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Structure and enumeration of permutation classes

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Title: Structure and enumeration of permutation classes

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Abstract: We define the operation of composing two hereditary classes of permutations using the standard composition of permutations as functions and we explore properties and structure of permutation classes considering this operation. We mostly concern ourselves with the problem of whether permutation classes can be composed from their proper subclasses. We provide examples of classes which can be composed from two proper subclasses, classes which can be composed from three but not from two proper subclasses and classes which cannot be composed from any finite number of proper subclasses.

Keywords: permutation class composition

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Contents

Introduction

Permutations of numbers or other finite sets are a very deeply and frequently studied combinatorial and algebraic object. There are two main structures on permutations investigated in modern mathematics: groups, closed under the composition operator, and hereditary pattern-avoiding classes, closed under the relation of containment. This thesis is one of several texts exploring the edge between the two notions by applying the composition operator to permutation classes. That is, given two classes $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathcal{A} \circ \mathcal{B}$ the class of all permutations which can be written as a composition of a permutation from $\mathcal{A}$ and a permutation from $\mathcal{B}$.

The oldest results combining permutation classes and groups that we know of are due to Atkinson and Beals [1], who consider the permutation classes whose permutations of length n form a subgroup of S_n for every n and completely characterise the types of groups which may occur this way. These results were recently refined and extended by Lehtonen and Pöschel in [2] and [3]. In an earlier version of their paper, Atkinson and Beals [4] also deal with composing permutation classes, showing that compositions of many pairs of finitely based classes are again finitely based.

Some permutation classes characterise permutations which can be sorted by some sorting machine such as a stack. In this view, a composition of two permutation classes can characterise permutations sortable by two corresponding sorting machines connected serially. For example, Atkinson and Stitt [5, Section 6.4] introduce the pop-stack, a sorting machine which sorts precisely the layered permutations (see Chapter 4 for a definition), and consider the class of permutations which can be sorted by two pop-stacks in series, i.e. which can be written as a composition of two layered permutations. Using their more general results they calculate its generating function and enumerate its basis.

Albert et al. [6] give more enumerative results on compositions of classes in terms of sorting machines.

In the present thesis, we study a different question connected to compositions of classes; namely whether a permutation of a given class $\mathcal{C}$ can always be written as a composition of two or more permutations from its subclasses, i.e. whether $\mathcal{C} \subseteq \mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$ for some $\mathcal{C}_1, \ldots, \mathcal{C}_k \subsetneq \mathcal{C}$. If this is true, we say that the class $\mathcal{C}$ is *composable* and we refer to this property of $\mathcal{C}$ as *composability*.

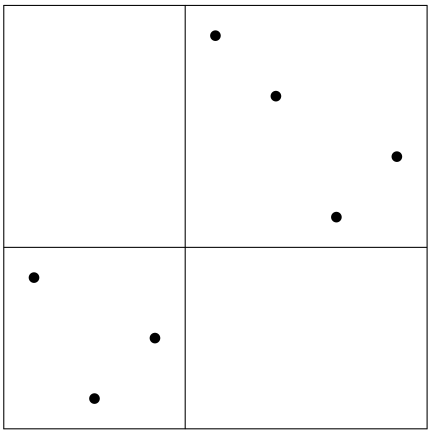
The thesis is organised as follows. In Chapter 1 we supply all the necessary definitions and facts about permutation classes. In Chapter 2 we introduce composability and give some basic results. In Chapter 3 we explore composability of the class $\mathrm{Av}(k \cdots 21)$. In Chapter 4 we explore composability of various classes of layered patterns. Finally in Chapter 5 we give several additional miscellaneous results.

1. Preliminaries

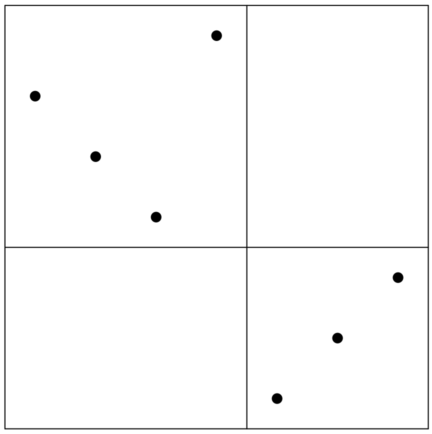
For a positive integer n we let $[n]$ denote the set $\{1, 2, \dots, n\}$. A *permutation of order* n is a bijective function $\pi\colon [n] \longrightarrow [n]$. We denote the order of a permutation π by $|\pi|$. We may also interpret a permutation π as a sequence $\pi(1), \pi(2), \dots, \pi(n)$ of distinct elements of $[n]$, or as a scheme in an $n \times n$ square in the plane, namely the set of points $\{(i, \pi(i)); 1 \leq i \leq n\}$. For $n \geq 0$ let S_n denote the set of all permutations of order n. We let $|S_0| = 1$, i.e. there is exactly one empty permutation. We denote the set of all permutations by $\mathcal{S} = \bigcup_{i=0}^{\infty} S_i$.

If π and σ are two permutations of order n we define their *composition* $\pi \circ \sigma$ as $(\pi \circ \sigma)(i) = \pi(\sigma(i))$ for every $i \in [n]$. Since the composition is used as the group multiplication operator on groups of permutations, throughout the text we might refer to composition also as multiplication.

We define two more permutation operators. The *sum* $\pi \oplus \sigma$ of permutations $\pi \in S_k$ and $\sigma \in S_l$ is the permutation $\pi(1), \pi(2), \dots, \pi(k), \sigma(1) + k, \sigma(2) + k, \dots, \sigma(l)+k$. The *skew sum* $\pi \ominus \sigma$ is the permutation $\pi(1)+l, \pi(2)+l, \dots, \pi(k)+l, \sigma(1), \sigma(2), \dots, \sigma(l)$. For example, $3127645 = 312 \oplus 4312$ and $6547123 = 3214 \ominus 123$ (see Figure 1.1).



(a) $3127645 = 312 \oplus 4312$



(b) $6547123 = 3214 \ominus 123$

Figure 1.1: An example of sums and skew sums

In addition, we will sometimes write $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ as $\bigoplus_{i=1}^{k} \pi_i$.

1.1 Permutation classes

Two sequences of numbers $s_1, s_2, \dots, s_n$ and $r_1, r_2, \dots, r_n$ are *order-isomorphic* if for any two indices $i, j \in [n]$ it holds that $s_i < s_j$ if and only if $r_i < r_j$.

We define the following partial ordering on the set of all permutations. We say that π is *contained in* σ and write $\pi \leq \sigma$ if σ has a subsequence of length $|\pi|$ order-isomorphic to π. See the example of containment in Figure 1.2. On the other hand, if $\pi \not\leq \sigma$, we say that σ *avoids* π.

A set $\mathcal{C}$ of permutations is called a *permutation class* if for every $\pi \in \mathcal{C}$ and every $\sigma \leq \pi$ we have $\sigma \in \mathcal{C}$. We say that $\mathcal{C}$ *avoids* a permutation σ if every $\pi \in \mathcal{C}$

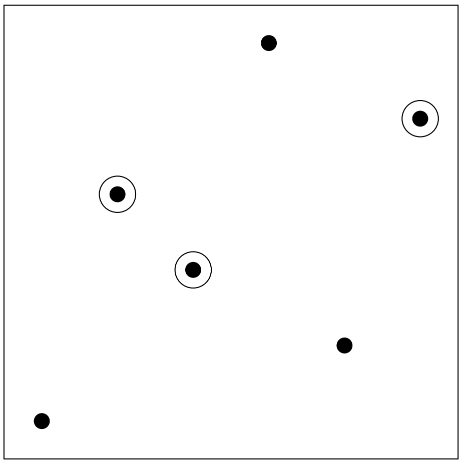


Figure 1.2: The permutation 213 is contained in 143625.

avoids σ. Permutation classes are often described by the patterns they avoid. If B is any set of permutations, we denote by $\mathrm{Av}(B)$ the set of all permutations avoiding every element of B. Observe that $\mathcal{C}$ is a permutation class if and only if $\mathcal{C} = \mathrm{Av}(B)$ for some set B. Indeed, if $\mathcal{C}$ is a permutation class, then $\mathcal{C} = \mathrm{Av}(\mathcal{S}\backslash\mathcal{C})$, and if $\sigma \leq \pi \in \mathcal{C}$, then π avoids all permutations of B and clearly σ avoids them too. If $\mathcal{C} = \mathrm{Av}(B)$ and B is an anti-chain with respect to containment, we call B the *basis* of $\mathcal{C}$. Also if $B = \{\pi_1, \pi_2, \ldots, \pi_k\}$ is finite, we write just $\mathrm{Av}(\pi_1, \ldots, \pi_k)$ instead of $\mathrm{Av}(\{\pi_1, \ldots, \pi_k\})$. Finally, if $\mathcal{C} = \mathrm{Av}(\pi)$ for a single permutation π, we say that $\mathcal{C}$ is a *principal class*.

Let $s_1, s_2, \ldots, s_k$ be k finite sequences of numbers. We denote their concatenation by $s_1 s_2 \cdots s_k$. If a sequence s can be constructed by interleaving $s_1, s_2, \ldots, s_k$ in some (not necessarily unique) way, we say that s is a *merge of* or it is *merged from* $s_1, s_2, \ldots, s_k$.

We define $\mathcal{I}_k$ resp. $\mathcal{D}_k$ to be the class of all permutations merged from at most k increasing resp. decreasing subsequences. Also let $\mathcal{I} = \mathcal{I}_1$ and $\mathcal{D} = \mathcal{D}_1$, i.e. $\mathcal{I} = \mathrm{Av}(21)$ is the set of all increasing permutations and $\mathcal{D} = \mathrm{Av}(12)$ is the set of all decreasing permutations, and for convenience let $\mathcal{I}_0 = \mathcal{D}_0 = \mathcal{S}_0$.

The classes $\mathcal{I}_k$ and $\mathcal{D}_k$ are well-known examples of principal classes.

Theorem 1.1. *$\mathcal{I}_{k-1} = \mathrm{Av}(k \cdots 21)$ and $\mathcal{D}_{k-1} = \mathrm{Av}(12 \cdots k)$ for any positive integer k.*

Proof. We will show the proof only for $\mathcal{I}_{k-1}$, the proof of the second equality is identical.

If π can be partitioned into $k-1$ increasing sequences, then out of any k elements there are at least two belonging to the same increasing sequence, so no k elements can form a decreasing sequence.

On the other hand if π avoids $k \cdots 21$, label elements of π left to right by the length of the longest decreasing subsequence of π ending in that element. Clearly we have used at most $k-1$ different labels since otherwise we would have used a label of size at least k and therefore we would have found a copy of $k \cdots 21$. Notice now that elements marked by the same label form an increasing sequence: for any two elements forming a decreasing sequence the label of the smaller element is greater by at least 1 than the label of the larger element since we may add the smaller element to the longest decreasing sequence ending in the larger element. $\square$

Now we state state and prove a known and important property of infinite permutation classes which will become useful in the upcoming chapters.

Lemma 1.2 (Atkinson, Beals [1]). *Let $\mathcal{C}$ be an infinite permutation class. Then either $\mathcal{I} \subseteq \mathcal{C}$ or $\mathcal{D} \subseteq \mathcal{C}$.*

Proof. Suppose that $\mathcal{I} \not\subseteq \mathcal{C}$ and $\mathcal{D} \not\subseteq \mathcal{C}$, then $\mathcal{C}$ avoids $12 \cdots k$ and $l \cdots 21$ for some positive integers k, l. According to the Erdős-Szekeres Theorem (see [7]), every permutation of length at least $(k-1)(l-1)+1$ contains $12 \cdots k$ or $l \cdots 21$, therefore $\mathcal{C}$ contains only permutations of length at most $(k-1)(l-1)$ and thus is finite. $\square$

Given a description of a permutation class, e.g. by a list of avoided patterns, a natural and fundamental question is how big the class is, i.e. how many permutations of fixed order it contains. For a permutation class $\mathcal{C}$ let $\mathcal{C}_n$ be the set of permutations of $\mathcal{C}$ of order n. In this notation, we are interested in the enumeration of the sequence $\{|\mathcal{C}_n|\}_{n=1}^{\infty}$. This can be attempted using standard approaches, e.g. finding the exact formula or computing the generating function, but this may often be difficult and we would then be content with computing at least the asymptotic behaviour of the sequence. We define the *upper growth rate* $\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ and the *lower growth rate* $\underline{\mathrm{gr}}(\mathcal{C}) = \liminf_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$. If these values are identical for a class $\mathcal{C}$ we say that $\mathcal{C}$ has the *growth rate* $\mathrm{gr}(\mathcal{C}) = \overline{\mathrm{gr}}(\mathcal{C}) = \underline{\mathrm{gr}}(\mathcal{C})$. It is not known whether the lower and upper growth rate are equal for every class $\mathcal{C}$, however if $\mathcal{C} = \mathrm{Av}(\pi)$ for any single permutation π, then $\mathrm{gr}(\mathcal{C})$ exists. This fact is due to Arratia [8] and we omit its proof.

Theorem 1.3 (Arratia [8]). *For any permutation π the limit $\lim_{n\to\infty} \sqrt[n]{\mathrm{Av}(\pi)_n}$ exists.* $\square$

Let us demonstrate a simple example of determining the basis and growth rate of two fairly small permutation classes which we will use in following chapters. We denote by $\mathcal{V}$ the class of permutations which are concatenations of two increasing sequences, e.g. $1346257 \in \mathcal{V}$, and we denote by $\mathcal{H}$ the class of permutations merged from two sequences of consecutive integers, e.g. $4125637 \in \mathcal{H}$.

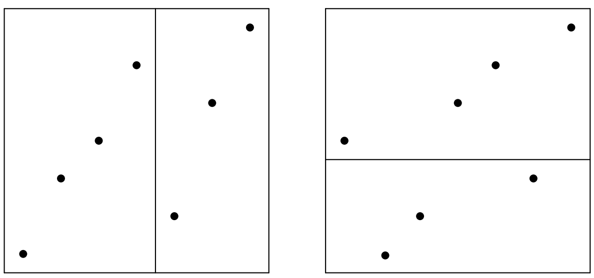


Figure 1.3: $1346257 \in \mathcal{V}$ (left) and $4125637 \in \mathcal{H}$ (right).

We will use the following simple lemma.

Lemma 1.4. *Consider k nonempty permutations $\pi_1, \dots, \pi_k$.*
Then $\mathrm{Av}(\pi_1^{-1}, \dots, \pi_k^{-1}) = \{\pi^{-1}; \pi \in \mathrm{Av}(\pi_1, \dots, \pi_k)\}$.

Proof. Observe that $\sigma \leq \pi$ if and only if $\sigma^{-1} \leq \pi^{-1}$. $\square$

Proposition 1.5. *The basis of* $\mathcal{V}$ *is the set* $\{321, 2143, 3142\}$*, the basis of* $\mathcal{H}$ *is the set* $\{321, 2143, 2413\}$ *and* $\mathrm{gr}(\mathcal{V}) = \mathrm{gr}(\mathcal{H}) = 2$.

Proof. We can easily see that permutations 321, 2143 and 3142 do not belong to $\mathcal{V}$, i.e. $\mathcal{V} \subseteq \mathrm{Av}(321, 2143, 3142)$. We will prove the opposite inclusion by induction. For permutations of length at most 3 the inclusion is trivial. Next consider $\pi \in \mathrm{Av}(321, 2143, 3142)$ of length at least 4 and consider the first two elements a, b of π and the last two elements c, d of π.

Now if $a < b$, we may ignore the first element, apply the induction hypothesis on the relabeled remaining permutation, get the concatenation of two increasing sequences and add a back, finishing the proof. The same can be done if $d > c$. Therefore assume $a > b$ and $c > d$. Now $d > b$ and $c > a$, otherwise abd or acd would form a copy of 321. Finally if $d > a$ then $abcd$ is a copy of 2143 and if $d < a$ then $abcd$ is a copy of 3142, which is a contradiction, completing the proof.

The enumeration of basis of $\mathcal{H}$ follows from the simple observation that $\mathcal{H} = \{\pi^{-1}; \pi \in \mathcal{V}\}$ and Lemma 1.4.

It is easy to see that $|\mathcal{V}_n| = 2^n - n$ since we can choose the left increasing sequence as any subset of $[n]$, but we count the identical permutation $n+1$ times this way, hence subtracting n. This implies $\mathrm{gr}(\mathcal{V}) = 2$. The growth rate of $\mathcal{H}$ follows from the observation that $\pi \longrightarrow \pi^{-1}$ is a bijection between $\mathcal{V}$ and $\mathcal{H}$. $\square$

1.2 Splittability

In this section we shortly introduce another concept which has been recently used to derive enumerative results on permutation classes and which we will also utilize in our work.

A permutation π is *merged from permutations* α and β if we can color the elements of π with red and blue such that the red subsequence is order-isomorphic to α and the blue sequence is order-isomorphic to β. Given two permutation classes $\mathcal{A}$ and $\mathcal{B}$ we define their *merge* denoted by $\mathcal{A} \odot \mathcal{B}$ as the class of all permutations which can be merged from a (possibly empty) permutation from $\mathcal{A}$ and a (possibly empty) permutation from $\mathcal{B}$. For example, it is easy to see that

$$\mathcal{I}_k = \underbrace{\mathcal{I} \odot \mathcal{I} \odot \cdots \odot \mathcal{I}}_{k\times}.$$

We say that a class $\mathcal{C}$ is *splittable* if it has two proper subclasses $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{C} \subseteq \mathcal{A} \odot \mathcal{B}$. For instance, the classes $\mathcal{V}$ and $\mathcal{H}$ from the previous section are clearly splittable since $\mathcal{V} \subseteq \mathcal{I} \odot \mathcal{I}$ and $\mathcal{H} \subseteq \mathcal{I} \odot \mathcal{I}$. The class of layered permutations (permutations which can be written as a sum of decreasing sequences) serves as an example of an unsplittable class. We refer the reader to [9] for an exhaustive study of splittability.

2. The notion of composability

In the following sections we provide definitions of the key notions of this work as well as basic facts and observations.

2.1 Composing permutation classes

We define the *composition* of two permutation classes $\mathcal{A}$ and $\mathcal{B}$ as the set $\mathcal{A} \circ \mathcal{B} = \{\pi \circ \varphi; \pi \in \mathcal{A}, \varphi \in \mathcal{B}, |\pi| = |\varphi|\}$.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary permutation classes.*

(a) $\mathcal{A} \circ \mathcal{B}$ is again a permutation class.

(b) Composing permutation classes is associative, i.e. $(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$.

Proof. Let $\alpha \circ \beta = \pi \in \mathcal{A} \circ \mathcal{B}$, so that $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$. Then a permutation contained in π at indices $i_1 < \cdots < i_r$ is composed of $\alpha' \leq \alpha$ and $\beta' \leq \beta$ such that β' is contained at indices $i_1, \ldots, i_r$ in β and α' is contained at indices $\beta(i_1), \ldots, \beta(i_2)$ in α. Associativity follows from associativity of permutation composition. □

Having verified associativity of the composition operator we can now define the composition of more than two classes in a natural inductive way:

$$\mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k = (\mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_{k-1}) \circ \mathcal{C}_k.$$

We will also sometimes use the power notation $\underbrace{\mathcal{C} \circ \mathcal{C} \circ \cdots \circ \mathcal{C}}_{k\times} = (\mathcal{C})^k$.

We continue by proving several simple lemmas about composing permutations merged from few increasing sequences.

Lemma 2.2. *$\mathcal{I}_k \circ \mathcal{I}_l \subseteq \mathcal{I}_{kl}$ for any integers $k, l \geq 0$.*

Proof. Choose $\pi \in \mathcal{I}_k$ and $\varphi \in \mathcal{I}_l$, partition φ into l increasing sequences and choose one of them at indices $i_1 < \cdots < i_r$. Then $\varphi(i_1) < \cdots < \varphi(i_r)$ and so $\pi(\varphi(i_1)), \ldots, \pi(\varphi(i_r))$ is a subsequence of π and therefore it can be partitioned into at most k increasing sequences since that is the property of π. This is true for the image of each of the l increasing subsequences in φ and therefore $\pi \circ \varphi$ can be partitioned into at most $k \cdot l$ increasing subsequences. □

Since $\mathcal{D} \circ \mathcal{D} = \mathcal{I}$, the argument of the previous proof can be repeated to show that $\mathcal{D}_k \circ \mathcal{D}_l \subseteq \mathcal{I}_{kl}$. We can generalise this even more.

Lemma 2.3. *Let k, l, m, n be any non-negative integers. Then*

$$(\mathcal{I}_k \odot \mathcal{D}_m) \circ (\mathcal{I}_l \odot \mathcal{D}_n) \subseteq \mathcal{I}_{kl+mn} \odot \mathcal{D}_{kn+ml}.$$

Proof. Use the approach identical to that of Lemma 2.2. □

We finish this section with one more variation of the preceding lemmas. We call a subsequence s of a permutation π a *block* if s is either an increasing or a decreasing contiguous subsequence of consecutive integers. We then call π a *k-block* if it is a concatenation of at most k blocks.

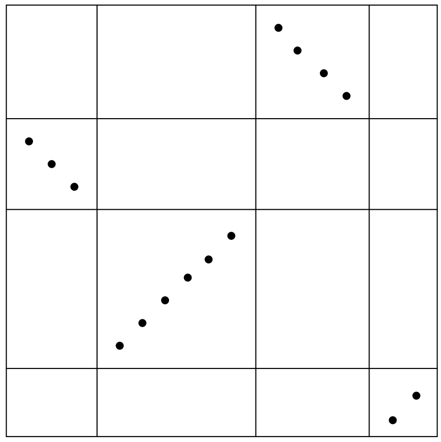


Figure 2.1: An example of a 4-block

Lemma 2.4. *Let $\pi \in S_n$ be a k-block and let $\sigma \in S_n$ be an l-block. Then $\pi \circ \sigma$ is a $(k \cdot l)$-block.*

Proof. Choose a block of σ at indices $a, a+1, \ldots, a+b$. Then the sequence $\pi(\sigma(a)), \pi(\sigma(a+1)), \ldots, \pi(\sigma(a+b))$ is a contiguous subsequence of either $\pi(1), \pi(2), \ldots, \pi(n)$ or $\pi(n), \ldots, \pi(2), \pi(1)$ and therefore is a concatenation of at most k blocks since π itself is a k-block. This is true for each of the l blocks of σ, therefore $\pi \circ \sigma$ is a $(k \cdot l)$-block. □

2.2 Composability

The main problem we are addressing in this work is whether permutations in a given permutation class can be constructed by composing permutations from two or more smaller classes. We formalise this as follows. A permutation class $\mathcal{C}$ is said to be *composable from classes* $\mathcal{C}_1, \ldots, \mathcal{C}_k$ if $\mathcal{C} \subseteq \mathcal{C}_1 \circ \cdots \circ \mathcal{C}_k$. A class $\mathcal{C}$ is *k-composable*, if it is composable from its k proper subclasses $\mathcal{C}_1, \ldots, \mathcal{C}_k$. A class $\mathcal{C}$ is *composable*, if it is k-composable for some $k \geq 2$. Using this terminology, our goal is thus answering the question whether a given permutation class is composable.

Clearly, for every class $\mathcal{C}$ we have $\mathcal{C} \subseteq \mathcal{C} \circ \mathcal{I}$. For an infinite class we have either $\mathcal{I} \subseteq \mathcal{C}$, which implies $\mathcal{C} \subseteq \mathcal{C} \circ \mathcal{C}$, or $\mathcal{D} \subseteq \mathcal{C}$, which implies $\mathcal{I} \subseteq \mathcal{C} \circ \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C} \circ \mathcal{C} \circ \mathcal{C}$. Restricting ourselves to proper subclasses in the definition of a composable class is motivated by these trivial inclusions.

We begin the exploration of composability by proving the following result which implies that unlike splittability, k-composability for $k > 2$ does not imply 2-composability.

Theorem 2.5. *Let $\mathcal{C}$ be an infinite permutation class such that $\mathcal{I} \nsubseteq \mathcal{C}$. Then $\mathcal{C}$ is not $2k$-composable for any positive integer k.*

Proof. Since $\mathcal{C}$ does not contain $\mathcal{I}$, there is an integer n such that $\mathcal{C}$ avoids $12 \cdots n(n+1)$ and therefore $\mathcal{C} \subseteq \mathcal{D}_n$ by Theorem 1.1.

Now let $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2, \ldots, \mathcal{A}_k, \mathcal{B}_k$ be proper subclasses of $\mathcal{C}$ and suppose that $\mathcal{C} \subseteq \mathcal{A}_1 \circ \mathcal{B}_1 \circ \mathcal{A}_2 \circ \mathcal{B}_2 \circ \cdots \circ \mathcal{A}_k \circ \mathcal{B}_k$. Since all these classes are subsets of $\mathcal{D}_n$, Lemma 2.3 implies $\mathcal{A}_i \circ \mathcal{B}_i \subseteq \mathcal{I}_{n^2}$ for every $i \in [k]$. Using Lemma 2.3 again we get that

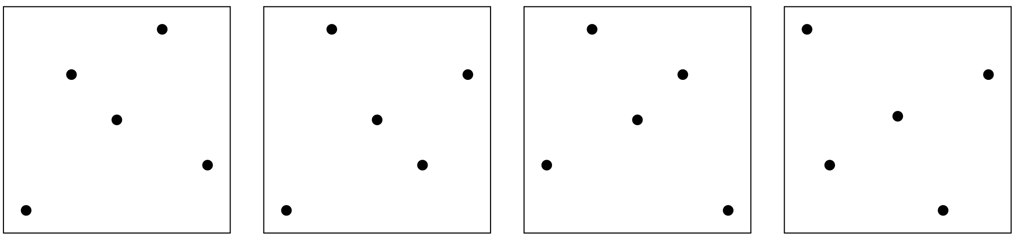
$$\mathcal{A}_1 \circ \mathcal{B}_1 \circ \mathcal{A}_2 \circ \mathcal{B}_2 \circ \cdots \circ \mathcal{A}_k \circ \mathcal{B}_k \subseteq \mathcal{I}_{n^2} \circ \cdots \circ \mathcal{I}_{n^2} \subseteq \mathcal{I}_{n^{2k}},$$

therefore, according to our assumption, $\mathcal{C} \subseteq \mathcal{I}_{n^{2k}}$, which means that $\mathcal{C}$ does not contain a decreasing permutation of length $n^{2k}+1$ by Theorem 1.1. But since $\mathcal{C}$ is infinite and does not contain $\mathcal{I}$, it has to contain $\mathcal{D}$ according to Lemma 1.2, which is a contradiction. □

2.3 Properties of symmetries

In the final section of this chapter we explore how composability is preserved under some of the usual symmetrical maps.

For a permutation π of length n we define π^r to be the *reverse* of π, i.e. $\pi^r(k) = \pi(n-k+1)$, and π^c to be the *complement* of π, i.e. $\pi^c(k) = n - \pi(k) + 1$. For a permutation class $\mathcal{A}$ we define the *inverse class* $\mathcal{A}^{-1} = \{\pi^{-1}; \pi \in \mathcal{A}\}$, the *reverse class* $\mathcal{A}^r = \{\pi^r; \pi \in \mathcal{A}\}$, and the *complementary class* $\mathcal{A}^c = \{\pi^c; \pi \in \mathcal{A}\}$.



(a) 14352 (b) $(14352)^{-1}$ (c) $(14352)^r$ (d) $(14352)^c$

Figure 2.2: Symmetries of the permutation 14352

It is clear that all these class operators are involutory, i.e. $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$, $(\mathcal{A}^r)^r = \mathcal{A}$ and $(\mathcal{A}^c)^c = \mathcal{A}$. The following simple lemma describes how these operators relate to composition.

Lemma 2.6. *Let $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be permutation classes. Then*

(a) $(\mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_k)^{-1} = \mathcal{A}_k^{-1} \circ \cdots \circ \mathcal{A}_2^{-1} \circ \mathcal{A}_1^{-1}$,

(b) $\mathcal{A}^r = \mathcal{A} \circ \mathcal{D}$ and $\mathcal{A}^c = \mathcal{D} \circ \mathcal{A}$,

(c) $(\mathcal{A}^r)^c = (\mathcal{A}^c)^r = \mathcal{D} \circ \mathcal{A} \circ \mathcal{D}$.

Proof. (a): If $\pi_i \in \mathcal{A}_i$ for every $i \in [k]$, then by the property of inverse elements in a group we have $(\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k)^{-1} = \pi_k^{-1} \circ \cdots \circ \pi_2^{-1} \circ \pi_1^{-1}$.

(b): Let $\alpha \in \mathcal{A}$ and $\delta \in \mathcal{D}$ be permutations of order n. By definition $\delta(k) = n - k + 1$ for every $k \in [n]$. Therefore $\alpha(\delta(k)) = \alpha(n-k+1) = \alpha^r(k)$ and $\delta(\alpha(k)) = n - \alpha(k) + 1 = \alpha^c(k)$ for every $k \in [n]$.

(c): Apparent from (b). □

Using this lemma we derive several composability criteria for symmetries of a given class, the first of which requires no further proof as it is an immediate consequence of Lemma 2.6.

Corollary 2.7. *Let $\mathcal{A}$ be a permutation class. Then the following statements are equivalent:*

(a) $\mathcal{A}$ is composable,

(b) $\mathcal{A}^{-1}$ is composable,

(c) $(\mathcal{A}^r)^c$ is composable.

The case of the reverse and complementary operators is more complicated and requires additional assumptions.

Lemma 2.8. *If $\mathcal{A}$ is a k-composable class and $\mathcal{I} \subsetneq \mathcal{A}$, then both $\mathcal{A}^r$ and $\mathcal{A}^c$ are $(2k-1)$-composable.*

Proof. Let $\mathcal{A}$ be composable from its proper subclasses $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. Then

$$\mathcal{A}^r = \mathcal{A} \circ \mathcal{D} \subseteq \mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_k \circ \mathcal{D} = (\mathcal{A}_1^r \circ \mathcal{D}) \circ (\mathcal{A}_2^r \circ \mathcal{D}) \circ \cdots \circ (\mathcal{A}_k^r \circ \mathcal{D}) \circ \mathcal{D}.$$

It holds that $\mathcal{D} \circ \mathcal{D} = \mathcal{I}$, so we have

$$\mathcal{A}^r \subseteq \mathcal{A}_1^r \circ \mathcal{D} \circ \mathcal{A}_2^r \circ \mathcal{D} \circ \cdots \circ \mathcal{A}_k^r.$$

Clearly $\mathcal{A}_i^r \subsetneq \mathcal{A}^r$ and since $\mathcal{I} \subsetneq \mathcal{A}$, we have $\mathcal{D} \subsetneq \mathcal{A}^r$, so the proper subclass criterion is met and $\mathcal{A}^r$ is therefore $(2k-1)$-composable. Analogously we show that

$$\mathcal{A}^c \subseteq \mathcal{A}_1^c \circ \mathcal{D} \circ \mathcal{A}_2^c \circ \mathcal{D} \circ \cdots \circ \mathcal{A}_k^c.$$

□

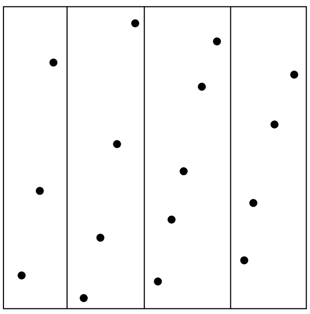
3. On permutations avoiding a decreasing sequence

Recall that $\mathcal{I}_k = \mathrm{Av}((k+1)\cdots 21)$ is the class of permutations merged from k increasing sequences, or equivalently those avoiding a decreasing sequence of length $k+1$. In this chapter, prove that $\mathcal{I}_k$ is 2-composable and show several examples of how $\mathcal{I}_k$ can be composed from two proper subclasses. For that we utilize a generalised version of the classes $\mathcal{V}$ and $\mathcal{H}$ from Chapter 1.

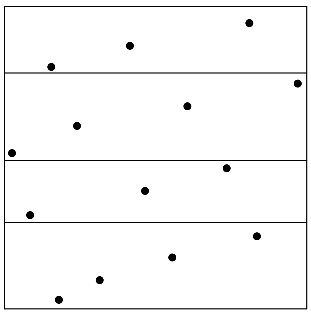
3.1 Vertical and horizontal merge

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be any permutation classes. We define the *vertical merge* of these classes as the class of permutations that can be written as a concatenation $s_1 s_2 \cdots s_k$ of k (possibly empty) sequences such that s_i is order-isomorphic to a permutation of $\mathcal{C}_i$. We denote this class by $\mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$. In addition, if $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_k = \mathcal{I}$, we denote $\mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ by $\mathcal{V}_k$. We define the *horizontal merge* of these classes as the class of permutations that can be written as a merge of k (possibly empty) sequences $s_1, s_2, \ldots, s_k$ such that each s_i is order-isomorphic to $\pi_i \in \mathcal{C}_i$ and $s_1 s_2 \cdots s_k = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$. In other words, each s_i uses a set of consecutive integers and every element of s_i is smaller than every element of s_{i+1}. We denote this class by $\mathcal{H}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ and if $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_k = \mathcal{I}$ we denote it by $\mathcal{H}_k$.

Alternatively, we can observe that $\pi \in \mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ resp. $\pi \in \mathcal{H}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ if and only if its plot in $\mathbb{R}^2$ can be separated by vertical resp. horizontal lines into at most k parts, i-th of them containing a sequence order-isomorphic to a permutation in $\mathcal{C}_i$, hence the names of the classes.



(a) An element of the vertical merge $\mathcal{V}_k$



(b) An element of the horizontal merge $\mathcal{H}_k$

Figure 3.1: Examples of vertical and horizontal merges

According to this definition we have $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{V} = \mathcal{V}_2$. In Chapter 1 we noticed that $\mathcal{H} = \mathcal{V}^{-1}$. We continue by proving a generalised version of this observation.

Lemma 3.1. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be any permutation classes. Then*

$$\mathcal{H}(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \left(\mathcal{V}(\mathcal{C}_1^{-1}, \ldots, \mathcal{C}_k^{-1})\right)^{-1}.$$

Proof. If $\pi \in \mathcal{V}(\mathcal{C}_1^{-1}, \ldots, \mathcal{C}_k^{-1})$, we have that $\pi = s_1 s_2 \cdots s_k$ such that s_i is order-isomorphic to $\pi_i \in \mathcal{C}_i^{-1}$. For every $i \in [k]$, π^{-1} contains a set of consecutive integers on indices $(s_i)_1, (s_i)_2, \ldots, (s_i)_{|s_i|}$ and the sequence at these indices is order-isomorphic to $\pi_i^{-1} \in \mathcal{C}_i$.

The opposite inclusion is equally straightforward. □

When composed with any other class $\mathcal{A}$, the classes $\mathcal{H}_k$, $\mathcal{V}_k$ and $\mathcal{I}_k$ can be viewed as a unary operator transforming $\mathcal{A}$ in a specific way. We formalise this approach in the following important lemma.

Lemma 3.2. *Let $\mathcal{A}$ be an arbitrary permutation class. Then*

(a) $\mathcal{A} \circ \mathcal{H}_k$ is precisely the class of permutations which can be obtained from a permutation of $\mathcal{A}$ by dividing it into at most k contiguous subsequences and interleaving them in any way,

(b) $\mathcal{A} \circ \mathcal{V}_k$ is precisely the class of permutations which can be obtained from a permutation of $\mathcal{A}$ by dividing it into at most k subsequences and concatenating them,

(c) $\mathcal{A} \circ \mathcal{I}_k$ is precisely the class of permutations which can be obtained from a permutation of $\mathcal{A}$ by dividing it into at most k subsequences and interleaving them in any way.

Proof. Let $\alpha \in \mathcal{A}$, $\eta \in \mathcal{H}_k$, $\nu \in \mathcal{V}_k$ and $\iota \in \mathcal{I}_k$.

(a): Consider the permutation $\alpha \circ \eta \in \mathcal{A} \circ \mathcal{H}_k$. Then η is merged from k (possibly empty) sequences of consecutive integers $s_1, \ldots, s_k$. Now we define k sequences $r_1, \ldots, r_k$ such that $|s_i| = |r_i|$ and $(r_i)_j = \alpha((s_i)_j)$ for every $i \in [k]$ and $j \in [|s_i|]$. Every r_i is a contiguous subsequence of α and at the same time $\alpha \circ \eta$ is merged from $r_1, \ldots, r_k$.

On the other hand, if a permutation π is obtained from $\alpha \in \mathcal{A}$ by dividing it into k contiguous subsequences $r_1, \ldots, r_k$ and merging them in some way, we define k sequences of consecutive integers $s_1, \ldots, s_k$ such that $s_1 s_2 \cdots s_k = 123 \cdots |\pi|$ and $|s_i| = |r_i|$ for every i. From this definition we get that $\alpha((s_i)_j) = (r_i)_j$. Now define $\eta \in \mathcal{H}_k$ as $\eta(\pi^{-1}((r_i)_j)) = (s_i)_j$. By multiplying this equation by α from the left we get that $\alpha \circ \eta = \pi$.

(b): Consider the permutation $\alpha \circ \nu \in \mathcal{A} \circ \mathcal{V}_k$. The permutation ν is formed by concatenating k increasing sequences $s_1, \ldots, s_k$. Define k sequences $r_1, \ldots, r_k$ such that $|r_i| = |s_i|$ and $(r_i)_j = \alpha((s_i)_j)$. Each r_i is a subsequence of α and at the same time $\alpha \circ \nu = r_1 r_2 \cdots r_k$.

On the other hand, if a permutation π is obtained from $\alpha \in \mathcal{A}$ by dividing it into k subsequences $r_1, \ldots, r_k$ and then concatenating them, we define k sequences $s_1, \ldots, s_k$ such that $|s_i| = |r_i|$ and $(s_i)_j = \alpha^{-1}((r_i)_j)$ and let $\nu = s_1 s_2 \cdots s_k$. Since s_i is by definition increasing for every i, we get $\nu \in \mathcal{V}_k$ and since $\alpha((s_i)_j) = (r_i)_j$, we get $\pi = \alpha \circ \nu$.

(c): Consider the permutation $\alpha \circ \iota \in \mathcal{A} \circ \mathcal{I}_k$. The permutation ι is merged from k increasing subsequences $s_1, \ldots, s_k$. We define k sequences $r_1, \ldots, r_k$ such that $|s_i| = |r_i|$ and $(r_i)_j = \alpha((s_i)_j)$. Clearly $\alpha \circ \iota$ is merged from the sequences $r_1, \ldots, r_k$ and at the same time each r_i is a subsequence of α.

On the other hand, if a permutation π is obtained from $\alpha \in \mathcal{A}$ by dividing it into k subsequences $r_1, \ldots, r_k$ and the merging them, we define k sequences

$s_1, \ldots, s_k$ such that $|s_i| = |r_i|$ and $(s_i)_j = \alpha^{-1}((r_i)_j)$. We then define the permutation ι as $\iota(\pi^{-1}((r_i)_j)) = (s_i)_j$. Every s_i is by definition increasing and ι is merged from $s_1, \ldots, s_k$, therefore $\iota \in \mathcal{I}_k$. Also, since $\alpha((s_i)_j) = (r_i)_j$, we get $\pi = \alpha \circ \iota$ from the definition of ι by multiplying by α from the left. $\square$

3.2 Composability results

Using the machinery introduced in the previous section we now prove a key lemma which we will use to show several composability results.

Lemma 3.3. *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ be arbitrary permutation classes. Then*

$$\mathcal{C}_1 \odot \mathcal{C}_2 \odot \cdots \odot \mathcal{C}_k \subseteq \mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k) \circ \mathcal{H}_k.$$

Proof. Consider a permutation $\pi \in \mathcal{C}_1 \odot \cdots \odot \mathcal{C}_k$ and divide it into k sequences $s_1, \ldots, s_k$ such that s_i is isomorphic to a permutation from $\mathcal{C}_i$. The permutation $\nu = s_1 s_2 \cdots s_k$ then lies in $\mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$, which together with Lemma 3.2(a) implies $\pi \in \mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k) \circ \mathcal{H}_k$. $\square$

By reformulating the previous statement we immediately get the following.

Corollary 3.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be permutation classes such that $\mathcal{C} \subseteq \mathcal{A} \odot \mathcal{B}$. Then $\mathcal{C} \subseteq \mathcal{V}(\mathcal{A}, \mathcal{B}) \circ \mathcal{H}$.*

Using what already has been done in this chapter it is now elementary to show that $\mathcal{I}_k$ is 2-composable.

Theorem 3.5. *The class $\mathcal{I}_k$ is 2-composable for every $k \geq 2$. In particular, $\mathcal{I}_k \subseteq \mathcal{V}_k \circ \mathcal{H}_k$.*

Proof. Trivially $\mathcal{V}_k \subsetneq \mathcal{I}_k$ and $\mathcal{H}_k \subsetneq \mathcal{I}_k$. Next we recall that

$$\mathcal{I}_k = \underbrace{\mathcal{I} \odot \cdots \odot \mathcal{I}}_{k\times}$$

and use Lemma 3.3 for $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_k = \mathcal{I}$. $\square$

We proceed by proving a result in some sense opposite to that of Lemma 2.2, namely we show that $\mathcal{I}_k$ may be constructed from smaller $\mathcal{I}_a, \mathcal{I}_b$ using composition.

Theorem 3.6. *$\mathcal{I}_{k+l-1} \subseteq \mathcal{I}_k \circ \mathcal{I}_l$ for all integers $k, l \geq 2$.*

Proof. Consider a permutation $\pi \in \mathcal{I}_{k+l-1}$, merged from two sequences a and b such that a is merged from k increasing sequences $s_1, \ldots, s_k$ and b is merged from $l-1$ increasing sequences $s_{k+1}, \ldots, s_{k+l-1}$. Let c be the increasing sequence created by sorting the elements of b. Consider a permutation σ created by merging the sequences a and c so that c and s_k form a single increasing sequence. Clearly $\sigma \in \mathcal{I}_k$ and sequences $s_{k+1}, \ldots, s_{k+l-1}$ are subsequences of σ, since they are increasing and therefore were not affected by sorting b.

According to Lemma 3.2(c) the class $\mathcal{I}_k \circ \mathcal{I}_l$ contains all permutations we can create from σ by dividing it into l subsequences and merging them in any way. It is therefore enough to find a way to divide σ into l subsequences which can be merged into π. A simple choice of l such subsequences is $a, s_{k+1}, \ldots, s_{k+l-1}$. $\square$

This theorem raises the question whether we could construct a bigger class from given $\mathcal{I}_k$ and $\mathcal{I}_l$.

Question. *Given positive integers k and l, what is the largest integer $m = m(k,l)$ such that $\mathcal{I}_m \subseteq \mathcal{I}_k \circ \mathcal{I}_l$?*

So far we have shown that $m(a,b) \leq kl$ (Lemma 2.2) and that $m(k,l) \geq k + l - 1$ (Theorem 3.6).

3.3 Upper bound on growth rate

The work of Regev [10] implies that $\mathrm{gr}(\mathcal{I}_k) = k^2$. A simple proof of the upper bound $\mathrm{gr}(\mathcal{I}_k) \leq k^2$ can also be found in [11, Section 1.4]. We re-derive this bound using results of the previous section and thus demonstrating how composability may be used to estimate growth rates.

Before we proceed, we state and prove two lemmas describing upper growth rate preserving properties of composition and vertical merge of permutation classes.

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two permutation classes satisfying $\overline{\mathrm{gr}}(\mathcal{A}) = a$ and $\overline{\mathrm{gr}}(\mathcal{B}) = b$ for non-negative real numbers a, b. Then $\overline{\mathrm{gr}}(\mathcal{A} \circ \mathcal{B}) \leq ab$.*

Proof. Fix a constant K such that for every $\varepsilon > 0$ and every n there are at most $K(a+\varepsilon)^n$ permutations of order n in $\mathcal{A}$ and at most $K(b+\varepsilon)^n$ permutations of order n in $\mathcal{B}$.

Then there are at most

$$K^2\left((a+\varepsilon)(b+\varepsilon)\right)^n = K^2(ab + (a+b+\varepsilon)\varepsilon)^n$$

permutations of order n in $\mathcal{A} \circ \mathcal{B}$ for every $\varepsilon > 0$ and every n, hence $\overline{\mathrm{gr}}(\mathcal{A} \circ \mathcal{B}) \leq ab$. ☐

Lemma 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two permutation classes satisfying $\overline{\mathrm{gr}}(\mathcal{A}) = a$ and $\overline{\mathrm{gr}}(\mathcal{B}) = b$ for non-negative real numbers a, b. Then $\overline{\mathrm{gr}}(\mathcal{V}(\mathcal{A},\mathcal{B})) \leq a + b$.*

Proof. We fix a constant K such that for every $\varepsilon > 0$ and every n there are at most $K(a+\varepsilon)^n$ permutations of order n in $\mathcal{A}$ and at most $K(b+\varepsilon)^n$ permutations of order n in $\mathcal{B}$.

For every integer k between 0 and n there are at most

$$\binom{n}{k} K(a+\varepsilon)^k K(b+\varepsilon)^{n-k}$$

permutations of order n such that their first k elements are order-isomorphic to a permutation of $\mathcal{A}$ and last $n-k$ elements are order-isomorphic to a permutation of $\mathcal{B}$. Summing over all values of k we get that there are at most

$$K^2 \sum_{k=0}^{n} \binom{n}{k} (a+\varepsilon)^k (b+\varepsilon)^{n-k} = K^2(a+b+2\varepsilon)^n$$

permutations of order n in $\mathcal{V}(\mathcal{A},\mathcal{B})$, therefore $\overline{\mathrm{gr}}(\mathcal{V}(\mathcal{A},\mathcal{B})) \leq a + b$. ☐

Since $\mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \mathcal{V}(\mathcal{V}(\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}), \mathcal{C}_k)$, the previous lemma is easily extended by an induction argument to the following.

Corollary 3.9. *Let* $\overline{\mathrm{gr}}(\mathcal{C}_i) = g_i$ *for* $i \in [k]$*. Then*

$$\overline{\mathrm{gr}}(\mathcal{V}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k)) \leq g_1 + g_2 + \ldots + g_k.$$

Specifically, since $\mathcal{H}_k = \mathcal{V}_k^{-1}$, $\overline{\mathrm{gr}}(\mathcal{H}_k) = \overline{\mathrm{gr}}(\mathcal{V}_k) \leq k$*.*

The proof of the desired upper bound of $\mathrm{gr}(\mathcal{I}_k)$ now becomes simple.

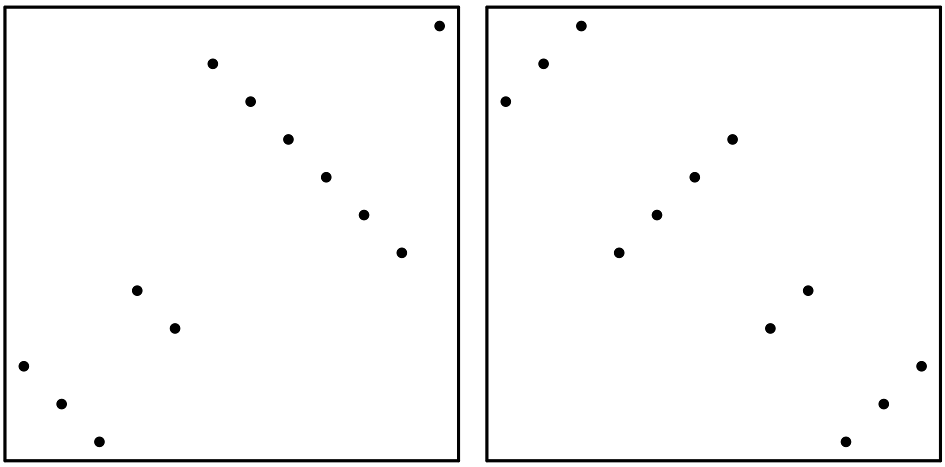
Theorem 3.10. *The growth rate of* $\mathcal{I}_k$ *is at most* k^2*.*

Proof. Recall that $\mathcal{I}_k \subseteq \mathcal{V}_k \circ \mathcal{H}_k$ by Theorem 3.5 and use Corollary 3.9 and Lemma 3.7. $\square$

4. On layered and related classes

In this chapter we cover classes of permutations which can be written as a sum or as a skew sum of increasing or decreasing permutations. Among these classes we provide infinitely many examples of composable classes as well as several examples of classes which are not composable.

Let ι_k denote the increasing permutation of order k and δ_k denote the decreasing permutation of order k. A permutation is *layered* if it is a sum of decreasing permutations which are then called *layers*. We shall denote the class of all layered permutations by $\mathcal{L}$. We shall denote by $\mathcal{L}_k$ the class of permutations which are sums of at most k layers. The complement of a layered permutation is clearly a skew sum of increasing permutations and we shall call such a permutation *co-layered*. The class $\mathcal{L}^c$ consists of precisely the co-layered permutations.



(a) Layered permutation (b) Co-layered permutation

Figure 4.1: Examples of layered and co-layered patterns

4.1 Layered permutations

We start by proving that $\mathcal{L}_2$ is not composable using a counting argument. As it turns out, proper subclasses of $\mathcal{L}_2$ are asymptotically too small to build the entire $\mathcal{L}_2$ class using composition.

Theorem 4.1. *The class $\mathcal{L}_2$ is not composable.*

Proof. Suppose that $\mathcal{L}_2 \subseteq \mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$ such that $\mathcal{C}_i \subsetneq \mathcal{L}_2$ for every $i \in [k]$. Each of these subclasses avoids at least one permutation of $\mathcal{L}_2$. In other words for every $\mathcal{C}_i$ there is a $\pi_i \in \mathcal{L}_2$ such that $\mathcal{C}_i \subseteq \mathcal{L}_2 \cap \mathrm{Av}(\pi_i)$. Considering a sufficiently large n so that $\pi_i \leq \delta_n \oplus \delta_n$ for every $i \in [k]$ we get that $\mathcal{C}_i \subseteq \mathcal{L}_2 \cap \mathrm{Av}(\delta_n \oplus \delta_n)$ for every i, in other words every permutation in these subclasses has one of its two layers shorter than n. It follows that for a fixed integer N there are at most $2(n-1)$ permutations of order N in any $\mathcal{C}_i$, therefore there are at most $(2n-2)^k$ permutations in $\mathcal{C}_1 \circ \cdots \circ \mathcal{C}_k$. But $\mathcal{L}_2$ contains N permutations of order N for any N, therefore we obtain a contradiction by choosing $N > (2n-2)^k$. $\square$

The number of permutations of order n in $\mathcal{L}_2$ is linear in n while any proper subclass contains only constantly many permutations of fixed order. We can use the same approach using the asymptotic jump from polynomial to exponential functions to show that a different class of permutations cannot be composable. Namely, let $\mathcal{F}_2$ be the class of layered permutations with layers of size 1 or 2.

Theorem 4.2. *The class $\mathcal{F}_2$ is not composable.*

Proof. Suppose $\mathcal{F}_2$ is composable from k of its proper subclasses $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$. We choose a permutation from $\mathcal{F}_2 \setminus C_i$ for every i and we select n large enough so that every chosen permutation is contained in $\pi = \sum_{i=1}^{n} \delta_2$. Then if $\mathcal{C} = \mathcal{F}_2 \cap \mathrm{Av}(\pi)$, we get that $\mathcal{F}_2 \subseteq (\mathcal{C})^k$. Every permutation in $\mathcal{C}$ contains fewer than n layers of size 2, otherwise it would contain π. Clearly there are at most N^a permutations of $\mathcal{F}_2$ that have order N and exactly a layers of size 2. Therefore $\mathcal{C}$ contains at most $N^1 + N^2 + \cdots + N^{n-1} \le nN^n$ permutations of order N and the composition $(\mathcal{C})^k$ then contains at most $n^k N^{nk}$ permutations of order N, which is a number polynomial in N. As mentioned in [11, Chapter 4], the number of permutations of order N of $\mathcal{F}_2$ is counted by the Fibonacci numbers which grow exponentially, therefore there is N large enough so that $\mathcal{F}_2$ has more permutations of order N than $(\mathcal{C})^k$.

Note that this result also follows immediately from the theorem of Kaiser and Klazar ([12, 3.4]), which states that if the number of permutations of order n in a permutation class is less than the n-th Fibonacci number for at least one value of n, then it is eventually polynomial in n. This implies that every class counted by the Fibonacci numbers is uncomposable. $\square$

The argument used in the proofs above cannot be used for $\mathcal{L}_3$, so we need a different approach to show that this class too is not composable. We will make use of the following property of $\mathcal{L}_2 \cup \mathcal{L}_2^r$.

Lemma 4.3. *$(\mathcal{L}_2 \cup \mathcal{L}_2^r) \cap S_n$ is a subgroup of S_n for every n, i.e. it is closed under composition.*

Proof. In this proof, we consider an additive group structure on the set $[n]$ with the neutral element n and an operator $+_n$ defined as

$$a +_n b = 1 + (a + b - 1) \mod n.$$

First we prove that $\mathcal{L}_2^r \cap S_n$ by itself is a subgroup of S_n. Observe that $\mathcal{L}_2^r \cap S_n$ contains exactly permutations π such that there is a shifting number k with $\pi(i) = i +_n k$ for every $i \in [n]$. Indeed, if $\pi = \iota_a \ominus \iota_b$ then for any $i \in [n]$ we have $\pi(i) = i +_n b$ and conversely if $\pi(i) = i +_n k$ for every $i \in [n]$ then $\pi = \iota_{n-k} \ominus \iota_k$. Now for two permutations $\pi, \sigma \in \mathcal{L}_2^r$ with shifting numbers k, l respectively we have $\pi(\sigma(i)) = i +_n l +_n k$ for any $i \in [n]$, therefore $\pi \circ \sigma \in \mathcal{L}_2^r$ since it has a shifting number $k +_n l$.

It trivially holds that $\mathcal{L}_2 \circ \mathcal{D} = \mathcal{L}_2^r = \mathcal{L}_2^c = \mathcal{D} \circ \mathcal{L}_2$. Considering $\pi, \sigma \in (\mathcal{L}_2 \cup \mathcal{L}_2^r) \cap S_n$ it remains to distinguish the following four cases:

(i) $\pi \in \mathcal{L}_2^r$ and $\sigma \in \mathcal{L}_2^r$, then $\pi \circ \sigma \in \mathcal{L}_2^r$ by discussion above,

(ii) $\pi \in \mathcal{L}_2^r$ and $\sigma \in \mathcal{L}_2$, then $\pi \circ \sigma = (\pi \circ \sigma^r) \circ \delta_n \in \mathcal{L}_2$,

(iii) $\pi \in \mathcal{L}_2$ and $\sigma \in \mathcal{L}_2^r$, then $\pi \circ \sigma = (\delta_n \circ \pi^c) \circ \sigma = \delta_n \circ (\pi^r \circ \sigma) \in \mathcal{L}_2$,

(iv) $\pi \in \mathcal{L}_2$ and $\sigma \in \mathcal{L}_2$, then $\pi \circ \sigma = \pi^r \circ (\delta_n \circ \delta_n) \circ \sigma^c = \pi^r \circ \sigma^r \in \mathcal{L}_2^r$. □

Theorem 4.4. *The class $\mathcal{L}_3$ is not composable.*

Proof. Suppose that $\mathcal{L}_3 \subseteq \mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$ such that $\mathcal{C}_i \subsetneq \mathcal{L}_3$ for any i. Using the same initial argumentation as in the proof of Theorem 4.1 we get that there is an n such that $\mathcal{L}_3 \subseteq (\mathcal{L}_3 \cap \mathrm{Av}(\delta_n \oplus \delta_n \oplus \delta_n))^k$, meaning that every permutation of $\mathcal{L}_3$ can be composed from k permutations having at least one of the three layers shorter than n.

Let $\pi_i \in \mathcal{C}_i$ for $1 \le i \le k$ and $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_k$. We now claim that it is possible to remove at most $(n-1)k$ elements from π to obtain a two-layered or a two-co-layered permutation. We will prove this by induction on k. The case $k = 1$ is easy since $\pi = \pi_1$ avoids $\delta_n \oplus \delta_n \oplus \delta_n$, so it has a layer of length shorter than n whose removal creates a two-layered pattern.

For $k > 1$ let $\sigma = \pi_1 \circ \cdots \circ \pi_{k-1}$ and $\pi = \sigma \circ \pi_k$. Let all these permutations have the order N. By the induction hypothesis, there are a indices $i_1, \ldots i_a$ such that $a \ge N - (n-1)(k-1)$ and σ restricted to these indices has the two-layer or the two-co-layer pattern. Also there are b indices $j_1, \ldots, j_b$ such that $b \ge N - (n-1)$ and π_k restricted to these indices forms the two-layer or the two-co-layer pattern.

Let us now restrict the function $\sigma \circ \pi_k$ to the set $S = \{\pi_k^{-1}(i_1), \ldots, \pi_k^{-1}(i_a)\} \cap \{j_1, \ldots, j_b\}$ whose size is at least $N - (n-1)k$. Then both $\pi_k(S)$ and $\sigma(\pi_k(S))$ are still two-layer or two-co-layer patterns, which implies the same for their composition according to Lemma 4.3. Therefore π restricted to S forms a two-layer or two-co-layer pattern and $N - |S| \le (n-1)k$ which completes the induction step.

Consequently, any permutation of order N in $\mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$ has a two-layered or a two-co-layered pattern except for $k(n-1)$ elements. But choosing $N = 3(k(n-1)+1)$ and considering the permutation $\bigoplus_{i=1}^{3} \delta_{k(n-1)+1} \in \mathcal{L}_3$ we obtain a contradiction. □

If we allow more than three but still constantly many layers, we always get a composable class.

Theorem 4.5. *The class $\mathcal{L}_k$ is 3-composable for every $k \ge 4$.*

Proof. We will show that $\mathcal{L}_k \subseteq \mathcal{L}_{k-1} \circ \mathcal{L}_{k-2} \circ \mathcal{L}_{k-1}$.

If $\pi \in \mathcal{L}_k$ of order n has fewer than k layers, then $\pi = \pi \circ \delta_n \circ \delta_n \in \mathcal{L}_{k-1} \circ \mathcal{L}_{k-2} \circ \mathcal{L}_{k-1}$. Otherwise π has at least 4 layers and has the form $\pi = \delta_a \oplus \delta_b \oplus \delta_c \oplus \delta_d \oplus \pi'$ for some positive a, b, c, d. Since for every layered σ we have $\sigma \circ \sigma \circ \sigma = \sigma$ it is not hard to check that

$$\pi = (\delta_{a+b} \oplus \delta_c \oplus \delta_d \oplus \pi') \circ (\delta_{a+b} \oplus \delta_{c+d} \oplus \pi') \circ (\delta_a \oplus \delta_b \oplus \delta_{c+d} \oplus \pi') \in \mathcal{L}_{k-1} \circ \mathcal{L}_{k-2} \circ \mathcal{L}_{k-1}.$$

The situation is represented in the figure below. □

This theorem raises the question whether $\mathcal{L}_k$ could be 2-composable for $k \ge 4$. Our work from Chapter 2 quickly determines that this is not the case.

Proposition 4.6. *$\mathcal{L}_k$ for $k \ge 4$ is not 2-composable. In particular, it is not n-composable for any even number n.*

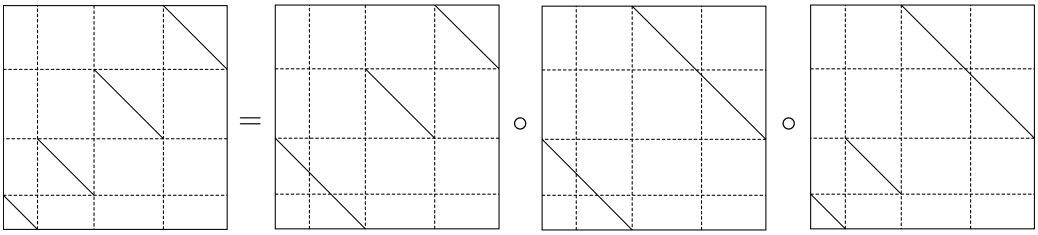


Figure 4.2: $\delta_a \oplus \delta_b \oplus \delta_c \oplus \delta_d = (\delta_{a+b} \oplus \delta_c \oplus \delta_d) \circ (\delta_{a+b} \oplus \delta_{c+d}) \circ (\delta_a \oplus \delta_b \oplus \delta_{c+d})$

Proof. Since $\mathcal{L}_k$ is an infinite class which does not contain $\mathcal{I}$ the statement directly follows from Theorem 2.5. $\square$

We have now covered the classes $\mathcal{L}_k$ for all $k \geq 2$. It remains to consider permutations with an unbounded number of layers.

Theorem 4.7. *The class $\mathcal{L}$ is not composable.*

Proof. Every subclass of $\mathcal{L}$ is determined by one or more forbidden layered permutations. If $\mathcal{L}$ is composable from k subclasses, we may choose one forbidden layered permutation from each of them and then choose n large enough so that $\pi = \bigoplus_{i=1}^{n+1} \delta_{n+1}$ contains all of the chosen patterns. That way, $\mathcal{L} \subseteq \mathcal{C}^k$ where $\mathcal{C} = \mathrm{Av}(\pi) \cap \mathcal{L}$.

Clearly every permutation in $\mathcal{C}$ has at most n layers longer than n, otherwise it would contain π. Our goal is to show that permutations in $\mathcal{C}^k$ are somehow very close to patterns composed from permutations that have a constant number of non-trivial layers and all other layers are just of size 1. We say that two permutations are (c,l)-close if you can transform one into another by changing at most l values arbitrarily and all other by at most c. For $\sigma \in \mathcal{C}$ we denote by $N(\sigma)$ the permutation created from σ by replacing all layers of length at most n by the corresponding number of layers of size 1.

We can now formally state our goal: we shall prove that for any $\sigma_1, \sigma_2, \ldots, \sigma_k \in \mathcal{C}$ the permutations $\sigma_k \circ \cdots \circ \sigma_2 \circ \sigma_1$ and $N(\sigma_k) \circ \cdots \circ N(\sigma_2) \circ N(\sigma_1)$ are $(nk, 4n^2k^2)$-close. We will prove this by induction on k.

If $k = 1$, we have to show that σ_1 and $N(\sigma_1)$ are $(n, 4n^2)$-close. Since $N(\sigma)$ is created by manipulating layers of σ of length at most n in place, every value is shifted by at most n, so they are even $(n, 0)$-close.

If $k \geq 2$, suppose that $\sigma = \sigma_k \circ \cdots \circ \sigma_2 \circ \sigma_1$ and $\nu = N(\sigma_k) \circ \cdots \circ N(\sigma_2) \circ N(\sigma_1)$ are $(nk, 4n^2k)$-close and we shall prove the statement for $k+1$.

Given a layered permutation and one of its layers of size $l+1$ at indices $i, i+1, \ldots, i+l$, we say that a number u is *in the area of influence* of this layer if $i \leq u \leq i+l$.

Consider those numbers u of $[|\sigma|]$ such that $\sigma(u)$ and $\nu(u)$ differ by at most kn. Then for a given big layer of $N(\sigma_{k+1})$ there are at most $2nk$ such numbers u such that $\nu(u)$ is in the area of influence of this layer and $\sigma(u)$ is not: at most nk from each side of the layer. The same goes the other way around: for each big layer of σ_{k+1} there are at most nk numbers u such that $\nu(u)$ is not in the area of influence of this layer but $\sigma(u)$ is. Since there are at most n big layers, we get that there are at most $4n^2k$ such numbers u which leave or enter the area of influence of a big layer. For all these numbers we allow the values $\sigma_{k+1}(\sigma(u))$ and

$N(\sigma_{k+1})(\nu(u))$ to differ arbitrarily. Together with at most $4n^2k^2$ numbers from the previous step of induction we get $4n^2k^2 + 4n^2k = 4n^2k(k+1) \leq 4n^2(k+1)^2$ values which satisfies the condition of the statement. It remains to show that all other values change by at most $n(k+1)$.

For other values $u \in [|\sigma|]$ not considered so far it holds that $|\nu(u) - \sigma(u)| \leq nk$ and that either $\nu(u)$ and $\sigma(u)$ are both in the area of influence of the same big layer of σ_{k+1} or they are in areas of influence of short or trivial layers. The latter case immediately implies $|N(\sigma_{k+1})(\nu(u)) - \sigma_{k+1}(\sigma(u))| \leq nk + n \leq n(k+1)$. In the former case it is enough to realise that $\delta_a(x \pm y) = \delta_a(x) \mp y$ so if $\sigma(u)$ differs by y from $\nu(u)$ and they are in the same big layer, after applying σ_{k+1} (or $N(\sigma_{k+1})$, which is the same for the big layers) the values still differ by $y \leq nk \leq n(k+1)$, which finishes the induction step.

Notice that for $\sigma \in \mathcal{C}$ the permutation $N(\sigma)$ can be written is a $(2n)$-block according to the notation used in Chapter 2. Thus by Lemma 2.4 we get that by composing k such permutations we get a permutation which is a $(2n)^k$-block. As a result we get that each permutation from $(\mathcal{C})^k$ is (c,l)-close to a C-block for suitable fixed constants c, l, C. Notice now that every C-block avoids the $(C+1)$-block $\gamma = 214365 \cdots (2C+2)(2C+1)$, so every permutation from $(\mathcal{C})^k$ is (c,l)-close to a permutation avoiding γ. We can construct a layered permutation which is not (c,l)-close to any γ-avoider as follows. Choose a layered permutation with $C+1$ layers of length $l+2c+1$ and consider a permutation (c,l)-close to it. Then in every layer there are at least $2c+1$ elements whose value changed by at most c; therefore there exist at least two elements which remained in decreasing order. Choosing these two elements from every layer forms an occurrence of γ. Since $\mathcal{L}$ contains a permutation which is not (c,l)-close to γ and $(\mathcal{C})^k$ does not contain such permutations, we get that $\mathcal{L} \nsubseteq (\mathcal{C})^k$, achieving contradiction. $\square$

Preceding results and Lemma 2.8 imply the following corollary.

Corollary 4.8. *The classes of co-layered permutations $\mathcal{L}_2^c$, $\mathcal{L}_3^c$ and $\mathcal{L}^c$ are not composable.*

4.2 More subclasses of layered permutations

In this section, we explore slightly generalized versions of the class $\mathcal{L}_k$ created by grouping layers of size 1 into one increasing layer. In the rest of this chapter, we refer to both increasing and decreasing layers as blocks as defined in Chapter 2. Let $\mathcal{L}_k^*$ denote the class of permutations which can be written as a sum of at most k blocks. Note that a permutation $\pi \in \mathcal{L}_k^*$ *has l blocks* if it is not a sum of less than l blocks. We introduce one more piece of notation: for classes $\mathcal{A}, \mathcal{B}$ let $\mathcal{A} \oplus \mathcal{B} = \{\alpha \oplus \beta; \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$. We will utilize an additional subclass of $\mathcal{L}$, denoted by $\mathcal{X}_3$ and defined as $\mathcal{X}_3 = \mathcal{I} \oplus \mathcal{D} \oplus \mathcal{I}$.

Theorem 4.9. *$\mathcal{L}_k^*$ is composable for $k \geq 2$.*

Proof. If $k = 2$, let $\mathcal{A} = \mathcal{I} \oplus \mathcal{D}$ and $\mathcal{B} = \mathcal{D} \oplus \mathcal{I}$. Then a permutation $\pi \in \mathcal{L}_2^*$ is either in $\mathcal{A}$ or in $\mathcal{B}$ or there are positive a, b such that $\pi = \delta_a \oplus \delta_b$, in which case $\pi = (\iota_a \oplus \delta_b) \circ (\delta_a \oplus \iota_b)$. Therefore $\mathcal{L}_2^* \subseteq \mathcal{A} \circ \mathcal{B}$ and it is 2-composable.

If $k \geq 3$, then $\mathcal{X}_3 \subsetneq \mathcal{L}_k^*$. Let $\pi \in \mathcal{L}_k^*$ have k (possibly empty) blocks $\pi_1, \pi_2, \ldots, \pi_k$ of sizes $l_1, l_2, \ldots, l_k$, i.e. $\pi = \bigoplus_{i=1}^k \pi_i$. Define $\iota_{<j} = \iota_{l_1+l_2+\cdots+l_{j-1}}$ and $\iota_{>j} = \iota_{l_{j+1}+l_{j+1}+\cdots+l_k}$. Then

$$\bigoplus_{i=1}^{k} \pi_i = (\pi_1 \oplus \iota_{>1}) \circ \cdots \circ (\iota_{<a} \oplus \pi_a \oplus \iota_{>a}) \circ \cdots \circ (\iota_{<k} \oplus \pi_k),$$

therefore $\pi \in (\mathcal{X}_3)^k$. Since π was chosen arbitrarily it follows that $\mathcal{L}_k^* \subseteq (\mathcal{X}_3)^k$. $\square$

The class $\mathcal{L}_k$ contains only sums of decreasing blocks, the class $\mathcal{L}_k^*$ contains all combinations of increasing and decreasing blocks. In the rest of this section we shall explore subclasses of $\mathcal{L}$ with bounded number of blocks for which we prescribe which blocks should be increasing or decreasing.

Let k be a positive integer and $\varphi : [k] \longrightarrow \{\mathcal{I}, \mathcal{D}\}$ a mapping satisfying the condition that two consecutive elements of $[k]$ cannot be both mapped to $\mathcal{I}$. We then define the class $\mathcal{L}_k^\varphi$ as

$$\mathcal{L}_k^\varphi = \varphi(1) \oplus \varphi(2) \oplus \cdots \oplus \varphi(k).$$

In other words, the class $\mathcal{L}_k^\varphi$ contains layered permutations made of up to k increasing or decreasing blocks in the order prescribed by the mapping φ. If two consecutive blocks were increasing, we could join them into one block, hence the condition for φ. Notice that if φ maps all elements of $[k]$ to $\mathcal{D}$, then $\mathcal{L}_k^\varphi = \mathcal{L}_k$.

We begin our exploration of these classes with the case $k = 2$.

Theorem 4.10. *The class $\mathcal{L}_2^\varphi$ is not composable for any φ.*

Proof. This can be easily proved by repeating the proof of Theorem 4.1, but we will show another possible approach here.

We will present this proof for the case $\varphi(1) = \mathcal{D}, \varphi(2) = \mathcal{I}$, i.e. $\mathcal{A} = \mathcal{L}_2^\varphi = \mathcal{D} \circ \mathcal{I}$. As in the proof of Theorem 4.1 we suppose that $\mathcal{A}$ is k-composable and we deduce that there is a sufficiently large n and a permutation $\pi = \delta_n \oplus \iota_n$ such that $\mathcal{A} \subseteq (\mathcal{C})^k$ where $\mathcal{C} = \mathcal{A} \cap \mathrm{Av}(\pi)$.

Let $N \geq 2kn + 2n$ be an even number and $\pi_1, \pi_2, \ldots, \pi_k \in \mathcal{C}$ arbitrary permutations of order N. We will show by induction on k that

$$N/2 - nk \leq (\pi_k \circ \cdots \circ \pi_2 \circ \pi_1)(N/2) \leq N/2 + n(k-1),$$

in other words, permutations of $\mathcal{C}$ can only map $N/2$ constantly far from its value.

If $k = 1$, let $\pi_1 = \delta_a \oplus \iota_b$. If $N/2 > a$, then $\pi_1(N/2) = N/2$ and the statement holds. If $N/2 \leq a$, then $\pi_1(N/2) = a - N/2 = N/2 - b \leq N/2$. Also one of a, b has to be smaller than n and since $a \geq N/2 \geq kn + n$, it must be b, therefore $\pi_1(h) = N/2 - b \geq N/2 - n$.

Now let the statement hold for $1, 2, \ldots, k$ and let $x = (\pi_k \circ \cdots \circ \pi_2 \circ \pi_1)(N/2)$. By induction hypothesis we have that

$$N/2 - nk \leq x \leq N/2 + n(k-1).$$

Let $\pi_{k+1} = \delta_a \oplus \iota_b$. If $x > a$, then $\pi_{k+1}(x) = x$ and we are done. If $a \geq x$, we have to have $a \geq n$, for otherwise $n > a \geq x \geq N/2 - nk$ which implies $N/2 < nk + n$, contradicting the choice of N. Therefore $b < n$ and $\pi_{k+1}(x) = a - x = N - b - x$. Finally we have

$$\pi_{k+1}(x) = N - b - x \leq N - x \leq N - (N/2 - nk) = N/2 + nk$$

and

$$\pi_{k+1}(x) = N - b - x \geq N - n - N/2 - n(k-1) = N/2 - nk \geq N/2 - n(k+1),$$

which finishes the second step of the induction.

We have shown that for sufficiently large N the composition $(\mathcal{C})^k$ contains only permutations which move the middle element by a constant depending only on n and k, but the class $\mathcal{A}$ contains permutations π of order N satisfying $\pi(N/2) = 1$, which is a contradiction. □

Theorem 4.11. *The class $\mathcal{L}_k^\varphi$ is composable for every $k \geq 4$ and for every mapping φ.*

Proof. We distinguish three cases.

(i) φ does not prescribe any increasing blocks. Then $\mathcal{L}_k^\varphi = \mathcal{L}_k$ and the theorem follows from Theorem 4.5.

(ii) φ prescribes at least two increasing blocks, i.e. $\varphi(i) = \varphi(j) = \mathcal{I}$ for $i < j$. Then by definition there is a k such that $i < k < j$ and $\varphi(k) = \mathcal{D}$, therefore $\mathcal{X}_3 \subsetneq \mathcal{L}_k^\varphi$. In this case, we can show that $\mathcal{L}_k^\varphi \subseteq (\mathcal{X}_3)^k$ using the approach of Theorem 4.9.

(iii) φ prescribes exactly one increasing block. As in the second case we easily get that $\mathcal{L}_k^\varphi \subseteq (\mathcal{X}_3)^k$, so it is enough to construct $\mathcal{X}_3$ from proper subclasses of $\mathcal{L}_k^\varphi$.

   Let $\mathcal{A} = \mathcal{D} \oplus \mathcal{D} \oplus \mathcal{I}$ and $\mathcal{B} = \mathcal{I} \oplus \mathcal{D} \oplus \mathcal{D}$. Since $k \geq 4$, the one increasing block is either preceded by at least two blocks, in which case $\mathcal{A} \subsetneq \mathcal{L}_k^\varphi$, or it is followed by at least two blocks, in which case $\mathcal{B} \subsetneq \mathcal{L}_k^\varphi$. Nevertheless we clearly have $\mathcal{X}_3 \subseteq \mathcal{A} \circ \mathcal{A}$ and $\mathcal{X}_3 \subseteq \mathcal{B} \circ \mathcal{B}$, therefore either $\mathcal{L}_k^\varphi \subseteq (\mathcal{A})^{2k}$ or $\mathcal{L}_k^\varphi \subseteq (\mathcal{B})^{2k}$. □

We omitted the case $k = 3$, which remains open for most choices of φ except the obvious case $\mathcal{D} \oplus \mathcal{I} \oplus \mathcal{D} \subseteq (\mathcal{D} \oplus \mathcal{I}) \circ (\mathcal{I} \oplus \mathcal{D})$. We believe that other choices of φ create classes which cannot be composed from their proper subclasses.

Question. *Is the class $\mathcal{L}_3^\varphi$ composable for various choices of φ?*

5. Other results

The final chapter of this work collects several miscellaneous results concerning composability. The first section provides more examples of composable classes, the second section offers an alternative proof of a known upper bound on the growth rate of Av(1324) and the third section presents several additional examples of uncomposable classes.

5.1 Composable principal classes

In this section, we use results of Chapter 3 and of [9] to prove that many classes avoiding a single decomposable pattern (a permutation which can be written as a non-trivial sum of smaller permutations) are composable.

We will base our proof on the following splittability result of Jelínek and Valtr [9].

Lemma 5.1 (Jelínek, Valtr [9]). *Let α, β, γ be three nonempty permutations and let $\pi \in \mathrm{Av}(\alpha \oplus \beta \oplus \gamma)$. Then π can be merged from two sequences $(a)_{i=1}^{n}$ and $(c)_{i=1}^{m}$ such that a avoids $\alpha \oplus \beta$, c avoids $\beta \oplus \gamma$ and for any $i \in [n]$ and $j \in [m]$ either $\pi^{-1}(a_i) < \pi^{-1}(c_j)$ or $a_i < c_j$.*

Theorem 5.2. *If α and γ are any non-empty permutations and $\beta = \delta_n$ for a positive integer n, then*

$$\mathrm{Av}(\alpha \oplus \beta \oplus \gamma) \subseteq (\mathcal{V}(\mathrm{Av}(\alpha \oplus \beta), \mathrm{Av}(\beta \oplus \gamma)) \cap \mathrm{Av}(\alpha \oplus \beta \oplus \gamma)) \circ \mathcal{H}.$$

In particular, $\mathrm{Av}(\alpha \oplus \beta \oplus \gamma)$ is 2-composable whenever $\alpha \oplus \beta \oplus \gamma \notin \mathcal{H}$.

Proof. Let $\mathcal{C} = \mathrm{Av}(\alpha \oplus \beta \oplus \gamma)$, $\mathcal{A} = \mathrm{Av}(\alpha \oplus \beta)$ and $\mathcal{B} = \mathrm{Av}(\beta \oplus \gamma)$. Lemma 5.1 and Corollary 3.4 immediately imply that $\mathcal{C} \subseteq \mathcal{V}(\mathcal{A}, \mathcal{B}) \circ \mathcal{H}$.

Let $\pi \in \mathcal{C}$ be merged from sequences a and c as in Lemma 5.1 and let $\sigma = ac$. We have to show that $\sigma \in \mathcal{C}$. Suppose for a contradiction that σ contains a copy of $\alpha \oplus \beta \oplus \gamma$. Let b the decreasing subsequence of σ representing the occurrence of β. Then b cannot be contained entirely in a or in b since that would create a copy of $\alpha \oplus \beta$ in a or of $\beta \oplus \gamma$ in b. Thus if $\beta = 1$ the contradiction is reached immediately.

If $|\beta| > 1$, we would like to show that b is also a subsequence of π. Assume it is not, therefore there are elements b_i and b_j with $i < j$ such that they appear in reverse order in π. That can only be achieved if b_i is in a and b_j is in c, which together with $b_i > b_j$ contradicts the properties of a and c from Lemma 5.1.

It follows that the entire occurrence of $\alpha \oplus \beta \oplus \gamma$ is also contained in π, which is a contradiction, thus $\mathcal{C} \subseteq (\mathcal{V}(\mathcal{A}, \mathcal{B}) \cap \mathcal{C}) \circ \mathcal{H}$.

To prove that $\mathcal{C}$ is really 2-composable for $\alpha \oplus \beta \oplus \gamma \notin \mathcal{H}$ it remains to verify that $\mathcal{V}(\mathcal{A}, \mathcal{B}) \cap \mathcal{C}$ and $\mathcal{H}$ are proper subclasses of $\mathcal{C}$. For the former consider the permutation $(\alpha \oplus \beta) \ominus (\beta \oplus \gamma)$ which is clearly in $\mathcal{C}$ and not in $\mathcal{V}(\mathcal{A}, \mathcal{B})$, and for the latter recall that by Proposition 1.5 the class $\mathcal{H}$ has a basis of size 3 and therefore it cannot be equal to a principal class. $\square$

Note that for the case $\beta = 1$ we get

$$\mathcal{V}(\mathrm{Av}(\alpha \oplus 1), \mathrm{Av}(1 \oplus \gamma)) \subsetneq \mathrm{Av}(\alpha \oplus 1 \oplus \gamma),$$

and thus we may omit the intersection with $\mathrm{Av}(\alpha \oplus 1 \oplus \gamma)$ in the formula of Theorem 5.2. Indeed, if a permutation is concatenated of two parts, first avoiding $\alpha \oplus 1$ and the second avoiding $1 \oplus \gamma$, such a permutation cannot contain an occurrence of $\alpha \oplus 1 \oplus \gamma$ since one of the two parts would contain the middle 1 and thus the forbidden pattern.

5.2 Growth rate of Av(1324)

In [13], Claesson, Jelínek and Steingrímsson proved one of the first upper bounds on the growth rate of Av(1324), namely they showed that $\mathrm{gr}(\mathrm{Av}(1324)) \leq 16$. We provide an alternative and short proof of this result based on our work in compositions and vertical and horizontal merges.

Theorem 5.3. *The growth rate of* Av(1324) *is at most 16.*

Proof. Plugging $\alpha = 1$, $\beta = 21$ and $\gamma = 1$ into Theorem 5.2 we get that

$$\mathrm{Av}(1324) \subseteq \mathcal{V}(\mathrm{Av}(132), \mathrm{Av}(231)) \circ \mathcal{H}.$$

It is a well-known fact that permutations avoiding a single pattern of length 3 are counted by the Catalan numbers (see for instance Knuth [14, 2.2.1]) and therefore $\mathrm{gr}(\mathrm{Av}(132)) = \mathrm{gr}(\mathrm{Av}(231)) = 4$. Thus, by Lemma 3.8,

$$\overline{\mathrm{gr}}(\mathcal{V}(\mathrm{Av}(132), \mathrm{Av}(231))) \leq 8.$$

Since $\mathrm{gr}(\mathcal{H}) = 2$, as shown in Chapter 1, we get that $\mathrm{gr}(\mathrm{Av}(1324)) \leq 16$ by Lemma 3.7. ☐

5.3 More uncomposable classes

So far we have used classes such as $\mathcal{V}$ or $\mathcal{H}$ to prove that other classes are composable. In this section, we will show that these classes, and classes similar to them, are themselves uncomposable.

We call a permutation $\eta \in \mathcal{H}$ *alternating* if $\eta(2i-1) < \eta(2i) > \eta(2i+1)$ for all possible values of i. We will use the following simple observation about alternating permutations in $\mathcal{H}$.

Observation 5.4. *Every permutation from $\mathcal{H}$ is contained in an alternating permutation from $\mathcal{H}$.*

Theorem 5.5. *The classes $\mathcal{V}$, $\mathcal{V}^c$, $\mathcal{V}(\mathcal{D}, \mathcal{I})$, $\mathcal{V}(\mathcal{I}, \mathcal{D})$, $\mathcal{H}$, $\mathcal{H}^c$, $\mathcal{H}(\mathcal{I}, \mathcal{D})$ and $\mathcal{H}(\mathcal{D}, \mathcal{I})$ are not composable.*

Proof. We will show the proof for the class $\mathcal{H}$, the same approach can be applied to every mentioned horizontal merge and the result is transferred by inversion to the vertical merges according to Corollary 2.7.

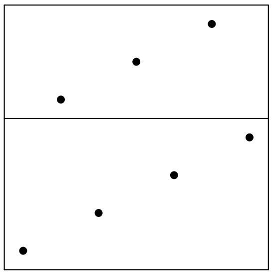


Figure 5.1: The alternating permutation of length 7

Suppose that $\mathcal{H}$ is composable from its proper subclasses $\mathcal{C}_1, \ldots, \mathcal{C}_k$. Each of $\mathcal{C}_1, \ldots, \mathcal{C}_k$ does not contain some permutation of $\mathcal{H}$, thus according to Observation 5.4 there is an alternating permutation $\eta \in \mathcal{H}$ such that $\mathcal{C}_i \subseteq \mathrm{Av}(\eta)$ and therefore if $\mathcal{C} = \mathrm{Av}(\eta) \cap \mathcal{H} \subsetneq \mathcal{H}$ we have $\mathcal{H} \subseteq (\mathcal{C})^k$.

Any permutation $\pi \in \mathcal{C}$ is merged from two sequences a and b of consecutive integers. We label elements of π by a or b depending on which sequence they belong to. A sequence of elements with alternating labels forms a copy of an alternating permutation in π. The length of the longest sequence of alternating labels in π is thus limited by a constant N determined by the order of η, thus π can be broken into at most N contiguous parts each having one label. Since elements labeled with a single label form a sequence of consecutive integers, this implies that π is in fact an N-block. Since the choice of π was arbitrary, every permutation of $\mathcal{C}$ is an N-block and by Lemma 2.4 every permutation of $(\mathcal{C})^k$ is an (N^k)-block. But a long enough alternating permutation from $\mathcal{H}$ is not an (N^k)-block, therefore $\mathcal{H} \not\subseteq (\mathcal{C})^k$ and the proof is finished. $\square$

Conclusion

This thesis studies the previously unexplored concept of composability of permutation classes. Given a permutation class, our main goal is to show, how it can be constructed using smaller permutation classes and the composition operator, or to prove that this cannot be done. Throughout the paper, we present both types of results.

On the positive side, Theorems 3.5 and 3.6 show two distinct ways of constructing the class $\mathrm{Av}(k \cdots 21)$, Theorems 4.5, 4.9 and 4.11 provide examples of many classes of layered patterns which can be constructed from simpler subclasses and Theorem 5.2 shows that many principal classes avoiding a decomposable pattern are composable.

On the negative side, in Theorems 4.1, 4.2, 4.4 and 4.7 we present four different classes of layered patterns which cannot be constructed from any number of proper subclasses using composition, and in Theorem 5.5 we provide 8 more examples of uncomposable classes.

Composability is similar to splittability in that both these properties describe how a bigger class is built from smaller ones. We do not know whether these two properties are somehow connected; however, our research suggests that this may be the case, since every composable class we have found so far is also splittable. We have found examples of splittable yet uncomposable classes, namely the classes $\mathcal{L}_2$ and $\mathcal{L}_3$ introduced in Chapter 4. The class of all layered permutations is an example of a both uncomposable and unsplittable class. The last case remains open and we pose it as a question for future work.

Question. *Is there a permutation class which is composable and unsplittable?*

Our work may find applications in enumerating permutation classes. We have shown how to use composability to estimate growth rates by re-deriving known upper bounds on growth rates of classes $\mathrm{Av}(k \cdots 21)$ and $\mathrm{Av}(1324)$ in Theorems 3.10 and 5.3 respectively. We believe that by refining our ideas, more and tighter bounds could be found. The current lowest upper bound of 13.74 on $\mathrm{gr}(\mathrm{Av}(1324))$ is due to Bóna [15] and the highest lower bound of 9.81 is due to Bevan [16].

Question. *Is the class* $\mathrm{Av}(1324)$ *composable from two or more classes small enough to give a tight upper bound on its growth rate?*

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