Using algebra in geometry

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1 Abstract

In this thesis, we develop a technique that combines algebra, algebraic topology and combinatorial arguments and provides non-embeddability results. The novelty of our approach is to examine non-embeddability arguments from a homological point of view. We illustrate its strength by proving two interesting theorems.

The first one states that k-dimensional skeleton of $\left(b\binom{2k+2}{k} + k + 3\right)$-dimensional simplex does not embed into any $2k$-dimensional manifold M with Betti number $\beta_k(M; \mathbb{Z}_2) \leq b$. It is the first finite upper bound for Kühnel's conjecture of non-embeddability of simplices into manifolds.

The second one is a very general topological Helly type theorem for sets in $\mathbb{R}^d$: There exists a function $h(b, d)$ such that the following holds. If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ such that $\tilde{\beta}_i \left(\bigcap \mathcal{G}; \mathbb{Z}_2\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$, then $\mathcal{F}$ has Helly number at most $h(b, d)$. If we are only interested whether the Helly numbers are bounded or not, the theorem subsumes a broad class of Helly types theorems for sets in $\mathbb{R}^d$.

Keywords:

Homological Non-embeddability, Helly Type Theorem, Kühnel's conjecture of non-embeddability of skeleta of simplices into manifolds