelsohn quasigroups satisfy the flexible law.

A *directed triple system* of order $n$, DTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of transitively ordered triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$ [13]. Given a DTS($n$), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$ where $z$ is the third element in the transitive triple containing the ordered pair $(x, y)$. However the structure obtained need not necessarily be a quasigroup. If $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$ then $u \cdot x = v \cdot x = y$. But some DTS($n$)s do yield quasigroups. Such a DTS($n$) will be called a *Latin directed triple system*, denoted by LDTS($n$), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup.*

In [7] the following two theorems were proved.

**Theorem 1.1.** *Let $D = (V, \mathcal{B})$ be a DTS(n). Then $D$ is an LDTS(n) if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle w, y, x\rangle \in \mathcal{B}$ for some $w \in V$.*

**Theorem 1.2.** *A DTS-quasigroup obtained from an LDTS(n), $D = (V, \mathcal{B})$, satisfies the flexible law if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$.*

Let $(V, \mathcal{B})$ be a pure LDTS($n$). Denote by $F$, the set of all unordered triples $\{x, y, z\}$, where $\langle x, y, z\rangle$ runs through all triples of $\mathcal{B}$. Now consider $F$ as a set of faces. Each edge $\{x, y\}$ is incident to two faces and hence we get a generalized pseudosurface. By separating pinch points we obtain a set of one or more components which are an invariant of the LDTS($n$) and are very useful in determining whether two DTS-quasigroups are isomorphic.

Consider a transitive triple $\langle z_1, x, z_0\rangle \in \mathcal{B}$. Then, using Theorem 1.1, there exists $k \geq 3$ and points $z_0$, $z_1$, $z_2$, …, $z_{k-1}$ such that

$$\langle z_1, x, z_0\rangle, \langle z_2, x, z_1\rangle, \ldots, \langle z_{k-1}, x, z_{k-2}\rangle, \langle z_0, x, z_{k-1}\rangle \in \mathcal{B}.$$

If $(V, \mathcal{B})$ is also flexible, using Theorem 1.2,

$$\langle z_1, y, z_2\rangle, \langle z_2, y, z_3\rangle, \ldots, \langle z_{k-1}, y, z_0\rangle, \langle z_0, y, z_1\rangle \in \mathcal{B}$$

where $y = z_0 \cdot z_1 = z_1 \cdot z_2 = \cdots = z_{k-2} \cdot z_{k-1} = z_{k-1} \cdot z_0$. These $2k$ transitive triples define a $k$-gonal bipyramid; denoted by $O_k$, i.e. a graph of $k+2$ vertices with a cycle of length $k$, the points of which can be thought of as situated around the equator of a sphere, and two middle vertex points which are connected to all points of the cycle and which can be thought of as situated at the poles of the sphere. Thus we have the following important result.

**Theorem 1.3.** *A pure flexible LDTS(n) exists if and only if the complete graph $K_n$ can be decomposed into $k$-gonal bipyramid graphs $O_k$, $k \geq 3$.*

Unlike Steiner and Mendelsohn triple systems and their algebraic counterparts, there is not a one-one correspondence between Latin directed triple systems and DTS-quasigroups. This is because if the LDTS($n$) is not pure, then it will contain a pair of triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$. Replacing these with the pair of triples $\langle y, z, x\rangle$ and $\langle x, z, y\rangle$ gives a system which yields the same DTS-quasigroup as the first and yet the two LDTS($n$)s may be non-isomorphic [7, Example 2.4]. However if the LDTS($n$) is pure then this situation does not arise and there is a one-one correspondence between pure Latin directed triple systems and anti-commutative DTS-quasigroups.

## 2. Recursive constructions

In this section we present some recursive constructions for pure Latin directed triple systems. We start with two elementary recursive constructions adapted from standard design-theoretic techniques and appropriate for our purposes.

**Proposition 2.1.**

(i) *If there exists a pure LDTS(n), then there exists a pure LDTS(3n).*
(ii) *If there exists a pure LDTS(n), then there exists a pure LDTS(3n − 2).*

*Proof.*

(i) Take three copies of the LDTS($n$) on point sets $\{0_i, 1_i, \ldots, (n-1)_i\}$, where $i \in \{0, 1, 2\}$ respectively. Then take two disjoint Latin squares $L(i, j)$ and $M(i, j)$ of order $n$ on the set $\{0, 1, \ldots, n-1\}$ and adjoin all transitive triples $\langle i_0, j_1, L(i, j)_2\rangle$ and $\langle M(i, j)_2, j_1, i_0\rangle$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$.
(ii) Take three copies of the LDTS($n$) on point sets $\{\infty, 0_i, 1_i, \ldots, (n-2)_i\}$, where $i \in \{0, 1, 2\}$ respectively. Then take two disjoint Latin squares $L(i, j)$ and $M(i, j)$ of order $n-1$ on the set $\{0, 1, \ldots, n-2\}$ and adjoin all transitive triples $\langle i_0, j_1, L(i, j)_2\rangle$ and $\langle M(i, j)_2, j_1, i_0\rangle$, $0 \leq i \leq n-2$, $0 \leq j \leq n-2$.

□

We now present some recursive constructions for pure flexible LDTSs. The following is a doubling construction which employs a Hamiltonian decomposition of the complete graph $K_{2n+1}$. In the proof we represent the $k$-gonal bipyramids $O_k$ by the notation $[N : E_1, E_2, \ldots, E_k : S]$ where $N$ and $S$ are the poles and $(E_1, E_2, \ldots, E_k)$ is the equator cycle.

**Proposition 2.2.** *If there exists a pure LDTS($2n$), then there exists a pure LDTS($4n+1$). The LDTS($4n+1$) is flexible if and only if the LDTS($2n$) is flexible.*

*Proof.* Let $D = (V, \mathcal{B})$ be a pure LDTS($2n$) where $V = \{1, 2, \ldots, 2n\}$ and $K_{2n+1}$ be the complete graph on the set $W$, disjoint from $V$. Take a decomposition of $K_{2n+1}$ into $n$ disjoint Hamiltonian cycles $H_i$, $1 \le i \le n$. For each $i$, construct a $(2n+1)$-gonal bipyramid $[2i-1 : H_i : 2i]$ and let $\mathcal{B}'$ be the set of transitive triples obtained from these bipyramids. Then $D' = (V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($4n+1$). Since $D'$ contains $D$ as a subsystem, $D'$ is flexible only if $D$ is flexible. Conversely, whenever $D$ is flexible, $D'$ will be flexible as well, because $\mathcal{B}'$ consists of bipyramid components which satisfy the flexible law. □

**Proposition 2.3.** *If there exists a pure LDTS($2n$), then there exists a pure LDTS($4n+19$). The LDTS($4n+19$) is flexible if and only if the LDTS($2n$) is flexible.*

*Proof.* Let $(V, \mathcal{B})$ be a pure LDTS($2n$) where $V = \{\infty_1, \infty_2, \ldots, \infty_{2n}\}$. Construct a set of triples $\mathcal{B}'$ on the point set $\mathbb{Z}_{2n+19} \cup V$ from the following set of starter blocks under the action of the mapping $i \mapsto i+1$ with the elements of $V$ as fixed points.

$$\{\langle 2,0,6\rangle, \langle 6,0,9\rangle, \langle 9,0,2\rangle, \langle 2,1,9\rangle, \langle 9,1,6\rangle, \langle 6,1,2\rangle\} \cup \{\, \langle 0, \infty_r, 9{+}r\rangle : r = 1, \ldots, 2n \,\}$$

Then $(\mathbb{Z}_{2n+19} \cup V, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($4n+19$). To see that the constructed system is flexible whenever $(V, \mathcal{B})$ is flexible, it suffices to check that the triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

**Proposition 2.4.** *If there exists a pure LDTS($6n+1$), then there exists a pure LDTS($12n+22$). The LDTS($12n+22$) is flexible if and only if the LDTS($6n+1$) is flexible.*

*Proof.* Let $(V, \mathcal{B})$ be a pure LDTS($6n+1$) where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n}\}$ and let $W = \{\, i_j : \; i \in \mathbb{Z}_{2n+7},\; j = 0,1,2 \,\}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of $V$ as fixed points.
$\langle 0_0, 0_2, 1_1\rangle$, $\langle 1_1, 0_2, 2_2\rangle$, $\langle 2_2, 0_2, 0_0\rangle$, $\langle 0_0, 3_2, 2_2\rangle$, $\langle 2_2, 3_2, 1_1\rangle$, $\langle 1_1, 3_2, 0_0\rangle$, $\langle 0_0, 3_0, 1_0\rangle$, $\langle 1_0, 3_0, 0_2\rangle$, $\langle 0_2, 3_0, 2_1\rangle$, $\langle 2_1, 3_0, 0_1\rangle$, $\langle 0_1, 3_0, 0_0\rangle$, $\langle 0_0, 3_1, 0_1\rangle$, $\langle 0_1, 3_1, 2_1\rangle$, $\langle 2_1, 3_1, 0_2\rangle$, $\langle 0_2, 3_1, 1_0\rangle$, $\langle 1_0, 3_1, 0_0\rangle$, $\langle 0_1, 3_2, 0_2\rangle$, $\langle 0_2, 3_2, 2_0\rangle$, $\langle 2_0, 3_2, 0_1\rangle$, $\langle 0_1, \infty_0, 2_0\rangle$, $\langle 0_2, \infty_0, 0_1\rangle$, $\langle 2_0, \infty_0, 0_2\rangle$,

$$\begin{array}{lll}
\langle 0_0, \infty_r, (3+r)_0\rangle, & \langle 0_1, \infty_r, (3+r)_1\rangle, & \langle 0_2, \infty_r, (3+r)_2\rangle, \\
\langle 0_0, \infty_{2n+r}, (3+r)_1\rangle, & \langle 0_1, \infty_{2n+r}, (3+r)_2\rangle, & \langle 0_2, \infty_{2n+r}, (3+r)_0\rangle, \\
\langle (3+r)_1, \infty_{4n+r}, 0_0\rangle, & \langle (3+r)_2, \infty_{4n+r}, 0_1\rangle, & \langle (3+r)_0, \infty_{4n+r}, 0_2\rangle,
\end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS($12n+22$). The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

**Proposition 2.5.** *If there exists a pure LDTS($6n+1$), then there exists a pure LDTS($12n+28$). The LDTS($12n+28$) is flexible if and only if the LDTS($6n+1$) is flexible.*

*Proof.* Let $(V, \mathcal{B})$ be a pure LDTS($6n+1$) where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n}\}$ and let $W = \{\, i_j : \; i \in \mathbb{Z}_{2n+9},\; j = 0,1,2 \,\}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of $V$ as fixed points.

$\langle 1_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 4_1\rangle$, $\langle 4_1, 0_0, 4_2\rangle$, $\langle 4_2, 0_0, 1_1\rangle$, $\langle 0_2, 3_2, 1_2\rangle$, $\langle 1_2, 3_2, 3_0\rangle$, $\langle 3_0, 3_2, 0_2\rangle$, $\langle 1_2, 0_0, 4_0\rangle$, $\langle 4_0, 0_0, 2_2\rangle$, $\langle 2_2, 0_0, 1_2\rangle$, $\langle 0_1, 3_1, 4_2\rangle$, $\langle 4_2, 3_1, 1_0\rangle$, $\langle 1_0, 3_1, 0_1\rangle$, $\langle 1_1, 0_2, 4_2\rangle$, $\langle 4_2, 0_2, 4_1\rangle$, $\langle 4_1, 0_2, 3_1\rangle$, $\langle 3_1, 0_2, 1_1\rangle$, $\langle 0_1, 4_0, 3_0\rangle$, $\langle 0_0, 1_0, 3_0\rangle$, $\langle 3_0, 1_0, 1_1\rangle$, $\langle 4_1, 4_0, 0_1\rangle$, $\langle 3_0, 2_2, 0_1\rangle$, $\langle 0_1, 2_2, 4_1\rangle$, $\langle 2_1, 0_2, 4_0\rangle$, $\langle 4_0, 0_2, 1_0\rangle$, $\langle 0_1, \infty_0, 1_0\rangle$, $\langle 1_0, \infty_0, 4_2\rangle$, $\langle 4_2, \infty_0, 0_1\rangle$,

$$\begin{array}{lll} \langle 0_0, \infty_r, (4+r)_0\rangle, & \langle 0_1, \infty_r, (4+r)_1\rangle, & \langle 0_2, \infty_r, (4+r)_2\rangle, \\ \langle 0_0, \infty_{2n+r}, (4+r)_1\rangle, & \langle 0_1, \infty_{2n+r}, (4+r)_2\rangle, & \langle 0_2, \infty_{2n+r}, (4+r)_0\rangle, \\ \langle (4+r)_1, \infty_{4n+r}, 0_0\rangle, & \langle (4+r)_2, \infty_{4n+r}, 0_1\rangle, & \langle (4+r)_0, \infty_{4n+r}, 0_2\rangle, \end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n + 28)$. The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

**Proposition 2.6.** *If there exists a pure LDTS$(6n + 3)$, then there exists a pure LDTS$(12n + 24)$. The LDTS$(12n + 24)$ is flexible if and only if the LDTS$(6n + 3)$ is flexible.*

*Proof.* Let $(V, \mathcal{B})$ be a pure LDTS$(6n+3)$ where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n+2}\}$ and let $W = \{ i_j : i \in \mathbb{Z}_{2n+7}, j = 0, 1, 2 \}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of $V$ as fixed points.
$\langle 0_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 1_1\rangle$, $\langle 2_1, 1_0, 0_0\rangle$, $\langle 0_0, 1_0, 3_0\rangle$, $\langle 2_0, 0_0, 2_2\rangle$, $\langle 2_2, 0_0, 3_2\rangle$, $\langle 3_2, 0_0, 0_1\rangle$, $\langle 0_1, 0_2, 3_2\rangle$, $\langle 3_2, 0_2, 2_2\rangle$, $\langle 2_2, 0_2, 2_0\rangle$, $\langle 3_0, 1_2, 0_0\rangle$, $\langle 0_0, 1_2, 2_1\rangle$, $\langle 1_1, 0_2, 3_1\rangle$, $\langle 3_1, 0_2, 0_1\rangle$, $\langle 1_1, 0_1, 2_2\rangle$, $\langle 2_2, 0_1, 3_0\rangle$, $\langle 3_0, 0_1, 1_1\rangle$, $\langle 0_1, \infty_0, 2_0\rangle$, $\langle 2_0, \infty_0, 1_2\rangle$, $\langle 1_2, \infty_0, 0_1\rangle$, $\langle 0_2, \infty_1, 2_1\rangle$, $\langle 2_1, \infty_1, 3_0\rangle$, $\langle 3_0, \infty_1, 0_2\rangle$, $\langle 0_2, \infty_2, 3_0\rangle$, $\langle 3_0, \infty_2, 2_1\rangle$, $\langle 2_1, \infty_2, 0_2\rangle$,

$$\begin{array}{lll} \langle 0_0, \infty_{r+2}, (3+r)_0\rangle, & \langle 0_1, \infty_{r+2}, (3+r)_1\rangle, & \langle 0_2, \infty_{r+2}, (3+r)_2\rangle, \\ \langle 0_0, \infty_{2n+r+2}, (3+r)_1\rangle, & \langle 0_1, \infty_{2n+r+2}, (3+r)_2\rangle, & \langle 0_2, \infty_{2n+r+2}, (3+r)_0\rangle, \\ \langle (3+r)_1, \infty_{4n+r+2}, 0_0\rangle, & \langle (3+r)_2, \infty_{4n+r+2}, 0_1\rangle, & \langle (3+r)_0, \infty_{4n+r+2}, 0_2\rangle, \end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n + 24)$. The triples in $\mathcal{B}'$ define a set of bipyramids satisfying the flexible law. □

The last recursive construction we present can only be used to produce non-flexible LDTSs.

**Proposition 2.7.** *If there exists a pure LDTS$(6n + 3)$, then there exists a pure non-flexible LDTS$(12n + 18)$.*

*Proof.* Let $(V, \mathcal{B})$ be a pure LDTS$(6n+3)$ where $V = \{\infty_0, \infty_1, \ldots, \infty_{6n+2}\}$ and let $W = \{ i_j : i \in \mathbb{Z}_{2n+5}, j = 0, 1, 2 \}$. Construct a set of triples $\mathcal{B}'$ from the following set of starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$ with the elements of $V$ as fixed points.
$\langle \infty_2, 0_0, 0_2\rangle$, $\langle 0_0, 0_1, 2_2\rangle$, $\langle 2_2, 0_1, 1_0\rangle$, $\langle 1_1, 0_2, 2_2\rangle$, $\langle 2_1, 0_2, 1_1\rangle$, $\langle 2_2, 0_2, 2_1\rangle$, $\langle 1_2, 1_0, 0_2\rangle$, $\langle 2_0, 1_1, 0_0\rangle$, $\langle 0_1, 2_0, 0_2\rangle$, $\langle 0_2, 2_0, \infty_2\rangle$, $\langle 1_2, 2_0, 0_1\rangle$, $\langle 0_0, 2_1, 2_0\rangle$, $\langle 1_0, 2_1, 0_0\rangle$, $\langle 0_0, \infty_0, 1_0\rangle$, $\langle 0_1, \infty_0, 1_1\rangle$, $\langle 0_2, \infty_0, 1_2\rangle$, $\langle 0_0, \infty_1, 1_2\rangle$, $\langle 0_1, \infty_1, 2_1\rangle$, $\langle 2_2, \infty_1, 0_0\rangle$, $\langle 2_1, \infty_2, 0_1\rangle$,

$$\begin{array}{lll} \langle 0_0, \infty_{r+2}, (2+r)_0\rangle, & \langle 0_1, \infty_{r+2}, (2+r)_1\rangle, & \langle 0_2, \infty_{r+2}, (2+r)_2\rangle, \\ \langle 0_0, \infty_{2n+r+2}, (2+r)_1\rangle, & \langle 0_1, \infty_{2n+r+2}, (2+r)_2\rangle, & \langle 0_2, \infty_{2n+r+2}, (2+r)_0\rangle, \\ \langle (2+r)_1, \infty_{4n+r+2}, 0_0\rangle, & \langle (2+r)_2, \infty_{4n+r+2}, 0_1\rangle, & \langle (2+r)_0, \infty_{4n+r+2}, 0_2\rangle, \end{array}$$

where $r = 1, \ldots, 2n$. Then $(V \cup W, \mathcal{B} \cup \mathcal{B}')$ is a pure LDTS$(12n + 18)$. The triples in $\mathcal{B}'$ do not satisfy the flexible law. For example $(0_0 \cdot 2_1) \cdot 0_0 = 2_0 \cdot 0_0 = 1_1$, whilst $0_0 \cdot (2_1 \cdot 0_0) = 0_0 \cdot 1_0 = \infty_0$. □

## 3. Pure non-flexible Latin directed triple systems

In this section we determine the existence spectrum of pure non-flexible LDTS($n$). It was shown in [7] that there is no pure LDTS($n$) for $3 \le n \le 12$. Part of the existence proof in this section uses a standard technique known as Wilson's fundamental construction for which we need the concept of a *group divisible design* (GDD). A 3-GDD of type $g^u$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where $V$ is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of $V$ into $u$ subsets of cardinality $g$ called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from <u>different</u> groups occur in precisely one block but no pair of elements from the <u>same</u> group occur at all. We will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set $V$ being of cardinality $v = gu + m$ and the partition $G$ being into $u$ subsets of cardinality $g$ and one subset of cardinality $m$. Necessary and sufficient conditions for 3-GDDs of type $g^u$ were determined in [12] and for 3-GDDs of type $g^u m^1$ in [4]; a convenient reference is [9] where the existence of all the GDDs that are used can be verified.

We will assume that the reader is familiar with this construction but briefly the basic idea is as follows. Begin with a 3-GDD of cardinality $v = gu$ or $gu + m$, usually called the *master GDD*. Each point is then assigned a weight, usually the same weight, say $w$. In effect, each point is replaced by $w$ points. Each block of the master GDD is then replaced by a 3-GDD of type $w^3$, called a *slave GDD*. We will only need to use the two values $w = 2$ and $w = 3$, and instead of slave GDDs we will use partial Latin directed triple systems. When $w = 2$ we will employ the partial LDTS(6), say $\mathcal{P}$, whose blocks are $\langle a, b, c\rangle$, $\langle a, y, z\rangle$, $\langle x, b, z\rangle$, $\langle x, y, c\rangle$, $\langle z, y, x\rangle$, $\langle c, b, x\rangle$, $\langle c, y, a\rangle$, $\langle z, b, a\rangle$ and the sets $\{a, x\}$, $\{b, y\}$, $\{c, z\}$ play the role of the groups. As a component of an LDTS($n$), it satisfies the flexible law. When $w = 3$ we will use the partial LDTS(9), say $\mathcal{Q}$, whose blocks are $\langle a, p, x\rangle$, $\langle b, q, y\rangle$, $\langle c, r, z\rangle$, $\langle a, q, z\rangle$, $\langle b, r, x\rangle$, $\langle c, p, y\rangle$, $\langle a, r, y\rangle$, $\langle b, p, z\rangle$, $\langle c, q, x\rangle$, $\langle x, q, a\rangle$, $\langle y, r, b\rangle$, $\langle z, p, c\rangle$, $\langle z, r, a\rangle$, $\langle x, p, b\rangle$, $\langle y, q, c\rangle$, $\langle y, p, a\rangle$, $\langle z, q, b\rangle$, $\langle x, r, c\rangle$ and $\{a, b, c\}$, $\{p, q, r\}$, $\{x, y, z\}$ play the role of the groups. It does not satisfy the flexible law, e.g. $(a \cdot p) \cdot a = x \cdot a = q$ but $a \cdot (p \cdot a) = a \cdot y = r$. To complete the construction we then "fill in" the groups of the expanded master GDD, sometimes adjoining an extra point, say $\infty$, to all of the groups. Thus we may need pure non-flexible Latin directed triple systems of orders $gw$, $mw$, $gw + 1$ or $mw + 1$ as appropriate. For a more elaborate explanation of this construction see, for example, the proof of Proposition 4.3 in [7].

| Type of master GDD | Orders of LDTS($n$) needed | Residue classes covered modulo 36 | Missing values |
|---|---|---|---|
| $6^s$, $s \ge 3$ | 13 | 1, 13, 25 | 25 |
| $9^{2s+1}$, $s \ge 1$ | 19 | 19 | |
| $9^{2s}\, 15^1$, $s \ge 2$ | 19, 31 | 31 | 67 |
| $9^{2s}\, 21^1$, $s \ge 2$ | 19, 43 | 7 | 79 |

Table 1. Schema for pure non-flexible LDTS($n$), $n \equiv 1 \pmod 6$.

Schema of the master GDDs and Latin directed triple systems needed to construct pure non-flexible LDTS($n$) for $n \equiv 1 \pmod 6$ is given in Table 1. We always weight with 2 and adjoin an extra point $\infty$. Pure non-flexible LDTS($n$) for $n = 13$, 19, 25 and 31 are given as Examples NO.1, NO.3, NO.5 and NO.7 respectively in the Appendix. The LDTSs of orders $n = 43$ and 79 can be constructed using part (ii) of Proposition 2.1 from the LDTS(15) and LDTS(27) which are given as Examples NO.2 and NO.6 in the Appendix. This just leaves the value $n = 67$ which can be constructed using a master GDD of type $4^4\,6^1$ or $6^3\,4^1$, assigning weight 3, adjoining the point $\infty$ and using the pure non-flexible LDTS(13) and LDTS(19).

We can now use these systems to construct pure non-flexible LDTS($n$) of order $n \equiv 4 \pmod 6$. By Proposition 2.5 there exists a pure non-flexible LDTS($n$) for all $n \equiv 4 \pmod{12}$, $n \geq 52$. Pure non-flexible systems of orders 16, 28 and 40 are given as Examples NE.1, NE.5 and NE.9 in the Appendix. By Proposition 2.4 there exists a pure non-flexible LDTS($n$) for all $n \equiv 10 \pmod{12}$, $n \geq 46$. Pure non-flexible systems of orders 22 and 34 are given as Examples NE.3 and NE.7 in the Appendix.

These systems may in turn be used to construct pure non-flexible LDTS($n$) of order $n \equiv 3 \pmod 6$. By Proposition 2.2 there exists a pure non-flexible LDTS($n$) for all $n \equiv 9 \pmod{12}$, $n \geq 33$. A pure non-flexible LDTS(21) is given as Example NO.4 in the Appendix. By Proposition 2.3 there exists a pure non-flexible LDTS($n$) for all $n \equiv 3 \pmod{12}$, $n \geq 51$. Pure non-flexible systems of orders 15 and 27 are given as Examples NO.2 and NO.6 in the Appendix and a pure non-flexible LDTS(39) can be constructed from the LDTS(13) using part (i) of Proposition 2.1.

Finally we construct pure non-flexible LDTS($n$) of order $n \equiv 0 \pmod 6$. By Proposition 2.6 there exists a pure non-flexible LDTS($n$) for all $n \equiv 0 \pmod{12}$, $n \geq 48$. Pure non-flexible systems of orders 24 and 36 are given as Examples NE.4 and NE.8 in the Appendix. By Proposition 2.7 there exists a pure non-flexible LDTS($n$) for all $n \equiv 6 \pmod{12}$, $n \geq 42$. Pure non-flexible systems of orders 18 and 30 are given as Examples NE.2 and NE.6 in the Appendix.

Collecting all the results together gives the following theorem.

**Theorem 3.1.** *The existence spectrum of pure non-flexible LDTS(n)s is* $n \equiv 0$ *or* $1 \pmod 3$, $n \geq 13$.

## 4. Pure flexible Latin directed triple systems

In this section we discuss the existence of pure flexible Latin directed triple systems. The further requirement of flexibility adds another constraint to the constructions. We are still able, and indeed do, use Wilson's fundamental construction but we cannot use weight $w = 3$ and the partial Latin directed triple system $\mathcal{Q}$ because as shown in the previous section it does not satisfy the flexible law. Another difficulty is that, as was shown in [7], there is no pure flexible LDTS($n$) for $3 \leq n \leq 15$ or $n = 18$. In particular there is no pure flexible LDTS(13) which was very useful in the non-flexible case. However if the above factors are against us, then we do have a feature of pure flexible LDTS($n$) to help us. This is their geometric structure as described in Theorem 1.3.

In the case where all the bipyramids have $k = 3$, this is a decomposition of $K_n$ into graphs $K_5$ but missing one edge, so-called $(K_5 \setminus e)$-designs. The spectrum of $n$ for which these designs exist has been fully determined [10, 15, 16, 20], see also [3]. It is $n \equiv 0$ or $1 \pmod 9$, $n \geq 19$. When all the bipyramids have $k = 4$, this is a decomposition of $K_n$ into Pasch configurations. The spectrum of $n$ for which this is true has also been determined [11, 1]. It is $n \equiv 1$ or $9 \pmod{24}$, $n \geq 25$.

We first complete the existence spectrum for the residue class $n \equiv 1 \pmod 6$. Table 2 gives the schema for $n \equiv 7$ or $13 \pmod{18}$. We use weight $w = 2$ and adjoin an extra point. The pure flexible LDTS($n$)s of orders $n \equiv 1 \pmod{18}$ can be constructed from $(K_5 \setminus e)$-designs, this includes, in particular, the LDTS(19)

| Type of master GDD | Orders of LDTS($n$) needed | Residue classes covered modulo 36 | Missing values |
|---|---|---|---|
| $9^{2s}\,15^1,\ s \geq 2$ | 19, 31 | 31 | 67 |
| $9^{2s}\,21^1,\ s \geq 2$ | 19, 43 | 7 | 79 |
| $18^{s}\,12^1,\ s \geq 3$ | 25, 37 | 25 | 61, 97 |
| $18^{s}\,24^1,\ s \geq 3$ | 37, 49 | 13 | 85, 121 |

TABLE 2. Schema for pure flexible LDTS($n$), $n \equiv 7$ or $13 \pmod{18}$.

and LDTS(37). Pure flexible LDTS($n$)s for $n = 31$, 43 and 67 are given as Examples FO.2, FO.4 and FO.5 respectively in the Appendix. For $n = 25$, 49, 97 and 121 we can use the decompositions of $K_n$ into Pasch configurations. The remaining missing systems can be obtained using 3-GDD constructions. For $n = 61$ use type $10^3$, for $n = 79$ use type $13^3$ and for $n = 85$ use type $10^3\,12^1$. To do this we need systems of orders 21, 25 and 27. A pure flexible LDTS(21) is given as Example FO.1 and the pure flexible LDTS(27) can be constructed from a $(K_5 \setminus e)$-design.

We next consider the residue class $n \equiv 4 \pmod 6$. By Proposition 2.4 there exists a pure flexible LDTS($n$) for $n = 22$ and for all $n \equiv 10 \pmod{12}$, $n \geq 58$. An LDTS(34) is given as Example FE.1 in the Appendix and an LDTS(46) can be constructed from a $(K_5 \setminus e)$-design. By Proposition 2.5 there exists a pure flexible LDTS($n$) for $n = 28$ and for all $n \equiv 4 \pmod{12}$, $n \geq 64$. A pure flexible LDTS(16) is given in [7, Example 3.9]. Systems of orders $n = 40$ and 52 are given as Examples FE.2 and FE.3 respectively in the Appendix.

The results for $n \equiv 4 \pmod 6$ now enable us to deal with the residue class $n \equiv 3 \pmod 6$. By Proposition 2.2 there exists a pure flexible LDTS($n$) for all $n \equiv 9 \pmod{12}$, $n \geq 33$. A pure flexible LDTS(21) is given as Example FO.1 in the Appendix. By Proposition 2.3 there exists a pure flexible LDTS($n$) for all $n \equiv 3 \pmod{12}$, $n \geq 51$. A pure flexible LDTS(27) can be constructed from a $(K_5 \setminus e)$-design and a pure flexible LDTS(39) is given as Example FO.3 in the Appendix.

This just leaves the residue class $n \equiv 0 \pmod 6$ to consider. By Proposition 2.6 there exists a pure flexible LDTS($n$) for all $n \equiv 0 \pmod{12}$, $n \geq 60$. A pure flexible LDTS(36) can be constructed from a $(K_5 \setminus e)$-design and a pure flexible LDTS(48) can be obtained using a 3-GDD of type $8^3$. This leaves $n = 24$ unresolved. Table 3 gives

| Type of master GDD | Orders of LDTS($n$) needed | Residue classes covered modulo 108 | Missing values |
|---|---|---|---|
| $27^{2s}\,33^1,\ s \geq 2$ | 54, 66 | 66 | 174 |
| $27^{2s}\,51^1,\ s \geq 2$ | 54, 102 | 102 | 210 |
| $27^{2s}\,69^1,\ s \geq 2$ | 54, 138 | 30 | 30, 246 |
| $27^{2s}\,93^1,\ s \geq 3$ | 54, 186 | 78 | 78, 294, 402 |
| $27^{2s}\,111^1,\ s \geq 3$ | 54, 222 | 6 | 114, 330, 438 |
| $27^{2s}\,129^1,\ s \geq 3$ | 54, 258 | 42 | 42, 150, 366, 474 |

TABLE 3. Schema for pure flexible LDTS($n$), $n \equiv 6$ or $30 \pmod{36}$.

the schema for $n \equiv 6$ or $30 \pmod{36}$. Again we use weight $w = 2$. The pure flexible LDTS($n$)s of orders $n \equiv 18 \pmod{36}$ can be constructed from $(K_5 \setminus e)$-designs, this includes, in particular, the LDTS(54). Pure flexible LDTSs of the following orders can be constructed using 3-GDDs: 66 (use $11^3$), 102 (use $17^3$), 138 (use $23^3$), 174 (use $29^3$), 186 (use $11^6\,27^1$), 210 (use $11^8\,17^1$), 222 (use $11^6\,45^1$), 246 (use $41^3$), 258 (use $17^6\,27^1$), 294 (use $17^6\,45^1$), 330 (use $11^{15}$), 366 (use $23^6\,45^1$), 402 (use $23^6\,63^1$), 438 (use $23^6\,81^1$) and 474 (use $17^{12}\,33^1$). This leaves $n = 30$, 42, 78, 114 and 150 unresolved.

Collecting all the results together gives the following theorem.

**Theorem 4.1.** *A pure flexible LDTS(n) exists for all $n \equiv 0$ or 1 $\pmod 3$, $n \geq 16$ and $n \neq 18$, possibly except $n = 24$, 30, 42, 78, 114 and 150.*

## Appendix. Examples of pure LDTSs

The following examples were obtained by computer with the help of the model builder Mace4 [17] using an algebraic description of a DTS-quasigroup, see [5]. We denote the elements $(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n$ as $i_j$. For simplicity, we omit commas from the triples.

**Example NO.1.** Pure non-flexible LDTS(13).
$V = \mathbb{Z}_{13}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
$\langle 1\,0\,5\rangle$, $\langle 5\,0\,7\rangle$, $\langle 7\,0\,3\rangle$, $\langle 3\,0\,1\rangle$.
The system is non-flexible, for example $(0\cdot 2)\cdot 0 = 8\cdot 0 = 9$, whilst $0\cdot(2\cdot 0) = 0\cdot 12 = 4$.

**Example NO.2.** Pure non-flexible LDTS(15).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i + 1)_j$.
$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,5_1\rangle$, $\langle 5_1\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,4_1\rangle$, $\langle 4_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 0_0\,\infty\,4_0\rangle$, $\langle 0_1\,\infty\,3_1\rangle$.
The system is non-flexible, for example $(0_0\cdot 1_0)\cdot 0_0 = 3_0\cdot 0_0 = \infty$, whilst $0_0\cdot(1_0\cdot 0_0) = 0_0\cdot 0_1 = 5_0$.

**Example NO.3.** Pure non-flexible LDTS(19).
$V = \mathbb{Z}_{19}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
$\langle 1, 0, 5\rangle$, $\langle 5, 0, 11\rangle$, $\langle 11, 0, 7\rangle$, $\langle 7, 0, 9\rangle$, $\langle 9, 0, 3\rangle$, $\langle 3, 0, 1\rangle$.
The system is non-flexible, for example $(0\cdot 6)\cdot 0 = 14\cdot 0 = 15$, whilst $0\cdot(6\cdot 0) = 0\cdot 16 = 17$.

**Example NO.4.** Pure non-flexible LDTS(21).
$V = \mathbb{Z}_7 \times \mathbb{Z}_3$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i + 1)_j$.
$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 2_0\,0_1\,6_0\rangle$, $\langle 6_0\,0_1\,6_2\rangle$, $\langle 6_2\,0_1\,5_1\rangle$, $\langle 5_1\,0_1\,2_0\rangle$, $\langle 2_0\,0_2\,5_1\rangle$, $\langle 5_1\,0_2\,4_1\rangle$, $\langle 4_1\,0_2\,0_1\rangle$, $\langle 0_1\,0_2\,3_0\rangle$, $\langle 3_0\,0_2\,6_2\rangle$, $\langle 6_2\,0_2\,6_0\rangle$, $\langle 6_0\,0_2\,5_2\rangle$, $\langle 5_2\,0_2\,2_0\rangle$, $\langle 6_0\,1_2\,3_1\rangle$, $\langle 3_1\,1_2\,4_1\rangle$, $\langle 4_1\,1_2\,5_2\rangle$, $\langle 5_2\,1_2\,6_0\rangle$.
The system is non-flexible, for example $(6_2\cdot 6_0)\cdot 6_2 = 0_2\cdot 6_2 = 3_0$, whilst $6_2\cdot(6_0\cdot 6_2) = 6_2\cdot 0_1 = 5_1$.

**Example NO.5.** Pure non-flexible LDTS(25).
$V = \mathbb{Z}_{25}$.

The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 1,0,5\rangle$, $\langle 5,0,16\rangle$, $\langle 16,0,12\rangle$, $\langle 12,0,19\rangle$, $\langle 19,0,8\rangle$, $\langle 8,0,10\rangle$, $\langle 10,0,3\rangle$, $\langle 3,0,1\rangle$.
The system is non-flexible, for example $(0{\cdot}2){\cdot}0 = 17{\cdot}0 = 11$, whilst $0{\cdot}(2{\cdot}0) = 0{\cdot}24 = 4$.

**Example NO.6.** Pure non-flexible LDTS(27).
$V = (\mathbb{Z}_{13} \times \mathbb{Z}_2) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 12_1\rangle$, $\langle 12_1\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 11_1\rangle$, $\langle 11_1\, 0_0\, 1_1\rangle$, $\langle 1_1\, 0_0\, 7_1\rangle$, $\langle 7_1\, 0_0\, 9_1\rangle$, $\langle 9_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 7_0\rangle$, $\langle 7_0\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$, $\langle 5_1\, 1_0\, 6_1\rangle$, $\langle 6_1\, 1_0\, 11_1\rangle$, $\langle 11_1\, 1_0\, 5_1\rangle$, $\langle 0_0\, \infty\, 2_0\rangle$, $\langle 0_1\, \infty\, 11_1\rangle$.
The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = \infty \cdot 0_0 = 11_0$, whilst $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 12_0 = 4_0$.

**Example NO.7.** Pure non-flexible LDTS(31).
$V = \mathbb{Z}_{31}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 1,0,5\rangle$, $\langle 5,0,19\rangle$, $\langle 19,0,10\rangle$, $\langle 10,0,18\rangle$, $\langle 18,0,20\rangle$, $\langle 20,0,6\rangle$, $\langle 6,0,15\rangle$, $\langle 15,0,7\rangle$, $\langle 7,0,3\rangle$, $\langle 3,0,1\rangle$.
The system is non-flexible, for example $(0{\cdot}2){\cdot}0 = 13{\cdot}0 = 23$, whilst $0{\cdot}(2{\cdot}0) = 0{\cdot}30 = 4$.

**Example NE.1.** Pure non-flexible LDTS(16).
$V = \mathbb{Z}_8 \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 2_0, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 2_0\rangle$.
The system is non-flexible, for example $(7_1 \cdot 7_0) \cdot 7_1 = 0_0 \cdot 7_1 = 3_1$, whilst $7_1 \cdot (7_0 \cdot 7_1) = 7_1 \cdot 0_1 = 2_1$.

**Example NE.2.** Pure non-flexible LDTS(18).
$V = \mathbb{Z}_3 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 1_0\rangle$, $\langle 1_0\, 2_1\, 0_1\rangle$, $\langle 0_1\, 2_1\, 0_5\rangle$, $\langle 0_5\, 2_1\, 1_2\rangle$, $\langle 1_2\, 2_1\, 0_2\rangle$, $\langle 0_2\, 2_1\, 1_0\rangle$, $\langle 0_1\, 0_3\, 2_5\rangle$, $\langle 2_5\, 0_3\, 0_5\rangle$, $\langle 0_5\, 0_3\, 0_1\rangle$, $\langle 0_1\, 0_4\, 0_2\rangle$, $\langle 0_2\, 0_4\, 2_5\rangle$, $\langle 2_5\, 0_4\, 0_1\rangle$, $\langle 0_2\, 0_5\, 2_3\rangle$, $\langle 2_3\, 0_5\, 2_2\rangle$, $\langle 2_2\, 0_5\, 0_2\rangle$, $\langle 1_2\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 0_5\rangle$, $\langle 0_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 1_2\rangle$, $\langle 0_4\, 2_0\, 1_5\rangle$, $\langle 1_5\, 2_0\, 0_5\rangle$, $\langle 0_5\, 2_0\, 0_4\rangle$, $\langle 1_3\, 0_1\, 2_4\rangle$, $\langle 2_4\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 1_3\rangle$, $\langle 1_3\, 0_2\, 1_4\rangle$, $\langle 1_4\, 0_2\, 2_4\rangle$, $\langle 2_4\, 0_2\, 1_3\rangle$.
The system is non-flexible, for example $(2_5 \cdot 0_1) \cdot 2_5 = 0_4 \cdot 2_5 = 0_2$, whilst $2_5 \cdot (0_1 \cdot 2_5) = 2_5 \cdot 0_3 = 0_5$.

**Example NE.3.** Pure non-flexible LDTS(22).
$V = \mathbb{Z}_{11} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 10_1\rangle$, $\langle 10_1, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 0_1\rangle$, $\langle 0_1, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$, $\langle 2_0, 0_1, 9_0\rangle$, $\langle 9_0, 0_1, 6_1\rangle$, $\langle 6_1, 0_1, 2_0\rangle$, $\langle 8_0, 0_1, 10_0\rangle$, $\langle 10_0, 0_1, 3_1\rangle$, $\langle 3_1, 0_1, 2_1\rangle$, $\langle 2_1, 0_1, 8_0\rangle$.
The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = 3_1 \cdot 0_0 = 5_1$, whilst $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 10_0 = 4_0$.

**Example NE.4.** Pure non-flexible LDTS(24).
$V = \mathbb{Z}_4 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 0_2\, 0_0\, 2_2\rangle$, $\langle 2_2\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 0_5\rangle$, $\langle 0_5\, 0_0\, 3_5\rangle$, $\langle 3_5\, 0_0\, 1_5\rangle$, $\langle 1_5\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 1_2\rangle$, $\langle 1_2\, 0_0\, 0_2\rangle$, $\langle 3_1\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 3_4\rangle$, $\langle 3_4\, 0_1\, 3_3\rangle$, $\langle 3_3\, 0_1\, 3_5\rangle$, $\langle 3_5\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 3_1\rangle$, $\langle 0_0\, 1_1\, 3_0\rangle$, $\langle 3_0\, 1_1\, 1_0\rangle$, $\langle 1_0\, 1_1\, 2_0\rangle$, $\langle 2_0\, 1_1\, 1_2\rangle$, $\langle 1_2\, 1_1\, 1_4\rangle$, $\langle 1_4\, 1_1\, 2_2\rangle$, $\langle 2_2\, 1_1\, 3_2\rangle$, $\langle 3_2\, 1_1\, 0_0\rangle$, $\langle 1_3\, 0_2\, 1_4\rangle$, $\langle 1_4\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 1_3\rangle$, $\langle 3_2\, 1_3\, 3_5\rangle$, $\langle 3_5\, 1_3\, 3_4\rangle$, $\langle 3_4\, 1_3\, 3_2\rangle$, $\langle 3_2\, 1_4\, 2_3\rangle$, $\langle 2_3\, 1_4\, 3_5\rangle$, $\langle 3_5\, 1_4\, 3_2\rangle$, $\langle 3_1\, 0_5\, 2_2\rangle$, $\langle 2_2\, 0_5\, 1_4\rangle$, $\langle 1_4\, 0_5\, 3_1\rangle$, $\langle 1_1\, 1_5\, 2_3\rangle$, $\langle 2_3\, 1_5\, 2_2\rangle$, $\langle 2_2\, 1_5\, 3_1\rangle$, $\langle 3_1\, 1_5\, 1_1\rangle$.
The system is non-flexible, for example $(3_5 \cdot 1_4) \cdot 3_5 = 3_2 \cdot 3_5 = 1_3$, whilst $3_5 \cdot (1_4 \cdot 3_5) = 3_5 \cdot 2_3 = 0_1$.

**Example NE.5.** Pure non-flexible LDTS(28).
$V = \mathbb{Z}_{14} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 12_1\rangle$, $\langle 12_1, 0_0, 4_1\rangle$, $\langle 4_1, 0_0, 6_1\rangle$, $\langle 6_1, 0_0, 13_1\rangle$, $\langle 13_1, 0_0, 9_1\rangle$, $\langle 9_1, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$.
The system is non-flexible, for example $(3_1 \cdot 3_0) \cdot 3_1 = 0_0 \cdot 3_1 = 9_1$, whilst $3_1 \cdot (3_0 \cdot 3_1) = 3_1 \cdot 0_1 = 1_1$.

**Example NE.6.** Pure non-flexible LDTS(30).
$V = \mathbb{Z}_5 \times \mathbb{Z}_6$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 0_5\, 0_0\, 3_5\rangle$, $\langle 3_5\, 0_0\, 4_5\rangle$, $\langle 4_5\, 0_0\, 1_5\rangle$, $\langle 1_5\, 0_0\, 0_5\rangle$, $\langle 0_0\, 0_1\, 1_0\rangle$, $\langle 1_0\, 0_1\, 4_0\rangle$, $\langle 4_0\, 0_1\, 3_0\rangle$, $\langle 3_0\, 0_1\, 0_0\rangle$, $\langle 3_1\, 0_0\, 4_2\rangle$, $\langle 4_2\, 0_0\, 1_2\rangle$, $\langle 1_2\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 3_2\rangle$, $\langle 3_2\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_4\, 3_2\rangle$, $\langle 3_2\, 0_4\, 4_2\rangle$, $\langle 4_2\, 0_4\, 3_1\rangle$, $\langle 2_2\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 4_3\rangle$, $\langle 4_3\, 0_0\, 4_4\rangle$, $\langle 4_4\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 2_2\rangle$, $\langle 3_1\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 2_2\rangle$, $\langle 2_2\, 0_1\, 2_5\rangle$, $\langle 2_5\, 0_1\, 3_1\rangle$, $\langle 4_1\, 0_1\, 3_5\rangle$, $\langle 3_5\, 0_1\, 0_4\rangle$, $\langle 0_4\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 4_2\rangle$, $\langle 4_2\, 0_1\, 4_3\rangle$, $\langle 4_3\, 0_1\, 2_3\rangle$, $\langle 2_3\, 0_1\, 4_1\rangle$, $\langle 2_1\, 0_2\, 2_5\rangle$, $\langle 2_5\, 0_2\, 3_4\rangle$, $\langle 3_4\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 0_4\rangle$, $\langle 0_4\, 0_2\, 2_1\rangle$, $\langle 4_1\, 0_3\, 3_4\rangle$, $\langle 3_4\, 0_3\, 4_4\rangle$, $\langle 4_4\, 0_3\, 0_5\rangle$, $\langle 0_5\, 0_3\, 1_2\rangle$, $\langle 1_2\, 0_3\, 4_5\rangle$, $\langle 4_5\, 0_3\, 4_1\rangle$, $\langle 2_2\, 0_3\, 1_5\rangle$, $\langle 1_5\, 0_3\, 3_2\rangle$, $\langle 3_2\, 0_3\, 2_4\rangle$, $\langle 2_4\, 0_3\, 2_5\rangle$, $\langle 2_5\, 0_3\, 2_2\rangle$, $\langle 4_1\, 0_5\, 2_3\rangle$, $\langle 2_3\, 0_5\, 3_4\rangle$, $\langle 3_4\, 0_5\, 4_1\rangle$.
The system is non-flexible, for example $(2_5 \cdot 3_4) \cdot 2_5 = 0_2 \cdot 2_5 = 2_1$, whilst $2_5 \cdot (3_4 \cdot 2_5) = 2_5 \cdot 0_0 = 2_4$.

**Example NE.7.** Pure non-flexible LDTS(34).
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$, $\langle 6_0, 0_0, 1_1\rangle$, $\langle 1_1, 0_0, 8_0\rangle$, $\langle 8_0, 0_0, 2_1\rangle$, $\langle 2_1, 0_0, 7_1\rangle$, $\langle 7_1, 0_0, 5_1\rangle$, $\langle 5_1, 0_0, 0_1\rangle$, $\langle 0_1, 0_0, 6_0\rangle$, $\langle 3_0, 1_1, 7_1\rangle$, $\langle 7_1, 1_1, 4_0\rangle$, $\langle 4_0, 1_1, 14_1\rangle$, $\langle 14_1, 1_1, 10_0\rangle$, $\langle 10_0, 1_1, 9_1\rangle$, $\langle 9_1, 1_1, 3_0\rangle$, $\langle 5_0, 1_1, 8_1\rangle$, $\langle 8_1, 1_1, 9_0\rangle$, $\langle 9_0, 1_1, 15_1\rangle$, $\langle 15_1, 1_1, 0_1\rangle$, $\langle 0_1, 1_1, 5_0\rangle$.
The system is non-flexible, for example $(0_0 \cdot 2_0) \cdot 0_0 = 12_0 \cdot 0_0 = 13_0$, whilst $0_0 \cdot (2_0 \cdot 0_0) = 0_0 \cdot 16_0 = 4_0$.

**Example NE.8.** Pure non-flexible LDTS(36).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_5) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.

$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 2_0\,0_1\,6_0\rangle$, $\langle 6_0\,0_1\,1_2\rangle$, $\langle 1_2\,0_1\,5_2\rangle$, $\langle 5_2\,0_1\,5_1\rangle$, $\langle 5_1\,0_1\,2_0\rangle$, $\langle 3_0\,0_3\,4_4\rangle$, $\langle 4_4\,0_3\,5_2\rangle$, $\langle 5_2\,0_3\,3_0\rangle$, $\langle 0_2\,0_4\,1_3\rangle$, $\langle 1_3\,0_4\,4_2\rangle$, $\langle 4_2\,0_4\,0_2\rangle$, $\langle 4_1\,0_0\,4_2\rangle$, $\langle 4_2\,0_0\,0_3\rangle$, $\langle 0_3\,0_0\,5_4\rangle$, $\langle 5_4\,0_0\,2_4\rangle$, $\langle 2_4\,0_0\,0_2\rangle$, $\langle 0_2\,0_0\,4_1\rangle$, $\langle 2_3\,1_1\,3_4\rangle$, $\langle 3_4\,1_1\,1_4\rangle$, $\langle 1_4\,1_1\,4_4\rangle$, $\langle 4_4\,1_1\,5_4\rangle$, $\langle 5_4\,1_1\,0_4\rangle$, $\langle 0_4\,1_1\,6_4\rangle$, $\langle 6_4\,1_1\,\infty\rangle$, $\langle \infty\,1_1\,2_4\rangle$, $\langle 2_4\,1_1\,2_3\rangle$, $\langle 2_0\,0_2\,5_1\rangle$, $\langle 5_1\,0_2\,1_1\rangle$, $\langle 1_1\,0_2\,0_3\rangle$, $\langle 0_3\,0_2\,1_2\rangle$, $\langle 1_2\,0_2\,5_4\rangle$, $\langle 5_4\,0_2\,4_0\rangle$, $\langle 4_0\,0_2\,\infty\rangle$, $\langle \infty\,0_2\,2_0\rangle$, $\langle 4_2\,2_2\,3_4\rangle$, $\langle 3_4\,2_2\,0_3\rangle$, $\langle 0_3\,2_2\,4_2\rangle$, $\langle 0_3\,\infty\,1_3\rangle$, $\langle 1_2\,0_0\,3_4\rangle$, $\langle 3_4\,0_0\,6_2\rangle$, $\langle 6_2\,0_0\,3_3\rangle$, $\langle 3_3\,0_0\,5_3\rangle$, $\langle 5_3\,0_0\,0_4\rangle$, $\langle 0_4\,0_0\,6_3\rangle$, $\langle 6_3\,0_0\,6_4\rangle$, $\langle 6_4\,0_0\,1_3\rangle$, $\langle 1_3\,0_0\,4_4\rangle$, $\langle 4_4\,0_0\,2_3\rangle$, $\langle 2_3\,0_0\,1_2\rangle$, $\langle 1_1\,0_1\,4_2\rangle$, $\langle 4_2\,0_1\,3_3\rangle$, $\langle 3_3\,0_1\,0_3\rangle$, $\langle 0_3\,0_1\,1_1\rangle$, $\langle 2_3\,0_1\,5_3\rangle$, $\langle 5_3\,0_1\,4_3\rangle$, $\langle 4_3\,0_1\,2_3\rangle$.
The system is non-flexible, for example $(0_3 \cdot 4_2) \cdot 0_3 = 2_2 \cdot 0_3 = 3_4$, whilst $0_3 \cdot (4_2 \cdot 0_3) = 0_3 \cdot 0_0 = 5_4$.

**Example NE.9.** Pure non-flexible LDTS(40).
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0, 0_0, 5_0\rangle$, $\langle 5_0, 0_0, 1_1\rangle$, $\langle 1_1, 0_0, 11_0\rangle$, $\langle 11_0, 0_0, 18_1\rangle$, $\langle 18_1, 0_0, 8_1\rangle$, $\langle 8_1, 0_0, 12_0\rangle$, $\langle 12_0, 0_0, 3_1\rangle$, $\langle 3_1, 0_0, 5_1\rangle$, $\langle 5_1, 0_0, 14_0\rangle$, $\langle 14_0, 0_0, 14_1\rangle$, $\langle 14_1, 0_0, 7_0\rangle$, $\langle 7_0, 0_0, 3_0\rangle$, $\langle 3_0, 0_0, 1_0\rangle$.
The system is non-flexible, for example $(14_1 \cdot 14_0) \cdot 14_1 = 0_1 \cdot 14_1 = 5_0$, whilst $14_1 \cdot (14_0 \cdot 14_1) = 14_1 \cdot 0_0 = 7_0$.

**Example FO.1.** Pure flexible LDTS(21).
$V = (\mathbb{Z}_5 \times \mathbb{Z}_4) \cup \{\infty\}$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
$\langle 1_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,1_0\rangle$, $\langle 1_0\,3_2\,0_1\rangle$, $\langle 0_1\,3_2\,2_1\rangle$, $\langle 2_1\,3_2\,1_0\rangle$, $\langle 2_0\,0_0\,0_3\rangle$, $\langle 0_3\,0_0\,\infty\rangle$, $\langle \infty\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,0_2\rangle$, $\langle 0_2\,0_0\,2_0\rangle$, $\langle 2_0\,3_2\,0_2\rangle$, $\langle 0_2\,3_2\,3_1\rangle$, $\langle 3_1\,3_2\,\infty\rangle$, $\langle \infty\,3_2\,0_3\rangle$, $\langle 0_3\,3_2\,2_0\rangle$, $\langle 1_0\,0_3\,0_2\rangle$, $\langle 0_2\,0_3\,4_2\rangle$, $\langle 4_2\,0_3\,0_1\rangle$, $\langle 0_1\,0_3\,1_1\rangle$, $\langle 1_1\,0_3\,2_3\rangle$, $\langle 2_3\,0_3\,1_0\rangle$, $\langle 1_0\,3_3\,2_3\rangle$, $\langle 2_3\,3_3\,1_1\rangle$, $\langle 1_1\,3_3\,0_1\rangle$, $\langle 0_1\,3_3\,4_2\rangle$, $\langle 4_2\,3_3\,0_2\rangle$, $\langle 0_2\,3_3\,1_0\rangle$.

**Example FO.2.** Pure flexible LDTS(31).
$V = \mathbb{Z}_{31}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 8, 7, 13\rangle$, $\langle 13, 7, 30\rangle$, $\langle 30, 7, 19\rangle$, $\langle 19, 7, 10\rangle$, $\langle 10, 7, 8\rangle$, $\langle 8, 23, 10\rangle$, $\langle 10, 23, 19\rangle$, $\langle 19, 23, 30\rangle$, $\langle 30, 23, 13\rangle$, $\langle 13, 23, 8\rangle$.

**Example FO.3.** Pure flexible LDTS(39).
$V = \mathbb{Z}_{13} \times \mathbb{Z}_3$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 10_0\,4_0\,4_2\rangle$, $\langle 4_2\,4_0\,5_1\rangle$, $\langle 5_1\,4_0\,10_0\rangle$, $\langle 10_0\,3_1\,5_1\rangle$, $\langle 5_1\,3_1\,4_2\rangle$, $\langle 4_2\,3_1\,10_0\rangle$, $\langle 2_1\,8_0\,11_1\rangle$, $\langle 11_1\,8_0\,5_2\rangle$, $\langle 5_2\,8_0\,2_1\rangle$, $\langle 2_1\,8_1\,5_2\rangle$, $\langle 5_2\,8_1\,11_1\rangle$, $\langle 11_1\,8_1\,2_1\rangle$, $\langle 0_0\,10_0\,11_0\rangle$, $\langle 11_0\,10_0\,2_0\rangle$, $\langle 2_0\,10_0\,12_1\rangle$, $\langle 12_1\,10_0\,0_0\rangle$, $\langle 0_0\,11_1\,12_1\rangle$, $\langle 12_1\,11_1\,2_0\rangle$, $\langle 2_0\,11_1\,11_0\rangle$, $\langle 11_0\,11_1\,0_0\rangle$, $\langle 3_1\,11_0\,12_2\rangle$, $\langle 12_2\,11_0\,6_2\rangle$, $\langle 6_2\,11_0\,3_2\rangle$, $\langle 3_2\,11_0\,3_1\rangle$, $\langle 3_1\,8_1\,3_2\rangle$, $\langle 3_2\,8_1\,6_2\rangle$, $\langle 6_2\,8_1\,12_2\rangle$, $\langle 12_2\,8_1\,3_1\rangle$, $\langle 3_0\,9_2\,7_1\rangle$, $\langle 7_1\,9_2\,0_2\rangle$, $\langle 0_2\,9_2\,10_0\rangle$, $\langle 10_0\,9_2\,1_2\rangle$, $\langle 1_2\,9_2\,3_0\rangle$, $\langle 3_0\,12_2\,1_2\rangle$, $\langle 1_2\,12_2\,10_0\rangle$, $\langle 10_0\,12_2\,0_2\rangle$, $\langle 0_2\,12_2\,7_1\rangle$, $\langle 7_1\,12_2\,3_0\rangle$.

**Example FO.4.** Pure flexible LDTS(43).
$V = \mathbb{Z}_{43}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.

$\langle 6,26,37\rangle$, $\langle 37,26,12\rangle$, $\langle 12,26,6\rangle$, $\langle 6,28,12\rangle$, $\langle 12,28,37\rangle$, $\langle 37,28,6\rangle$, $\langle 4,3,8\rangle$, $\langle 8,3,16\rangle$, $\langle 16,3,6\rangle$, $\langle 6,3,4\rangle$, $\langle 4,23,6\rangle$, $\langle 6,23,16\rangle$, $\langle 16,23,8\rangle$, $\langle 8,23,4\rangle$.

**Example FO.5.** Pure flexible LDTS(67).
$V = \mathbb{Z}_{67}$.
The triples are obtained from the following starter blocks under the action of the mapping $i \mapsto i+1$.
$\langle 16,0,25\rangle$, $\langle 25,0,18\rangle$, $\langle 18,0,26\rangle$, $\langle 26,0,30\rangle$, $\langle 30,0,32\rangle$, $\langle 32,0,33\rangle$, $\langle 33,0,22\rangle$, $\langle 22,0,28\rangle$, $\langle 28,0,31\rangle$, $\langle 31,0,21\rangle$, $\langle 21,0,16\rangle$, $\langle 16,45,21\rangle$, $\langle 21,45,31\rangle$, $\langle 31,45,28\rangle$, $\langle 28,45,22\rangle$, $\langle 22,45,33\rangle$, $\langle 33,45,32\rangle$, $\langle 32,45,30\rangle$, $\langle 30,45,26\rangle$, $\langle 26,45,18\rangle$, $\langle 18,45,25\rangle$, $\langle 25,45,16\rangle$.

**Example FE.1.** Pure flexible LDTS(34).
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 5_0\, 8_1\, 12_0\rangle$, $\langle 12_0\, 8_1\, 16_1\rangle$, $\langle 16_1\, 8_1\, 5_0\rangle$, $\langle 5_0\, 10_1\, 16_1\rangle$, $\langle 16_1\, 10_1\, 12_0\rangle$, $\langle 12_0\, 10_1\, 5_0\rangle$, $\langle 12_0\, 2_1\, 14_1\rangle$, $\langle 14_1\, 2_1\, 1_1\rangle$, $\langle 1_1\, 2_1\, 12_0\rangle$, $\langle 12_0\, 4_1\, 1_1\rangle$, $\langle 1_1\, 4_1\, 14_1\rangle$, $\langle 14_1\, 4_1\, 12_0\rangle$, $\langle 0_0\, 8_0\, 11_0\rangle$, $\langle 11_0\, 8_0\, 9_0\rangle$, $\langle 9_0\, 8_0\, 13_0\rangle$, $\langle 13_0\, 8_0\, 8_1\rangle$, $\langle 8_1\, 8_0\, 0_0\rangle$, $\langle 0_0\, 10_1\, 8_1\rangle$, $\langle 8_1\, 10_1\, 13_0\rangle$, $\langle 13_0\, 10_1\, 9_0\rangle$, $\langle 9_0\, 10_1\, 11_0\rangle$, $\langle 11_0\, 10_1\, 0_0\rangle$.

**Example FE.2.** Pure flexible LDTS(40).
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 6_0\, 9_1\, 16_0\rangle$, $\langle 16_0\, 9_1\, 7_1\rangle$, $\langle 7_1\, 9_1\, 17_1\rangle$, $\langle 17_1\, 9_1\, 6_0\rangle$, $\langle 2_0\, 11_0\, 3_1\rangle$, $\langle 3_1\, 11_0\, 14_0\rangle$, $\langle 14_0\, 11_0\, 12_0\rangle$, $\langle 12_0\, 11_0\, 16_0\rangle$, $\langle 16_0\, 11_0\, 2_0\rangle$, $\langle 2_0\, 12_1\, 16_0\rangle$, $\langle 16_0\, 12_1\, 12_0\rangle$, $\langle 12_0\, 12_1\, 14_0\rangle$, $\langle 14_0\, 12_1\, 3_1\rangle$, $\langle 3_1\, 12_1\, 2_0\rangle$, $\langle 14_0\, 2_0\, 19_1\rangle$, $\langle 19_1\, 2_0\, 15_0\rangle$, $\langle 15_0\, 2_0\, 17_1\rangle$, $\langle 17_1\, 2_0\, 1_1\rangle$, $\langle 1_1\, 2_0\, 8_1\rangle$, $\langle 8_1\, 2_0\, 14_0\rangle$, $\langle 14_0\, 2_1\, 8_1\rangle$, $\langle 8_1\, 2_1\, 1_1\rangle$, $\langle 1_1\, 2_1\, 17_1\rangle$, $\langle 17_1\, 2_1\, 15_0\rangle$, $\langle 15_0\, 2_1\, 19_1\rangle$, $\langle 19_1\, 2_1\, 14_0\rangle$.

**Example FE.3.** Pure flexible LDTS(52).
$V = \mathbb{Z}_{26} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 15_0\, 8_0\, 12_1\rangle$, $\langle 12_1\, 8_0\, 25_0\rangle$, $\langle 25_0\, 8_0\, 15_0\rangle$, $\langle 15_0\, 18_1\, 25_0\rangle$, $\langle 25_0\, 18_1\, 12_1\rangle$, $\langle 12_1\, 18_1\, 15_0\rangle$, $\langle 5_0\, 20_0\, 23_0\rangle$, $\langle 23_0\, 20_0\, 21_0\rangle$, $\langle 21_0\, 20_0\, 25_0\rangle$, $\langle 25_0\, 20_0\, 5_0\rangle$, $\langle 5_0\, 21_1\, 25_0\rangle$, $\langle 25_0\, 21_1\, 21_0\rangle$, $\langle 21_0\, 21_1\, 23_0\rangle$, $\langle 23_0\, 21_1\, 5_0\rangle$, $\langle 4_0\, 12_1\, 3_1\rangle$, $\langle 3_1\, 12_1\, 18_0\rangle$, $\langle 18_0\, 12_1\, 9_1\rangle$, $\langle 9_1\, 12_1\, 23_0\rangle$, $\langle 23_0\, 12_1\, 11_0\rangle$, $\langle 11_0\, 12_1\, 24_0\rangle$, $\langle 24_0\, 12_1\, 10_0\rangle$, $\langle 10_0\, 12_1\, 22_1\rangle$, $\langle 22_1\, 12_1\, 5_0\rangle$, $\langle 5_0\, 12_1\, 16_1\rangle$, $\langle 16_1\, 12_1\, 17_0\rangle$, $\langle 17_0\, 12_1\, 0_1\rangle$, $\langle 0_1\, 12_1\, 20_0\rangle$, $\langle 20_0\, 12_1\, 4_1\rangle$, $\langle 4_1\, 12_1\, 23_1\rangle$, $\langle 23_1\, 12_1\, 10_1\rangle$, $\langle 10_1\, 12_1\, 17_1\rangle$, $\langle 17_1\, 12_1\, 7_0\rangle$, $\langle 7_0\, 12_1\, 13_1\rangle$, $\langle 13_1\, 12_1\, 4_0\rangle$.

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# ANTIFLEXIBLE LATIN DIRECTED TRIPLE SYSTEMS

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Abstract. It is well known that given a Steiner triple system one can define a quasigroup operation $\cdot$ upon its base set by assigning $x \cdot x = x$ for all $x$ and $x \cdot y = z$, where $z$ is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems, where $(x, y)$ is considered to be ordered. But this is not necessarily the case for directed triple systems. However there do exist directed triple systems, which induce a quasigroup under this operation and these are called Latin directed triple systems. The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $y \cdot (x \cdot y) = (y \cdot x) \cdot y$ but those associated with Latin directed triple systems need not. In this paper we study the Latin directed triple systems where the flexible identity holds for the least possible number of ordered pairs $(x, y)$. We describe their geometry, present a surprisingly simple cyclic construction and prove that they exist if and only if the order $n$ is $n \equiv 0$ or $1 \pmod 3$ and $n \geq 13$.

## 1. Introduction

A *Steiner triple system* of order $n$, STS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of triples of distinct points taken from $V$ such that every pair of distinct points from $V$ appears in precisely one triple. Given an STS $(V, \mathcal{B})$ one can define a binary operation $\cdot$ on the set $V$ by assigning $x \cdot x = x$ for all $x \in V$ and $x \cdot y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The induced operation satisfies the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x$$

for all $x$ and $y$ in $V$. Any binary operation satisfying these three identities is called an *idempotent totally symmetric quasigroup.* The process described above is reversible. Given an idempotent totally symmetric quasigroup one can obtain an STS by assigning $\{x, y, x \cdot y\} \in \mathcal{B}$ for all $x$, $y \in V$, $x \neq y$. Thus there is a one-to-one correspondence between Steiner triple systems and idempotent totally symmetric quasigroups or *Steiner quasigroups*, as they are commonly known. All Steiner quasigroups satisfy the *flexible law* $y \cdot (x \cdot y) = (y \cdot x) \cdot y$.

If we consider oriented triples then there are two possibilities. A *cyclic triple* $(x, y, z)$ contains the ordered pair $(x, y)$, $(y, z)$ and $(z, x)$. A *transitive triple* $\langle x, y, z \rangle$ contains the ordered pair $(x, y)$, $(y, z)$ and $(x, z)$.

A *Mendelsohn triple system* of order $n$, MTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of cyclic triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Quasigroups can be obtained from Mendelsohn triple systems by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$, where $z$ is the third element in the transitive triple containing the ordered pair $(x, y)$. These so called *Mendelsohn quasigroups* satisfy the same algebraic properties as their Steiner

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counterparts with the exception of commutativity. Similarly there is a one-to-one correspondence between Mendelsohn triple systems and Mendelsohn quasigroups. Again, all Mendelsohn quasigroups satisfy the flexible law.

A *directed triple system* of order $n$, DTS($n$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $n$ points and $\mathcal{B}$ is a collection of transitive triples of distinct points taken from $V$ such that every ordered pair of distinct points from $V$ appears in precisely one triple. Given a DTS($n$), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$, where $z$ is the third element in the transitive triple containing the ordered pair $(x, y)$. However the structure obtained need not necessarily be a quasigroup. If for example $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$, then $u \cdot x = v \cdot x = y$, but $u \neq v$. There do however exist DTSs that yield quasigroups. Such a DTS($n$) is called a *Latin directed triple system*, denoted by LDTS($n$), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*. The binary operation will sometimes be replaced with juxtaposition, for example $x \cdot yz$ meaning $x \cdot (y \cdot z)$.

Latin directed triple systems were introduced in [3], where it was shown that an LDTS($n$) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 4$, 6 or 10. The algebraic and geometrical aspects of LDTSs were studied in [4]. Together these two papers also give enumeration results for all orders less than or equal to 13.

The following theorem was proved in [4].

**Theorem 1.1.** *Let $(V, \mathcal{B})$ be a directed triple system. Define a binary operation $\cdot$ on $V$ in such a way that $x \cdot y = z$, $y \cdot z = x$ and $x \cdot z = y$ whenever $\langle x, y, z\rangle \in \mathcal{B}$, and that $x \cdot x = x$ for all $x \in V$. Then $V(\cdot)$ is a quasigroup if and only if for all $\langle x, y, z\rangle \in \mathcal{B}$ there exist $x', y', z' \in V$ such that*

$$\langle z', y, x\rangle,\ \langle z, y', x\rangle,\ \langle z, y, x'\rangle \in \mathcal{B}.$$

*In such a case $z' = y \cdot x$, $y' = z \cdot x$ and $x' = z \cdot y$.*

It is now easy to see that in an LDTS, $(V, \mathcal{B})$,

$$\langle x, y, x \cdot y\rangle \in \mathcal{B} \quad \Rightarrow \quad y \cdot (x \cdot y) = (y \cdot x) \cdot y, \tag{1}$$

since if $\langle x, y, z\rangle \in \mathcal{B}$ then $\langle z', y, x\rangle \in \mathcal{B}$ for some $z'$ and the ordered pair $(x, y)$ satisfies the flexible identity $y \cdot (x \cdot y) = y \cdot z = x = z' \cdot y = (y \cdot x) \cdot y$. However, the flexible identity need not be satisfied for all ordered pairs of points from $V$. The following theorem proved in [3] gives the necessary and sufficient condition for an LDTS to be flexible.

**Theorem 1.2.** *A DTS-quasigroup obtained from an LDTS(n), $(V, \mathcal{B})$, satisfies the flexible law if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$.*

In [5] it was shown that a flexible LDTS($n$) exists for all $n \equiv 0$ or $1 \pmod 3$, $n \neq 4, 6, 10, 12$.

In this paper we study the LDTSs whose binary operation satisfies the reverse of (1), i.e. for all $x, y \in V$, $x \neq y$,

$$y \cdot (x \cdot y) = (y \cdot x) \cdot y \quad \Rightarrow \quad \langle x, y, x \cdot y\rangle \in \mathcal{B}.$$

An LDTS satisfying this condition is called *antiflexible*. In other words an antiflexible DTS-quasigroup is one where the flexible identity $(y \cdot x) \cdot y = y \cdot (x \cdot y)$ holds for the least possible number of ordered pairs $(x, y) \in V \times V$. Thus, in a sense, antiflexible LDTSs are the LDTSs which are as distant from STSs as possible.

At first glance antiflexible LDTSs may appear to be a very artificial construct. However, there exists a surprisingly simple cyclic construction of LDTSs which naturally produces antiflexible LDTSs, see Theorem 3.1.

## 2. Properties

Let $(V, \mathcal{B})$ be a DTS and denote by $\mathcal{F}$ the set of all $\{x, y, z\}$ such that $\langle x, y, z\rangle \in \mathcal{B}$. This set is known as the *underlying twofold triple system* of $(V, \mathcal{B})$. Consider now $\mathcal{F}$ as a set of faces. Each edge $\{a, b\}$ is incident to two faces, hence the faces can be sewn together along common edges to form a pseudosurface. Note that we can orient a face $\{x, y, z\} \in \mathcal{F}$ as a cycle $(x, y, z)$ whenever $\langle x, y, z\rangle \in \mathcal{B}$. It follows from Theorem 1.1 that this defines a coherent orientation. Hence $\mathcal{F}$ is an orientable pseudosurface.

A DTS is said to be *pure* if its underlying twofold triple system contains no repeated blocks. It is easy to see that every antiflexible LDTS is pure. If for some antiflexible LDTS, $(V, \mathcal{B})$, the triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$ both belonged to $\mathcal{B}$, then $x \cdot (y \cdot x) = x \cdot z = y = z \cdot x = (x \cdot y) \cdot x$, which would imply that $\langle y, x, y \cdot x\rangle \in \mathcal{B}$. But this is a contradiction since $\langle z, y, x\rangle$ and $\langle y, x, y \cdot x\rangle$ cannot both belong to $\mathcal{B}$.

With each point $x \in V$ we can associate a partition of $V \setminus \{x\}$ into a set of cycles $(y_{1,1}, \ldots, y_{1,k_1})(y_{2,1}, \ldots, y_{2,k_2}) \cdots (y_{m,1}, \ldots, y_{m,k_m})$, such that $(x, y_{i,j}, y_{i,j+1})$ and $(x, y_{i,k_i}, y_{i,1})$ are oriented faces of $\mathcal{F}$ for all $1 \le j \le k_i - 1$ and $1 \le i \le m$. If $m > 1$ then $x$ is said to be a *pinch point*. A pseudosurface can be turned into a surface by separating each pinch point into several new points, called *vertices*, such that every vertex is associated with a single cycle. The length of the associated cycle is called the *degree* of the vertex. Thus we obtain an orientable surface. It follows from Theorem 1.1 that there are two types of vertices in this surface. A vertex may be associated with a point $x$ and a cycle $(y_1, \ldots, y_k)$ such that

$$\langle y_2, x, y_1\rangle, \langle y_3, x, y_2\rangle, \ldots, \langle y_1, x, y_k\rangle \in \mathcal{B}.$$

This type of vertex is called a *middle vertex* to reflect the fact that $x$ appears in the middle position of each of the $k$ transitive triples. Alternatively, a vertex may be associated with a point $x$ and a cycle $(y_1, z_1, y_2, z_2, \ldots, y_k, z_k)$ such that

$$\langle x, y_1, z_1\rangle, \langle z_1, y_2, x\rangle, \langle x, y_2, z_2\rangle, \langle z_2, y_3, x\rangle, \ldots, \langle x, y_k, z_k\rangle, \langle z_k, y_1, x\rangle \in \mathcal{B}.$$

This type of vertex is called a *residual vertex* in accordance with [4]. The degree of a residual vertex is always even.

**Example 2.1.** Let $V = \mathbb{Z}_{13}$ and define the set of starter triples $\mathcal{S} = \{\langle 1, 4, 0\rangle, \langle 0, 6, 1\rangle, \langle 2, 6, 0\rangle, \langle 0, 5, 2\rangle\}$. Let $\mathcal{B} = \{\, \langle x + i, y + i, z + i\rangle : \langle x, y, z\rangle \in \mathcal{S},\ i \in \mathbb{Z}_n \,\}$. Then $(V, \mathcal{B})$ is an antiflexible LDTS(13). As one can see from the triples listed below, the set of cycles associated with the point 0 is $(7, 9, 10, 8)(5, 2, 6, 1, 4, 11, 3, 12)$. Thus the point 0 separates into two vertices. The vertex associated with the cycle $(7, 9, 10, 8)$ is a middle vertex and it is formed by the triples $\langle 9, 0, 7\rangle$, $\langle 10, 0, 9\rangle$, $\langle 8, 0, 10\rangle$, $\langle 7, 0, 8\rangle$ in $\mathcal{B}$. The vertex associated with the cycle $(5, 2, 6, 1, 4, 11, 3, 12)$ is a residual vertex and it is formed by the triples $\langle 0, 5, 2\rangle$, $\langle 2, 6, 0\rangle$, $\langle 0, 6, 1\rangle$, $\langle 1, 4, 0\rangle$, $\langle 0, 4, 11\rangle$, $\langle 11, 3, 0\rangle$, $\langle 0, 3, 12\rangle$, $\langle 12, 5, 0\rangle$ in $\mathcal{B}$.

**Theorem 2.2.** *Let $(V, \mathcal{B})$ be an LDTS. Then the following conditions are equivalent:*

(i) $(V, \mathcal{B})$ *is antiflexible;*
(ii) $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, zx, yx\rangle \notin \mathcal{B}$;

(iii) *every residual vertex has degree at least 6.*

*Proof.* First assume that $(V, \mathcal{B})$ is antiflexible and let $\langle x, y, z\rangle \in \mathcal{B}$. Then using Theorem 1.1 the triple $\langle yx, y, x\rangle$ belongs to $\mathcal{B}$ as well. If it were the case that $\langle x, zx, yx\rangle \in \mathcal{B}$, then we would have $x \cdot yx = zx = xy \cdot x$. Then by assumption $\langle y, x, yx\rangle \in \mathcal{B}$. But this is a contradiction since $\langle y, x, yx\rangle$ and $\langle yx, y, x\rangle$ cannot both belong to $\mathcal{B}$. Thus $\langle x, zx, yx\rangle \notin \mathcal{B}$. We see that (i) implies (ii).

Assume that condition (ii) holds. If the cycle about a residual vertex corresponding to a point $x$ were of length 2, say $(y_1, z_1)$, then we would have $\langle x, y_1, z_1\rangle$, $\langle z_1, y_1, x\rangle \in \mathcal{B}$. But then $\mathcal{B}$ would contain $\langle x, z_1 \cdot x, y_1 \cdot x\rangle$, since this is the triple $\langle x, y_1, z_1\rangle$. Similarly if the cycle were of length 4, say $(y_1, z_1, y_2, z_2)$, then we would have $\langle x, y_1, z_1\rangle$, $\langle z_1, y_2, x\rangle$, $\langle x, y_2, z_2\rangle$, $\langle z_2, y_1, x\rangle \in \mathcal{B}$. But then $\mathcal{B}$ would again contain $\langle x, z_1 \cdot x, y_1 \cdot x\rangle$, since this is the triple $\langle x, y_2, z_2\rangle$. Thus (ii) implies (iii).

Finally assume that condition (iii) holds. Let $x, y \in V$ such that $x \neq y$ and $y \cdot xy = yx \cdot y$. Now either $\langle xy, x, y\rangle$, $\langle x, xy, y\rangle$ or $\langle x, y, xy\rangle$ lies in $\mathcal{B}$. However, the first two of these possibilities violate the assumption. If $\langle xy, x, y\rangle \in \mathcal{B}$, then $\langle y, x, yx\rangle$, $\langle yx, yx \cdot y, y\rangle$, $\langle y, y \cdot xy, xy\rangle \in \mathcal{B}$, i.e. there exists a residual vertex associated with the point $y$ and the cycle $(x, yx, y \cdot xy, xy)$. If $\langle x, xy, y\rangle \in \mathcal{B}$, then $\langle y, yx, x\rangle$, $\langle y, xy, y \cdot xy\rangle$, $\langle yx \cdot y, yx, y\rangle \in \mathcal{B}$, i.e. there exists a residual vertex associated with the point $y$ and the cycle $(yx, x, xy, y \cdot xy)$. Thus (iii) implies (i). □

## 3. Existence

In this section we investigate the existence spectrum of antiflexible LDTS($n$). It was shown in [3] that there is no pure LDTS($n$) for $3 \le n \le 12$. We start with a cyclic construction. An LDTS($n$) is said to be *cyclic* if it admits an automorphism which permutes its points in a single cycle of length $n$. In [11] it was shown that a pure cyclic LDTS($n$) exists if and only if $n \equiv 1 \pmod 6$ and $n \ge 13$. The following theorem shows that the construction used in [11] can always be used to produce antiflexible LDTSs. It is interesting to note, however, that for certain orders the construction can also be used to produce flexible LDTSs.

**Theorem 3.1.** *A cyclic antiflexible LDTS(n) exists if and only if* $n \equiv 1 \pmod 6$ *and* $n \ge 13$.

*Proof.* Let $n = 6k + 1$ and $k \ge 2$. Set $V = \mathbb{Z}_n$ and define the set of starter triples

$$\mathcal{S} = \big\{ \langle r, k + 2r, 0\rangle,\ \langle 0, 3k - r + 1, r\rangle : r = 1, 2, \ldots, k \big\}.$$

If $k \equiv 1 \pmod 3$, then replace the starter triples

$$\langle \tfrac{2k+1}{3},\ k + 2\tfrac{2k+1}{3},\ 0\rangle, \quad \langle 0,\ 3k - \tfrac{2k+1}{3} + 1,\ \tfrac{2k+1}{3}\rangle, \quad \langle k,\ 3k,\ 0\rangle$$

in $\mathcal{S}$ with the starter triples

$$B_1 = \langle 4k{+}1,\ 0,\ \tfrac{1}{3}(5k{+}1)\rangle,\ B_2 = \langle \tfrac{1}{3}(5k{+}1),\ 0,\ \tfrac{1}{3}(2k{+}1)\rangle,\ B_3 = \langle \tfrac{1}{3}(2k{+}1),\ 0,\ 3k{+}1\rangle.$$

Let $\mathcal{B} = \{ \langle x{+}i, y{+}i, z{+}i\rangle : \langle x, y, z\rangle \in \mathcal{S},\ i \in \mathbb{Z}_n \}$. Then $(V, \mathcal{B})$ is an LDTS($n$).

We check that condition (ii) of Theorem 2.2 is satisfied for each starter triple. To begin with let us assume that $k \not\equiv 1 \pmod 3$. Let $1 \le s \le k$ and consider the starter triple $\langle x, y, z\rangle = \langle s, k + 2s, 0\rangle$. We have $zx = 0 \cdot s = 3k - s + 1$. If $s$ is even, then $\langle \frac{3}{2}s, k + 2s, s\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}s$ and $i = s$) i.e. $yx = \frac{3}{2}s$, and if $s$ is odd, then $\langle \frac{1}{2}(3s - 2k - 1), k + 2s, s\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(2k + 1 - s)$ and $i = \frac{1}{2}(3s - 2k - 1)$), i.e. $yx = \frac{1}{2}(3s - 2k - 1)$. If $s \le \frac{1}{2}k$ then $\langle s, 3k - s + 1, 3s\rangle \in \mathcal{B}$ (use $r = 2s$ and $i = s$), and if $s > \frac{1}{2}k$ then $\langle s, 3k - s + 1, 3s - 2k - 1\rangle \in \mathcal{B}$ (use $r = 2k + 1 - 2s$

and $i = 3s - 2k - 1$). The first two points in these two triples are $x$ and $zx$ respectively, but one can check that the third point is not equal to $yx$ for any $s$. Thus $\langle x, zx, yx\rangle \notin \mathcal{B}$.

Now consider the starter triple $\langle x, y, z\rangle = \langle 0, 3k-s+1, s\rangle$. We have $zx = s\cdot 0 = k + 2s$. If $s$ is odd, then $\langle k - \frac{1}{2}(s-1), 3k - s + 1, 0\rangle \in \mathcal{B}$ (use $r = k - \frac{1}{2}(s-1)$ and $i = 0$), i.e. $yx = k - \frac{1}{2}(s-1)$, and if $s$ is even, then $\langle -\frac{1}{2}s, 3k - s + 1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}s$ and $i = -\frac{1}{2}s$), i.e. $yx = -\frac{1}{2}s$. If $s \le \frac{1}{2}k$, then $\langle 0, k + 2s, -2s\rangle \in \mathcal{B}$ (use $r = 2s$ and $i = -2s$), and if $s > \frac{1}{2}k$, then $\langle 0, k + 2s, 2k - 2s + 1\rangle \in \mathcal{B}$ (use $r = 2k - 2s + 1$ and $i = 0$). We come to the same conclusion as above.

When $k \equiv 1 \pmod 3$ the statements above remain valid for all starter triples except for those that took part in the replacement, the case $s = \frac{1}{2}(k+1)$ discussed in the second paragraph and the cases $s \in \{1, \frac{1}{2}k, k\}$ discussed in the third paragraph. We prove that condition (ii) of Theorem 2.2 is satisfied for these triples as well:

For $\langle x, y, z\rangle = \langle 4k+1, 0, \frac{1}{3}(5k+1)\rangle$ we have $\langle \frac{1}{3}(5k+1), k, 4k+1\rangle \in \mathcal{B}$ (use $B_3$ and $i = k$), i.e. $zx = k$. If $k$ is odd, then $\langle \frac{1}{2}(3k+1), 0, 4k+1\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(7k+1)$ and $i = 4k+1$), if $k$ is even, then $\langle \frac{1}{2}k, 0, 4k+1\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(7k+2)$ and $i = \frac{1}{2}k$). Thus $yx \in \{\frac{1}{2}(3k+1), \frac{1}{2}k\}$ but $\langle 4k+1, k, 4k+2\rangle \in \mathcal{B}$ (use $r = 1$ and $i = 4k+1$).

For $\langle x, y, z\rangle = \langle \frac{1}{3}(5k+1), 0, \frac{1}{3}(2k+1)\rangle$ we have $\langle \frac{1}{3}(2k+1), \frac{4}{3}(2k+1), \frac{1}{3}(5k+1)\rangle \in \mathcal{B}$ (use $r = k$ and $i = \frac{1}{3}(2k+1)$), i.e. $zx = \frac{4}{3}(2k+1)$, and from $B_1$ we have $yx = 4k+1$. But $\langle \frac{1}{3}(5k+1), \frac{4}{3}(2k+1), \frac{1}{3}(5k-2)\rangle \in \mathcal{B}$ (use $r = 1$ and $i = \frac{1}{3}(5k-2)$).

For $\langle x, y, z\rangle = \langle \frac{1}{3}(2k+1), 0, 3k+1\rangle$ we have $\langle 3k+1, -k, \frac{1}{3}(2k+1)\rangle \in \mathcal{B}$ (use $B_1$ and $i = -k$), i.e. $zx = -k$, and from $B_2$ we have $yx = \frac{1}{3}(5k+1)$. But $\langle \frac{1}{3}(2k+1), -k, \frac{1}{3}(1-k)\rangle \in \mathcal{B}$ (use $B_2$ and $i = -k$).

If $k$ is odd, then for $\langle x, y, z\rangle = \langle \frac{1}{2}(k+1), 2k+1, 0\rangle$ we have $\langle 0, \frac{1}{2}(5k+1), \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(k+1)$ and $i = 0$), i.e. $zx = \frac{1}{2}(5k+1)$. If $k \equiv 1 \pmod 4$, then $\langle \frac{1}{4}(1-k), 2k+1, \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(3k+1)$ and $i = \frac{1}{4}(1-k)$), if $k \equiv 3 \pmod 4$, then $\langle \frac{3}{4}(k+1), 2k+1, \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(k+1)$ and $i = \frac{1}{2}(k+1)$). Thus $yx \in \{\frac{1}{4}(1-k), \frac{3}{4}(k+1)\}$ but $\langle \frac{1}{2}(k+1), \frac{1}{2}(5k+1), \frac{5}{6}(5k+1)\rangle \in \mathcal{B}$ (use $B_1$ and $i = \frac{1}{2}(5k+1)$).

For $\langle x, y, z\rangle = \langle 0, 3k, 1\rangle$ we have $zx = k+2$ as before and $\langle \frac{1}{3}(11k+1), 3k, 0\rangle \in \mathcal{B}$ (use $B_3$ and $i = 3k$), i.e. $yx = \frac{1}{3}(11k+1)$. But $\langle 0, k+2, 6k-1\rangle \in \mathcal{B}$ (use $r = 2$ and $i = -2$).

For $\langle x, y, z\rangle = \langle 0, \frac{5}{2}k+1, \frac{1}{2}k\rangle$ we have $zx = 2k$ as before. If $k \equiv 0 \pmod 4$, then $\langle -\frac{1}{4}k, \frac{1}{2}(5k+2), 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}k$ and $i = -\frac{1}{4}k$), and if $k \equiv 2 \pmod 4$, then $\langle \frac{1}{4}(3k+2), \frac{1}{2}(5k+2), 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(3k+2)$ and $i = 0$). Thus $yx \in \{-\frac{1}{4}k, \frac{1}{4}(3k+2)\}$ but $\langle 0, 2k, \frac{1}{3}(11k+1)\rangle \in \mathcal{B}$ (use $B_1$ and $i = 2k$).

For $\langle x, y, z\rangle = \langle 0, 2k+1, k\rangle$ we have $\langle k, -\frac{1}{3}(2k+1), 0\rangle \in \mathcal{B}$ (use $B_2$ and $i = -\frac{1}{3}(2k+1)$), i.e. $zx = -\frac{1}{3}(2k+1)$. If $k$ is odd, then $\langle \frac{1}{2}(k+1), 2k+1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(k+1)$ and $i = 0$), if $k$ is even, then $\langle -\frac{1}{2}k, 2k+1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}k$ and $i = -\frac{1}{2}k$). Thus $yx \in \{\frac{1}{2}(k+1), -\frac{1}{2}k\}$ but $\langle 0, -\frac{1}{3}(2k+1), \frac{1}{3}(7k+2)\rangle \in \mathcal{B}$ (use $B_3$ and $i = -\frac{1}{3}(2k+1)$). □

In [4] all LDTSs of order 13 were enumerated and classified by their automorphism group. Out of the total of 1 206 969 non-isomorphic LDTS(13)s 8 444 are pure, but only two of them are antiflexible. They are the two pure cyclic systems. The starter triples for these two systems are $\langle 1, 4, 0\rangle$, $\langle 0, 6, 1\rangle$, $\langle 2, 6, 0\rangle$, $\langle 0, 5, 2\rangle$

for one and $\langle 1,10,0\rangle$, $\langle 0,8,1\rangle$, $\langle 2,9,0\rangle$, $\langle 0,10,2\rangle$ for the other. In comparison, there are 924 flexible LDTS(13)s up to isomorphism.

Next is an elementary recursive construction adapted from standard design-theoretic techniques.

**Proposition 3.2.** *If there exists an antiflexible LDTS($n$), $n > 3$, then there exists*

(i) *an antiflexible LDTS($3n$), and*
(ii) *an antiflexible LDTS($3n-2$).*

*Proof.*

(i) Take three copies of the LDTS($n$) on point sets $\{\, i_j : i \in \mathbb{Z}_n \,\}$, where $j \in \{0, 1, 2\}$ respectively, then adjoin all transitive triples $\langle i_0, j_1, (i+j)_2\rangle$ and $\langle (i+j-1)_2, j_1, i_0\rangle$, $i, j \in \mathbb{Z}_n$. The adjoined transitive triples create one new residual vertex of degree $2n$ for each of the points in the first and third copies of the LDTS($n$). For any point $i_0$, where $i \in \mathbb{Z}_n$, the newly created residual vertex corresponds to the cycle
$$(0_1, i_2, 1_1, (i+1)_2, \ldots, (n-1)_1, (i-1)_2).$$
For any point $i_2$, where $i \in \mathbb{Z}_n$, the newly created residual vertex corresponds to the cycle
$$(0_1, (i+1)_0, (n-1)_1, (i+2)_0, \ldots, 1_1, i_0).$$
Thus the resulting system is antiflexible as long as $n > 2$.

(ii) Take three copies of the LDTS($n$) on point sets $\{\, i_j : i \in \mathbb{Z}_{n-1} \,\} \cup \{\infty\}$, $j \in \{0, 1, 2\}$ respectively, then adjoin all transitive triples $\langle i_0, j_1, (i+j)_2\rangle$ and $\langle (i+j-1)_2, j_1, i_0\rangle$, $i, j \in \mathbb{Z}_{n-1}$. Similarly this system is antiflexible as long as $n > 3$.

□

**Lemma 3.3.** *If $n \equiv 3 \pmod{18}$ and $n \neq 3$, then there exists an antiflexible LDTS($n$).*

*Proof.* It follows from Theorem 3.1 and part (i) of Proposition 3.2 that there exists an antiflexible LDTS($n$) for all $n \equiv 3 \pmod{18}$, $n \geq 39$. An antiflexible LDTS(21) is given as Example A.4 in the Appendix. □

**Proposition 3.4.** *If there exists an antiflexible LDTS($n$), $(V, \mathcal{B})$, and a quasi-group*
*$(V \cup \{\infty\}, *)$ satisfying*

(1) *$x * x = \infty$, and*
(2) *$(x * y = y * z \wedge z * y = y * x) \Rightarrow x = y = z$,*

*then there exists an antiflexible LDTS($2n+1$).*

*Proof.* Let $W = V \cup \{\, x' : x \in V \,\} \cup \{\infty'\}$. Form a set of transitive triples $\mathcal{D}$ by starting with the set $\mathcal{B}$ and adjoining all triples $\langle x', x * y, y'\rangle$, where $x$, $y \in V \cup \{\infty\}$, $x \neq y$. Then $(W, \mathcal{D})$ is an LDTS. We verify that $(W, \mathcal{D})$ satisfies condition (ii) of Theorem 2.2. Let $\langle x, y, z\rangle \in \mathcal{D}$. If $\langle x, y, z\rangle \in \mathcal{B}$, then $\langle x, z \cdot x, y \cdot x\rangle$ does not lie in $\mathcal{D}$, since if it did, then it would have had to come from $\mathcal{B}$, which would be a contradiction. It remains to deal with the case when $\langle x, y, z\rangle$ is of the form $\langle u', u * v, v'\rangle$, for some $u$, $v \in V \cup \{\infty\}$. Clearly $z \cdot x = v * u$. There exists $w \in V \cup \{\infty\}$ such that $\langle w', u * v, u'\rangle \in \mathcal{D}$, i.e. $y \cdot x = w'$. Now since $w * u = u * v$, by assumption $v * u \neq u * w$, and thus $\langle x, z \cdot x, y \cdot x\rangle = \langle u', v * u, w'\rangle$ does not lie in $\mathcal{D}$. □

A quasigroup of order $n$ satisfying conditions (1) and (2) of Proposition 3.4 will be referred to as a *unipotent locally self-orthogonal quasigroup*, ULSOQ($n$).

The remainder of the existence proof in this section uses a standard technique known as Wilson's fundamental construction for which we need the concept of a *group divisible design* (GDD). Let $K$ be a set of positive integers. A $K$-GDD of type $g^u$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where $V$ is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of $V$ into $u$ subsets of cardinality $g$ called *groups* and $\mathcal{B}$ is a family of subsets called *blocks* such that (1) $|B| \in K$ for all $B \in \mathcal{B}$, and (2) every pair of distinct elements of $V$ occurs in exacly one block or one group, but not both. We will also need $K$-GDDs of type $g^u m^1$. These are defined analogously, with the base set $V$ being of cardinality $v = gu + m$ and the partition $G$ being into $u$ subsets of cardinality $g$ and one set of cardinality $m$. If $K$ is a singleton, then instead of $\{k\}$-GDD we write simply $k$-GDD. Necessary and sufficient conditions for the existence of 3-GDDs of type $g^u$ were determined in [10] and for 3-GDDs of type $g^u m^1$ in [2]. The existence of the 4-GDDs that we will be using was determined in [1, 7, 8, 9]. A convenient reference is [6] where the existence of all the GDDs that are used can be verified.

We will assume that the reader is familiar with this construction but briefly the basic idea is as follows. Begin with a $k$-GDD of cardinality $v = gu$ or $gu+m$, usually called the *master GDD*. Each point is then assigned a weight, usually the same weight, say $w$. In effect, each point is replaced by $w$ points. Each inflated block of the master GDD is then replaced by a $k$-GDD of type $w^k$, called a *slave GDD*. We will only need to use the value $w = 3$, and instead of slave GDDs we will use partial Latin directed triple systems. When $k = 3$ we will employ the partial LDTS(9) whose blocks are $\langle a,p,x\rangle$, $\langle b,q,y\rangle$, $\langle c,r,z\rangle$, $\langle a,q,z\rangle$, $\langle b,r,x\rangle$, $\langle c,p,y\rangle$, $\langle a,r,y\rangle$, $\langle b,p,z\rangle$, $\langle c,q,x\rangle$, $\langle x,q,a\rangle$, $\langle y,r,b\rangle$, $\langle z,p,c\rangle$, $\langle z,r,a\rangle$, $\langle x,p,b\rangle$, $\langle y,q,c\rangle$, $\langle y,p,a\rangle$, $\langle z,q,b\rangle$, $\langle x,r,c\rangle$ and the sets $\{a,b,c\}$, $\{p,q,r\}$, $\{x,y,z\}$ play the role of the groups. When $k = 4$ we will use the partial LDTS(12) whose blocks are $\langle p,a,x\rangle$, $\langle s,a,p\rangle$, $\langle x,a,s\rangle$, $\langle q,b,y\rangle$, $\langle u,b,q\rangle$, $\langle y,b,u\rangle$, $\langle r,c,z\rangle$, $\langle t,c,r\rangle$, $\langle z,c,t\rangle$, $\langle c,p,u\rangle$, $\langle u,p,y\rangle$, $\langle y,p,c\rangle$, $\langle a,q,t\rangle$, $\langle t,q,z\rangle$, $\langle z,q,a\rangle$, $\langle b,r,s\rangle$, $\langle s,r,x\rangle$, $\langle x,r,b\rangle$, $\langle c,s,y\rangle$, $\langle q,s,c\rangle$, $\langle y,s,q\rangle$, $\langle b,t,x\rangle$, $\langle p,t,b\rangle$, $\langle x,t,p\rangle$, $\langle a,u,z\rangle$, $\langle r,u,a\rangle$, $\langle z,u,r\rangle$, $\langle c,x,q\rangle$, $\langle q,x,u\rangle$, $\langle u,x,c\rangle$, $\langle a,y,r\rangle$, $\langle r,y,t\rangle$, $\langle t,y,a\rangle$, $\langle b,z,p\rangle$, $\langle p,z,s\rangle$, $\langle s,z,b\rangle$ and the sets $\{a,b,c\}$, $\{p,q,r\}$, $\{s,t,u\}$, $\{x,y,z\}$ play the role of the groups. Note that both of these partial systems induce a closed surface with all residual vertices of degree 6. To complete the construction we then "fill in" the groups of the expanded master GDD, sometimes adjoining an extra point, to all of the groups. Thus we may need antiflexible Latin directed triple systems of orders $gw$, $mw$, $gw+1$ or $mw+1$ as appropriate.

In several cases we use a $\{3,4\}$-GDD as the master GDD which requires that when we replace the inflated blocks, we employ both of the partial systems given above. Before continuing the existence proof of the antiflexible LDTSs, let us establish the existence of the $\{3,4\}$-GDDs we will be using.

**Proposition 3.5.** *If $g \notin \{2,6\}$ and $0 \le m \le g$, then there exists a $\{3,4\}$-GDD of type $g^3 m^1$.*

*Proof.* Take a 4-GDD of type $g^4$ with groups $G_i = \{1_i, \ldots, g_i\}$, where $i \in \{0,1,2,3\}$. To get a $\{3,4\}$-GDD of type $g^3 m^1$ simply remove each of the points $(m+1)_3$, $(m+2)_3$, …, $g_3$ from the design. In other words replace every block $\{x_0, y_1, z_2, w_3\}$ such that $m < w \le g$ with the block $\{x_0, y_1, z_2\}$ to obtain a $\{3,4\}$-GDD with groups $G_1$, $G_2$, $G_3$ and $G'_4 = \{1_3, \ldots, m_3\}$. □

**Example 3.6.** $\{3,4\}$-GDD of type $6^3\,5^1$.
The groups are $G_j = \{\, i_j : i \in \mathbb{Z}_6 \,\}$, where $j \in \{0,1,2\}$, and

$$G_3 = \{\, i_3 : i \in \mathbb{Z}_2 \,\} \cup \{\infty_0, \infty_1, \infty_2\}.$$

To obtain the blocks develop the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty_0$, $\infty_1$ and $\infty_2$ as fixed points: $\{0_0, 0_1, 0_2, \infty_0\}$, $\{0_0, 1_1, 2_2, \infty_1\}$, $\{0_0, 2_1, 4_2, \infty_2\}$, $\{0_0, 3_1, 1_2\}$, $\{0_0, 4_1, 3_2\}$, $\{0_0, 5_1, 0_3\}$, $\{0_0, 5_2, 1_3\}$, $\{0_1, 3_2, 0_3\}$.

**Lemma 3.7.** *If $n \equiv 0 \pmod 6$ and $n \geq 18$, then there exists an antiflexible LDTS(n).*

*Proof.* Table 1 gives the schema for antiflexible LDTS($n$), $n \equiv 0 \pmod 6$. No extra points are adjoined in this case. The missing antiflexible LDTSs of orders 36 and 42 as well as the systems of orders 18, 24 and 30 which are needed to construct the infinite classes are all given in the Appendix. The missing antiflexible LDTS(48) and LDTS(66) can be obtained using part (i) of Proposition 3.2 from the LDTS(16) and LDTS(22) given in the Appendix. The antiflexible LDTS(60) can be constructed by taking a master 4-GDD of type $5^4$, inflating each point by a factor of 3 and using the antiflexible LDTS(15) given in the Appendix. □

| Type of master 3-GDD | Orders of LDTS($n$) needed | Residue classes covered modulo 18 | Missing values |
|---|---|---|---|
| $6^s$, $s \geq 3$ | 18 | 0 | 36 |
| $6^s\,8^1$, $s \geq 3$ | 18, 24 | 6 | 42, 60 |
| $6^s\,10^1$, $s \geq 3$ | 18, 30 | 12 | 48, 66 |

Table 1. Schema for antiflexible LDTS($n$), $n \equiv 0 \pmod 6$.

**Lemma 3.8.** *If $n \equiv 16 \pmod{18}$, then there exists an antiflexible LDTS(n).*

*Proof.* It follows from the previous lemma and part (ii) of Proposition 3.2 that there exists an antiflexible LDTS($n$) for all $n \equiv 16 \pmod{18}$, $n \geq 52$. Antiflexible LDTSs of orders 16 and 34 are given in the Appendix. □

**Lemma 3.9.** *If $n \equiv 15 \pmod{18}$, then there exists an antiflexible LDTS(n).*

*Proof.* Table 2 gives the schema for antiflexible LDTS($n$), $n \equiv 15 \pmod{18}$. Once again, no extra points are adjoined in this case. The required antiflexible LDTS($n$)s of orders $n = 15$ and 27 are given in the Appendix. The antiflexible LDTS(33) can be obtained by taking an antiflexible LDTS(16) given in the Appendix together with the quasigroup given in Example A.14 and applying Proposition 3.4. Similarly the antiflexible LDTS(51) can be obtained by taking a (cyclic) antiflexible LDTS(25) together with the quasigroup given in Example A.15. The missing antiflexible LDTS(69) can be constructed using a master $\{3,4\}$-GDD of type $6^3\,5^1$ given in Example 3.6 and the antiflexible LDTS(15) and LDTS(18) given in the Appendix. The antiflexible LDTS(87) can be constructed using a master 3-GDD of type $5^4\,9^1$ together with the antiflexible LDTS(15) and LDTS(27) and the antiflexible LDTS(105) can be constructed using a master 3-GDD of type $5^7$ and the LDTS(15). □

**Lemma 3.10.** *If $n \equiv 4$, 9 or $10 \pmod{18}$ and $n \geq 22$, then there exists an antiflexible LDTS(n).*

| Type of master 3-GDD | Orders of LDTS($n$) needed | Residue classes covered modulo 54 | Missing values |
|---|---|---|---|
| $9^{2s}\, 5^1$, $s \geq 2$ | 15, 27 | 15 | 69 |
| $9^{2s}\, 11^1$, $s \geq 2$ | 27, 33 | 33 | 87 |
| $9^{2s}\, 17^1$, $s \geq 2$ | 27, 51 | 51 | 105 |

Table 2. Schema for antiflexible LDTS($n$), $n \equiv 15 \pmod{18}$.

*Proof.* Table 3 gives the schema for antiflexible LDTS($n$), $n \equiv 4, 9$ or $10 \pmod{18}$. The required antiflexible LDTS($n$)s of orders $n = 18$, 22, 27, 28 and 40 are given in the Appendix and the ones of orders 13 and 19 exist by Theorem 3.1. For the missing $n = 45$, 63 and 81 use part (i) of Proposition 3.2 and for $n = 46$, 64 and 82 use part (ii) of Proposition 3.2. To do this we need systems of orders 15, 21, 27, 16, 22 and 28, respectively, all of which are given in the Appendix. The missing antiflexible LDTS(58) and LDTS(76) can be constructed using master $\{3,4\}$-GDDs of types $5^3\, 4^1$ and $7^3\, 4^1$, respectively, adjoining an extra point and taking the antiflexible LDTSs of orders 13, 16 and 22. The missing antiflexible LDTS(112) can be constructed using a master 3-GDD of type $5^6\, 7^1$, adjoining an extra point and taking the antiflexible LDTSs of orders 16 and 22. □

| Type of master 4-GDD | Points adjoined | Orders of LDTS($n$) needed | Residue classes covered modulo 36 | Missing values |
|---|---|---|---|---|
| $4^{3s}\, 7^1$, $s \geq 2$ | 1 | 13, 22 | 22 | 58 |
| $4^{3s}\, 13^1$, $s \geq 3$ | 1 | 13, 40 | 4 | 76, 112 |
| $6^s\, 9^1$, $s \geq 4$ | 0 | 18, 27 | 9, 27 | 45, 63, 81 |
| $6^s\, 9^1$, $s \geq 4$ | 1 | 19, 28 | 10, 28 | 46, 64, 82 |

Table 3. Schema for antiflexible LDTS($n$), $n \equiv 4, 9$ or $10 \pmod{18}$.

**Theorem 3.11.** *An antiflexible LDTS(n) exists if and only if* $n \equiv 0$ *or* $1 \pmod 3$ *and* $n \geq 13$.

## Appendix. Examples of antiflexible LDTSs

The following examples were obtained by computer with the help of the model builder Mace4 [12] using an algebraic description of a DTS-quasigroup, see [4]. We denote the elements $(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$ as $i_j$. For simplicity, we omit commas from the triples.

**Example A.1.** Antiflexible LDTS(15).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
$\langle 2_0\, 0_0\, 2_1 \rangle$, $\langle 2_1\, 0_0\, 1_1 \rangle$, $\langle 1_1\, 0_0\, 5_1 \rangle$, $\langle 5_1\, 0_0\, 3_1 \rangle$, $\langle 3_1\, 0_0\, 4_1 \rangle$, $\langle 4_1\, 0_0\, 6_1 \rangle$, $\langle 6_1\, 0_0\, 6_0 \rangle$, $\langle 6_0\, 0_0\, 2_0 \rangle$, $\langle 0_0\, \infty\, 4_0 \rangle$, $\langle 0_1\, \infty\, 3_1 \rangle$.

**Example A.2.** Antiflexible LDTS(16).
$V = \mathbb{Z}_8 \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 2_0\, 0_0\, 7_1 \rangle$, $\langle 7_1\, 0_0\, 7_0 \rangle$, $\langle 7_0\, 0_0\, 2_0 \rangle$, $\langle 0_0\, 2_1\, 4_0 \rangle$, $\langle 4_0\, 2_1\, 4_1 \rangle$, $\langle 4_1\, 2_1\, 1_0 \rangle$, $\langle 1_0\, 2_1\, 6_0 \rangle$, $\langle 6_0\, 2_1\, 1_1 \rangle$, $\langle 1_1\, 2_1\, 5_1 \rangle$, $\langle 5_1\, 2_1\, 0_0 \rangle$.

**Example A.3.** Antiflexible LDTS(18).
$V = \mathbb{Z}_3 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 1_0\rangle$, $\langle 1_0\, 2_1\, 1_5\rangle$, $\langle 1_5\, 2_1\, 2_5\rangle$, $\langle 2_5\, 2_1\, 0_2\rangle$, $\langle 0_2\, 2_1\, 1_0\rangle$, $\langle 0_1\, 0_3\, 2_4\rangle$, $\langle 2_4\, 0_3\, 1_5\rangle$, $\langle 1_5\, 0_3\, 0_1\rangle$, $\langle 0_1\, 0_4\, 2_3\rangle$, $\langle 2_3\, 0_4\, 1_2\rangle$, $\langle 1_2\, 0_4\, 2_4\rangle$, $\langle 2_4\, 0_4\, 0_1\rangle$, $\langle 1_0\, 1_4\, 0_1\rangle$, $\langle 0_1\, 1_4\, 1_5\rangle$, $\langle 1_5\, 1_4\, 1_0\rangle$, $\langle 1_2\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 1_5\rangle$, $\langle 1_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 1_2\rangle$, $\langle 2_1\, 0_1\, 2_2\rangle$, $\langle 2_2\, 0_1\, 1_3\rangle$, $\langle 1_3\, 0_1\, 2_1\rangle$, $\langle 2_2\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 0_4\rangle$, $\langle 0_4\, 0_2\, 2_2\rangle$, $\langle 0_2\, 0_5\, 1_3\rangle$, $\langle 1_3\, 0_5\, 0_3\rangle$, $\langle 0_3\, 0_5\, 0_2\rangle$.

**Example A.4.** Antiflexible LDTS(21).
$V = (\mathbb{Z}_{10} \times \mathbb{Z}_2) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
$\langle 2_0\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 9_1\rangle$, $\langle 9_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 4_1\rangle$, $\langle 4_1\, 0_0\, 5_1\rangle$, $\langle 5_1\, 0_0\, 1_1\rangle$, $\langle 1_1\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 7_1\rangle$, $\langle 7_1\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 4_0\rangle$, $\langle 4_0\, 0_0\, 9_0\rangle$, $\langle 9_0\, 0_0\, 2_0\rangle$, $\langle 0_0\, \infty\, 7_0\rangle$, $\langle 0_1\, \infty\, 3_1\rangle$.

**Example A.5.** Antiflexible LDTS(22).
$V = \mathbb{Z}_{11} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$, $\langle 0_0\, 1_1\, 2_0\rangle$, $\langle 2_0\, 1_1\, 9_0\rangle$, $\langle 9_0\, 1_1\, 5_1\rangle$, $\langle 5_1\, 1_1\, 7_0\rangle$, $\langle 7_0\, 1_1\, 2_1\rangle$, $\langle 2_1\, 1_1\, 4_1\rangle$, $\langle 4_1\, 1_1\, 8_0\rangle$, $\langle 8_0\, 1_1\, 6_1\rangle$, $\langle 6_1\, 1_1\, 0_0\rangle$.

**Example A.6.** Antiflexible LDTS(24).
$V = \mathbb{Z}_4 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 3_3\rangle$, $\langle 3_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 1_0\rangle$, $\langle 2_3\, 1_0\, 2_4\rangle$, $\langle 2_4\, 1_0\, 0_5\rangle$, $\langle 0_5\, 1_0\, 2_5\rangle$, $\langle 2_5\, 1_0\, 1_5\rangle$, $\langle 1_5\, 1_0\, 0_4\rangle$, $\langle 0_4\, 1_0\, 1_4\rangle$, $\langle 1_4\, 1_0\, 3_3\rangle$, $\langle 3_3\, 1_0\, 2_3\rangle$, $\langle 2_3\, 0_1\, 0_4\rangle$, $\langle 0_4\, 0_1\, 1_5\rangle$, $\langle 1_5\, 0_1\, 2_3\rangle$, $\langle 3_3\, 0_1\, 2_4\rangle$, $\langle 2_4\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 3_4\rangle$, $\langle 3_4\, 0_1\, 3_3\rangle$, $\langle 0_0\, 0_2\, 2_0\rangle$, $\langle 2_0\, 0_2\, 1_1\rangle$, $\langle 1_1\, 0_2\, 3_1\rangle$, $\langle 3_1\, 0_2\, 2_1\rangle$, $\langle 2_1\, 0_2\, 2_5\rangle$, $\langle 2_5\, 0_2\, 0_4\rangle$, $\langle 0_4\, 0_2\, 3_2\rangle$, $\langle 3_2\, 0_2\, 0_0\rangle$, $\langle 3_0\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 2_4\rangle$, $\langle 2_4\, 0_2\, 3_3\rangle$, $\langle 3_3\, 0_2\, 1_3\rangle$, $\langle 1_3\, 0_2\, 0_5\rangle$, $\langle 0_5\, 0_2\, 0_1\rangle$, $\langle 0_1\, 0_2\, 3_0\rangle$, $\langle 0_2\, 0_3\, 1_4\rangle$, $\langle 1_4\, 0_3\, 0_5\rangle$, $\langle 0_5\, 0_3\, 1_5\rangle$, $\langle 1_5\, 0_3\, 2_2\rangle$, $\langle 2_2\, 0_3\, 0_2\rangle$, $\langle 2_0\, 0_4\, 1_2\rangle$, $\langle 1_2\, 0_4\, 0_5\rangle$, $\langle 0_5\, 0_4\, 2_0\rangle$, $\langle 1_1\, 0_5\, 2_1\rangle$, $\langle 2_1\, 0_5\, 2_3\rangle$, $\langle 2_3\, 0_5\, 1_1\rangle$.

**Example A.7.** Antiflexible LDTS(27).
$V = (\mathbb{Z}_{13} \times \mathbb{Z}_2) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$, $\langle 2_0\, 0_1\, 11_0\rangle$, $\langle 11_0\, 0_1\, 8_1\rangle$, $\langle 8_1\, 0_1\, 2_0\rangle$, $\langle 0_0\, 1_1\, 2_0\rangle$, $\langle 2_0\, 1_1\, 9_0\rangle$, $\langle 9_0\, 1_1\, 0_1\rangle$, $\langle 0_1\, 1_1\, 5_0\rangle$, $\langle 5_0\, 1_1\, 11_1\rangle$, $\langle 11_1\, 1_1\, 7_0\rangle$, $\langle 7_0\, 1_1\, 10_1\rangle$, $\langle 10_1\, 1_1\, 3_1\rangle$, $\langle 3_1\, 1_1\, 0_0\rangle$, $\langle 0_0\, \infty\, 6_0\rangle$, $\langle 0_1\, \infty\, 7_1\rangle$.

**Example A.8.** Antiflexible LDTS(28).
$V = \mathbb{Z}_{14} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 12_1\rangle$, $\langle 12_1\, 0_0\, 4_1\rangle$, $\langle 4_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 13_1\rangle$, $\langle 13_1\, 0_0\, 9_1\rangle$, $\langle 9_1\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$.

**Example A.9.** Antiflexible LDTS(30).
$V = \mathbb{Z}_5 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.

$\langle 0_5\, 0_0\, 3_5\rangle, \langle 3_5\, 0_0\, 4_5\rangle, \langle 4_5\, 0_0\, 1_5\rangle, \langle 1_5\, 0_0\, 0_5\rangle, \langle 0_0\, 0_1\, 1_0\rangle, \langle 1_0\, 0_1\, 4_0\rangle, \langle 4_0\, 0_1\, 3_0\rangle, \langle 3_0\, 0_1\, 0_0\rangle,$
$\langle 3_1\, 0_0\, 4_2\rangle, \langle 4_2\, 0_0\, 1_2\rangle, \langle 1_2\, 0_0\, 0_2\rangle, \langle 0_2\, 0_0\, 3_2\rangle, \langle 3_2\, 0_0\, 3_1\rangle, \langle 2_1\, 0_4\, 1_5\rangle, \langle 1_5\, 0_4\, 4_2\rangle, \langle 4_2\, 0_4\, 0_2\rangle,$
$\langle 0_2\, 0_4\, 2_1\rangle, \langle 3_1\, 4_4\, 0_2\rangle, \langle 0_2\, 4_4\, 2_5\rangle, \langle 2_5\, 4_4\, 1_3\rangle, \langle 1_3\, 4_4\, 1_5\rangle, \langle 1_5\, 4_4\, 3_1\rangle, \langle 0_1\, 0_5\, 1_3\rangle, \langle 1_3\, 0_5\, 0_2\rangle,$
$\langle 0_2\, 0_5\, 3_1\rangle, \langle 3_1\, 0_5\, 1_2\rangle, \langle 1_2\, 0_5\, 0_1\rangle, \langle 2_2\, 0_0\, 3_3\rangle, \langle 3_3\, 0_0\, 4_3\rangle, \langle 4_3\, 0_0\, 4_4\rangle, \langle 4_4\, 0_0\, 1_4\rangle, \langle 1_4\, 0_0\, 0_4\rangle,$
$\langle 0_4\, 0_0\, 3_4\rangle, \langle 3_4\, 0_0\, 2_5\rangle, \langle 2_5\, 0_0\, 2_4\rangle, \langle 2_4\, 0_0\, 1_3\rangle, \langle 1_3\, 0_0\, 0_3\rangle, \langle 0_3\, 0_0\, 2_3\rangle, \langle 2_3\, 0_0\, 2_2\rangle, \langle 3_1\, 0_1\, 1_5\rangle,$
$\langle 1_5\, 0_1\, 3_3\rangle, \langle 3_3\, 0_1\, 2_4\rangle, \langle 2_4\, 0_1\, 3_1\rangle, \langle 4_1\, 0_1\, 4_2\rangle, \langle 4_2\, 0_1\, 4_3\rangle, \langle 4_3\, 0_1\, 0_4\rangle, \langle 0_4\, 0_1\, 0_3\rangle, \langle 0_3\, 0_1\, 4_1\rangle,$
$\langle 2_3\, 0_2\, 3_5\rangle, \langle 3_5\, 0_2\, 3_3\rangle, \langle 3_3\, 0_2\, 1_5\rangle, \langle 1_5\, 0_2\, 2_4\rangle, \langle 2_4\, 0_2\, 3_4\rangle, \langle 3_4\, 0_2\, 4_3\rangle, \langle 4_3\, 0_2\, 2_3\rangle, \langle 3_1\, 0_3\, 2_4\rangle,$
$\langle 2_4\, 0_3\, 2_5\rangle, \langle 2_5\, 0_3\, 3_1\rangle.$

**Example A.10.** Antiflexible LDTS(34).
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 5_0\rangle, \langle 5_0\, 0_0\, 7_0\rangle, \langle 7_0\, 0_0\, 3_0\rangle, \langle 3_0\, 0_0\, 1_0\rangle, \langle 6_0\, 0_0\, 1_1\rangle, \langle 1_1\, 0_0\, 8_0\rangle, \langle 8_0\, 0_0\, 2_1\rangle, \langle 2_1\, 0_0\, 9_1\rangle,$
$\langle 9_1\, 0_0\, 5_1\rangle, \langle 5_1\, 0_0\, 0_1\rangle, \langle 0_1\, 0_0\, 6_0\rangle, \langle 1_1\, 2_0\, 5_1\rangle, \langle 5_1\, 2_0\, 10_1\rangle, \langle 10_1\, 2_0\, 16_1\rangle, \langle 16_1\, 2_0\, 9_1\rangle,$
$\langle 9_1\, 2_0\, 1_1\rangle, \langle 3_1\, 5_0\, 11_1\rangle, \langle 11_1\, 5_0\, 9_1\rangle, \langle 9_1\, 5_0\, 3_1\rangle, \langle 4_0\, 0_1\, 14_1\rangle, \langle 14_1\, 0_1\, 16_1\rangle, \langle 16_1\, 0_1\, 4_0\rangle.$

**Example A.11.** Antiflexible LDTS(36).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_5) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with $\infty$ as a fixed point.
$\langle 2_0\, 0_0\, 2_1\rangle, \langle 2_1\, 0_0\, 6_1\rangle, \langle 6_1\, 0_0\, 6_0\rangle, \langle 6_0\, 0_0\, 2_0\rangle, \langle 2_0\, 0_1\, 6_0\rangle, \langle 6_0\, 0_1\, 1_2\rangle, \langle 1_2\, 0_1\, 5_2\rangle, \langle 5_2\, 0_1\, 5_1\rangle,$
$\langle 5_1\, 0_1\, 2_0\rangle, \langle 4_1\, 0_0\, 4_2\rangle, \langle 4_2\, 0_0\, 0_3\rangle, \langle 0_3\, 0_0\, 0_4\rangle, \langle 0_4\, 0_0\, 6_3\rangle, \langle 6_3\, 0_0\, 5_4\rangle, \langle 5_4\, 0_0\, 3_2\rangle, \langle 3_2\, 0_0\, 6_4\rangle,$
$\langle 6_4\, 0_0\, 3_4\rangle, \langle 3_4\, 0_0\, 0_2\rangle, \langle 0_2\, 0_0\, 4_1\rangle, \langle 0_2\, 1_0\, 4_3\rangle, \langle 4_3\, 1_0\, 6_3\rangle, \langle 6_3\, 1_0\, 0_2\rangle, \langle 1_3\, 0_1\, 2_4\rangle, \langle 2_4\, 0_1\, 0_4\rangle,$
$\langle 0_4\, 0_1\, 3_4\rangle, \langle 3_4\, 0_1\, 4_4\rangle, \langle 4_4\, 0_1\, 6_4\rangle, \langle 6_4\, 0_1\, 5_4\rangle, \langle 5_4\, 0_1\, \infty\rangle, \langle \infty\, 0_1\, 1_4\rangle, \langle 1_4\, 0_1\, 1_3\rangle, \langle 2_3\, 0_1\, 5_3\rangle,$
$\langle 5_3\, 0_1\, 4_3\rangle, \langle 4_3\, 0_1\, 2_3\rangle, \langle 2_1\, 1_1\, 5_2\rangle, \langle 5_2\, 1_1\, 4_3\rangle, \langle 4_3\, 1_1\, 1_3\rangle, \langle 1_3\, 1_1\, 2_1\rangle, \langle 2_0\, 0_2\, 5_1\rangle, \langle 5_1\, 0_2\, 1_1\rangle,$
$\langle 1_1\, 0_2\, 0_3\rangle, \langle 0_3\, 0_2\, 6_0\rangle, \langle 6_0\, 0_2\, 0_4\rangle, \langle 0_4\, 0_2\, 2_2\rangle, \langle 2_2\, 0_2\, 1_4\rangle, \langle 1_4\, 0_2\, 2_3\rangle, \langle 2_3\, 0_2\, 6_2\rangle, \langle 6_2\, 0_2\, 4_4\rangle,$
$\langle 4_4\, 0_2\, 2_0\rangle, \langle 3_0\, 0_3\, 5_4\rangle, \langle 5_4\, 0_3\, 6_2\rangle, \langle 6_2\, 0_3\, 2_2\rangle, \langle 2_2\, 0_3\, 4_4\rangle, \langle 4_4\, 0_3\, 3_0\rangle, \langle 3_0\, 0_4\, 4_3\rangle, \langle 4_3\, 0_4\, 5_3\rangle,$
$\langle 5_3\, 0_4\, 3_0\rangle, \langle 0_0\, \infty\, 2_3\rangle, \langle 2_3\, \infty\, 5_2\rangle, \langle 5_2\, \infty\, 3_0\rangle.$

**Example A.12.** Antiflexible LDTS(40).
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0\, 0_0\, 5_0\rangle, \langle 5_0\, 0_0\, 1_1\rangle, \langle 1_1\, 0_0\, 11_0\rangle, \langle 11_0\, 0_0\, 18_1\rangle, \langle 18_1\, 0_0\, 8_1\rangle, \langle 8_1\, 0_0\, 12_0\rangle, \langle 12_0\, 0_0\, 3_1\rangle,$
$\langle 3_1\, 0_0\, 5_1\rangle, \langle 5_1\, 0_0\, 14_0\rangle, \langle 14_0\, 0_0\, 14_1\rangle, \langle 14_1\, 0_0\, 7_0\rangle, \langle 7_0\, 0_0\, 3_0\rangle, \langle 3_0\, 0_0\, 1_0\rangle.$

**Example A.13.** Antiflexible LDTS(42).
$V = \mathbb{Z}_7 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$.
$\langle 0_1\, 0_5\, 1_5\rangle, \langle 1_5\, 0_5\, 5_1\rangle, \langle 5_1\, 0_5\, 2_5\rangle, \langle 2_5\, 0_5\, 1_1\rangle, \langle 1_1\, 0_5\, 4_5\rangle, \langle 4_5\, 0_5\, 0_1\rangle, \langle 2_0\, 0_0\, 2_1\rangle, \langle 2_1\, 0_0\, 6_1\rangle,$
$\langle 6_1\, 0_0\, 6_0\rangle, \langle 6_0\, 0_0\, 2_0\rangle, \langle 2_0\, 0_1\, 6_0\rangle, \langle 6_0\, 0_1\, 1_2\rangle, \langle 1_2\, 0_1\, 5_2\rangle, \langle 5_2\, 0_1\, 5_1\rangle, \langle 5_1\, 0_1\, 2_0\rangle, \langle 5_0\, 0_4\, 1_5\rangle,$
$\langle 1_5\, 0_4\, 4_3\rangle, \langle 4_3\, 0_4\, 5_3\rangle, \langle 5_3\, 0_4\, 4_2\rangle, \langle 4_2\, 0_4\, 5_0\rangle, \langle 3_0\, 0_5\, 2_4\rangle, \langle 2_4\, 0_5\, 4_3\rangle, \langle 4_3\, 0_5\, 5_2\rangle, \langle 5_2\, 0_5\, 3_0\rangle,$
$\langle 4_1\, 0_0\, 4_2\rangle, \langle 4_2\, 0_0\, 5_3\rangle, \langle 5_3\, 0_0\, 3_4\rangle, \langle 3_4\, 0_0\, 0_4\rangle, \langle 0_4\, 0_0\, 6_3\rangle, \langle 6_3\, 0_0\, 0_5\rangle, \langle 0_5\, 0_0\, 0_3\rangle, \langle 0_3\, 0_0\, 2_3\rangle,$
$\langle 2_3\, 0_0\, 6_5\rangle, \langle 6_5\, 0_0\, 4_4\rangle, \langle 4_4\, 0_0\, 0_2\rangle, \langle 0_2\, 0_0\, 4_1\rangle, \langle 1_2\, 5_0\, 6_3\rangle, \langle 6_3\, 5_0\, 6_5\rangle, \langle 6_5\, 5_0\, 3_4\rangle, \langle 3_4\, 5_0\, 0_5\rangle,$
$\langle 0_5\, 5_0\, 1_2\rangle, \langle 1_1\, 0_1\, 4_2\rangle, \langle 4_2\, 0_1\, 3_3\rangle, \langle 3_3\, 0_1\, 0_3\rangle, \langle 0_3\, 0_1\, 1_1\rangle, \langle 1_3\, 0_1\, 2_4\rangle, \langle 2_4\, 0_1\, 0_4\rangle, \langle 0_4\, 0_1\, 3_4\rangle,$
$\langle 3_4\, 0_1\, 4_4\rangle, \langle 4_4\, 0_1\, 6_4\rangle, \langle 6_4\, 0_1\, 5_4\rangle, \langle 5_4\, 0_1\, 5_5\rangle, \langle 5_5\, 0_1\, 1_4\rangle, \langle 1_4\, 0_1\, 1_3\rangle, \langle 0_3\, 2_1\, 6_3\rangle, \langle 6_3\, 2_1\, 4_3\rangle,$
$\langle 4_3\, 2_1\, 0_3\rangle, \langle 2_0\, 0_2\, 5_1\rangle, \langle 5_1\, 0_2\, 1_1\rangle, \langle 1_1\, 0_2\, 0_3\rangle, \langle 0_3\, 0_2\, 0_4\rangle, \langle 0_4\, 0_2\, 6_0\rangle, \langle 6_0\, 0_2\, 4_5\rangle, \langle 4_5\, 0_2\, 3_3\rangle,$
$\langle 3_3\, 0_2\, 2_4\rangle, \langle 2_4\, 0_2\, 5_5\rangle, \langle 5_5\, 0_2\, 5_4\rangle, \langle 5_4\, 0_2\, 0_5\rangle, \langle 0_5\, 0_2\, 6_2\rangle, \langle 6_2\, 0_2\, 2_2\rangle, \langle 2_2\, 0_2\, 1_4\rangle, \langle 1_4\, 0_2\, 2_0\rangle,$
$\langle 3_0\, 0_3\, 4_4\rangle, \langle 4_4\, 0_3\, 5_2\rangle, \langle 5_2\, 0_3\, 6_5\rangle, \langle 6_5\, 0_3\, 3_0\rangle, \langle 4_0\, 0_3\, 3_2\rangle, \langle 3_2\, 0_3\, 2_5\rangle, \langle 2_5\, 0_3\, 4_0\rangle, \langle 4_2\, 0_5\, 1_4\rangle,$
$\langle 1_4\, 0_5\, 2_3\rangle, \langle 2_3\, 0_5\, 4_2\rangle.$

**Example A.14.** ULSOQ(17).
$Q = \mathbb{Z}_4 \times \mathbb{Z}_4 \cup \{\infty\}$.
The quasigroup is obtained by defining $\infty * x = x$ and developing the following partial Cayley table under the action of the automorphism $i_j \mapsto (i+1)_j$ with $\infty$ as a fixed point:

| $*$ | $\infty$ | $0_0$ | $1_0$ | $2_0$ | $3_0$ | $0_1$ | $1_1$ | $2_1$ | $3_1$ | $0_2$ | $1_2$ | $2_2$ | $3_2$ | $0_3$ | $1_3$ | $2_3$ | $3_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0_0$ | $1_0$ | $\infty$ | $3_0$ | $0_1$ | $0_0$ | $3_3$ | $3_1$ | $2_0$ | $0_3$ | $2_1$ | $2_2$ | $1_1$ | $1_2$ | $0_2$ | $2_3$ | $3_2$ | $1_3$ |
| $0_1$ | $1_1$ | $3_0$ | $0_1$ | $0_2$ | $3_1$ | $\infty$ | $3_3$ | $2_3$ | $1_2$ | $1_3$ | $1_0$ | $0_3$ | $2_2$ | $2_1$ | $3_2$ | $2_0$ | $0_0$ |
| $0_2$ | $1_2$ | $1_1$ | $0_2$ | $0_3$ | $3_2$ | $1_0$ | $0_1$ | $0_0$ | $2_0$ | $\infty$ | $1_3$ | $3_1$ | $2_3$ | $3_3$ | $3_0$ | $2_1$ | $2_2$ |
| $0_3$ | $1_3$ | $0_3$ | $2_2$ | $3_3$ | $2_3$ | $1_2$ | $0_2$ | $3_1$ | $3_2$ | $0_1$ | $3_0$ | $1_0$ | $0_0$ | $\infty$ | $2_1$ | $1_1$ | $2_0$ |

**Example A.15.** ULSOQ(26).
$Q = \mathbb{Z}_5 \times \mathbb{Z}_5 \cup \{\infty\}$.
The quasigroup is obtained by defining $\infty * x = x$ and developing the following partial Cayley table under the action of the automorphism $i_j \mapsto (i+1)_j$ with $\infty$ as a fixed point:

| $*$ | $\infty$ | $0_0$ | $1_0$ | $2_0$ | $3_0$ | $4_0$ | $0_1$ | $1_1$ | $2_1$ | $3_1$ | $4_1$ | $0_2$ | $1_2$ | $2_2$ | $3_2$ | $4_2$ | $0_3$ | $1_3$ | $2_3$ | $3_3$ | $4_3$ | $0_4$ | $1_4$ | $2_4$ | $3_4$ | $4_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0_0$ | $1_0$ | $\infty$ | $0_1$ | $4_0$ | $3_1$ | $0_0$ | $1_1$ | $4_4$ | $3_4$ | $3_0$ | $2_0$ | $2_2$ | $2_1$ | $3_2$ | $1_2$ | $4_1$ | $1_3$ | $0_3$ | $4_3$ | $0_2$ | $4_2$ | $2_3$ | $2_4$ | $1_4$ | $0_4$ | $3_3$ |
| $0_1$ | $1_1$ | $3_0$ | $0_2$ | $4_1$ | $2_0$ | $0_1$ | $\infty$ | $4_3$ | $1_4$ | $0_4$ | $4_4$ | $2_4$ | $1_0$ | $4_0$ | $3_4$ | $0_3$ | $3_1$ | $2_1$ | $0_0$ | $4_2$ | $2_2$ | $1_3$ | $1_2$ | $3_2$ | $3_3$ | $2_3$ |
| $0_2$ | $1_2$ | $3_1$ | $2_2$ | $0_2$ | $0_3$ | $4_2$ | $1_0$ | $0_1$ | $4_0$ | $1_1$ | $3_0$ | $\infty$ | $1_3$ | $4_3$ | $4_4$ | $2_3$ | $3_3$ | $3_4$ | $2_4$ | $1_4$ | $0_4$ | $0_0$ | $3_2$ | $2_1$ | $4_1$ | $2_0$ |
| $0_3$ | $1_3$ | $2_2$ | $0_3$ | $2_3$ | $4_3$ | $0_4$ | $2_1$ | $0_2$ | $3_2$ | $1_2$ | $4_2$ | $3_4$ | $0_1$ | $1_0$ | $2_4$ | $3_3$ | $\infty$ | $2_0$ | $1_4$ | $3_0$ | $1_1$ | $4_0$ | $4_4$ | $4_1$ | $0_0$ | $3_1$ |
| $0_4$ | $1_4$ | $3_3$ | $0_4$ | $4_4$ | $3_4$ | $2_4$ | $2_2$ | $0_3$ | $2_3$ | $4_3$ | $1_3$ | $2_1$ | $4_1$ | $3_0$ | $1_0$ | $3_2$ | $4_2$ | $1_1$ | $4_0$ | $2_0$ | $3_1$ | $\infty$ | $0_2$ | $0_1$ | $1_2$ | $0_0$ |

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# THE CENTRE OF A STEINER LOOP AND THE MAXI-PASCH PROBLEM

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Abstract. A binary operation $\cdot$ which satisfies the identities $x{\cdot}e = x$, $x{\cdot}x = e$, $(x \cdot y) \cdot x = y$ and $x \cdot y = y \cdot x$ is called a Steiner loop. In this paper the necessary and sufficient conditions for the existence of a Steiner loop of order $n$ with centre of order $m$ are derived. The paper also discusses the connection of this problem to the question of the maximum number of Pasch configurations which can occur in a Steiner triple system of order $v$. We find that the Steiner loops of all known maxi-Pasch Steiner triple systems have centre of maximum possible order.

## 1. Introduction

A *Steiner triple system* of order $v$, STS($v$), is a pair $(V, \mathcal{B})$ where $V$ is a set of $v$ points and $\mathcal{B}$ is a collection of triples of distinct points taken from $V$ such that every pair of distinct points from $V$ appears in precisely one triple. Such systems exist if and only if $v \equiv 1$ or $3 \pmod 6$ [6]. Given an STS $(V, \mathcal{B})$ one can define a binary operation $\cdot$ on the set $L = V \cup \{e\}$ by assigning $x \cdot e = e \cdot x = x$, $x \cdot x = e$ for all $x \in L$ and $x \cdot y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The induced operation satisfies the identities

$$x \cdot e = x, \quad x \cdot x = e, \quad (x \cdot y) \cdot x = y, \quad x \cdot y = y \cdot x \tag{1}$$

for all $x$ and $y$ in $L$. Any binary operation satisfying these four identities is called a *Steiner loop*. The process described above is reversible. Given a non-trivial Steiner loop one can obtain an STS by assigning $\{x, y, x \cdot y\} \in \mathcal{B}$ for all $x$, $y \in V$, $x \neq y$. There is therefore a one-to-one correspondence between Steiner triple systems and non-trivial Steiner loops. Thus a Steiner loop of order $n$ exists if and only if $n = 1$ or $n \equiv 2$ or $4 \pmod 6$. In the remainder of this paper we replace the loop operation $\cdot$ with juxtaposition.

The most well known examples of Steiner triple systems come from finite geometry. Let $V = \mathbb{F}_2^k \setminus \{\mathbf{0}\}$ and let $\mathcal{B}$ be the collection of all $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ are pairwise distinct and $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. Then $(V, \mathcal{B})$ is a *projective* STS($2^k - 1$). Its corresponding Steiner loop is $(\mathbb{F}_2^k, +)$. A Steiner loop is associative if and only if it is isomorphic to $(\mathbb{F}_2^k, +)$ [2].

In a Steiner triple system, a collection of four triples on six points is called a *Pasch configuration* or *quadrilateral*. It is easily seen that this structure necessarily has the form $\{a, b, c\}$, $\{a, d, e\}$, $\{b, e, f\}$, $\{c, d, f\}$. For example an STS(7) contains seven distinct Pasch configurations. A Steiner triple system is said to be *anti-Pasch* if it does not contain a Pasch configuration.

**Theorem 1.1** ([3, 7]). *An anti-Pasch STS($v$) exists if and only if* $v \equiv 1$ *or* $3$ $\pmod 6$ *and* $v \neq 7$, $13$.

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The *left nucleus* $N_\lambda$, *middle nucleus* $N_\mu$ and *right nucleus* $N_\rho$ of a loop $L$ are defined as

$$N_\lambda(L) = \{\, x \in L : x(yz) = (xy)z \text{ for all } y, z \in L \,\},$$
$$N_\mu(L) = \{\, y \in L : x(yz) = (xy)z \text{ for all } x, z \in L \,\},$$
$$N_\rho(L) = \{\, z \in L : x(yz) = (xy)z \text{ for all } x, y \in L \,\}.$$

The *nucleus* of $L$, defined as $N(L) = N_\lambda(L) \cap N_\mu(L) \cap N_\rho(L)$, is a subgroup of $L$. The *centre* of a loop $L$ is defined as

$$Z(L) = N(L) \cap \{\, x \in L : xy = yx \text{ for all } y \in L \,\}.$$

If $L$ is a Steiner loop, then the three nuclei coincide [9] and $N(L) = Z(L)$. Because the centre of a Steiner loop is an associative Steiner loop, its cardinality is a power of 2.

## 2. The centre of a Steiner loop

A subloop $K$ of $L$ is said to be *normal* in $L$ if $xK = Kx$, $x(yK) = (xy)K$ and $(xK)y = x(Ky)$ for all $x$, $y \in L$. The *factor loop* $L/K$ is then defined in the usual way. Clearly, for any loop $L$ the centre $Z(L)$ is normal in $L$.

**Lemma 2.1.** *Let $L$ be a Steiner loop of order $n$ with centre of order $m$ and let $k$ be the largest integer such that $2^k$ divides $n$. Then $m = 2^i$, where $i \in \{0, 1, \dots, k\}$. If $n \neq 2^k$, then $m \neq 2^k$.*

*Proof.* As noted in the introduction, $m$ is a power of 2. Since the factor loop $L/Z(L)$ satisfies the identities (1), it is also a Steiner loop, and we either have $n/m = 1$ or we have $n/m \equiv 2$ or $4 \pmod 6$. In the former case the loop is associative, thus $n = 2^k$ and $m = 2^k$. In the latter case, in order for $n/m$ to be even, $m$ must be at most $2^{k-1}$. □

**Lemma 2.2.** *If there exists a Steiner loop of order $n$ with centre of order $m$, then there exists a Steiner loop of order $2n$ with centre of order $2m$.*

*Proof.* Let $L$ be a Steiner loop of order $n$. Then $L \times \mathbb{F}_2$ is also a Steiner loop, since it satisfies the identities (1), and its centre is $Z(L) \times \mathbb{F}_2$. □

**Proposition 2.3.** *A Steiner loop of order $n$ with a non-trivial centre exists if and only if $n \equiv 4$ or $8 \pmod{12}$ or $n = 2$.*

*Proof.* If $n \equiv 4$ or $8 \pmod{12}$ or $n = 2$, then there exists a Steiner loop of order $n/2$. By Lemma 2.2 there then exists a Steiner loop of order $n$ with centre of order at least 2. If $n \equiv 2$ or $10 \pmod{12}$ and $n \neq 2$, then by Lemma 2.1 the centre of every Steiner loop of order $n$ is trivial. □

With the help of a computer running the model builder Mace4 [8], we can obtain a census of the centres of Steiner loops of order up to 20. The three unique Steiner triple systems of orders 1, 3 and 7 are projective, thus their corresponding loops all satisfy $Z(L) = L$. The Steiner loops of the unique STS(9) and of both STS(13)s all have trivial centre. There are only two STS(15)s up to isomorphism that induce a loop with a non-trivial centre. One is the projective STS(15) and the other is the system with automorphism group of order 192, i.e. System # 2 in [1]. The latter has centre of order 2. There are only three STS(19)s up to isomorphism that induce a loop with a non-trivial centre. They are the unique systems with automorphism groups of orders 108, 144 and 432. Each has centre of order 2. In light of the following theorem, it does not come as a big surprise that these are precisely the three systems with 84 Pasch configurations, which is the maximum possible for any STS(19) [5].

**Theorem 2.4.** *Let $(V, \mathcal{B})$ be an STS and let $L$ be its corresponding Steiner loop. For any $x \in L$ the following conditions are equivalent:*

(1) *The element $x$ lies in the centre of $L$.*
(2) *If $x$, $y$ and $z$ are pairwise distinct elements of $V$, then the set $\{x, y, z\}$ generates a sub-STS(3) or a sub-STS(7) in $(V, \mathcal{B})$.*
(3) *For each $y$, $z \in L$, the subloop $\langle x, y, z\rangle$ is of order at most 8.*

*Proof.* Let $x \in Z(L) \setminus \{e\}$ and $y$, $z \in V$ be pairwise distinct elements such that $\{x, y, z\}$ does not lie in $\mathcal{B}$. By definition $\{x, y, xy\}$, $\{x, z, xz\}$, $\{y, z, yz\} \in \mathcal{B}$, and since $(xy)(xz) = ((xy)x)z = yz$, we also have $\{xy, xz, yz\} \in \mathcal{B}$. Furthermore $\{x, yz, xyz\}$, $\{y, xz, xyz\}$, $\{z, xy, xyz\} \in \mathcal{B}$. These seven triples form a sub-STS(7). Thus (1) implies (2).

Assume that (2) holds and let $x$, $y$, $z \in L$. If these three points are not pairwise distinct elements of $V$ or if $\{x, y, z\} \in \mathcal{B}$, then $\langle x, y, z\rangle$ is a subloop of order 1, 2 or 4 in $L$. Otherwise, by assumption, $\langle x, y, z\rangle$ is a subloop of order 8 in $L$. Thus (2) implies (3).

In a Steiner loop every subloop of order at most 8 is necessarily a group. Thus (3) implies (1). □

It immediately follows from the previous theorem that every anti-Pasch STS($n$), $n > 3$, gives rise to a Steiner loop with trivial centre. Taking into account the census above, we have the following:

**Corollary 2.5.** *A Steiner loop of order $n$ with trivial centre exists if and only if $n = 1$ or $n \equiv 2$ or $4 \pmod 6$ and $n \notin \{2, 4, 8\}$.*

**Lemma 2.6.** *Let $L$ be a non-associative Steiner loop of order $n$ with centre of order $m$. Then $m < \frac{1}{4}n$.*

*Proof.* Since $L$ is non-associative, it follows from Theorem 2.4 that there exist points $x$, $y$, $z \in L$ such that the order of the subloop $\langle x, y, z\rangle$ is strictly greater than 8. None of these three points lie in the centre and neither does the point $xy$, because $\langle xy, x, z\rangle = \langle x, y, z\rangle$. For any $u \in Z(L)$ we have $\langle u, x, y\rangle = \{e, x, y, xy, u, xu, yu, (xy)u\}$, where only $e$ and $u$ lie in the centre. Thus if $u$, $v \in Z(L)$, $u \neq v$, then $\langle u, x, y\rangle \neq \langle v, x, y\rangle$ and therefore $\langle u, x, y\rangle \cap \langle v, x, y\rangle = \langle x, y\rangle$. If $u \in Z(L) \setminus \{e\}$, then $\langle u, x, y\rangle$ is of order 8 and $\langle x, y\rangle$ is of order 4, thus the set $\bigcup_{u \in Z(L)} \langle u, x, y\rangle$ has cardinality $4(m-1)+4$. Finally, note that the point $z$ does not lie in $\langle u, x, y\rangle$ for any $u \in Z(L)$. Thus there are at least $4m+1$ pairwise distinct points in $L$. □

**Theorem 2.7.** *Let $n$ be a positive integer and let $k$ be the largest integer such that $2^k$ divides $n$. A non-trivial Steiner loop of order $n$ with centre of order $m$ exists if and only if $n \equiv 2$ or $4 \pmod 6$, and*

(1) *$n \neq 2^k$ and $m = 2^i$, where $i \in \{0, 1, \ldots, k-1\}$, or*
(2) *$n = 2^k$, $(n, m) \neq (8, 1)$ and $m = 2^i$, where $i \in \{0, 1, \ldots, k-3\} \cup \{k\}$.*

*Proof.* The necessity of the conditions follows from Lemmas 2.1 and 2.6 and from the fact that the unique STS(7) is projective.

If the integers $n$ and $m = 2^i$ satisfy the conditions given above, then $n/m = 1$ or $n/m \equiv 2$ or $4 \pmod 6$ but $n/m \notin \{2, 4\}$. If $n/m \neq 8$, then by Corollary 2.5 there exists a Steiner loop of order $n/m$ with trivial centre, thus by iterating

$i$ times Lemma 2.2 we obtain a Steiner loop of order $n$ with centre of order $m$. If $n/m = 8$, then start instead with the Steiner loop of order 16 that has centre of order 2 and iterate $i-1$ times Lemma 2.2. □

## 3. Maxi-Pasch Steiner triple systems

Denote the number of Pasch configurations in an STS($v$), $S$, by $P(S)$. Define

$$P(v) = \max\{\, P(S) : S \text{ is an STS}(v)\,\}.$$

An STS($v$), $S$, is said to be *maxi-Pasch* if $P(S) = P(v)$. In [10] Stinson and Wei undertook a preliminary investigation of the bounds on $P(v)$. An elementary counting argument yields $P(v) \le v(v-1)(v-3)/24$. The authors show that an STS($v$) achieves this bound if and only if it is projective. They then present several recursive lower bounds on $P(v)$. In [4] Gray and Ramsay present another recursive lower bound on $P(v)$:

**Theorem 3.1** ([4])**.** *If $v = 2u+1 \equiv 3$ or $7 \pmod{12}$, $u \ge 7$, then*

$$P(v) \ge \frac{7(v-1)(v-3)}{24} + 8P(u).$$

In [11] Grannell and Lovegrove give lower bounds on $P(v)$ for $v$ of the form $2^{2k}+3$ or $2^{2k}+5$. The only known values when $v \ne 2^k - 1$ appear to be $P(9) = 0$, $P(13) = 13$ and $P(19) = 84$.

Theorem 2.4 indicates that a Steiner loop with large centre corresponds to a Steiner triple system with a large number of STS(7) subsystems and thus a large number of Pasch configurations. We start by obtaining a lower bound on the number of sub-STS(7)s.

**Proposition 3.2.** *Let $L$ be a Steiner loop of order $n$ with centre of order $m$. Then the number of sub-STS(7)s in the Steiner triple system corresponding to $L$ is at least*

$$\frac{m-1}{168}\big((m-2)(m-4) + 7(n-m)(n-m-2)\big).$$

*Proof.* Let $(V, \mathcal{B})$ be the Steiner triple system which corresponds to $L$. By $\mathcal{F}_i$ denote the set of all sub-STS(7)s in $(V, \mathcal{B})$ such that exactly $i$ points of the subsystem lie in the centre of $L$. The only admissible values of $i$ are 0, 1, 3 and 7. Consider three pairwise distinct points $x$, $y$, $z \in V$, which do not lie in a common block. These three points generate a system in $\mathcal{F}_7$ if and only if they all lie in the centre. The number of ways of choosing three points from $Z(L)\setminus\{e\}$, so that they do not lie in a common block, is $(m-1)(m-2)(m-4)$. This way each of the systems in $\mathcal{F}_7$ is counted 168 times, thus

$$|\mathcal{F}_7| = (m-1)(m-2)(m-4)/168.$$

It follows from Theorem 2.4 that if one of the points $x$, $y$ or $z$ lies in the centre and the other two do not, then they generate a system in $\mathcal{F}_1$ or $\mathcal{F}_3$. In fact every system in $\mathcal{F}_1 \cup \mathcal{F}_3$ can be generated in this manner. The number of ways of choosing three points such that the first is from $Z(L)\setminus\{e\}$ and the remaining two are from $V\setminus Z(L)$, but do not all lie in a common block, is $(m-1)(n-m)(n-m-2)$. This way each of the systems in $\mathcal{F}_1$ and $\mathcal{F}_3$ is counted 24 times, thus

$$|\mathcal{F}_1| + |\mathcal{F}_3| = (m-1)(n-m)(n-m-2)/24.$$

The sum $|\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_7|$ gives a lower bound on the number of sub-STS(7)s in $(V, \mathcal{B})$. □

To obtain a lower bound on the maximum number of sub-STS(7)s in a Steiner triple system of order $v$, we can set the order $m$ of the centre in the previous proposition to the maximum value as given by Theorem 2.7. Multiplying the resulting bound by 7 gives a lower bound on the number of Pasch configurations, because there are seven Pasch configurations in each sub-STS(7) and no two sub-STS(7)s share a common Pasch configuration. This yields the following result:

**Corollary 3.3.** *Let* $v \equiv 1$ *or* $3 \pmod 6$ *and let* $k$ *be the largest integer such that* $2^k$ *divides* $v+1$. *Then*

$$P(v) \geq \frac{2^{k-1}-1}{24}\big(2^{k-1}(2^{k-1}-6)+7(v-2^{k-1})^2+1\big).$$

Call a Steiner triple system *maxi-central* if its corresponding Steiner loop has centre of maximum possible order (see Theorem 2.7).

The doubling construction of Steiner loops used in Lemma 2.2, which is the source of the results above, is equivalent to the standard $v \to 2v+1$ construction for Steiner triple systems. In most cases it is possible to refine the bound on $P(v)$ by using the $v \to 2v+1$ construction and focusing directly on the Pasch configurations rather than on the sub-STS(7)s as we did above. That is the method which was used to obtain most of the results in [4]. The present paper therefore does not improve the known lower bounds on $P(v)$, but shows that all known maxi-Pasch STSs are maxi-central and that for some values of $v$ the maximum known lower bound is attained by a maxi-central Steiner triple system. These would most notably be the cases $v = 27$, 39, 43, 51 and 55 in [4] and trivially all $v$ of the form $2^k-1$. For example the STS(27) in [4] with 286 Pasch configurations is obtained by using the standard $v \to 2v+1$ construction on the STS(13) with 13 Pasch configurations. Thus the corresponding Steiner loop of order 28 has centre of order 2, which is the maximum possible.


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# List of abbreviations

| | |
|---|---|
| DTS | Directed Triple System |
| GDD | Group Divisible Design |
| HTS | Hybrid Triple System |
| KTS | Kirkman Triple System |
| LDTS | Latin Directed Triple System |
| LHTS | Latin Hybrid Triple System |
| MTS | Mendelsohn Triple System |
| PBD | Pairwise Balanced Design |
| STS | Steiner Triple System |