Charles University in Prague

Faculty of Social Sciences

Institute of Economic Studies

DISSERTATION THESIS

Estimation of Financial Agent-Based Models

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Academic Year: **2015/2016**

Acknowledgments

I am especially indebted to my supervisor Dr. Jozef Barunik, whose research enthusiasm, never-ending personal support, and willingness to devote his attention to my research questions and discuss my work almost at any time steered my effort in the right direction and opened for me the door into the amazing world of science and academia. No words written can fully express my warmest gratitude. Many many many thanks, Jozo!

This work has benefited much from the exhaustive and thoughtful suggestions and detailed comments by the thesis opponents Dr. Gerba (LSE), Dr. Vacha (CUNI, CAS), and Dr. Zwinkels (VU Amsterdam). I am honoured that they agreed to complete my Ph.D. committee.

The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement No. FP7-SSH- 612955 (FinMaP). Support from the Czech Science Foundation under the P402/12/G097 DYME - 'Dynamic Models in Economics' project and from the Grant Agency of Charles University under the 588912 and 192215 projects is gratefully acknowledged.

Research related to this dissertation thesis has been regularly presented at scientific events such as WEHIA 2012–2015, CEF 2012 and 2015, CFE 2012, 2014, and 2015, Latsis Symposium 2012, Bordeaux-Milano Agent-Based Modelling events 2013–2015, AESCS 2013, and Econophysics Colloquium 2014 and 2015. I also gained lots of inspiration from participating at WEHIA-related Summer Schools, 4^{th} Summer School of the European Social Simulation Association 2013, Agent-Based Modelling teaching course 2014, CEF 2015-related workshops on ABM and Complexity, and the First Ancona-Milano Summer School on AB Economics 2015. I truly appreciate the time and effort devoted by organizers of these events and I wish to thank discussants for many constructive remarks. Developing working results were periodically debated during the doctoral seminar Nonlinear Dynamic Economic Systems. Honest thanks to my colleagues, especially opponents of my presentations Lucie, Jan, Tomas A., and Tomas K., for all their comments. My warm acknowledgements also head to the editors and anonymous referees who provided me unselfishly with valuable suggestions. All errors contained herein are mine and mine alone.

Last but definitely not least, I am grateful to my parents for their caring support, without which I cannot even imagine my entire decade of university studies, and to my beloved one Veru for her tolerance within periods of writing and her kind understanding during my ‘home-office’ research weekends.

Thank you!

Bibliographic Record

KUKACKA, J. (2016): *Estimation of Financial Agent-Based Models.* Dissertation thesis, Charles University in Prague, Faculty of Social Sciences, Institute of Economic Studies, pages 210. Supervisor: Dr. Jozef Barunik.

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Typeset in $\LaTeX\,2_\varepsilon$ using the IES Thesis Template by Dr. Tomas Havranek.

Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature. The thesis was not used to obtain an academic degree before.

Chapter 3 has already been published under the full title "Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility" in *Quantitative Finance*, 2015, 15 (6), pp. 959-973. It is a joint work with the thesis supervisor Jozef Barunik and both authors contributed equally to this work. The rest of the (yet unpublished) dissertation thesis text was composed solely by Jiri Kukacka under the standard academic guidance of the thesis supervisor.

The approximate contribution to the text of the dissertation thesis can be divided to 90% by Jiri Kukacka and 10% by Jozef Barunik. There are no other co-authors, collaborators, or students involved.

Prague, February 26, 2016

Signature

Abstract

This thesis proposes computational framework for empirical estimation of Financial Agent-Based Models (FABMs) that does not rely upon restrictive theoretical assumptions.

First, we develop a two-step estimation methodology for one of the historically first FABMs—the stochastic cusp catastrophe model. Our method allows us to apply catastrophe theory to stock market returns with time-varying volatility and to model stock market crashes. The methodology is empirically tested on nearly 27 years of U.S. stock market returns. We find that the U.S. stock market's downturns were more likely to be driven by the endogenous market forces during the first half of the studied period, while during the second half of the period, the exogenous forces seem to be driving the market's instability. The results suggest that the proposed methodology provides an important shift in the application of catastrophe theory to stock markets.

Second, we customise a recent methodology of the Non-Parametric Simulated Maximum Likelihood Estimator (NPSMLE) based on kernel methods by Kristensen & Shin (2012) and elaborate its capability for FABMs estimation purposes. To start with, we apply the methodology to the most famous and widely analysed model of Brock & Hommes (1998). We extensively test finite sample properties of the estimator via Monte Carlo simulations and show that important theoretical features of the estimator, the consistency and asymptotic efficiency, also hold in small samples for the model. We also verify smoothness of the simulated log-likelihood function and identification of parameters. Main empirical results of our analysis are the statistical insignificance of the switching coefficient β but markedly significant belief parameters defining heterogeneous trading regimes with an absolute superiority of trend-following over contrarian strategies and a slight proportional dominance of fundamentalists over trend following chartists.

Finally, we apply the NPSMLE to a stylised herding FABM developed by Alfarano *et al.* (2008). Empirical estimates of parameters governing opinion switching indicate unimodal distribution of the market sentiment variable. Model behaviour is thus characterised by a general tendency to gradually revert back to a balanced sentiment and theoretically expected performance of the estimator. Rolling window estimation reveals interesting model dynamics and clearly captures jumps in the 'herding-based' opinion switching parameter and elevated fundamental volatility in turbulent times.

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Acronyms

AA	Autonomous Agents
AB	Agent-Based
ABM	Agent-Based Model
ABS	Adaptive Belief System
ACF	Agent-Based Computational Finance
ANT	Ant type of system
AR	Autoregressive
ARCH	Autoregressive Conditional Heteroscedasticity
ARED	Adaptive Rational Equilibrium Dynamics
ARIMA	Autoregressive Integrated Moving Average
CDS	Credit Default Swap
CHF	Swiss Franc
CSI 300	Shanghai Shenzhen CSI 300 Index
CT	Catastrophe Theory
DAX	Deutscher Aktien IndeX (German stock index)
DJIA	Dow Jones Industrial Average
DM	Deutsche Mark (former German currency)
DSGE	Dynamic Stochastic General Equilibrium
EMH	Efficient Market Hypothesis
EMM	Efficient Method of Moments
EMS	Empirical Martingale Simulation
EUR	euro (€)
FABM	Financial Agent-Based Model
FOREX	Foreign Exchange

FX Foreign Exchange

FTSE Financial Times Stock Exchange 100 (London Stock Exchange index)

GARCH Generalized Autoregressive Conditional Heteroscedasticity

GMM Generalized Method of Moments

HA Heterogeneous Agent

HAM Heterogeneous Agent Model

HSI Hang Seng Index (Hong Kong stock market index)

IAH Interactive Agent Hypothesis

IEC Interactive Evolutionary Computation

IMF International Monetary Fund

i.i.d. independent identically distributed

JY Japanese Yen (¥)

MA Moving Average

MCMC Markov Chain Monte Carlo

ML Maximum Likelihood

MLE Maximum Likelihood Estimator

MM Method of Moments

MSM Method of Simulated Moments

NASDAQ National Association of Securities Dealers Automated Quotations Composite Index

NLS Nonlinear Least Squares

NPSML Non-Parametric Simulated Maximum Likelihood

NPSMLE Non-Parametric Simulated Maximum Likelihood Estimator

NIKKEI 225 Nikkei 225 (Tokyo Stock Exchange index)

OLS Ordinary Least Squares

QML Quasi Maximum Likelihood

RE Rational Expectations

RV Realised Volatility

RW Random Walk

SMD Simulated Minimum Distance

STAR Smooth Transition Autoregressive

S&P500 Standard & Poor's 500 Index

USD United States Dollar ($)

U.S. United States

VAR Vector Autoregression

VECM Vector Error Correction Model

WTI West Texas Intermediate

Chapter 1

Introduction

Financial markets are one of the fundamental motivative forces of the economic development but the Global Financial Crisis pointed again at the deficiency of knowledge of how this important segment of the global economy works. After the failure of traditional financial models in the Global Financial Crisis of 2007–2008, the Agent-Based (AB) approaches in Finance denoted as Financial Agent-Based Models (FABMs)[1] have attracted attention both of academicians as well as practitioners and hence gradually replace traditional financial models in the recent financial literature. This advancement emphasises that although the serious macroeconomic consequences of market fluctuations are worldwide, the essence of problems remains at the level of individual market agents with their heterogeneous expectations. The FABMs reflect this well documented and systematic human departure from the representative agent's full rationality towards reasonably realistic bounded, limited rationality (Simon 1957). An essential achievement of the FABM methodology is the ability to replicate so called stylised facts of financial data[2] and account for emergence of asset market bubbles followed by sudden crashes. Neither observed empirical regularities, nor explosive bubbles (Evans 1991) can be reasonably explained by traditional

[1]The notation 'Financial Agent-Based Model' (FABM) is a more general version of a term 'Heterogeneous Agent Model' (HAM). In fact, both terms are essentially equivalent, but FABM seems better understandable within the markedly diverse community of economists. In this thesis, both terms are used interchangeably according to the context. The notation HAM has been anchored after publication of a seminal paper by Brock & Hommes (1998) when the agent-based approaches in Economics were focused mainly on the field of Finance. Therefore, it seems reasonable to terminologically distinguish between financial and e.g. macroeconomics Agent-Based Models (ABMs). For a general overview of the financial agent-based modelling and its development, Chen *et al.* (2012), Hommes (2006), or LeBaron (2006) provide excellent surveys.

[2]A term coined by Kaldor (1961, pg. 178) as view of the facts concentrated "on broad tendencies, ignoring individual detail", for comprehensive surveys consult Cont (2001; 2007).

financial models. Recently, number of projects propose a courageous attempt to complement or even alternate current mainstream policy making approaches through the use of ABMs, typically at the level of central banks. For this to happen, it is, however, essential to estimate these models on the empirical data in order to use them for forecasting.

Traditional models in Economics and Finance are based on the hypothesis of Rational Expectations (RE) (Muth 1961; Lucas 1972) and approximation of market population by a representative agent. Under RE, agents form expectation using all available information, however, they may be individually incorrect. Nonetheless, agents must not be systematically biased, i.e. the forecasting errors agents make must be random. The representative agent, which notion dates back to Edgeworth (1881), thus behaves in a perfectly rational (i.e. model consistent) manner according to solution of a maximisation problem under full information (involving also information about behaviour of all other agents) and no computational constraints. Especially in Finance, mostly simple linear, stable equilibrium models driven by exogenous random news about fundamentals have been developed under this paradigm. A 'textbook example' is the Capital Asset Pricing Model (CAPM, e.g. Sharpe 1964). RE is a necessary condition for the striking Efficient Market Hypothesis (EMH) (Fama 1970), dominating the filed in the past, according to which asset prices reflect all relevant information about economic fundamentals available to economic agents. As a consequence, securities prices follow Random Walk (RW). Irrational traders thus in average receive lower profits that rational agents and in the process of the 'evolutionary market pressure' are driven out of the market, a statement called the 'Friedman Hypothesis'.

Despite many criticisms, RE constitute an important milestone in the history of economic modelling as it opened a mathematically elegant way to a controversial problem how to model expectations in models studying large number of agents (individuals, firms) facing uncertainty. Put simply, RE guarantee internal mathematical consistency of aggregate dynamic stochastic models in Economics, with a high level of theoretical coherence, that can be utilised as a 'lab' for policy experiments. Macroeconomic models are largely founded on assumption of perfect competition, prioritising analytical tractability at the expense of empirical evidence (LeBaron & Tesfatsion 2008). "Potentially important real-world factors such as incomplete markets, ...strategic behavioural interactions, and open-ended learning that tremendously complicate analytical formulations are typically not incorporated" [pg. 246]. But hence, on the other hand, these simplifications allow for unambiguous theoretical conclusions and strong policy implication. RE provide a conceptual solution to a variety of theoretical economic problems in which the future expectations determine current behaviour. RE Hypothesis has largely influenced dynamic macroeconomic research as it constitutes a crucial building block of many macroeconomic concepts: e.g. the 'Permanent Income Hypothesis' (Friedman 1957), alternative 'life-long' theories of consumption, or short-term monetary, fiscal, and regulatory macroeconomic stabilization policies. Especially for such kind of a robust large-scale policy modelling, RE is often considered to be a sufficiently good approximation tool. Actually, as claimed by Levine *et al.* (2007, pg. 2), "forecasting of rational agents' behaviour has been seen as a step in resolving the 'Lucas Critique' issues (Lucas 1976).

This thesis focuses on the field of Agent-Based Computational Finance (ACF) that has experienced an extensive development during the last three decades. The departure of FABMs from the RE paradigm has proceeded from the 1980s ensued by first macroeconomic ABMs from the 1990s. Recently, many Macro ABMs have been developed sharing similar modelling concepts with FABMs but also following the Dynamic Stochastic General Equilibrium (DSGE) literature as many challenges within these two fields overlap. A rapid development of Macro ABMs was substantially accelerated by events in 2008, known as the Global Financial Crisis of 2007–2008, followed by the period of so called 'Great Recession'. Fagiolo & Roventini (2012, pg. 67, 69) comment that "the Great Recession seems to be a natural experiment for macroeconomics showing the inadequacy of the predominant theoretical framework—the New Neoclassical Synthesis—grounded on the DSGE model" and draw attention to the fact that "an increasing number of leading economists claim that the current economic crisis is a crisis for economic theory". Canova *et al.* (2014, pg. 1029) argue that "linear Gaussian specifications [of DSGE models][3] are inadequate to describe the 2008–2009 Great Recession, the subsequent episode of zero nominal interest rates and the events during the subsequent sovereign debt crisis in Europe".

However, turning almost three decades back, yet in late 1980s and early 1990s, empirical micro studies reported heterogeneity as an empirically significant phenomena, suggested various methods how market agents might form beliefs about the future, and revealed quantitatively compelling evidence for heterogeneity in preferences, skills, discount rates, risk aversion parameters,

[3] A note added by the authors.

and many other aspects—e.g. Branch (2004, pg. 592) summarises studies documenting "failure of the RE Hypothesis to account for survey data on inflationary expectations" and Hansen & Heckman (1996, pg. 101) indicate a "considerable interest in heterogeneous agent models in the real business cycle literature research…to narrow the range of specification errors in calibrating with microeconomic data". Brock & Hommes (1997; 1998) theoretically prove that it may be individually 'rational' for agents not to follow RE—and instead to behave according to simple predictors—in situations when acquiring information implicates additional costs or in periods when markets are populated by a majority of 'non-RE' traders. Heterogeneity thus arises when agents evaluate benefits of costly RE compared to simple costless rules resulting in time-varying distribution of market participants across the set of all possible trading strategies. Browning *et al.* (1999) claim that heterogeneous preferences can be considered as a "major finding of modern microeconomic data analysis", draw attention to the fact that macroeconomics models are often incompatible with the empirical microeconomic evidence, and suggest exploring DSGE models with explicit heterogeneity. Evans & Honkapohja (2001) explain that agents lack the required sophistication to form expectations rationally and Sims (2003, pg. 687), the author of the 'Rational Inattention Theory', supports his work by saying that RE "that postulate a common information set for all agents at all times imply quick, error-free reactions of all prices and all kinds of agent behaviour to every kind of new information" and thus contrast strongly with the empirical data. Branch (2004, pg. 592, 594, 620) summarises another frequently criticised drawback of RE, namely that the RE Hypothesis "requires agents to possess too much knowledge", i.e. it assumes agents to have full information about the true structure of the economy and all related probability distributions. The author aptly argues that "even econometricians must approximate the true structure of the economy" and are not able to estimate models perfectly. It is therefore not realistic to expect all market agents to possess this kind of ability, however, the empirical rejection of RE in survey data does not mean that agents act 'blindly irrationally' using completely random rules. It is rather optimal for them not to invest too much resources and base their behaviour on simple 'rules of thumb' trading strategies. The author concludes the discussion about the RE Hypothesis stating that expectations of agents "are boundedly rational and consistent with optimising behaviour"—i.e. that each temporary trading strategy choice is optimal for themselves—and suggests a concept of 'Rationally Heterogeneous Expectations' motivated by Adaptive

Rational Equilibrium Dynamics (ARED) by Brock & Hommes (1997). Empirical data support his model, "but not completely so". Branch & Evans (2006, pg. 265) point out that in practice "econometricians often misspecify their models. . . If agents are expected to behave like econometricians then they can also be expected to misspecify their models". Authors confront agents with a set of misspecified underparameterised models and let them to determine boundedly correct, however, optimally computed, parametrisation. Under this specific setting, they propose a new concept of the 'Misspecification Equilibrium' that exhibits so called 'Intrinsic Heterogeneity'. Crucially importantly for economic policy making, a study of Branch & McGough (2004) finds that "if policy makers unwittingly assume agents have rational expectations they may destabilize an already stable system" and Branch & McGough (2009, pg. 1048) reveal that "whether heterogeneity stabilizes or destabilizes depends on the distribution of agents across rational and adaptive expectations, and how strongly agents project past data in the adaptive predictor". Carroll (2003) and Mankiw *et al.* (2004) draw attention to statistically significant disagreement in survey data on inflation expectation even among professional economists and to the evolution of the level of heterogeneity reflecting market volatility. Branch (2007, pg. 246) stresses limitations of RE and asserts that "recent approaches impose bounded rationality at the primitive level". Branch & McGough (2010, pg. 1497) argue that incorporating full rationality and perfect foresight of agents into decision-making "motivates the literature's assumption that agents treat the forecasting. . . issue as a statistical problem distinct from their optimization" and remark that the choice between rational and adaptive behaviour is the ideal modelling "scenario discussed extensively in the monetary policy literature". Finally, Levine *et al.* (2012), partially quoting Evans & Honkapohja (2009), simply state that "economic agents should be assumed to be about as smart as, but no smarter than good economists".

AB approaches in Economics thus departure from models with perfectly rational representative agent[4] and model-consistent homogeneous expectations in reaction to unrealistic assumptions of the RE paradigm, equilibrium conditions, lack of microeconomics foundation when applying the RE Hypothesis in macroeconomic research,[5] and the inability of asset pricing models derived from

[4]An important early criticism of the representative agent paradigm is provided by Kirman (1991).

[5]So called 'Aggregation Problem' refers to a theoretical fact that the assumption of rationality at the individual level does not imply aggregate rationality (e.g. Janssen 1993). Fagiolo & Roventini (2012) correctly point out that "RE is a property of the economic system as a

the EMH to replicate empirically observed stylised facts and explain speculative bubbles. This modern approach builds on direct interactions of boundedly rational economic agents (Simon 1955; 1957; Sargent 1993) with limited cognitive and information processing capacities, disposing insufficient computational power, and incomplete information. Nonetheless, agents do not act irrationally, but follow simple behavioural heuristics, that may be the most 'rational' choice given objective constrains they face and costs of gathering information. Agents' actions are not solutions of a maximisation problems, but are selected according to adaptive updating rules and their relative profitability. According to Branch (2004, pg. 592), they in fact "behave as if they were econometricians". Agents are assumed to behave according to psychological and sociological evidence to better reflect the real world phenomena, i.e. ABMs often embrace findings from market psychology (e.g. Kahneman & Tversky 1974; 1979) and herding behaviour (Keynes 1936). Dynamics of these economic systems is not generated via exogenous shock mechanisms but prices are driven endogenously based on boundedly rational expectations of agents resulting in direct interactions. Any equilibrium condition is not required, that means, markets may be found even continually out of equilibrium without violating model assumptions. Another important theoretical viewpoints in favour of ABMs is revealed e.g. by Browning *et al.* (1999) who remark that representative preferences mostly cannot govern model behaviour asymptotically, or by Fagiolo *et al.* (2008) who rightly point out that RE prevent models to address distributional issues in situations when many macroeconomic time series distributions can be well-approximated by fat tail densities.

Although the notion of market heterogeneity is intellectually satisfying, researchers should consider so called 'endogeneity problem of heterogeneity' addressing the question how much heterogeneity truly exists and how much might be—possibly artificially—imposed by scientists. Or, conversely, is the level of heterogeneity within a model sufficient enough to describe essential features of the reality reasonably well? To what extent do we benefit from finely differentiated agents and what level of detail is technically optimal for extracting highly aggregate information from empirical data?

Real market investors differ in a large number of aspects. W.r.t. practical issues of the AB modelling, the crucial are following: expectation concepts forming specific trading strategies, attitudes to risk, sources, completeness,

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whole, individual rationality is not a sufficient condition for letting the system converge to the RE fixed-point equilibrium (Howitt 2012)".

and costs of information, and in more detailed models for instance also the investment scale, possibly different types of interactions, memory and various learning capabilities, etc. Another essential modelling concept defining the overall complexity of models and directly influencing interactions of agents is the organisation of the artificial market and resulting rules for the price formation, ranging from a simple weighted average of beliefs (Brock & Hommes 1998), through the microeconomic concept of the 'Walrasian Auctioneer' (e.g. in De Grauwe & Grimaldi 2006b), ended up with limit order book systems aiming at replicating the real market trading dynamics architecture.

Design of FABMs is to a large extent motivated by empirical evidence on behaviour of real financial agents accumulated at the turn of 1980s and 1990s (Allen & Taylor 1990; Frankel & Froot 1990). These studies conclude that interactions of the two main types of expectations govern the dynamics of financial markets. So called fundamental traders, who believe that possible mispricing is likely to be corrected over short periods by arbitrageurs and the market price thus tends to revert to its fundamental value, characterise a stabilising market force. Technical analysts, often called chartists, who believe that a currently observable trend will continue also in the short-run, constitute a destabilising market force responsible for emergence of speculative bubbles. These trader-types might be rather understood as possible trading strategy-types as an intelligent market agent is likely to adapt his or her strategy over time based on its relative historical performance. The time-varying evolution of market fraction between these two investor-types is thus an essence of many artificial markets. In the seminal Brock & Hommes (1998) discrete-choice model this is embodied via a switching parameter of the intensity of choice defining the overall willingness of market agents to switch between potential trading strategies. 'N-type models', in which the autonomy of agents is constrained by a predetermined class of strategies, i.e. models consisting of fundamentalists and few types of chartistic strategies, have been found successful in mimicking many financial stylised facts (Chen *et al.* 2012). By virtue of a relatively simple design, especially the 2-type and 3-type versions have been subject of empirical estimation so far and therefore occur in the spotlight of this thesis. Compared to so called Autonomous Agents (AA) models[6] with rich system complexity, advanced individual learning, and idiosyncratic distributions of uncertainty, the heterogeneity in these simple models is restrained: first, the agents are

[6] A typical representative is the 'Santa Fe Artificial Stock Market Model' (Holland & Miller 1991; Palmer *et al.* 1994).

in fact homogeneous ex ante, but randomness of their stochastic choices gives rise to evolving heterogeneity in time; second, the evolutionary selection of strategies is based only on social, not individual learning. In case of the 2-type, 3-type, or generally 'N-type' models, we therefore believe that the phenomenon of supra-imposed artificial heterogeneity does not play an important role, actually, researcher rather balance on the other side of the problem when designing relatively trivial models that account only for the most robust heterogeneous features of real markets.

Although the empirical estimation is an important part of the modelling cycle and seems crucial for model validation, one cannot find many attempts on empirical estimation of FABMs. Moreover, looking ten years back in the financial literature, we neither observe any general consensus on the estimation methodology, nor conclusive results. Fagiolo *et al.* (2007, pg. 202) even emphasise "no consensus at all about how (and if) AB models should be empirically validated". Generally, there are two essential difficulties, or rather challenges, in estimating the FABMs. First, a highly nonlinear and complex nature of these systems prohibits researchers of using classical estimation methods as the objective function often has no analytical expression. Second, a possible overparametrisation, high number of degrees of freedom, and optional model settings together with the stochastic dynamics further escalate the complexity of the problem. The emerging properties of these models cannot be analytically deduced, a Method of Moments (MM), "while fine in theory, might be too computationally costly to undertake" (LeBaron & Tesfatsion 2008, pg. 249), and thus a considerable simulation capacity for the numerical analysis is required.

This thesis makes a step forward and proposes more general methodological framework for empirical validation of FABMs. First, we develop an original two-step estimation methodology for one of the very first FABMs—the stochastic cusp catastrophe model. Although theoretical research regarding the cusp catastrophe model is broad, there is a minimum of empirical applications, especially for stock markets. Our method allows us to apply catastrophe theory to stock market returns with time-varying volatility and to model stock market crashes. In the first step, we utilise high-frequency data to estimate daily realised volatility from returns. Then, we apply stochastic cusp catastrophe to data normalised by the estimated volatility in the second step to study possible discontinuities in the markets. We support our methodology through simulations in which we discuss the importance of stochastic noise and volatility in a deterministic cusp catastrophe model. The methodology is empirically tested on nearly 27 years of U.S. stock market returns covering several important recessions and crisis periods.

Second, we propose a general computational framework for empirical validation of full-fledged FABMs. We base the estimation methodology on a recently developed Non-Parametric Simulated Maximum Likelihood Estimator (NPSMLE) by Kristensen & Shin (2012). The main advantage of this framework is that for many FABMs one cannot analytically derive the likelihood function to estimate the model parameters via Maximum Likelihood Estimator (MLE). However, the observations from the model can be numerically simulated and utilised for the kernel estimation of the conditional density of the data-generating process. Thus, the likelihood function can be replaced by the simulated likelihood. NPSMLE is an estimation framework that functions under very general conditions met by many FABMs. Hence its theoretical properties hold and it can be transferred to the FABM literature. Indeed, recently Grazzini & Richiardi (2015, pg. 151) suggest to employ the NPSMLE methodology on ABMs in general. We extensively test capability of the method for the FABMs estimation purposes on particular models via a complex Monte Carlo analysis. To start with, we apply the methodology to the most famous and widely analysed model of Brock & Hommes (1998) for which we customise the general framework of Kristensen & Shin (2012). The key feature of the model is an evolutionary switching of agents between simple trading strategies based on past realised profits—so called Adaptive Belief System (ABS)—governed by the switching parameter of the intensity of choice β. This parameter is responsible for high nonlinearity of the system and possibly chaotic price motion. We presuppose that if the NPSMLE method succeeds to estimate this generally challenging FABM framework and the switching parameter β, it is likely to appear more general and useful for other ABMs in the future.

Finally, we further apply the NPSMLE to another simple financial ABM developed by Alfarano *et al.* (2008). The model is based on asymmetric herding towards investment strategies and can function under two different modes of the market sentiment. The bimodal version of the model is characterised by abrupt changes in majority opinion and by extreme sentiment dynamics. The unimodal setting reveals general tendency to gradually revert back to the balanced sentiment and fluctuate around the mean value. The possible bimodality of the model generally introduces a serious challenge for the estimation procedure. The model is estimated using three stock market indices, price of gold in USD, and three exchange rates, and both the full sample static estimation as well as the rolling window approach is utilised.

The thesis is organised as follows. After the Introduction, in Chapter 2 we provide a literature survey on FABMs and DSGE estimation methods, an innovative categorisation of findings, and a conceptual comparison between calibration and estimation. Next, Chapter 3 develops a two-step estimation methodology for one of the first FABMs—the cusp catastrophe model. Chapter 4 introduces theoretical background of the NPSMLE method by Kristensen & Shin (2012) and describes the Brock & Hommes (1998) Heterogeneous Agent Model (HAM) framework. Findings of the Monte Carlo simulation study of NPSMLE application to the HAM are reported in Chapter 5 and Chapter 6 presents empirical estimation results. The penultimate Chapter 7 is devoted to another application of the NPSMLE method to a stylised herding FABM by Alfarano *et al.* (2008). Finally, Chapter 8 draws overall conclusions and presents our thoughts about the future of a broader field of AB modelling in Economics and empirical estimation of FABMs.

Chapter 2

Literature review: methods and results

Over last decades, a large number of various Heterogeneous Agent Models (HAMs) have been developed and analysed. However, although the empirical estimation is an important validation part of the modelling cycle, one cannot find many examples on empirical estimation of HAMs using empirical data—typical HAM studies mostly employ simulation techniques to confirm ability to replicate stylised facts of financial data. Additionally, only several of those attempts provide a rigorous comparison of forecasting performance or in terms of fitting empirical market data with 'mainstream' approaches such as ARIMA, GARCH 'family' or other 'competing' econometrics models. In existing empirical papers, estimation methods are often chosen ad hoc or the models are ex ante designed or substantially simplified in a way that a particular estimation method can be used. For the reason, as de Jong *et al.* (2010, pg. 1653) point out: "although the heterogeneity of agents approach is intellectually satisfying, the heterogeneity model has hardly been estimated with empirical financial data because of the non-linear nature of the model that mainly arises from the existence of the mechanism that governs the switching between beliefs". Furthermore, Westerhoff & Reitz (2005, pg. 642) highlight the fact that "one has to sacrifice certain real-life market details. If the setup is too complicated, econometric analysis is precluded". In any case, since the complexity of HAMs often does not allow for analytical solutions, the empirical validation of agent-based systems together with simulation analyses remain the crucial tools of HAMs verification.

2.1 Estimation vs. calibration

In one of the most recent contributions, Grazzini & Richiardi (2015, pg. 148) pointedly remark that empirical grounding of ABMs "is often limited to some ad hoc calibration of the relevant parameters; this resembles the state of the art in DSGE modelling a few years ago, which has now moved forward toward more formal estimation". Authors argue that while recent advancements in the DSGE literature favour proper estimation over calibration dominating the DSGE field in the past, the ABMs "are still lagging behind" [pg. 149]. Whilst we agree with the authors that empirical validation of ABMs via estimation is an essential step in the modelling cycle and an important premise for models' comparison and resulting policy recommendations, we have to inherently acknowledge the role and importance of calibration not only as a starting point of ABM empirical analysis or a passed 'evolutionary' step but as a complementary—rather than alternative—method composing an optimal toolkit for ABMs validation procedures and sensitivity analyses. Actually, to quote from Hansen & Heckman (1996, pg. 91) "the distinction drawn between calibrating and estimating the parameters of a model is artificial at best" and the justification of the two terms is vague, confusing, and in part arbitrary based on the scientific field and individual authors. Truly, for estimation of DSGE models the Bayesian methods[1] are currently commonly used and priors are often set very narrow. As reported e.g. by Fernández-Villaverde (2010, pg. 36), "some researchers prefer to select loose priors and let the likelihood dominate the posterior as much as feasible. Other researchers favour tighter priors that sharpen our inference and guide the posterior to plausible regions". The estimation then strongly resembles calibration and there is not much difference between these otherwise different concepts. It is also evident that the evolution of the DSGE methodology in the last decade constitutes a crucial source of knowledge for current development of validation methods for ABMs as many challenges clearly overlap. Issues discussed below are thus shared also by the ABM field to the similar extent as the subject matter.

2.1.1 Calibration

Starting from the older practice, the main benefit of calibration is that it allows a researcher to study a model consistently. Via calibration a theoretical

[1] A pioneering employment of Bayesian techniques in ABMs is Grazzini *et al.* (2015).

model is specified and can be repeatedly simulated. The calibration might be based on an external knowledge, e.g. related literature comprising specific micro-studies[2] regarding individual parameters or previous studies focused on the same modelling framework. Another option is a calibration founded on a relevant theory or some empirical criteria. The output of simulations can then be compared with empirical data, mostly various stylised facts of financial or macroeconomic data are compared with patterns and regularities in the simulation outputs. Another benefit of calibration is that it avoids difficulties related to imperfect estimates. This econometric issue may arise from many potential sources comprising problematic identification of parameters, possible estimation bias, lack of data, unmet theoretical assumptions, misspecification of the theoretical model, etc. On grounds of the aforementioned benefits, one might consider simulation studies based on calibration more credible and scientifically valid.

On the other hand, for many reasons, calibration might be viewed further away from trustworthiness than estimation. First, as aptly mentioned by Hansen & Heckman (1996), the process of calibration in Economics is often different than in natural or technical sciences where precise data are available from experiments or measurements. This can simply lead to interdisciplinary misunderstandings. Originally, "calibration referred to a graduation of measurement instruments" as a Celsius thermometer (Kydland & Prescott 1996, pg. 74). In natural sciences, the term 'calibration' refers to a process "of tuning the model—that is, the manipulation of the independent variables to obtain a match between the observed and simulated distribution or distributions of a dependent variable or variables" [pg. 91]. This can of course also be practised in Economics, i.e. the model parameters can be manipulated until the model mimics well a carefully defined number of dimensions (stylised facts or moments), but mostly 'calibration denotes a relatively simple process of setting the model parameters before running simulations without the subsequent iterations.

In natural sciences 'calibration', available data are divided into two parts. The first set is used for 'calibrating' the model and using the remaining data the model is 'verified'. In Econometrics a comparable approach consists of (in-sample) estimation and (out-of-sample) 'testing'. The process of calibration might be often seen vague and arbitrary as no coherent methodology for

[2] "Micro data offer one potential avenue for resolving the identification problem" (Hansen & Heckman 1996, pg. 88).

determination of individual parameters (from microeconomic data) usually exists. For instance Fernández-Villaverde & Rubio-Ramírez (2010, pg. 23) point out that following Method of Simulated Moments (MSM), "it is not obvious which moments to select to calibrate the parameters" and that "the experience from many years of methods of moments estimations is that choosing different moments may lead to rather different point estimates". Kydland & Prescott (1996, pg. 80) admit that in calibration, the model economy is sometimes made "inconsistent with the data on one dimension so that it will be consistent on another" and Hansen & Heckman (1996, pg. 97) allude "weak standards for verification imposed by the calibrators". Next, mostly no comparison with other differently calibrated versions of the same model is discussed.[3] Little emphasis is often devoted to assessing the quality of the resulting calibration: "formalizing the criteria for calibration and verification via loss functions makes the principle by which a particular model is chosen easier to understand" Hansen & Heckman (1996, pg. 93).

It can also be "very misleading to plug microeconometric parameter estimates into a macroeconomic model when the economic environments for the two models are fundamentally different" (Hansen & Heckman 1996, pg. 97). Simply, coefficients used for calibration might have been derived under assumptions incompatible with the modelling framework resulting in a theoretical mismatch of the final model. Also individual micro-studies are different from every other, they may condition on different assumptions and thus implicate different economic interpretations—micro-evidence synthesis is therefore a challenging task. "The researcher should be careful, though, translating this micro evidence into macro priors. Parameter values do not have an existence of their own, like a Platonic entity waiting to be discovered. They are only defined within the context of a model, and changes in the theory, even if minor, may have a considerable impact on the parameter values.", as emphasised by Fernández-Villaverde (2010, pg. 10). An extensive discussion of issues emerging from application of microeconomics evidence and its incompatibility with macro models is provided by Browning *et al.* (1999). Authors highlight the heterogeneity of preferences—an essential finding of modern empirical Microeconomics—that is, however, often abstracted from in DSGE models to gain computational tractability. A caution is recommended against weak micro

[3]One of exceptions is the analysis by Aruoba *et al.* (2006) who devote a specific attention to the robustness of the calibration and compare a benchmark version with different 'unrealistics' combinations.

empirical foundations of many DSGE models and possibly improper use of synthesised microeconomic evidence within an internally consistent framework that "may alter the structure and hence the time series implications of the model". Another area of problematic application is the notion of uncertainty, where aggregating may cause a disconnect between its various individual sources—"different economic agents may confront fundamentally different risks"—and alter the predictability of the model. Calibrating the distributions of individual shocks thus becomes crucially important. In the same vein, ignoring some real-world inefficiencies can then lead to under/overstate of the true risk faced by agents.

2.1.2 Estimation

Unlike calibration, estimation is used to assess the approximate size of some phenomenon and offers a traditional rigorous optimisation approach for which loss function is unambiguously defined.[4] Already Kydland & Prescott (1996, pg. 74) emphasise that compared do estimation, in calibration "the parameter values selected are not the ones that provide the best fit in some statistical sense" and that the procedure often involves only one year's data or "the simple task of computing a few averages" over years: e.g. in a standard Cobb-Douglass production function, the parameter determining output elasticities can be simply computed as the average of labour share of total output. According to Grazzini & Richiardi (2015, pg. 148), "estimation (as opposed to calibration) involves the attainment of clearly specified scientific standards in the way models are confronted with the data" and they emphasise the acute need of such an approach "to gain confidence and ultimately support from the wider scientific community and the policy circles in the use of a new tool". As a differentiating principle authors mention—to some extent philosophically—distinct goals of the two methods but at the same moment they add that a clear classification is problematic. Whilst calibration primarily aims a optimally tracking and describing the observed empirical data, the ultimate goal of estimation is recovering via parameters' values the true data generating process beyond them, thus it is more "concerned with the properties of the estimators and the quantification of the uncertainty around the estimates".

Moreover, a crucial theoretical distinction between estimation and calibra-

[4]Various methods such as Ordinary Least Squares (OLS) or MSM and different loss functions weight various features of the data differently and thus selection of a proper loss function might help to solve various modelling issues.

tion lies in the the notion of convergence. Selection of appropriate statistics of a stationary and ergodic series ensures estimates consistent in time. Conversely, when using cross-sectional statistics to describe a model as in many calibration exercises, the consistency in population size cannot be assured due to possible autocorrelation between cross-sectional moment, that does not disappear with increasing sample size. Another important task is the final choice of variables used for the empirical estimation. On one hand, an ABM may be too stylised to accommodate estimation of all the variables in the model. On the other hand, over-parametrization brings serious difficulties related to identification and practical aspects of estimation procedures such as high computational burden. Additionally, in recent attempts to use DSGE models to forecasting, the estimation seems favourable as it often produces smaller mean squared errors than calibration.[5]

Compared to DSGE models and Macro ABMs, a distinguishing feature of FABMs is the structure of the output which almost always is a single time series. Thus, there are no cross-correlations observed and usual autocorrelations are employed in a standard set together with other financial stylised facts. The modellers are thus likely to avoid possibly arbitrary decisions what information should be used for calibration and what reserved for model testing. On the other hand, FABMs often share with DSGE models problems related to flat likelihood function—most frequently likelihood encompasses little information in a direction of a particular parameter. In such cases, according to Fagiolo & Roventini (2012, pg. 81), informal calibration might be "a more honest and internally consistent strategy to set up a model". When it comes to estimation, opposed to DSGE models and Macro ABMs with a relative large number of parameters, where empirical estimation might not be feasible or advisable, simple FABMs might contain only a few parameters that often do not have any obvious empirically measurable counterparts. It might even seem that some researchers 'deliberately' design simple stylised or highly aggregated models to facilitate computations and estimation of a small set of crucial parameters, but such approach is fully scientifically legitimate and follows so called KISS[6] modelling principle.

This thesis focuses on three well-known FABMs. In the Cusp model (Zeeman 1974, Thom 1975, see Chapter 3), the parameters are to a large extent

[5]Del Negro & Schorfheide (2012) provide an extensive study on forecasting performance of DSGE models and related literature review.

[6]An acronym for "Keep it Short and Simple", "Keep it Simple and Straightforward", or an impolite original meaning by U.S. Navy "Keep It Simple, Stupid".

artificial without any apparent economic interpretation. They rather represent weights of control variables that need to be optimised to fit the data. I can hardly imagine a reasonable calibration procedure in this case. For the Brock & Hommes (1998) model (see Chapters 4, 5, 6), the crucial switching coefficient—the intensity of choice β—needs to be retrieved from data also using some optimisation technique. Since the literature lacks a general consensus (see Sections 2.2 and 2.3) either on existence of behavioural switching on various markets or its intensity, the calibration approach would not be of much help in this situation. In some ways similar uncertainty hinders calibration also for the market noise intensity that we estimate in Chapter 6. Conversely, some of model parameters allow for calibration based on micro-studies or literature surveys, e.g. the overall market risk aversion or specifications of trading strategies on specific markets. Finally, in the Alfarano *et al.* (2008) model (see Chapter 7), the autonomous and herding switching intensities do also not have any reasonable empirical proxies. Clearly, the model allows for calibration using the 'natural sciences' approach to replicate a set of financial stylised facts—that was perhaps done by the original authors as well as by Chen & Lux (2015) and Ghonghadze & Lux (2015) resulting in the proposed simulation setting—but then an interesting scientific question appears whether one can estimate comparable values from the market data.

2.2 The use of econometric techniques

Utilisation of econometrics to empirically validate or estimate HAMs dates more than one decade back in the financial literature history. Within this stage of development of the Heterogeneous Agent (HA) modelling, the central concern embraces the determination of appropriate values of model parameters and assessment of their statistical significance. However, as summarised in Table 2.1 and Table 2.2, looking ten years back in literature, we neither observe any general consensus on the estimation methodology, nor conclusive results. Fagiolo *et al.* (2007, pg. 199, 202) assert that "a strongly heterogeneous set of approaches to empirical validation is to be found in the AB literature". Given different origins as well as various modelling concepts, the estimation methodology also varies. As depicted in the third column of Tables 2.1 and 2.2, the three estimation methods—the Nonlinear Least Squares (NLS), Quasi Maximum Likelihood (QML), and the MSM—prevail among others. When moving to the fourth column of Tables 2.1 and 2.2 we can see how the choice of estimated parameters is affected by various model designs. Nonetheless, we can observe a general tendency to estimate mainly parameters related to 'behavioural rules' of agents: belief coefficients defining individual trading strategies and the intensity of choice or its corresponding concepts in different types of models (mutation, herding tendency, and switching thresholds). All these parameters are apparently meaningful from the economic interpretation point of view.

Various direct and indirect estimation methods have already been employed. However, for the use of direct methods, instead of the usual OLS or Maximum Likelihood (ML) methods, the NLS and QML methods are applied in most of the cases. In these applications, crucial HAM structural features, e.g. the evolutionary switching between trading strategies—one of the key concepts of the HA modelling, are sometimes restrained or even sacrificed to obtain simplified approach which can be estimated using suggested methods. However, for many HAMs the aggregation equation, which would contain all parameters of interest, cannot be derived analytically and therefore the application of direct estimation techniques is not feasible. Indirect estimation methods thus overcome this problematic issue by simulating artificial data from the model through which the aggregation concepts such as moments for the MSM are derived. These simulation-based econometric methods "are very applicable and may dramatically open the empirical accessibility of agent-based models in the future" as suggested by Chen *et al.* (2012, pg. 204). Simulation-based econometric methods already used for the HAMs estimation include the MSM, the Efficient Method of Moments (EMM), or generally the Simulated Minimum Distance (SMD). All these methods are based on minimising the (weighted) distance between two sets of simulated and observed moments. So far, however, the use of simulation-based econometric methods for validation of HAMs is relatively rare.

2.3 Literature on FABMs estimation

This section provides additional brief description of papers summarised in Tables 2.1, 2.2, and 2.3. A special focus is devoted to attempts on estimating the switching coefficient—the key concept of the HA modelling. The estimated switching coefficient, however, cannot be directly compared across various models, assets, or time periods. It is a unit-free variable and its magnitude is conditional on the specific model design or the specific dataset. On the other hand, the intensity of choice is a crucial and a very robust driver of the data

Table 2.1: Estimation methods of FABMs I.

Models	Origin	Methods	Parameters estimated	#	Data	Type	Fit	|IOC|
Alfarano *et al.* (2005)	IAH	ML	Herding tendency	2	d:5034–9761 o.	s,fx,g	-	-
Alfarano *et al.* (2006)	IAH	ML	Herding tendency	2	d:5495,6523 o.	s,fx	-	-
Alfarano *et al.* (2007)	IAH	ML	Herding tendency	2	d:1975–2001	s	-	-
Amilon (2008)	ABS	EMM/ML	Intensity of choice[a]	15	d:1980–2000	s	$p\text{-}v$=0%	1.99(i),1.91(s)
Boswijk *et al.* (2007)	ABS	NLS	Belief coefficients/Intensity of choice	3	a:132 o.	s	R^2=.82	10.29(i),7.54(i)
de Jong *et al.* (2009b)	ABS	NLS	Belief coefficients/Intensity of choice	5	w:102 o.	fx	$adjR^2$=.14	1.52(i)
de Jong *et al.* (2010)	ABS	Quasi ML	Belief coefficients/Intensity of choice	7	m:238 o.	fx	-	.0007(i)–6.29(s)
Diks & Weide (2005)	ABS	ML	(G)ARCH relations/Sign of MA(1) c.	3	d:3914 o.	fx	-	-
Ecemis *et al.* (2005)	AA	IEC	Market fractions/Behavioural rules	3	-	s	-	-
Gilli & Winker (2003)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	NA	-
Manzan & Westerhoff (2007)	ABS	OLS	Reaction coeffs./Switching threshold	4	m:1/74–12/98	fx	NA	-
Reitz & Westerhoff (2007)	ABS	Quasi ML	Behavioural rules/Intensity of choice	6	m:365 o.	c	-	.17(s)–.47(s)
Westerhoff & Reitz (2003)	ABS	Quasi ML	Behavioural rules/Intensity of choice	7	d:4431 o.	fx	-	.02(s)–.17(s)
Winker & Gilli (2001)	ANT	MSM	Mutation/Conviction rate	2	d:1991–2000	fx	NA	-
Winker *et al.* (2007)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	$p\text{-}v$<1%[b]	-

Note: The Table is adopted from Chen *et al.* (2012, pg. 203) and amended by the authors. Authors are alphabetised. The full meaning of the acronyms under 'Origin': AA stands for Autonomous Agents, ABS for Adaptive Belief System, ANT for the Ant type of system, and IAH for Interactive Agent Hypothesis. The full meaning of the acronyms under 'Methods': ML stands for Maximum Likelihood, EMM for Efficient Method of Moments, NLS for Nonlinear Least Squares, OLS for Ordinary Least Squares, IEC for Interactive Evolutionary Computation, and MSM (SMM) for Method of Simulated Moments. '#' displays total number of estimated parameters; 'Data' describes data frequency: 'd/w/m/q/a' for daily/weekly/monthly/quarterly/annual, and number of observations (when a specific figure is not provided, we report starting and final years); 'Type' shows the type of data: 's/fx/c/g/re' for stock markets/FX/commodity markets/gold/real estate; 'Fit' reports the statistical fit of the estimation (R^2, its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and '|IOC|' displays the absolute estimated value of the 'intensity of choice'—the switching parameter from the multinomial logit model, see Equation 4.16 (where relevant), furthermore 's'/'i' denotes its statistical significance/insignificance at 5% level. Figures are rounded to 2 decimal digits.

Source: The Table is adopted from Chen *et al.* (2012, pg. 203) and amended by the authors.

[a] Chen *et al.* (2012) do not report other important parameters estimated: belief coefficients, intensities of exogenous noises, risk aversion, information costs for fundamentalists, forgetting factors, and memory in the fitness measure.

[b] While p-val for GARCH(1,1) model > 5%.

Table 2.2: Estimation methods of FABMs II. a)

Models	Origin	Methods	Parameters estimated
Barunik & Vosvrda (2009)	Cusp	ML	Asymmetry and bifurcation factors/Location and scale coefficient
Barunik & Kukacka (2015)	Cusp	RV/ML	Asymmetry and bifurcation factors/Polynomial data approximation
Bolt *et al.* (2011)	ABS	NLS(?)	Expectations' bias/Discount factor/Belief coefficients/Intensity of choice
Bolt *et al.* (2014)	ABS	NLS	Belief coefficients/A-synchronous updating ratio/Intensity of choice
Cornea *et al.* (2013)	ABS	VAR/NLS	Fundamentalists' belief coefficient/Intensity of choice
Chen & Lux (2015)	IAH	MSM	Standard deviation of innovations/Herding tendency
Chiarella *et al.* (2014)	ABS	Quasi ML	Belief & market maker coefficients/Memory decay rate/Intensity of choice
Chiarella *et al.* (2015)	ABS	Quasi ML	Belief coefficients/Variance risk premium/Intensity of choice
de Jong *et al.* (2009a)	ABS	ML	Belief coefficients/Intensity of choice
ter Ellen & Zwinkels (2010)	ABS	Quasi ML	Belief coefficients/Intensity of choice
ter Ellen *et al.* (2013)	ABS	OLS/NLS	Behavioural rules/Intensity of choice
Franke (2009)	ABS	MSM	Reaction coefficients/Switching threshold
Frijns *et al.* (2010)	ABS	EMS	Local volatility/Belief coefficients/Intensity of choice
Franke & Westerhoff (2011)	IAH	MSM	Behavioural rules/Flexibility/Predisposition coefficients
Franke & Westerhoff (2012)	ABS/IAH	MSM	Behavioural rules/Wealth/Predisposition/Misalignment coefficients
Ghonghadze & Lux (2015)	IAH	GMM	Standard deviation of innovations/Herding tendency
Grazzini *et al.* (2013)	Bass (1969)	ML, MSM	Probability of independent adoption/Peer pressure/Population size
Grazzini & Richiardi (2015)	-	SMD	-
Goldbaum & Zwinkels (2014)	?	OLS (iterative)	Belief coefficients
Hommes & Veld (2015)	ABS	NLS(?)	Belief coefficients/A-synchronous updating ratio(?)/Intensity of choice
Huisman *et al.* (2010)	ABS	Quasi ML	Belief coefficients/Intensity of choice
Kouwenberg & Zwinkels (2014)	ABS	Quasi ML	Belief coefficients/Intensity of choice
Kouwenberg & Zwinkels (2015)	ABS	Quasi ML	Price elasticity/Belief coefficients/Intensity of choice
Lof (2012)	ABS	NLS	Belief coefficients/Intensity of choice
Lof (0)	ABS	VAR/NLS	Discount factors/Belief coefficients/Intensity of choice
Reitz & Slopek (2009)	ABS	Quasi ML	GARCH coefficients/Belief coefficients/Transition parameter
Recchioni *et al.* (2015)	ABS	calibration	Belief coefficient/Intensity of choice/Risk aversion/Fundamental value/Memory
Verschoor & Zwinkels (2013)	ABS	ML	Belief coefficients/Intensity of choice

Note: The Table follows the logic of Table 2.1 and summarises recent research not covered there. Authors are alphabetised. The full meaning of the acronyms under 'Origin': Cusp stands for the cusp catastrophe model, ABS for Adaptive Belief System, and IAH for Interactive Agent Hypothesis. The full meaning of the acronyms under 'Methods': ML stands for Maximum Likelihood, RV for Realised Volatility, NLS for Nonlinear Least Squares, VAR for Vector Autoregression, MSM (SMM) for Method of Simulated Moments, OLS for Ordinary Least Squares, EMS for Empirical Martingale Simulation by Duan & Simonato (1998), GMM for Generalized Method of Moments, and SMD for Simulated Minimum Distance. '?' means that given information is unclear to authors.

Source: The Table has been completed by the authors.

Table 2.3: Estimation methods of FABMs II. b)

Models	#	Data	Type	Fit	|IOC|
Barunik & Vosvrda (2009)	8,17	d:1987–1988,2001–2002	s	*pseudo*-R^2 up to .8	-
Barunik & Kukacka (2015)	10	d:6739,409 o.	s	*pseudo*-R^2=.8, .86	-
Bolt *et al.* (2011)	5	q:164 o.	re	NA	2716(i),12420(i)
Bolt *et al.* (2014)	4	q:178 o.	re	NA	795(i)–26333(i)
Cornea *et al.* (2013)	2	q:204 o.	U.S. inflation	R^2=.78, .94	4.78(s)
Chen & Lux (2015)	3	d:1/1980–12/2010	s/fx/g	$p\text{-}v \in \langle 4.6\%, 45.5\% \rangle$	-
Chiarella *et al.* (2014)	6	m:502,251 o.	s	-	.44(s),.54(s),.69(s)
Chiarella *et al.* (2015)	5	w:2007–4/2013	CDS spreads	-	.74(i)–6.84(s)
de Jong *et al.* (2009a)	10	q:112 o.	s	-	1.03(s),2.87(s)
ter Ellen & Zwinkels (2010)	7	m:295,319	c (crude oil)	-	1.19(s),1.36(s)
ter Ellen *et al.* (2013)	2–5	w:1/2003–2/2008	fx	$adj R^2$ up to .7	7.72(i)–454.4(i)
Franke (2009)	6	d:4115–6867 o.	s,fx	$p\text{-}v \in \langle 0\%, 2\% \rangle$	-
Frijns *et al.* (2010)	5	d:01–12/2000	s (index options)	-	107.34(i)
Franke & Westerhoff (2011)	6	d:6866,6861 o.	s,fx	$p\text{-}v$=12.8%, 27.7%	-
Franke & Westerhoff (2012)	9	d:6866 o.	s	$p\text{-}v$=12.7%–32.6%	-
Ghonghadze & Lux (2015)	3	d:1/1980–12/2009	s/fx/g	$p\text{-}v \in \langle .3\%, 67\% \rangle$	-
Grazzini *et al.* (2013)	3	-	-	-	-
Grazzini & Richiardi (2015)	1	d:400 o.	s	-	-
Goldbaum & Zwinkels (2014)	4	m:2825–2941 o.	fx (experts' forecasts)	$adj R^2$=.55–.79	-
Hommes & Veld (2015)	4	q:252 o.	re	R^2=.95	2.44(i)
Huisman *et al.* (2010)	4	d:694,753,1038 o.	c (electricity futures)	-	1.06(s),1.77(s),15.87(i)
Kouwenberg & Zwinkels (2014)	4	q:127,198 o.	re	-	2.98(s),1.36(s)
Kouwenberg & Zwinkels (2015)	5	q:204 o.	re	-	2.18(s)
Lof (2012)	7	q:208 o.	s	R^2=.97	7.45(s), 4.74(s)
Lof (0)	5	a:140 o.	s	R^2=.55	type-specific: .8(i),1.13(s),5.18(i)
Reitz & Slopek (2009)	6	m:252 o.	c (crude oil)	-	-
Recchioni *et al.* (2015)	4	d:245 o.	s	-	2.14(s),.59(i),.03(s),.36(i)
Verschoor & Zwinkels (2013)	5	m:107 o.	fx	-	2.64(i),14.51(i)

Note: The Table complements information in Table 2.2 following the logic of Table 2.1. Authors are alphabetised. '#' displays total number of estimated parameters; 'Data' describes data frequency: 'd/w/m/q/a' for daily/weekly/monthly/quarterly/annual, and number of observations (when a specific figure is not provided, we report starting and final years); 'Type' shows the type of data: 's/fx/c/g/re' for stock markets/FX/commodity markets/gold/real estate; 'Fit' reports the statistical fit of the estimation (R^2, its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and '|IOC|' displays the absolute estimated value of the 'intensity of choice'—the switching parameter from the multinomial logit model, see Equation 4.16 (where relevant), furthermore 's'/'i' denotes its statistical significance/insignificance at 5% level. Figures are rounded to 2 decimal digits.

Source: The Table has been completed by the authors.

generating process behind switching HAMs and to a large extent determines the behaviour of the system in a very consistent manner: zero intensity of choice fixes market fractions and does not allow for any evolutionary switching, high values implicate a wild switching for vast majority of model specifications, assets, or periods. Relatively small positive intensity of choice is associated with a presence of some detectable behavioural switching. Thus we mainly intend to avail the general knowledge of previous estimation results from literature for setting meaningful simulation grids in Chapter 5 or to constrain random generation of initial points in Chapter 6.

Winker & Gilli (2001), Gilli & Winker (2003), and Winker *et al.* (2007) extend the exchange rate HAM originally suggested in the seminal work of Kirman (1991; 1993). Authors develop computational algorithms to deal with high complexity of empirical validation and estimate their model using daily DM/USD exchange rates from 1991 to 2000. As the main result they argue that "the foreign exchange market can be better characterised by switching moods of the investors than by assuming that the mix of fundamentalists and chartists remains rather stable over time" (Gilli & Winker 2003, pg. 310). In addition, Winker *et al.* (2007) employs the bootstrap method to analyse properties of the objective function.

Westerhoff & Reitz (2003) develop a STAR-GARCH exchange rate HAM and estimate the model using daily rates of main world currencies to the USD from 1980 to 1996. Their results indicate the existence of fundamentalist- and chartist-driven exchange rate dynamics and provide evidence for a substantial fluctuations of market fractions with fundamental traders leaving the market after increase in the deviation from the fundamental value. Switching parameter is estimated between 0.021 and 0.172 and statistically significant. Westerhoff & Reitz (2005) suggest a STAR-GARCH HAM to analyse cycles in commodity prices. Employing the QML estimation to the U.S. monthly corn price index data between May 1973 an May 2003 by the U.S. Department of Labour, they reveal that technical and fundamental speculations are to a great extent able to explain cycles in commodity prices. Switching parameter in their model is found 0.199 with t-statistics 2.22. Reitz & Westerhoff (2007) introduce another commodity HAM. Authors again utilise the STAR-GARCH approach and analyse monthly USD prices of cotton, soybeans, lead, sugar, rice, and zinc over the period from January 1973 to May 2003. The estimated model can again account for the cyclicality on commodity markets as the switching parameter is found between 0.17 and 0.47 and statistically significant.

Diks & Weide (2005) estimate their MA-(G)ARCH specification of a HAM via daily log returns of exchange rates of the USD against six local currencies from November 1987 to November 2002.

Alfarano *et al.* (2005) estimate a HAM based also on Kirman's ANT process (Kirman 1991; 1993) but allow for asymmetry in the attractiveness of trading strategies. A two-step ML estimation is used. Daily data contains returns of gold between 1974 and 1998, stock returns of Siemens and the Deutsche Bank from 1974 to 2001, and DAX returns covering the period form 1959 to 1998. The model is able to mimic the crucial financial stylised facts and authors therefore conclude that asymmetric herding appears useful for explaining patterns in financial returns. Alfarano *et al.* (2006) estimate a similar model on the exchange rate data of the Australian dollar against the USD covering the period from December 1983 to December 2004 and on the Australian index data ranging form January 1980 to December 2004. Alfarano *et al.* (2007) follow the previous works in estimating the model parameters for a large daily dataset of 100 stocks from the Japanese market between January 1975 and December 2001.

Manzan & Westerhoff (2007) estimate a HAM using monthly exchange rates of five major world currencies against the USD covering the period from the January 1974 to December 1998. The model demonstrates reasonable explanatory power for the in-sample estimation but fits significantly only for two of analysed currencies for the out-of-sample prediction.

Boswijk *et al.* (2007) present one of the first attempts to estimate a full-fledged behavioural HAM which is a reformulation of the Brock & Hommes (1998) model in terms of price-to-cash flow ratios. Using NLS and annual S&P500 data between 1871 and 2003, three parameters of the model: coefficients characterising two trading strategies and the intensity of choice β, are jointly estimated. While two significantly different expectation regimes are found, β remains significantly not different from zero, followed by a discussion why this is not worrying as long as heterogeneity in regimes is confirmed.

Amilon (2008) presents a more general version of the Brock & Hommes (1998) model amended via nonconstant risk aversion, information costs of fundamental traders, and time-varying risk of the portfolio. Finally, he estimates a simple version of the model by ML and EMM using detrended daily S&P500 returns ranging from January 1980 to December 2000. The most important point of the work, however, lies in a discussion how the stochastic noise term influences the ability of the model to fit real data and to generate stylised facts of financial returns—the author argues that "it is not a very realistic way to model noise that is not normalized to the price level... if the non-linearities of the system are turned off and prices are just generated from a random walk $p_t = p_{t-1} + \varepsilon_t$, the resulting return series, $r_t = \varepsilon_t / p_{t-1}$, will actually show signs of all the stylised facts mentioned above, such as large autocorrelation in squared or absolute returns, and fat tails!" [pg. 349]. In the estimation part of the paper the intensity of choice β is estimated with ambiguous results: 1.91 and statistically strongly significant by MLE and 1.99 and markedly insignificant using EMM.

Franke (2009) empirically estimates the model introduced Manzan & Westerhoff (2007) using the MSM. The dataset contains stock market indices between January 1980 to March 2007: S&P500, DJIA, DAX, NIKKEI 225, as well as exchange rates: USD/DM and USD/JY. The author states that the MSM "estimation results lend the model a remarkable explanatory power" [pg. 814]. Franke & Westerhoff (2011) and Franke & Westerhoff (2012) develop an elementary HAM based on herding mechanism and estimate it via MSM on the S&P500 returns from January 1980 to March 2007. In both papers authors take the advantage of the block bootstrap method in order to better account for the small-sample properties of the estimators.

Cyclical behaviour of oil price is assessed in Reitz & Slopek (2009) who propose an empirical oil market STAR-GARCH HAM and estimate it on the WTI crude oil prices obtained from the IMF International Financial Statistics database. The data comprise 252 monthly observations covering the period from January 1986 to December 2006. The model provides significant evidence for activities of prices speculators represented by chartistic traders who are thus likely to cause price movements' amplification in the last part of the dataset. The transition parameter is found 1.94 and statistically significant.

de Jong *et al.* (2009a) expand the Brock & Hommes (1998) approach by adding a new belief type—internationalists—and introducing a two-market model to explain the shift-contagion within the Asian crisis. They allow for multiple asset trading and model market fractions using a VECM with time-varying coefficients employing data ranging form 1980 to 2007. Equation-by-equation ML estimation is performed because of difficulties caused by non-linear nature of the system. The switching parameter is found positive and statistically significant, namely 1.031 for the Stock Exchange of Thailand, and 2.869 for the HSI. de Jong *et al.* (2009b; 2010) further analyse the function of the European Monetary System from its launch in March 1979 to December 1998, when it was superseded by the European Exchange Rate Mechanism II. The model is estimated with NLS, MLE, respectively, and shows significant evidence of the behavioural heterogeneity using a weekly and monthly dataset containing exchange rates of seven major world currencies against DM. de Jong *et al.* (2009b) report the switching parameter positive but statistically insignificant: 1.52. In the de Jong *et al.* (2010) framework the interpretation of the intensity of choice is a partly different as a negative value assures the positive feedback effect, i.e. agents switch to the more profitable strategy. The switching parameter is statistically significant for three countries and insignificant for four countries, in absolute values between 0.0007 and 6.29.

Barunik & Vosvrda (2009) and Barunik & Kukacka (2015) estimate the cusp catastrophe HAM using MLE and a dataset consisting of S&P500 stock market returns, S&P500 futures, and OEX put/call ratios. While Barunik & Vosvrda (2009) take use of only relatively short periods around 1987 and 2001 stock market crashes, Barunik & Kukacka (2015) develop a two-step estimation methodology based on the theory of RV and estimate the model on high-frequency data covering almost 27 years from February 1984 to November 2010 (for details see Chapter 3).

A housing market application appears in Bolt *et al.* (2011) whose model is based on the framework of Boswijk *et al.* (2007). Authors employ OECD housing market dataset containing the U.S. and the Netherlands data ranging from 1970 Q1 to 2010 Q4 and report the intensity of choice statistically insignificant for both countries. Authors follow their previous work in Bolt *et al.* (2014) and estimate a modified version of the model for eight different countries employing the same OECD dataset that covers period from 1970 Q1 to 2014 Q2. The intensity of choice β in the model is found insignificant for all countries and authors conclude that "the fact that β is found to be insignificant is merely an indication that the model's forecast accuracy is not very sensitive to the exact value of β, and the other parameters can to a large extent compensate for changes in β" [pg. 15]. Hommes & Veld (2015) estimate a similar model on quarterly S&P500 data between 1950 and 2013. Executing a Monte Carlo simulation they demonstrate a very flat shape of the likelihood function for the intensity of choice β selection that hampers validity of the test to reject the null hypothesis of switching, especially for small samples.

ter Ellen & Zwinkels (2010) develop a simple oil market HAM based on Brock & Hommes (1998) framework and validate it on a similar dataset as Reitz & Slopek (2009). The data covers Brent and WTI Cushing oil monthly USD prices over the period from January 1984 to August 2009 taken from DataStream. Estimation is carried out using QML and the results indicate a significant price effect of both fundamental and chartistic traders. Switching between strategies is also statistically significant—the intensity of choice is estimated 1.19 for Brent and 1.36 for WTI. Huisman *et al.* (2010) introduce a HAM based on ter Ellen & Zwinkels (2010) design and study the European historical forward electricity prices over three years for the base-load year 2008 contracts. The model is estimated using QML and the intensity of choice is found statistically significant for two of three analysed markets, in absolute values[7] 1.06 and 1.77. An innovative usage of HAMs appears in Frijns *et al.* (2010) who focus on option market and propose a model for the volatility trading and pricing of options. Daily DAX prices covering year 2000 provided by the European Futures and Options Exchange are utilised. Results of the EMS estimation give support to hypothesised heterogeneity and the evidence of the traders' switching behaviour is stronger than in a stock market case by Boswijk *et al.* (2007). The intensity of choice is found "positive and of considerable magnitude throughout the sample" with mean value 107.34 [pg. 2281]. Verschoor & Zwinkels (2013) originally estimate a HAM on two currency trader indices, namely the Parker FX Index and the Barclay Currency Manager Index covering monthly data between October 2000 and August 2009. Using ML method, the intensity of choice is reported -2.64 and insignificant for the Parker FX Index and -14.51 and significant at 10% level for the Barclay Currency Manager Index. The negative sign implies prevailing negative feedback trading strategy, i.e. the contrarian-type behaviour. ter Ellen *et al.* (2013) employ the FX Week survey dataset containing weekly forecasts of a large number of wholesale FOREX investors for four exchange rates and three forecast horizons. The intensity of choice is found statistically insignificant for ten of twelve assessed combinations, amounting to both negative and positive absolute values. This suggests generally ambiguous traders' switching both to historically profitable strategies, but also to strategies performing the worst in the previous period, i.e. the contrarian-type of speculation. Goldbaum & Zwinkels (2014) construct a fundamental-chartist type of HAM without the switching feature. The model is estimated likewise the work of ter Ellen *et al.* (2013) taking the advantage of FOREX survey-based monthly forecast data provided by the Consensus Economics of London. Two currencies, EUR and JY, and three forecast horizons are analysed. Kouwenberg

[7] The interpretation of the expected negative intensity of choice is now inverse as agents switch to the strategy with the smallest forecasting error.

& Zwinkels (2014; 2015) are in fact pioneers of HAM estimation using housing market data by QML. Their dataset covers quarterly U.S. records on prices and rents from 1960Q1 to 2013Q3, 2014Q1, respectively. For all estimated sub-periods (1963–2013, 1963–1995, and 1960–2014) a significant presence of fundamentalists and chartists, and dynamic evolution of market fractions is reported. The intensity of choice is estimated 2.98, 1.36, 2.18, respectively for given periods and statistically significant in all cases.

Lof (2012) estimates a STAR version of a HAM following the Boswijk *et al.* (2007) framework using quarterly S&P500 price-dividends and price-earnings ratios since 1960. For the model with univariate transition function the intensity of choice is reported unrealistically high but for the model with multivariate transition function it is lower (7.45 and 4.74 for two different model specifications) and statistically significant. Lof (0) innovatively allows for strategy type-specific intensity of choice and estimates a three-type HAM using annual S&P500 data from 1872 to 2011. Except one specific case, the intensities of choice are found statistically insignificant.

Chiarella *et al.* (2014) follow and enrich the Boswijk *et al.* (2007) approach and estimate a modified ABS employing monthly S&P500 prices and earnings over the period from January 1970 to October 2015. Interesting discussion is devoted to indirect estimation of the market maker coefficient and the intensity of choice that are not uniquely identified and must be retrieved in combination with belief coefficients. The implied intensity of choice derived from the reduced form model is finally reported 0.44. Chiarella *et al.* (2015) estimate a HAM for credit default swap spreads of 13 European countries using QML. Switching coefficients are found between -6.84 and 5.66 for individual countries both statistically significant and insignificant.

A calibration procedure for validating ABMs is developed by Recchioni *et al.* (2015) and applied to Brock & Hommes (1998) model. Although the authors do not estimate the model to all intents and purposes but calibrate the parameters to optimally describe short-run dynamics of daily closing stock prices or index values, their work in terms of the model setting is very closely related to ours. For the empirical experiment, data from four stock market indices representing four different geographical areas are employed: S&P500, Euro Stoxx 50, NIKKEI 225, and CSI 300. The data span ranges form February 2011 to February 2012 (245 observations) and five parameters of the model are calibrated (see Table 2.2). The intensity of choice β varies markedly for individual regions ordered as above: 2.044, 0.642, 0.001, 0.078 for the deterministic model, and

2.140, 0.586, 0.032, 0.363 for the model adjusted for stochastic noise. Authors also assess forecasting performance of calibrated models for one and two days ahead predictions. They conclude that both versions "show good performance in predicting the trend" [pg. 23] and almost always outperform the RW benchmark.

In two the most recent working papers, Chen & Lux (2015) and Ghonghadze & Lux (2015) estimate the Alfarano *et al.* (2008) financial herding ABM employing MSM and GMM, respectively, on a cross section of stock markets, exchange rates, and commodity data starting in 1980.

To mention another than purely financial application, one of the first attempts to estimate a macroeconomic ABM based on a 'tradition' of the Brock & Hommes (1998) framework is presented by Cornea *et al.* (2013) who develop a model to capture possible heterogeneity in inflation dynamics. Using quarterly U.S. data on the inflation rate and other macroeconomic variables for the first step VAR specification covering period from 1960 Q1 to 2010 Q4, they subsequently estimate their model using NLS. The intensity of choice β is reported around 4.8 for various VAR specifications and strongly statistically significant. Grazzini *et al.* (2013) and Grazzini & Richiardi (2015) provide a comprehensive theoretical discussion on macroeconomic ABMs estimation via simulation based methods, mainly SMD and MSM. In a simple Monte Carlo analysis they apply suggested approach to an elementary Bass (1969) model of innovation diffusion.

2.4 Categorisation of findings

2.4.1 Basic taxonomy of FABMs

An extensive survey by Chen *et al.* (2012) suggests a general basic taxonomy of FABMs based on their complexity. Overall complexity is assessed simultaneously in terms on three fundamental characteristics: heterogeneity, learning, and interactions. Models are classified primarily according to number of trading strategies and degree of agents' autonomy into two main subgroups: relatively simpler 'N-type models' where autonomy of agents is constrained by a predetermined class of strategies, and 'AA models' with rich system complexity, where agents are allowed e.g. to develop new strategies or to adapt their prediction rules based on learning and using genetic algorithms (Koza 1992). The former type is in the spotlight of this thesis as almost solely simple N-type models, mainly the 2-type and the 3-type versions have been subject of empirical estimation to date.

Typical representatives of the 'N-type' class are models derived from the ABS by Brock & Hommes (1998), models developed in the tradition of Kirman's (1991; 1993) Ant type of system (ANT), and models based on Interactive Agent Hypothesis (IAH) by Lux (1995; 1997; 1998). So called 'Santa Fe Artificial Stock Market Model' (Holland & Miller 1991; Palmer *et al.* 1994) represents the latter type. Chen *et al.* (2012) further attempt to provide additional forms of classification. First, they divide 50 analysed models according their origin into 9 exclusive subcategories: 19 models are derived from ABS, 12 AA models (all assessed) follow the Santa Fe Institute tradition, 5 models are based on the IAH, and 3 on ANT, threshold principle, or minority game principle. Second, a switching of agents' beliefs between a predetermined set of trading strategies is a feature of 25 models. Third, models are also classified in terms of ability to reproduce stylised features of financial time series. From an exhaustive list of 30 stylised facts only 12 can be replicated by at least one of 50 models [pg. 200, 201]: 41 models are reported to reproduce fat tails, 37 models mimic volatility clustering, and 27 are able to reproduce the absence of autocorrelation of returns.

In this section we aim at providing a less robust and possibly incomplete, but deeper and more quantitatively based alternative categorisation based on findings from this chapter and focused on empirical estimation aspects of the analysed models. It is now important to emphasize two aspects. First, Tables 2.1, 2.2, and 2.2 contain only very aggregated information because it seems counterproductive and also technically difficult to comprise full diversity and complete heterogeneity of all evaluated studies. In many cases, authors present results of various model settings and different combinations of estimated parameters. In such situations, we often report only the most successful or most representative results. If reasonable, we report the lowest and the highest values or an interval formulation. Second, the two sets of models—the original by Chen *et al.* (2012) and ours—only partially overlap: the models presented in Table 2.1 belong to both sets, works collected in Table 2.2 are newly analysed in this thesis. We are not aware of any attempt of empirical estimation of the remaining models from the original 50.

2.4.2 Model origin, methods, and parameters

First, as depicted in Tables 2.1 and 2.2, we observe a strong dominance of models derived from ABS: 28 from 43 models to some extent follow the tradition of the Brock & Hommes (1998) original framework, 7 is based on IAH and 3 (the very same models as in the original sample) on ANT. This seems to strongly reflect the fact that according to Chen *et al.* (2012, pg. 207), "the Lux model was rejected, similar to the rejection of the ANT model" based on its empirical validation in favour of ABS, that seems feasible (however, challenging) for empirical estimation. Indeed, there are only 4 new empirical studies in Table 2.2 based on the IAH and none derived from ANT. The strong dominance of the ABS in the recent FABM literature is the principal reason why we analyse the original Brock & Hommes (1998) model in the core part of the thesis (Chapters 4, 5, 6).

Second, we might observe some regularities in the use of particular estimation methods. ANT models are estimated solely using MSM, nonetheless, all 3 works are considerably related and elaborated by the same group of authors. The 3 original IAH models from Table 2.1 are intentionally designed so that authors are able to theoretically derive the likelihood function and use ML, but the 4 'new' models are again estimated solely via simulations based on the MM. The empirical rejection of these two sources of models' origin thus might be somewhat related to inability to estimate these models via other then purely computational simulation-based techniques, but this is a kind of a strong hypothesis that is difficult to assess. On the other hand, all ABS models are estimated via relatively less complicated direct techniques: NLS and Quasi ML or in some specific simplified cases even by OLS and ML.

Third, in terms of parameters estimated, we do not observe anything surprising. Authors dominantly make efforts to minimise the set of estimated parameters (see the '#' column in Tables 2.1, 2.3) to a small number of the most relevant ones governing dynamics of the model and determining heterogeneous behaviour of agents: in 22 cases 4 and less coefficients are estimated and only 9 works consider 7 and more parameters. For ABS models, belief coefficients are always estimated and where relevant, authors always pay specific attention to the sign and statistical significance of the switching coefficient of the intensity of choice. This is a justified approach and as aptly summarised by Chen *et al.* (2012, pg. 202), "supposing that we are given the significance of the intensity of choice in generating some stylized facts [via simulation studies],[8] then the next legitimate question will be: can this intensity be empirically determined, and if so, how big or how small is it?".

2.4.3 Datasets

Another interesting aspect is hidden in the utilised datasets (columns 'Data'). Almost one-half of studies, 20 (out of 43), use daily datasets providing thousands of observations. This is an important aspect both from the statistical point of view, as datasets are long enough for sound statistical inference, as well as from the viewpoint of model stability because number of periods is generally sufficiently long for the model dynamics to stabilise. Especially for simulation-based methods the stabilisation period deserves adequate consideration, e.g. we always discard first 100 observations from the HAM where the model dynamics is being established (see Section 5.1). Having e.g. only quarterly or only annual data, we would be forced to fiercely shorten the stabilisation period, possibly affecting the relevance of results. Availability of long historical daily-frequency datasets is one of important distinguishing features of FABMs compared to Macro ABMs. Conversely, surprisingly many studies employ low-frequency data ranging from weekly to annual observations. Regarding the type of the data (columns 'Type'), stock market (20 studies) and Foreign Exchange (FX) data (17 studies) largely prevail, followed by housing market data (5 studies), various commodities (4 studies), and gold (3 studies). In the past (see Table 2.1), almost all models were estimated using stock market and FX data mainly of daily frequency, more recent estimation attempts (see Table 2.3) offer considerably richer composition of markets and frequencies.

2.4.4 Performance of OLS, NLS, and MSM

One of the most intriguing areas of possible analysis is the relative performance of particular models and estimation methods. However, we need to admit that strong mutual heterogeneity of particular models besides variability of estimation methods make eventual conclusions rather hypothetical and possible candidates for further research. Starting with the performance, in columns 'Fit' we report R^2 or its various alternatives for OLS, NLS, and possibly ML, and the p-value of the general specification J-test of overidentifying restrictions to accept the structural model as a possible data generating process. Having p-

[8] A note added by the authors.

value higher than a chosen significance level, the null hypothesis of the empirical data being possibly generated by the analysed model of the 'true' moment-generating process cannot be rejected.

Regarding the R^2 type of fit, the vast majority of models (8 out of 10) exhibit almost suspiciously very good fit ranging between 70–97%. Nonetheless, it is important to add that these results are most often[9] based on low-frequency datasets and therefore relatively straightforward estimation methods might not be very challenged to find a well fitting model for maximum of a few hundreds observations.

A different situation is observable for various MMs: roughly one-half of models are rejected based on the J-test as the data generating processes for given data/moments at a usual 5% level. Only the most recent contributions show acceptance of a particular model but sometimes only for selected datasets from several options (Chen & Lux 2015; Ghonghadze & Lux 2015). As all these studies are based on daily stock market or FX data, no conjunction with frequency or type of the data can be made. As a matter of interest, we can, however, compare the performance of models and selected moments based on the Standard & Poor's 500 Index (S&P500) dataset that is shared by all 6 stock market models using MMs and covering with a small exception of the older work of Amilon (2008) almost the similar span of data. We can simply follow Franke & Westerhoff (2012, pg. 1208) arguing by a large number of existent FABMs and the 'wilderness of bounded rationality problem' and calling for a general guidance and a model contest to "judge which models can mimic the stylized facts, say, 'fairly well' and which are even 'very good' in this way". Authors suggest the MSM as the "superb tool to serve this purpose". If we compare the 6 above mentioned models, the two most successful are those by Franke & Westerhoff (2012) based on a rich set of 9 moments reaching the J-test p-value 32.6%, and by Ghonghadze & Lux (2015) employing a partly different set of 8 moments, with p-value amounting to 50.2% for S&P500.

In summary, the application of the MMs, mainly the MSM version, offers a tool for mutual comparison of models and estimation frameworks, however, its application struggles with practical technical issues that require a further development of the method. The most recent contribution of Chen & Lux (2015, pg. 16) explains the main problem prohibiting proper identification that is shared by all these studies: "we have to cope with multiple local minima as

[9] With an exception of Cusp models that are, however, estimated via ML and only so called pseudo-R^2 (see Subsection 3.2.1) is reported.

well as with relatively flat surfaces in certain regions of the parameter space. Any standard optimisation algorithm could, thus, not be expected to converge to a unique solution from different initial conditions". Another closely related problem is a very rugged surface of the objective function, further embarrassing standard methods of optimisation search. To tackle all these issues, authors suggest a strategy of a preliminary rough grid search followed by a fine-tuning on a considerably restricted subset of the parameter space. We generally follow these recommendations also within the empirical application of NPSMLE, but opt for a strategy of multiple random starting points. Finally, MMs inevitably challenge researchers to partially arbitrarily select a set of moments that must be representative enough to hope in capturing well the most crucial features of the data, but at the same time reasonably bounded to accommodate computational burden and possible clashes in explanatory potential of particular moments. Such a task does not seem to be satisfactorily resolved for FABMs yet, however, the progress in time is evident also within the very limited sample of only several analysed papers.

2.4.5 Performance of ML and Quasi ML

Application of methods based on ML principle shares a relatively similar problem with MM: the objective function is often very flat in direction of some parameters—typically the switching parameter of the intensity of choice. Problematic identification of given parameters is then reflected in large standard deviations of estimates preventing from contributive interpretation of results. In the majority, especially older, studies a discussion about the shape of the log-likelihood function is missing and the reader might only guess from insignificant estimates of the switching coefficient. A few most recent studies report that the likelihood is not very informative and the model accuracy is not sensitive for given parameter, and "the other parameters can to a large extent compensate for changes in β", the switching coefficient (Bolt *et al.* 2014, pg. 15). However, the shape of the objective function is almost never rigorously studied. An exception is e.g. Hommes & Veld (2015), who emphasise a very flat shape of the likelihood function for the intensity of choice selection that hampers validity of the test to reject the null hypothesis of switching, especially for small samples. On the other hand, smoothness of the objective function does not seem to be an issue for ML methods in comparison with MMs—this finding is further confirmed through Chapters 5, 6, and partially in 7 also for Non-Parametric Simulated Maximum Likelihood (NPSML).

2.4.6 Switching

Finally, a high importance is devoted to the existence of behavioural switching, that is, to the sign, magnitude, and the statistical significance of the intensity of choice. Following the question of Chen *et al.* (2012, pg. 202), “how big or how small is it?”, we, however, need to emphasise that the magnitude of the switching coefficient cannot be directly compared across various models, assets, or time periods, as it is a unit free variable and its effect on the model dynamics is conditional on the particular model design and data. Four studies find a very large switching coefficients (Bolt *et al.* 2011; 2014; ter Ellen *et al.* 2013; Frijns *et al.* 2010), however, statistically insignificant in all cases. In other relevant studies (20 out of 23), the estimated values are mostly found single-digit and often close to zero, that well resembles the economic intuition of some, but realistically low switching frequency between major types of trading strategies. Although the sign of the parameter is of a crucial importance, in Tables 2.1 and 2.3 we present absolute values because the interpretation of the positive/negative sign also depends on the specific design of the model. Almost all studies report the theoretically expected sign of the effect, nonetheless, we do not observe any conclusive results regarding the statistical significance of the intensity of choice—no connection can be observed w.r.t. the ‘#’ number of estimated parameters or the frequency and length of the data. Statistically significant and insignificant findings are reported across these categories without any clear pattern. On the other hand, some but definitely hypothetical relation might be observed based on the ‘Type’ of the data: statistically significant estimates strongly dominate for commodities and weakly prevail for stock markets; insignificant estimates prevail for real estate markets and dominate for FX. However, as the sample of studies is rather small and often problematically mutually comparable, these findings should be interpreted with a high caution.

2.5 Best practices from DSGE estimation

Many analogies can be found between ABMs and DSGEs models, both in terms of methodological approaches as well as validation strategies and econometric consequences. Not only a possibly large dimension of the parameter space, but most importantly the aspect of system complexity involving a variety of nonlinear feedback effects between agents is shared. As several econometric issues evidently overlap, the evolution of the DSGE methodology constitutes an important source of guidance, experience, hints, but also caution, that may be potentially utilised or adapted within the field of ABMs estimation. Researchers are thus likely to largely benefit from studying empirical DSGE literature before launching their own ABM validation research efforts. Indeed we have inspired ourselves by several below mentioned suggestions in further chapters of this work.

Grazzini & Richiardi (2015, pg. 151) report the ML as the most standard procedure to estimate DSGE models. A common solution concept of linearisation via a Taylor approximation (and possible detrending in the presence of nonstationarity) around the steady state is, however, not feasible for inherently nonlinear AB systems potentially exhibiting even chaotic motion. Authors emphasise two main problems: the first one is "stochastic singularity, which arises when a small number of structural shocks is used to generate predictions about a large number of observable variables" leading to zero likelihood with probability 1. A possible treatment may be based on decreasing the number of parameters, adding further shocks, or introducing measurement errors. Second, "ML estimation is very sensitive to model misspecification, which often leads to absurd parameter estimates. The reason is the very flat shape of the likelihood function, which might display a multitude of local optima". A standard solution is to define Bayesian priors for distributions of parameters—these are likely to increase curvature of the objective function and smooth out its surface—and obtain the posterior distributions using Markov Chain Monte Carlo (MCMC) methods. Indeed, Grazzini *et al.* (2015) are the pioneers of application of Bayesian techniques in the field of ABM estimation. However, if the likelihood function is very flat, the posterior distribution will resemble the prior and the method is not likely to reveal much of further understanding. In such situation, as already mentioned in Subsection 2.1.2, statistically reasonable estimates are obtained only because artificial restriction increase the likelihood of the data. Consequently, estimation in fact turns into a sophisticated calibration exercise. Grazzini & Richiardi (2015) further discuss practical applicability of this relatively complicated procedure within the ABM field due to absence of an analytical closed-form solutions of the objective functions, priors based on the non-AB literature, significant nonlinearities, and likely non-Gaussianity.

More sophisticated methods might be used to overcome some of these technical problems, known as Sequential Monte Carlo methods or 'particle filtering' (see e.g. Fernández-Villaverde 2010; Fernández-Villaverde & Rubio-Ramírez 2010; 2007a;b; 2005; 2004), that allow for estimation of nonlinear economies and solution of the optimisation problem in the presence of non-Gaussian shocks. A substantial disadvantage is a large computational burden and thus basic particle filtering works conveniently only for lower dimensions of the parameter space. Fernández-Villaverde & Rubio-Ramírez (2004, pg. 31, 1) conclude that the Sequential Monte Carlo "procedure works superbly in delivering accurate and consistent estimates" and suggest marginal likelihood ratio test to nonlinear and linearised models. They also comment on the use of alternative methods claiming that "MMs may suffer strong biases resulting from using small samples and may not use efficiently all the existing information", but avoid the problem of evaluating the likelihood function "by moving away from full information approaches to inference". However, we are not aware of any application of particle filtering to ABMs to date.

Del Negro & Schorfheide (2012, pg. 31, 63) summarise that "DSGE model forecasts are comparable to standard autoregressive or vector autoregressive models but can be dominated by more sophisticated univariate or multivariate time series" and focus also on "point forecasts generated from DSGE models and on how to improve their accuracy by using external information", e.g. how to optimally design priors.

Levine *et al.* (2012, pg. 1299) design a model composing of rational and adaptive agents and show that imperfect information in the New Keynesian model significantly supports its empirical fit. Authors report that "all behavioural models decisively, in fact very decisively, dominate the purely rational models with very large LL differences of around 20" but at the same moment they highlight an essential limitation of the marginal likelihood race, that is, the outperforming model is only better than its rivals and may be still potentially misspecified.

Fagiolo & Roventini (2012, pg. 80) comment on a common strategy to deal with identification problems, that is, to follow the limited-information approach consisting in "calibrating the parameters hard to identify and then estimating the others". However, this approach works only when calibrated parameters are set close enough to their true values, otherwise the subsequent estimation cannot deliver correct results.

Fernández-Villaverde & Rubio-Ramírez (2010, pg. 21) sum up the important advantages of ML, i.e. "that the likelihood is a coherent procedure that respects the likelihood principle and allows us to back up all the parameters of interest, and that has good small and large sample properties. Furthermore, the likelihood function can be easily complemented with presample information in the form of priors, which are particularly useful in macroeconomics, where we have short samples". Moreover, according to Aruoba *et al.* (2006, pg. 2479), linearisation "methods deliver an interesting compromise between accuracy, speed, and programming burden". On the other hand, Fernández-Villaverde (2010, pg. 12) focus also on disadvantages and claim that "many non-parametric and semiparametric approaches sound more natural when set up in a classical framework", e.g. the clear and intuitive Generalized Method of Moments (GMM) or MMs in general, that offer "a good way to estimate models with multiple equilibria, since all of those equilibria need to satisfy certain first order conditions that we can exploit to come up with a set of moments".

Canova & Sala (2009) thoroughly discuss the identification issue inherent to DSGE models. Suggestions, that can be to a large extent shared also by ABMs, comprise the following: to decrease the dimension of estimated parameters, to calibrate some parameters in the first step before estimating the others, to reveal by means of a sensitivity analysis what parameters do not make much difference in the model outcome and fix them to some reasonable values, or to use real data in the model. In order to abandon these mixed validation approaches, authors further suggest several relatively obvious hints such as avoiding inappropriate priors, employing robust estimation methods, or rethink the model design, but also two considerably important concepts: a) exploring the curvature, smoothness, and possible local optima of the objective function in advance before estimation, e.g. depicting its shape via simulations and check its properties visually, b) performing Monte Carlo simulations to check the estimation performance robustly.

Ruge-Murcia (2007, pg. 2600) contrasts and compares the performance of MSM and GMM with ML within the framework of a simple real business cycle model. His simulation results show that both MMs are less severely affected by stochastic singularity as well as less deteriorated by potential model misspecification. The reason is that "ML estimation is limited by the number of linearly independent variables while moment-based estimation by... linearly independent moments. The latter is a weaker restriction because it is possible to find independent moments that incorporate information about more variables than those... linearly independent".

2.6 Concluding remarks

To sum up, although we claim in the Introduction that so far neither general consensus on the estimation methodology, nor conclusive results are clearly observable, we might, however, conclude the most important findings, issues, and tendencies observed within the field. First, surprisingly no considerable effect of the curse of dimensionality is directly observable across the sample of assessed studies. Models with only a few parameters estimated show a balanced result in terms of the estimation performance as well as the ability to reveal the switching parameters of the intensity of choice. Surprisingly, studies where a relatively large number of parameters is estimated reveal mostly favourable results in both aspects. ABS constitutes a largely dominant modelling framework and seems to be relatively successfully estimable via NLS and Quasi ML, but the models need to be specifically designed and often simplified accordingly to allow for taking advantage of these methods. The flatness of the log-likelihood function, especially in the direction of the switching coefficient, seems to be the main, but not sufficiently studied, handicap of the ML based estimation methods. Simulation-based methods are generally applicable and not constrained by strict theoretical assumptions, but yet not developed enough to bring unambiguous conclusions. Most importantly, they suffer from issues related to flatness and roughness of the surface of the objective function. However, a future progress of simulation-based methods, especially via solving related rather technical issues, is likely to be largely encouraged and fostered by recent rapid development of high-speed computational facilities.

Chapter 3

Estimation of the cusp catastrophe model under time-varying volatility

Financial inefficiencies such as under- or over-reactions to information as the causes of extreme events in the stock markets attract researchers across all fields of economics. In one recent contribution, Levy (2008) highlighted the endogeneity of large market crashes as a result of the natural conformity of investors with their peers and the heterogeneity of the investor population. The stronger the conformity and homogeneity across the market, the more likely the existence of multiple equilibria in the market, which is a prerequisite for a market crash to occur. Gennotte & Leland (1990) presented a model that shares the same notions as Levy (2008) in terms of the effect of small changes when the market is close to a crash point as well as the volatility amplification signaling. In another work, Levy *et al.* (1994) considered the signals produced by dividend yields and assessed the effect of computer trading, which is blamed for making the market more homogeneous and thus more conducive to a crash. Kleidon (1995) summarized and compared several older models from the 1980s

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The text of this chapter (except minor changes based on opponents' reports) has been published under the full title *'Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility'* in *Quantitative Finance*, 2015, 15 (6), pp. 959-973. The paper is a joint work with the thesis supervisor Jozef Barunik and both authors contributed equally to this work. The paper is available at: `http://dx.doi.org/10.1080/14697688.2014.950319`. We are grateful to the editors and two anonymous referees for many useful comments and suggestions. The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement No. FP7-SSH- 612955 (FinMaP). Support from the Czech Science Foundation under the 402/09/0965, and 13-32263S projects is gratefully acknowledged. J. Kukacka gratefully acknowledges financial support from the Grant Agency of Charles University under the 588912 project.

and 1990s, and Barlevy & Veronesi (2003) proposed a model based on rational but uninformed traders who can unreasonably panic. Again with this approach, abrupt declines in stock prices can occur without any real change in the underlying fundamentals. Lux (1995) linked the phenomena of market crashes to the process of phase transition from thermodynamics and modeled the emergence of bubbles and crashes as a result of herd behavior among heterogeneous traders in speculative markets. Finally, a strand of literature documenting precursory patterns and log-periodic signatures years before the largest crashes in the modern history suggested that crashes have an endogenous origin in 'crowd' behavior and through the interactions of many agents (Sornette & Johansen 1998; Johansen *et al.* 2000; Sornette 2002; 2003). In contrast to many commonly shared beliefs, Didier Sornette and his colleagues argued that exogenous shocks can only serve as triggers and not as the direct causes of crashes and that large crashes are 'outliers'.

Catastrophe theory provides a very different theoretical framework to understand how even small shifts in the speculative part of the market can trigger a sudden, discontinuous effect on prices. Catastrophe theory was proposed by French mathematician Thom (1975) with the aim of shedding some light on the 'mystery' of biological morphogenesis. Despite its mathematical virtues, the theory was promptly heavily criticized by Zahler & Sussmann (1977) and Sussmann & Zahler (1978a;b) for its excessive utilization of qualitative approaches, the improper usage of certain statistical methods and for violations of necessary mathematical assumptions in many of its applications. Due to these criticisms, the intellectual bubble and the heyday of the cusp catastrophe approach declined rapidly after the 1970s, although the theory was defended by some researchers, e.g., by Boutot (1993) and the extensive, gradually updated work of Arnold (2004). Nonetheless, the 'fatal' criticism was ridiculed by Rosser (2007, p. 3275 & 3257), who stated that "the baby of catastrophe theory was largely thrown out with the bathwater of its inappropriate applications", and the author suggested that "economists should reevaluate the former fad and move it to a more proper valuation".

From the field of mathematical biology the area of application flourished especially towards other natural sciences. Catastrophe theory was utilized e.g. in mathematics and statistics (Sussmann 1978; Novak 1986), operational research (Bonanno & Zeeman 1988), and informatics (Thornton & Hung 1996; Sethi & King 1998). The new method was also utilised in other branches of biology (summaries of older applications were written by Rosen 1979; Deakin

1990; van Harten 2000; Torres 2001), chemistry (Calo & Chang 1980), and physics (Puu 1988; Beckmann & Puu 1990; Aerts *et al.* 2003; Tamaki *et al.* 2003; Kostomarov *et al.* 2012). A survey covering older applications across the field of physics was written by Stewart (1981). In geography, we have to mention the contradiction about the usefulness of catastrophe theory between Wagstaff (1978; 1979) and Baker (1979), and as an example of the application in geology we refer to Henley (1976). From the most recent applications, frequent contributions occurred especially within the field of building and construction industry (Kounadis 2002; Yiu & Cheung 2006; Yang *et al.* 2010; Xiaoping *et al.* 2010) and ecology (Roopnarine 2008; Wang *et al.* 2011; Piyaratne *et al.* 2013).

The application of catastrophe theory in the social sciences has not been as extensive as in the natural sciences, although it was utilized early in its existence. Zeeman's (1974) cooperation with Thom and his own popularization of the theory through the use of nontechnical examples (Zeeman 1975; 1976) led to the development of many applications in the fields of economics, psychology, sociology, political studies, and others. A catastrophe theory approach to the theory of economic equilibria is presented by Balasko (1978a;b). Jammernegg & Fischer (1986) and Fischer & Jammernegg (1986) survey utilization of the cusp model in economics and analyze a catastrophe theory model of the Phillips Curve, which they further empirically estimate using U.S. data. One of the most recent contributions to the economic theory is suggested by Accinelli & Anyul (2005). In the field of finance, Ho & Saunders (1980) apply the theory of catastrophes to model bank failures as the interaction between bank management, regulatory bodies, and depositors, Scapens *et al.* (1981) use the cusp catastrophe approach within accounting and corporate finance, and to mention one of the most actual theoretical contribution, we refer to Pleten (2012). In the neighbouring social sciences, perhaps the most frequently catastrophe models were developed for explaining various psychological phenomena. From many let us mention the survey of older works by Flay (1978), applications in clinical psychology such as anorexia nervosa, schizophrenia, aggressiveness, and others by Scott (1985), catastrophe approach to cognitive development by van der Maas & Molenaar (1992), empirical application for predicting adolescent alcohol use by Clair (1998) as the interaction between ones disposition and the situational pressure, or a very recent research on the effect of human emotions on academic performance by Vitasari *et al.* (2011). Within the field of sociology, we refer for instance to Holyst *et al.* (2000) and their research on the social impact of opinion formation, or to van der Maas *et al.* (2003) and their work focused on methodological discussion as well as empirical testing of people attitude formation. In political studies, we mention the applications to the study of international conflicts (Holt *et al.* 1978; Dockery & Chiatti 1986), analysis of political changes and development (Adelman & Hihn 1982), or political opinion formation (Weidlich & Huebner 2008). Catastrophe theory approach was also utilized within the area of education (Stamovlasis & Tsaparlis 2012), marketing (Oliva *et al.* 1995; Dou & Ghose 2006), and even linguistics (Petitot 1989; Bernardez 1995).

Zeeman (1974) also proposed the application of the cusp catastrophe model to stock markets. Translating seven qualitative hypotheses about stock exchanges to the mathematical terminology of catastrophe theory produced one of the first heterogeneous agent models for two main types of investors: fundamentalists and chartists. Heterogeneity and the interactions between these two distinct types of agents attracted wider attention in the behavioral finance literature. Fundamentalists base their expectations about future asset prices on their beliefs about fundamental and economic factors such as dividends, earnings, and the macroeconomic environment. In contrast, chartists do not consider fundamentals in their trading strategies at all. Their expectations about future asset prices are based on finding historical patterns in prices. While Zeeman's work was only one qualitative description of observed bull and bear markets, it contained a number of important behavioral elements that were later used in the large volume of literature that focused on heterogeneous agent modeling.[2] Today, the statistical theory is well developed, and parameterized cusp catastrophe models can be evaluated quantitatively based on data.

The biggest difficulty in the application of catastrophe theory arises from the fact that it stems from deterministic systems. Thus, it is difficult to apply it directly to systems that are subject to random influences, which are common in the behavioral sciences. Cobb & Watson (1980); Cobb (1981) and Cobb & Zacks (1985) provided the necessary bridge and took catastrophe theory from determinism to stochastic systems. While this was an important shift, there are further complications in the theory's application to stock market data. The main restriction of Cobb's method of estimation was the requirement of a constant variance, which forces researchers to assume that the volatility of the stock markets (as the standard deviation of the returns) is constant.

[2]For a recent survey of heterogeneous agent models, see Hommes (2006). A special issue on heterogeneous interacting agents in financial markets edited by Lux & Marchesi (2002) also provides interesting contributions.

Quantitative verification of Zeeman's (1974) hypotheses about the application of the theory to stock market crashes was pioneered by Barunik & Vosvrda (2009), where authors fit the cusp model to two separate, large stock market crashes. However, the successful application Barunik & Vosvrda (2009) brought only preliminary results in a restricted environment. Application of the cusp catastrophe theory on stock market data deserves much more attention. In the current work, we propose an improved method of application that we believe brings us closer to an answer regarding whether cusp catastrophe theory is capable of explaining stock market crashes.

Time-varying volatility has become an important stylized fact for stock market data, and researchers have recognized that it is an important feature of any modeling strategy. One of the most successful early works of Engle (1982); Bollerslev (1986) proposed including volatility as a time-varying process in a (generalized) autoregressive conditional heteroskedasticity framework. From that beginning, many models have been developed in the literature to improve the original frameworks. As early as the late 1990s, high frequency data became available to researchers, and this led to another important shift in volatility modeling — realized volatility. A very simple, intuitive approach to compute daily volatility using the sum of squared high-frequency returns was formalized by Andersen *et al.* (2003); Barndorff-Nielsen & Shephard (2004). While the volatility literature is immense,[3] several researchers have also studied volatility and stock market crashes. For example, Shaffer (1991) argued that volatility might be the cause of a stock market crash. In contrast, Levy (2008) argued that volatility increases before a crash, even when no dramatic information is revealed.

In this study, we utilize the availability of high-frequency data and the popular realized volatility approach to propose a two-step method of estimation that overcomes the difficulties in the application of cusp catastrophe theory to stock market data.[4] Using realized volatility, we estimate stock market returns' volatility, and then we apply the stochastic cusp catastrophe model on volatility-adjusted returns with constant variance. This approach is motivated by the confirmed bimodal distributions of such standardized data in some periods, and it allows us to study whether stock markets are driven into catastrophe

[3] Andersen *et al.* (2004) provide a very useful and complete review of the methodologies.

[4] In fact, a two-step estimation method can be applied also using other possible models for volatility, e.g. the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) family. However, the popular realized volatility approach provides us with a supreme methodology.

endogenously or whether it is simply an effect of volatility. We also run simulations that provide strong support for the methodology. The simulations also illustrate the importance of stochastic noise and volatility in the deterministic cusp model.

Using a unique dataset covering almost 27 years of the U.S. stock market evolution, we empirically test the stochastic cusp catastrophe model in a time-varying volatility environment. Moreover, we develop a rolling regression approach to study the dynamics of the model's parameters over a long period, covering several important recessions and crises. This approach allows us to localize the bifurcation periods.

We need to mention several important works that provided similar results to ours. Creedy & Martin (1993); Creedy *et al.* (1996) developed a framework for the estimation of non-linear exchange rate models, and they showed that swings in exchange rates can be attributed to bimodality even without the explicit use of catastrophe theory. More recently, Koh *et al.* (2007) proposed using Cardan's discriminant to detect bimodality and confirm the predictive ability of currency pairs for emerging countries. In our work, we bring new insight to the non-linear phenomena by including time-varying volatility in the modeling strategy.

The chapter is organized as follows. The second and the third sections examine the theoretical framework of the stochastic catastrophe theory under time-varying volatility and describe the model's estimation. The fourth section presents the simulations that support our two-step method of estimation, and the fifth section presents the empirical application of the theory on the modeling of stock market crashes. Finally, the last section concludes.

3.1 Theoretical framework

Catastrophe theory was developed as a deterministic theory for systems that may respond to continuous changes in control variables by a discontinuous change from one equilibrium state to another. A key idea is that the system under study is driven toward an equilibrium state. The behavior of the dynamical systems under study is completely determined by a so-called potential function, which depends on behavioral and control variables. The behavioral, or state, variable describes the state of the system, while control variables determine the behavior of the system. The dynamics under catastrophe models can become extremely complex and according to the classification theory of

Thom (1975), there are seven different families based on the number of control and dependent variables. We focus on the application of catastrophe theory to model sudden stock market crashes, as qualitatively proposed by Zeeman (1974). In his work, Zeeman used the so-called cusp catastrophe model, which is the simplest specification that gives rise to sudden discontinuities.

3.1.1 Deterministic dynamics

Let us suppose that the process y_t evolves over $t = 1, \ldots, T$ as

$$dy_t = -\frac{dV(y_t; \alpha, \beta)}{dy_t} dt, \tag{3.1}$$

where $V(y_t; \alpha, \beta)$ is the potential function describing the dynamics of the state variable y_t controlled by parameters α and β determining the system. When the right-hand side of Eq. (3.1) equals zero, $-dV(y_t; \alpha, \beta)/dy_t = 0$, the system is in equilibrium. If the system is at a state of non-equilibrium, it will move back to equilibrium where the potential function takes the minimum values with respect to y_t. While the concept of potential function is very general, i.e., it can be a quadratic function yielding equilibrium of a simple flat response surface, one of the most applied potential functions in behavioral sciences, a cusp potential function, is defined as

$$-V(y_t; \alpha, \beta) = -1/4 y_t^4 + 1/2 \beta y_t^2 + \alpha y_t, \tag{3.2}$$

with equilibria at

$$-\frac{dV(y_t; \alpha, \beta)}{dy_t} = -y_t^3 + \beta y_t + \alpha \tag{3.3}$$

being equal to zero. The two dimensions of the control space, α and β, further depend on realizations from $i = 1 \ldots, n$ of the independent variables $x_{i,t}$. Thus, it is convenient to think about α and β as functions

$$\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \ldots + \alpha_n x_{n,t} \tag{3.4}$$

$$\beta_x = \beta_0 + \beta_1 x_{1,t} + \ldots + \beta_n x_{n,t}. \tag{3.5}$$

The control functions α_x and β_x are called normal and splitting factors, or asymmetry and bifurcation factors, respectively (Stewart & Peregoy 1983), and they determine the predicted values of y_t given $x_{i,t}$. Therefore, for each combination of values of independent variables, there could be up to three predicted values of the state variable given by roots of

$$-\frac{dV(y_t;\alpha_x,\beta_x)}{dy_t} = -y_t^3 + \beta_x y_t + \alpha_x = 0. \tag{3.6}$$

This equation has one solution if

$$\delta_x = 1/4\alpha_x^2 - 1/27\beta_x^3 \tag{3.7}$$

is greater than zero, $\delta_x > 0$, and three solutions if $\delta_x < 0$. This construction was first described by the 16^{th} century mathematician Geronimo Cardan and can serve as a statistic for bimodality, one of the catastrophe flags. The set of values for which Cardan's discriminant is equal to zero, $\delta_x = 0$, is the bifurcation set that determines the set of singularity points in the system. In the case of three roots, the central root is called an "anti-prediction" and is the least probable state of the system. Inside the bifurcation, when $\delta_x < 0$, the surface predicts two possible values of the state variable, which means that in this case, the state variable is bimodal. For an illustration of the deterministic response surface of cusp catastrophe, we borrow from Figure 3.2 in the simulations section, where the deterministic response surface is a smooth pleat.

3.1.2 Stochastic dynamics

Most of the systems in behavioral sciences are subject to noise stemming from measurement errors or the inherent stochastic nature of the system under study. Thus, for real-world applications, it is necessary to add non-deterministic behavior into the system. Because catastrophe theory was primarily developed to describe deterministic systems, it may not be obvious how to extend the theory to stochastic systems. An important bridge was provided by Cobb & Watson (1980); Cobb (1981) and Cobb & Zacks (1985), who used the Itô stochastic differential equations to establish a link between the potential function of a deterministic catastrophe system and the stationary probability density function of the corresponding stochastic process. This approach in turn led to the definition of a stochastic equilibrium state and bifurcation that was compatible with the deterministic counterpart. Cobb and his colleagues simply added a stochastic Gaussian white noise term to the system

$$dy_t = -\frac{dV(y_t;\alpha_x,\beta_x)}{dy_t}dt + \sigma_{y_t} dW_t, \tag{3.8}$$

where $-dV(y_t; \alpha_x, \beta_x)/dy_t$ is the deterministic term, or drift function representing the equilibrium state of the cusp catastrophe, σ_{y_t} is the diffusion function and W_t is a Wiener process. When the diffusion function is constant, $\sigma_{y_t} = \sigma$, and the current measurement scale is not to be nonlinearly transformed, the stochastic potential function is proportional to the deterministic potential function, and the probability distribution function corresponding to the solution from Eq. (3.8) converges to a probability distribution function of a limiting stationary stochastic process because the dynamics of y_t are assumed to be much faster than changes in $x_{i,t}$ (Cobb 1981; Cobb & Zacks 1985; Wagenmakers *et al.* 2005). The probability density that describes the distribution of the system's states at any t is then

$$f_s(y|x) = \psi \exp\left(\frac{(-1/4)y^4 + (\beta_x/2)y^2 + \alpha_x y}{\sigma}\right). \tag{3.9}$$

The constant ψ normalizes the probability distribution function, so its integral over the entire range equals one. As the bifurcation factor β_x changes from negative to positive, the $f_s(y|x)$ changes its shape from unimodal to bimodal. Conversely, α_x causes asymmetry in $f_s(y|x)$.

3.1.3 Cusp catastrophe under time-varying volatility

Stochastic catastrophe theory works only under the assumption that the diffusion function is constant, $\sigma_{y_t} = \sigma$, and the current measurement scale is not to be nonlinearly transformed. While this assumption may be reliable in some applications in the behavioral sciences, it may cause crucial difficulties in others. One of the problematic applications is in modeling stock market crashes because the diffusion function σ, called the volatility of stock market returns, has strong time-varying dynamics, and it clusters over time, which is documented by strong dependence in the squared returns. To illustrate the volatility dynamics, let us borrow the dataset used later in this study. Figure 3.4 shows the evolution of the S&P 500 stock index returns over almost 27 years and documents how volatility strongly varies over time. One of the possible and very simple solutions in applying cusp catastrophe theory to the stock markets is to consider only a short time window and to fit the catastrophe model to data where volatility can be assumed to be constant (Barunik & Vosvrda 2009). Although Barunik & Vosvrda (2009) were the first to quantitatively apply stochastic catastrophes to explain stock market crashes on localized periods of crashes, this assumption is generally very restrictive.

Here, we propose a more rigorous solution to the problem by utilizing the recently developed concept of realized volatility. This approach allows us to use the previously introduced concepts after estimating the volatility from the returns process consistently, and we are able to estimate the catastrophe model on the process that fulfills the assumptions of the stochastic catastrophe theory. Thus, we assume that stock markets can be described by the cusp catastrophe process subject to time-varying volatility. While this approach represents a great advantage that allows us to apply cusp catastrophe theory to different time periods conveniently, the disadvantage is that the method cannot be generalized to other branches of the behavioral sciences where high-frequency data are not available and therefore realized volatility cannot be computed. Thus, our generalization is mainly restricted to applications on financial data. Still, our main aim is to study stock market crashes, and therefore the advocated approach is very useful in the field of behavioral finance. We now describe the theoretical concept, and in the next sections, we will present the full model and the two-step estimation procedure.

Suppose that the sample path of the corresponding (latent) logarithmic price process p_t is continuous over $t = 1, \ldots, T$ and determined by the stochastic differential equation

$$dp_t = \mu_t dt + \sigma_t dW_t, \tag{3.10}$$

where μ_t is a drift term, σ_t is the predictable diffusion function, or instantaneous volatility, and W_t is a standard Brownian motion. A natural measure of the ex-post return variability over the $[t-h, t]$ time interval, $0 \leq h \leq t \leq T$ is the integrated variance

$$IV_{t,h} = \int_{t-h}^{t} \sigma_\tau^2 d\tau, \tag{3.11}$$

which is not directly observed, but as shown by Andersen *et al.* (2003) and Barndorff-Nielsen & Shephard (2004), the corresponding realized volatilities provide its consistent estimates. While it is convenient to work in the continuous time environment, empirical investigations are based on discretely sampled prices, and we are interested in studying h-period continuously compounded discrete-time returns $r_{t,h} = p_t - p_{t-h}$. Andersen *et al.* (2003) and Barndorff-Nielsen & Shephard (2004) showed that daily returns are Gaussian, conditional on an information set generated by the sample paths of μ_t and σ_t, and integrated volatility normalizes the returns as

$$r_{t,h}\left(\int_{t-h}^{t}\sigma_{\tau}^{2}d\tau\right)^{-1/2}\sim N\left(\int_{t-h}^{t}\mu_{\tau}d\tau,1\right). \qquad (3.12)$$

This result of quadratic variation theory is important to us because we use it to study stochastic cusp catastrophe in an environment where volatility is time varying. In the modern stochastic volatility literature, it is common to assume that stock market returns follow the very general semi-martingale process (as in Eq. 3.10), where the drift and volatility functions are predictable and the rest is unpredictable. In the origins of this stream of literature, one of the very first contributions published regarding stochastic volatility by Taylor (1982) assumed that daily returns are the product of a volatility and autoregression process. In our application, we also assume that daily stock market returns are described by a process that is the product of volatility and the cusp catastrophe model.

To formulate the approach, we assume that stock returns normalized by their volatility

$$y_t^{*}=r_t\left(\int_{t-h}^{t}\sigma_{\tau}^{2}d\tau\right)^{-1/2} \qquad (3.13)$$

follow a stochastic cusp catastrophe process

$$dy_t^{*}=-\frac{dV(y_t^{*};\alpha_x,\beta_x)}{dy_t^{*}}dt+dW_t. \qquad (3.14)$$

It is important to note the difference between Equation (3.14) and Equation (3.8) because there is no longer any diffusion term in the process. Because the diffusion term of y_t^{*} is constant and now equal to one, Cobb's results can conveniently be used, and we can use the stationary probability distribution function of y_t^{*} for the parameter estimation using the maximum likelihood method.

As noted previously, the integrated volatility is not directly observable. However, the now-popular concept of realized volatility and the availability of high-frequency data provide a simple method to accurately measure integrated volatility, which helps us propose a simple and intuitive method to estimate the cusp catastrophe model on stock market returns under highly dynamic volatility.

3.2 Estimation

A simple, consistent estimator of the integrated variance under the assumption of no microstructure noise contamination in the price process is provided by the well-known realized variance (Andersen *et al.* 2003; Barndorff-Nielsen & Shephard 2004). The realized variance over $[t-h, t]$, for $0 \leq h \leq t \leq T$, is defined by

$$\widehat{RV}_{t,h} = \sum_{i=1}^{N} r^2_{t-h+\left(\frac{i}{N}\right)h}, \tag{3.15}$$

where N is the number of observations in $[t-h, t]$, and $r_{t-h+\left(\frac{i}{N}\right)h}$ is i−th intraday return in the $[t-h, t]$ interval. $\widehat{RV}_t$ converges in probability to the true integrated variance of the process as $N \to \infty$

$$\widehat{RV}_{t,h} \xrightarrow{p} \int_{t-h}^{t} \sigma^2_\tau d\tau. \tag{3.16}$$

As observed, the log-prices are contaminated with microstructure noise in the real world, and the literature has developed several estimators. While it is important to consider both jumps and microstructure noise in the data, our main interest is in estimating the catastrophe theory and addressing the question whether it can be used to explain the deterministic portion of stock market returns. Thus, we restrict ourselves to the simplest estimator, which uses sparse sampling to deal with the microstructure noise. The extant literature showed support for this simple estimator; most recently, Liu *et al.* (2012) ran a horse race for the most popular estimators and concluded that when simple realized volatility is computed using 5-minute sampling, it is very difficult to outperform.

In the first step, we estimate realized volatility from the stock market returns using 5-min. data as proposed by the theory, and we normalize the daily returns to obtain returns with constant volatility. While using the daily returns, $h = 1$, and we henceforth drop h for ease of notation.

$$\tilde{r}_t = r_t \widehat{RV}_t^{-1/2} \tag{3.17}$$

In the second step, we apply the stochastic cusp catastrophe to model the normalized stock market returns. While the state variable of the cusp is a canonical variable, it is an unknown smooth transformation of the actual state variable of the system. As proposed by Grasman *et al.* (2009), we allow for the first order approximation to the true, smooth transition allowing the measured $\tilde{r}$ to be a

$$y_t = \omega_0 + \omega_1 \tilde{r}_t, \tag{3.18}$$

with ω_1 as the first order coefficient of a polynomial approximation. The independent variables are

$$\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \ldots + \alpha_n x_{n,t} \tag{3.19}$$
$$\beta_x = \beta_0 + \beta_1 x_{1,t} + \ldots + \beta_n x_{n,t}, \tag{3.20}$$

Hence, the statistical estimation problem is to estimate $2n + 2$ parameters $\{\omega_0, \omega_1, \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \ldots, \beta_n\}$. We estimate the parameters using the maximum likelihood approach of Cobb & Watson (1980) as augmented by Grasman *et al.* (2009). The negative log-likelihood for a sample of observed values $(x_{1,t}, \ldots, x_{n,t}, y_t)$ for $t = 1, \ldots, T$ is simply the logarithm of the probability distribution function in Eq. (3.9).

3.2.1 Statistical evaluation of the fit

To assess the fit of the cusp catastrophe model to the data, a number of diagnostic tools have been suggested. Stewart & Peregoy (1983) proposed a pseudo-R^2 as a measure of the explained variance. However, a difficulty arises here because for a given set of values of the independent variables, the model may predict multiple values for the dependent variable. Because of bimodal density, the expected value is unlikely to be observed because it is an unstable solution at equilibrium. For this reason, two alternatives for the expected value as the predictive value can be used. The first method chooses the mode of the density closest to the state values, which is known as the delay convention; the second method uses the mode at which the density is highest, which is known as the Maxwell convention. Cobb & Watson (1980) and Stewart & Peregoy (1983) suggested using the delay convention where the variance of the error is defined as the variance of the difference between the estimated states and then using the mode of the distribution that is closest to this value. The pseudo-R^2 is defined as $1 - Var(\epsilon)/Var(y)$, where ϵ is error.

While pseudo-R^2 is problematic due to the nature of the cusp catastrophe model, it should be used in a complementary fashion to other alternatives. To rigorously test the statistical fit of the cusp catastrophe model, we use following steps. First, the cusp fit should be substantially better than multiple linear regression. The cusp fit could be tested by means of a likelihood ratio test, which is asymptotically chi-squared distributed with degrees of freedom equal to the difference in degrees of freedom for two compared models. Second, the ω_1 coefficient should deviate significantly from zero. Otherwise, the y_t in Eq. (3.18) would be constant, and the cusp model would not describe the data. Third, the cusp model should show a better fit than the following logistic curve:

$$y_t = \frac{1}{1 + e^{-\alpha_t/\beta_t^2}} + \epsilon_t, \tag{3.21}$$

for $t = 1, \dots, T$, where ϵ_t are zero mean random disturbances. The rationale for choosing to compare the cusp model to this logistic curve is that this function does not possess degenerate points, while it possibly models steep changes in the state variable as a function of changes in the independent variables mimicking the sudden transitions of the cusp. Thus, a comparison of the cusp catastrophe model to the logistic function serves as a good indicator of the presence of bifurcations in the data. While these two models are not nested, Wagenmakers *et al.* (2005) suggested comparing them via information criteria, where a stronger Bayesian Information Criterion (BIC) should be required for the decision.

3.3 Monte Carlo study

To validate our assumptions about the process of generating stock market returns and our two-step estimation procedure, we conduct a Monte Carlo study where we simulate the data from the stochastic cusp catastrophe model, allow for time-varying volatility in the process and estimate the parameters to see whether we can recover the true values.

We simulate the data from the stochastic cusp catastrophe model subject to time-varying volatility as

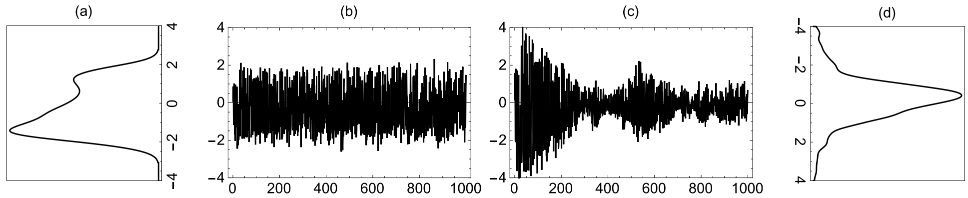
$$r_t = \sigma_t y_t \tag{3.22}$$

$$d\sigma_t^2 = \kappa(\omega - \sigma_t^2)dt + \gamma dW_{t,1}, \tag{3.23}$$

$$dy_t = (\alpha_t + \beta_t y_t - y_t^3)dt + dW_{t,2} \tag{3.24}$$

where $dW_{t,1}$ and $dW_{t,2}$ are standard Brownian motions with zero correlation, $\kappa = 5$, $\omega = 0.04$ and $\gamma = 0.5$. The volatility parameters satisfy Feller's condition $2\kappa\omega \geq \gamma^2$, which keeps the volatility process away from the zero boundary. We

Figure 3.1: An example of a simulated time series



Note: An example of a simulated time series where the cusp surface is subject to noise only, y_t, and noise together with volatility, r_t. (b) simulated returns y_t (a) kernel density estimate of y_t (c) simulated returns r_t contaminated with volatility (d) kernel density estimate of r_t.

Source: Authors' own computations in *R* and *Wolfram Mathematica*.

set the parameters to values that are reasonable for a stock price, as in Zhang *et al.* (2005).

In the cusp equation, we use two independent variables

$$\alpha_t = \alpha_0 + \alpha_1 x_{t,1} + \alpha_2 x_{t,2} \quad (3.25)$$

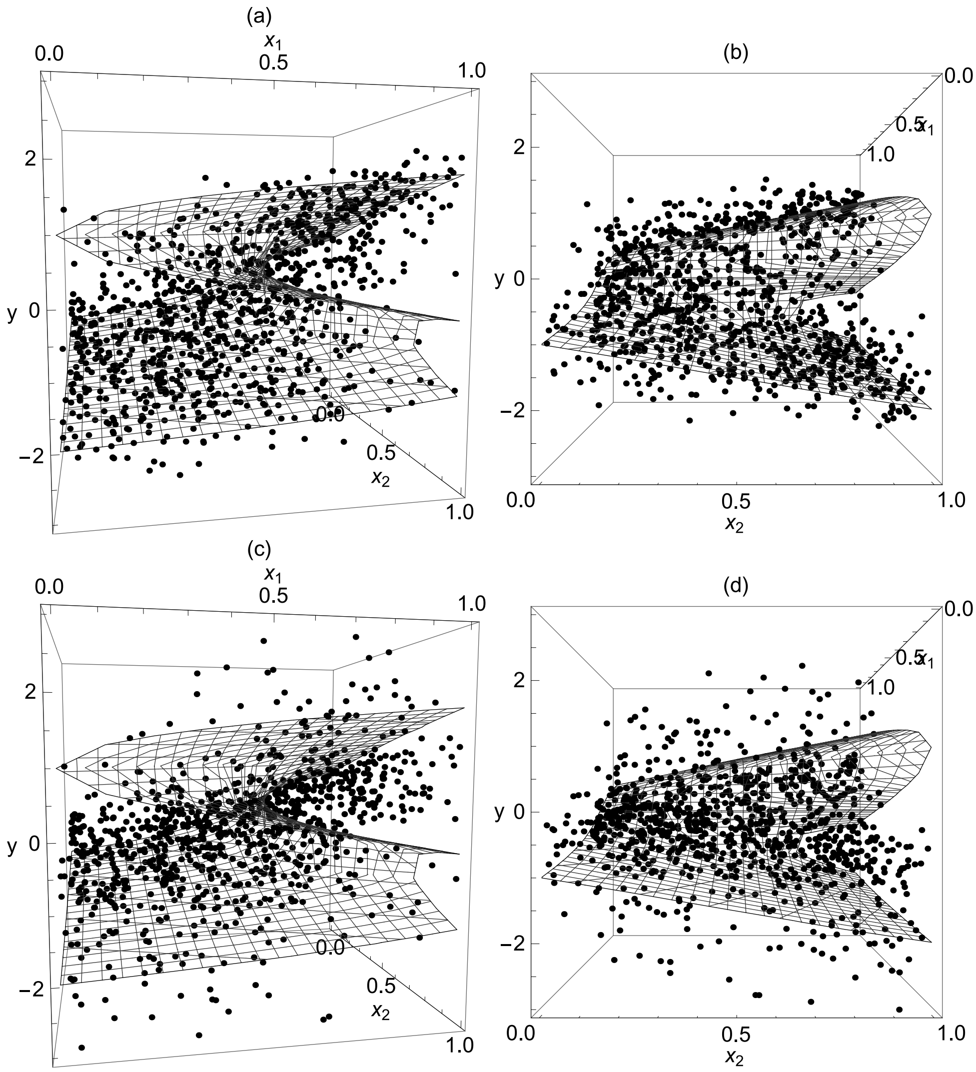
$$\beta_t = \beta_0 + \beta_1 x_{t,1} + \beta_2 x_{t,2} \quad (3.26)$$

with coefficients $\alpha_2 = \beta_1 = 0$. Hence $x_{t,1}$ drawn from the $U(0,1)$ distribution drives the asymmetry side, and $x_{t,2}$ drawn from the $U(0,1)$ distribution drives the bifurcation side of the model. The parameters are set as $\alpha_0 = -2$, $\alpha_1 = 3$, $\beta_0 = -1$ and $\beta_2 = 4$.

In the simulations, we are interested in determining how the cusp catastrophe model performs under time-varying volatility. Thus, we estimate the coefficients on the processes $y_t = r_t/\sigma_t$ and r_t. Figure 3.1 shows one realization of the simulated returns y_t and r_t. While y_t is the cusp catastrophe subject to noise, r_t is subject to time-varying volatility as well. It is noticeable how time-varying volatility causes the shift from bimodal density to unimodal. More illustrative is Figure 3.2, which shows the cusp catastrophe surface of both processes. While the solution from the deterministic cusp catastrophe is contaminated with noise in the first case, the volatility process in the second case makes it much more difficult to recognize the two states of the system in the bifurcation area. This result causes difficulty in recovering the true parameters.

Table 3.1 shows the results of the simulation. We simulate the processes

Figure 3.2: An example of simulated data



Note: An example of simulated data where the cusp surface is subject to noise only, y_t, and noise with volatility, r_t. Parts (a-b) show the cusp deterministic pleat simulated $\{x_1, x_2, y_t\}$ from two different perspectives, and parts (c-d) show the cusp deterministic pleat simulated $\{x_1, x_2, r_t\}$ from two different perspectives.

Source: Authors' own computations in *R* and *Wolfram Mathematica*.

100 times and report the mean and standard deviations from the mean.[5] We

[5]The distribution of the parameters of our main interest using $y_t = r_t/\sigma_t$ is Gaussian, which makes it possible to compare the results within the means and standard deviations. Figures A.1 and A.2 in Appendix A depict smooth histogram kernel approximations of the probability densities of respective estimates contrasted to normal distribution. Based on the Jarque-Bera ALM test at the 5% level, the null hypothesis of normality is only rejected for ω_0 and α_2 of the unrestricted model. Moreover, ω_0 is a constant term and α_2 is left out in the restricted model which further testifies the Gaussianity of the key parameters. For the restricted model normality of parameters is not rejected in any case.

estimate all parameters of the cusp equation, i.e. ω_0 and ω_1, defining the first order approximation of a smooth transformation of the actual state variable, together with six parameters $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ determining two independent variables. The true parameters are easily recovered in the simulations from y_t when the cusp catastrophe is subject to noise only because the mean values are statistically indistinguishable from the true simulated values $\alpha_0 = -2$, $\alpha_1 = 3$, $\beta_0 = -1$ and $\beta_2 = 4$. The fits are reasonable because they explain approximately 60% of the data variation in the noisy environment. Moreover, in the cusp model, we first estimate the full set of parameters $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$, and then we restrict the parameters $\alpha_2 = \beta_1 = 0$. The estimation easily recovers the true parameters in both cases, while in the unrestricted case, the estimates $\alpha_2 = \beta_1 = 0$ and fits are statistically the same. In comparison, both cusp models perform much better than logistic regression and linear models, which was expected. It is also interesting to note that $\omega_0 = 0$ and $\omega_1 = 1$, which means that the observed data are the true data, and no transformation is needed. These results are important because they confirm that the estimation of the stochastic cusp catastrophe model is valid, and it can be used to quantitatively apply the theory to the data.

The results of the estimation on the r_t process, which is subject to time-varying volatility, reveal that the addition of the volatility process makes it difficult for the maximum likelihood estimation to recover the true parameters. The variances of the estimated parameters are very large, and the means are far away from the true simulated values. Moreover, the fits are statistically weaker, as they explain no more than 38% of the variance in the data. It is also interesting to note that the logistic fit and the linear fit are much closer to the cusp fit.

In conclusion, the simulation results reveal that time-varying volatility in the cusp catastrophe model destroys the ability of the maximum likelihood estimator to recover the cusp potential.

3.4 Empirical modeling of stock market crashes

Armed with the results from the simulations, we move to the estimation of the cusp catastrophe model on the real-world data from stock markets. We use long time span for the S&P 500, a broad U.S. stock market index, that covers almost 27 years, from February 24, 1984 to November 17, 2010. Figure 3.3 plots the prices and depicts the several recessions and crisis periods. According to

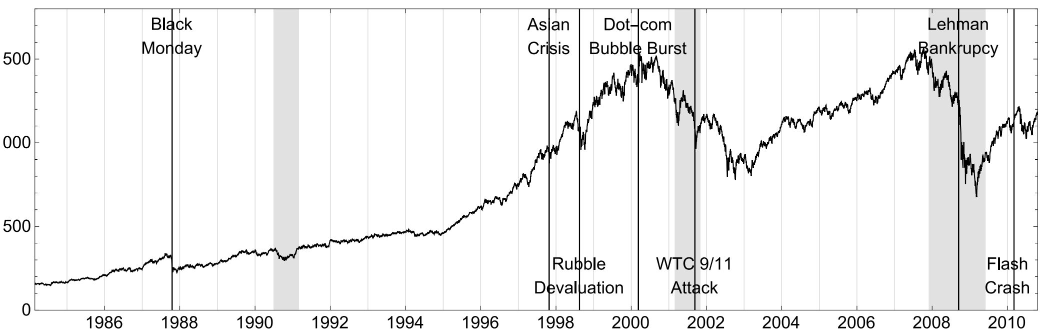
Table 3.1: Simulation results

	(a) Estimates using $y_t = r_t/\sigma_t$				(b) Estimates using r_t			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	-2.059 (0.143)	-2.066 (0.141)			-1.614 (3.352)	-3.111 (0.531)		
α_1	3.083 (0.195)	3.084 (0.203)			4.355 (1.382)	4.540 (0.690)		
α_2	-0.009 (0.118)				-1.126 (2.121)			
β_0	-1.016 (0.237)	-1.018 (0.187)			-7.088 (3.287)	-5.428 (2.420)		
β_1	0.001 (0.319)				2.394 (1.806)			
β_2	4.003 (0.232)	4.006 (0.223)			5.821 (1.430)	5.544 (1.293)		
ω_0	-0.005 (0.035)	-0.006 (0.025)			0.163 (0.305)	-0.038 (0.053)		
ω_1	1.003 (0.021)	1.003 (0.021)			0.539 (0.155)	0.552 (0.154)		
R^2	0.609 (0.022)	0.608 (0.023)	0.397 (0.025)	0.462 (0.027)	0.378 (0.043)	0.379 (0.044)	0.345 (0.034)	0.401 (0.038)
LL	-888.6 (24.3)	-889.4 (24.1)	-1323.3 (21.4)	-1266.4 (23.5)	-1109.4 (52.7)	-1117.6 (57.6)	-1471.7 (212.6)	-1426.5 (211.4)
AIC	1793.1 (48.5)	1790.9 (48.3)	2654.7 (42.7)	2546.9 (47.0)	2234.8 (105.5)	2247.3 (115.3)	2951.3 (425.1)	2867.0 (422.9)
BIC	1832.4 (48.5)	1820.3 (48.3)	2674.3 (42.7)	2581.2 (47.0)	2274.0 (105.5)	2276.7 (115.3)	2970.9 (425.1)	2901.4 (422.9)

Note: Simulation results (a) according to the stochastic cusp catastrophe model and (b) according to the stochastic cusp catastrophe model with process in volatility. The total sample based on 100 random simulations is used, and the sample means and standard deviations (in parentheses) for each value are reported. All figures are rounded to one or three decimal digits.

Source: Authors' own computations in *R*.

Figure 3.3: S&P 500 price data



Note: The figure highlights several important recession periods as grey periods and crash events as black lines. The periods are more closely described in the text.

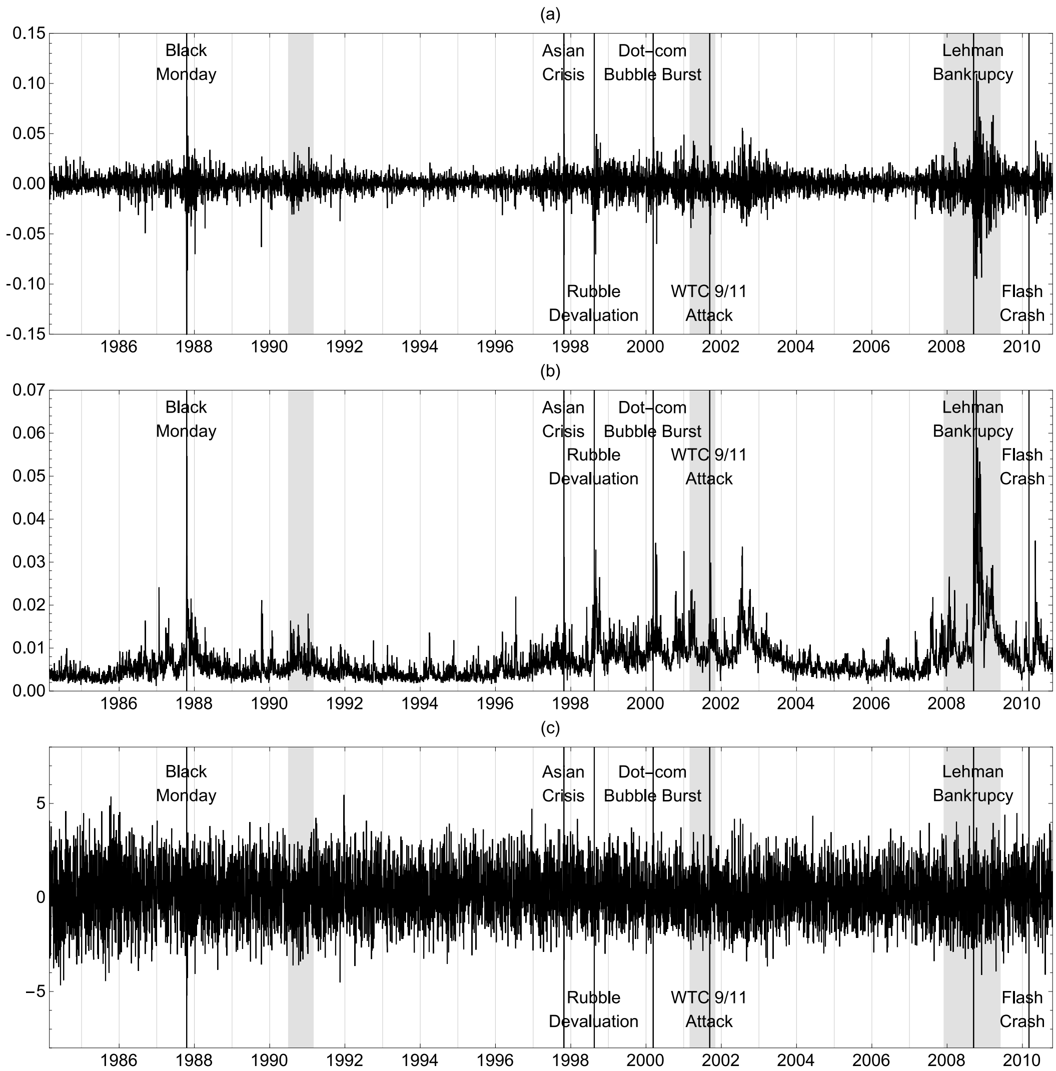
Source: Authors' own computations in *R* and *Wolfram Mathematica*.

the National Bureau of Economic Research (NBER), there were three U.S. recessions during the periods of July 1990 – March 1991, March 2001 – November 2001 and December 2007 – June 2009. These recessions are depicted as grey periods. Black lines depict one-day crashes associated with large price drops. Namely, these include Black Monday 1987 (October 19, 1987), the Asian Crisis Crash (October 27, 1997), the Ruble Devaluation of 1998 (August 17, 1998), the Dot-com Bubble Burst (March 10, 2000), the World Trade Center Attacks (September 11, 2001), the Lehman Brothers Holdings Bankruptcy (September 15, 2008), and finally the Flash Crash (March 6, 2010). Technically, the largest one day drops in the studied period occurred on October 19, 1987, October 26, 1987, September 29, 2008, October 9, 2008, October 15, 2008, and December 1, 2008, recording declines of 20.47%, 8.28%, 8.79%, 7.62%, 9.03%, and 8.93%, respectively.

Let us now look closer at the crashes depicted by Figure 3.3 and discuss their nature. The term Black Monday refers to Monday, October 19, 1987 when stock markets around the world from Hong Kong to Europe and the U.S. crashed in a very short time and recorded the largest one-day drop in history. After this unexpected, severe event, many analysts predicted the most troubled years since the 1930s. However, stock markets gained the losses back and closed the year positively. There has been no consensus opinion on the cause of the crash. Potential causes include program trading, overvaluation, illiquidity, and market psychology. Thus, this crash seems to have had an endogenous cause. Stock markets did not record any large shocks for the next several years until 1996, when the Asian Financial Crisis driven by investors deserting overheated, emerging Asian markets resulted in the October 27, 1997 mini crash of the U.S. markets. The next year, the Russian government devalued the ruble, defaulted on its domestic debt and declared a moratorium on payments to foreign creditors. These actions caused another international crash on August 17, 1998. These last two shocks are believed to be exogenous to the U.S. stock markets. During the period from 1997 – 2000, the so-called dot-com bubble emerged, when a group of internet-based companies entered the markets and attracted many investors who were confident in the companies' profits, overlooking their fundamental values. The result was a collapse, or burst bubble, during the period from 2000 – 2001. Another exogenous shock was brought to stock markets in the 2001 when the World Trade Center (WTC) was attacked and destroyed. While the markets recorded a sudden drop, it should not be attributed to internal forces of the markets. The recent financial crisis of 2007 – 2008, also known as the Global Financial Crisis, emerged from the bursting of the U.S. housing bubble, which peaked in 2006. In a series of days in September and October 2008, stock markets saw successive large declines. Many analysts believe that this crash was mainly driven by the housing markets, but there is no consensus about the real causes. Finally, our studied period also covers the May 6, 2010 Flash Crash, also known as The Crash of 2:45, in which the Dow Jones Industrial Average plunged approximately 1,000 points (9%), only to recover its losses within a few minutes. It was the biggest intraday drop in history, and one of its main possible causes may have been the impact of high-frequency traders or large directional bets.

In terms of Zeeman's (1974) hypotheses, cusp catastrophe theory proposes to model the crashes as endogenous events driven by speculative money. Employing our two-step estimation method, we estimate the cusp model to quantitatively test the theory on the period that covers all of these crashes to determine whether the theory can explain the crashes using our data. An interesting discussion may stem from studying the causality between volatility and crashes. While Levy (2008) provided a modeling approach for increasing volatility before crash events, the crashes are driven endogenously by speculative money in our approach; thus, the sudden discontinuities are not connected to volatility.

Figure 3.4: S&P 500 returns and realized volatility



Note: S&P 500 (a) returns r_t, (b) realized volatility RV_t, and (c) standardized returns $r_t RV_t^{-1/2}$. The figure highlights several important recession periods as grey periods and crash events as black lines. The periods are more closely described in the text.

Source: Authors' own computations in *R* and *Wolfram Mathematica*.

3.4.1 Data description

For our two-step estimation procedure, we need two sets of data. The first set consists of high-frequency trading data related to S&P 500, which are used to estimate the volatility of returns according to Eq. (3.15). The second set consists of data on sentiment and contains control variables that drive the fundamental (asymmetry) and chartists (bifurcation) side of the model. Let us describe both datasets used. For the realized volatility estimation, we use the S&P 500 futures traded on the Chicago Mercantile Exchange (CME)[6]. The sample period extends from February 24, 1984 to November 17, 2010. Although after the introduction of the CME Globex(R) electronic trading platform on Monday, December 18, 2006, CME started to offer nearly continuous trading, we restrict the analysis to the intraday returns with 5-minute frequencies within the business hours of the New York Stock Exchange (NYSE) because the most liquidity of the S&P 500 futures came from the period when the U.S. markets were open. We eliminate transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2 because of the low activity on those days, which could lead to estimation bias.

Using the realized volatility estimator, we then measure the volatility of the stock market returns as the sum of the squared 5-minute intraday returns. In this way, we obtain 6,739 daily volatility estimates. Figure 3.4 shows the estimated volatility 3.4(b) together with the daily returns 3.4(a). It can be immediately observed that the volatility of the S&P 500 is strongly time varying over the very long period.

For the state (behavioral) variable of the cusp model, we use the S&P 500 daily returns standardized by the estimated daily realized volatility according to Eq. (3.17). By standardization, we obtain stationary data depicted in Figure 3.4(c).

In choosing the control variables, we follow the successful method from the previous application in Barunik & Vosvrda (2009), where authors compared several measures of control variables and showed that fundamentalists, or the asymmetry side of the market, are best described by the ratio of advancing and declining stock volume, and chartists, or the bifurcation side of the model, are best described by the OEX put/call ratio[7]. The variables related to the trading volume generally correlate with the volatility and therefore are considered good measures of the trading activity of large funds and other institutional investors. Trading volume also relates to market liquidity and a major part of trading volume are fundamental money. Trading volume indicators thus represent the fundamental side of the market and can be used as a good proxy for fundamental investors. Therefore, the ratio of advancing and declining stock volume should mainly contribute to the asymmetry side of the model. Conversely, the activity of market speculators and technical traders should be well captured by the measures of sentiment, precisely the OEX put/call ratio, which is the ratio

[6]The data were provided by Tick Data, Inc.

[7]The data were provided by Pinnacle Data Corp.

Table 3.2: Descriptive statistics of the data

Data	Mean	Med.	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC r_t^2
1984–2010:											
r_t	.000	.001	-.229	.110	.012	-1.327	32.444	-.023	.022	-.036	.131
RV_t	.007	.006	.001	.076	.005	3.553	26.960	.002	.019	.824	.697
$r_t RV_t^{-1/2}$	.155	.110	-5.220	5.457	1.425	.137	2.973	-2.503	3.065	.012	-.064
x_1	1.671	1.127	.002	44.068	2.216	6.717	79.354	.129	6.609	.021	.005
x_2	1.166	1.110	.300	4.560	.362	1.374	7.871	.620	2.040	.489	.433
x_3	.020	.003	-.760	2.196	.214	2.385	20.366	-.322	.455	-.286	.098
1987 crash:											
r_t	.000	.001	-.229	.087	.018	-4.911	63.734	-.027	.027	.025	.097
RV_t	.008	.006	.002	.055	.005	4.350	33.829	.003	.019	.703	.647
$r_t RV_t^{-1/2}$	.182	.140	-5.220	4.080	1.550	-.123	2.844	-2.674	3.034	.057	-.092
x_1	2.005	1.254	.002	21.691	2.565	3.903	23.797	.098	8.026	.039	.015
x_2	.886	.860	.330	4.270	.259	5.714	73.304	.540	1.310	.340	.096
x_3	.030	.002	-.661	2.114	.261	1.933	13.743	-.382	.565	-.365	.064
2008 crash:											
r_t	-.001	.000	-.095	.110	.024	-.035	6.358	-.053	.043	-.147	.123
RV_t	.015	.012	.003	.076	.010	2.219	9.739	.006	.041	.812	.628
$r_t RV_t^{-1/2}$	-.035	.042	-4.122	4.027	1.435	.105	3.062	-2.705	3.063	-.126	-.155
x_1	2.341	.990	.012	41.617	4.112	4.358	29.870	.039	14.302	-.064	-.034
x_2	1.086	1.070	.320	1.950	.332	.314	2.614	.510	1.790	.270	.232
x_3	.023	-.003	-.760	1.245	.221	1.134	8.007	-.362	.562	-.339	.175

Note: The sample period extends from February 24, 1984 to November 17, 2010 for the full sample 1984–2010, from January 5, 1987 to August 15, 1988 for the 1987 crash (409 observations), and from November 11, 2007 to July 8, 2009 for the 2008 crash (409 observations). The S&P 500 stock market returns r_t, realized volatility RV_t, daily returns normalized by the realized volatility $r_t RV_t^{-1/2}$ and data for the independent variables $\{x_1, x_2, x_3\}$ are the ratio of advancing and declining stock volume, the OEX put/call options and the change in total volume, respectively. Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, autocorrelations (AC), and autocorrelations of squared data (AC^2) are reported. Figures are rounded to 3 decimal digits.

Source: Authors' own computations in *Wolfram Mathematica*.

of daily put and daily call option volume with the underlying S&P 500 index. Financial options are widely used and are the most popular instruments for speculative purposes. Therefore, they serve as a good measure of speculative money in capital markets (see e.g. Bates (1991), Finucane (1991), or Wang *et al.* (2006)) because they represent the data about extraordinary premiums and excessive greed or fear on the market. Thus, they should represent the internal forces that lead the market to bifurcation within the cusp catastrophe model. Overall, we assume the OEX put/call option ratio mainly contributes to the bifurcation side of the model. Moreover, we use a third control variable, the daily change in total trading volume, as a driver for both the fundamental and speculative money in the market. The daily change in the total volume indicator is generally related to continuous fundamental trading activity, but it may also reflect elevated speculative activity on the market as well. Therefore, we expect this variable to help the regression not only on the asymmetry side but also on the bifurcation side. The time span for all of these data matches the time span of the S&P 500 returns, i.e., February 24, 1984 to November 17, 2010. The descriptive statistics for all of the data are summarized in the upper part of Table 3.2.

3.4.2 Full sample static estimates

In the estimation, we primarily aim to test whether the cusp catastrophe model is able to describe the stock market data in the time-varying volatility environment and therefore that stock markets show signs of bifurcations. In doing so, we follow the statistical testing described earlier in the text. At first, we estimate all parameters of the cusp equation and contrast these result to estimation results of the restricted model with $\alpha_2 = \beta_1 = 0$ according to our hypothesis about primary driving forces of the asymmetry and bifurcation sides of the model.

Table 3.3 shows the estimates of the cusp fits. Let us concentrate of the left side of Table 3.3(a), where we fit the cusp catastrophe model to the standardized returns $r_t \widehat{RV}_t^{-1/2}$. First, we do not make any restrictions, and we use all three control variables; thus, $\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \alpha_2 x_{2,t} + \alpha_3 x_{3,t}$ and $\beta_x = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t}$, where x_1 is the ratio of advancing and declining stock volume, x_2 is the OEX put/call option ratio and x_3 is the rate of change of the total volume. The state variable is $\tilde{r}_t$, and the returns are normalized with estimated realized volatility. In terms of log likelihood, the cusp

Table 3.3: Estimation results on the S&P 500 stock market data

	(a) Estimates using $r_t\widehat{RV}_t^{-1/2}$				(b) Estimates using r_t			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	-2.378 ***	1.777 ***			5.008 ***	5.000 ***		
α_1	4.974 ***	3.881 ***			4.426 ***	2.730 ***		
α_2	0.062 ***				0.771 ***			
α_3	0.300 ***	-0.055 ***			-0.463 ***	-0.534 ***		
β_0	-4.654 ***	-1.521 ***			-5.011 ***	-5.000 ***		
β_1	-5.011 ***				-1.201 ***			
β_2	0.139 ***	0.086 **			-1.138 ***	-0.208 ***		
β_3	0.422 ***	0.300 ***			0.644 ***	0.683 ***		
ω_0	-0.700 ***	0.439 ***			0.786 ***	0.826 ***		
ω_1	0.492 ***	0.849 ***			0.407 ***	0.402 ***		
R^2	0.800	0.783	0.385	0.823	0.637	0.530	0.405	0.687
LL	-4786.161	-6408.378	-7634.756	-5885.983	-7728.981	-8174.635	-7811.134	-5648.799
AIC	9592.321	12832.760	15279.512	11789.970	15477.960	16365.270	15632.270	11311.600
BIC	9660.209	12887.070	15313.456	11851.070	15545.820	16419.800	15666.350	11359.310

Note: The full sample extends from February 24, 1984 to November 17, 2010. The left side of the table presents the estimation results on the normalized returns $r_t\widehat{RV}_t^{-1/2}$, and the right side of the table presents the results for the original S&P 500 stock market returns r_t.
Note: ***, **, *, '.', and ' ' denote significance levels of 0%, 0.001%, 0.01%, 0.05%, and 0.1%, respectively.

Source: Authors' own computations in *R*.

model describes the data much better than the linear regression model. The ω_1 coefficient is far away from zero, although some degree of transformation of the data is needed. All of the other coefficients are strongly significant at the 99% level. Most importantly, when the cusp fit is compared with the logistic fit in terms of AIC and BIC, we can see that the cusp model strongly outperforms the logistic model.

Our hypothesis is that the ratio of advancing and declining stock volume only contributes to the asymmetry side, and the OEX put/call ratio, representing the measure of speculative money in the market, contributes to the bifurcation side of the model. To test this hypothesis, we set the parameters $\alpha_2 = \beta_1 = 0$ and refer to it as a restricted model. From Table 3.3(a), we can see that the log likelihood of the restricted model naturally decreases in comparison with the unrestricted model because the log likelihood of the restricted model is always lower (or equal) than of the unrestricted model. All of the parameters are again strongly significant, and we can see that they change considerably. This result can be attributed to the fact that $x_{1,t}$ seems to contribute strongly to both sides of the market in the unrestricted model. Although the β_2 coefficient representing the speculative money is quite small in comparison with the other coefficients, it is still strongly significant. Because this coefficient is the key for the model in driving the stock market to bifurcation, we further investigate its impact in the following sections. It is interesting to note that the ω_1 parameter increases to very close to one in the restricted model. This result means that the observed data are close to the state variable.

When moving to the right (b) side of Table 3.3, we repeat the same analysis, but this time, we use the original r_t returns as the state variable. We wish to compare the cusp catastrophe fit to the data with strongly varying volatility. In using the data's very long time span where the volatility varies considerably, we expect the model to deteriorate. Although the application of the cusp catastrophe model to the non-stationary data can be questioned, we provide these estimates to compare them with our modeling approach. We see an important result. While the linear and logistic models provide very similar fits in terms of the log likelihoods, the information criteria and R^2 deteriorate in both the unrestricted and restricted cusp models. The ω_1 coefficient, together with all of the other coefficients, is still strongly different from zero, but the important result is that the logistic model not only describes the data better, but also the presence of bifurcations in the raw return data cannot be claimed.

To conclude this section, the results suggest strong evidence that over the long period of almost 27 years, the stock markets are better described by the cusp catastrophe model. Using our two-step modeling approach, we have shown that the cusp model fits the data well and the fundamental and bifurcation sides are controlled by the indicators for the fundamental and speculative money, respectively. In contrast, when the cusp is fit to the original data with a strong variation in volatility, the model deteriorates. We should note that these results resemble the results from the simulation; thus, the simulation also strongly supports our modeling approach.

3.4.3 Examples of the 1987 and 2008 crashes

While the results from the previous section are supportive of the cusp catastrophe model, the sample period of almost 27 years may contain many structural changes. Thus, we wish to further investigate how the model performs over time. Therefore, we use the two very distinct crashes of 1987 and 2008 and compare them to the localized cusp fits. There are several reasons to study these particular periods. These crashes were distinct in time, as there were 21 years between them, so they offer us an opportunity to determine how the data describe the periods. On the one hand, the stock market crash of 1987 has not yet been explained, and many analysts believe it was an endogenous crash. Therefore, it constitutes a perfect candidate for the cusp model. On the other hand, the 2008 period covered a much deeper recession, so it was very different from 1987. Finally, the two periods contain all of the largest one-day drops, which occurred on October 19, 1987, October 26, 1987, September 29, 2008, October 9, 2008, October 15, 2008, and December 1, 2008, recording declines of 20.47%, 8.28%, 8.79%, 7.62%, 9.03%, and 8.93%, respectively. In the following estimations, we restrict ourselves to our newly proposed two-step approach for the cusp catastrophe fitting procedure, and we utilize samples covering one-half year. The descriptive statistics for both periods are summarized in Table 3.2.

When focusing on the estimation results for the 1987 crash, we can see that both the restricted and the unrestricted models fit the data much better than the linear regression. The ω_1 coefficients are significantly different from zero, and when the cusp models are compared with the logistic model, they seem to provide much better fits. Thus, the cusp catastrophe model explains the data very well, and we can conclude that the stock market crash of 1987 was led by internal forces. This result confirms previous findings in Barunik & Vosvrda (2009), although a comparison cannot be made directly because authors used

Table 3.4: Estimation results on the 1987 and 2008 crashes

	1987				2008			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	-0.970 *	-0.535 ***			-3.938 ***	-0.598 ***		
α_1	1.792 ***	1.794 ***			1.793 ***	1.798 ***		
α_2	-0.073				-1.322			
α_3	0.191	0.029			-0.814	0.005 ***		
β_0	0.562 *	0.322 *			-0.542 ***	-1.142 *		
β_1	-0.395 *				-1.803 ***			
β_2	-1.255 **	-0.731 *			-0.990 **	-0.031 ***		
β_3	1.648 ***	1.547 ***			-0.803	0.282		
ω_0	0.453 ***	0.771 ***			-0.637 ***	0.680 ***		
ω_1	0.561 ***	0.602 ***			0.476 ***	0.641 ***		
R^2	0.855	0.827	0.454	0.895	0.816	0.858	0.483	0.884
LL	-85.971	-90.659	-208.557	-104.045	-88.054	-103.437	-185.150	-90.148
AIC	191.943	197.317	427.114	226.089	196.108	222.875	380.300	198.296
BIC	220.385	220.071	441.335	251.687	224.550	245.628	394.521	223.894

Note: Estimation results for the two distinct periods of the S&P 500 normalized stock market returns, $r_t\widehat{RV}_t^{-1/2}$. Note: ***, **, *, '.', and ' ' denote significance levels of 0%, 0.001%, 0.01%, 0.05%, and 0.1%, respectively.

Source: Authors' own computations in *Wolfram Mathematica*.

a different sample length. When comparing the fits of the unrestricted and restricted models, we can see that they do not differ significantly within the log likelihoods, AIC and BIC. In addition, the coefficient estimates are close to each other. The reason for this result is that the unrestricted model estimates the α_2 coefficient, which cannot be distinguished from zero, and the β_1 coefficient is significant only at a 90% level of significance. Thus, x_1 is proven to drive the fundamentals, and x_2 drives the speculators. Note that the β_2 coefficient is much larger in magnitude than on the fit for the full sample in the previous section. Interestingly, x_3 seems to drive the speculative money in the 1987 crash, but it does not help to explain the 2008 behavior.

The data from the 2008 period present different results. While the cusp fits are much better in comparison with the linear regression, they cannot be statistically distinguished from the logistic model. Therefore, there is very weak evidence of discontinuities in this period. This result is interesting because it may suggest that the large drops in 2008 were not driven endogenously by stock market participants but exogenously by the burst of the housing market bubble.

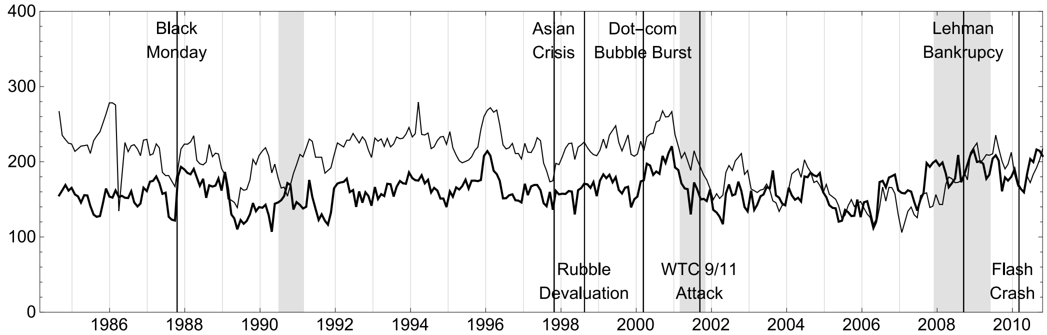
3.4.4 Rolling regression estimates

While the 1987 data are explained by the cusp catastrophe model very well and the 2008 data are not, we would like to further investigate how the cusp catastrophe fit changes over time. With almost 27 years of data needed for our two-step method of estimation, we estimate the cusp catastrophe model on one-half year rolling samples with a step of one month. The one-half year period is reasonable because it represents enough data for a sound statistical fit, but it is not a very long period, so we can uncover any structural breaks in the data.[8] In the estimation, we again restrict ourselves to our two-step estimation procedure. To keep the results under control, we use the final restricted model, where we assume that x_1 controls the asymmetry side of the model, and x_2 controls the bifurcation side solely, while x_3 contributes to both sides. Thus, $\alpha_2 = \beta_1 = 0$.

Before we proceed to interpreting the rolling regression results, let us discuss the bimodality of the rolling samples. Stochastic catastrophe corresponds to a

[8] Various combinations of rolling sample lengths and steps had been used in the preliminary analysis without affecting the overall aggregated results, e.g. comparing one day, and one month steps. The outcomes of the preliminary analysis are available from authors upon request.

Figure 3.5: Rolling values of BIC



Note: Rolling values of BIC information criteria for the cusp catastrophe (in bold black) and the logistic (in black) models.

Source: Authors' own computations in *R* and *Wolfram Mathematica*.

transition from a unimodal to a bimodal distribution. Thus, we first need to test for bimodality to be able to draw any conclusions from our analysis. To do this, we use the dip test of unimodality developed by Hartigan (1985); Hartigan & Hartigan (1985). The dip statistic is the maximum difference between the empirical distribution function and the unimodal distribution function, and it measures the departure of the sample from unimodality. Asymptotically, the dip statistics for samples from a unimodal distribution approach zero, and for samples from any multimodal distribution, the dip statistics approach a positive constant. We use bootstrapped critical values for the small rolling sample sizes to assess the unimodality. Figure A.3 shows the histogram of all of the dip statistics, together with its bootstrapped critical value 0.0406 at the 90% significance level. The results suggest that unimodality is rejected at the 90% significance level for several periods, but for most of the periods, unimodality cannot be rejected. Thus, we observe a transition from unimodal to bimodal (or possibly multimodal) distributions several times during the studied period.

Encouraged by the knowledge that the bifurcations could be present in our dataset, we move to the rolling cusp results. Figure A.4 shows the rolling coefficient estimates together with their significances. The ω_1 is significantly different from zero in all periods, and the α_1 coefficient is strongly significant over the whole period, although it becomes lower in magnitude during the latest years. Thus, the ratio of advancing and declining volume is a good measure for fundamentalists driving the asymmetry portion of the model. Much more important, however, is the β_2 parameter, which drives the bifurcations in the model. We can observe that until 1996, β_2 was significant, and its value changed considerably over time, but after 1996, it cannot be distinguished statistically from zero (except for some periods). This result is very interesting because it shows that the OEX put/call ratio was a good measure of speculative money in the market, and it controlled the bifurcation side of the model. During the first period, the OEX put/call ratio drove the stock market into bifurcations, but in the second period, the market was rather stable under the model. The parameter only started to play a role in the model again in the last few years and during the recent 2008 recession. However, its contribution was still relatively small.

This result is also confirmed when we compare the cusp model to the logistic model. Figure 3.5 compares the Bayesian Information Criteria (BIC) of the two models. Because the BIC cannot be directly compared across various time periods, we do not intend to track their dynamic evolution in time but to contrast the criteria of the cusp model and the logistic model in every single rolling one-half year period. We can see that the cusp catastrophe model was a much better fit for the data up to 2003, while for roughly 2003-2009, the cusp catastrophe model cannot be distinguished from the logistic model or logistic model strongly outperforms the cusp model. In the last period after 2009 and before the Flash Crash, the cusp again explains the data better, but the difference is not as strong as in the pre-2003 period. This result shows that before 2003, the stock markets showed signs of bifurcation behavior according to the cusp model, but after 2003, in the period of stable growth when participants believed that stock markets were stable, the markets no longer showed signs of bifurcation behavior.

To conclude this section, we find that despite the fact that we modeled volatility in the first step, the stock markets showed signs of bistability over several crisis periods.

3.5 Concluding remarks

In this chapter, we contribute to the literature on the modeling of stock market crashes and the quantitative application of the stochastic cusp catastrophe theory. We develop a two-step estimation procedure and estimate the cusp catastrophe model under time-varying stock market volatility. This approach allows us to test Zeeman's (1974) qualitative hypotheses on cusp catastrophe and bring new empirical results to previous work in this area (Barunik & Vosvrda 2009).

In the empirical testing, we use high frequency and sentiment data on the U.S. stock market covering almost 27 years. The results suggest that over a long period, stock markets are well described by the stochastic cusp catastrophe model. Using our two-step modeling approach, we show that the cusp model fits the data well and that the fundamental and bifurcation sides are controlled by the indicators for fundamental and speculative money, respectively. In contrast, when the cusp model is fit to the original data with strong variations in volatility, the model deteriorates. We should note that these results are similar to the results from a Monte Carlo study that we ran; thus, our simulation strongly supports our analysis. Furthermore, we develop a rolling estimation, and we find that until 2003, the cusp catastrophe model explains the data well, but this result changes during the period of stable growth from 2003–2008.

In conclusion, we find that despite the fact that we modeled volatility in the first step, the stock markets showed signs of bistability during several crisis periods. An interesting venue of future research will be to translate these results to a probability of the crash occurrence and its possible prediction.

Chapter 4

Simulation-based estimation of FABMs: the case of Brock & Hommes HAM

This chapter describes an innovative general computational framework for empirical estimation of full-fledged FABMs. While for the cusp catastrophe model the likelihood function is theoretically known, for many FABMs we lack this convenient feature. Some few authors thus apply simulation-based estimation methods of moments (for details see Chapter 2). We follow the Kristensen & Shin (2012) concept of simulated MLE based on nonparametric kernel methods. The methodology has been developed for dynamic models where no closed-form representation of the likelihood function exists and thus we cannot derive the usual MLE. Therefore it constitutes an opportune estimation method for general class of FABMs.

In Chapters 5 and 6 we adopt the NPSMLE method to the FABM literature and test its capability on the most famous and widely analysed model developed by Brock & Hommes (1998) for which we customise the general framework of Kristensen & Shin (2012). Chapter 2 summarises other attempts to estimate models derived from Brock & Hommes (1998) approach that builds on evolutionary switching between trading strategies.

In Chapter 7 we further apply the NPSMLE to a simple herding FABM developed by Alfarano *et al.* (2008). Although we focus on two specific implementations in this work, we presuppose that if it succeeds in estimation of these two rather challenging FABM frameworks, the NPSMLE methodology is likely to appear useful for other ABMs in the future.

4.1 The Brock & Hommes (1998) model

Our modelling framework is within the Brock & Hommes (1998) HAM. The model is a financial market application of the ABS—the endogenous, evolutionary selection of heterogeneous expectation rules following the framework of Lucas (1978) and proposed in Brock & Hommes (1997; 1998). We consider an asset pricing model with one risk free and one risky asset. The dynamics of the wealth is as follows:

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \tag{4.1}$$

where W_{t+1} stands for the total wealth at time $t+1$, p_t denotes the ex-dividend price per share of the risky asset at time t, and $\{y_t\}$ denotes its stochastic dividend process. The risk-free asset is perfectly elastically supplied at constant gross interest rate $R = 1+r$, where r is the interest rate. Finally, z_t denotes the number of shares of the risky asset purchased at time t. The type of utility function considered is essential for each economic model and determines its nature and dynamics. The utility for each[1] investor (trader or agent alternatively) h is given by $U(W) = -exp(-aW)$, where $a > 0$ denotes the risk aversion, which is assumed to be equal for all investors.[2] For determining the market prices in this model, the Walrasian auction scenario is assumed. I.e. the market clearing price p_t is defined as the price that makes demand for the risky asset equal to supply at each trading period t and investors are 'price takers'. The detailed description of the price formation mechanism is offered further in this section and finally summarised by Equation 4.13 and Equation 4.14.

Let E_t, V_t denote the conditional expectation and conditional variance op-

[1]This is a crucial assumption without which the original model of Brock & Hommes (1998) loses one of its greatest advantages of analytical tractability.

[2]The generalised version of the model with the aim to study the model behaviour after relaxing a number of assumptions—especially homogeneous risk aversion—has been proposed by Chiarella & He (2002). The authors allow agents to have different risk attitudes by generalising Equation 4.2 and letting the risk aversion coefficient a_h differ among particular traders. The paper is focused mainly on a study of two-belief systems. Typically, fundamentalists are expected to be more risk averse than chartists and thus the relative risk ratio $a_{cf} = \frac{a_{chart.}}{a_{fund.}} < 1$. The authors offer an exuberant analysis of many specific setting combinations and conclude that relaxing some assumptions of the original Brock & Hommes (1998) model leads to a markedly enriched system with some significant differences (e.g. stability of the model equilibrium might depend directly on a_{cf}; decreasing a_{cf} may trigger chaotic fluctuations around the fundamental price, i.e. when fundamentalists are more risk averse, the market becomes more chaotic; or that adding noise has a small effect when a_{cf} is large, but the opposite is true when a_{cf} is small). On the other hand, many of the original results are robust enough with regard to suggested generalisations.

erators, respectively, based on a publicly available information consisting of past prices and dividends, i.e. on the information set $\mathcal{F}_t = \{p_t, p_{t-1}, \dots; y_t, y_{t-1}, \dots\}$. Let $E_{h,t}$, $V_{h,t}$ denote the beliefs of investor type h (trader type h alternatively) about the conditional expectation and conditional variance. For analytical tractability, beliefs about the conditional variance of excess returns are assumed to be constant and the same for all investor types, i.e. $V_{h,t}(p_{t+1} + y_{t+1} - Rp_t) = \sigma^2$. Thus the conditional variance of total wealth $V_{h,t}(W_{t+1}) = z_t^2\sigma^2$.

Each investor is assumed to be a myopic[3] mean variance maximiser, so for each investor h the demand for the risky asset $z_{h,t}$ is the solution of:

$$\max_{z_t} \left\{ E_{h,t}[W_{t+1}] - \frac{a}{2} V_{h,t}[W_{t+1}] \right\}. \tag{4.2}$$

Thus

$$E_{h,t}[p_{t+1} + y_{t+1} - Rp_t] - a\sigma^2 z_{h,t} = 0, \tag{4.3}$$

$$z_{h,t} = \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2}. \tag{4.4}$$

Let $n_{h,t}$ be the fraction of investors of type h at time t and its sum is one, i.e. $\sum_{h=1}^{H} n_{h,t} = 1$. Let $z_{s,t}$ be the overall supply of outside risky shares. The Walrasian temporary market equilibrium for demand and supply of the risky asset then yields:

$$\sum_{h=1}^{H} n_{h,t} z_{h,t} = \sum_{h=1}^{H} n_{h,t} \left\{ \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2} \right\} = z_{s,t}, \tag{4.5}$$

where H is the number of different investor types. In the simple case $H = 1$ we obtain the equilibrium pricing equation and for the specific case of zero supply of outside risky shares, i.e. $z_{s,t} = 0$ for all t, the market equilibrium then satisfies:

$$Rp_t = \sum_{h=1}^{H} n_{h,t} \{E_{h,t}[p_{t+1} + y_{t+1}]\}. \tag{4.6}$$

In a completely rational market Equation 4.6 reduces to $Rp_t = E_t[p_{t+1} + y_{t+1}]$ and the price of the risky asset is completely determined by economic fundamentals and given by the discounted sum of its future dividend cash flow:

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{(1+r)^k}, \tag{4.7}$$

[3] To be 'myopic' means to have a lack of long run perspective in planning. Roughly speaking, it is the opposite expression to 'intertemporal' in economic modelling.

where p_t^* depends upon the stochastic dividend process $\{y_t\}$ and denotes the fundamental price which serves as a benchmark for asset valuation based on economic fundamentals under rational expectations. In the specific case where the process $\{y_t\}$ is independent and identically distributed, $E_t\{y_{t+1}\} = \bar{y}$ is a constant. The fundamental price, which all investors are able to derive, is then given by the simple formula:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{4.8}$$

For the further analysis it is convenient to work not with the price levels, but with the deviation x_t from the fundamental price p_t^*:

$$x_t = p_t - p_t^*. \tag{4.9}$$

4.1.1 Heterogeneous beliefs

Now we introduce the heterogeneous beliefs about future prices. We follow the Brock & Hommes (1998) approach and assume the beliefs of individual trader types in the form:

$$E_{h,t}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + f_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t, \tag{4.10}$$

where p_{t+1}^* denotes the fundamental price (Equation 4.7), $E_t(p_{t+1}^* + y_{t+1})$ denotes the conditional expectation of the fundamental price based on the information set $\mathcal{F}_t = \{p_t, p_{t-1}, \dots ; y_t, y_{t-1}, \dots\}$, $x_t = p_t - p_t^*$ is the deviation from the fundamental price (Equation 4.9), f_h is some deterministic function which can differ across trader types h and represents a 'h-type' model of the market, and L denotes the number of lags.

It is now important to be very precise about the class of beliefs. From the expression in Equation 4.10 it follows that beliefs about future dividends flow:

$$E_{h,t}(y_{t+1}) = E_t(y_{t+1}), \quad h = 1, \dots H, \tag{4.11}$$

are the same for all trader types and equal to the true conditional expectation. In the case where the dividend process $\{y_t\}$ is i.i.d., from Equation 4.8 we know that all trader types are able to derive the same fundamental price p_t^*.

On the other hand, traders' beliefs about future price abandon the idea of perfect rationality and move the model closer to the real world. The form of this class of beliefs:

$$E_{h,t}(p_{t+1}) = E_t(p^*_{t+1}) + f_h(x_{t-1}, \ldots, x_{t-L}), \quad \text{for all } h, t, \tag{4.12}$$

allows prices to deviate from their fundamental value p^*_t, which is a crucial step in heterogeneous agent modelling. f_h allows individual trader types to believe that the market price will differ from its fundamental value p^*_t.

An important consequence of the assumptions above is that heterogeneous market equilibrium from Equation 4.6 can be reformulated in the deviations form, which can be conveniently used in empirical and experimental testing. We thus use Equation 4.9, 4.10 and the fact that $\sum_{h=1}^{H} n_{h,t} = 1$ to obtain:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}[x_{t+1}] = \sum_{h=1}^{H} n_{h,t} f_h(x_{t-1}, \ldots, x_{t-L}) \equiv \sum_{h=1}^{H} n_{h,t} f_{h,t}, \tag{4.13}$$

where $n_{h,t}$ is the value related to the beginning of period t, before the equilibrium price deviation x_t has been observed. The actual market clearing price p_t might then be calculated simply using Equation 4.9 as $p_t = x_t + p^*_t$, expressed more precisely, combining Equation 4.8, Equation 4.9, and Equation 4.13 as:

$$p_t = x_t + p^*_t = \frac{\sum_{h=1}^{H} n_{h,t} f_{h,t}}{R} + \frac{\bar{y}}{r}. \tag{4.14}$$

4.1.2 Selection of strategies

Beliefs of individual trader types are updated evolutionary and thus create the ABS, where the selection is controlled by endogenous market forces (Brock & Hommes 1997). It is actually an expectation feedback system as variables depend partly on the present values and partly on the future expectations.

The profitability (performance) measures for strategies h, $h = 1, \ldots H$ are derived from past realised profits as:[4]

$$U_{h,t} = (x_t - Rx_{t-1}) \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}. \tag{4.15}$$

[4] Additional memory can be introduced into the profitability measure (Equation 4.15) e.g. as a weighted average of past realised values $U_{m,h,t} = U_{h,t} + \eta U_{m,h,t-1}$, where $0 \leq \eta \leq 1$ denotes the 'dilution parameter' of the past memory in the profitability measure. Nonetheless, for the majority of examples, Brock & Hommes (1998) use $\eta = 0$ to keep derivations analytically tractable and work with models without memory, i.e. Equation 4.15 specification is used directly.

Market fractions of trader types $n_{h,t}$ are then given by the discrete choice probability—the multinomial logit model:[5]

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{Z_t}, \tag{4.16}$$

$$Z_t \equiv \sum_{h=1}^{H} \exp(\beta U_{h,t-1}), \tag{4.17}$$

where the one-period-lagged timing of $U_{h,t-1}$ ensures that all information for the market fraction $n_{h,t}$ updating is available at the beginning of period t, β is the intensity of choice parameter measuring how fast traders are willing to switch between different strategies. Z_t is then normalisation ensuring $\sum_{h=1}^{H} n_{h,t} = 1$.

4.1.3 Basic belief types

In the original paper by Brock & Hommes (1998), the authors analyse the behaviour of the artificial market consisting of a few simple belief types (trader types or strategies). The aim of investigating the model with only two, three, or four belief types is to describe the role of each particular belief type in deviation from fundamental price and to investigate the complexity of the simple model dynamics with the help of the bifurcation theory.

All beliefs have the simple linear form:

$$f_{h,t} = g_h x_{t-1} + b_h, \tag{4.18}$$

where g_h denotes the trend parameter and b_h is the bias of trader type h. This form comes from the argument that only very simple forecasting rules can have a real impact on equilibrium prices as complicated rules are unlikely to be learned and followed by sufficient number of traders. Hommes (2006) also notices another important feature of Equation 4.18, which is that x_{t-1} is used to forecast x_{t+1}, because Equation 4.5 has not revealed equilibrium p_t yet when p_{t+1} forecast is estimated.

The first belief type are fundamentalists or rational 'smart money' traders. They believe that the asset price is determined solely by economic fundamen-

[5] With regard to Macro ABMs, Branch & Evans (2006, pg. 266) point out that "the multinomial logit has proven to be an important approach to modelling economic choices, and has been increasingly employed in recent work in dynamic macroeconomics".

tals according to the EMH introduced in Fama (1970) and computed as the present value of the discounted future dividends flow. Fundamentalists believe that prices always converge to their fundamental values. In the model, fundamentalists comprise the special case of Equation 4.18 where $g_h = b_h = f_{h,t} = 0$. It is important to note that fundamentalists' demand also reflects market actions of other trader types. Fundamentalists have all past market prices and dividends in their information set $\mathcal{F}_{h,t}$, but they are not aware of the fractions $n_{h,t}$ of other trader types. Fundamentalists might pay costs $C \geq 0$ to learn how fundamentals work and to obtain market information. However, Brock & Hommes (1998) themselves mostly set $C = 0$ to keep simplicity of the analysis.

Chartists or technical analysts, sometimes called 'noise traders' represent another belief type. They believe that asset price is not determined by economic fundamentals only, but it can be partially predicted using simple technical trading rules, extrapolation techniques or taking various patterns observed in the past prices into account. If $b_h = 0$, trader h is called a pure trend chaser if $0 < g_h \leq R$ and a strong trend chaser if $g_h > R$. Additionally, if $-R \leq g_h < 0$, the trader h is called contrarian or strong contrarian if $g_h < -R$.

Next, if $g_h = 0$ trader h is considered to be purely upward biased if $b_h > 0$ or purely downward biased if $b_h < 0$.

4.2 Construction of the NPSMLE

This section introduces the estimation framework for the Brock & Hommes (1998) model. Let us assume processes (x, v), $x : t \mapsto \mathbb{R}^k$, $v : t \mapsto \mathcal{V}_t$, $t = 1, \ldots, \infty$. The space $\mathcal{V}_t$ can be time-varying. Suppose that we have T realisations $\{(x_t, v_t)\}_{t=1}^T$. Let us further assume the time series $\{x_t\}_{t=1}^T$ has been generated by a fully parametric model:

$$x_t = q_t(v_t, \varepsilon_t, \theta), \quad t = 1, \ldots, T, \tag{4.19}$$

where a function $q : \{v_t, \varepsilon_t, \theta\} \mapsto \mathbb{R}^k$, $\theta \in \Theta \subseteq \mathbb{R}^l$ is an unknown parameter vector, and ε_t is an independent identically distributed (i.i.d.) sequence with known distribution $\mathcal{F}_\varepsilon$, which is (without loss of generality) assumed not to depend on t or θ. In general, the processes (x, v) can be non-stationary and v_t is allowed to contain other exogenous variables than lagged x_t. We also assume the model to have an associated conditional density $c_t(x|v; \theta)$, i.e.

$$C(x \in A|v_t = v) = \int_A c_t(x|v;\theta)dx, \quad t = 1, \ldots, T, \tag{4.20}$$

for any Borel set $A \subseteq \mathbb{R}^k$.

Let us now suppose that $c_t(x|v;\theta)$ from Equation 4.20 does not have a closed-form representation. In such situation, we are not able to derive the exact likelihood function of the model in Equation 4.19 and thus a natural estimator of θ—the maximiser of the conditional log-likelihood:

$$\tilde{\theta} = arg \underbrace{max}_{\theta \in \Theta} L_T(\theta), \quad L_T(\theta) = \sum_{t=1}^{T} log\ c_t(x_t|v_t;\theta) \tag{4.21}$$

is not feasible.

In such situation, however, we are still able to simulate observations from the model in Equation 4.19 numerically.[6] The method presented allows us to compute a simulated conditional density, which we further use to gain a simulated version of the MLE.

To obtain a simulated version of $c_t(x_t|v_t;\theta)\ \forall\ t \in \langle 1, \ldots, T\rangle, x \in \mathbb{R}^k, v \in \mathcal{V}_t$, and $\theta \in \Theta$, we firstly generate $N \in \mathbb{N}$ i.i.d. draws from $\mathcal{F}_\varepsilon$, $\{\varepsilon_i\}_{i=1}^N$, which are used to compute:

$$X_{t,i}^\theta = q_t(v_t, \varepsilon_i, \theta), \quad i = 1, \ldots, N. \tag{4.22}$$

These N simulated i.i.d. random variables, $\{X_{t,i}^\theta\}_{i=1}^N$, follow the target distribution by construction: $X_{t,i}^\theta \sim c_t(\cdot|v_t;\theta)$, and therefore can be used to estimate the conditional density $c_t(x|v;\theta)$ with kernel methods—we define:

$$\hat{c}_t(x_t|v_t;\theta) = \frac{1}{N}\sum_{i=1}^{N} K_\eta(X_{t,i}^\theta - x_t), \tag{4.23}$$

where $K_\eta(\psi) = K(\psi/\eta)/\eta^k$, $K : \mathbb{R}^k \mapsto \mathbb{R}$ is a generic kernel and $\eta > 0$ is a bandwidth. Under regularity conditions on c_t and K, we get:

$$\hat{c}_t(x_t|v_t;\theta) = c_t(x_t|v_t;\theta) + O_P(1/\sqrt{N\eta^k}) + O_P(\eta^2), \quad N \longrightarrow \infty, \tag{4.24}$$

[6] For cases in which the model in Equation 4.19 is itself intractable and thus we cannot generate observations from the exact model, Kristensen & Shin (2012) suggest a methodology for approximate simulations and define regularity conditions for the associated approximate NPSMLE $\hat{\theta}_M$ to have the same asymptotic properties as the simulated MLE $\hat{\theta}$ defined in Equation 4.25.

where the last two terms are $o_P(1)$ if $\eta \longrightarrow 0$ and $N\eta^k \longrightarrow \infty$.

Having obtained the simulated conditional density $\hat{c}_t(x_t|v_t;\theta)$ from Equation 4.23, we can now derive the simulated MLE of θ:

$$\hat{\theta} = arg \underbrace{max}_{\theta\in\Theta} \hat{L}_T(\theta), \quad \hat{L}_T(\theta) = \sum_{t=1}^{T} log\ \hat{c}_t(x_t|v_t;\theta). \tag{4.25}$$

The same draws are used for all values of θ and we may also use the same set of draws from $\mathcal{F}_\varepsilon(\cdot)$, $\{\varepsilon_i\}_i^N$, across t. Numerical optimization is facilitated if $\hat{L}_T(\theta)$ is continuous and differentiable in θ. Considering Equation 4.23, if K and $\theta \mapsto q_t(v,\varepsilon,\theta)$ are $s \geq 0$ continuously differentiable, the same holds for $\hat{L}_T(\theta)$.

Under the regularity condition, the fact that $\hat{c}_t(x_t|v_t;\theta) \xrightarrow{P} c_t(x_t|v_t;\theta)$ implies that also $\hat{L}_T(\theta) \xrightarrow{P} L_T(\theta)$ as $N \longrightarrow \infty$ for a given $T \geq 1$. Thus, the simulated MLE, $\hat{\theta}$, retains the same properties as the infeasible MLE, $\tilde{\theta}$, as $T, N \longrightarrow \infty$ under suitable conditions.

4.3 Advantages and disadvantages

To quote from Kristensen & Shin (2012, pg. 85), "one of the merits of NPSML is its general applicability". Authors also provide three examples of application of the methodology in their article. The first comprises an estimation of the short-term interest rate model of Cox *et al.* (1985). The second applies the methodology to a jump-diffusion model of daily S&P500 returns by Andersen *et al.* (2002). In the third example the general capabilities of the NPSMLE to estimate a generic Markov decision processes are examined.

Kristensen & Shin (2012) also report several advantages and disadvantages of the proposed estimator. Starting with the former, the estimator works whether the observations x_t are i.i.d. or non-stationary because the density estimator based on i.i.d. draws is not affected by the dependence structures in the observed data. Second, the estimator does not suffer from the curse of dimensionality, which is usually associated with kernel estimators. In general, high dimensional models, i.e. with larger $k \equiv dim(x_t)$ as we smooth only over x_t here, require larger number of simulations to control the variance component of the resulting estimator. However, the summation in Equation 4.25 reveals an additional smoothing effect and the additional variance of $\hat{L}_T(\theta)$ caused by simulations retains the standard parametric rate $1/N$.

Conversely, the simulated log-likelihood function is a biased estimate of the actual log-likelihood function for fixed N and $\eta > 0$. To obtain consistency, we need $N \longrightarrow \infty$ and $\eta \longrightarrow 0$. Thus, the parameter η needs to be properly chosen for given sample and simulation size. In the stationary case, the standard identification assumption is:

$$\mathbb{E}[log\ c(x_t|v_t, \theta)] < \mathbb{E}[log\ c(x_t|v_t, \theta_0)]\ \forall\ \theta \neq \theta_0. \tag{4.26}$$

Under stronger identification assumptions, the choice of the parameter η might be less important and one can prove the consistency of the estimator for any fixed $0 < \eta < \bar{\eta}$ for some $\bar{\eta} > 0$ as $N \longrightarrow \infty$ (Altissimo & Mele 2009). In practice this still requires us to know the threshold level $\bar{\eta} > 0$ but from the theoretical viewpoint this ensures that parameters can be well identified in large finite samples after a given $\bar{\eta} > 0$ is set. Moreover, it suggests that proposed methodology is fairly robust to the choice of η. In their simulation study, Kristensen & Shin (2012) show indeed that the NPSMLE performs well using broad range of bandwidth choices.

4.4 Asymptotic properties

As the theoretical convergence of the simulated conditional density towards the true density is met, we would expect the NPSMLE $\hat{\theta}$ to have the same asymptotic properties as the infeasible MLE $\tilde{\theta}$ for a properly chosen sequence $N = N(T)$ and $\eta = \eta(N)$. Kristensen & Shin (2012) show that $\hat{\theta}$ is first-order asymptotic equivalent to $\tilde{\theta}$ under set a general conditions, allowing even for non-stationary and mixed discrete and continuous distribution of the response variable. Further, using additional assumptions, including stationarity, they provide results regarding the higher-order asymptotic properties of $\hat{\theta}$ and derive expressions of the bias and variance components of the NPSMLE $\hat{\theta}$ compared to the actual MLE due to kernel approximation and simulations.

Therefore, a set of general conditions, satisfied by most models, need to be verified so that $\hat{c} \longrightarrow c$ sufficiently fast to ensure asymptotic equivalence of $\hat{\theta}$ and $\tilde{\theta}$. Kristensen & Shin (2012) define a set of regularity conditions on the model and its associated conditional density that satisfy these general conditions for uniform rates of kernel estimators defined in Kristensen (2009).

The kernel K from Equation 4.23 has to belong to a broad class of so-called bias high-order or bias reducing kernels. E.g. the Gaussian kernel, which we use in Chapter 5, satisfy this condition if $r \geq 2$, where r is the number of derivatives of c. Higher r causes faster rate of convergence and determines the degree of $\hat{c}$ bias reduction. Moreover, general versions of conditions usually required for consistency and well-defined asymptotic distribution (asymptotic normality) of MLEs in stationary and ergodic models are imposed on actual log-likelihood function and the associated MLE to ensure the actual MLE $\tilde{\theta}$ in Equation 4.21 is asymptotically well-behaved.

Chapter 5

Monte Carlo study: NPSMLE of the HAM

This chapter analyses the capability of the NPSMLE methodology for the HAMs estimation purposes and evaluates small sample properties of the estimator via an extensive Monte Carlo study. We simulate data from the HAM and employ the NPSMLE of selected model parameters to analyse how well and under what conditions is the estimation method able to recover true values of parameters in the controlled environment. For its conceptual importance, a detailed focus is devoted to the switching parameter—intensity of choice β.

5.1 Simulation setup for the HAM

In the simulation setup, we follow the previous works of Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013). The joint setup for the basic HAM model (see Section 4.1) is used for all (if not explicitly stated otherwise) conducted simulations in this chapter and is defined as follows. The model we use to generate observations is a very stylised simple version compactly described in Hommes (2006, pg. 1169) and consisting of three mutually dependent equations:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} f_{h,t} + \epsilon_t \equiv \sum_{h=1}^{H} n_{h,t}(g_h x_{t-1} + b_h) + \epsilon_t, \tag{5.1}$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}, \tag{5.2}$$

$$\begin{aligned} U_{h,t-1} &= (x_{t-1} - Rx_{t-2}) \frac{f_{h,t-2} - Rx_{t-2}}{a\sigma^2} \\ &\equiv (x_{t-1} - Rx_{t-2}) \frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}, \end{aligned} \tag{5.3}$$

where ϵ_t (which coincides with ε_t in Equation 4.19) is an i.i.d. noise term sequence with given distribution[1] representing the market uncertainty and unpredictable market events.

In order to run the model in various different settings, we inevitably need to fix several variables less important for the dynamics of the model to enable estimation of the key parameters. First, we set the constant gross interest rate $R = 1+r = 1.0001$ to resemble real market risk free rate. Assuming 250 trading days per year and daily compounding, this daily value represents circa 2.5% annual risk free interest rate which is a reasonable approximation. Although this figure is not based on any rigorous calibration or taken from a specific study, similar values are largely used in various financial and macroeconomic works. Moreover, as we show in further analysis, the model exhibits considerable robustness w.r.t. various reasonable risk free values and thus there is no need for more precise derivation of this parameter. We further fix the linear term $1/a\sigma^2$ (comprising the risk aversion coefficient $a > 0$ and the beliefs about the conditional variance of excess returns σ^2) to 1. The similar setting has already been succesfully used in previous works of Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013). It is important to note that a and σ^2 are only scale factors for the profitability measure U. Their magnitudes do not affect relative proportions of $U_{h,t}$ and thus do not influence the dynamics of the model output, that is on the contrary usually characterised by time-varying variance. In other words, although we assume constant σ^2, the output time series generated by the model does not have constant variance. Strategy-specific a_h or time-varying $\sigma^2_{h,t}$ are appealing concepts mainly for simulation analyses of HAMs (see e.g. Gaunersdorfer 2000; Chiarella & He

[1] Various specifications of normal and uniform distributions are utilised in Chapter 5, standard deviation of a normal distribution is estimated in Chapter 6.

2002; Amilon 2008). Moreover, we intentionally use relatively small number of possible trading strategies following (Kukacka & Barunik 2013), $H = 5$, for the general model setting or $H \in \{2, 3\}$ for so called 2-type and 3-type model, respectively (Chen *et al.* 2012). Following Hommes (2006) via Equation 5.3, neither 'dilution parameter' of the past memory η nor information costs C for fundamentalists are implemented into the basic model setup to keep the dynamics of the model and impacts of assessed modification as clear as possible. Indeed, also Brock & Hommes (1998) mostly set $C = 0$ to keep simplicity of the analysis and work with models without memory, i.e. they set the $\eta = 0$ (see Subsection 4.1.2) to keep derivations analytically tractable.

Within the Monte Carlo method, several parameters are repeatedly randomly generated to obtain statistically valid inference. Following the previous works by Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013), trend parameters g_h are drawn from the normal distribution $N(0, 0.4^2)$ and bias parameters b_h are drawn from the normal distribution $N(0, 0.3^2)$. 'Strict' fundamental strategy in the sense of the original Brock & Hommes (1998, pg. 1245) article appears in the market by default, i.e. the first strategy is always defined as $g_1 = b_1 = 0$ and therefore fundamentalists are always present on the market.

In the Monte Carlo simulations, we first study the capabilities of the NPSMLE under various levels of the switching parameter—intensity of choice β. As discussed in Section 2.3, literature estimating β using real marked data is relatively scarce because of difficulties arising from the nonlinear nature of the HAM. Thus, β still remains a rather theoretical concept. Larger β implies higher willingness of agents to switch between available trading strategies based on their relative profitability—the best strategy attracts the most agents at each period. On the one hand, comprising the large variety of possible β values might seem as a dominant simulation strategy, on the other hand one has to consider computational burden of the simulation process in real time. What is perhaps even more important is to consider intensity of choice β from the economic viewpoint. First, high values give rise to unrealistically high switching frequency, which is hardly to be observed among market agents in reality. Next, negative β does not make any economic sense in the presented model framework as it causes inverse illogical switching towards less profitable strategies. Although the intensity of choice β cannot be directly rigorously compared across various models, assets, or time periods (see discussion in Section 2.3), we utilise the general knowledge of previous estimation efforts for models sharing similar framework to set meaningful simulation grids in this chapter. When referring to literature review (see Section 2.3 and Subsection 2.4.6), in vast majority of research articles sharing the ABS framework derived from Brock & Hommes (1998), β is found single-digit and often close to zero, that well resembles the economic intuition of some, but realistically low switching frequency between major types of trading strategies. Thus, we employ relatively rich, but reasonable discrete range of βs in our simulations: $\{0, 0.1, 0.5, 1, 3, 5, 10\}$. It is far beyond the scope of this work to provide a deep analysis of the model behaviour, e.g. how the intensity of choice β influences the dynamics of the model that can under some setting even generate purely chaotic behaviour. Many studies have been devoted to this generally difficult issue in past two decades. In this context we refer the interested reader to the original paper of Brock & Hommes (1998) containing comprehensive model dynamics analysis, extensive studies by Hommes (2006), Hommes & Wagener (2009), Chiarella *et al.* (2009), or the recent book summarising 20 years of research on the Heterogeneous Expectations Hypothesis by Hommes (2013).

Next, as discussed by Amilon (2008), the magnitude of noise term has to be considered carefully. Noise is an inevitable part of the model as it represents the market uncertainty and unpredictable market events, but it must not overshadow the effect of variables under scrutiny. As mentioned in Kukacka & Barunik (2013), although varying noise variance can cause some minor changes in model outcomes, the analysed HAM embodies major similarities across various noise. Although theoretically the $\mathcal{F}_\varepsilon$ from which the $\{\varepsilon_i\}_{i=1}^N$ are drawn to simulate $\{X_{t,i}^\theta\}_{i=1}^N$ (Equation 4.22) is a generic known distribution, the assumptions about market noise can play crucial role in the NPSMLE application to real world data. Therefore we test the model sensitivity and robustness of proposed methodology using 30 stochastic noise specifications from an extensive range drawn a) from various normal distributions and b) from the uniform distributions

1. that cover the same intervals as are covered by their respective normal counterparts by the 99.74% of the probability mass;
2. with the same variances as their respective normal counterparts.

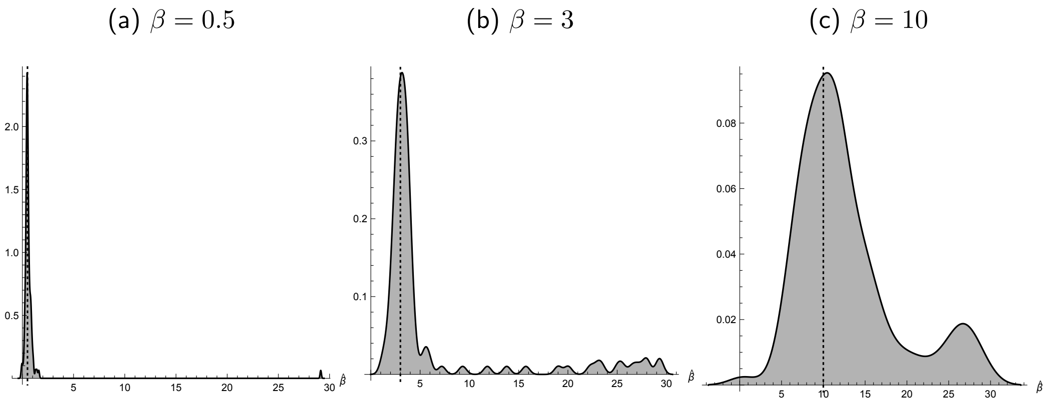
Detailed description of all 30 stochastic noise specifications can be found in Table 5.1 (specification for normal distributions), Table 5.4 (specification for uniform distributions of the 1. type), and Table 5.5 (specification for uniform distributions of the 2. type). Basically, for the normally distributed noise the range extends from a 'miniscule' standard deviations $SD = 10^{-8}$, 10^{-7}, 10^{-6} [a value used by Hommes (2013, pg. 170, 174, 177) in a similar model setting], or 10^{-5}, followed by 'small' standard deviations $SD = 10^{-4}$, 10^{-3}, 0.01 [another value used by Hommes (2013, e.g. pg. 171) in a similar setting], standard normal $SD = 1$, and finally a relatively large 'experimental' standard deviation $SD = 2$. The sensitivity analysis of the NPSMLE method to the stochastic noise specification is based on the normality assumption. The normal distribution of market noise seems reasonably realistic and similar assumption has already been used in related studies, where "the non-linear models are fed with an exogenous stochastic process, but the noise process is 'nice', which in this case means that it is normally distributed", as pointed out by Amilon (2008, pg. 344). We also utilise the favourable theoretical properties of the Gaussian kernel (Kristensen & Shin 2012, pg. 81) in Equation 4.23. To check the robustness of the method, we concur the previous research in Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013)—where uniform stochastic noise specification is utilised—and compare and contrast the results based on normally distributed noise to the two rather extreme and economically unrealistic uniform variants defined above. We intentionally do not consider any at first sight soliciting heavy-tailed noise distribution. The fact that financial data are heavy-tailed does not suggest any specific distribution of the market noise. In fact, the situation is opposite. The attractiveness of the HAM is based on its ability to produce heavy-tailed distribution of model output although we input normally distributed stochastic noise. Thus the HAM explains one of the most important stylised facts of financial time series via endogenous interactions of fundamentalists and boundedly rational chartists, not as an effect of a specific distribution of noise input. Finally, five lengths of the resulting series entering the NPSMLE algorithm are used: 100, 500, 1000, 5000, and 10000, and first 100 observations from the HAM are always discarded[2] as initial period, where the model dynamic is being established.

5.2 Simulation setup for the NPSMLE

We follow the Kristensen & Shin (2012) methodology used for the estimation of the Cox *et al.* (1985) short-term interest rate model. However, we examine and

[2]We always simulate 100 extra observations to be discarded so that we finally get the intended length of the series with stable dynamics without the initialisation period.

Figure 5.1: Pre-estimation performance for selected βs



Note: Results based on 100 random runs, number of observations $t = 1000$, and the kernel estimation precision $N = 100$, initial point drawn from uniform distribution $U(0, 30)$. Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from standard normal distribution $N(0, 1)$. Black dotted vertical lines depict the true βs. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

adapt the setup for the purposes of the HAM. As discussed in Section 4.3, there are two main trade-offs: between the precision of the kernel estimation and the computational burden, and between the smoothness of the kernel estimation and the bias.

As we are the first to apply a very recent NPSMLE methodology on a well-known HA modelling framework, we can only partially base our simulation setup on some results from literature. For this reason, we elaborate an extensive Monte Carlo simulation testing of the robustness of our findings. First, to analyse statistically valid results, we start from the benchmark in Kristensen & Shin (2012) and compare the simulations of 100, 500, and 1000 runs. Moreover, three levels of the kernel estimation precision are considered, namely $N = 100$, $N = 500$, and $N = 1000$. It is important to note that the same draws $\{\varepsilon_i\}_{i=1}^N$ are used to generate the simulations $\{X_{t,i}^{\theta}\}_{i=1}^N$ over time.

Second, the numerical algorithm is designed to find an optimum of either unconstrained or constrained multivariable function. As discussed in Section 5.1, we expect β to be non-negative but rather small, i.e. single-digit. Using pre-estimation step with unconstrained parameter space we can obtain reasonably sufficient preliminary knowledge about the approximate true value of estimated parameters even for computationally feasible setting. This general principle of a preliminary rough search followed by a fine-tuning on a considerably restricted subset of the parameter space is successfully applied e.g. in Chen & Lux (2015) for MSM estimation. In Figure 5.1 we demonstrate the pre-estimation performance via smooth histograms of $\hat{\beta}$ based on a setting which can be easily computed using a personal computer within several minutes: $\beta \in \{0.5, 3, 10\}$, 100 runs, number of observations $t = 1000$, kernel estimation precision $N = 100$, ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from standard normal distribution, and the initial point drawn from uniform distribution covering a broad interval $\langle 0, 30\rangle$. We can observe how the peak of the distribution approximately detects the true value of β which helps us to constrain the parameter space in the next step. In subsequent—this time computationally very extensive—optimisation of the constrained function to fine-tune the precision of estimates we can therefore opt for relatively narrow bounds of the parameter space set as $\langle -\beta, 3\beta\rangle$ for $\beta \geq 0$ and $\langle -0.5, 0.5\rangle$ for $\beta = 0$.[3] For the 2-type model simulation estimation study (see Subsection 5.3.2), we use even wider and off-centered interval for the bounds of the parameter space set as $\langle -3|g_2|, 3|g_2|\rangle$ and $\langle -3|b_2|, 3|b_2|\rangle$, respectively, to allow for possible negative values, and $\langle -0.5, 0.5\rangle$ for $\beta = 0$. For the 3-type model simulation estimation study (see Subsection 5.3.3), it is, however, important to limit bounds by zero from one side, i.e. $\langle 0, 3|g_2|\rangle$, $\langle -3|g_3|, 0\rangle$ to avoid problems with insufficient specification of the model leading to ambiguous bimodal distributions of estimated parameters. The same intervals are used for a random draw of a single[4] starting point of the optimisation search procedure which is drawn from the uniform distributions.

Third, to estimate the conditional density $c_t(x|v;\theta)$ with the kernel method (Equation 4.23), the Gaussian kernel and the Silverman's (1986) rule of thumb for finding the optimal size of the bandwidth:

$$\eta = \left(\frac{4}{3N}\right)^{1/5} \widehat{\sigma}, \tag{5.4}$$

where $\widehat{\sigma}$ denotes the standard deviation of $\{X_{t,i}^{\theta}\}_{i=1}^N$, are employed.

Additionally, Kristensen & Shin (2012, pg. 82) suggest undersmoothing option for the bandwidth size selection concluding that "simulation results

[3] This extended range of the parameter space bounds covering also economically irrelevant negative values is used to ensure the robustness of the method and not imposing excessive demands on the precision of the unconstrained pre-estimation. Moreover, not allowing for negative values might naturally lead to an upward bias of the simulated estimator especially for βs close to 0.

[4] Kristensen & Shin (2012) use multiple starting points for the numerical optimisation but for the HAM estimation purposes in a simulated environment the single starting point is sufficient bringing the merits of markedly reduced computational time and burden.

indicate that standard bandwidth selection rules together with a bit of undersmoothing in general deliver satisfactory results". Moreover, as found by Jones *et al.* (1996), smaller bandwidths are better for larger kernel approximation precision, "because the estimator should be 'more local' when more information is present, and when the density is rougher, because the bias effect is stronger". However, we do not use the undersmoothing option in our numerical algorithm as for HAM the methodology is robust in this aspect and various levels of undersmoothing do not change the outcomes.[5]

5.3 Monte Carlo results

In all simulations we are concerned in questions how accurately is the method able to recover the true values and how robust is the method with respect to various settings. For this reason, all tables (but not figures) in this chapter always report results based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$ i.i.d. draws from given distribution. Sample medians and means of the estimated values together with standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.[6]

5.3.1 β estimation in the general model

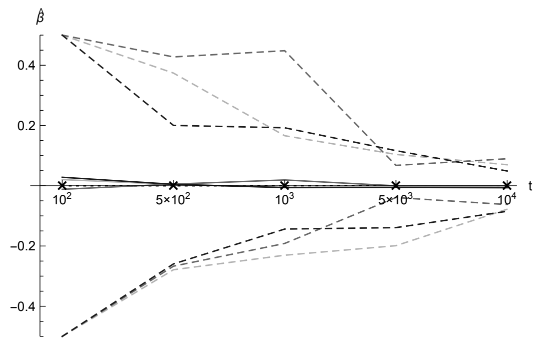
First and foremost, we assess the simplest but also the most crucial case of β estimation in the general model with $H = 5$ possible trading strategies. The intensity of choice β is the most important parameter influencing the dynamics of the system through the multinomial logit model of a continuous adaptive evolution of market fractions in Equation 4.16. Not only its magnitude between two extreme cases $\beta = 0$ and $\beta = \infty$ is important, but β also determines the type of the model equilibrium that can generally take the form of a (multiple) steady state(s), cycles, or even chaotic behaviour. The intensity of choice β is also crucial for its conceptual importance—it represents the dominant approach how the boundedly rational choices of agents are mathematically modelled in the current literature (see discussion in Subsection 2.4.2 and the ABS origin of models in Tables 2.1 and 2.2).

[5] Results of this testing are available upon request from authors.

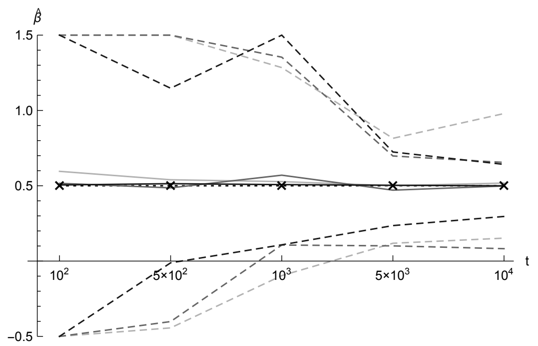
[6] We comment more on the issue of possible occurrence of 'NaN' outcome from the NPSMLE procedure in the following Subsection 5.3.1.

Figure 5.2: Simulation results for various number of runs and βs I.

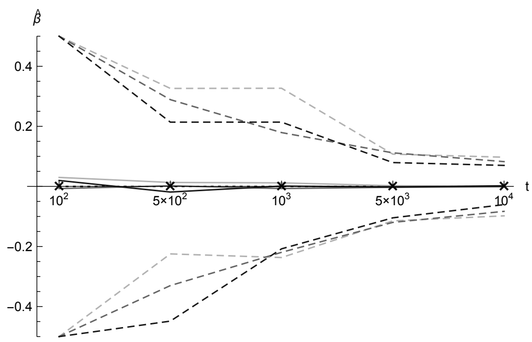
(a) 100 runs, $\beta = 0$, $N(0, 2^2)$



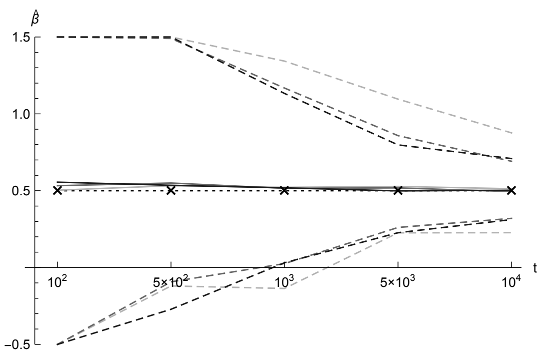
(b) 100 runs, $\beta = 0.5$, $N(0, 1)$



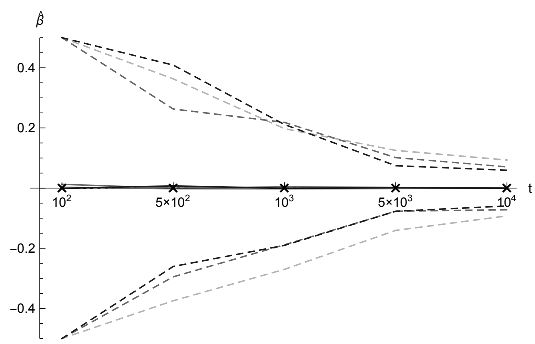
(c) 500 runs, $\beta = 0$, $N(0, 2^2)$



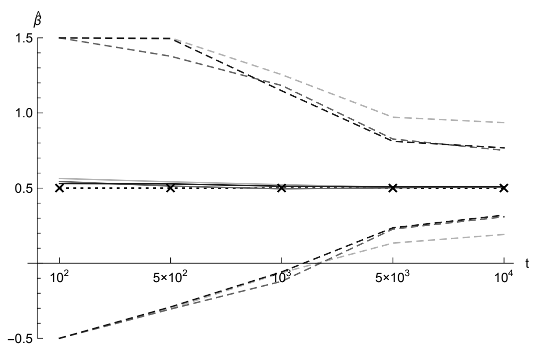
(d) 500 runs, $\beta = 0.5$, $N(0, 1)$



(e) 1000 runs, $\beta = 0$, $N(0, 2^2)$



(f) 1000 runs, $\beta = 0.5$, $N(0, 1)$

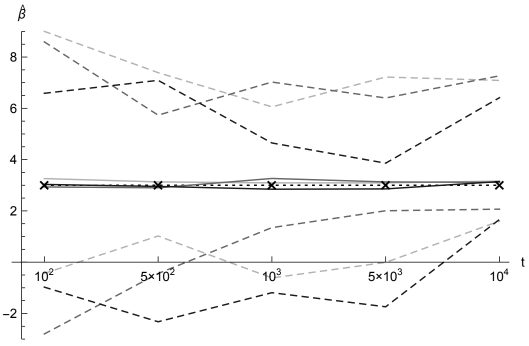


Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from given normal distributions. Black dotted lines with $\times$ depict the true β. Grey full lines depict sample means of estimated β. Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for $N = 100$, normal grey for $N = 500$, and dark grey for $N = 1000$. 't' (horizontal axis) stands for the length of generated time series.

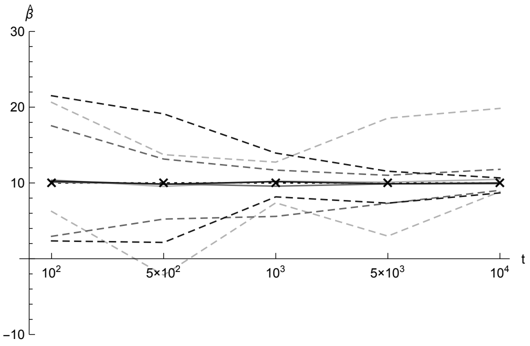
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 5.3: Simulation results for various number of runs and βs II.

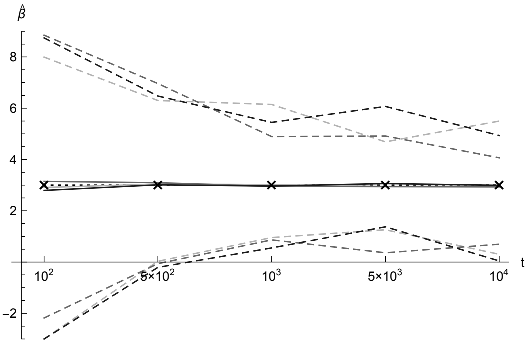
(a) 100 runs, $\beta = 3$, $N(0, 0.1^2)$



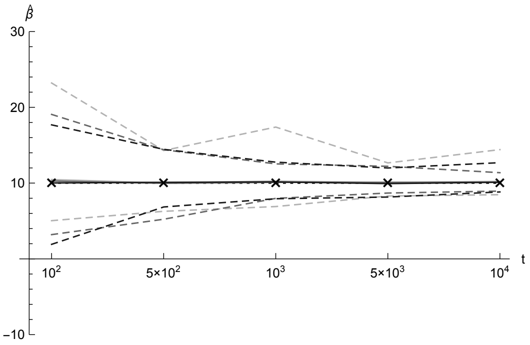
(b) 100 runs, $\beta = 10$, $N(0, 0.1^2)$



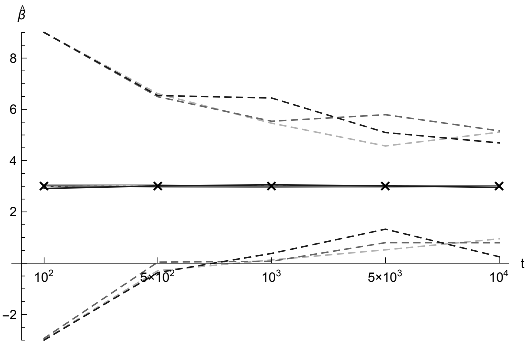
(c) 500 runs, $\beta = 3$, $N(0, 0.1^2)$



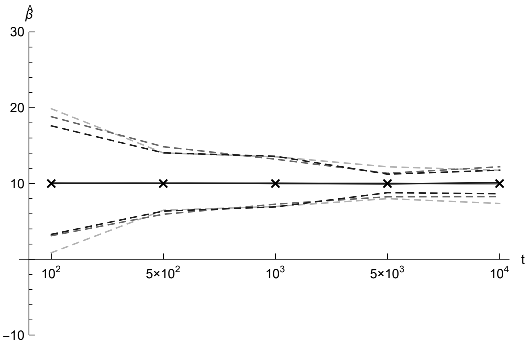
(d) 500 runs, $\beta = 10$, $N(0, 0.1^2)$



(e) 1000 runs, $\beta = 3$, $N(0, 0.1^2)$



(f) 1000 runs, $\beta = 10$, $N(0, 0.1^2)$



Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from given normal distributions. Black dotted lines with $\times$ depict the true β. Grey full lines depict sample means of estimated β. Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for $N = 100$, normal grey for $N = 500$, and dark grey for $N = 1000$. 't' (horizontal axis) stands for the length of generated time series.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Despite of its relative simplicity, the setting is otherwise very challenging as capturing the effect of the switching coefficient β is generally difficult (see Section 2.3). Moreover, algorithm with a single starting point for the numerical optimisation and new random draws of the parameters g_h and b_h, $h \in \{2, 3, 4, 5\}$ for each independent run require very robust performance of the search procedure.

Qualitative results

We primarily aim to verify whether important theoretical properties of the estimator, the consistency and asymptotic efficiency, also hold in small samples for the model. In Figures 5.2 and 5.3 we depict and describe a 'snapshot' of simulation results for four interesting values of the intensity of choice $\beta \in \{0, 0.5, 3, 10\}$ combined with three specifications of the stochastic noise: $\epsilon_t \sim N(0, 2^2)$, $\epsilon_t \sim N(0, 1)$ and $\epsilon_t \sim N(0, 0.1^2)$. First, we can clearly observe how the method is able to reveal the true value of β demonstrated by black dotted lines with $\times$. Grey full lines depict the sample means of estimated βs and closely follow the true line. The small departures are naturally mainly observable for the smallest considered number of runs 100 [subfigures (a) and (b)] and for the smallest considered length of generated time series (number of observations) $t = \{100, 500\}$. Equally importantly, we can clearly observe the consistency of the estimator and how the efficiency of the mean estimate increases simultaneously with increasing length of generated time series t as well as the precision of the kernel estimation N. The precision is demonstrated by different shades of grey dashed lines depicting 2.5% and 97.5% quantiles of estimated parameters β. The shift from a relatively orderless pattern observed for 100 runs [subfigures (a) and (b)] to very exemplary theoretically expected pattern with grey dashed lines nearly aligned according to increasing kernel estimation precision (from light grey farther and dark grey closer to the true/mean value) in (e) and (f) is obvious. The difference is much evident between $N = 100$ (light grey) and two higher values, for $N = \{500, 1000\}$ dashed lines are often very close to one another. The line representing $N = 500$ actually appears closer to the mean value in some cases, indicating the sufficiency of the $N = 500$ kernel approximation precision. Increasing statistical validity of results is apparent via rising number of runs starting at 100. Number of runs 500 seems sufficient in terms of only small differences compared to 1000 runs. In the right column of Figure 5.3 for $\beta = 10$ the shift from the case of 100 runs (b) to the case of 1000 runs (f) is not so substantive as in the previous cases and we assign this to a more stable behaviour of the NPSMLE method under various settings caused by value of β farther from 0. But on the other hand, consistency of the estimator and the growth of efficiency of the mean estimate with increasing lengths of generated time series t and the precision of the kernel estimation N is well observable also in this setting.

Focusing further on the right column of Figure 5.2 and the left column of Figure 5.3, we moreover observe an important result from the economic interpretation point of view of the β parameter value. Although only the non-negative values of β have an economic interpretation,[7] in simulations we also allow for negative estimated values (for details of the simulation setup please refer to Section 5.1 and Section 5.2) to test the capability of the method even for such extreme values and to avoid the upward bias of the estimator. However, the most important result is that using a reasonably robust setting [e.g. number of runs 500 and precision of the kernel estimation $N = 500$] we obtain more than 97.5% non-negative observation (represented by the bound of the 2.5% quantile to be found in the positive half-plane) for $\beta = 10$ even when length of generated time series $t = 100$, for $\beta = 3$ when $t \doteq 500$, and for $\beta = 0.5$ when $t \doteq 1000$. At the same moment 95% of observation appear reasonably close to the true value, far from the numerical bounds of the parameter space imposed to make the constrained optimisation computationally feasible. These features have important favourable consequences for application of the method to datasets of various lengths—we should be able to detect even very weak signs of behavioural switching in long-span daily financial data, but also stronger signs of switching in macroeconomic data where typically lower-frequency time series of shorter lengths are available. W.r.t. the complexity of the estimation issue in the nonlinear HAM setting with five repeatedly randomly generated strategies (as well as to many other estimation attempts from Chapter 2 that have found the switching coefficient insignificant), we consider our results very promising. The most important property of the estimation method in the current setting is the ability to distinguish between statistically significant and insignificant β and this objective is well achieved.

Figures 5.2 and 5.3 also allow for comparison between estimation of models with and without switching. The left column of Figure 5.2 represents the model

[7] $\beta < 0$ for which we technically allow implies switching of agent towards less profitable strategies, unambiguously economically irrational behaviour.

without switching ($\beta = 0$), the right column and Figure 5.3 illustrate estimation performance for models with switching ($\beta > 0$).

Quantitative results

Now we move from the graphical to the quantitative description of simulation results. We now consider only the second most robust setting combination which proved optimal, i.e. results based on 1000 random runs, length of generated time series (number of observations) $t = 5000$, and the kernel estimation precision $N = 1000$ i.i.d. draws from given distribution. First we comment on the robustness of the method w.r.t. various noise specifications used both for generating the stochastic term ϵ_t in Equation 5.3 as well as for N i.i.d. draws, $\{\varepsilon_i\}_{i=1}^N$, to simulate N i.i.d. random variables, $\{X_{t,i}^\theta\}_{i=1}^N$, used for the kernel estimation of the conditional density. Again, following results offer a direct comparison between estimation of models with and without switching as first rows of all panels in all tables in Subsection 5.3.1 always represent the model setting without switching ($\beta = 0$).

Interpreting results in Table 5.1, the big picture seems promising for the NPSMLE method. We can observe relatively stable results over a reasonable grid of noise specifications and therefore the important issue of the robustness of the method is verified. Focusing on first columns containing the sample medians and means of the estimated values (denoted 'Med.' and 'Mean'), we reveal the ability of the method to recover the true values of the intensity of choice β coefficient with very high precision over all noise specification. Median value is generally more precisely estimating the true value but the difference is negligible in majority of cases. Only for two most intensive noises in combination with higher values of the β coefficient, the mean estimate gives considerably better results. Comparing the third columns displaying related standard deviations we observe statistical significance of estimates for majority of combinations of the true β and the magnitude of noise. Generally the specifications with the smallest noises [subparts (a) and especially (b)] appear markedly more precise in estimating the lowest $\beta s = \{0, 0.1, 0.5\}$ with only noise specification (b) having real ability estimate zero β with reasonable precision as we can observe using 2.5 (LQ) and 97.5 (HQ) quantile figures. On the other hand, the specifications with almost largest noises [subparts (i) and especially (h)] appear the most precise in estimating higher $\beta s = \{1, 3, 5, 10\}$. Values of the intensity of choice β very close to zero, $\beta = \{0.1, 0.5\}$, are the most difficult to estimate.

Table 5.1: Results for β estimation with normal noise

β	(a) $\widehat{\beta},\ N(0,10^{-16})$						(b) $\widehat{\beta},\ N(0,10^{-14})$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	-.00	.00	.03	-.02	.04	41%	-.00	-.00	.07	-.15	.11	2%
.1	.10	.10	.02	.08	.12	17%	.10	.10	.05	-.03	.21	0%
.5	.50	.50	.05	.47	.54	54%	.50	.50	.11	.40	.67	7%
1	1.00	.99	.08	.91	1.05	70%	1.00	1.00	.18	.84	1.15	22%
3	3.00	3.02	.17	2.88	3.38	86%	3.00	3.01	.35	2.77	3.35	43%
5	5.00	5.08	.90	4.88	5.20	89%	5.00	4.99	.29	4.74	5.16	59%
10	10.00	9.98	.07	9.78	10.10	95%	10.00	9.99	.29	9.57	10.54	72%
	(c) $\widehat{\beta},\ N(0,10^{-12})$						(d) $\widehat{\beta},\ N(0,10^{-10})$					
0	.00	-.01	.17	-.46	.41	0%	.00	.01	.24	-.49	.49	0%
.1	.10	.10	.11	-.10	.30	0%	.09	.09	.13	-.10	.30	0%
.5	.50	.49	.26	-.19	1.15	0%	.50	.49	.36	-.37	1.29	0%
1	1.00	.99	.33	.28	1.78	0%	1.00	1.00	.50	-.22	2.25	0%
3	3.00	3.01	.60	2.21	3.78	3%	3.01	3.02	1.07	.39	5.31	0%
5	5.00	4.97	.62	4.12	5.59	10%	5.01	4.97	1.39	2.51	6.83	2%
10	10.00	9.96	1.04	8.91	11.00	33%	10.01	9.80	2.10	5.93	11.29	20%
	(e) $\widehat{\beta},\ N(0,10^{-8})$						(f) $\widehat{\beta},\ N(0,10^{-6})$					
0	-.00	-.00	.23	-.48	.48	0%	-.00	-.01	.23	-.46	.46	0%
.1	.11	.11	.12	-.10	.30	0%	.10	.10	.12	-.10	.30	0%
.5	.50	.49	.33	-.30	1.21	0%	.49	.47	.35	-.37	1.26	0%
1	1.01	1.04	.50	-.09	2.36	0%	1.01	1.04	.51	-.10	2.43	0%
3	3.01	3.03	.91	1.33	5.07	0%	2.99	3.00	.95	.93	5.03	0%
5	4.99	5.01	1.27	3.13	6.90	2%	5.00	4.98	1.19	2.52	6.52	2%
10	10.00	10.02	2.28	7.85	12.48	19%	9.99	9.94	2.00	6.83	11.61	19%
	(g) $\widehat{\beta},\ N(0,0.01^2)$						(h) $\widehat{\beta},\ N(0,0.1^2)$					
0	.01	-.00	.23	-.49	.45	0%	.01	.01	.22	-.45	.46	0%
.1	.09	.10	.12	-.10	.30	0%	.11	.11	.12	-.10	.30	0%
.5	.50	.49	.34	-.32	1.26	0%	.50	.50	.35	-.35	1.30	0%
1	.99	.99	.52	-.35	2.31	0%	.99	1.00	.50	-.19	2.46	0%
3	3.01	3.00	.89	1.04	4.74	0%	2.99	3.05	1.00	1.48	5.86	0%
5	5.01	5.01	1.26	2.39	7.11	2%	4.99	5.05	1.21	3.75	6.81	1%
10	10.00	9.85	2.42	5.97	11.90	13%	9.99	9.99	2.22	7.57	11.64	3%
	(i) $\widehat{\beta},\ N(0,1)$						(j) $\widehat{\beta},\ N(0,2^2)$					
0	.00	.00	.11	-.24	.23	0%	-.00	-.00	.05	-.08	.08	0%
.1	.11	.11	.08	-.09	.30	0%	.10	.10	.04	.01	.20	0%
.5	.50	.51	.14	.23	.81	0%	.50	.51	.11	.33	.72	2%
1	1.00	1.01	.23	.66	1.45	1%	1.01	1.05	.27	.71	1.76	4%
3	3.07	3.59	1.41	2.35	7.93	3%	3.34	4.01	1.69	2.14	8.49	35%
5	5.61	7.23	3.30	3.82	14.41	8%	4.96	5.01	1.64	2.57	8.44	64%
10	11.20	13.43	6.31	5.16	28.13	23%	7.77	5.63	5.87	-9.53	10.64	96%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distributions of given parameters, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

This is, however, almost the extreme case of no switching of agents among possible strategies in which the dynamics of the model is restrained as there is only a small difference of the model behaviour compared to the agents' absolute inertia case with $\beta = 0$. These are crucial findings highlighting the necessity of a proper noise specification within the estimation procedure. Larger noises seem to stabilise the system but overshadow the effect of switching under low βs and therefore favour estimation of higher βs. Lower βs require small noises for the effect of switching to be detectable. A puzzling result is then the subpart (h) with the largest noise intensity $N(0, 2^2)$ estimating the lowest βs with high precision.

Another emerging issue is the occurrence of 'Not a Number' outcomes from some runs of the NPSMLE procedure. This is reported via the NN column as the percentage of the 'NaN' outcomes. Technical reason behind the 'NaN' emergence is that the HAM algorithm does not converge into a stationary series in the particular run and the NPSMLE algorithm therefore produces a 'NaN' outcome. As we can see in Table 5.1, this situation is typical mainly for small distribution intervals of the stochastic noise ϵ_t that do not always suffice to stabilise the system [see subparts (a) and (b)] and for highest values of the intensity of choice β [see subparts (i) and especially (j)] increasing switching dynamics in the model, particularly when these two effects combine together. We can interpret this as a specific kind of censorship of results as these runs are not considered for calculation of reported values. We do not consider results with high number of 'NaN' outcomes relevant within this analysis, however, we keep displaying them to retain the completeness of provided information as well as an optimal warning signal of an improper behaviour of the system under scrutiny. We can also observe signs of upward and downward biases of the estimates in cases of largest distribution intervals of the stochastic noise ϵ_t when combined with highest βs [see subparts (i) and (j)] but this might just be an effect of this data censorship as all such cases are accompanied with occurrence of 'NaN' outcomes. Finally, one might notice that our grid of noise specifications in Table 5.1 to a great extent covers the entire range of reasonable values w.r.t. the 'NaN' emergence issue affecting estimation results crucially both for the smallest noises [see subparts (a) and (b)] as well as for the noise specifications with the largest standard deviation [subpart (j)]. Decreasing or increasing the noise intensity behind these bounds in specified setup leads to even less relevant results and therefore is not considered. As the 'NaN' emergence censoring estimation results might be a serious issue, it definitely needs more assessment in the next part of the study (see Section 5.3.1 and Subsection 5.3.2 for further discussion).

As we need to reduce the large grid of stochastic noise specification for other applications in Chapter 5, taking all discussed aspects into account, we select the two most 'successful' specifications, namely $N(0, 0.1^2)$—especially for larger $\beta = \{3, 5, 10\}$—and $N(0, 1)$—especially for smaller $\beta s = \{0.5, 1\}$).[8] They produce estimates with low standard errors for majority of βs considered, they are not accompanied with excessive number of 'NaN' outcomes, and they appear reasonably realistic w.r.t. the empirical application in Chapter 6 where we analyse time series of price deviations implying higher standard deviations of the assumed stochastic market noise. Figure 5.4 depicts smooth histograms of selected estimated βs based on these three noise specifications. One can clearly observe how the noise specification $N(0, 0.1^2)$ performs best for $\beta = \{3, 10\}$, $N(0, 1)$ for $\beta = 0.5$, and $N(0, 2^2)$ for $\beta = 0$.

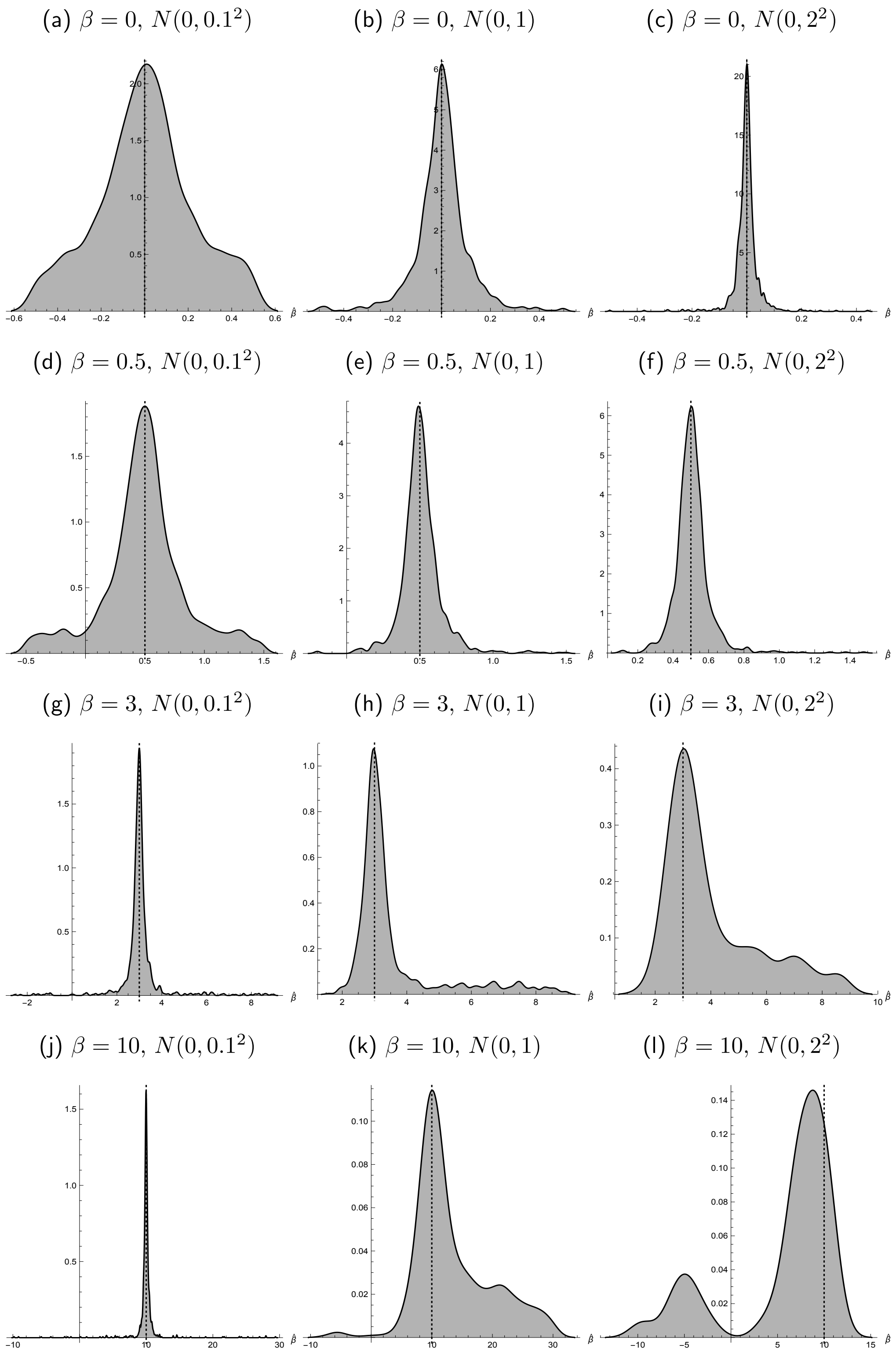
Behaviour of the simulated log-likelihood function

Kristensen & Shin (2012, pg. 80–81) define a set of regularity conditions A.1–A.4 regarding the model and its associated conditional density that ensure sufficiently fast convergence of $\hat{c} \longrightarrow c$ and thus asymptotic equivalence of $\hat{\theta}$ and $\tilde{\theta}$. These conditions basically impose restrictions on the data-generating functions and the conditional density that is being estimated. With regard to data-generating functions, authors argue that the "smoothness conditions are rather weak, and satisfied by most models". For the conditional density function, they state that "the assumptions are quite weak and are satisfied by many models". However, for the HAM, we are not able to verify these conditions analytically and we must rely on graphical computational tools. Another important issue regards the identification of parameters in the model assuring uniqueness of the set of estimates.

For both purposes, we draw the simulated log-likelihood function and verify an existence of a unique maximum. We depict simulated log-likelihood functions for the same four interesting values of the intensity of choice $\beta \in \{0, 0.5, 3, 10\}$ combined with three specifications of the stochastic noise: $\epsilon_t \sim N(0, 2^2)$, $\epsilon_t \sim N(0, 1)$ and $\epsilon_t \sim N(0, 0.1^2)$ as in Section 5.3.1. In Figure 5.5 we clearly observe very smooth shape of the functions over the entire assessed domain with a unique maximum generally shared for all of 100 random runs.

[8] Moreover, for extreme cases $\beta s = \{0, 0.1\}$ noise $N(0, 2^2)$ seems optimal.

Figure 5.4: Smooth histograms for selected $\widehat{\beta}s$

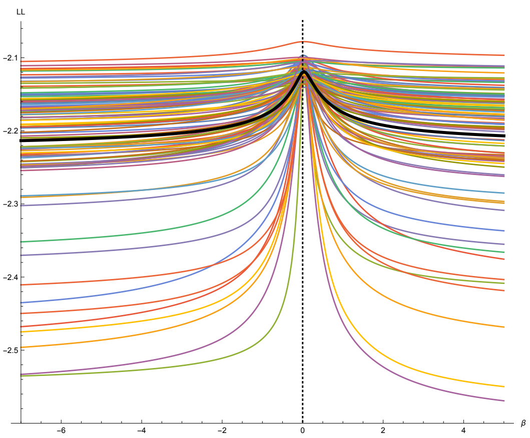


Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from given normal distributions, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Black dotted lines depict the true βs. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

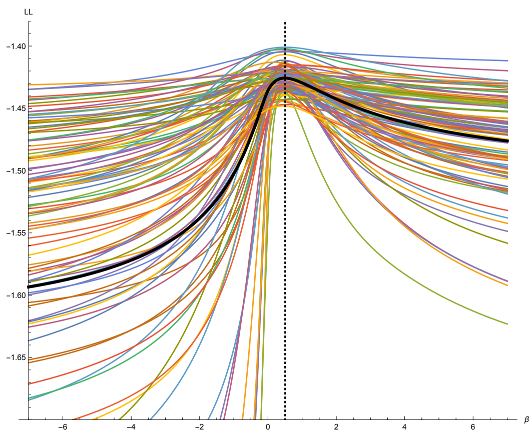
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 5.5: Shape of the simulated log-likelihood function

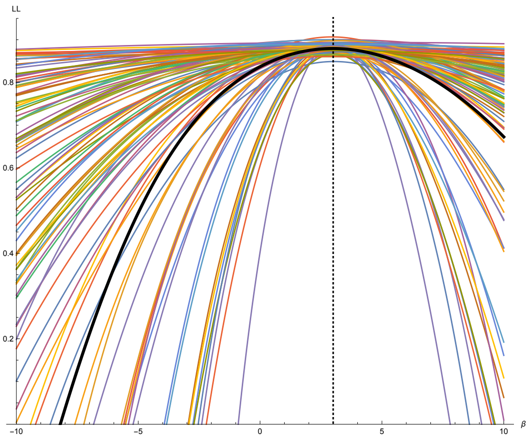
(a) $\beta = 0,\ N(0, 2^2)$



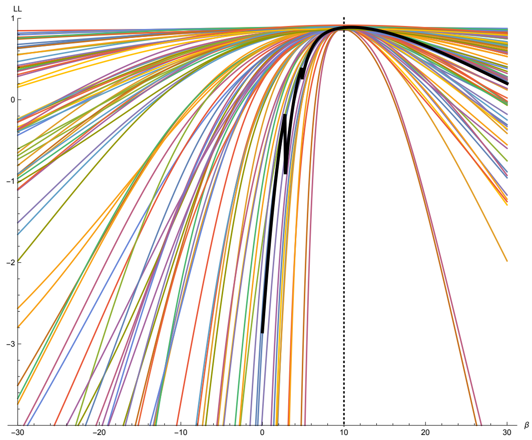
(b) $\beta = 0.5,\ N(0, 1)$



(c) $\beta = 3,\ N(0, 0.1^2)$



(d) $\beta = 10,\ N(0, 0.1^2)$



Note: Results based on 100 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from given normal distribution. Black dotted vertical lines depict the true βs. Bold black full lines depict sample averages.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Bold black full lines then represent sample averages over these 100 runs and brings the aggregate information,[9] which is, however, obvious also directly from the set of 100 original simulated log-likelihood functions. Based on generally smooth shapes and unique optima of the simulated log-likelihood functions we assume that the regularity conditions are met for the HAM and the identification

[9]The only violation of the smoothness of the averaged function appears in subpart (d) with relatively high value $\beta = 10$ where for several runs the model diverges. Disruptions of the averaged function are thus only of the technical origin when depicting, not the feature of the function itself.

of parameters is also assured.

Robustness check

To assess the robustness of our general model setting, we contrast the results in Table 5.1 with several setup modifications. First, in Table 5.2 we consider 10 times higher gross interest rate $R = 1 + r = 1.001$ representing real market risk free rate. This daily value unrealistically represents circa 28.4% annual risk free interest rate that can nonetheless serve as a useful robustness check. The only considerable effect appears in the increased probability of a 'NaN' outcome for the smallest distribution intervals of the stochastic noise ϵ_t [see subparts (a), (b), and (c)]. Conversely, for larger distribution intervals of the stochastic noise ϵ_t the results are comparable and largely similar. This is another important finding mainly for the empirical application in Chapter 6 where time series of price deviations are likely to be associated with higher standard deviation of the assumed stochastic market noise. The robustness of the method w.r.t. assumption of the real market risk free rate therefore relaxes the need of a very precise derivation of this parameter for various countries and historical periods and the reasonable approximation $R = 1 + r = 1.0001$ representing circa 2.5% annual risk free interest rate can be generally used in Chapter 6.

Second, we test the ability of the estimation method to provide unbiased estimates even if bounds of the parameter space are off-centered, more specifically shifted up by 50% of actual β to $\langle -0.5\beta, 3.5\beta \rangle$ for $\beta > 0$ and to $\langle -0.375, 625 \rangle$ for $\beta = 0$. When results of this testing summarized in Table 5.3 are compared to the original results in Table 5.1, we clearly observe expected shift in the 2.5 (LQ) and 97.5 (HQ) quantiles of the estimate distribution but the ability of the NPSMLE method to reveal true parameter with high precision remains unaffected and the standard deviations are to a great extent similar to the original settings. We therefore verified that there is no need of an excessive precision of the unconstrained pre-estimation via which we define bounds of the parameter space for the constrained optimisation.

We further test how is the NPSMLE method performance affected by assumption of another than normal distribution of the stochastic noise ϵ_t. For that purpose we select uniform distribution both for its simplicity as well as for its feature of being the maximum entropy probability distribution of its family of symmetric probability distributions. As the assumption of normal distribution of stock market noise seems reasonably realistic (see discussion in Section 5.1),

Table 5.2: Results for β estimation with normal noise, $R = 1.001$

β	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
	(a) $\widehat{\beta},\ N(0,10^{-16})$						(b) $\widehat{\beta},\ N(0,10^{-14})$					
0	.00	.00	.04	-.00	.01	83%	-.00	-.00	.02	-.03	.02	39%
.1	.10	.10	.01	.10	.11	72%	.10	.10	.02	.08	.12	17%
.5	.50	.50	.01	.49	.52	90%	.50	.50	.05	.47	.55	56%
1	1.00	1.01	.10	.97	1.03	93%	1.00	1.00	.10	.95	1.11	69%
3	3.00	3.00	.03	2.94	3.15	96%	3.00	3.00	.03	2.94	3.07	86%
5	5.00	5.03	.23	4.49	5.84	98%	5.00	5.05	.44	4.86	5.28	90%
10	10.00	10.01	.01	10.00	10.04	99%	10.00	9.65	2.50	9.07	10.08	94%
	(c) $\widehat{\beta},\ N(0,10^{-12})$						(d) $\widehat{\beta},\ N(0,10^{-10})$					
0	.00	.00	.08	-.12	.16	2%	.00	.00	.18	-.46	.44	0%
.1	.10	.10	.05	-.07	.24	0%	.10	.10	.10	-.10	.30	0%
.5	.50	.50	.11	.34	.66	7%	.50	.49	.24	-.13	1.01	0%
1	1.00	.99	.15	.85	1.17	19%	1.00	1.00	.28	.49	1.58	0%
3	3.00	2.99	.29	2.77	3.21	43%	3.00	2.98	.41	2.18	3.60	3%
5	5.00	5.01	.21	4.74	5.38	59%	5.00	4.98	.84	4.19	5.76	10%
10	10.00	9.97	.47	9.60	10.24	73%	10.00	9.96	.94	9.08	10.88	30%
	(e) $\widehat{\beta},\ N(0,10^{-8})$						(f) $\widehat{\beta},\ N(0,10^{-6})$					
0	-.00	-.00	.25	-.50	.50	0%	.00	.00	.23	-.46	.48	0%
.1	.09	.09	.13	-.10	.30	0%	.10	.10	.12	-.10	.30	0%
.5	.50	.50	.34	-.25	1.32	0%	.50	.51	.34	-.29	1.35	0%
1	.98	.97	.52	-.32	2.18	0%	1.02	1.03	.52	-.37	2.41	0%
3	3.00	2.97	.87	.79	4.40	0%	3.00	2.98	1.01	.44	4.86	0%
5	5.00	4.97	1.22	3.10	6.74	3%	4.99	4.96	1.56	1.59	7.41	3%
10	9.99	9.95	1.55	7.98	11.45	19%	9.99	10.04	2.04	8.29	11.97	17%
	(g) $\widehat{\beta},\ N(0,0.01^2)$						(h) $\widehat{\beta},\ N(0,0.1^2)$					
0	-.00	-.00	.23	-.47	.45	0%	-.01	-.01	.22	-.47	.43	0%
.1	.10	.10	.12	-.10	.30	0%	.10	.10	.12	-.10	.30	0%
.5	.50	.50	.35	-.30	1.30	0%	.50	.50	.36	-.32	1.35	0%
1	1.00	.99	.55	-.43	2.40	0%	1.01	1.02	.52	-.28	2.38	0%
3	3.01	3.04	1.02	.87	5.48	0%	3.00	3.00	.90	1.16	4.85	0%
5	5.00	4.98	1.51	2.77	7.55	3%	4.99	4.98	1.41	2.90	7.09	0%
10	10.01	9.87	2.31	4.97	11.58	16%	10.01	10.07	1.83	8.82	12.20	1%
	(i) $\widehat{\beta},\ N(0,1)$						(j) $\widehat{\beta},\ N(0,2^2)$					
0	.00	.00	.10	-.23	.22	0%	.00	.00	.05	-.08	.10	0%
.1	.10	.10	.09	-.10	.30	0%	.10	.10	.04	-.00	.18	0%
.5	.50	.50	.14	.24	.80	0%	.50	.51	.10	.36	.73	2%
1	1.00	1.02	.20	.66	1.43	1%	1.01	1.06	.31	.71	2.03	4%
3	3.08	3.58	1.38	2.36	7.79	4%	3.40	3.93	1.57	2.11	8.36	34%
5	5.54	7.21	3.25	3.77	14.39	6%	5.03	5.25	1.67	2.80	9.13	63%
10	10.87	12.84	6.09	4.87	27.55	25%	6.14	3.97	6.82	-9.84	11.22	97%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from normal distributions of given parameters, $R = 1.001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$. and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table 5.3: Results for β estimation with normal noise, off-centered

β	(a) $\widehat{\beta},\ N(0,10^{-16})$						(b) $\widehat{\beta},\ N(0,10^{-14})$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	-.00	-.00	.03	-.02	.02	40%	-.00	.00	.07	-.12	.12	2%
.1	.10	.10	.02	.09	.11	18%	.10	.10	.05	-.04	.21	0%
.5	.50	.50	.09	.46	.54	57%	.50	.51	.12	.36	.69	9%
1	1.00	1.00	.04	.95	1.04	72%	1.00	.99	.13	.80	1.13	22%
3	3.00	3.00	.31	2.89	3.07	85%	3.00	2.99	.31	2.78	3.25	45%
5	5.00	5.00	.06	4.88	5.19	90%	5.00	5.00	.60	4.56	5.30	59%
10	10.00	10.07	.49	9.93	10.17	94%	10.00	10.01	.26	9.44	10.57	74%
	(c) $\widehat{\beta},\ N(0,10^{-12})$						(d) $\widehat{\beta},\ N(0,10^{-10})$					
0	-.00	.01	.17	-.38	.52	0%	.03	.05	.25	-.38	.60	0%
.1	.10	.11	.10	-.05	.35	0%	.13	.13	.13	-.05	.35	0%
.5	.50	.52	.26	-.02	1.34	0%	.50	.53	.36	-.20	1.49	0%
1	1.00	1.01	.31	.37	1.79	0%	1.00	1.05	.55	-.08	2.69	0%
3	3.00	3.01	.58	2.26	3.85	4%	3.00	3.05	1.00	1.25	5.50	0%
5	5.00	5.02	.87	4.13	5.81	12%	5.00	5.08	1.39	3.08	9.45	2%
10	10.00	10.09	1.53	9.01	11.66	34%	10.00	10.16	2.47	7.95	13.38	20
	(e) $\widehat{\beta},\ N(0,10^{-8})$						(f) $\widehat{\beta},\ N(0,10^{-6})$					
0	.00	.01	.23	-.38	.54	0%	.03	.06	.24	-.37	.59	0%
.1	.11	.12	.12	-.05	.35	0%	.12	.13	.12	-.05	.35	0%
.5	.50	.53	.34	-.15	1.47	0%	.51	.56	.36	-.14	1.55	0%
1	1.01	1.03	.49	-.02	2.34	0%	1.01	1.07	.51	.03	2.52	0%
3	3.01	3.07	.97	1.21	5.96	0%	3.00	3.08	1.01	1.14	5.88	0%
5	5.00	5.05	1.37	2.68	7.51	3%	4.99	5.08	1.28	3.51	7.24	2%
10	10.01	10.18	2.38	7.95	14.78	20%	10.00	10.24	2.60	8.40	13.99	20%
	(g) $\widehat{\beta},\ N(0,0.01^2)$						(h) $\widehat{\beta},\ N(0,0.1^2)$					
0	.02	.03	.22	-.38	.52	0%	.01	.03	.23	-.36	.56	0%
.1	.11	.13	.12	-.05	.35	0%	.12	.13	.12	-.05	.35	0%
.5	.51	.55	.35	-.13	1.50	0%	.50	.54	.35	-.15	1.54	0%
1	.99	1.05	.53	-.05	2.60	0%	1.01	1.07	.55	-.06	2.81	0%
3	3.00	3.13	1.08	1.75	6.87	0%	3.00	3.07	1.05	1.20	5.77	0%
5	5.00	5.12	1.55	2.96	8.68	3%	5.01	5.19	1.58	3.89	9.97	0%
10	9.99	9.93	2.46	5.26	12.04	17%	10.00	10.09	1.75	8.70	11.34	1%
	(i) $\widehat{\beta},\ N(0,1)$						(j) $\widehat{\beta},\ N(0,2^2)$					
0	.00	.00	.10	-.21	.20	0%	.00	.00	.05	-.08	.10	0%
.1	.10	.11	.08	-.05	.32	0%	.10	.10	.04	.02	.20	0%
.5	.50	.51	.15	.23	.89	0%	.50	.51	.10	.34	.71	2%
1	1.01	1.03	.25	.73	1.41	0%	1.01	1.09	.40	.71	2.70	5%
3	3.17	4.17	2.08	2.39	9.67	3%	3.54	4.26	1.89	2.07	9.01	40%
5	6.39	8.36	4.08	4.03	16.81	10%	5.01	5.28	1.70	2.87	9.55	67%
10	11.82	14.99	7.33	6.69	32.74	33%	6.30	6.46	3.08	-4.79	11.83	97%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distributions of given parameters, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

the uniform distribution provides a less realistic candidate for the robustness check. We consider two principles for comparison of our original results with the results assuming the uniform distribution—the first based on almost identical covered intervals (Table 5.4) and the second based on the same variance (Table 5.5). We therefore define a grid of 10 uniform noise specifications

1. that cover the same intervals as are covered by their respective normal counterparts by the 99.74% of the probability mass;[10]

2. with the same variances as their respective normal counterparts.[11]

Contrasting original results in Table 5.1 with results based on the uniform distribution of the stochastic noise, we basically verify the assumption of Kristensen & Shin (2012) that $\mathcal{F}_\varepsilon$ can be any known distribution. The overall result are largely similar and the observed differences can be attributed mainly to different shapes of the normal and uniform distribution. In the case of identical covered intervals (Table 5.4) we observe slightly lower probability of 'NaN' outcome occurrence for smallest intervals [subparts (a), (b), and (c)], but higher for largest intervals [subparts (i) and (j)]. In the case of identical distribution (Table 5.5), the 'NaN' outcome occurrence is comparable with the normal distribution. In both cases the efficiency of estimates tends to be higher than using the normal distribution and intervals between the 2.5 (LQ) and 97.5 (HQ) quantiles are narrower in majority of specifications. We can assign this to differences in shape of compared distribution—the highest probability density of the normal distribution around the zero mean combined with possibility of extreme observations, both apparently negatively affecting the efficiency of estimates.

One of the most challenging concepts in the estimation method setting is the repeated random generation of parameters g_h and b_h for each from 1000 runs. In Table 5.6 we abandon this setup feature, fix parameters g_h and b_h randomly before the loop and use the very same figures for all 1000 runs. Although the repeated random generation of trend and bias parameters for each run is one of the robustness cornerstones of the analysis in Chapter 5, it is of our interest to provide the comprehensive picture of the NPSMLE method performance.

[10] I.e. $\langle -3SD, 3SD \rangle$, where SD stands for the standard deviation of the respective normal noise specification.

[11] Variance (or the second centralized moment) of the continuous uniform distribution is defined as $\frac{(b-a)^2}{12}$, where a, b are the minimum and maximum values of the distribution's domain.

Table 5.4: Results for β estimation with uniform noise I...

... covering *equal intervals* as their respective normal counterparts by 99.74% of the probability mass

β	(a) $\widehat{\beta},\ U(-3\times10^{-8},3\times10^{-8})$						(b) $\widehat{\beta},\ U(-3\times10^{-7},3\times10^{-7})$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	-.00	-.00	.04	-.03	.03	26%	.00	.00	.07	-.12	.13	0%
.1	.10	.10	.02	.09	.12	10%	.10	.10	.05	-.01	.24	0%
.5	.50	.50	.04	.47	.53	42%	.50	.50	.11	.30	.73	2%
1	1.00	1.00	.09	.94	1.04	56%	1.00	1.00	.10	.86	1.12	9%
3	3.00	3.01	.11	2.96	3.14	80%	3.00	3.01	.16	2.81	3.15	31%
5	5.00	5.00	.05	4.91	5.12	85%	5.00	5.00	.39	4.85	5.22	44%
10	10.00	10.02	.20	9.88	10.07	92%	10.00	10.02	.35	9.79	10.41	62%
	(c) $\widehat{\beta},\ U(-3\times10^{-6},3\times10^{-6})$						(d) $\widehat{\beta},\ U(-3\times10^{-5},3\times10^{-5})$					
0	-.00	-.01	.15	-.41	.34	0%	-.00	-.01	.20	-.45	.45	0%
.1	.10	.10	.10	-.10	.30	0%	.10	.10	.11	-.10	.30	0%
.5	.50	.50	.21	.02	.98	0%	.49	.48	.27	-.21	1.14	0%
1	1.00	1.01	.28	.49	1.59	0%	1.01	1.00	.38	-.08	1.92	0%
3	3.00	3.02	.50	2.57	3.74	1%	2.99	2.98	.79	1.76	4.19	0%
5	5.00	5.03	.60	4.44	5.69	5%	5.00	4.96	.86	4.02	5.64	2%
10	10.00	10.02	.71	9.50	10.91	28%	10.00	10.05	1.25	9.31	10.79	22%
	(e) $\widehat{\beta},\ U(-3\times10^{-4},3\times10^{-4})$						(f) $\widehat{\beta},\ U(-3\times10^{-3},3\times10^{-3})$					
0	.00	-.00	.17	-.39	.39	0%	.00	.00	.17	-.39	.41	0%
.1	.10	.10	.10	-.09	.28	0%	.10	.10	.09	-.09	.29	0%
.5	.50	.49	.26	-.20	1.16	0%	.50	.49	.29	-.33	1.16	0%
1	1.00	1.00	.40	-.06	2.03	0%	1.01	1.03	.35	.29	1.96	0%
3	2.99	3.00	.73	2.13	3.69	0%	3.00	3.02	.80	2.01	4.38	0%
5	5.00	4.95	1.07	3.23	6.01	2%	4.99	5.00	1.11	4.14	5.84	2%
10	10.00	10.02	1.55	9.38	11.20	19%	10.00	9.91	1.72	8.76	10.67	17%
	(g) $\widehat{\beta},\ U(-0.03,0.03)$						(h) $\widehat{\beta},\ U(-0.3,0.3)$					
0	-.00	-.01	.17	-.41	.37	0%	.00	-.00	.16	-.40	.35	0%
.1	.10	.10	.10	-.08	.29	0%	.10	.10	.09	-.09	.29	0%
.5	.50	.50	.26	-.21	1.11	0%	.50	.51	.25	-.15	1.14	0%
1	1.00	.99	.40	-.21	1.88	0%	1.00	.99	.40	-.08	2.17	0%
3	3.00	3.03	.72	1.97	4.56	0%	2.99	3.00	.64	2.28	3.73	0%
5	5.00	5.02	1.13	3.85	6.25	2%	5.00	5.03	.89	4.44	5.81	0%
10	10.00	10.04	1.65	9.34	10.64	12%	10.01	10.10	1.18	9.40	10.89	1%
	(i) $\widehat{\beta},\ U(-3,3)$						(j) $\widehat{\beta},\ U(-6,6)$					
0	-.00	.00	.04	-.06	.06	0%	-.00	.00	.01	-.02	.02	0%
.1	.10	.10	.03	.03	.16	0%	.10	.10	.01	.08	.13	0%
.5	.50	.50	.05	.42	.59	1%	.50	.50	.06	.43	.61	7%
1	1.00	1.01	.15	.86	1.20	3%	1.01	1.06	.27	.81	1.99	20%
3	3.09	3.95	1.77	2.56	8.49	11%	3.14	3.40	.99	2.03	5.90	67%
5	5.56	7.28	3.17	3.95	14.41	26%	4.24	4.05	2.02	-3.68	7.20	88%
10	10.26	11.46	4.78	5.44	24.34	56%	-	-	-	-	-	100%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from uniform distributions of given parameters, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table 5.5: Results for β estimation with uniform noise II...

... with *equal variances* as their respective normal counterparts

β	(a) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-8}, \frac{\sqrt{12}}{2}\times 10^{-8})$						(b) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-7}, \frac{\sqrt{12}}{2}\times 10^{-7})$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.00	.00	.02	-.01	.01	39%	-.00	-.00	.05	-.09	.10	1%
.1	.10	.10	.01	.09	.11	18%	.10	.10	.03	.04	.18	0%
.5	.50	.50	.02	.48	.53	57%	.50	.50	.08	.43	.59	6%
1	1.00	1.00	.04	.97	1.04	68%	1.00	1.00	.10	.92	1.11	20%
3	3.00	2.93	.54	2.93	3.06	87%	3.00	3.00	.15	2.84	3.12	45%
5	5.00	5.01	.07	4.95	5.11	88%	5.00	4.99	.18	4.72	5.18	55%
10	10.00	10.00	.05	9.89	10.18	94%	10.00	10.01	.17	9.85	10.29	72%
	(c) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-6}, \frac{\sqrt{12}}{2}\times 10^{-6})$						(d) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-5}, \frac{\sqrt{12}}{2}\times 10^{-5})$					
0	-.00	.00	.13	-.29	.39	0%	-.00	-.00	.20	-.48	.44	0%
.1	.10	.10	.09	-.10	.30	0%	.10	.10	.11	-.10	.30	0%
.5	.50	.49	.19	.00	.89	0%	.50	.51	.27	-.16	1.25	0%
1	1.00	1.00	.25	.45	1.43	0%	1.00	.98	.41	-.12	2.01	0%
3	3.00	3.00	.35	2.62	3.42	3%	3.00	3.01	.74	2.15	4.04	0%
5	5.00	4.97	.54	4.44	5.41	12%	5.01	5.00	.86	3.84	5.84	2%
10	10.00	10.01	1.04	9.49	10.46	32%	10.00	10.02	1.38	9.23	10.90	21%
	(e) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-4}, \frac{\sqrt{12}}{2}\times 10^{-4})$						(f) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-3}, \frac{\sqrt{12}}{2}\times 10^{-3})$					
0	.00	.00	.18	-.40	.42	0%	.00	.00	.17	-.39	.42	0%
.1	.10	.10	.10	-.09	.29	0%	.10	.10	.09	-.09	.29	0%
.5	.50	.50	.28	-.25	1.22	0%	.50	.50	.27	-.23	1.25	0%
1	1.00	1.02	.44	-.03	2.36	0%	1.00	1.00	.39	-.06	2.08	0%
3	3.00	3.00	.80	1.48	4.29	0%	3.00	2.99	.74	2.01	3.85	0%
5	5.00	4.95	1.03	3.79	5.72	3%	5.00	5.03	.87	4.25	6.43	2%
10	10.01	9.93	1.79	9.41	10.74	19%	10.00	9.97	1.76	9.11	10.69	19%
	(g) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-2}, \frac{\sqrt{12}}{2}\times 10^{-2})$						(h) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}\times 10^{-1}, \frac{\sqrt{12}}{2}\times 10^{-1})$					
0	.00	.00	.17	-.40	.39	0%	-.00	-.00	.17	-.40	.39	0%
.1	.10	.10	.09	-.08	.29	0%	.10	.10	.10	-.09	.29	0%
.5	.50	.52	.27	-.15	1.21	0%	.50	.49	.27	-.21	1.19	0%
1	1.00	.99	.45	-.23	2.21	0%	1.00	.99	.36	.13	1.91	0%
3	3.00	3.01	.72	2.15	4.44	0%	3.01	3.02	.80	1.71	4.55	0%
5	5.00	4.98	.90	4.08	5.87	3%	5.00	5.05	.97	4.51	5.81	1%
10	9.99	9.99	1.34	9.27	10.74	14%	10.01	10.02	1.51	9.44	10.71	2%
	(i) $\widehat{\beta}$, $U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$						(j) $\widehat{\beta}$, $U(-2\frac{\sqrt{12}}{2}, 2\frac{\sqrt{12}}{2})$					
0	.00	.00	.06	-.11	.15	0%	.00	.00	.02	-.04	.04	0%
.1	.10	.10	.06	-.02	.25	0%	.10	.10	.03	.06	.16	0%
.5	.50	.50	.09	.34	.64	0%	.50	.50	.06	.42	.59	2%
1	1.00	1.00	.08	.82	1.16	1%	1.00	1.01	.11	.87	1.18	3%
3	3.02	3.18	.82	2.64	6.53	3%	3.16	4.25	1.96	2.54	8.67	18%
5	5.13	6.62	2.89	4.34	13.75	6%	5.49	6.81	2.86	3.49	13.82	38%
10	11.05	14.78	6.59	8.05	29.42	17%	9.46	9.02	4.31	-5.54	17.48	68%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from uniform distributions of given parameters, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table 5.6: Results for β estimation with normal noise, fixed g_h & b_h

β	(a) $\widehat{\beta},\ N(0,0.1^2)$						(b) $\widehat{\beta},\ N(0,1)$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	-.00	.00	.13	-.26	.28	0%	-.00	-.00	.05	-.10	.09	0%
.1	.11	.10	.12	-.10	.30	0%	.10	.10	.06	-.02	.22	0%
.5	.49	.49	.11	.26	.71	0%	.50	.51	.17	.19	.85	0%
1	1.00	1.00	.07	.86	1.14	0%	1.00	1.01	.18	.67	1.40	0%
3	3.01	3.00	.13	2.75	3.23	0%	3.03	3.12	.57	2.49	5.28	0%
5	5.00	5.00	.23	4.56	5.47	0%	5.19	6.42	2.70	4.25	13.65	4%
10	10.00	10.00	.12	9.77	10.23	0%	11.04	12.28	4.64	6.07	24.11	19%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distributions of given parameters, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Next in this section, we only compute and depict results for two specification of the stochastic noise which appeared the most useful based on results in Table 5.1, namely $N(0, 0.1^2)$ and $N(0, 1)$. Comparing results in Table 5.6 with respective counterparts Table 5.1 [subparts (h) and (i)], aside a minor reduction of the 'NaN' emergence probability we observe overall significant reduction of standard deviations of β estimates. Fixing trend a bias parameters thus naturally makes the system more predictable and leads to more efficient estimates.

Tables 5.7 and 5.8 report results of another robustness check of the methodology focused on various distributions of belief parameters g_h and b_h (see Equation 4.18). To recap, in the general model we follow the previous work of Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013) and thus trend parameters g_h are drawn from the normal distribution $N(0, 0.4^2)$ and bias parameters b_h are drawn from the normal distribution $N(0, 0.3^2)$. Here we relax this assumption using a range of reasonable variances defining the distribution of beliefs' parameters: $\{0.1^2, 0.2^2, 0.3^2, 0.4^2, 0.6^2, 0.8^2, 1, 1.2^2\}$.

Analysing results in Tables 5.7 and 5.8, we observe the general ability of the method to reveal accurately the true value of the intensity of choice β for the vast majority of combinations of the simulation grid as well as for both noise specifications. On the other hand, in specific cases we can observe signs of an upward bias, values on the border of statistical significance, or considerable probability of 'NaN' outcome emergence. The upward bias is observable only in Table 5.8 associated with the larger specification of the stochastic noise $N(0, 1)$ and for higher values of β. However, the upward bias tendency dis-

Table 5.7: Results for β estimation w.r.t. various dist. of g_h & b_h I.

β	Med.	Mean	SD	LQ	HQ	NN
	(a) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.1^2)$					
0	.02	.01	.29	-.47	.48	0%
.1	.10	.10	.12	-.09	.29	0%
.5	.50	.51	.57	-.42	1.44	0%
1	.87	.94	1.15	-.95	2.91	0%
3	2.95	2.95	3.25	-2.67	8.69	0%
5	4.99	5.07	5.04	-4.26	14.17	0%
10	10.02	10.15	9.03	-8.22	27.82	0%
	(c) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.3^2)$					
0	.00	.00	.23	-.47	.47	0%
.1	.10	.10	.12	-.10	.30	0%
.5	.50	.50	.34	-.34	1.27	0%
1	1.01	1.01	.54	-.40	2.33	0%
3	3.00	3.01	.83	1.57	4.93	0%
5	5.00	5.00	1.35	3.02	6.84	0%
10	9.99	10.01	1.87	8.19	11.55	1%
	(e) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.6^2)$					
0	.00	-.00	.11	-.34	.25	0%
.1	.10	.10	.08	-.09	.29	0%
.5	.50	.50	.13	.28	.75	0%
1	1.00	1.00	.15	.71	1.27	0%
3	3.00	3.00	.19	2.72	3.29	7%
5	5.00	4.99	.31	4.62	5.26	16%
10	10.01	10.04	.88	9.55	10.36	28%
	(g) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1)$					
0	.00	.00	.05	-.06	.10	2%
.1	.10	.10	.04	.02	.18	2%
.5	.50	.50	.07	.39	.58	9%
1	1.00	1.00	.09	.91	1.13	20%
3	3.00	3.00	.07	2.84	3.13	52%
5	5.00	5.01	.26	4.87	5.20	64%
10	10.00	10.21	1.78	9.59	11.98	73%

β	Med.	Mean	SD	LQ	HQ	NN
	(b) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.2^2)$					
0	-.02	-.01	.28	-.48	.48	0%
.1	.10	.10	.12	-.10	.30	0%
.5	.51	.51	.48	-.40	1.39	0%
1	.99	.99	.80	-.73	2.73	0%
3	3.00	2.97	1.81	-1.65	7.43	0%
5	4.97	4.77	2.60	-3.23	10.94	0%
10	9.98	9.92	3.65	.31	18.84	0%
	(d) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.4^2)$					
0	-.01	-.00	.17	-.38	.43	0%
.1	.10	.10	.11	-.10	.30	0%
.5	.50	.50	.24	-.09	1.17	0%
1	1.00	1.00	.37	.20	1.85	0%
3	2.99	2.95	.78	1.24	3.76	0%
5	5.00	5.00	.85	4.17	5.65	1%
10	10.00	10.02	1.33	9.11	10.77	6%
	(f) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.8^2)$					
0	.00	.00	.07	-.14	.16	0%
.1	.10	.10	.05	-.00	.25	1%
.5	.50	.50	.08	.39	.62	2%
1	1.00	1.00	.11	.84	1.16	6%
3	3.00	3.01	.27	2.84	3.18	29%
5	5.00	5.01	.29	4.80	5.27	42%
10	10.00	10.04	1.05	9.47	10.28	54%
	(h) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1.2^2)$					
0	.00	.00	.03	-.05	.06	7%
.1	.10	.10	.03	.05	.15	4%
.5	.50	.50	.04	.43	.56	20%
1	1.00	1.00	.05	.94	1.09	40%
3	3.00	2.99	.14	2.85	3.09	69%
5	5.00	5.04	.67	4.59	5.18	75%
10	10.01	10.42	2.59	9.72	21.17	83%

Note: Belief parameters g_h and b_h drawn from various normal distributions of given parameter, stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0, 0.1^2)$, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

appears with increasing variances of the distribution of beliefs' parameters. This is another example of somehow puzzling behaviour for which we cannot find any obvious explanation. The problem with statistical insignificance of estimates is naturally mostly evident for distribution specification with small variances and small values of β, both technically inhibiting the dynamics of the model. Increasing the variance of beliefs' distribution associated with higher values of randomly generated belief parameters g_h and b_h, we generally obtain a richer model dynamics which can be more simply and more efficiently

Table 5.8: Results for β estimation w.r.t. various dist. of g_h & b_h II.

β	(a) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.1^2)$						(b) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.2^2)$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.02	.01	.29	-.47	.48	0%	-.00	-.00	.24	-.47	.49	0%
.1	.10	.10	.12	-.09	.29	0%	.10	.10	.13	-.10	.30	0%
.5	.51	.52	.57	-.44	1.46	0%	.51	.53	.37	-.34	1.33	0%
1	1.02	1.05	1.11	-.85	2.92	0%	1.02	1.08	.58	-.12	2.64	0%
3	3.62	3.55	3.22	-2.50	8.75	0%	3.49	4.28	2.17	.83	8.59	0%
5	6.74	6.39	5.62	-4.34	14.58	0%	6.59	7.86	3.89	2.14	14.57	0%
10	12.73	12.32	11.46	-8.31	29.60	1%	12.51	13.98	8.89	-5.54	29.58	2%
	(c) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.3^2)$						(d) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.4^2)$					
0	.00	.00	.16	-.39	.34	0%	-.00	-.00	.09	-.21	.19	0%
.1	.10	.10	.10	-.10	.30	0%	.10	.10	.07	-.06	.26	0%
.5	.51	.52	.19	.17	.97	0%	.50	.50	.11	.31	.73	0%
1	1.00	1.02	.30	.60	1.70	0%	1.00	1.01	.19	.75	1.32	1%
3	3.17	3.78	1.57	2.33	8.23	0%	3.05	3.43	1.25	2.48	7.72	4%
5	5.82	7.43	3.42	3.71	14.46	1%	5.35	6.92	3.15	3.89	14.39	8%
10	11.49	13.91	6.99	-.28	28.97	9%	11.16	13.21	5.83	6.10	28.03	25%
	(e) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.6^2)$						(f) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.8^2)$					
0	-.00	-.00	.04	-.08	.08	0%	.00	-.00	.02	-.05	.04	0%
.1	.10	.10	.04	.02	.20	0%	.10	.10	.02	.05	.14	7%
.5	.50	.50	.06	.39	.63	12%	.50	.50	.04	.42	.59	30%
1	1.00	1.00	.09	.83	1.23	16%	1.00	1.00	.07	.87	1.15	41%
3	3.02	3.17	.78	2.56	6.19	26%	3.02	3.08	.53	2.57	3.64	62%
5	5.29	6.67	2.84	4.20	13.73	42%	5.10	6.14	2.51	4.16	13.49	64%
10	10.52	12.26	4.88	6.59	25.70	62%	10.29	11.94	4.29	7.88	24.38	84%
	(g) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1)$						(h) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1.2^2)$					
0	-.00	.00	.01	-.03	.03	3%	.00	.00	.01	-.02	.02	9%
.1	.10	.10	.01	.07	.13	20%	.10	.10	.01	.08	.13	38%
.5	.50	.50	.03	.45	.58	55%	.50	.50	.02	.45	.54	68%
1	1.00	1.00	.06	.89	1.10	64%	1.00	1.00	.04	.91	1.09	77%
3	2.99	3.00	.30	2.61	3.34	75%	2.98	3.01	.31	2.65	3.41	87%
5	5.14	6.19	2.56	4.24	13.26	82%	5.04	6.17	2.69	4.36	13.96	93%
10	10.26	11.42	3.19	8.08	20.32	92%	9.85	10.75	2.88	8.47	19.13	96%

Note: Belief parameters g_h and b_h drawn from various normal distributions of given parameter, stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0, 1)$, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

estimated. However, there is a crucial trade-off in form of model divergence and results censorship by 'NaN' outcomes reported in the NN column. This situation is generally more likely for higher values of the intensity of choice β as well as for the higher potential values of belief parameters g_h and b_h. For instance, the two highest values of our simulation range: $\{1, 1.2^2\}$ induce serious results censorship (up to 96%) for the majority of values from the discrete range of βs, particularly when combined the stochastic noise $N(0, 1)$ that is generally associated with considerably higher probability of 'NaN' outcome emergence. Small stochastic noise specifications generally do not exhibit signs of upward biases but can be also largely affected by 'NaN' outcomes. We refer the reader to additional results of the stochastic noise specification $N(0, 10^{-12})$ and $N(0, 10^{-14})$ summarized in Table B.1 and Table B.2 in Appendix B. The inference based on such filtered result is invalid and we concur the discussion about the 'NaN' outcomes originated in Section 5.3.1 via describing this third[12] aspect triggering the divergence of the model leading to 'NaN' outcome of the NPSMLE procedure. We observe a somewhat nontrivial complex interplay between the intensity of the stochastic noise, estimation efficiency, and probability of 'NaN' outcomes. Higher probabilities are associated both with tiny noises $N(0, 10^{-14})$ and $N(0, 10^{-12})$ (see Tables B.1 and B.2) as well as with larger noise $N(0, 1)$ (Table 5.8) when compared to $N(0, 0.1^2)$ (Table 5.7). However, the mitigating impact on the results censorship in specific setups is far from solving the censorship issue. When comparing results of $N(0, 0.1^2)$ and $N(0, 1)$ in terms of efficiency, one would conclude that the wider distribution interval $\epsilon_t \sim N(0, 1)$ increases the efficiency of estimates. But the opposite holds when comparing $N(0, 10^{-14})$ and $N(0, 10^{-12})$ in Appendix B. We observe this partly ambiguous relation between stochastic noise distributions intervals and values of β also in the original results in Table 5.1.

To sum up hitherto findings regarding the robustness of the NPSMLE method w.r.t. various setup specifications, we face an interesting 'two-sided' trade-off. Basically, we are able to estimate relatively well a model exhibiting reasonably rich dynamics. This is, however, on the one hand inhibited assuming:

1. low values of the intensity of choice β,
2. small distribution intervals of the stochastic noise ϵ_t,
3. or distribution specifications of belief parameters g_h and b_h with small variances,

producing insufficient dynamics or fragile stability of the system. But the other hand also by:

1. too high values of the intensity of choice β
2. large distribution intervals of the stochastic noise ϵ_t,

[12] Along with very small distribution intervals of the stochastic noise ϵ_t and high values of the intensity of choice β.

Table 5.9: Results for β estimation with various combined noises I.

β	(a) $\widehat{\beta},\ \epsilon_t \sim N(0, 0.1^2)$, $\{\varepsilon_i\}_{i=1}^N \sim N(0,1)$						(b) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.00	.00	.29	-.47	.47	0%	-.01	-.00	.29	-.48	.47	0%
.1	.10	.10	.11	-.09	.29	0%	.10	.10	.11	-.09	.29	0%
.5	.50	.50	.58	-.47	1.45	0%	.51	.51	.58	-.47	1.46	0%
1	.97	.99	1.13	-.86	2.89	0%	.99	.98	1.12	-.90	2.87	0%
3	2.87	2.84	3.30	-2.67	8.60	0%	2.87	2.86	3.43	-2.77	8.72	0%
5	4.89	4.87	5.31	-4.48	14.35	0%	5.24	5.32	5.70	-4.69	14.63	0%
10	10.96	11.42	10.56	-8.40	28.99	0%	11.19	10.21	11.65	-9.95	29.17	1%
	(c) $\widehat{\beta},\ \epsilon_t \sim N(0,1)$, $\{\varepsilon_i\}_{i=1}^N \sim N(0, 0.1^2)$						(d) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$					
						100%						100%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from the same distributions with different variances, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

3. or distribution specifications of belief parameters g_h and b_h with large variances,

potentially causing model to diverge and therefore censoring the results. This seems to be another of challenging issues for the empirical application of the NPSMLE method.

The last part of Section 5.3.1 addresses an important question of what happens if wrong stochastic noise assumption is used to perform the NPSMLE? This can either be a right distribution, but with wrong parameters, or a completely different distribution. This question is especially important w.r.t. empirical application in Chapter 6 because in real world data we are rarely able to ascertain proper noise. For the purpose of analysing this issue, we present results of various combinations of different distributions used for random generation of stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$. In Table 5.9 we report the case where stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from the same distributions with different variances, Table 5.10 then displays results when different distributions with various variances (same as well as different) are combined together. Basically, we use combinations of normal and uniform distributions and for different variances we use specifications with 10 time higher or lower values. Conclusions for this robustness check are very clear and can be summarized into several points:

Table 5.10: Results for β estimation with various combined noises II.

β	(a) $\widehat{\beta},\ \epsilon_t \sim N(0,1)$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$						(b) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$, $\{\varepsilon_i\}_{i=1}^N \sim N(0,1)$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.00	-.01	.14	-.38	.29	0%	.00	.01	.12	-.25	.24	0%
.1	.10	.10	.10	-.10	.30	0%	.10	.10	.10	-.10	.30	0%
.5	.50	.51	.16	.19	.89	0%	.50	.51	.14	.22	.82	0%
1	.99	1.00	.25	.54	1.56	1%	1.00	1.01	.23	.63	1.48	1%
3	3.05	3.22	.95	2.14	6.41	5%	3.07	3.56	1.44	2.25	7.97	3%
5	5.41	6.68	2.97	3.44	14.03	9%	5.55	7.27	3.28	3.79	14.34	6%
10	11.95	13.88	5.88	6.94	27.99	24%	11.39	14.03	6.84	5.51	29.30	13%
	(c) $\widehat{\beta},\ \epsilon_t \sim N(0,0.1^2)$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}\times 10^{-1}, \frac{\sqrt{12}}{2}\times 10^{-1})$						(d) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2}\times 10^{-1}, \frac{\sqrt{12}}{2}\times 10^{-1})$, $\{\varepsilon_i\}_{i=1}^N \sim N(0,0.1^2)$					
0	.02	.01	.27	-.50	.50	0%	-.00	-.00	.22	-.47	.46	0%
.1	.10	.10	.14	-.10	.30	0%	.10	.10	.12	-.10	.30	0%
.5	.50	.50	.40	-.43	1.44	0%	.50	.50	.33	-.27	1.27	0%
1	1.01	1.01	.53	-.21	2.39	0%	1.00	1.03	.52	-.31	2.45	0%
3	3.02	3.03	.86	1.46	4.70	0%	3.00	3.03	1.06	.64	6.23	0%
5	5.01	5.01	1.18	3.05	6.76	0%	5.00	5.00	1.26	3.32	6.79	0%
10	10.00	10.01	1.43	8.13	11.88	2%	10.00	10.08	1.67	8.83	11.70	1%
	(e) $\widehat{\beta},\ \epsilon_t \sim N(0,0.1^2)$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$						(f) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2}\times 10^{-1}, \frac{\sqrt{12}}{2}\times 10^{-1})$, $\{\varepsilon_i\}_{i=1}^N \sim N(0,1)$					
0	.01	.01	.28	-.47	.47	0%	-.00	-.00	.29	-.48	.48	0%
.1	.09	.09	.12	-.09	.29	0%	.10	.09	.12	-.09	.29	0%
.5	.46	.49	.58	-.46	1.43	0%	.46	.48	.58	-.46	1.45	0%
1	.95	.98	1.18	-.94	2.88	0%	1.06	1.04	1.12	-.89	2.90	0%
3	3.28	3.07	3.42	-2.75	8.68	0%	3.07	3.06	3.34	-2.73	8.81	0%
5	4.98	5.11	5.73	-4.77	14.78	0%	5.02	5.20	5.28	-4.13	14.36	0%
10	12.01	10.84	11.87	-10.00	29.41	1%	12.23	12.13	10.62	-8.21	29.17	1%
	(g) $\widehat{\beta},\ \epsilon_t \sim N(0,1)$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}\times 10^{-1}, \frac{\sqrt{12}}{2}\times 10^{-1})$						(h) $\widehat{\beta},\ \epsilon_t \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$, $\{\varepsilon_i\}_{i=1}^N \sim N(0,0.1^2)$					
						100%						100%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from different distributions with various variance, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

1. when a distribution with (10 times) higher variance is used for generating stochastic noise ϵ_t then for kernel approximation of the conditional density $c_t(x|v;\theta)$ via $\{\varepsilon_i\}_{i=1}^N$, the NPSMLE method is inapplicable as this situation leads to complete 'NaN' outcome. The reason is very different from a usual 'NaN' occurrence caused by HAM divergence. In this case, apparently, the method itself is not able to approximate the true conditional density using $\{\varepsilon_i\}_{i=1}^N$ generated from 10 times 'smaller' distribution. This holds irrespective of whether same or different distributions are used [see subparts (c) and (d) of Table 5.9 and (g) and (h) of Table 5.10];

2. when a distribution with (10 times) lower variance is used for generating stochastic noise ϵ_t then for kernel approximation of the conditional density $c_t(x|v;\theta)$ via $\{\varepsilon_i\}_{i=1}^N$, the NPSMLE method works from the technical point of view but produces completely statistically insignificant estimates and very random uniform distribution of estimated values as captured in HQ and LQ columns where the quantile values almost copy the borders of the parameter space, this also holds irrespective of whether same or different distributions are used;

3. when different distributions with the same variances are used [compare subparts (a), (b) and (c), (d) of Table 5.10], the order and shapes of distributions matter considerably. In our case, if stochastic noise ϵ_t is generated from the normal distribution and $\{\varepsilon_i\}_{i=1}^N$ is generated from the uniform distribution [subparts (c) and (d)], we obtain considerably better estimates with markedly lower standard deviations then if the order is opposite [subparts (a) and (b)]. As both distributions have identical mean and variance, there is not much than shape of the distribution defined by higher moments to make difference.

These findings very strongly confirm the need of a proper noise specification for the empirical application of NPSMLE which is at the same moment one of the most important findings from the analysis of the original results from Table 5.1.

5.3.2 2-type model estimation

An important advantage of FABMs is that their dynamics is mostly driven by a few crucials parameters. As a result, we might promisingly attempt to estimate all essential coefficients simultaneously and thus we do not need any rigorous criteria for selection. In the Brock & Hommes (1998) setting we select estimated parameters consistently with the current literature (see Tables 2.1 and 2.2), i.e. the key switching parameter β and the behavioural belief coefficients. The other coefficients, e.g. the risk aversion a, the conditional variance of excess returns σ^2, or the risk free rate R are simplified already in the original model as constants and shared by all investor types. The model is then theoretically derived based on those assumptions. These parameters only influence the absolute values of the profitability measures U_h but not their relative proportions (R additionally a little bit adjusts the model output x_t).

Thus we can naturally consider them not influencing dynamics of the model as described in Section 5.1.

A natural subsequent step of the NPSMLE method testing is thus a multiple parameter estimation in which we simultaneously estimate the intensity of choice β and agents' belief coefficients g_h and b_h defining individual trading strategies in the 2-type and the 3-type models for which both theoretical as well as empirical rationale exists in the current literature as indicated in Chen *et al.* (2012, pg. 191, 207). With reference to Biondi *et al.* (2012, pg. 5534), "it has been advocated that the two broad categories of chartism and fundamentalism account for most of possible investment strategies". The aim of this analysis is to assess the performance of the NPSMLE method in estimating other model parameters then solely the intensity of choice β.

First, we study the most simple system consisting of two trading strategies, where fundamental strategy again appears in the market by default ($g_1 = b_1 = 0$). Based of the knowledge gained in Subsection 5.3.1, we define a discrete grid of combinations of the true intensity of choice β and the chartistic beliefs g_2 and b_2 representing the second-type trading strategy to cover a purposeful range of values w.r.t. issues studied in the previous sections. To keep a reasonable number of combinations and lucidity of results, we opt for $\beta = \{0, 0.5, 3, 10\}$. In defining a grid of chartistic beliefs, we also cover various combinations of trend following ($g_2 > 0$), contrarian ($g_2 < 0$), upward-biased ($b_2 > 0$), and downward-biased ($b_2 < 0$) strategies based on multiples of standard deviations from the general model setting: $0.5\times, 1\times, 2\times, 3\times$. We refer the reader to the first column of Table 5.11 for detailed specification. We again employ only two specification of noise, namely $\epsilon_t \sim N(0, 0.1^2)$ and $\epsilon_t \sim N(0, 1)$.

Quantitative results

In Table 5.11 and Table 5.12 we summarise the simulation results. Basically, we are able to confirm all main findings from the single parameter β estimation simulation analysis. First, the method is generally able to reveal accurately the true values (see columns reporting sample medians and means denoted 'Md' and 'Mn', respectively) of estimated parameters also in the 3-parameter simultaneous estimation case. Especially belief coefficients g_2 and b_2 [subparts (b) and (c)], that are of central importance in this section, are estimated overall significant and with almost surprisingly high precision. The estimation precision of the β parameter is not directly comparable to previous results as the

Table 5.11: Results of 3-parameter estimation of a 2-type model I.

True	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$		(c) $\widehat{b_2}$		(d) LL			
β, g_2, b_2	Md	Mn	SD	Mn	SD	Mn	SD	L-rat	$2\Delta LL$	p-v	NN
.0, .2, .15	-.01	-.00	.29	.20	.03	.15	.01	1	0	1	0%
.5, .2, .15	.49	.50	.57	.20	.03	.15	.01	1	0	1	0%
3, .2, .15	3.01	2.99	3.44	.20	.05	.15	.01	1	0	1	0%
10, .2, .15	11.60	12.19	9.46	.19	.10	.15	.01	1	0	1	0%
.0, -.2, -.15	.02	-.00	.28	-.20	.03	-.15	.01	1	0	1	0%
.5, -.2, -.15	.50	.48	.56	-.20	.03	-.15	.01	1	0	1	0%
3, -.2, -.15	3.11	2.97	3.39	-.20	.05	-.15	.01	1	0	1	0%
10, -.2, -.15	10.46	9.93	11.30	-.19	.10	-.15	.01	1	0	1	0%
.0, .2, -.15	.03	.01	.28	.20	.03	-.15	.01	1	0	1	0%
.5, .2, -.15	.56	.52	.56	.20	.03	-.15	.01	1	0	1	0%
3, .2, -.15	3.31	3.12	3.40	.20	.05	-.15	.01	1	0	1	0%
10, .2, -.15	11.97	12.23	9.60	.19	.10	-.15	.01	1	0	1	0%
.0, -.2, .15	-.00	-.01	.30	-.20	.03	.15	.01	1	0	1	0%
.5, -.2, .15	.53	.50	.60	-.20	.03	.15	.01	1	0	1	0%
3, -.2, .15	3.31	3.08	3.53	-.20	.04	.15	.01	1	0	1	0%
10, -.2, .15	11.37	10.62	11.69	-.19	.10	.15	.01	1	0	1	0%
.0, .4, .3	.00	.01	.33	.40	.03	.30	.01	1	0	1	0%
.5, .4, .3	.50	.49	.48	.40	.04	.30	.01	1	0	1	0%
3, .4, .3	2.99	3.04	.46	.40	.04	.30	.01	.99	.01	.92	0%
10, .4, .3	10.00	10.01	.61	.40	.03	.30	.01	.94	.13	.62	4%
.0, -.4, -.3	-.08	-.06	.30	-.40	.03	-.30	.01	1	0	1	0%
.5, -.4, -.3	.31	.35	.58	-.39	.04	-.30	.01	1	0	1	0%
3, -.4, -.3	2.96	2.84	1.47	-.39	.06	-.30	.01	1	0	1	0%
10, -.4, -.3	10.08	10.11	1.24	-.40	.04	-.30	.01	.99	.03	.86	2%
.0, .4, -.3	-.01	.00	.31	.40	.03	-.30	.01	1	0	1	0%
.5, .4, -.3	.51	.48	.48	.40	.04	-.30	.01	1	0	1	0%
3, .4, -.3	3.03	3.04	.46	.40	.04	-.30	.01	.99	.01	.92	1%
10, .4, -.3	9.98	10.02	.63	.40	.03	-.30	.01	.94	.13	.62	2%
.0, -.4, .3	-.06	-.05	.30	-.40	.03	.30	.01	1	0	1	0%
.5, -.4, .3	.23	.32	.58	-.40	.04	.30	.01	1	0	1	0%
3, -.4, .3	2.98	2.87	1.38	-.40	.06	.30	.01	1	0	1	0%
10, -.4, .3	9.98	10.00	1.13	-.40	.04	.30	.01	.99	.03	.86	2%
.0, .8, .6	-.00	-.00	.06	.80	.03	.60	.02	1	0	1	30%
.5, .8, .6	.50	.50	.06	.80	.03	.60	.02	.99	.02	.89	31%
3, .8, .6	2.99	3.00	.07	.80	.02	.60	.01	.59	1.07	.30	54%
10, .8, .6	9.99	9.99	.12	.80	.00	.60	.01	NA	NA	NA	95%
.0, -.8, -.6	.00	-.01	.27	-.80	.04	-.60	.01	1	0	1	17%
.5, -.8, -.6	.52	.51	.26	-.80	.04	-.60	.01	1	0	1	21%
3, -.8, -.6	3.00	3.00	.26	-.80	.04	-.60	.01	.99	.02	0	34%
10, -.8, -.6	9.98	9.99	.45	-.80	.03	-.60	.01	.89	.24	0	43%
.0, .8, -.6	.00	.00	.06	.80	.03	-.60	.02	1	0	1	32%
.5, .8, -.6	.50	.50	.06	.80	.03	-.60	.02	.99	.02	.89	33%
3, .8, -.6	3.00	3.00	.07	.80	.02	-.60	.01	.58	1.08	.30	58%
10, .8, -.6	9.97	9.99	.13	.80	.00	-.60	.00	NA	NA	NA	96%
.0, -.8, .6	-.00	-.00	.27	-.80	.04	.60	.01	1	0	1	16%
.5, -.8, .6	.52	.51	.26	-.80	.04	.60	.01	1	0	1	20%
3, -.8, .6	3.01	3.01	.27	-.80	.04	.60	.01	.99	.02	.89	31%
10, -.8, .6	10.04	10.02	.43	-.80	.03	.60	.01	.89	.24	.62	42%
.0, 1.2, .9	.00	.00	.01	1.20	.03	.90	.03	1	0	1	67%
.5, 1.2, .9	.50	.50	.01	1.20	.01	.90	.02	.70	.70	.40	77%
3, 1.2, .9											100%
10, 1.2, .9											100%
.0, -1.2, -.9	-.00	-.00	.11	-1.20	.04	-.90	.01	1	0	1	47%
.5, -1.2, -.9	.49	.50	.11	-1.20	.04	-.90	.01	1	0	1	49%
3, -1.2, -.9	2.99	3.00	.14	-1.20	.03	-.90	.01	.96	.09	.76	56%
10, -1.2, -.9	9.99	9.99	.29	-1.20	.02	-.90	.01	.60	1.03	.31	72%
.0, 1.2, -.9	.00	.00	.01	1.20	.03	-.90	.03	1	0	1	67%
.5, 1.2, -.9	.50	.50	.01	1.20	.02	-.90	.02	.70	.71	.40	76%
3, 1.2, -.9											100%
10, 1.2, -.9											100%
.0, -1.2, .9	.01	.00	.11	-1.20	.04	.90	.01	1	0	1	50%
.5, -1.2, .9	.51	.50	.11	-1.20	.04	.90	.01	1	0	1	52%
3, -1.2, .9	3.00	3.00	.13	-1.20	.04	.90	.01	.96	.09	.76	60%
10, -1.2, .9	10.00	10.00	.30	-1.20	.02	.90	.01	.60	1.03	.31	72%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from normal distribution $N(0, 0.1^2)$, $R = 1.0001$. Each sample is based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians (Md), means (Mn), and standard deviations (SD) are reported. Figures are rounded to 2 decimal digits. 'L-rat' denotes the likelihood ratio of the null static (i.e. restricted) model vs. the alternative switching model, '$2\Delta LL$' is the test statistics of the log-likelihood ratio test being approximately χ^2 distributed with 1 degree of freedom, and 'p-v' is related p-value of the test. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers. 'NA' typically means that the static model is associated with 100% of 'NaN' outcomes.

Source: Author's own computations in *MATLAB*.

Table 5.12: Results of 3-parameter estimation of a 2-type model II.

True	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$		(c) $\widehat{b_2}$		(d) LL			
β, g_2, b_2	Md	Mn	SD	Mn	SD	Mn	SD	L-rat	$2\Delta LL$	p-v	NN
.0, .2, .15	.03	.02	.30	.20	.03	.15	.07	1	0	1	0%
.5, .2, .15	.62	.60	.56	.19	.03	.15	.07	1	0	1	0%
3, .2, .15	3.99	4.13	2.75	.19	.03	.15	.06	1	0	1	0%
10, .2, .15	11.42	11.23	11.08	.18	.04	.14	.05	1	0	1	1%
.0, -.2, -.15	-.06	-.04	.31	-.20	.03	-.15	.07	1	0	1	0%
.5, -.2, -.15	.46	.46	.57	-.20	.03	-.15	.07	1	0	1	0%
3, -.2, -.15	3.14	3.10	3.37	-.18	.04	-.13	.07	1	0	1	0%
10, -.2, -.15	9.54	9.53	11.40	-.16	.06	-.13	.06	1	0	1	1%
.0, .2, -.15	.03	.02	.30	.20	.03	-.15	.07	1	0	1	0%
.5, .2, -.15	.62	.59	.57	.19	.03	-.15	.07	1	0	1	0%
3, .2, -.15	3.99	4.16	2.61	.19	.03	-.15	.06	1	0	1	0%
10, .2, -.15	11.33	11.43	10.87	.18	.04	-.14	.05	1	0	1	0%
.0, -.2, .15	-.05	-.04	.31	-.20	.03	.15	.07	1	0	1	0%
.5, -.2, .15	.49	.49	.56	-.20	.03	.15	.07	1	0	1	0%
3, -.2, .15	3.42	3.37	3.36	-.18	.04	.13	.06	1	0	1	0%
10, -.2, .15	10.43	10.25	11.40	-.16	.06	.12	.06	1	0	1	2%
.0, .4, .3	.00	.01	.15	.40	.03	.30	.07	1	0	1	0%
.5, .4, .3	.49	.55	.26	.40	.03	.30	.07	1	0	1	0%
3, .4, .3	3.95	4.54	2.13	.38	.04	.29	.04	.99	.02	.89	0%
10, .4, .3	12.48	13.14	9.41	.38	.05	.27	.09	.98	.04	.84	4%
.0, -.4, -.3	.00	-.01	.14	-.40	.03	-.30	.07	1	0	1	0%
.5, -.4, -.3	.49	.52	.20	-.40	.03	-.30	.07	1	0	1	0%
3, -.4, -.3	3.96	4.38	2.41	-.38	.06	-.28	.06	.99	.02	.89	0%
10, -.4, -.3	10.76	10.44	10.21	-.33	.13	-.27	.08	.99	.02	.89	11%
.0, .4, -.3	-.00	.00	.15	.40	.03	-.30	.07	1	0	1	0%
.5, .4, -.3	.50	.54	.25	.40	.03	-.30	.07	1	0	1	0%
3, .4, -.3	4.01	4.53	2.05	.38	.04	-.29	.04	.99	.02	.89	0%
10, .4, -.3	12.30	13.17	8.79	.38	.05	-.28	.08	.98	.04	.84	4%
.0, -.4, .3	-.01	-.01	.13	-.40	.03	.30	.07	1	0	1	0%
.5, -.4, .3	.49	.51	.19	-.40	.03	.30	.08	1	0	1	0%
3, -.4, .3	4.03	4.40	2.47	-.38	.06	.28	.06	.99	.02	.89	0%
10, -.4, .3	10.42	9.92	10.17	-.33	.14	.26	.09	.99	.02	.89	12%
.0, .8, .6	.00	.00	.03	.80	.03	.60	.07	1	0	1	0%
.5, .8, .6	.50	.50	.07	.80	.03	.60	.06	.97	.05	.82	3%
3, .8, .6	3.01	3.04	.37	.80	.02	.60	.03	.83	.36	.55	23%
10, .8, .6	10.24	10.54	2.16	.80	.01	.60	.02	.75	.56	.45	72%
.0, -.8, -.6	.00	.00	.02	-.80	.03	-.60	.07	1	0	1	0%
.5, -.8, -.6	.50	.51	.08	-.80	.03	-.60	.08	.98	.04	.84	0%
3, -.8, -.6	3.19	3.61	2.47	-.74	.18	-.57	.09	.94	.12	.73	4%
10, -.8, -.6	8.50	4.97	8.61	-.57	.33	-.44	.26	.95	.09	.76	44%
.0, .8, -.6	-.00	-.00	.03	.80	.03	-.60	.07	1	0	1	0%
.5, .8, -.6	.50	.50	.07	.80	.03	-.60	.06	.97	.05	.82	2%
3, .8, -.6	3.00	3.04	.36	.80	.02	-.60	.03	.83	.36	.55	23%
10, .8, -.6	10.19	10.51	2.22	.80	.01	-.60	.02	.75	.57	.45	69%
.0, -.8, .6	.00	.00	.02	-.80	.03	.60	.07	1	0	1	0%
.5, -.8, .6	.50	.50	.08	-.80	.03	.60	.07	.98	.04	.84	1%
3, -.8, .6	3.19	3.62	2.38	-.75	.17	.57	.08	.94	.12	.73	4%
10, -.8, .6	8.56	5.67	8.17	-.60	.32	.46	.25	.95	.11	.74	44%
.0, 1.2, .9	-.00	.00	.01	1.20	.02	.90	.07	1	0	1	25%
.5, 1.2, .9											100%
3, 1.2, .9											100%
10, 1.2, .9											100%
.0, -1.2, -.9	.00	.00	.01	-1.20	.03	-.90	.07	1	0	1	19%
.5, -1.2, -.9	.50	.51	.05	-1.20	.03	-.90	.07	.91	.19	.66	28%
3, -1.2, -.9	2.97	2.39	2.06	-1.07	.34	-.79	.34	.83	.38	.46	48%
10, -1.2, -.9	7.86	3.28	7.85	-.82	.50	-.54	.54	.88	.25	.62	88%
.0, 1.2, -.9	.00	.00	.01	1.20	.03	-.90	.07	1	0	1	23%
.5, 1.2, -.9											100%
3, 1.2, -.9											100%
10, 1.2, -.9											100%
.0, -1.2, .9	.00	.00	.01	-1.20	.03	.90	.07	1	0	1	18%
.5, -1.2, .9	.50	.50	.05	-1.20	.03	.90	.07	.91	.19	.66	26%
3, -1.2, .9	2.94	2.34	1.95	-1.08	.34	.81	.30	.83	.38	.46	50%
10, -1.2, .9	8.25	3.21	8.26	-.79	.52	.57	.50	.89	.23	.63	89%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0,1)$, $R = 1.0001$. Each sample is based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians (Md), means (Mn), and standard deviations (SD) are reported. Figures are rounded to 2 decimal digits. 'L-rat' denotes the likelihood ratio of the null static (i.e. restricted) model vs. the alternative switching model, '$2\Delta LL$' is the test statistics of the log-likelihood ratio test being approximately χ^2 distributed with 1 degree of freedom, and 'p-v' is related p-value of the test. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers. 'NA' typically means that the static model is associated with 100% of 'NaN' outcomes.

Source: Author's own computations in *MATLAB*.

setting of the 2-type model is different from the general model. Notwithstanding, we still get generally conformable figures. Second, the 'two-sided' trade-off (see Section 5.3.1) related to a reasonably rich model dynamics restricted from both sides plays an important role. We again clearly observe the relative estimation inefficiency in case of setting combinations with small values of the intensity of choice $\beta = 0, 0.5$, especially when combined with small values of belief coefficients g_2 and b_2 (the upper half of Table 5.11 and Table 5.12). On the other hand, for high values of belief coefficients g_2 and b_2 (the bottom half of Table 5.11 and Table 5.12) we experience considerable number of model overflows leading to significant censorship of results and thus generally confirming findings from β estimation exercise w.r.t. various distributions of beliefs parameters (Tables 5.7 and 5.8). These effects to a large extent prohibit serious results interpretation for reported combination with the smallest values of belief coefficients $g_2 = \pm 0.2$ and $b_2 = \pm 0.15$ as well as for the combination with the highest values of belief coefficients $g_2 = \pm 1.2$ and $b_2 = \pm 0.9$. We also observe the nontrivial complex interplay between the intensity of the stochastic noise, estimation efficiency, and probability of 'NaN' outcomes [in subparts (d), 'NN' columnd] detected within the analysis of the general model. Here, a stabilising effect is associated with the wider noise interval $N(0, 1)$ leading to a significant decrease of the number of model overflows (on the contrary, in the case of 3-type model estimation in Subsection 5.3.3, the stability is higher for the narrower noise $N(0, 0.1^2)$ specification). This may paradoxically lead to a seemingly lower efficiency as the divergent runs are not filtered out—this effect is e.g. observable in Table 5.12 for combinations with $\beta = 10$, $g_2 = \pm 0.4$, and $b_2 = \pm 0.3$ where β estimates are often biased upwards. The standard deviations of these empirical estimates is significantly larger compared to Table 5.11 due to the effect of estimates close to the upper bound of the parameter space that would otherwise be likely filtered out in the case of smaller stochastic noise in the system. We further observe a prevailing upward bias tendency in β estimates for smallest values of belief coefficients g_2 and b_2. When it comes to efficiency of g_2 and b_2 estimates, the setup with noise interval $N(0, 0.1^2)$ produces markedly more precise estimates but the overall statistical significance of estimates is apparent for both setups.

Although we report all combination of trend following ($g_2 > 0$), contrarian ($g_2 < 0$), upward-biased ($b_2 > 0$), and downward-biased ($b_2 < 0$) strategy specification mainly from technical reasons and the analysis of the model dynamics goes beyond the scope of this paper, we can observe some patterns regarding the beliefs' combination. For instance the efficiency of the estimation is considerably higher for trend following ($g_2 > 0$) beliefs than for contrarians ($g_2 < 0$), but setups with trend following beliefs are more vulnerable to overflows causing 'NaN' outcomes. On the other hand the direction of the bias parameter b_2 does not seem to play any important role based on reported results.

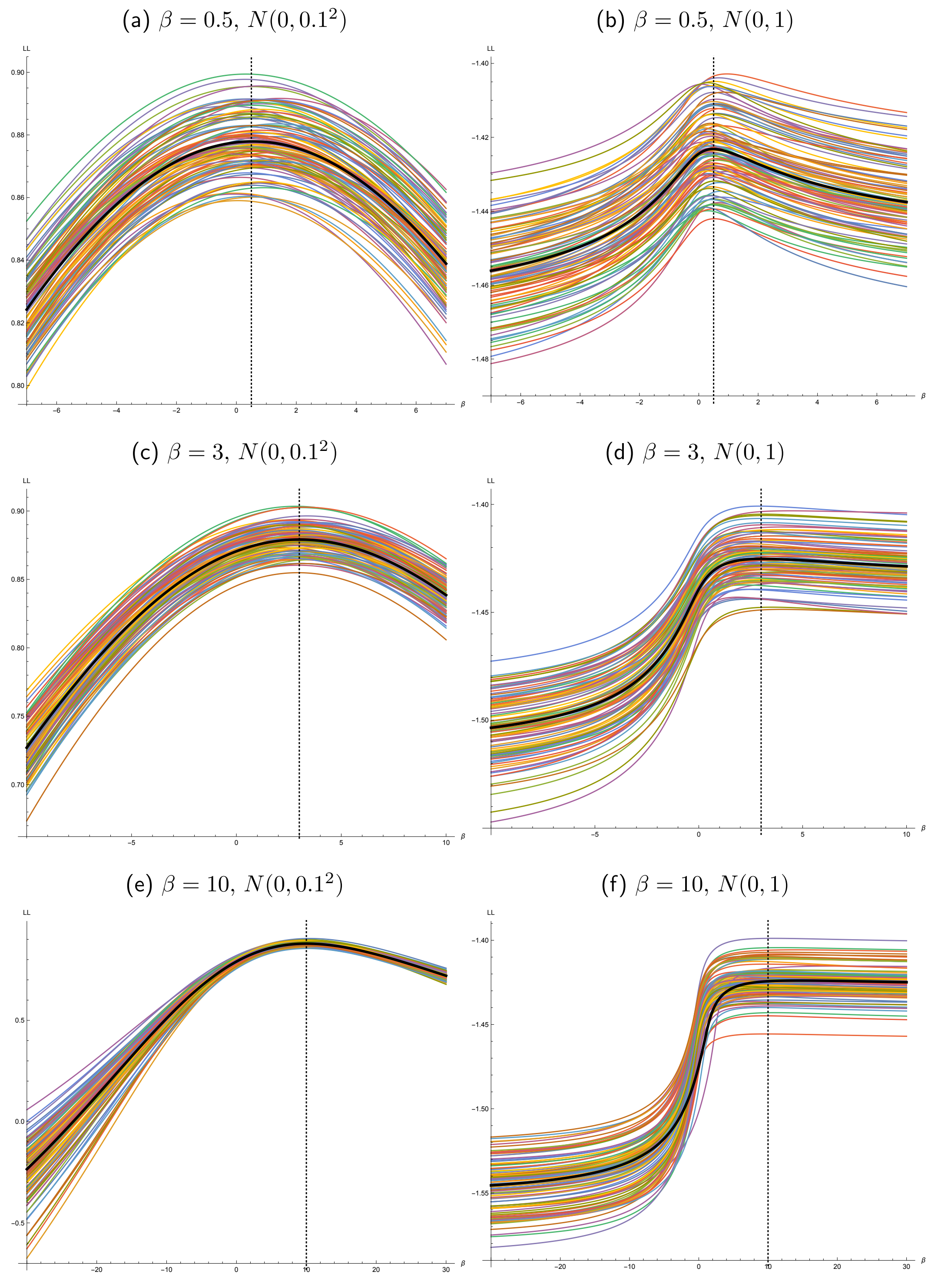
At the end of the 2-type model analysis, it is, however, important to stress that this is the extreme case of the most simple setting more vulnerable to potential extreme model dynamics. Nonetheless, this simple setting is the cornerstone in the current HAM estimation literature (see Chapter 2) and therefore it needs to be properly elaborated also for the new NPSMLE method. The favourable results presented above give promise for the function of the method also in more complex settings.

Behaviour of the simulated log-likelihood function

To verify smoothness conditions and identification of parameters in the 2-type model estimation case, we aim at depicting shape of simulated log-likelihood functions also for the simultaneous estimation of three parameters. As we can hardly demonstrate the 4D shape of the resulting simulated log-likelihood function, we depict sub-log-likelihood functions in 2D and 3D making out the global visualisation when combined together.

Figures 5.6 and 5.7 demonstrate simulated 2D sub-log-likelihood functions for the single parameters intensity of choice $\beta \in \{0.5, 3, 10\}$, trend coefficient $g_2 = 0.4$, and bias coefficient $b_2 = 0.3$ estimation (keeping the two others fixed) combined with the two most 'successful' specifications of the stochastic noise $\epsilon_t \sim N(0, 0.1^2)$ and $\epsilon_t \sim N(0, 1)$ found in Section 5.3.1. We again simply observe very smooth shapes and unique maxima generally shared for all random runs. Moreover, consistent to Section 5.3.1 findings, small $\beta = 0.5$ is more precisely detectable assuming stochastic noise $\epsilon_t \sim N(0, 1)$ [subfigure (b) of Figure 5.6], higher $\beta \in \{3, 10\}$ assuming stochastic noise $\epsilon_t \sim N(0, 0.1^2)$ [subfigures (c) and (e) of Figure 5.6], upward bias tendency of $\widehat{\beta}$ is clear for the stochastic noise $\epsilon_t \sim N(0, 1)$ in subfigures (d) and (f) of Figure 5.6 due to very flat shape of the log-likelihood function above the positive subpart of the domain, and finally the belief parameters $b_2 = 0.3$ and $g_2 = 0.4$ are very well detected in both cases, however the performance of estimators is more efficient for smaller stochastic noise $\epsilon_t \sim N(0, 0.1^2)$. Next, in Figure 5.8 we visualise 3D simulated sub-log-likelihood functions based on all possible combinations of

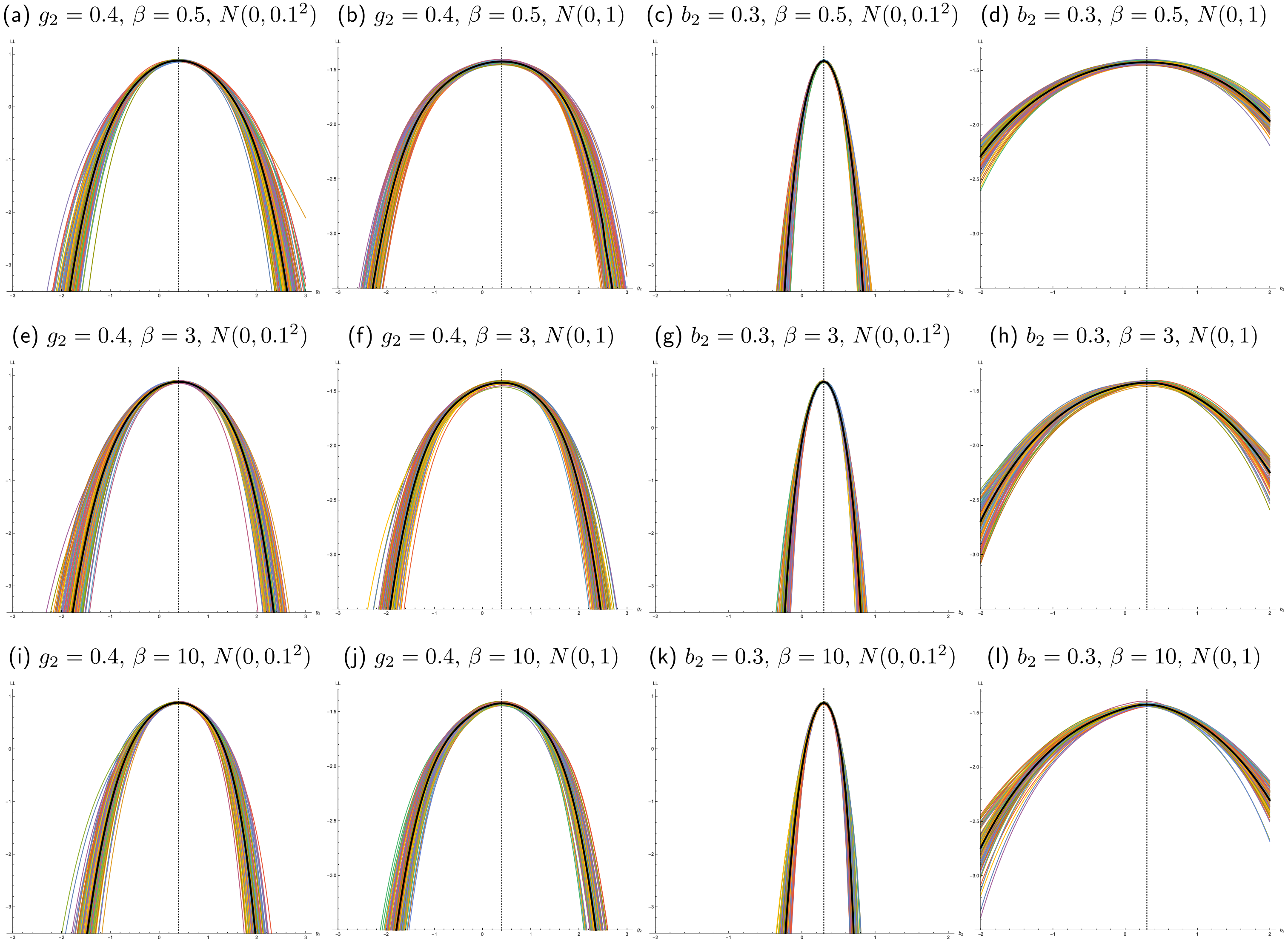
Figure 5.6: Simulated sub-log-likelihood functions for β estimation



Note: Results based on 100 random runs, $g_2 = 0.4$, $b_2 = 0.3$, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from given normal distribution. Black dotted vertical lines depict the true βs. Bold black full lines depict sample averages.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

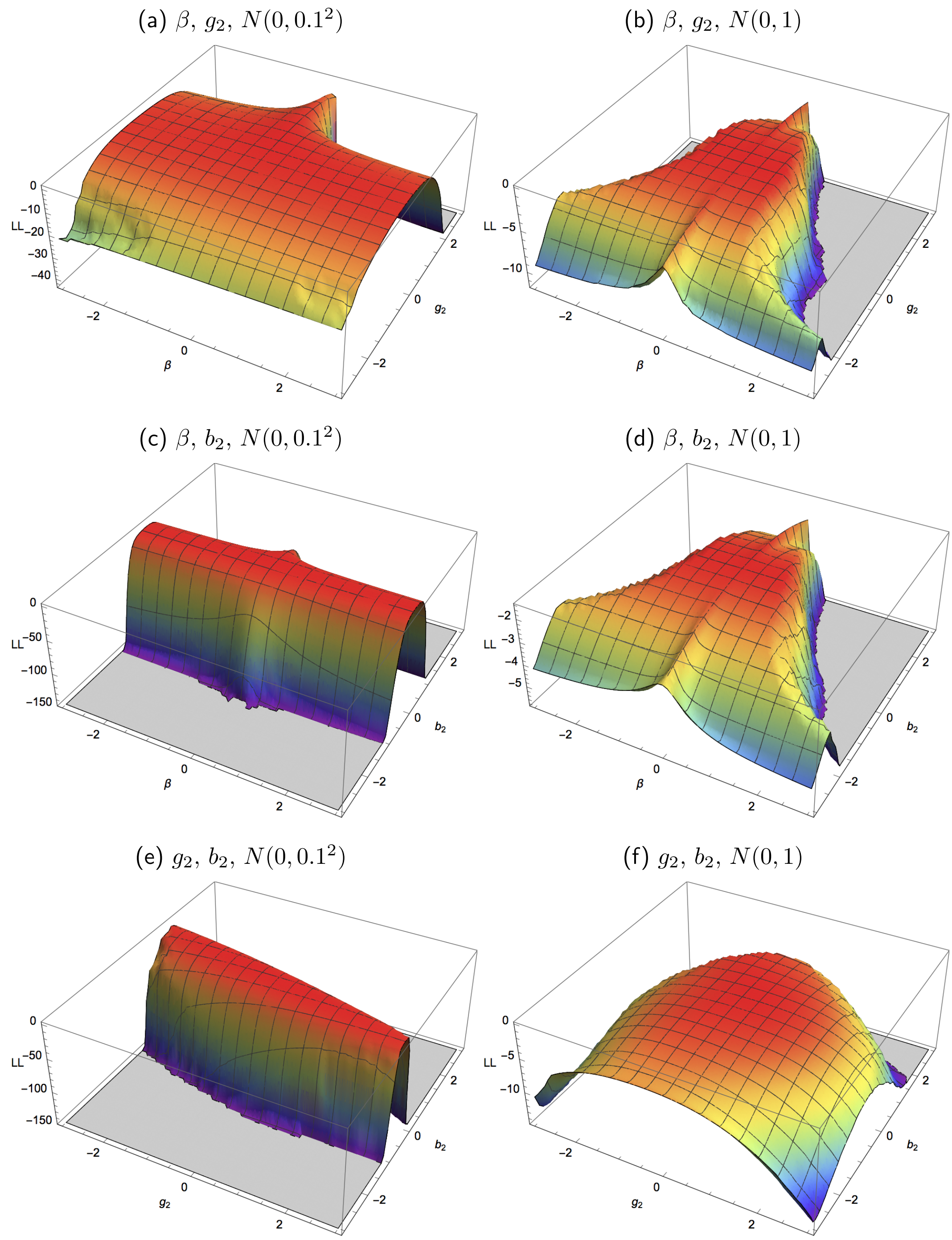
Figure 5.7: Simulated sub-log-likelihood fcns. for g_2 and b_2 estimation



Note: Results based on 100 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from given normal distribution. Black dotted vertical lines depict the true $g_2 = 0.4$ and $b_2 = 0.3$. Bold black full lines depict sample averages.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 5.8: Simulated sub-log-likelihood functions in 3D



Note: Results averaged over 30 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. The complete set of true parameters: $\beta = 0.5$, $g_2 = 0.4$, $b_2 = 0.3$. Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from given normal distribution.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

three parameters of interest, keeping one of them fixed, for the model setting with: $\beta = 0.5$, $g_2 = 0.4$, $b_2 = 0.3$ and both stochastic noise specifications. The smoothness of the surface generally keeps retained also in the 3D visualisations and regions of possible maxima are well detectable via red color although the 3D depiction cannot provide such detailed and 'zoomed' view as the 2D visualisations in Figures 5.6 and 5.7. For some combinations of parameters the model is numerically unstable and thus for specific subsets of the domain plane the shape is not well defined. However, these areas are always far from maxima regions. Parameters g_2 and b_2 seem to be relatively well identified which is further confirmed quantitatively in Section 5.3.2. As expected, the most challenging is revealing the β coefficient in which direction the surface is very flat for a large interval of the domain. These findings are largely in accord with conclusions of Bolt *et al.* (2014, pg. 15) and Hommes & Veld (2015) who claim that "the other parameters can to a large extent compensate for changes in β" and report very flat shape of the likelihood function for the intensity of choice selection. In any case, based on these results we again generally assume that the regularity conditions are met and the identification of parameters is assured also for the 2-type model estimation.

Likelihood-ratio test

Previous sections have shown that the NPSML estimation method does a fairly good job in distinguish between various βs. As a next step we might be interested how capable the estimation method is in a rigorous statistical comparison between static and switching models. As the static version of the model with $\beta = 0$ and the switching version are nested, i.e. the less complex static model is derived via a restriction on β from the switching model, we can apply the usual likelihood ratio test to assess the relative goodness of fit between models with and without switching. For this purpose Table 5.11 and Table 5.12 further display information about tests of model fit in subpart (d). 'L-rat' column denotes the likelihood ratio of the null static (i.e. restricted) model vs. the alternative switching model, '$2\Delta LL$' is the test statistics of the log-likelihood ratio test being approximately χ^2 distributed with 1 degree of freedom (because only the switching parameter is restricted), and 'p-v' is related p-value of the test. Application of the likelihood ratio test seems natural in this situation, nevertheless the Monte Carlo simulation framework brings several imperfections. On the one hand, simulated data smartly avoid the problem of model misspecification, on the other hand the goodness of fit test is designed rather for comparison based on a single empirical dataset. In our situation, new variables and a different dataset are randomly generated in each of 1000 random runs and the test is then based on the aggregate mean values over all runs. But since the standard deviation of individual maximum log-likelihoods is negligible compared to their value (around 1%), aggregation does not cause any appreciable loss of information. A second imperfection is related to the relative flatness of the log-likelihood function in the dimension of the restricted switching coefficient β for a large interval of the domain (see detailed discussion in Section 5.3.2). Although the estimation method detects precisely the true intensity of choice especially for combinations of higher values of β, stronger strategies, and lower stochastic noise specification, due to flat likelihood the test exhibits only a moderate capability of distinguishing between the restricted and the unrestricted model. Translated into p-values of the test to reject the null of the static model, in the most distinct cases the value reaches 30% which is far above usual econometric levels. Inspecting subpart (d) of Table 5.11 and Table 5.12, we observe expected behaviour but generally low power of the test. For all true $\beta = 0$, the likelihood ratio is equal to 1 and the p-value remains 100%. Increasing true β and strength of strategies, the likelihood ratio and the p-value naturally decrease (as the test statistics '$2\Delta LL$' increases), but the pace of the progress is low for the selected range of βs.

5.3.3 3-type model estimation

Results of simultaneous estimation of 5 parameters in the 3-type model including three basic strategies: fundamental represented by $g_1 = b_1 = 0$ and two chartistic defined in the Table B.3 and Table B.4, can be found in Appendix B. We keep the strategy of defining a grid of chartistic beliefs from Subsection 5.3.2 based on various combinations of trend following ($g_2 > 0$), contrarian ($g_2 < 0$), upward-biased ($b_2 > 0$), and downward-biased ($b_2 < 0$) strategies and the same multiples of standard deviations from the general model setting. Conclusion are generally in accord with the results of the 2-type model estimation, the difference is mainly in efficiency of estimates that is by nature lower than for the 2-type model. However, combining two chartistic strategies we also gain some new knowledge about the system behaviour. For instance, in case of a combination of two trend following strategies ($g_2 > 0, g_3 > 0$), it is rather complicated for the NPSMLE method to distinguish between impacts of these two strategies leading to lower statistical significance of both estimates compared to other combination of trend following and contrarian strategies. Conclusions regarding the nontrivial interplay between the intensity of the stochastic noise, estimation efficiency, and the probability of 'NaN' outcomes seem to be somewhat mixed in this markedly more complex case as the estimation efficiency is comparable for both stochastic noise specifications $N(0, 0.1^2)$ and $N(0, 1)$, and the stability of the system in terms of 'NaN' outcomes is higher for $N(0, 1)$ which is the opposite tendency than observed within the analysis of the 2-type model estimation.

5.3.4 Suggestions for future research

Although this section has proven that the NPSMLE does a fairly good job in retrieving true parameters of the HAM, two important methodological issues remain open for future research. First, a further development of an appropriate statistical framework for the goodness of fit testing to complement Section 5.3.2 and be able to distinguish between the 2-type and the 3-type model is advisable. For this purpose, we plan to consider a version of the Vuong's closeness test (Vuong 1989) which is an advanced likelihood-ratio test for model selection that allows also for overlapping or non-nested models. Second—and not only because the estimation method is relatively computationally intensive compared to traditional methods—an important area for further research is assessing performance benefits that the new method brings, i.e. whether NPSMLE yields more precise results then other simulation-based or traditional (see Tables 2.1 and 2.2) methods. However, this is a difficult task to tackle as traditional methods are largely infeasible for the HAM.

Chapter 6

HAM estimation on empirical data

Equipped with the knowledge from the Monte Carlo study in Chapter 5, we broaden the topic via an empirical application and estimate the Brock & Hommes (1998) model using cross section of world stock markets. We analyse S&P500 and NASDAQ for the U.S., DAX and FTSE and for Europe, NIKKEI 225 and HSI for Japan and Hong Kong, respectively.

6.1 The estimation setting

Compared to the simulation study in Chapter 5, the setting of models estimated using real data is less challenging and in terms of statistical validity also less computationally demanding. On the contrary, the estimation algorithm is a bit technically more complicated as the structure of the real world data is far away from the regularity of the simulated dataset. Concurring findings about the sufficient setting of the NPSMLE method from Chapter 5, we compute results for 1000 random runs, number of observations $t = 5000$, and the kernel approximation precision is set to $N = 500$. On the other hand, because of a problematic numerical stability of the model when real data analysis is introduced we increase the number of starting points for the numerical optimisation to 8.[1] The other setting remains the same as defined in Section 5.1. More importantly, as the effect of the more complicated structure of the code with multiple initial points, the paralellisation of the computational procedure cannot be maintained via the *MATLAB* because of technical assumptions of

[1] Based on a preliminary analysis, 8 starting points are sufficient to filter out issues related to numerical instability on the system.

the paralellisation function. Therefore we are left with simple and relatively slow standard one-core type of calculations.

Moreover, based on main findings of Chapter 5, the proper intensity of stochastic market noise is crucial for the correct function of the NPSMLE method. A wrong stochastic noise specification is likely to influence the behaviour of the system and validity or results to a great extent. Hence we do not longer use the grid strategy to ensure the robustness of result as e.g. the stochastic noise intensity on various real markets is immensely unpredictable variable. Leaving this idea, we instead add the intensity of the stochastic market noise to the list of estimated parameters. Generally, we thus apply a simultaneous unconstrained multivariable function estimation of all interesting parameters: agents' belief coefficients defining individual trading strategies g_h and b_h, the intensity of choice β, and the intensity of market noise, which is defined as a fraction of the standard deviation of the noise term and the standard deviation of the data and denoted as *noise intensity*.

First, we estimate the most simple 2-type model including two basic strategies only—fundamental one represented by implicitly defined $g_1 = b_1 = 0$ and chartistic one which is to be estimated. Within this setting, we simultaneously estimate four parameters of interest—β, g_2, b_2, and the *noise intensity*. To support the numerical stability of the estimated system, we constrain the intervals for the starting points random generation to $\langle -0.5, 0.5\rangle$ for β, $\langle 1.3, 2.3\rangle$ for g_2, $\langle -0.2, 0.2\rangle$ for b_2, and $\langle 0.4, 0.9\rangle$ for the *noise intensity*. Nonetheless, as the algorithm is designed to find an optimum of an unconstrained multivariable function, it can freely leave these initial intervals during the search procedure.

We then continue with the estimation of the 3-type model including three basic strategies—fundamental and two different chartistic strategies which are to be estimated. Based on results of the 2-type model estimation, we assume zero bias of both the trend following as well as the contrarian strategy, i.e. $b_2 = b_3 = 0$. The simultaneous estimation of four parameters of interest—β, g_2, g_3, and the *noise intensity*—technically requires a modification of the algorithm setting to the constrained multivariable function estimation as the two different strategies—the trend following $g_2 > 0$ and the contrarian $g_3 < 0$ need to be strictly distinguished using the following constrains: $g_2 \in \langle 1.8, 2.8\rangle$, $g_3 \in \langle 0, -0.5\rangle$.

6.2 Fundamental price approximation

Approximation of the fundamental price is inevitably the most 'challenging' issue of the entire empirical chapter. Unfortunately, in the original framework of the Brock & Hommes (1998) asset pricing model of a simple stylised stock market there is no hint about how the empirical fundamental value might be derived.[2] Thus, we are left with the existing literature and following Winker *et al.* (2007); ter Ellen & Zwinkels (2010); Huisman *et al.* (2010), the fundamental price is approximated as a Moving Average (MA) value. Winker *et al.* (2007) assume as the proxy for the fundamental price a MA over the last 200 observations of the DM/USD exchange rate time series for the period 1991/11/11 to 2000/11/9. ter Ellen & Zwinkels (2010) use the MA of the Brent and WTI Cushing oil monthly USD prices over 24 months, i.e. from 1984/1 to 2009/8. Huisman *et al.* (2010) employ the MA of European forward electricity daily historical prices over 3 year for the base-load calendar year 2008 forward contracts. Authors set the MA window to 3 as a calibration result of the optimal length.

Long-term and short-term MAs are also commonly used by practitioners in trading to extrapolate divergence from the fundamental value in technical analysis. Since the fundamental value of stocks is essentially unknown, market practitioners often tend to at least estimate whether the stock is over or undervalued, whether the possible mispricing is small or large, and whether the gap is going to increase or whether a soon correction is more likely. As the Brock & Hommes (1998) model is also formulated in deviations from the fundamental price, the MA approach seems to be one of reasonable guidances. The MA filtering is the cornerstone of technical analysis and therefore widely used by active traders: Allen & Taylor (1990, pg. 50) present empirical evidence on the perceived importance of technical analysis among London foreign exchange dealers and refer to prevalent mechanical indicators such as trend-following rules: "buy when a shorter MA cuts a longer MA from below". Taylor & Allen (1992) survey chief foreign exchange dealers operating in London and report

[2] In contrast, another class of HAMs of FOREX markets successfully utilises the Purchasing Power Parity between two countries as the approximation of the fundamental value of the currency exchange rate [see e.g. Vigfusson (1997); Westerhoff & Reitz (2003); Manzan & Westerhoff (2007); Wan & Kao (2009); Goldbaum & Zwinkels (2014); Verschoor & Zwinkels (2013)]. Boswijk *et al.* (2007) and de Jong *et al.* (2009a) employ the static Gordon growth model for equity valuation proposed by Gordon (1962), which is, however, infeasible for the empirical validation of the original Brock & Hommes (1998) model. Some other papers simply use a RW formula to drive the fundamental price (De Grauwe & Grimaldi 2005; 2006b; Winker *et al.* 2007; Franke 2009).

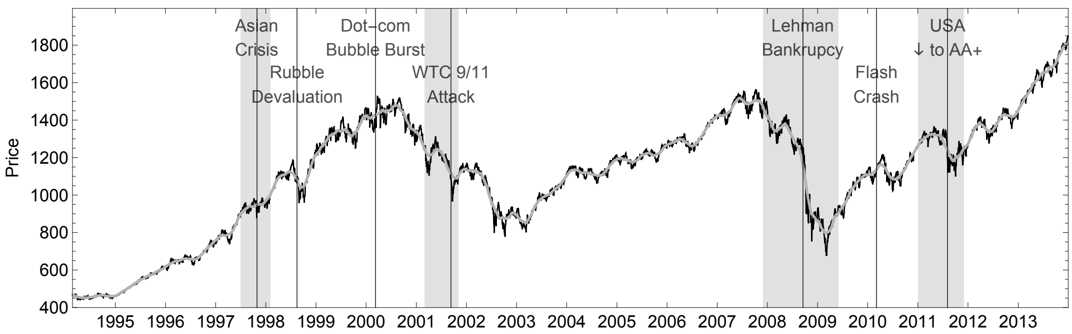
that 64.3% of organisations use MAs and/or other trend-following analytical techniques. Brock *et al.* (1992, pg. 1735) refer to MA technical rules as to one of the two simplest and most widely used: “when the short-period MA penetrates the long-period MA, a trend is considered to be initiated”. Lui & Mole (1998, pg. 541, 535) repeat largely analogical survey as Taylor & Allen (1992) among Hong Kong foreign exchange dealers and report that MAs “are seen to be the most useful technical technique at all three horizons” (i.e. intraday, intramonth, > 1 month) and that “technical analysis is considered . . . significantly more useful in predicting turning points”. Goldbaum (1999, pg. 70, 71) describes the way how in practice the MA trading rules translate into buy-sell indicators: “when the short period moving average, say the average price of the security over the last five trading days, rises above the long period moving average, say the average of the price over the last 200 trading days, this is a buy indicator. When the short period moving average drops below the long period moving average, this is a sell indicator”. Sullivan *et al.* (1999, 1999) summarise that “MA cross-over rules . . . are among the most popular and common trading rules discussed in the technical analysis literature”. To quote from Isakov & Hollistein (1999), “the most popular moving average rule used is (1,200), where the short period is one day (in fact it is the index itself) and the long period is 200 days (almost a year)”. According to authors, “the academic literature has shown that the best results were obtained when the short average is one day”. Closely related to our work, “motivated by the popularity of MA strategies in real markets and empirical studies” Chiarella *et al.* (2006, pg. 1748) propose a model in which the demand of chartists is determined by the difference between a long-term MA and current market price.

For the MA setting in this analysis, we keep to the strategy of a wide range of possible settings to ensure robustness of our findings. Within this work, we present results for two specific window lengths, namely 61 and 121 days. For the robustness check, we also tested other variants ranging from one month to two years, namely 21, 241, and 481 days, leading to comparable results.[3] Instead of usual ‘historical’ MA taking into account only the past information for given time, we use the ‘centred’ MA taking into account the same number of observation back as ahead. Both MA versions were analysed and found to produce to a large extent comparable results. The centred MA is therefore suggested to reduce the delay of the information flow. Moreover, the centred MA incorporates a convenient property that the price converges to it by definition

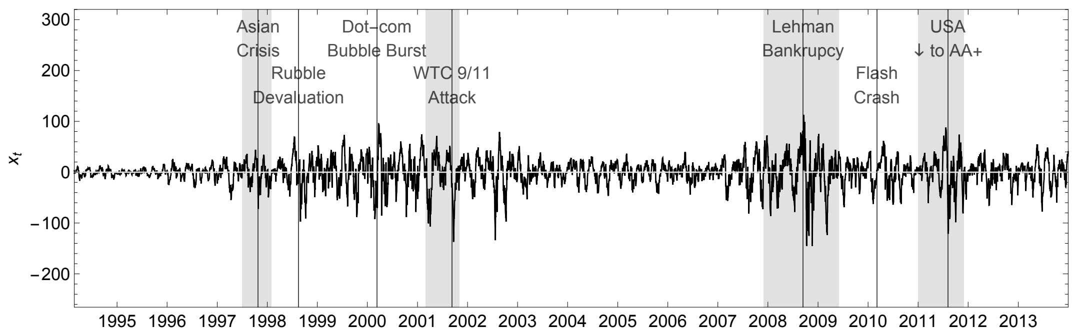
[3]Results of this robustness testing are available upon request from authors.

Figure 6.1: S&P500 fundamental price MA61 approximation

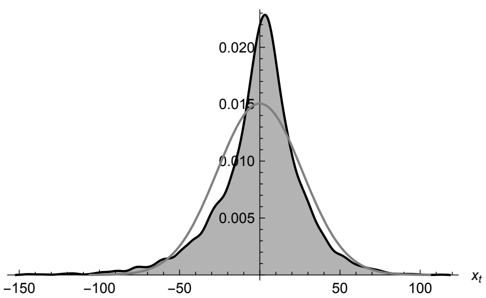
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

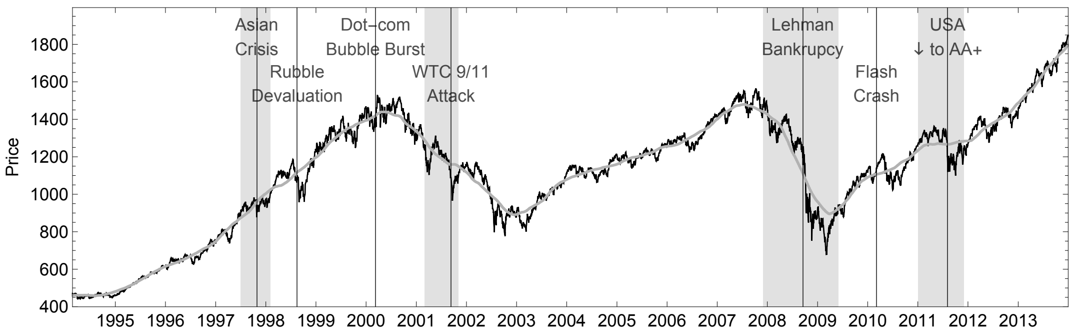


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

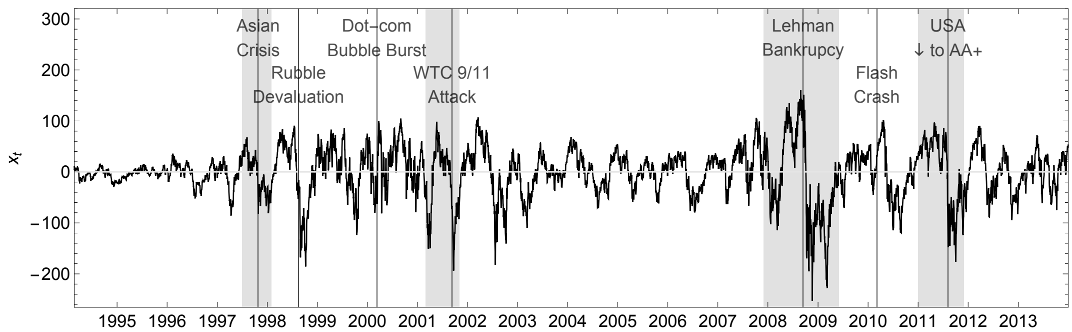
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 6.2: S&P500 fundamental price MA241 approximation

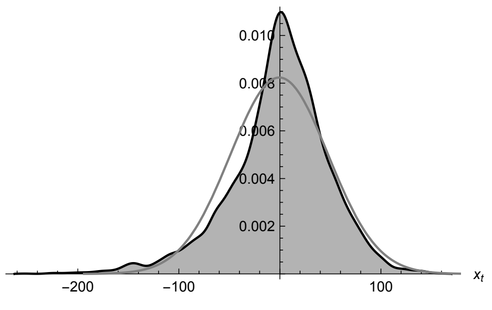
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram



Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

that is exactly a feature one would expect from the fundamental value. Although undoubtedly our fundamental price approximation differ from the true fundamental value, the MA filter produces a series of an anticipated structure as depicted in Figures 6.1 and 6.2 for S&P500 and in Figures C.1 and C.2, C.3 and C.4, C.5 and C.6, C.7 and C.8, C.9 and C.10 for other respective indices (in Appendix C).

6.3 Data description

We use daily closing prices of six world stock market indices as the base of our empirical dataset. For S&P500, we retrieve the closing prices of the index using the *Wolfram Mathematica* `FinancialData` function covering the period from 1994/02/23 to 2013/12/31, i.e. 5000 observations in total. For other indices, only the starting dates of the dataset vary by reason of different public holidays a calendar configurations around the world, i.e. 1994/04/22 for DAX,[4] 1994/11/02 for FTSE, 1994/02/23 for NASDAQ, 1993/09/03 for NIKKEI 225, and 1994/06/13 HSI. For each index we nonetheless also obtain comparable amount of 5000 observations with the same end date 2013/12/31. The fundamental price is simultaneously calculated as the centred MA as described in Section 6.2 and subtracted from the actual price following Equation 4.9.[5] Thus we obtain deviations x_t from the fundamental price that are the subject of further estimation. The span of the data is represented in Figure 6.1 and 6.2 where the original time series of prices p_t is depicted in the (a) part of the Figure together with the fundamental price p_t^* approximation and the (b) and (c) parts depict the implied series of deviations from the fundamental price $x_t = p_t - p_t^*$. Descriptive statistics of x_t series for all indices and two MA lengths for the fundamental value approximation are summarized in Table 6.1.

6.4 Static NPSMLE estimates

In the estimation, our main goal is to verify the HAM ability to describe world stock market data and whether we obtain estimates of a reasonable precision using the NPSMLE method. A special focus is also devoted to possible differ-

[4] For DAX the data are available from 1990/11/26, i.e. 5850 observations till 2013/12/31.

[5] For the calculation of the fundamental price we need some extra data points preceding and succeeding the defined period, the complete dataset retrieved and used therefore consists of '4999 + MA window length' observations.

Table 6.1: Descriptive statistics of empirical x_t time series

Data, MA period	Mean.	Med.	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC x_t^2
SP500, 61	-.03	2.0	-145.2	113.0	26.5	-.65	5.5	-61.4	50.7	.87	.73
NASDAQ, 61	-.10	2.4	-753.5	639.0	81.8	.03	13.6	-170.7	147.9	.89	.81
DAX, 61	-.28	10.1	-939.6	716.1	163.2	-.50	5.4	-361.8	311.6	.89	.80
FTSE, 61	-.12	7.5	-702.5	404.6	122.7	-.58	5.1	-275.4	235.7	.88	.73
HSI, 61	.35	24.3	-4253.2	3463.5	562.2	-.29	6.7	-1186.7	1130.0	.90	.74
NIKKEI 225, 61	.16	6.8	-2249.8	1900.2	434.7	-.30	4.2	-953.8	799.4	.89	.77
SP500, 241	-.60	2.8	-252.9	160.1	48.4	-.64	4.5	-112.7	85.7	.96	.90
NASDAQ, 241	-1.78	1.6	-756.4	1253.7	168.1	1.01	11.3	-350.9	322.0	.97	.96
DAX, 241	-3.35	1.4	-1531.3	1242.6	330.4	-.21	4.5	-728.0	669.1	.97	.94
FTSE, 241	-.73	11.9	-1072.4	721.0	210.6	-.56	4.4	-479.4	388.5	.96	.90
HSI, 241	2.26	-5.5	-6505.5	7099.5	1177.6	.18	5.7	-2424.1	2282.0	.98	.94
NIKKEI 225, 241	-6.38	-22.6	-3497.8	2872.4	860.0	-.20	3.3	-1821.4	1581.3	.97	.92

Note: Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, and autocorrelations (AC) are reported. Figures are rounded to 1 or 2 decimal digits.

Source: Author's own computations in *MATLAB*.

ences and similarities between particular indices. For reader's convenience we briefly repeat the structure of the estimated model model:

$$
\begin{aligned}
Rx_t &= \sum_{h=1}^{H} n_{h,t}(g_h x_{t-1} + b_h) + \epsilon_t, & (6.1)\\
n_{h,t} &= \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}, & (6.2)\\
U_{h,t-1} &= (x_{t-1} - Rx_{t-2})\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}, & (6.3)
\end{aligned}
$$

where ϵ_t is an i.i.d. noise term sequence with normal distribution $\mathcal{N}(0, sd^2)$. Please, consult details of the model setting in Section 5.1. To recap, in the 2-type model ($H = 2$) we simultaneously estimate parameters β, g_2, b_2, and the *noise intensity*. In the 3-type model ($H = 3$) we simultaneously estimate parameters β, g_2, g_3, and the *noise intensity*.

6.4.1 Full sample estimates of the 2-type model

We start with the full sample static estimation summarised in Table 6.2. Generally, we can observe broad similarities across all indices and markedly statistically significant estimates of a positive belief parameter g_2 revealing superiority of trend following over contrarian strategies on markets. In contrast, the estimates of the intensity of choice—the switching parameter β—and the bias parameter b_2 are largely statistically insignificant. While for the bias this is an expected result as there is no obvious reason why the trend following strategies should be somehow biased in the long-term, the insignificance of $\widehat{\beta}$ is an important and interesting result. We thus contrast a large subpart of the HAM estimation literature (see Chapter 2) but confirm the main results of e.g. Westerhoff & Reitz (2005); Boswijk *et al.* (2007); de Jong *et al.* (2009b); ter Ellen *et al.* (2013); Bolt *et al.* (2014). Since the heterogeneity in trading regimes is confirmed by the significance of g_2, this might not worrying as discussed in Boswijk *et al.* (2007, pg. 1995) or Hommes (2013, pg. 203) who emphasise that "this is a common result in non-linear switching regression models, where the parameter in the transition function is difficult to estimate and has a large standard deviation, because relatively large changes in β^* cause only small variation of the fraction n_t. Teräsvirta (1994) argues that this should not be worrying as long as there is significant heterogeneity in the estimated regimes." Further-

Table 6.2: Empirical results of the 2-type β model estimation

Data, MA period	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{noise\ intensity}$			(e) LL		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	.015	.040	.122	1.567	1.587	.233	.009	.003	.121	.653	.656	.108	-1.486	-1.491	.074
NASDAQ, 61	.002	.006	.146	1.717	1.715	.166	-.005	-.003	.092	.609	.609	.079	.117	.115	.038
DAX, 61	.001	.018	.112	1.640	1.646	.215	.008	.001	.117	.590	.601	.099	-1.259	-1.264	.081
FTSE, 61	-.000	.008	.113	1.671	1.668	.201	.001	.004	.117	.597	.602	.092	-.988	-.995	.059
HSI, 61	-.004	-.002	.150	1.724	1.727	.164	-.003	-.001	.098	.566	.570	.069	-.145	-.147	.036
N225, 61	.008	.021	.100	1.601	1.619	.210	.011	.007	.121	.593	.601	.103	-1.544	-1.546	.085
SP500, 241	-.021	-.009	.195	1.878	1.875	.127	.007	.001	.111	.388	.397	.064	.029	.019	.074
NASDAQ, 241	.038	.047	.189	1.882	1.869	.152	-.009	-.003	.105	.423	.419	.078	.422	.424	.111
DAX, 241	.014	.012	.247	1.932	1.928	.109	.007	.003	.103	.292	.315	.070	.328	.309	.096
FTSE, 241	.010	.012	.162	1.871	1.858	.143	.008	.004	.114	.408	.409	.071	-.153	-.157	.085
HSI, 241	.008	.006	.204	1.914	1.908	.130	.004	.002	.103	.351	.366	.080	.369	.363	.118
N225, 241	.003	.012	.147	1.865	1.850	.165	-.000	-.000	.125	.394	.392	.101	-.793	-.793	.160
Robustness check															
SP500 monthly, 13	-.016	-.031	.272	.886	.891	.238	.006	.002	.121	.863	.869	.104	-3.660	-3.666	.054
SP500 weekly, 13	-.085	-.103	.228	.953	.971	.224	-.003	-.001	.121	.865	.869	.098	-2.336	-2.349	.057
SP500 weekly, 49	.044	.089	.146	1.119	1.168	.241	.005	.002	.118	.724	.720	.103	-2.626	-2.615	.073
SP500 R=1.001, 61	.016	.035	.112	1.585	1.607	.241	.004	.004	.121	.656	.654	.110	-1.480	-1.485	.073
SP500 R=1.001, 241	.001	-.001	.196	1.889	1.880	.134	.005	.002	.111	.393	.397	.062	.029	.019	.073
SP500 m_h=40, 61	-.032	-.041	.149	1.659	1.673	.173	-.002	-.002	.114	.584	.587	.075	-1.393	-1.402	.038
SP500 m_h=40, 241	-.028	-.020	.298	1.908	1.905	.107	-.004	-.003	.099	.338	.355	.056	.100	.086	.054
SP500 m_h=80, 61	-.047	-.054	.177	1.644	1.664	.182	.003	.003	.115	.573	.576	.065	-1.385	-1.392	.032
SP500 m_h=80, 241	-.036	-.014	.309	1.908	1.906	.105	-.002	-.004	.097	.335	.352	.054	.106	.091	.056

Note: Results are based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians, means, and standard deviations (SD) are reported. 'LL' denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

Source: Author's own computations in *MATLAB*.

more, as Huisman *et al.* (2010, pg. 17, 20) point out, “the significance of the intensity of choice is not a necessary condition for the switching to have added value to the fit of the model” and “the non-significant intensity of choice, as this indicates that the switching does not occur systematically”. The magnitudes of trend parameter estimates $\widehat{g_2}$, that keep roughly between 1.6 and 1.9, might seem large, but it is important to note that they influence the price change only from circa 50% implied by the insignificance of the intensity of choice β keeping the population ratio of the two strategies stable around 0.5/0.5.

It is now important to contrast empirical findings with simulation results of Chapter 5. Based on the analysis of the confidence bands in Figure 5.2, our computational setting based on 1000 random runs, number of observations $t = 5000$, and the kernel approximation precision $N = 500$ provides us with a reasonably high precision of the intensity of choice β estimates. Figure 5.2 (bottom part) and further quantitative analysis clearly show that even for a very small $\beta = 0.5$, when estimating a time series of 5000 observations and considering 5% significance level, β is markedly statistically significant. Thus, if there is some behavioural switching present in our empirical data, it should have been detectable under similarly robust setting.

Differences across markets can be partly seen in the (d) column of Table 6.2 between well efficiently estimated values of the $\widehat{noise\ intensity}$. Although the difference are often on the border of statistical significance, it might be worth mentioning that the highest stock market noise intensity has been estimated for the U.S. indices, specifically S&P500 in case of MA61 based fundamental value and NASDAQ in case of MA241 based fundamental value. Conversely, the lowest values has been estimated for DAX and the difference is circa 30% in case of the MA241 based fundamental value.

The level differences in values between the upper part of Table 6.2 depicting results for the MA61 fundamental price approximation and the middle part with results for the MA241 is perhaps mainly the technical feature of different MA windows. It is therefore important to consider absolute values of estimated coefficient with this respect and compare both versions. Nevertheless, the main results concerning the positive sign and statistical significance of $\widehat{g_2}$ and insignificance of $\widehat{\beta}$ and $\widehat{b_2}$ keep similar as well as the main detected relative relationships between values of the $\widehat{noise\ intensity}$. Our most important results thus demonstrate robustness w.r.t. the choice of the fundamental value specification. The lower values of the $\widehat{noise\ intensity}$ might be explained by reason of a better fundamental value approximation using bigger MA window.

6.4.2 Behaviour of the simulated log-likelihood function

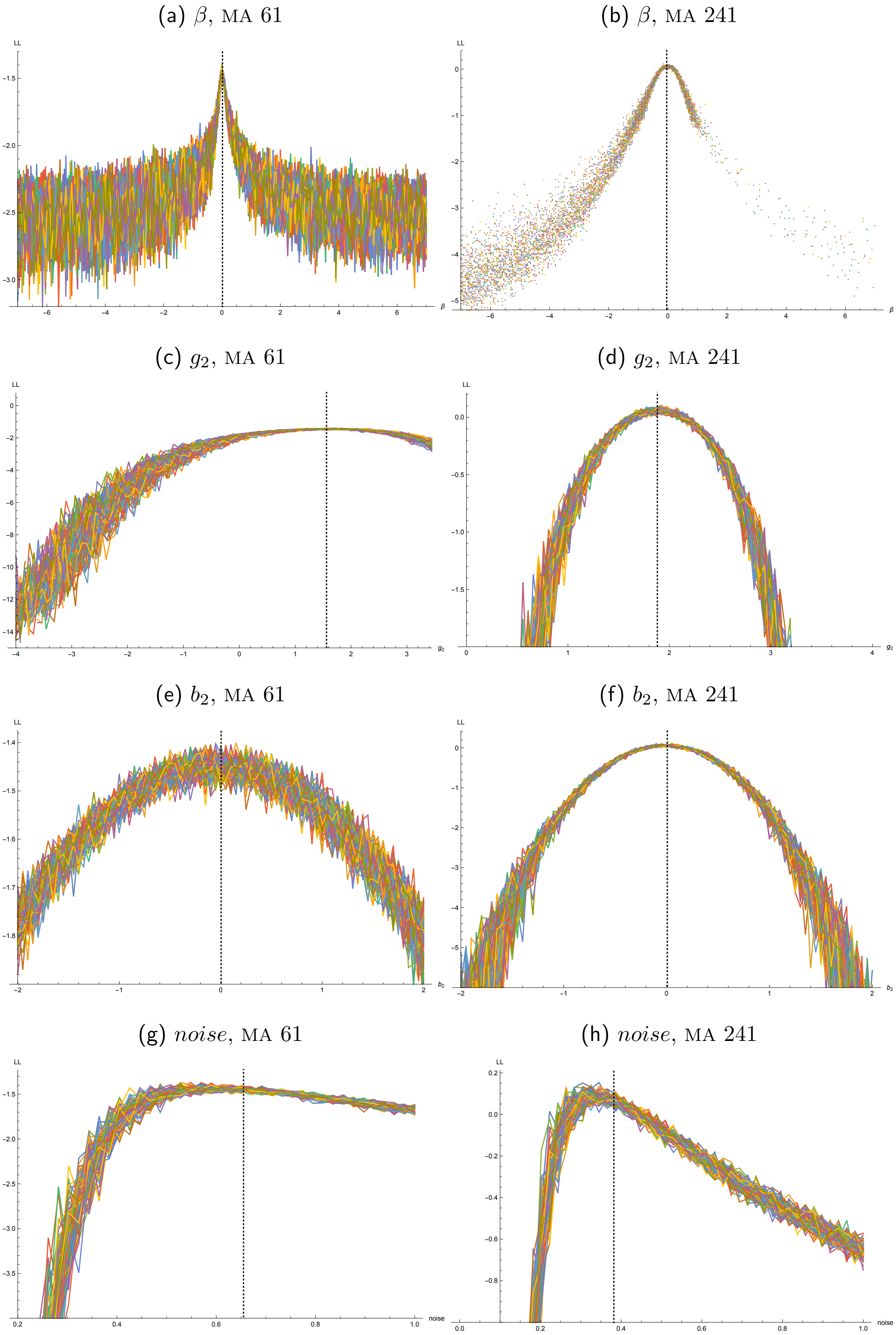
We verify the smoothness conditions and unique maxima presence of the simulated log-likelihood functions that are crucial for the parameter detection and identification also within the empirical application. However, the 5D surface resulting from simultaneous estimation of four parameters makes the graphical demonstration even more complicated than in Subsection 6.4.2. Therefore, we again depict sub-log-likelihood functions in 2D and 3D, assuming other parameters fixed at estimated values from Table 6.2.

Figure 6.3 demonstrates partial 2D shapes of the simulated sub-log-likelihood function in direction of individual parameters. Generally we observe a bit rough shape in detail, but very consistent performance of the estimation method over all 100 random runs leading to unique maxima consistent with the full sample estimates in all cases. The shape is affected by the structure of the real world data which is far away from the regularity of the simulated dataset. For this reason, we also adapted the computational algorithm by increasing the number of initial points (see details in Section 6.1) so that it is able to deal well with the not-completely-smooth surface of the log-likelihood function. Figure 6.4 then visualises 3D simulated sub-functions for all combinations of four estimated parameters, keeping the other two fixed. The relative smoothness of the surface and well detectable regions of possible maxima keep generally retained corresponding to simulation results from Section 5.3.2. For more 'extreme' combinations of parameters the model is again numerically unstable and we do not depict the surface for these regions. Equivalently to full sample estimation results, parameters β and g_2 seem to be well detectable, while in the b_2 direction [subfigures (b), (d), and (f)] the surface is very flat. Interestingly, based on visual inspection of subfigures (g) and (h) we suspect a small potential upward bias for the *noise intensity* estimates.

6.4.3 Robustness check of the 2-type model

For the robustness check of the validity of estimated values (results are reported in Table 6.2, bottom part), we not only use more than single MA specification of the fundamental value, but also consider several modification of the setup and even different frequency of the data. Equipped with the knowledge from previous analysis, we again only compute results for S&P500. Aside utilisation of weekly and monthly data, we also follow the robustness testing from Chapter 5 and estimate the model using

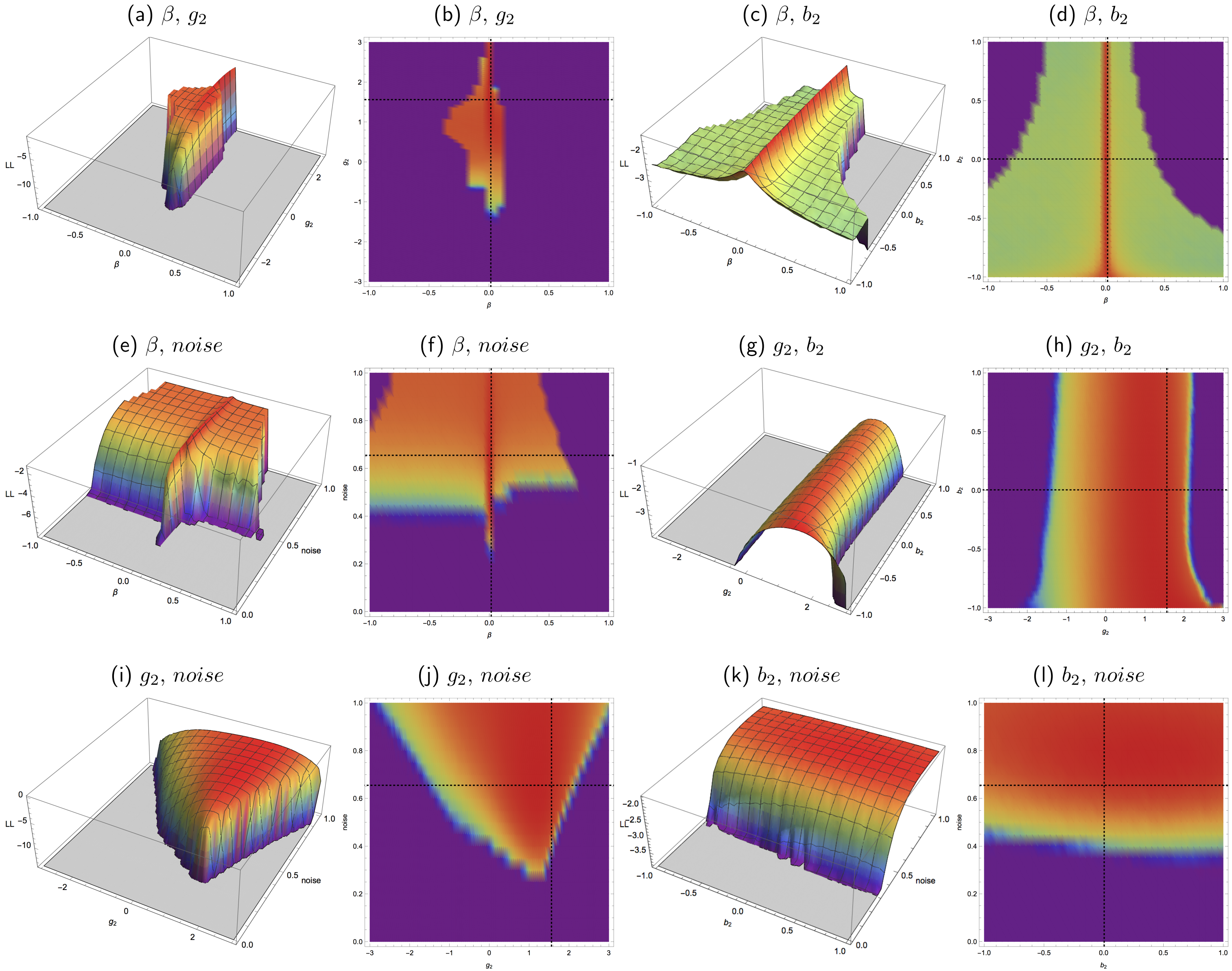
Figure 6.3: Simulated sub-log-likelihood fcns. for single parameters



Note: Results based on 100 random runs, S&P500 data, given MA fundamental price approximation, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. $\{\varepsilon_i\}_{i=1}^{N}$ drawn from normal distribution. Black dotted vertical lines depict estimated parameters (see Table 6.2).

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 6.4: Simulated sub-log-likelihood functions in 3D



Note: Results averaged over 100 random runs, S&P500 data, MA61 fundamental price approximation, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution. Black dotted lines in horizontal projections depict the true values of parameters.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

1. 10 times higher assumed market risk free rate $R = 1.001$;

2. nontrivial memory of agents defined via parameters $m_h = \{40, 80\}\ \forall\ h$.[6]

Three new dataset cover S&P500 weekly data from 1994/02/28 to 2013/12/30 (i.e. 1035 observations) and monthly data from 1994/03/01 to 2013/12/02 (i.e. 238 observations). Selected periods cover the same span as the original daily dataset. The MA lengths have been selected so that they resemble most closely the 61 and 241 days for the fundamental value specification, i.e. 13 and 49 weeks in case of weekly data and 13 months for monthly data. The assumed market risk free rate has been adjusted to reflect the modified data periodicity, namely to $R = 1.0005$ for weekly data and $R = 1.002$ for monthly data.

The most important findings of the original empirical analysis of six world stock market indices remain unaffected under the robustness burden. The $\widehat{\beta}$ parameter still reveals evident statistical insignificance, the same does the bias parameter $\widehat{b_2}$. Differences are basically observable at the level of trend parameters $\widehat{g_2}$ and $\widehat{noise\ intensity}$, but the behaviour keeps patterns uncovered within the original analysis—$\widehat{g_2}$ slightly increases and $\widehat{noise\ intensity}$ decreases moving from MA61 to MA241 fundamental value approximation. Results based on monthly and weekly data show considerably lower $\widehat{g_2}$, the values fall even under 1 for the MA13 fundamental value specification—instead of strong trend chasing strategy only the weak trend chasing strategy is detected—but this seems to be again an implied technical side-effect of a small MA window that produces more average dynamics of the price deviations series. Monthly data with insufficient length of the estimated series (238 observations) expectedly, based on findings of the NPSMLE method performance analysis in Chapter 5, perform the worst statistical fit compared to weekly and daily dataset. Memory only slightly increases the model fit and as the interconnected effect decreases the $\widehat{noise\ intensity}$. Nothing surprising is therefore found within the validity check

[6]Memory process is a substantial modification of the model structure. We employ the similar approach as Barunik *et al.* (2009), Vacha *et al.* (2012), and Kukacka & Barunik (2013), i.e. Equation 5.3 is extended via memory parameters m_h:

$$U_{h,t-1} \quad = \quad \frac{1}{m_h} \sum_{l=0}^{m_h-1} \left[(x_{t-1-l} - Rx_{t-2-l}) \frac{g_h x_{t-3-l} + b_h - Rx_{t-2-l}}{a\sigma^2} \right]. \tag{6.4}$$

The memory for each individual strategy is then randomly generated from the uniform distribution $U(0, m_h)$, therefore the average memory length resembles circa the one- or two-month period, i.e. 20 or 40 days.

of original results which prove robust to various data frequency specifications as well as modifications of interesting parameters in the model.

6.4.4 Full sample estimates of the 3-type model

Estimation of a more-flexible 3-type model reveals markedly similar big picture as the estimation of the 2-type model. For the matter of computational time, based on our knowledge from the 2-type model estimation revealing large similarities across all estimated stock market indices we again only compute results for S&P500. Estimated parameters are reported in Table 6.3. The only new conclusion is a statistical insignificance of the contrarian strategy represented by coefficient $\widehat{g}_3$ (exactly specified in the model and defined via the constraint $g_3 < 0$). Although point estimates reported via median and mean values are negative, this is only an effect of the enforced $g_3 < 0$ constraint. The distribution mass of estimates from all 500 runs concentrates close to 0 as depicted in Figure 6.5. The optimised function is likely to be very flat in the dimension of g_3 parameter because the effect of a very weak contrarian strategy is overshadowed if combined with a very strong trend following strategy. The estimate of the intensity of choice β keeps its statistical insignificance and the trend following strategy coefficient $\widehat{g}_2$ retains its positive sign as well as high statistical significance. The absolute value of $\widehat{g}_2$ is naturally higher because the trend following strategy impacts the price via only the 1/3 weight in the 3-type model compared to 1/2 weight in the 3-type model (in both cases conditional on insignificant $\widehat{\beta}$). Taking those weights into account, we obtain very similar impact of the trend following strategy in both models. Comparing results for the MA61 and MA241 fundamental price approximation shows the very same effects as within the 2-type estimation, under MA241 we reveal somewhat stronger trend following strategy and lower intensity of stochastic noise.

6.5 Rolling NPSMLE estimates

To confirm the robustness of the full sample static estimates in Section 6.4 over time, we further investigate how the HAM estimation results might possibly change between 1994 and 2014. Utilising almost 20 years of data (5000 observations) used for the static estimates, we now estimate the HAM on one year (240 days) rolling samples with steps of two months (40 days). Based on results of the simulation Monte Carlo study in Chapter 5, the one year pe-

Table 6.3: Empirical results of the 3-type β model estimation

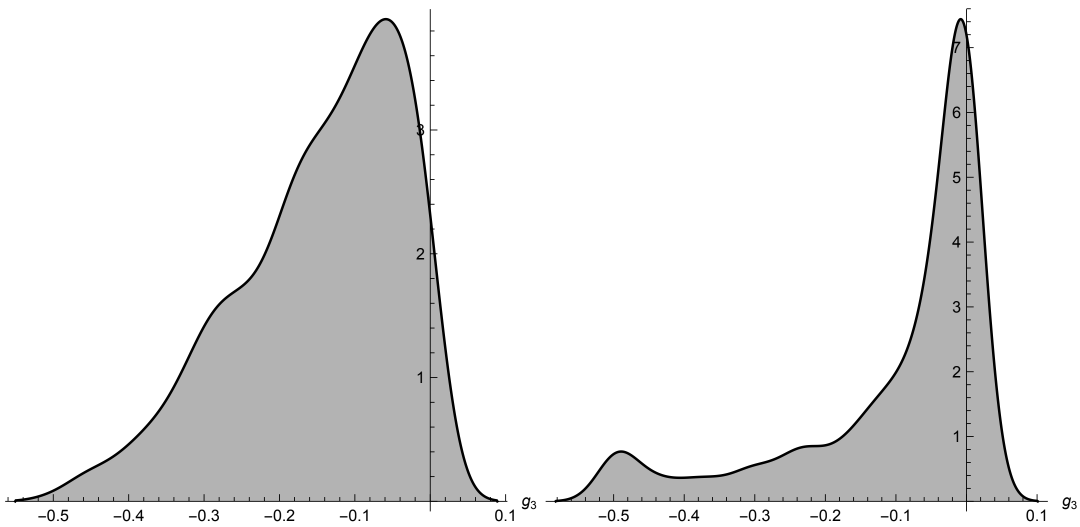
Data, MA p.	(a) $\widehat{\beta}$		(b) $\widehat{g_2}$		(c) $\widehat{g_3}$		(d) $\widehat{noise\ i.}$		(e) LL	
	Med.	SD	Med.	SD	Med.	SD	Med.	SD	Med.	SD
SP500, 61	-.003	.082	2.502	.175	-.123	.111	.550	.047	-.127	.022
SP500, 241	.007	.050	2.674	.217	-.032	.142	.403	.045	-.289	.094

Note: Results are based on 500 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians and standard deviations (SD) are reported. 'LL' denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

Source: Author's own computations in *MATLAB*.

Figure 6.5: Smooth histogram of the contrarian coefficient g_3

(a) S&P500 MA 61 (b) S&P500 MA 241



Note: Results are based on 500 random runs. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica.*

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica.*

riod still represents relatively sufficient length of the estimated time series for a reasonable statistical inference based on the NPSMLE method. At the same time it is a relatively short period to detect possible structural breaks in the data.[7] Due to high computational burden we have significantly decreased the robustness of the algorithm setting: we display results based on 200 runs and the number of initial points for the numerical optimisation has been decreased to 4. We also always discard only 10 observations as the initial stabilising periods. The 'cost' of such relaxation of the computational setting is reflected in the lower efficiency of estimates and the standard deviations of rolling estimates are expected higher compared to full sample static estimates. On the other hand, the rolling analysis still provides clear insight into the model dynamics and credible conclusion.

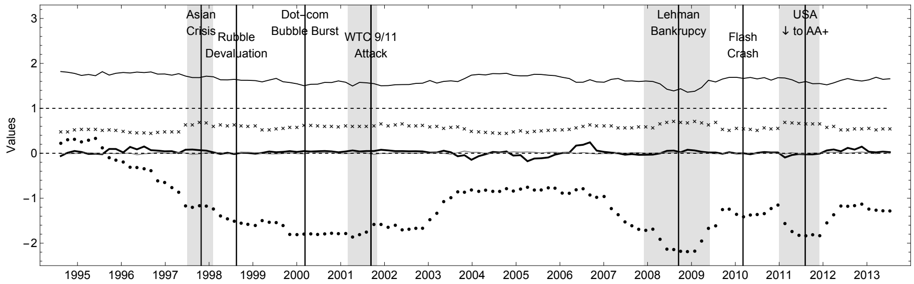
6.5.1 Rolling estimates of the 2-type model

We depict rolling estimation results of the S&P500 in Figure 6.6. Rolling estimate results for other indices are reported in Appendix D, Figures D.1, D.2, D.3, D.4, and D.5. Interpreting primarily results in Figure 6.6, we can observe relatively stable behaviour of the model throughout the entire investigated period. The $\widehat{g_2}$ and $\widehat{noise\ intensity}$ estimates keep steadily around their long-term static estimates and exhibit high statistical significance as traceable in the (c) and (d) subparts of the Figure depicting rolling standard deviation for all coefficients. The $\widehat{\beta}$ and $\widehat{b_2}$ keep to zero and are statistically insignificant all the time. This is something we generally expect to observe as the full sample $\widehat{\beta}$ estimates (see Table 6.2) are close to zero, hence the dynamics is restrained and the model in fact boils down to a simple model which we further analyse in Section 6.6. Nonetheless, we might detect some signs of dynamics of $\widehat{g_2}$ e.g. around the Lehman Brothers bankruptcy and related U.S. recession between December 2007 and June 2009 in the MA61 case, but these shifts are strongly bellow the level of statistical significance and also largely dependent on the window length of the MA fundamental price approximation as one can see when comparing the (a) and (b) part of Figure 6.6. Some slight dynamics is also detectable for $\widehat{noise\ intensity}$ which slightly increases in turbulent periods. The only clear dynamics seems to be observable at the level of the log-likelihood LL. With this respect we highlight the fact that a direct compar-

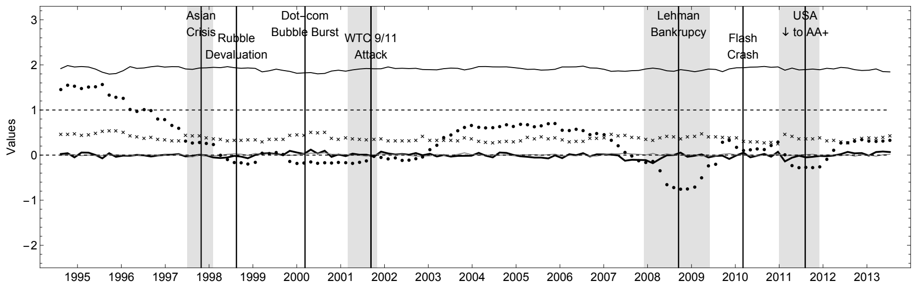
[7] Various combinations of rolling sample windows and steps had been used in the preliminary analysis without impacting the overall results, e.g. comparing one month and one-half year steps. The outcomes of the preliminary analysis are available from authors upon request.

Figure 6.6: Rolling estimates of the 2-type model for S&P500

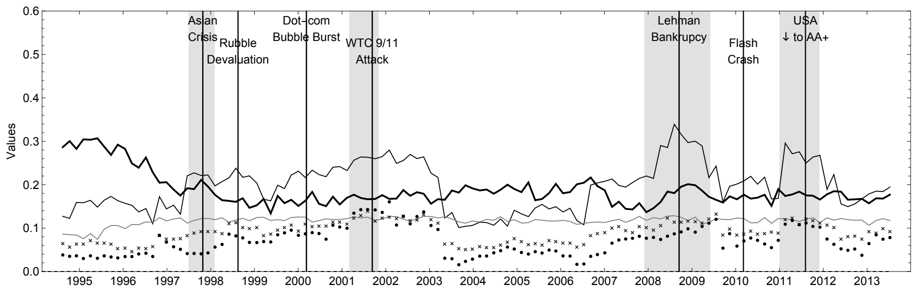
(a) MA61 fundamental price approximation



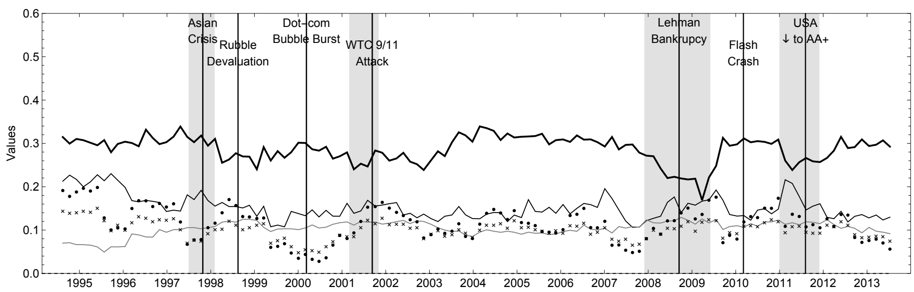
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations



Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

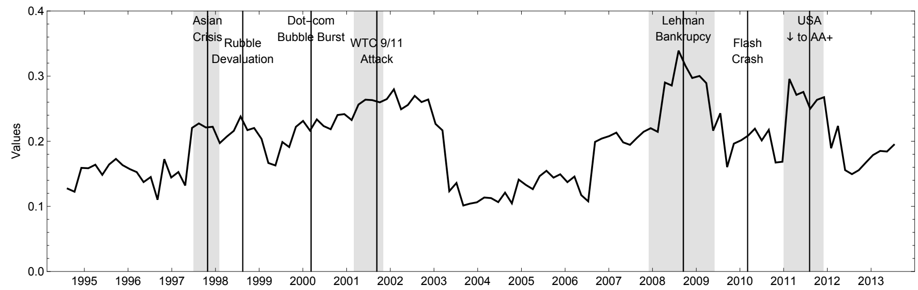
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

ison of rolling log-likelihoods is methodologically disputable because it is based on different rolling sub-samples. However, we argue that since rolling datasets keep the same length and overlap by circa 83% between adjacent steps, the overall evolution of the LL pattern provides us with a valuable information. The fit is relatively higher during rather tranquil periods (e.g. at the beginning and the end of the sample period or between 2003 and 2007) and on the contrary it generally decreases during volatile periods (well detectable in Figures 6.1 and 6.2), especially during highlighted recessions. This might seem puzzling as during the high volatile periods the trend following strategy is likely to attract attention of market participants, however such behaviour is in accord with the overall stability of the rolling estimates close to long-term values. A potential dynamics is, however, much more observable at the level of standard deviations [subparts (c) and (d) and detached Figure 6.7]. The standard deviation of the trend following coefficient g_2 clearly jumps up in volatile periods such as recession around the WTC 9/11 attack, Lehman Bankrupcy, or downgrade of USA ranking to AA+. This can be interpreted as a sign of increased presence of contrarians (nonetheless, still being a large minority) during such turbulent stock market periods. Although the method faces difficulties to adjust the average absolute value of the trend-following coefficient $\widehat{g_2}$ over 500 repeated runs, it detects increasing population of contrarians via less efficient estimates of the effect of trend followers. A similar pattern is observable for the $\widehat{noise\ intensity}$ and LL. This is again nothing surprising as econometrics models generally perform better in periods of market stability. Using larger MA241 fundamental value approximation naturally decreases the flexibility of the estimation to detect effects of single events and so the captured dynamics is considerably more stable.

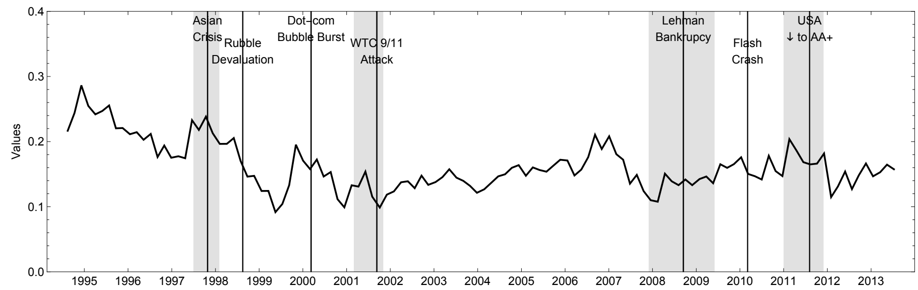
We also observe some interesting signs of a specific and economically well interpretable dynamic behaviour for other indices illustrated in Figure 6.8. For NASDAQ behaviour, although otherwise considerably more stable compared to S&P500, the worth mentioning is especially the drop of the standard deviation of $\widehat{\beta}$ and increase of the standard deviation of the trend parameter $\widehat{g_2}$ around the Asian Crisis in 1997 and the Dot-com Bubble Burst in 2000. Since NASDAQ is especially used for trading technological and IT companies, this makes somewhat sense as technological companies are often based or produce in Asia and IT companies were hit by the Dot-com Bubble much more than other sectors. Similar behaviour is not observed for the more general S&P500. For both European indices DAX and FTSE the model on the level of standard deviations does

Figure 6.7: Rolling behaviour of the SD of the $\widehat{g}_2$ estimate I.

(a) S&P500 MA61 fundamental price approximation



(b) NASDAQ MA61 fundamental price approximation



Note: Bold black full line depicts standard deviation of the $\widehat{g}_2$ estimate. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

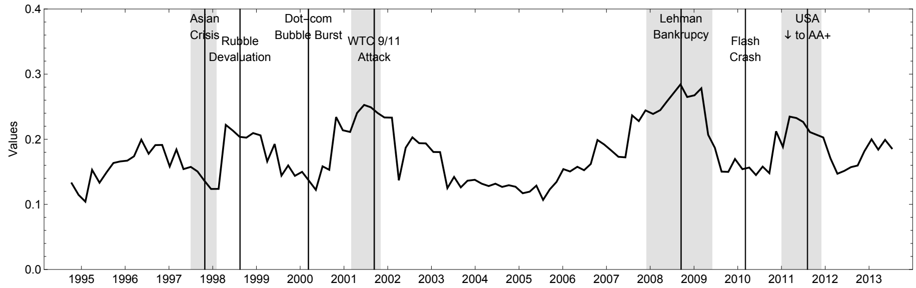
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

reflect the crisis around the Rubble devaluation in 1998, again not captured by S&P500. The effect of both worldwide crises in 2001 and 2008-2009 is also captured, but not the Asian Crisis of 1997. Interestingly, the HSI data representing Asia captures the effect of 1997 Asian Crisis as well as the worldwide cries of 2008-2009 and 2011, but does not reflect neither the 1998 Rubble devaluation, nor WTC 2001/9/11 attack. Behaviour of model under NIKKEI 225 data resembles much more the patterns observed for S&P500 than for HSI. All this can be attributed to increased presence of contrarians at specific periods on specific markets detected via less efficient estimates of the effect of trend followers. Finally for some indices, namely NASDAQ, FTSE and HSI (but not for the other three) we can observe some intriguing negative correlation between standard deviations of $\widehat{\beta}$ and $\widehat{g}_2$ in some turbulent periods.

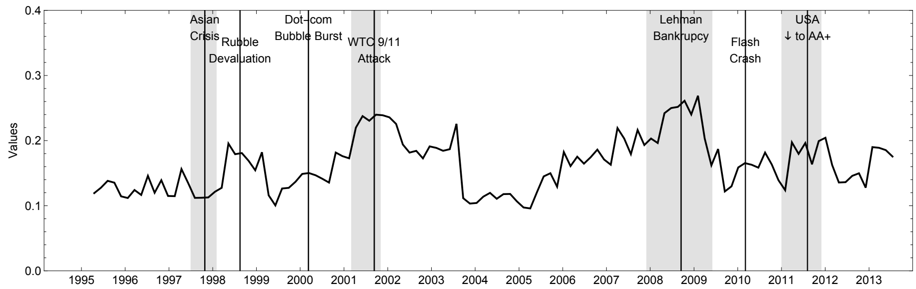
In general, although we can reveal some patterns of interpretable dynamics

Figure 6.8: Rolling behaviour of the SD of the $\widehat{g}_2$ estimate II.

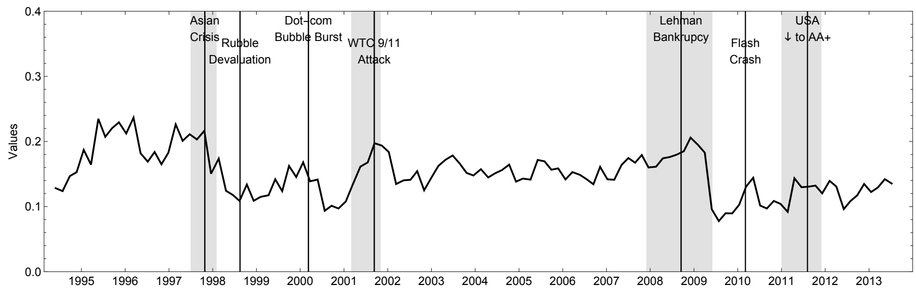
(a) DAX MA61 fundamental price approximation



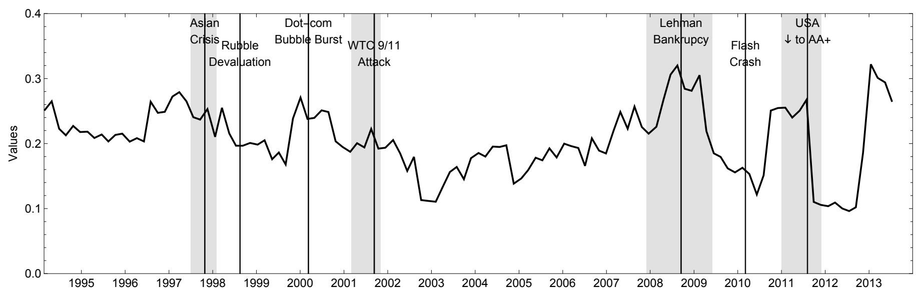
(b) FTSE MA61 fundamental price approximation



(c) HSI MA61 fundamental price approximation



(d) NIKKEI 225 MA61 fundamental price approximation



Note: Bold black full line depicts standard deviation of the $\widehat{g}_2$ estimate. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

for individual indices, the rolling approach in particular strongly supports the stability of model behaviour over time and thus confirms the validity of full sample estimation results from Subsection 6.4.1.

6.6 Estimation of market fractions

Our findings from the performed estimations in Section 6.4 and Section 6.5, mainly the overall statistical insignificance of the intensity of choice $\widehat{\beta}$, statistically insignificant sizes of $\widehat{g_3}$ coefficients of contrarian strategies, and stability of rolling coefficients, lead us to another, this once truly significant modification of the model. Interpreting these results, hitherto model specifications do not seem to correspond to the data fully. In the 2-type model, the insignificant $\widehat{\beta}$ coefficient implicates stable population ratio of trading strategies $n_{1,t}/n_{2,t} \doteq 0.5/0.5$, which means that the population of fundamentalists is forced to be of almost the same magnitude as the population of chartists throughout the entire span of the dataset. Thus the model in fact boils down to a simple weighted AR(1) process and different types of traders cannot be identified because they do not switch over time. In such a case the trend and bias parameters $\widehat{g_2}$ and $\widehat{b_2}$ (or $\widehat{g_3}$ in the 3-type model) can be viewed as nuisance parameters—they to a large extent lose the original model interpretation and we cannot fully trust the estimated magnitudes of these parameters. Although we understand that it is generally very complicated for the estimation method to detect some systematic evolutionary switching between trading strategies when it is exposed to the full dataset (and therefore the average zero $\widehat{\beta}$ coefficient seems reasonable), we cannot agree with such a strong assumption of similar population magnitudes for both strategies. Moreover, contrarians in the 3-type model, who technically account for 1/3 of the population size as the $\widehat{\beta}$ coefficient is steadily insignificant, in fact behave as fundamentalists in terms of their price impact because sizes of $\widehat{g_3}$ coefficients are small and statistically insignificant. These findings from the analysis of the 2-type and 3-type model estimation imply two important conclusions. First, the 3-type model does not really help us to capture additional features of the data-generating process and rather deviates the implied market fraction. Second, it suggests there might be more fundamentalists than chartists on real markets and therefore the almost fixed population ratio of trading strategies $n_{1,t}/n_{2,t} \doteq 0.5/0.5$ as the result of the 2-type model estimation is much likely not capturing the real market population proportions.

Therefore, as a consequence of previous findings, we trivialise the simulated model (Equation 5.1, Equation 5.2, and Equation 5.3) via disabling the evolutionary switching behaviour and fixing the population ratio of trading strategies to $n_{1,t}/n_{2,t} = const$. A rationale of this step is further supported by overall stability of rolling estimates. Equation 5.2 and Equation 5.3 are now replaced by Equation 6.6 and the coefficient n_1, which we further call percentage $fraction$ of fundamentalists, is to be estimated instead of the switching coefficient β:

$$Rx_t = \sum_{h=1}^{H} n_h f_{h,t} + \epsilon_t \equiv \sum_{h=1}^{H} n_h(g_h x_{t-1} + b_h) + \epsilon_t, \tag{6.5}$$

$$n_1 = 1 - n_2, \tag{6.6}$$

where $H = 2$ in the 2-type model. Interval for the starting points random generation is constrained to $\langle 0.3, 0.9\rangle$ for *fraction* and to $\langle 1.5, 2.5\rangle$ for g_2, the other setting remains the same as in Section 6.4. The modified setup keeps the logic of aforementioned findings and does not distract the structure of the original model. On the other hand, the population ratio of trading strategies n_1/n_2 and implied percentage *fraction* of fundamentalists on the market is now a direct subject of the interest.

6.6.1 Full sample estimates of the 2-type fraction model

Outcomes of the full sample static estimation of all six stock market indices are reported in Table 6.4. The main interest lies in the behaviour of the new variable *fraction* representing the percentage market fraction of fundamentalists ($g_1 = b_1 = 0$). All other variables behave at average very similar as in the 2-type β model estimation, moreover we do not longer observe considerable distinctions caused by the MA window length for the fundamental value approximation.

The $\widehat{fraction}$ coefficient is strongly statistically significant with it value closely around 0.56, leaving only 44% of the market population to chartistic strategies. The model therefore suggests overall proportional dominance of the fundamental strategy on all investigated world stock markets. The stability of the $\widehat{fraction}$ coefficient is also confirmed via the rolling estimates in Figure 6.10 where it closely oscillates around the long-term static value. Estimates of the trend following coefficient g_2 are generally higher compared to values for the 2-type β model estimation reported in Table 6.4 but one must realise that within the 2-type *fraction* model the trend following strategy is

Table 6.4: Empirical results of the 2-type *fraction* model estimation

Data, MA period	(a) $\widehat{fraction}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{noise\ intensity}$			(e) LL		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	.569	.555	.088	1.963	1.986	.362	.006	.003	.121	.559	.558	.044	-4.022	-4.023	.013
NASDAQ, 61	.555	.541	.086	1.972	1.985	.344	-.001	-.001	.117	.565	.566	.054	-5.118	-5.119	.019
DAX, 61	.559	.547	.090	2.006	2.021	.387	.008	.005	.123	.489	.490	.035	-5.733	-5.733	.011
FTSE, 61	.556	.545	.085	1.980	1.997	.357	-.009	-.007	.121	.519	.520	.037	-5.507	-5.507	.011
HSI, 61	.559	.549	.082	1.993	2.020	.344	-.006	-.001	.120	.531	.531	.045	-6.999	-7.000	.016
N225, 61	.562	.553	.084	1.995	2.023	.361	.005	.001	.124	.482	.484	.032	-6.700	-6.701	.010
SP500, 241	.562	.556	.084	2.161	2.228	.425	.002	-.001	.133	.331	.348	.053	-4.073	-4.088	.049
NASDAQ, 241	.562	.556	.086	2.191	2.244	.434	-.002	-.001	.133	.311	.345	.077	-5.195	-5.231	.089
DAX, 241	.565	.559	.088	2.219	2.275	.455	.013	.003	.137	.256	.279	.069	-5.782	-5.814	.086
FTSE, 241	.562	.554	.087	2.179	2.227	.445	-.002	.003	.133	.322	.339	.053	-5.547	-5.563	.045
HSI, 241	.562	.554	.084	2.204	2.253	.446	-.009	-.006	.137	.276	.304	.070	-7.057	-7.095	.091
N225, 241	.570	.562	.087	2.232	2.281	.460	-.012	-.005	.138	.255	.272	.058	-6.744	-6.767	.069
Robustness check															
SP500 monthly, 13	.606	.592	.114	1.429	1.451	.306	-.000	.000	.116	.806	.801	.064	-5.150	-5.153	.014
SP500 weekly, 13	.635	.616	.128	1.404	1.422	.309	-.011	-.004	.115	.853	.848	.063	-4.598	-4.603	.028
SP500 weekly, 49	.485	.483	.093	1.591	1.624	.292	.012	.006	.117	.606	.606	.057	-4.721	-4.722	.017
SP500 R=1.001, 61	.566	.553	.089	1.967	1.984	.368	-.004	-.004	.121	.557	.557	.044	-4.022	-4.022	.013
SP500 R=1.001, 241	.570	.557	.085	2.220	2.239	.430	.003	-.001	.132	.332	.349	.054	-4.074	-4.089	.047

Note: Results are based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 500$. Sample medians, means, and standard deviations (SD) are reported. 'LL' denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

Source: Author's own computations in *MATLAB*.

relatively weaker in terms of impact to the market price (see Equation 4.13) because the proportion of these strong trend chasers is lower than 0.5. If we consider market proportions incorrectly implied by the 2-type $\widehat{\beta}$ coefficient and related $\widehat{g_2}$ and compare it to $\widehat{fraction}$ and related $\widehat{g_2}$ estimated in this section according to Equation 4.13, we deduce almost similar impact. This confirms our suspicion about an improper specification of the model with insignificant $\widehat{\beta}$ and we corrected for this misspecification introducing $\widehat{fraction}$ specification via Equation 6.6. Evolutionary switching between strategies can be now captured via changes in the $\widehat{fraction}$ coefficient in its smooth form using the rolling approach as asserted by Teräsvirta (1994, pg. 217): "if one assumes that the agents make only dichotomous decisions or change their behaviour discretely, it is unlikely that they do this simultaneously. Thus if only an aggregated process is observed, then the regime changes in that process may be more accurately described as being smooth rather than discrete." Nonetheless, the rolling approach does not reveal any significant dynamics in the behaviour of $\widehat{fraction}$ which again only confirms the validity of full sample estimation results from Section 6.6.

6.6.2 Behaviour of the simulated log-likelihood function

Conclusions for the smoothness conditions and unique maxima presence of the simulated log-likelihood functions for the β model (see Subsection 6.4.2) hold generally identically for the *fraction* model. Here we only depict sub-log-likelihood functions in 2D assuming other parameters fixed at estimated values from Table 6.4. In Figure 6.9 we demonstrate 2D shapes of the simulated sub-function in direction of individual parameters. Differences compared to the β model are threefold:

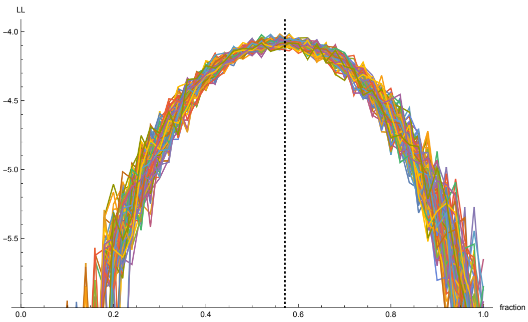
1. in the *fraction* direction [subfigures (a) and (b)] the function behaves more 'nicely';

2. in the b_2 direction [subfigures (e) and (f)] we do not observe any optimum, thus the identification of the parameter seems problematic;

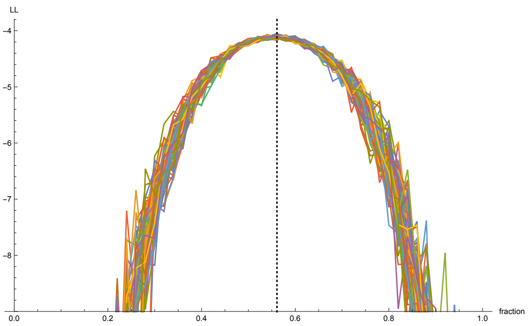
3. on the contrary, we do not longer suspect the potential upward bias for the *noise intensity* estimates [subfigures (g) and (h)] compared to Subsection 6.6.2.

Figure 6.9: Simulated sub-log-likelihood fcns. for single parameters

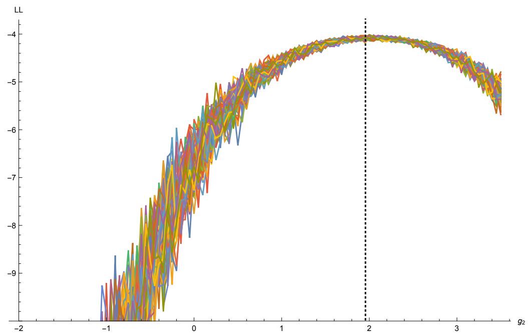
(a) *fraction*, MA61



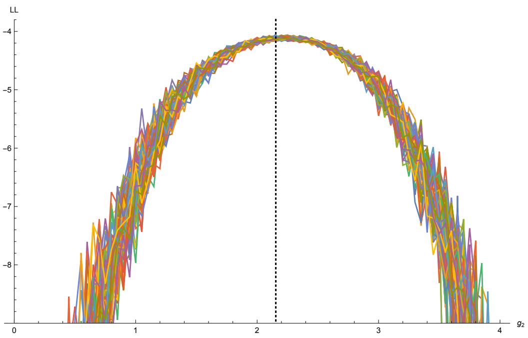
(b) *fraction*, MA241



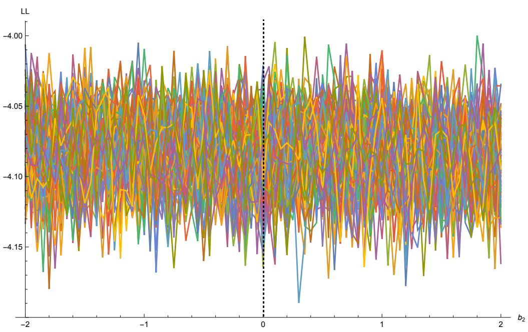
(c) g_2, MA61



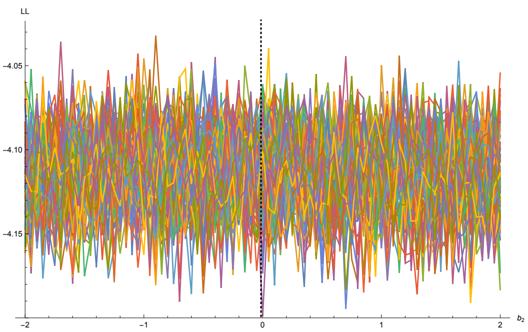
(d) g_2, MA241



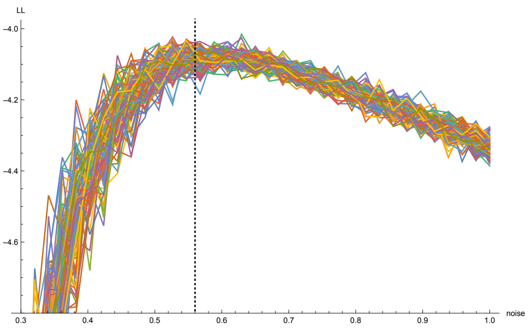
(e) b_2, MA61



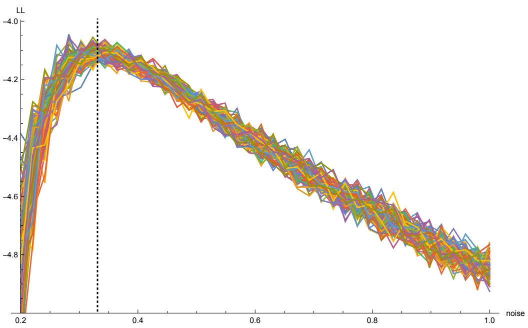
(f) b_2, MA241



(g) *noise*, MA61



(h) *noise*, MA241



Note: Results based on 100 random runs, S&P500 data, given MA fundamental price approximation, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. $\{\varepsilon_i\}_{i=1}^{N}$ drawn from normal distribution. Black dotted vertical lines depict estimated parameters (see Table 6.4).

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

6.6.3 Robustness check of the 2-type fraction model

We employ an identical (except for now irrelevant effect of memory) robustness check as in the previous case for the β model also for the *fraction* model. Results of the weekly and monthly data estimation and the model assuming higher market risk free rate are reported in Table 6.4 (bottom part). Basic conclusions for the robustness and validity check of the β model from Subsection 6.4.3 hold identically for the *fraction* model. The $\widehat{fraction}$, g_2, and $\widehat{noise\ intensity}$ generally reveal strong statistical significance, the opposite does the bias parameter $\widehat{b_2}$. Differences are again observable at the level of trend parameters $\widehat{g_2}$ and $\widehat{noise\ intensity}$ based on monthly and weekly data—results show lower $\widehat{g_2}$ and higher *noise intensity* compared to daily data and monthly data perform the worst statistical fit compared to weekly and daily dataset. These findings are once again likely to be an implied technical side-effect of small MA window.

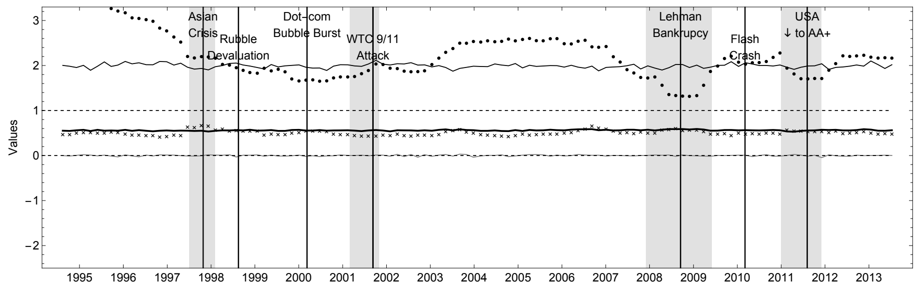
6.6.4 Rolling estimates of the 2-type fraction model

We also follow the same logic of the assessing validity of the full sample static estimates in Section 6.6 over time via rolling estimation on the entire investigated period between 1994 and 2014 with the same general setting an in the case of the β model. Results of the rolling estimation of the S&P500 are depicted in Figure 6.10. Rolling estimate results for other indices are reported in Appendix D, Figures Figure D.6, Figure D.7, Figure D.8, Figure D.9, Figure D.10. Comparing and contrasting rolling results with findings from the β model analysis in Subsection 6.5.1, at first sight we might conclude that the stability of the *fraction* model is markedly higher and we neither observe much dynamics around important crashes and turbulent periods from the stock market history nor in recessions. We also do not observe any obvious specific behaviour for individual indices.

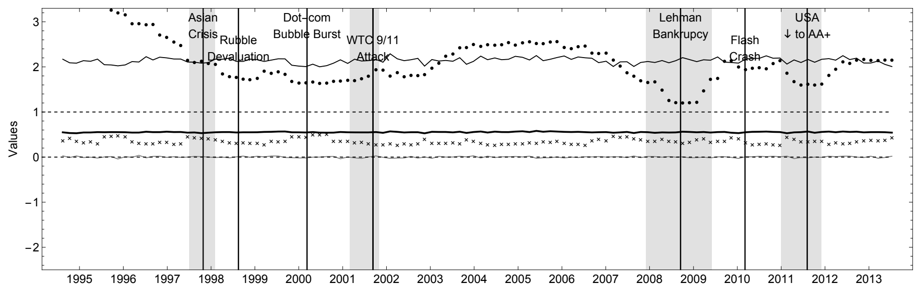
The $\widehat{fraction}$, $\widehat{g_2}$, $\widehat{noise\ intensity}$ estimates keep steadily around their long-term static estimates with high statistical significance traceable in the (c) and (d) subparts depicting rolling standard deviation. The $\widehat{b_2}$ keeps to zero and is associated with high standard deviation values over the entire data period. The fit is again relatively higher during rather tranquil periods and on the contrary it generally decreases during volatile periods. At the level of standard deviations [subparts (c) and (d)] signs of potential dynamics are detectable for the MA241 fundamental value specification which shows apparently more chaotic behaviour of standard deviations than the MA61 case. However, one can hardly detect

Figure 6.10: Rolling estimates of the 2-type *fraction* model for S&P500

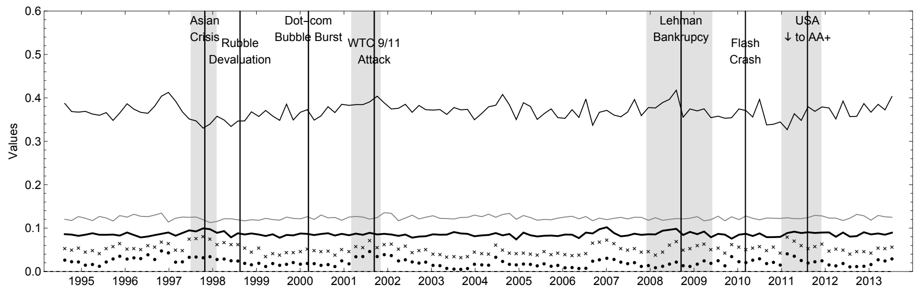
(a) MA61 fundamental price approximation



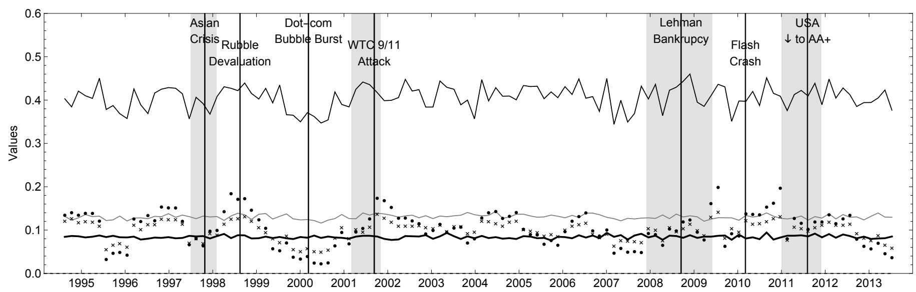
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations



Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $LL+6$ are represented by × and •, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

Source: Author's own computations in *MATLAB*.

any clear pattern in it. This is, nonetheless somewhat interesting if compared with the rolling results for the β model where larger MA241 fundamental value specification considerably stabilises captured dynamics of standard deviations. For some reason the *fraction* model behaves inversely in this respect.

Generally, the rolling approach again strongly supports the stability of the *fraction* model behaviour over the entire investigated period and therefore affirms the validity of full sample estimates from Subsection 6.6.1.

Chapter 7

Simulation-based estimation of FABMs: the case of Alfarano et al. model

In this chapter we apply the NPSMLE methodology to another simple FABM originally developed by Alfarano *et al.* (2008). The model is derived in the tradition of Kirman's ANT mechanics introduced by (Kirman 1991; 1993) that is amended by asymmetric herding towards investment strategies resulting in the IAH origin of the system. Unlike the most widely used discrete-choice multinomial logit switching rule approach (Brock & Hommes 1998, Equation 4.16) studied in previous chapters, the Alfarano *et al.* (2008) model is based on the other typical ingredient of FABMs—the herding behaviour. The concept of herding represents the second widely accepted principle of possible evolution of market fractions applied in FABMs that can trigger interesting nonlinear endogenous dynamics resulting in large aggregate price fluctuations. Therefore estimation of two models based on these two leading principles—switching and herding—is introduced in the thesis. A further motivation for the analysis and repetitious empirical testing of the Alfarano *et al.* (2008) model comes from somewhat puzzling conclusions of previous estimation attempts of similar concepts via MSM. As concluded by Chen *et al.* (2012, pg. 207), "the Lux model was rejected, similar to the rejection of the ANT model" based on its empirical validation (Winker *et al.* 2007) in favour of ABS. However, the most recent studies (Franke & Westerhoff 2012; Ghonghadze & Lux 2015) accept the model or systems based on the same origin as possible data generating processes with high p-values of the J-test (see Subsection 2.4.4).

Authors model the herding behaviour utilising the master equation approach from Physics and demonstrate that (and under what circumstances) herding mechanism gives rise to realistic time series that resemble well various stylised facts of financial data (see e.g. Figure 7.2 with illustrative examples or related descriptive statistics in Table 7.1). The model generically replicates the leptokurtic distributions of returns and volatility clustering that are directly linked to the herding component of the model which puts emphasis on "the importance of bounded rational behaviour as a potential explanation of the stylized facts" (Alfarano *et al.* 2008, pg. 125). Currently, two attempts on the empirical validation of this model have been published by Chen & Lux (2015) and Ghonghadze & Lux (2015) who estimate the model using MSM and GMM, respectively. Utilising the same dataset we estimate the Alfarano *et al.* (2008) model by NPSML.[1]

7.1 The model

In this section we briefly outline the computational design of the asset pricing model based on the work of Chen & Lux (2015). A comprehensive analytical derivation of the model can be found in the original article by Alfarano *et al.* (2008). The model assumes two distinct group of market participants—fundamentalists and noise traders. Fundamentalists make their investment decision based on the deviation from the fundamental value, i.e. buy/sell when the asset is under/over valued. The population of fundamentalists consists of N_f traders with average trading volume V_f and their excess demand is thus expressed as:

$$D^f = N_f V_f (F_t - p_t), \tag{7.1}$$

where F_t is the log fundamental value and p_t is the log price at time t. The other part of the artificial market is represented by N_c noise traders, each of them being during each period either in a positive (optimistic) state which is associated with buying V_c units of the asset, or in a negative (pessimistic) mood associated with selling the same amount. The number of optimistic noise

[1] The dataset and the *MATLAB* code of the model have been kindly provided by prof. Thomas Lux and Dr. Zhenxi Chen from the University of Kiel. However, for the purpose of this study we employ a different span of the data than Chen & Lux (2015).

traders in time t is denoted as n_t and we define the population sentiment index as:

$$x_t = \frac{2n_t}{N_c} - 1, \tag{7.2}$$

i.e. $x_t \in \langle -1, 1 \rangle$, it is equal to zero for balanced sentiment and gains positive or negative values when the majority of noise traders are optimistic or pessimistic in time t, respectively. The value of the population sentiment index directly translates into the excess demand of noise traders:

$$D^c = N_c V_c x_t. \tag{7.3}$$

The herding dynamics of the model is governed by a process of opinion changes within the population of noise traders who dynamically switch between the optimistic and pessimistic regimes. The transition probabilities are given by the Poisson intensity $a \geq 0$ inducing autonomous switches of opinion and the rate $b \geq 0$ implicate 'herding-based' switches caused by pair-wise communication among noise traders:

$$\pi_{x,t}(- \to +) = \frac{N_c - n_t}{N_c}\big(a + b\frac{n_t}{N_c}\big) = (1 - x_t)\left[\frac{2a}{N_c} + b(1 + x_t)\right]N_c^2, \tag{7.4}$$

$$\pi_{x,t}(+ \to -) = \frac{n_t}{N_c}\big(a + b\frac{N_c - n_t}{N_c}\big) = (1 + x_t)\left[\frac{2a}{N_c} + b(1 - x_t)\right]N_c^2, \tag{7.5}$$

where $\pi_{x,t}(- \to +)$ denotes the probability that a pessimistic noise trader switches to an optimistic mood and vice versa (note that the rate b needs to be multiplied by the fraction of noise traders of the opposite opinion).

Assuming Walrasian price adjustment mechanism with instantaneous market clearing, the price change is derived from the overall excess demand:

$$\frac{dp}{dt} = D^f + D^c = N_f V_f (F_t - p_t) + N_c V_c x_t. \tag{7.6}$$

The equilibrium market price is then formulated as:

$$p_t = F_t + \frac{N_c V_c}{N_f V_f} x_t. \tag{7.7}$$

The complete model in continuous-time version is finally summarized by four mutually dependent equations:

$$\begin{aligned}
dF_t &= \sigma_f dB_{1,t}, && (7.8)\\
dx_t &= -2a x_t dt + \sqrt{2b(1 - x_t^2) + \frac{4a}{N_c}} dB_{2,t}, && (7.9)\\
p_t &= F_t + \frac{N_c V_c}{N_f V_f} x_t, && (7.10)\\
r_t &= p_t - p_{t-1}, &&\\
&\equiv F_t - F_{t-1} + \frac{N_c V_c}{N_f V_f}(x_t - x_{t-1}), &&\\
&\equiv \sigma_f e_t + \frac{N_c V_c}{N_f V_f}(x_t - x_{t-1}), && (7.11)
\end{aligned}$$

where:

- the log fundamental value F_t follows a Brownian motion without drift, σ_f denotes the standard deviation of innovations of F_t;
- $B_{1,1}$ and $B_{2,t}$ stand for independent standard Wiener processes;
- r_t is the log price and log return at time t, and $e_t \sim$ i.i.d. standard normal distribution $N(0, 1)$ as the fundamental value for a unit time change associated with daily data, i.e. F_{t+1}, can be obtained by a normal distribution $N(F_t, \sigma_f)$.

The sentiment dynamics process is characterised by a mean-reverting drift. Parameter b relatively high compared to a brings about moderation of the random variation and thus results in strong majorities of optimistic or pessimistic noise traders over time, i.e. x_t generally tends to occur close to bounds of its stochastic process and the system is characterised by a bimodal distribution of the sentiment index x_t. In such case the pair-wise communication among noise traders leads the sentiment dynamics and is responsible for a persistent majority opinion. Conversely, if $a > b$, the sentiment index x_t is dominantly governed by autonomous opinion changes and therefore embodies unimodal unconditional distribution with its peak at 0, i.e. the balanced situation.

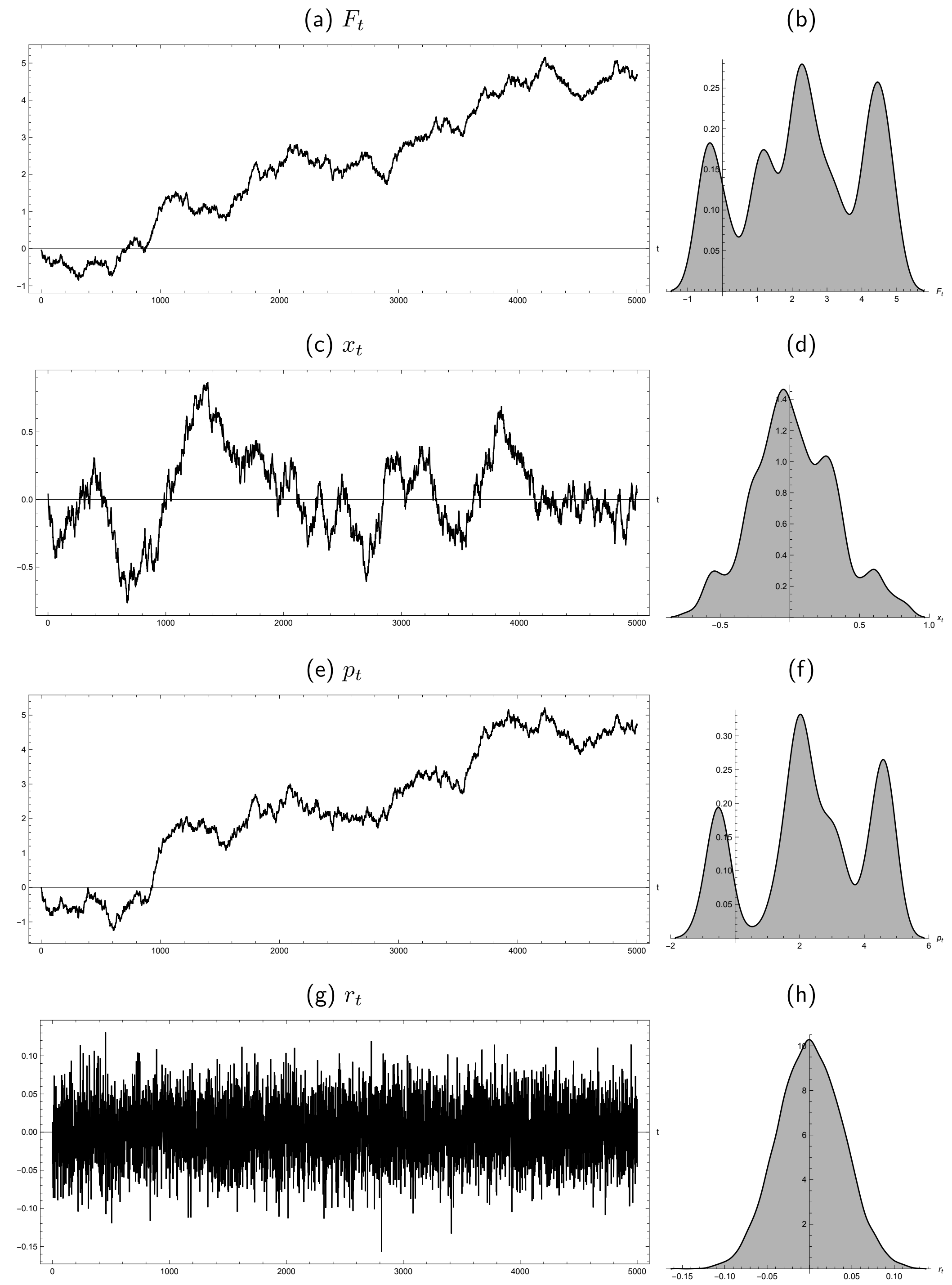
7.2 Computational setting

In our application we follow the Chen & Lux (2015) and Ghonghadze & Lux (2015) computational setting of the model. First, a moderate number of noise traders, $N_c = 100$, is assumed to reduce computational time. Second, to represent a specific unimodal and bimodal version of the model, a is fixed to 0.0014 and b to 0.0003 for the unimodal version and the reversed values, $a = 0.0003$ and $b = 0.0014$, define the bimodal setting. Third, the standard deviation of innovations of F_t is set to $\sigma_f = 0.03$ for both cases and the complete set of parameters $\theta = (\sigma_f, a, b)$ is being estimated. As argued by Chen & Lux (2015, pg. 11), "adding $\frac{N_c V_c}{N_f V_f}$ as a fourth parameter would deteriorate results by so much that the outcomes of our estimation would become almost useless". Therefore the $\frac{N_c V_c}{N_f V_f}$ term, which can, moreover, be interpreted as only a scale factor and thus omitted, is set to 1. Finally, as a reasonable technical assumption, initial values for F_t, x_t, p_t, and r_t are jointly set to 0.

In order to demonstrate differences between the unimodal and the bimodal version of the model, in Figures 7.1 and 7.2 we depict an illustrative example of model outcomes based on a single random run and the very same random seed so that the series of the log fundamental value F_t keeps identical for both cases and any difference between the unimodal and the bimodal version is solely due to the flipped setting of parameters a and b. Comparing bottom parts of Figures 7.1 and 7.2, we clearly observe strong mean reverting behaviour of the population sentiment x_t for the unimodal setting and the bimodal pattern of prevailing positive sentiment[2] for the other setting. When sub-figures (e) depicting log returns r_t are preliminarily visually compared in terms of similarity to real market returns, the bimodal version resembles the real data more accurately, for instance we observe clusters of higher volatility followed by periods of lower volatility—one of the most robust stylised fact of financial returns. Contrasting descriptive statistics of log returns r_t summarised in Table 7.1, one can detect higher extreme values and leptokurtic distribution for the bimodal version, both important symptoms of a financial-like type of distribution.

[2]The prevailing positive sentiment x_t as well as rather increasing fundamental value F_t are chance results only based on the selected random seed.

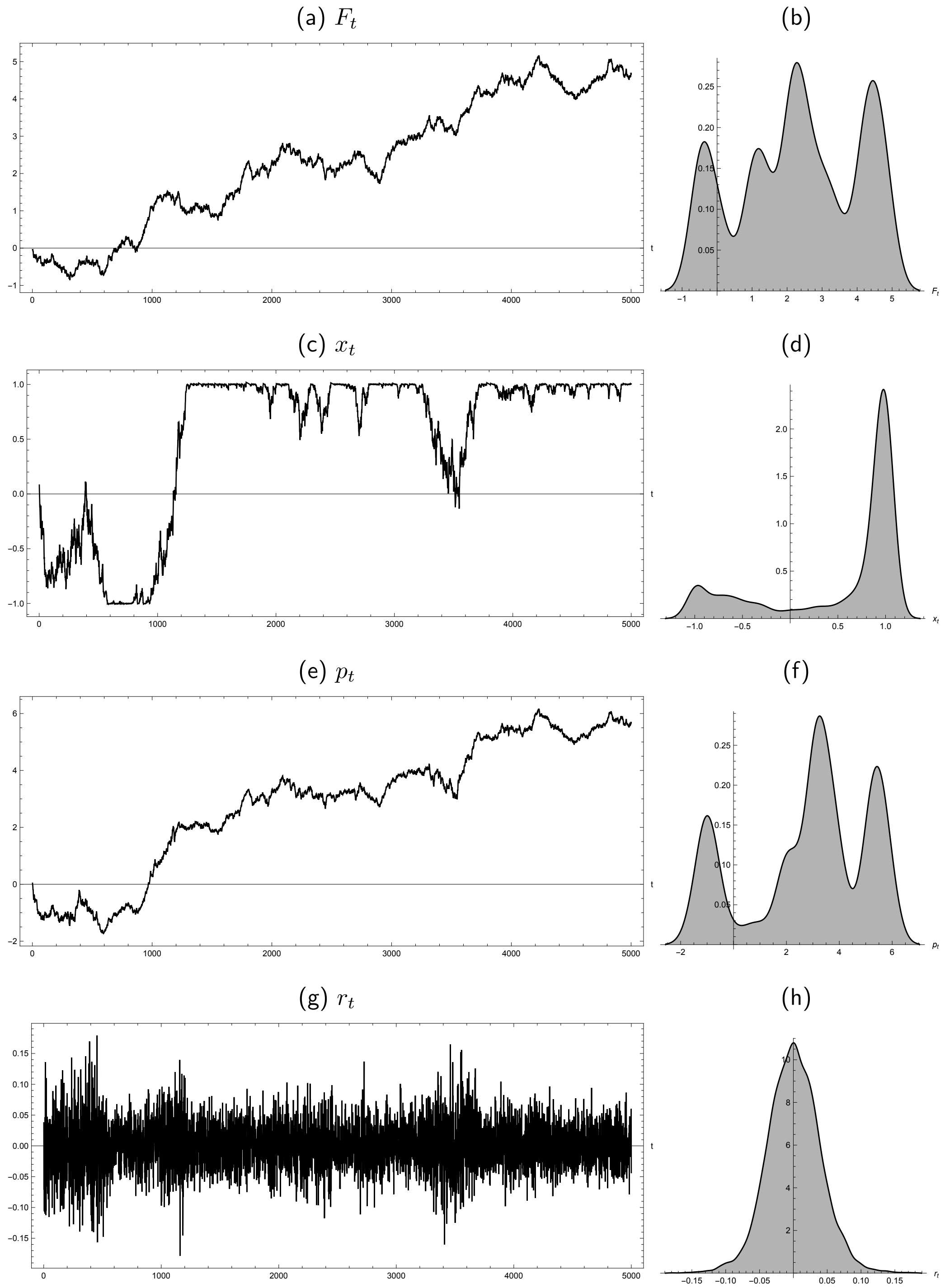
Figure 7.1: Illustrative example of model outcomes (unimodal v.)



Note: An illustrative example of simulated time series and related smooth histograms for the unimodal model version. Results based on a single random run and identical random seed as Figure 7.2, number of observations $t = 5000$. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.2: Illustrative example of model outcomes (bimodal v.)



Note: An illustrative example of simulated time series and related smooth histograms for the bimodal model version. Results based on a single random run and identical random seed as Figure 7.1, number of observations $t = 5000$. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Table 7.1: Descriptive statistics of simulated log-return r time series

Data	Mean.	Med.	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC r_t^2
unimodal r	.00094	.00061	-.16	.13	.038	-.0077	2.91	-.073	.076	-.0038	.0047
bimodal r	.0011	.00056	-.18	.18	.039	.037	3.84	-.075	.079	.0031	.12

Note: Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, and autocorrelations (AC) are reported. Results based on the identical random seed as Figure 7.1 and Figure 7.2. Figures are rounded to 2 valid decimal digits.

Source: Author's own computations in *MATLAB*.

7.3 Monte Carlo study

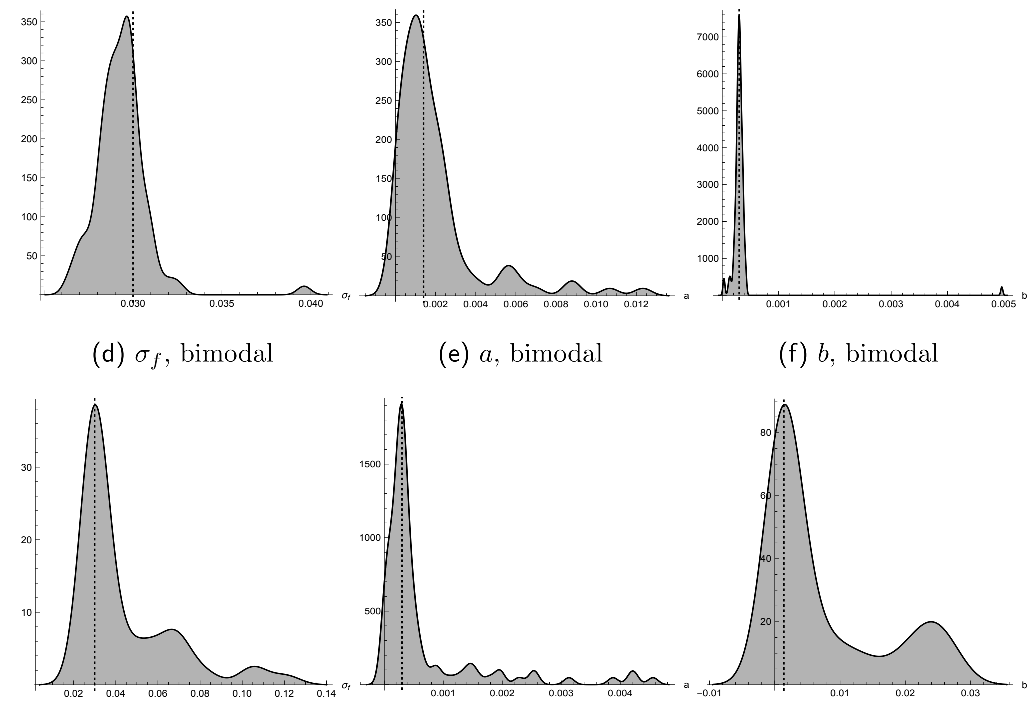
For the NPSMLE setting, we simply follow the 'best practice' from Chapter 5. Outputs of 100, 500, and 1000 random runs are compared, three levels of the kernel estimation precision, $N = 100$, $N = 500$, and $N = 1000$, are considered, and results based of five sample sizes ranging from 100 to 10000 are used for graphical depiction in Figures 7.4, 7.5, and 7.6. The Gaussian kernel and the Silverman's (1986) rule of thumb (Equation 5.4) for finding the optimal size of the bandwidth without the undersmoothing option are employed within the kernel approximation of the conditional density (Equation 4.23). Log returns are naturally used as the input of the NPSMLE procedure.

Omitting the scale factor $\frac{N_c V_c}{N_f V_f}$ (see discussion in Section 7.2) results in a system of three parameters essential for the model dynamics—the switching coefficients a and b, and the fundamental volatility σ_f. We estimate this parameter-triplet in all following sections.

Having all parameters of interest theoretically constrained by 0 from bellow, we firstly perform a computationally feasible pre-estimation step based on 100 random runs, number of observations $t = 1000$, the kernel estimation precision $N = 1000$, and with constrained parameter space set to $\langle 0, 20 \times \mathit{true\ value} \rangle$ for each parameter to gain preliminary knowledge about the approximate true value of estimated parameters. The same intervals are also used for generating single starting points drawn from the uniform distribution. Figure 7.3 depicts a very sufficient pre-estimation performance providing us with a general presumption about the magnitude of true parameters that we can utilize in the subsequent estimation step.

The algorithm to fine-tune final estimates is computationally very demanding, although based on the preliminary results we constrain the parameter space set to $\langle 0.1, 3 \times \mathit{true\ value} \rangle$. Left nonzero bound increases precision of the estimation considerably compared to zero bound and we also obtain a robust-

Figure 7.3: Pre-estimation performance



(a) σ_f, unimodal (b) a, unimodal (c) b, unimodal, const.

(d) σ_f, bimodal (e) a, bimodal (f) b, bimodal

Note: Results based on 100 random runs, number of observations $t = 1000$, and the kernel estimation precision $N = 1000$, initial point drawn from uniform distribution $U(0, 20 \times \textit{true value})$. Black dotted vertical lines depict the true values of parameters. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

ness check knowledge as a considerably off-centered parameter space is used. Again, the same intervals are used to randomly draw single starting points of the optimisation procedure from the uniform distribution.

Figures 7.4, 7.5, and 7.6 demonstrate a general overview of the estimation performance. Most importantly, we observe diametrically opposite patterns for the unimodal and bimodal setting of the model. Under the unimodal setting (left columns), the estimation reveals theoretically expected performance. For parameter σ_f we observe a very precise estimation performance even for small sample sizes with a general tendency of a tiny downward bias based on sample mean. Effects of theoretical properties of the estimator, the consistency and asymptotic efficiency, are well observable also in small samples for the model although the differences implied by increasing sample size as well as increasing kernel estimation precision are relatively small. Parameter a is associated with relatively worst estimation performance, nonetheless very satisfactory for larger samples, $t = \{5000, 10000\}$, and high kernel estimation precision $N = 1000$. For smaller samples, $t = \{100, 1000\}$, a persisting upward bias of the sample mean is caused by bounds of the parameter space. The exemplary performance is demonstrated for parameter b that is estimated very precisely with only a tiny downward bias based on sample mean for the smallest sample size $t = 100$. We observe a considerable effect of the sample length increase for the performance of the estimator, but on the other hand the effect the kernel estimation precision increase seems negligible—even the smallest option $N = 100$ performs almost the same as $N = 1000$. Moreover, we do not see any considerable impact of the number of random run—results based on 100 runs are almost identical to results based on 1000 runs. Table 7.2 and Figure 7.7 partially attempt to compare and contrast the estimation performance for both model versions quantitatively and depicts selected descriptive statistics of samples of estimated coefficients and related smooth histograms for 1000 random runs, the kernel estimation precision $N = 1000$, and two distinct numbers of observation, $t = \{500, 5000\}$. An important finding arising from Table 7.2 and Figure 7.7 is that sample medians generally provide considerably better estimates of the true parameters than sample means.

Under the bimodal setting (right columns) almost nothing holds from the description of the unimodal version. At first glance the estimation procedure reveals a pathological performance demonstrated by decreasing efficiency and increasing upward bias of estimates of σ_f and b with increasing sample size, markedly less clear effect of the kernel estimation precision for σ_f and a and a puzzling behaviour of parameter a estimation for larger sample sizes $t = \{1000, 5000, 10000\}$.

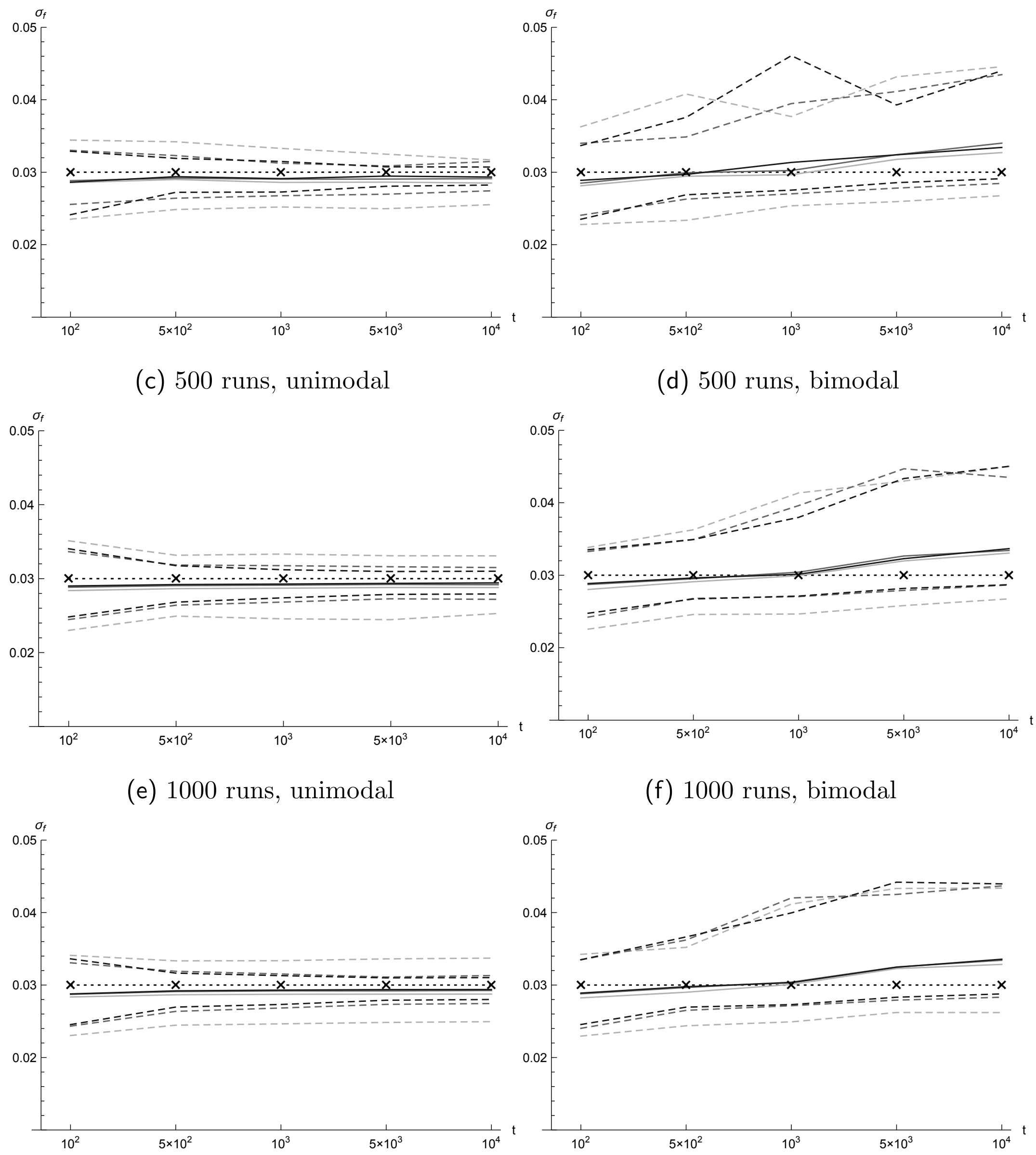
These somewhat strange outcomes of the estimation procedure are, however, well explainable by the nature of the bimodal version of the model. The bimodality of x_t directly translates into behaviour of r_t via Equation 7.11. This perhaps accounts for favourable aggregate behaviour of the time series in terms of resembling the 'big picture' of financial stylised facts, but on the other hand it is also likely to generate a specific pathological dynamics at the detailed inter-period level causing further problems with kernel estimation and parameter identification. This constitutes a serious challenge for the estimation method and amplifies estimation inaccuracies and divergence with increasing length of the time series. As Chen & Lux (2015, pg. 11) point out, even for the simultaneous estimation of three parameters only, "we do have to cope with

Figure 7.4: Simulation results of σ_f estimation

(a) 100 runs, unimodal (b) 100 runs, bimodal

(c) 500 runs, unimodal (d) 500 runs, bimodal

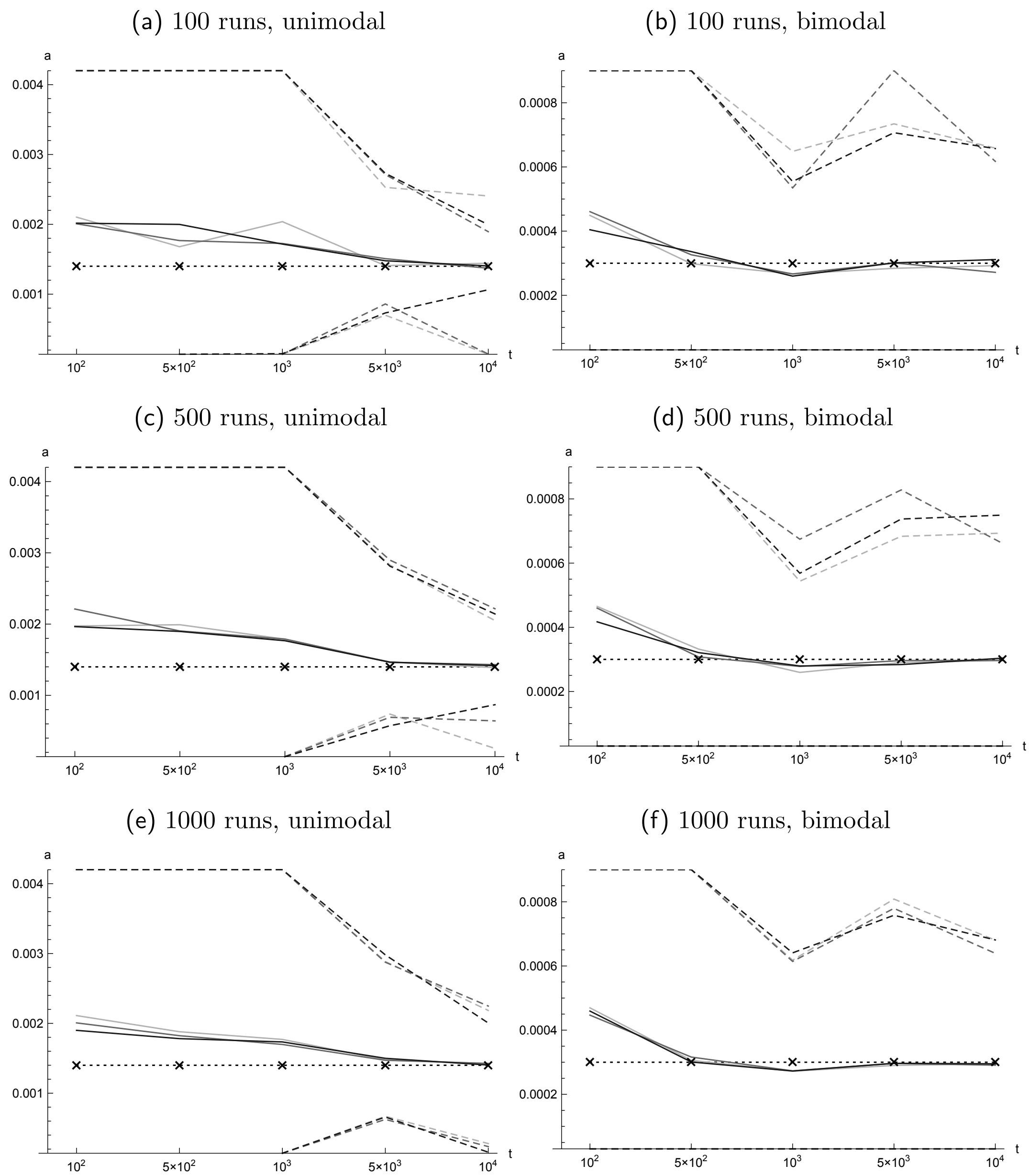
(e) 1000 runs, unimodal (f) 1000 runs, bimodal



Note: Black dotted lines with × depict the true parameter σ_f. Grey full lines depict sample means of estimated $\widehat{\sigma_f}$. Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for $N = 100$, normal grey for $N = 500$, and dark grey for $N = 1000$. ‘t’ (horizontal axis) stands for the length of generated time series.

Source: Author’s own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.5: Simulation results of a estimation

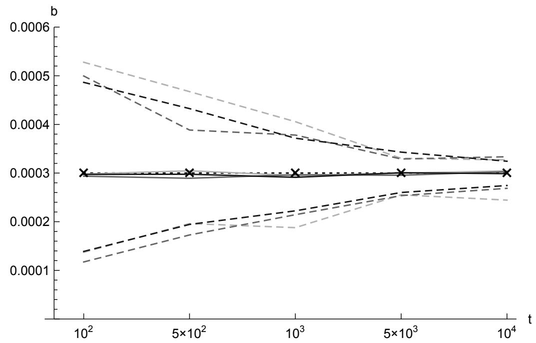


Note: Black dotted lines with × depict the true parameter a. Grey full lines depict sample means of estimated $\widehat{a}$. Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for $N = 100$, normal grey for $N = 500$, and dark grey for $N = 1000$. 't' (horizontal axis) stands for the length of generated time series.

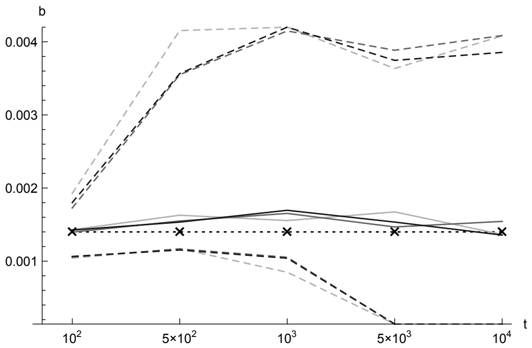
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.6: Simulation results of b estimation

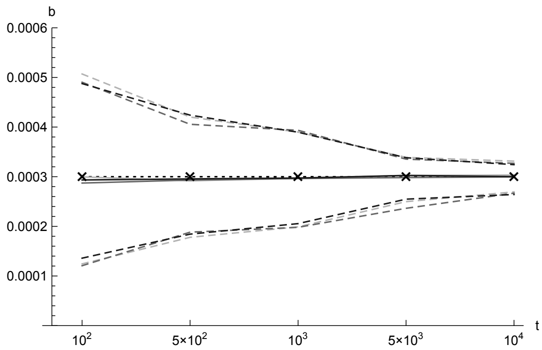
(a) 100 runs, unimodal



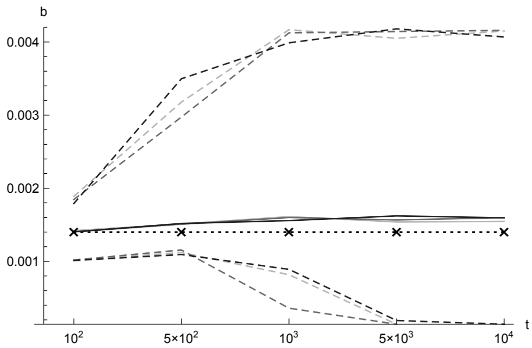
(b) 100 runs, bimodal



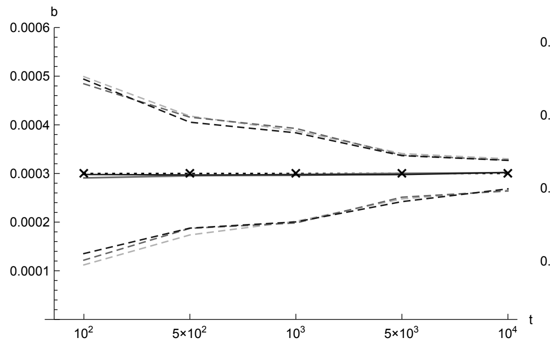
(c) 500 runs, unimodal



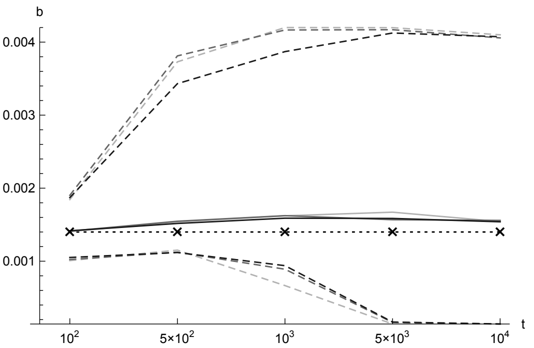
(d) 500 runs, bimodal



(e) 1000 runs, unimodal



(f) 1000 runs, bimodal



Note: Black dotted lines with $\times$ depict the true parameter b. Grey full lines depict sample means of estimated $\widehat{b}$. Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for $N = 100$, normal grey for $N = 500$, and dark grey for $N = 1000$. 't' (horizontal axis) stands for the length of generated time series.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Table 7.2: Quantitative results

	(a) unimodal setting, $t = 500$						(b) bimodal setting, $t = 500$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
$\widehat{\sigma_f} \times 10^3$	29.2	29.2	1.21	27.0	31.7	7%	29.5	29.8	2.19	27.0	36.6	6%
$\widehat{a} \times 10^3$	1.29	1.78	1.54	.14	4.20	7%	.27	.30	.23	.03	.90	6%
$\widehat{b} \times 10^3$	.30	.30	.06	.19	.41	7%	1.41	1.52	.50	1.12	3.43	6%
	(c) unimodal setting, $t = 5000$						(d) bimodal setting, $t = 5000$					
$\widehat{\sigma_f} \times 10^3$	29.3	29.4	.81	27.9	31.0	5%	31.1	32.5	4.01	28.3	44.2	14%
$\widehat{a} \times 10^3$	1.41	1.50	.54	.66	2.98	5%	.29	.30	.16	.03	.76	14%
$\widehat{b} \times 10^3$	.30	.30	.02	.24	.34	5%	1.39	1.59	.91	.17	4.13	14%

Note: Each sample is based on 1000 random runs, the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are multiplied by 1000 for better legibility and rounded to 1 or 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

the issue of weak identification due to high correlations of our parameters". Results of the NPSMLE simulation study indicate that the method is well able to cope with problematic identification of parameters for the unimodal case, however, it is likely to experience similar difficulties as the MSM for the bimodal version.

7.3.1 Behaviour of the simulated log-likelihood function

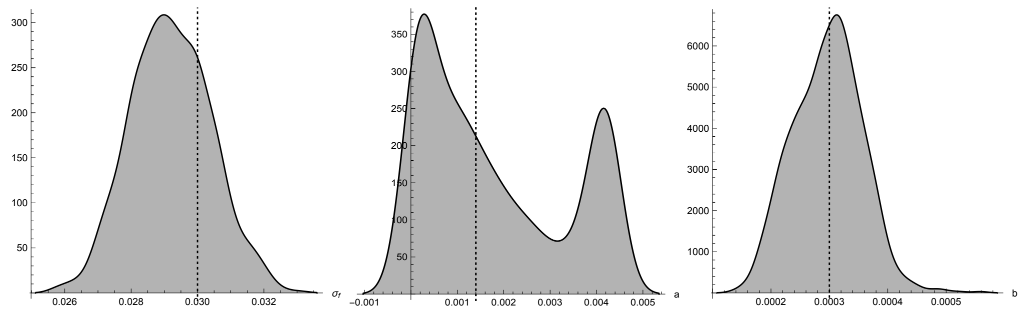
For simulation based verification of the smoothness conditions and identification of parameters for both versions of the model, we depicts shapes of simulated sub-log-likelihood functions in 2D and 3D in Figures 7.8 and 7.9.

Figure 7.8 provides the most artificial view as we assume the knowledge of two other parameters and depict only a 'slice' view of the likelihood function in direction of the third parameter. However, even based on such limited information we observe crucial differences between the unimodal and the bimodal version of the model. It is important to stress that Figure 7.8 is based on one specific setting, namely 100 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$, i.e. it especially cannot reveal the effect of sample size increase which seem crucial for estimation performance of the bimodal version of the model. Despite this, it still captures the situation when the unimodal setting produces very precise results and on the contrary the bimodal setting demonstrates unsatisfactory results.

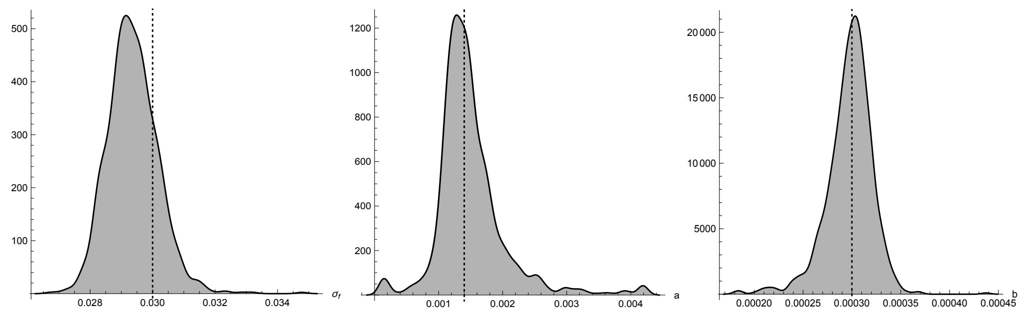
For parameter σ_f we observe an excellently smooth shape of all 100 randomly generated functions over the entire domain for both model versions. An

Figure 7.7: Smooth histograms of estimated simulated parameters

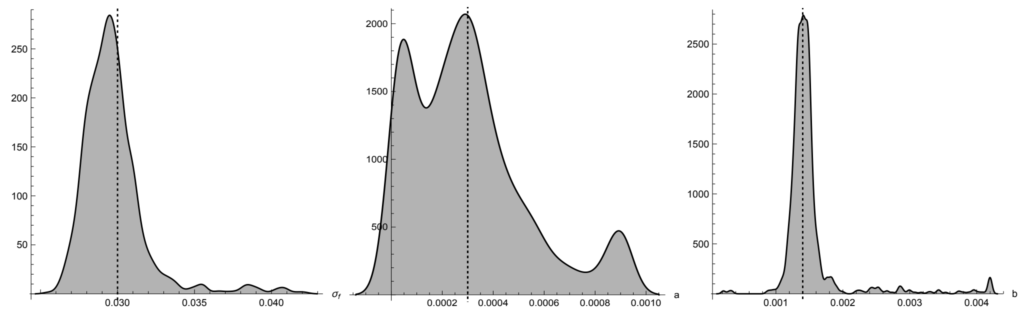
(a) $\widehat{\sigma_f}$, unimodal, $t = 500$ (b) $\widehat{a}$, unimodal, $t = 500$ (c) $\widehat{b}$, unimodal, $t = 500$



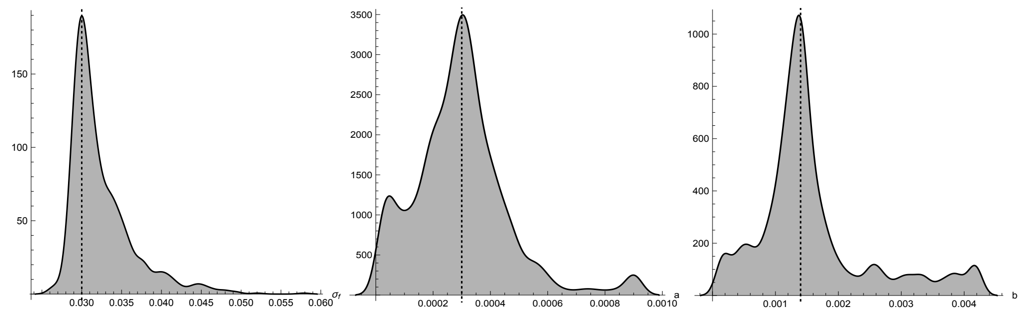
(d) $\widehat{\sigma_f}$, unimodal, $t = 5000$ (e) $\widehat{a}$, unimodal, $t = 5000$ (f) $\widehat{b}$, unimodal, $t = 5000$



(g) $\widehat{\sigma_f}$, bimodal, $t = 500$ (h) $\widehat{a}$, bimodal, $t = 500$ (i) $\widehat{b}$, bimodal, $t = 500$



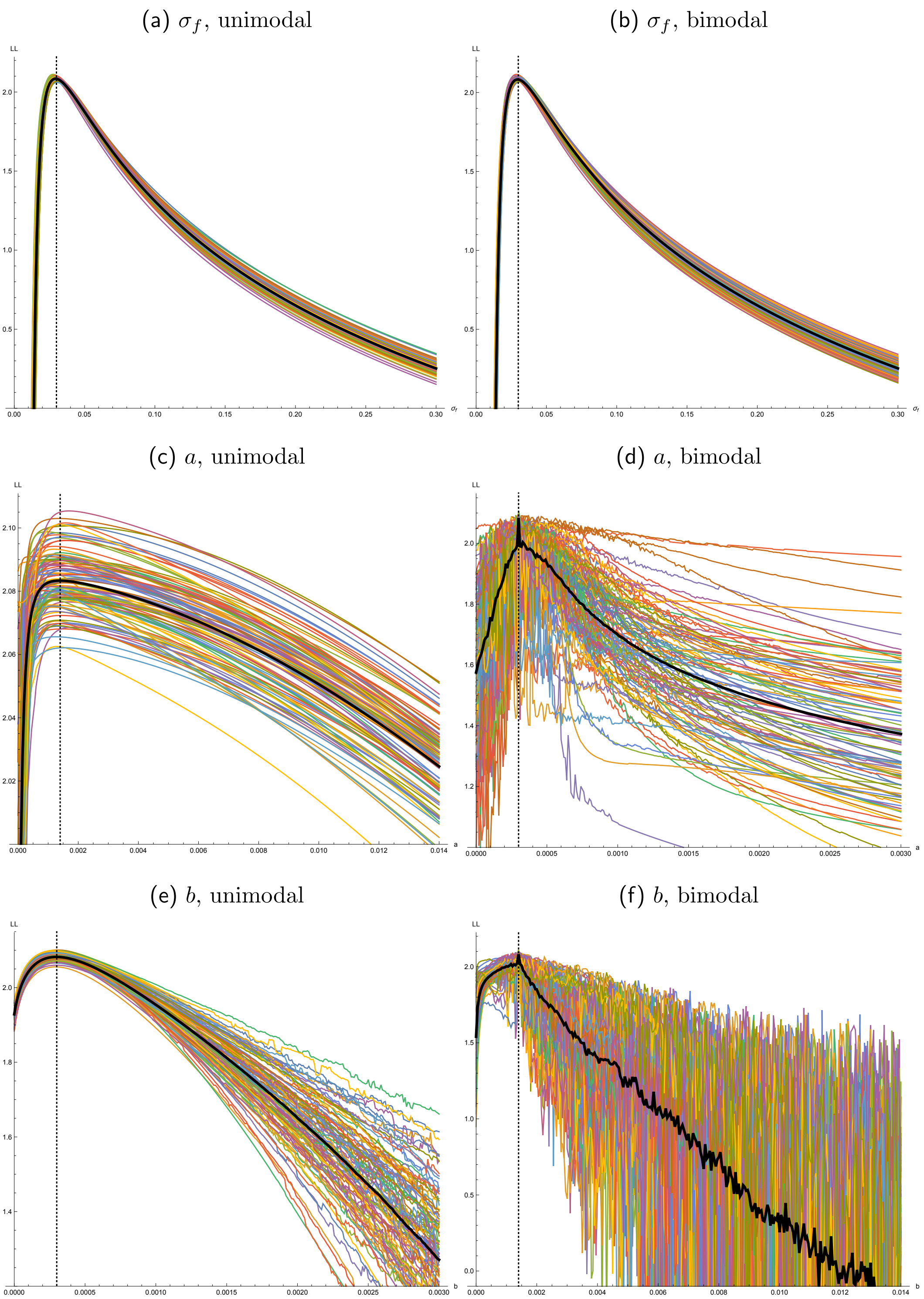
(j) $\widehat{\sigma_f}$, bimodal, $t = 5000$ (k) $\widehat{a}$, bimodal, $t = 5000$ (l) $\widehat{b}$, bimodal, $t = 5000$



Note: *Note:* Results based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Black dotted vertical lines depict the true values of parameters. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

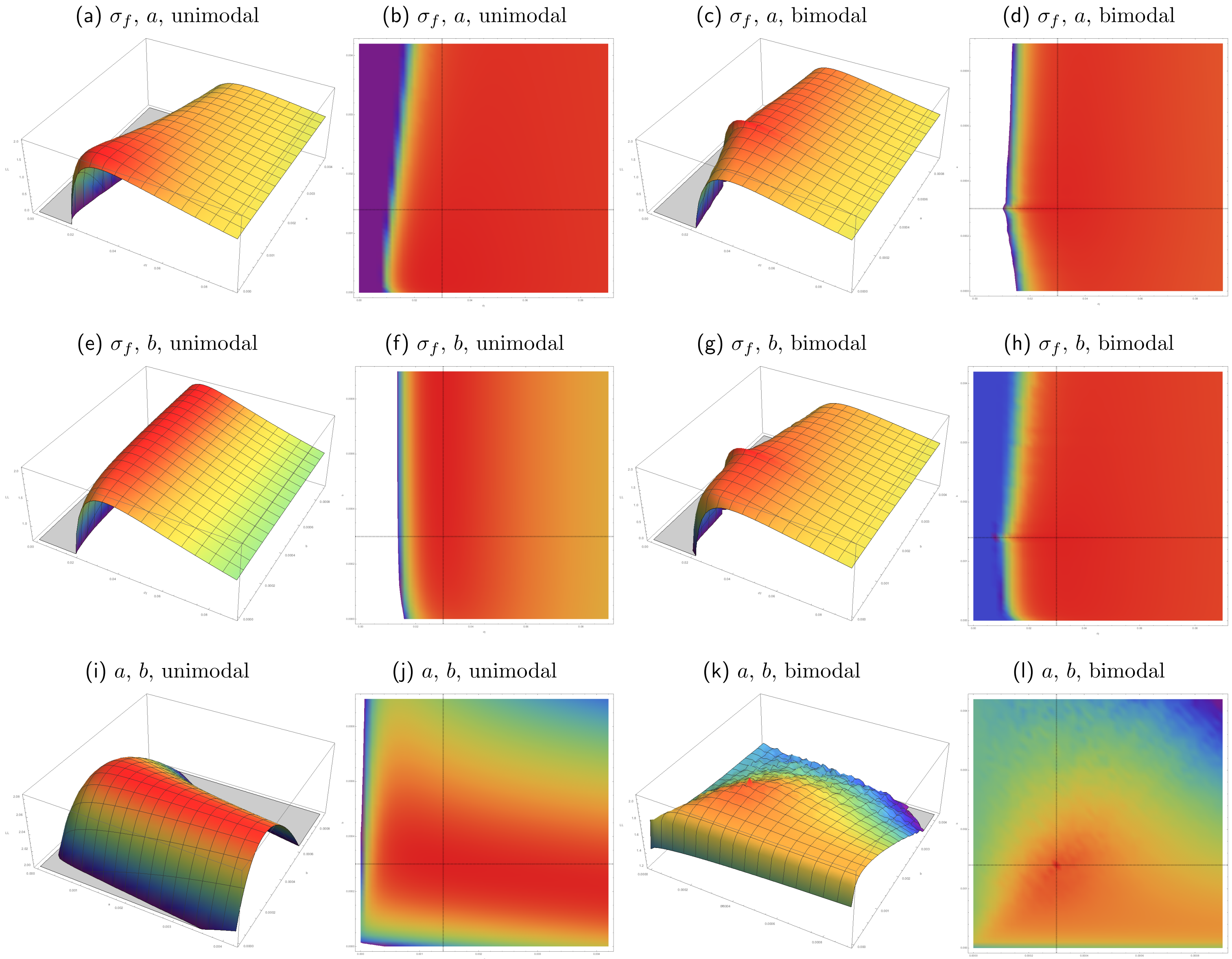
Figure 7.8: Simulated sub-log-likelihood functions



Note: Results based on 100 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Black dotted vertical lines depict the true values of parameters. Bold black full lines depict sample averages.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.9: Simulated sub-log-likelihood functions in 3D



Note: Results averaged over 100 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Black dotted lines in horizontal projections depict the true values of parameters.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

obvious unique maximum is shared over all 100 random runs. This is nothing surprising as based on knowledge gained from Figure 7.4, the estimation performance for σ_f is generally strong, notwithstanding lower for the bimodal setting compared to the unimodal case. The same can be concluded for the unimodal sub-log-likelihood function for parameter b [subpart (e)]. Although at the right side of the domain a bit rugged shape can be observed, around the true value and also within the entire constrained interval $\langle 0.1, 3 \times \textit{true value} \rangle$ the shape is absolutely smooth allowing the optimisation procedure to find the unique maximum easily. A more problematic situation can be seen in subpart (c) for unimodal parameter a. The set of 100 randomly generated functions are generally very nice behaving in terms of smoothness but they do not share a unique maximum. On average the maximum can be detected, but the function is relatively flat around the maxima region. Together, these two problematic features explain well the relative weak performance of the estimation procedure in Figure 7.5. However, for the a and b parameters in the bimodal setting [subparts (d) and (f)] the shapes of the sub-log-likelihood functions are very problematic w.r.t. both the smoothness conditions and identification of parameters. Although on average the true values seem to be detectable, an extremely rough shape of individual realisations in all 100 random runs implies substantial difficulties for the optimisation procedure and is also likely a consequence of a violation of the regularity conditions regarding either data generating process or its associated conditional density. This important finding is comparable to conclusions of Chen & Lux (2015, pg. 16) who report serious issues related to multiple local minima and a very rugged surface of the objective function, further embarrassing standard methods of optimisation search. Our suspicion is concurrently supported by the problematic performance of the estimation procedure revealed in Figures 7.4, 7.5, and 7.6. 3D depictions and horizontal projection in Figure 7.9 sketch in the smoothness of the the log-likelihood functions in the unimodal case with clear maxima regions. Conversely, we draw attention to the subpart (k & l) demonstrating the bimodal case where a very rough surface of the sub-log-likelihood is demonstrated. Multiple local minima and maxima thus impose a serious challenge for the estimation method. Although a clear global maximum is well observable in the horizontal projection, we need to stress that given depiction is not an outcome of any estimation process, but it is solely of a simulation origin, i.e. all combination of axes of the parameter space were simply input and the function was depicted.

To sum up, based on shapes of the simulated log-likelihood functions we

Table 7.3: Descriptive statistics of empirical log-return r time series

Data	Mean.	Med.	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC r_t^2
SP500	.00032	.0003	-.230	.11	.011	-1.18	30.59	-.022	.022	-0.031	0.14
DAX	.00034	.0004	-.140	.11	.013	-.32	9.98	-.027	.025	-0.001	0.19
NIKKEI 225	.00011	0	-.160	.13	.013	-.33	12.42	-.028	.027	-0.014	0.20
GOLD	.00009	0	-.180	.12	.012	-.46	18.75	-.025	.023	-0.033	0.15
USD/YEN	.00007	0	-.058	.08	.007	.42	9.89	-.013	.015	-0.027	0.15
USD/EUR	-.00001	0	-.038	.05	.006	.18	5.61	-.013	.013	0.015	0.04
EUR/CHF	.00012	0	-.079	.13	.005	5.18	201.5	-.008	.009	0.140	0.25

Note: Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, and autocorrelations (AC) are reported. Figures are rounded to 2 valid decimal digits.

Source: Author's own computations in *MATLAB*.

assume that the regularity conditions are met for the unimodal version of the model, but not for the bimodal case. The detection of true coefficients is expected very challenging for a and b parameters in the bimodal case, for which also the assumption ensuring the consistency and asymptotic efficiency of the estimator are likely not met.

7.4 Empirical estimation

7.4.1 Data description and estimation setting

We estimate the Alfarano *et al.* (2008) model utilising a cross section of three stock markets: S&P500, DAX, and NIKKEI 225, three exchange rates: USD/EUR, USD/JY, and EUR/CHF, and the price of the troy ounce of gold, covering periods 1980/01/01 to 2015/02/24 for stock indices and gold, 1999/01/01 to 2015/02/24 for USD/EUR, 1986/01/02 to 2015/02/24 for USD/JY, and finally 2003/07/15 to 2015/02/24 for EUR/CHF. All data has been retrieved from DataStream. Based on the design of the theoretical model, log returns are used as a time series input of the NPSML method. Table 7.3 offers descriptive statistics of the dataset. When simulation-based statistics from Table 7.1 are compared to those empirical data, noticeable similarities can be found, especially between simulated data and stock market indices/gold. The main differences appear in higher standard deviations and lower kurtosis produced by the model.

To set the empirical estimation algorithm, we follow the 'best practice' from Chapter 6 and compute the full sample static estimates to reveal robust average model behaviour as well as rolling estimates to capture possible dynamics

Table 7.4: Empirical estimation results

Data	(a) $\widehat{\sigma_f}$			(b) $\widehat{a}$		
	Med.	Mean	SD	Med.	Mean	SD
SP500	.0164	.0165	.0013	.000113	.000135	.000111
DAX	.0165	.0167	.0015	.000084	.000110	.000103
NIKKEI 225	.0169	.0171	.0014	.000097	.000122	.000108
GOLD	.0162	.0164	.0015	.000107	.000127	.000111
USD/JY	.0092	.0096	.0019	.000082	.000107	.000108
USD/EUR	.0079	.0083	.0021	.000087	.000107	.000110
EUR/CHF	.0106	.0111	.0022	.000104	.000120	.000116
	(c) $\widehat{b}$			(d) LL		
SP500	.0000023	.0000110	.0000328	2.819	2.815	.041
DAX	.0000008	.0000080	.0000317	2.762	2.758	.042
NIKKEI 225	.0000006	.0000043	.0000199	2.727	2.726	.034
GOLD	.0000007	.0000052	.0000218	2.791	2.788	.042
USD/JY	.0000013	.0000061	.0000252	3.357	3.335	.108
USD/EUR	.0000013	.0000046	.0000223	3.513	3.480	.136
EUR/CHF	.0000011	.0000034	.0000170	3.252	3.232	.102

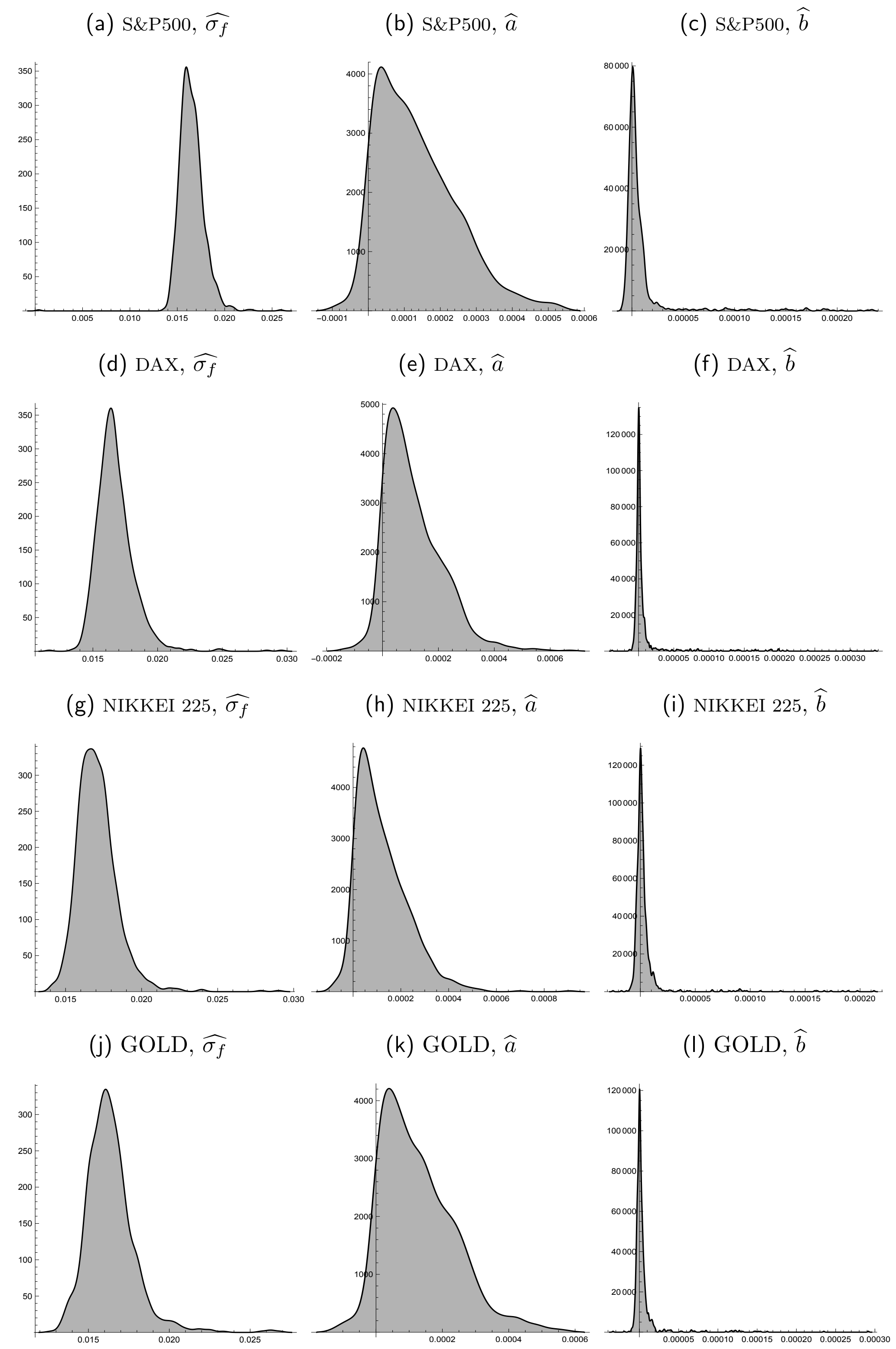
Note: Results are based on 1000 random runs, the kernel estimation precision $N = 500$. Sample medians, means, and standard deviations (SD) are reported. 'LL' denotes log-likelihoods of estimated models representing statistical fits.

Source: Author's own computations in *MATLAB*.

of the model specification in time. To cope with computational burden of the estimation procedure, for the full sample estimates we report results based on 1000 random runs and the kernel approximation precision $N = 500$. The number of starting points for the numerical optimisation is again set to 8. This on the one hand crucially increases the strength of the estimation algorithm, but on the other hand it sacrifices the parallel algorithm to speed up the computation. Moreover, it is the length of the series that generally imposes the highest computational burden on the estimation procedure. To support the numerical stability of the estimated system, we constrain the intervals for the starting points random generation to $\langle 0, 0.4\rangle$ for σ_f and $\langle 0, 0.0002\rangle$ for a and b. The empirical optimisation algorithm, unlike the algorithm for simulations, is however redesigned to the unconstrained version so that it can freely leave bounds for initial conditions during the optimisation procedure.

For the rolling analysis, we estimate the model on one year (240 days) rolling samples with two-month steps (40 days) and display results based on 100 random runs, 4 initial points for the numerical optimisation, and kernel estimation precision $N = 500$. Based on the simulation study in Section 7.3, the one year still period represents relatively sufficient length for the estimation. Moreover, it well allows for detection of possible structural dynamics in the data.

Figure 7.10: Smooth histograms of estimated empirical parameters

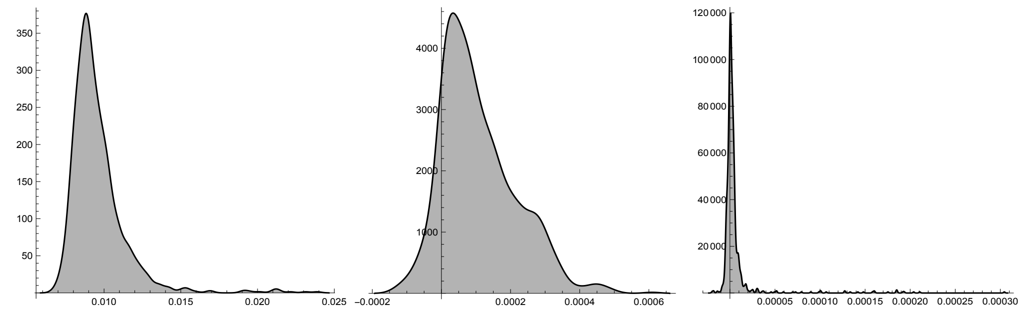


Note: Note: Results based on the same 1000 random runs as Table 7.4. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

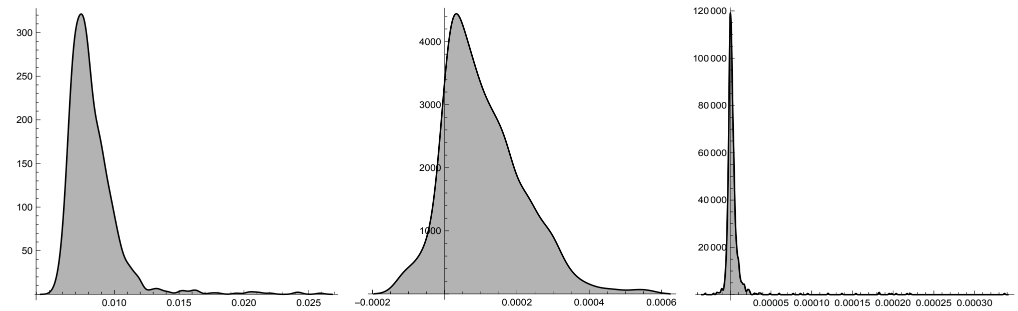
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.11: Smooth histograms of estimated empirical parameters

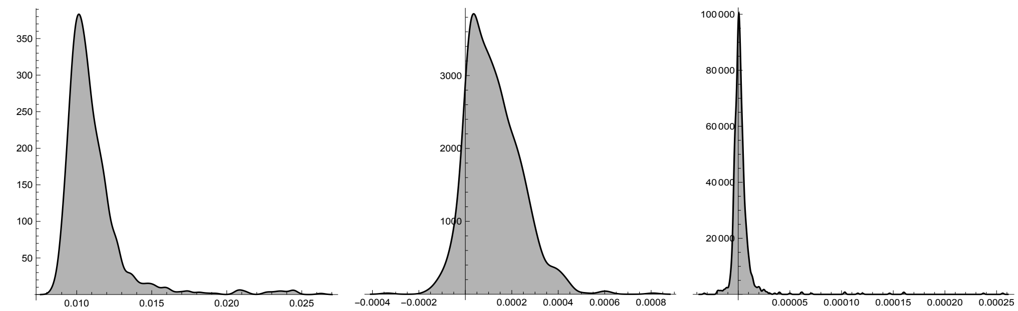
(a) U.S./JY, $\widehat{\sigma_f}$ (b) U.S./JY, $\widehat{a}$ (c) U.S./JY, $\widehat{b}$



(d) U.S./EUR, $\widehat{\sigma_f}$ (e) U.S./EUR, $\widehat{a}$ (f) U.S./EUR, $\widehat{b}$



(g) EUR/CHF, $\widehat{\sigma_f}$ (h) EUR/CHF, $\widehat{a}$ (i) EUR/CHF, $\widehat{b}$



Note: *Note:* Results based on the same 1000 random runs as Table 7.4. Produced using automatic `SmoothHistogram` kernel approximation function in *Wolfram Mathematica*.

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

7.4.2 Full sample static estimates

The full sample static estimation results are depicted in Table 7.4. Sample medians are mainly considered for the interpretation in the following text as they generally (based on findings of the Monte Carlo analysis in Section 7.3) provide considerably better estimates than sample means. In all cases, the volatility of the fundamental value $\widehat{\sigma_f}$ is estimated clearly statistically significant, parameter $\widehat{a}$ is by one or two digit places higher than $\widehat{b}$ but at the first glance statistically insignificant, and parameter $\widehat{b}$ is at the first glance statistically strongly insignificant.

However, it is important to highlight that for $\widehat{a}$ the vast majority of the sample estimates are positive, i.e. the 'illusive' statistical insignificance does not have its origin in negative estimates. The similar principle holds for $\widehat{b}$ in which case, nonetheless, the large standard deviations are predominantly caused by positive outliers but a relative high proportion (although still a minority) of sample estimates are negative. Related smooth histograms of estimated empirical parameters based on the same 1000 random runs as Table 7.4 are depicted in Tables 7.10 and 7.11. $\widehat{a} > \widehat{b}$ indicates globally unimodal distribution of the sentiment variable accompanied by a slow process of the opinion change, tendency to fluctuate around the mean value, and most importantly theoretically expected performance of the estimator based on findings from Section 7.3.

In a more detailed inspection, we can basically distinguish between the interpretation of the stock/commodity market results and FOREX based estimates. For stock market indices and gold the magnitude of $\widehat{\sigma_f}$ fluctuates between 0.016 and 0.017. For FOREX data we observe smaller magnitudes. While the autonomous switching rate $\widehat{a}$ estimates are generally comparable, some difference appears when contrasting the 'herding-based' switching coefficients $\widehat{b}$. Stock/commodity market $\widehat{b}$ estimates are generally (with an exception of the S&P500 that has the highest $\widehat{b}$) lower then currency $\widehat{b}$ estimates. An irrationally impetuous herding behaviour leading to locally extremely enhanced volatility, bubbles, and crashes is thus surprisingly found a bit stronger for FOREX markets. However, distinctions might be also to some extent caused by a different structure of the data—while log returns of stocks and gold are based on prices, FOREX log returns are derived from exchange rates. The upward bias tendency of sample means compared to sample medians observable especially for $\widehat{a}$ and $\widehat{b}$ illustrates majorities of positive sample estimates as well as the existence of positive outliers for $\widehat{b}$. The log-likelihoods embody very low standard deviations reflecting consistent optimisation performance over all random runs.

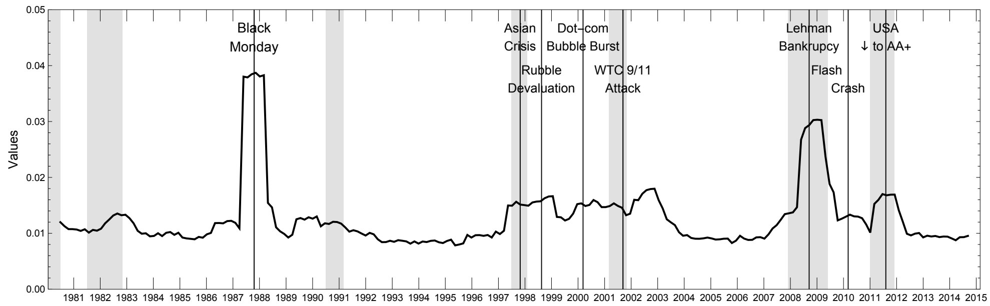
7.4.3 Rolling estimates

Rolling estimates reveals interesting dynamics of the model in time. A stock market examples, S&P500 and NIKKEI 225, and a currency example, USD/JY are depicted in Figures 7.12, 7.13, 7.14. Rolling estimates for DAX, GOLD, U.S./EUR, and EUR/CHF are depicted in Appendix E in Figures E.1, E.2, E.3, and E.4, respectively. While it seems very difficult for the model to detect signs of a herding behaviour when exposed to the complete data samples, rolling window estimation clearly captures jumps in the 'herding-based' switching parameter $\widehat{b}$ associated with some turbulent market periods and market crashes. We also observe elevated fundamental volatility coefficient $\widehat{\sigma_f}$ (notice e.g. the period around the Rubble 1998 devaluation for USD/JY). Rolling behaviour of the autonomous switching parameter $\widehat{a}$ does not reveal any interesting pattern, only a weak negative correlation with $\widehat{b}$ for stock markets in several most volatile periods (e.g. the Black Monday).

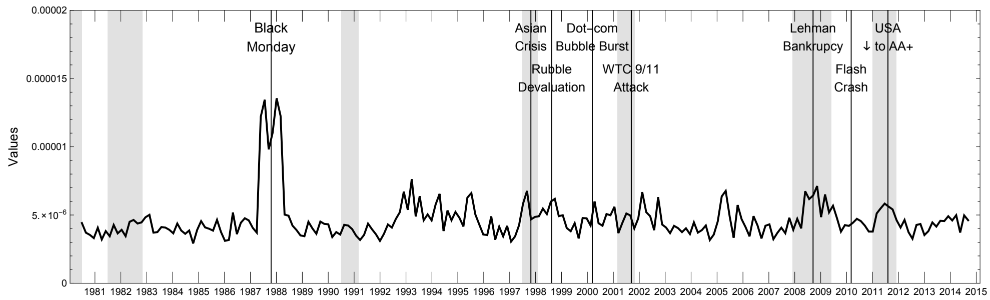
To interpret these results correctly, we need to discuss the nature of volatility in the model. The total volatility of the model output p_t is derived from so called fundamental value F_t and the effect of market sentiment x_t (see Equation 7.10). However, the fundamental volatility in this highly stylised simple model cannot be fully interpreted as the real world fundamental risk. When bringing the model to empirical data, the fundamental volatility term to a large extent represents all the remaining volatility that is not caused as the effect of noise traders' switching between the optimistic and pessimistic mood. E.g. based on rolling estimation results, the estimation method seems to predominantly assign the cause of the elevated market volatility to the fundamental value term, although e.g. Black Monday can be hardly denoted as a fundamental event. However, we also observe jumps in the estimates of the herding intensity b that definitely have a good economic interpretation for selected historical events. The NPSMLE in combination with a very simple stylised model are thus perhaps weak in distinguishing well between these two theoretical sources of volatility. Comparing the shapes of the simulated sub-log-likelihood-functions in Figure 7.8 and Figure 7.9 in dimensions of individual model parameters and their combinations, we clearly observe relative flatness of the resulting log-likelihood-function in dimensions of a and b w.r.t. the high-pitched shape in the dimension of σ_f. Thus, the market volatility amplification is likely to be

Figure 7.12: Rolling estimates for S&P500

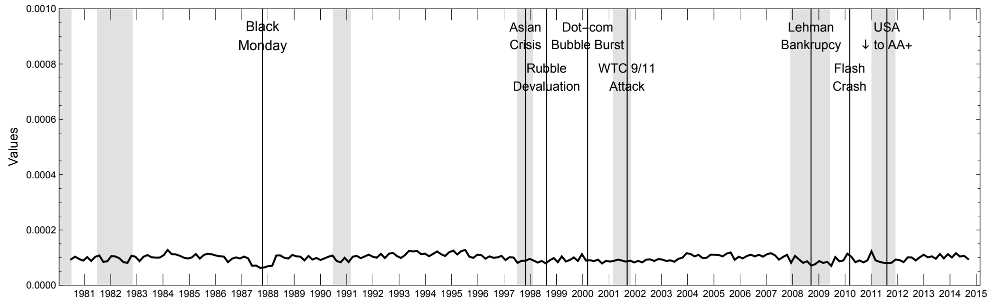
(a) $\widehat{\sigma_f},\ meanSD = .0027$



(b) $\widehat{b},\ meanSD = .0000132$



(c) $\widehat{a},\ meanSD = .000088$

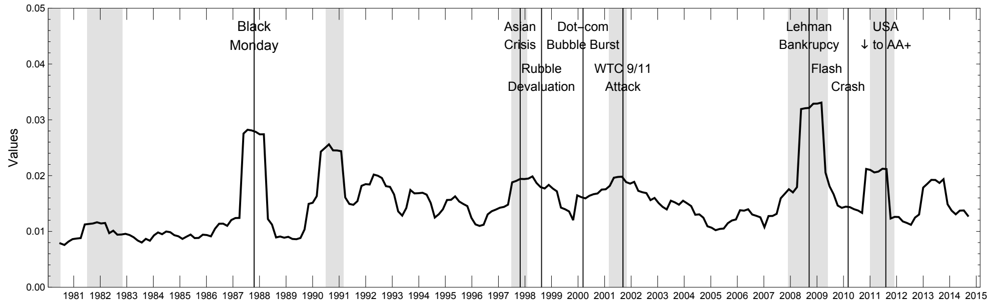


Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

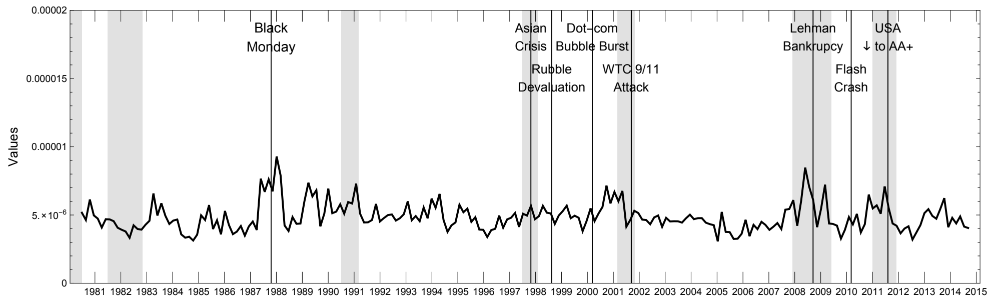
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.13: Rolling estimates for NIKKEI 225

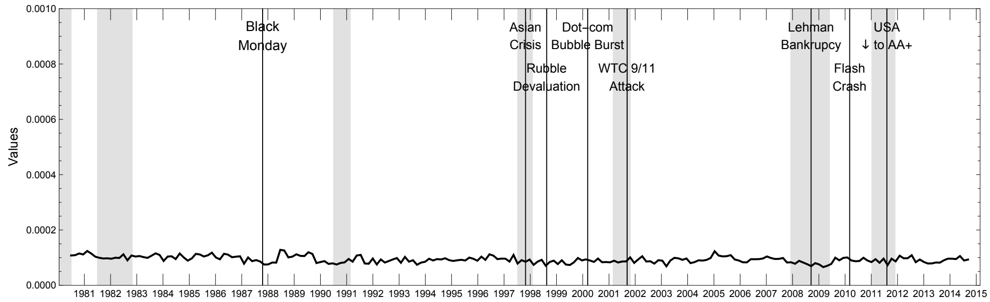
(a) $\widehat{\sigma_f}, \ meanSD = .0026$



(b) $\widehat{b}, \ meanSD = .0000130$



(c) $\widehat{a}, \ meanSD = .000088$

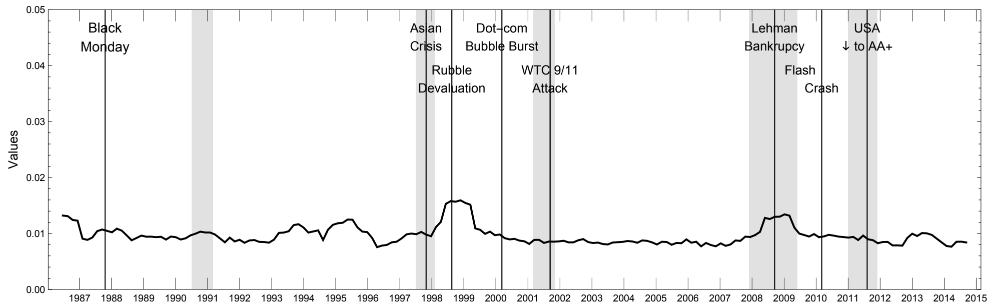


Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

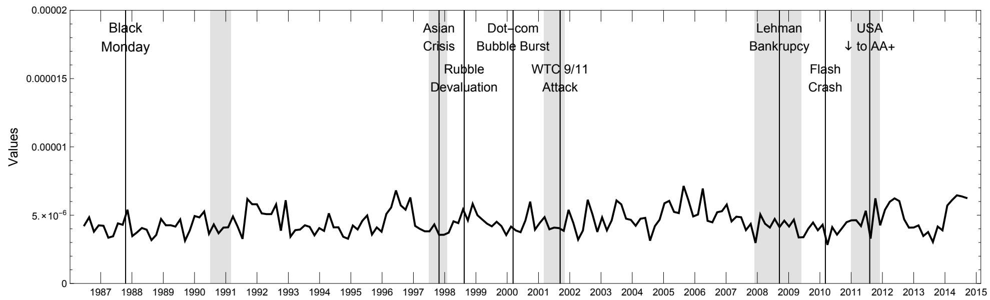
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure 7.14: Rolling estimates for U.S./JY

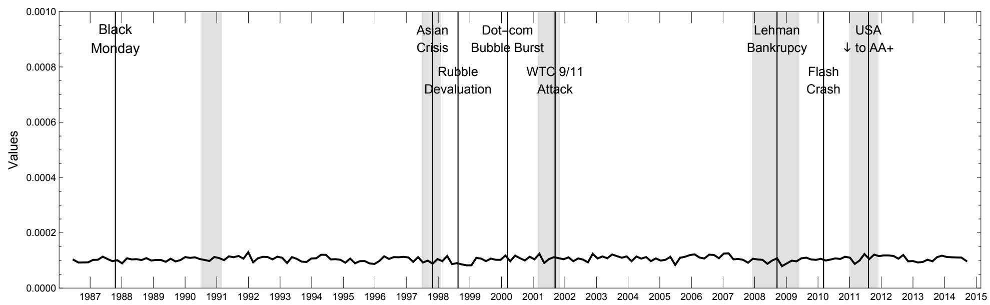
(a) $\widehat{\sigma_f},\ meanSD = .0028$



(b) $\widehat{b},\ meanSD = .0000135$



(c) $\widehat{a},\ meanSD = .000087$



Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

assigned based on rather technical optimisation criteria mainly to the fundamental volatility, in which dimension the optimisation algorithm search can work well better and which is likely to overshadow the effect of switching parameters a and b during the estimation procedure.

As expected, LL values describing the model fit generally decrease in turbulent periods. We highlight the important remark from Subsection 6.5.1 that direct comparison of rolling log-likelihoods is methodologically disputable because it is based on different rolling sub-samples. Nonetheless, since rolling datasets overlap by circa 83% between adjacent steps, the rolling pattern of LL provides us with a valuable information.

Standard deviations of the rolling estimates reveal an interesting behaviour. They are naturally expected to be markedly larger compared to full sample estimates, but this only holds for standard deviations of rolling $\widehat{\sigma_f}$, for which the mean standard deviation (computed as the mean of standard deviations for all rolling periods) is roughly twice as large. On the other hand, the mean standard deviation of rolling $\widehat{a}$ is a bit lower compared to full sample estimates for all analysed datasets and the mean standard deviation of rolling $\widehat{b}$ is even circa two times lower in average. Moreover, all single sample estimates for all three parameters for all datasets are positive. Conversely, we do not observe any considerable distinctions between the mean values of full sample estimates and average values of mean rolling estimates. Although results indicating that $\widehat{a} > \widehat{b}$ suggest globally unimodal distribution of the sentiment variable related to theoretically expected performance of the estimator and its consistency and asymptotic efficiency observable also in small samples for the model, the aggregate results of the rolling analysis seem to detect some globally pathological dynamics increasing estimation inaccuracies and divergence with increasing length of the time series. Therefore, conditional modelling and more elaborate time-varying empirical estimation, or possibly advanced data driven bandwidth selection methods (e.g. based on OLS or ML cross-validation) for kernel density estimation are candidates for future research on the NPSML estimation of the Alfarano *et al.* (2008) model and are likely to bring a new insight into the model behaviour.

7.5 Concluding remarks

In this penultimate chapter we show that the NPSMLE method generally works for various types of HAMs. We confirm that simulated MLE constitutes a very flexible method to estimate complicated nonlinear models for which traditional estimation approaches cannot be used and that historically remained to a large extent inestimable. Employing NPSMLE, we are generally able to estimate models for which the closed-form solution or theoretical approximation of the objective function does not exist. We also prove that using simulation-based nonparametric methods the parameters of such systems can be recovered reasonably well. Together with quickly increasing computational capabilities of personal computers, server clusters, and super-computers, we anticipate a bright future and rapid development of simulation-based methods in next years.

Chapter 8

Conclusion

This thesis proposes innovative computational framework for empirical estimation of FABMs. Motivated by the lack of general consensus on the estimation methodology, not many examples on structural estimation of FABMs, and inconclusive results in recent FABM literature, we aim at developing and testing more general methods for estimation of FABMs that significantly reduce the importance of restrictive theoretical assumptions.

In Chapter 3 we develop a two-step estimation procedure and estimate one of the historically first FABMs—the cusp catastrophe model—under time-varying stock market volatility. Utilising the availability of high-frequency data and the popular realised volatility approach, we estimate stock market returns' volatility in the first step and subsequently apply the stochastic cusp catastrophe model to volatility-adjusted returns with constant variance. In the empirical part, we use a high frequency and sentiment dataset and test the model on nearly 27 years of U.S. stock market returns covering several important recessions and crisis periods. The results suggest that over a long period, stock markets are well described by the stochastic cusp catastrophe model. Using our two-step modelling approach, we show that the cusp model fits the data well and that the fundamental and bifurcation sides are controlled by the indicators for fundamental and speculative money, respectively. While we find that the stock markets showed signs of bifurcation in the first half of the period, catastrophe theory was not able to confirm this behaviour in the second half. Translating the results, we find that the U.S. stock market's downturns were more likely to be driven by the endogenous market forces during the first half of the studied period, while during the second half of the period, the exogenous forces seem to be driving the market's instability. In conclusion, we find that despite the fact that we modelled volatility in the first step, the stock markets showed signs of bistability during several crisis periods. Our methodology thus overcomes the difficulties of the cusp catastrophe model estimation using financial data and provides an important shift in the application of catastrophe theory to stock markets.

Chapter 4 introduces a general computational framework for empirical estimation of full-fledged FABMs. Because for many FABMs no closed-form representation of the likelihood function exists, we follow the Kristensen & Shin (2012) framework of a simulated MLE based on nonparametric kernel methods. In situations when we cannot derive the usual MLE, simulated MLE constitutes an opportune estimation method for general class of FABMs.

In Chapter 5 we customise the NPSMLE methodology of Kristensen & Shin (2012) and elaborate its capability for FABMs estimation purposes. To start with, we apply the methodology to the most famous and widely analysed model of Brock & Hommes (1998). We extensively test small sample properties of the estimator via Monte Carlo simulations and confirm the ability of the NPSMLE method to reveal true parameters with high precision. We further show that theoretical properties of the estimator, the consistency and asymptotic efficiency, also hold in small samples for the model. Next, we assess the impact of the stochastic noise intensity in the system and investigate the robustness of the estimation method w.r.t. various modifications of the model as well as of the estimation algorithm. Finally, using graphical computational tools we analyse behaviour of simulated log-likelihood functions. Based on generally very smooth shape with a unique maximum we assume that the regularity conditions are met for the HAM and the identification of parameters is assured.

Chapter 6 presents estimation results of the 2-type and the 3-type Brock & Hommes (1998) model using cross section of world stock markets. We introduce the full sample static estimates to reveal robust average relationships as well as rolling window approach to detect possible dynamics in the behaviour of market coefficients over time and eventual structural breaks. The crucial result of our analysis is the statistical insignificance of the switching coefficient $\widehat{\beta}$. This is a common result in the existing literature, but on the other hand we contrast another part of the HAM estimation literature reporting significant $\widehat{\beta}s$ for various specific markets. In contrary, our estimation results of the 2-type model reveal markedly statistically significant belief parameters defining heterogeneous trading regimes with an absolute superiority of trend-following over contrarian strategies. Our findings further indicate robustness w.r.t. the fundamental value specification and remain largely unaffected under the robustness burden of different than daily data frequency, jumps in market risk free rate, or introducing of agents' memory. Graphical inspection of simulated log-likelihood functions reveals a bit rough surface, but very consistent performance of the estimation method over all random runs leading to unique a maxima. The adapted computational algorithm is, however, able to deal well with the not-completely-smooth surface of the simulated log-likelihood function and the important identification feature is thus verified also for the empirical application.

Both main results are also stable over the entire period confirmed via rolling estimation approach which primarily supports the validity of the full sample static estimates. This is an expected result considering the insignificant switching parameter $\widehat{\beta}$ from the full sample estimation. However, a clear dynamics is observable for the model fit which is markedly higher during tranquil periods and generally decreases during volatile and recession periods. Conversely, intensity of market noise expectedly slightly increases in turbulent period. Interesting signs of a specific behaviour are detectable for rolling estimation of individual indices on the level of standard deviations. S&P500 data detects turbulent periods around the WTC 9/11 attack, Lehman Bankruptcy, or downgrade of the U.S. ranking to AA+ in 2011 but also other tracked world events. NASDAQ reflects mainly the Asian Crisis in 1997 and the Dot-com Bubble Burst in 2000 which is, however, not observed for a more general S&P500. European indices DAX and FTSE can detect the crisis around the Rubble devaluation in 1998 but not the Asian Crisis of 1997. HSI conversely captures the effect of 1997 Asian Crisis but not the 1998 Rubble devaluation. This can be interpreted as a sign of increased presence of contrarians during such turbulent stock market periods detected via less efficient estimates of trend following coefficients. NIKKEI 225 interestingly behaves more closely to S&P500 than to its Asian 'fellow' HSI. Next, estimation of o more-flexible 3-type model with the mix of fundamental, trend following, and contrarian strategy further suggests redundancy of the contrarian strategy for the overall model fit.

All hitherto results lead us to a suspicion of a possible model misspecification as an important technical side-effect of the zero $\widehat{\beta}$ coefficient are equal and stable population magnitudes of the fundamental and trend following strategies. We therefore correct for this possible misspecification introducing fixed fraction of the fundamental strategy instead of switching coefficient β which seems hardly to be estimated. The fraction is, however, the subject of empirical estimation and is found strongly statistically significant for all analysed indices. A strong trend chasing strategy is then expressed via trend coefficient around 2 for all indices in the *fraction* model. The magnitude of the fundamentalists' population closely around 56% represents overall proportional dominance of fundamentalists over trend following chartists on world stock markets and even stable in time.

Finally, in Chapter 7 we apply the NPSMLE methodology to a stylised herding FABM developed by Alfarano *et al.* (2008). First, we analyse small sample properties of the estimator in a Monte Carlo study and show dissimilar estimation performance for the unimodal and bimodal market sentiment version of the model. Second, exploring behaviour of the simulated log-likelihood function we verify the identification of parameters and theoretical assumptions of the estimation method for the unimodal case. Next, we estimate the model using three stock market indices, price of gold in USD, and three exchange rates. The fundamental volatility σ_f is estimated strongly statistically significant, but parameters governing opinion switches and sentiment dynamics face difficulties with statistical significance mainly due to tiny magnitude very close to zero. However, the 'illusive' statistical insignificance does not have its main origin in negative estimates, rather in large standard deviations and positive outliers. Generally, for FOREX data we observe smaller magnitude of the fundamental volatility and parameter estimates $\widehat{a} > \widehat{b}$ indicate unimodal distribution of the sentiment variable accompanied by a general tendency of gradual reversion back to the balanced situation and theoretically expected performance of the estimator. Moreover, the 'herding-based' switching coefficients $\widehat{b}$ is the highest for S&P500 but comparing other stock markets and gold with FOREX, an irrationally impetuous herding behaviour leading to locally extremely enhanced volatility, bubbles, and crashes is surprisingly found a bit stronger for FOREX markets. While it seems very difficult for the model to detect signs of a herding behaviour when exposed to the complete data samples, rolling window estimation reveals interesting dynamics of model coefficients and clearly captures jumps in the 'herding-based' switching parameter $\widehat{b}$ and elevated fundamental volatility coefficient $\widehat{\sigma_f}$ in turbulent times.

After all, we would like to 'pick up' the gauntlet and summarise our thoughts about the future of a broader field of AB modelling in Economics. Let us approach this relatively unbounded and highly subjective area employing the SWOT matrix from strategic analysis. The following text connects our ideas about the current state and possible future development of both the broader picture of AB modelling in Economics as well as the empirical estimation of ABMs, the central topic of this thesis.

In our opinion, the main *strengths* of the current field are a constantly rising awareness of the AB approaches in Economics among broad audience comprising scholars, policy makers, central bankers, or opinion leaders—note e.g. a recent publication by Stiglitz & Gallegati (2011). As already mentioned in Chapter 1, a number of current research projects propose ideas of complementing current mainstream policy making approaches through the use of ABMs, new subjects comprising Behavioural Finance or Behavioural Macro are being developed worldwide, number of related conferences, workshops, and summers schools is growing. Based on our personal experience, students accept the critique of traditional mainstream Finance and Macro models well and young research get easily attracted by enhanced realistic features of ABMs and novel opportunities for economic analysis that these models bring. Together with increasing computational power of computers to study AB systems by simulation-based methods, the ABM estimation literature has a favourable opportunity to flourish.

On the other hand, the field seems to be constrained by many *weaknesses*. The biggest one can be according to our mind naturally found in the style how the current research is done. The research efforts are highly fragmented. Almost every researcher or small teams develop their own models. Numerical methods, computational analysis, or programming are rarely part of the university Economic curriculum even on the graduate/doctoral level. A huge effort of young researchers is thus often expended to design and analyse relatively elementary models, instead of extending some already successful and verified approaches. An often seen strategy is to develop a model, calibrate it with insufficient verification of the consistency of calibration (i.e. calibrated coefficients might have been derived under assumptions incompatible with the calibrated model, see Section 2.1 discussing advantages and disadvantages of calibration into details), possibly surrendering real empirical estimation, and relying instead on simulation analysis. To complete the critique of traditional models rigorously, a proper empirical validation of new AB approaches is needed. Nevertheless, the validation phase is often based only on a casual comparison with at least partially arbitrarily chosen set o stylised facts. The features that the model replicates well are accentuated, but the other not-so-well-replicated facts are either suppressed or might be left as an 'area for improvement and further research'. A considerable weaknesses is therefore the lack of proper empirical validation and successful estimation attempts of AB models—the situation which this thesis aims to contribute to.

We have also identified several crucial *opportunities* that lie in a synthesis of the current state of art and findings. Although the community itself is heterogeneous with regard to this issue, for me personally a consolidation of modelling approaches would be highly beneficial for the field. This might comprise identification of best modelling practices, e.g. in form of a standardised modelling cookbook/ABM textbook for student and young researchers or via developing new ABM software packages for the most popular programming languages. So far there is no JEL classification for FABMs or Macro ABM, which creation would definitely help to demarcate the field. We are personally aware of only two special journal issues devoted to empirical validation of ABMs in Economics, the special issue on Empirical Validation in Agent-based Models in *Computational Economics* (Volume 30, Issue 3, October 2007) and the special Issue on The Methodology of Simulation Models in *Journal of Artificial Societies and Social Simulation* (Volume 12, Issue 4, October 2009). A support from this direction would definitely be beneficial for the development of the field.

We are not able to resolve whether this really is a *threat*, but the field nowadays is highly concentrated and locked in several workplaces in Europe, the U.S., and Australia. Moreover, we personally feel that the ABM field has only a limited time to establish itself and defend its position and usefulness within Economics. Spreading collaboration among these workplaces and intensifying cooperation with other fields such as Mathematics or Computer Science would decrease the thread of locking the ABM field at the edge Economics. Furthermore, it would definitely bring benefits in the form of promoting the AB approaches among wider (not only) Economics community, but also synergies such as better publication opportunities, etc.

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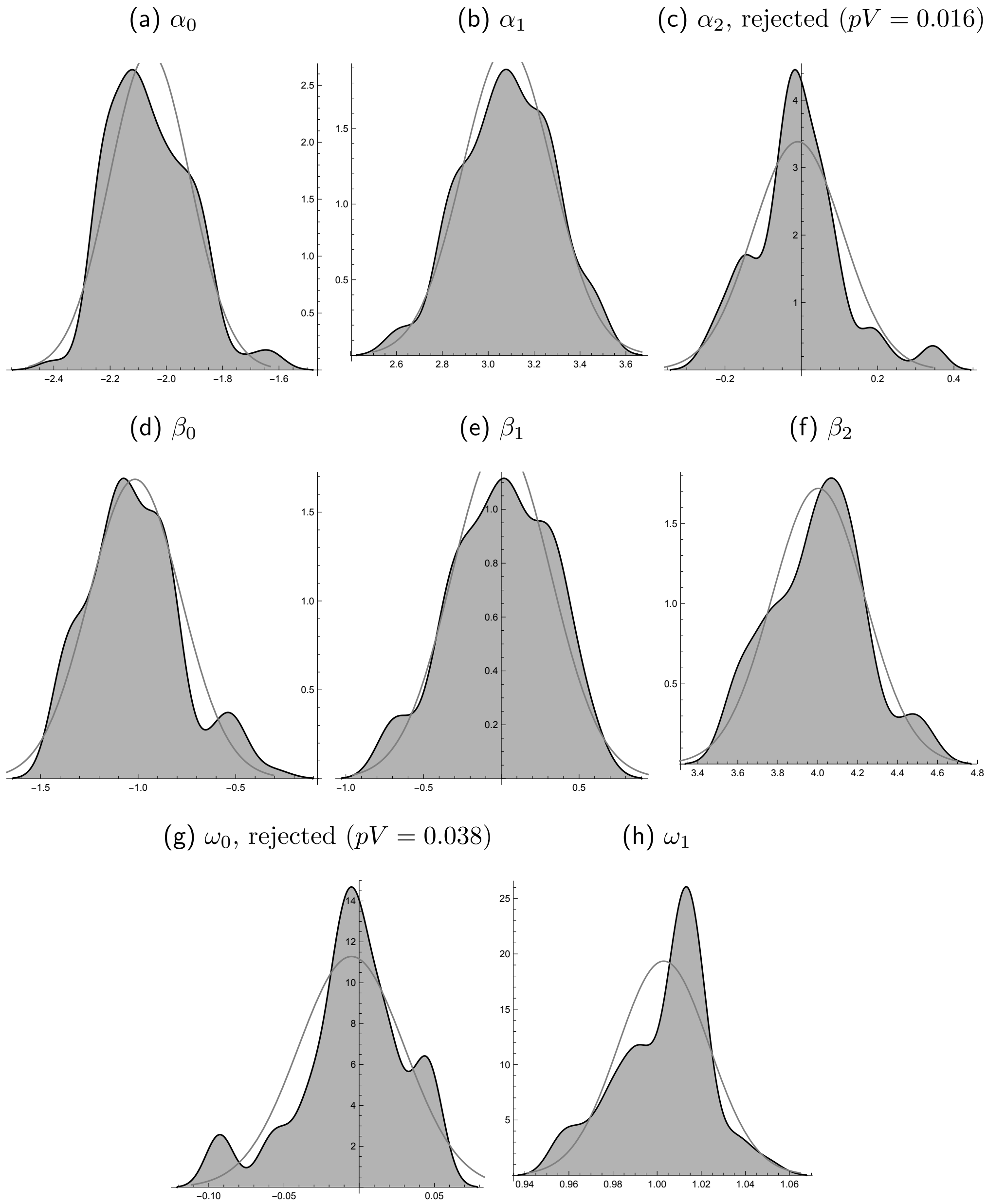
Appendix A

Cusp catastrophe model supplements

Figure A.1: Gaussianity of cusp coefficients I.



(a) α_0 (b) α_1 (c) α_2, rejected ($pV = 0.016$)

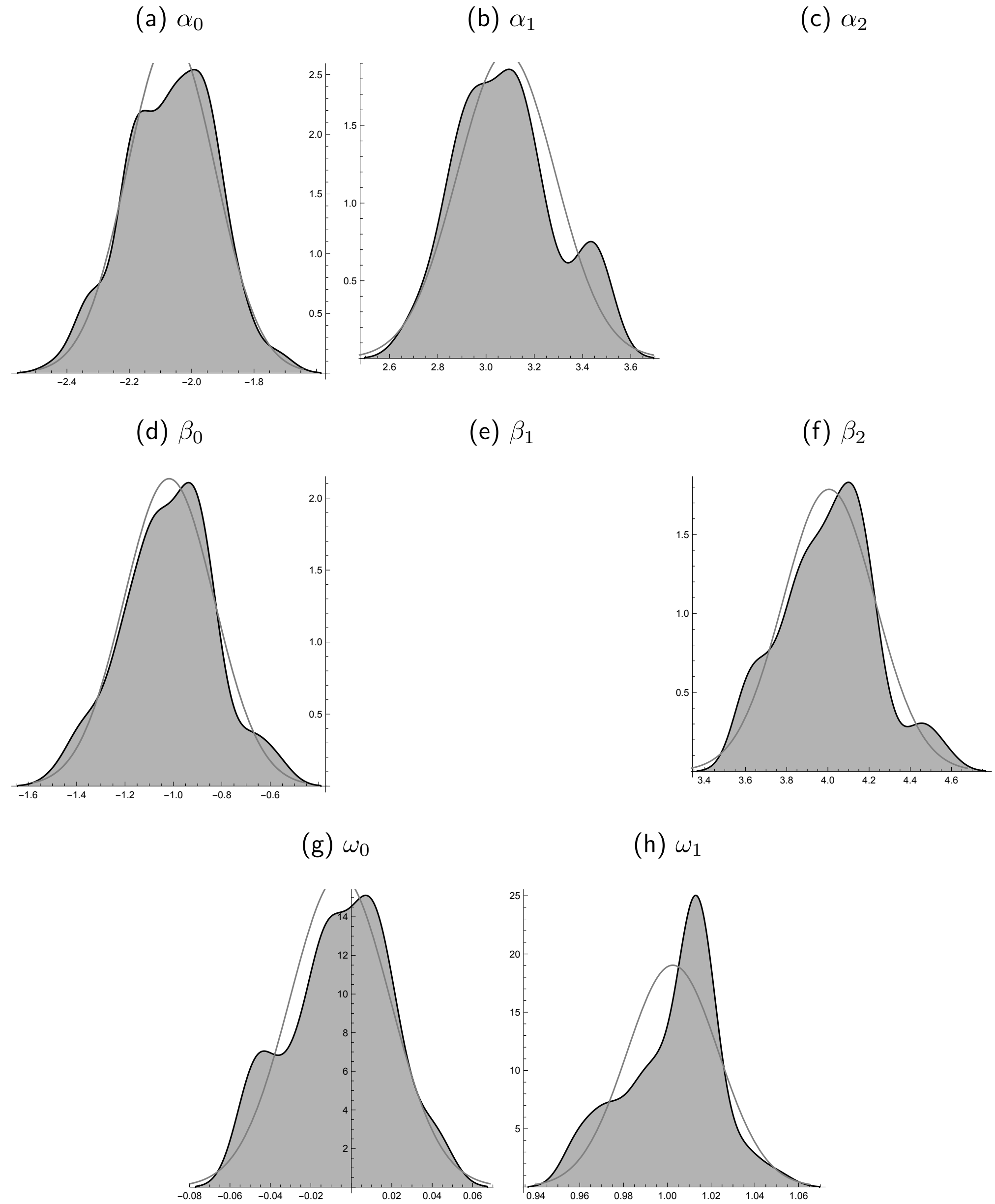
(d) β_0 (e) β_1 (f) β_2

(g) ω_0, rejected ($pV = 0.038$) (h) ω_1

Note: Results based on the total sample of 100 random simulations and the unrestricted cusp model estimates from Table 3.1 using $y_t = r_t/\sigma_t$. Figures depicts smooth histogram kernel approximations of the probability density in black together with the fit of $N(\mu, \sigma^2_{data})$ in grey. 'rejected' indicates rejection of the null hypothesis of normality based on the Jarque-Bera ALM test at 5% level (p-values in the two rejection cases are depicted in parentheses).

Source: Author's own computations in *R* and *Wolfram Mathematica*.

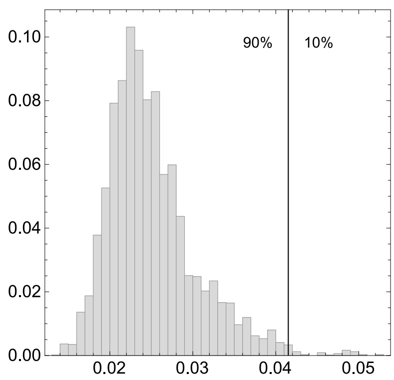
Figure A.2: Gaussianity of cusp coefficients II.



Note: Results based on the total sample of 100 random simulations and the restricted cusp model estimates from Table 3.1 using $y_t = r_t/\sigma_t$. Figures depicts smooth histogram kernel approximations of the probability density in black together with the fit of $N(\mu, \sigma^2_{data})$ in grey. The null hypothesis of normality based on the Jarque-Bera ALM test at 5% level is not rejected in any case.

Source: Author's own computations in *R* and *Wolfram Mathematica*.

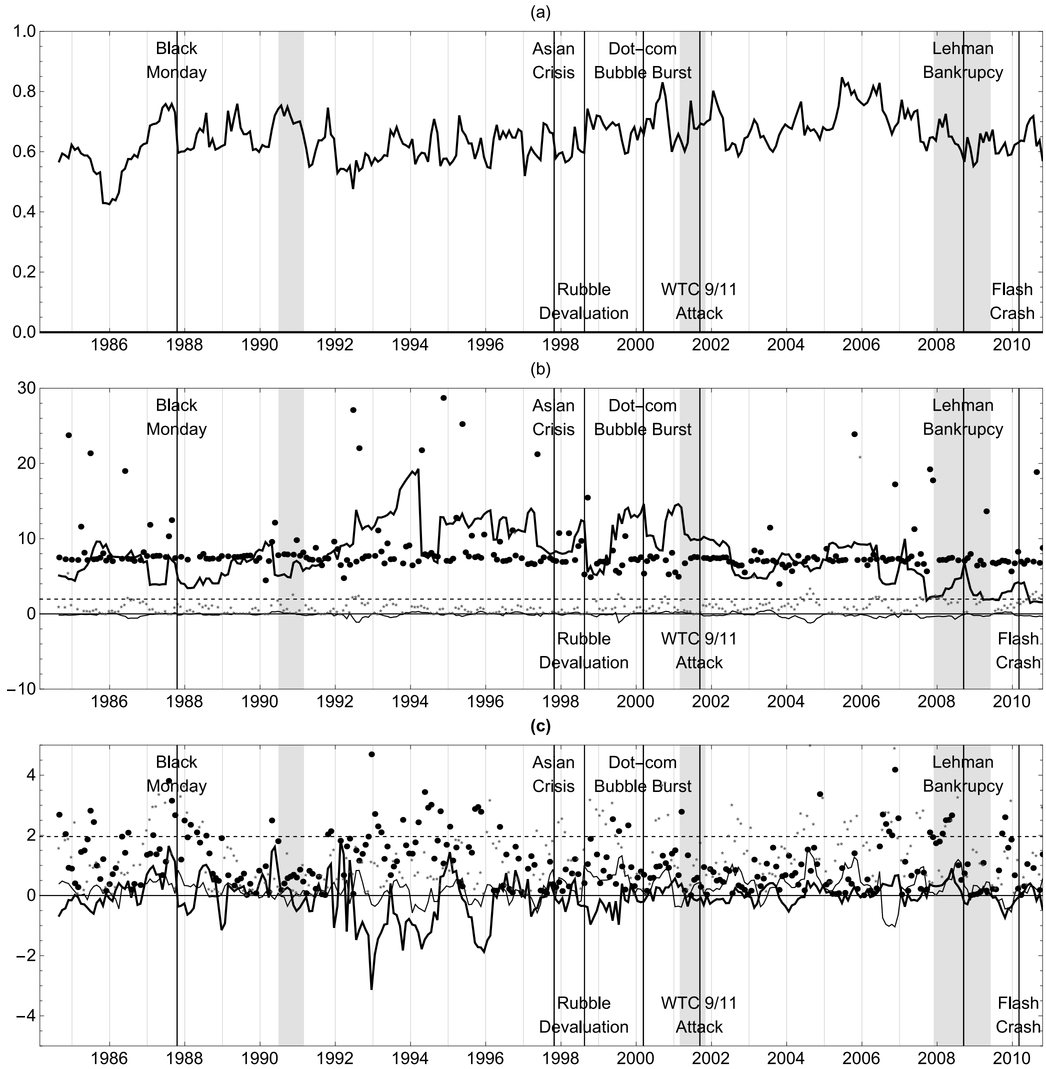
Figure A.3: Histogram of the dip statistics for bimodality



Note: Computed for all of the rolling window periods, together with the bootstrapped critical value 0.0406 for the 90% significance level plotted in bold black.

Source: Authors' own computations in *MATLAB* and *Wolfram Mathematica*.

Figure A.4: Rolling regression estimates



Note: Rolling coefficients with their |z-values|. (a) Estimated ω_1 coefficient values. (b) Estimated values of asymmetry coefficients, α_1 in bold black, α_3 in black. |z-values| related to both coefficient estimates are depicted as • and $*$, respectively. (c) Estimated values of bifurcation coefficients, β_2 in bold black, β_3 in black. |z-values| related to both coefficient estimates are as • and $*$, respectively. Plots (b) and (c) also contain the 95% reference z-value as a dashed black line.

Source: Authors' own computations in *MATLAB* and *Wolfram Mathematica*.

Appendix B

Supplementary Tables

On the following pages, a few supplementary tables are provided.

Table B.1: Results for β estim. w.r.t. various dist. of g_h and b_h III.

β	(a) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.1^2)$						(b) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.2^2)$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.00	.00	.27	-.50	.50	0%	.00	.00	.13	-.36	.39	0%
.1	.10	.10	.13	-.10	.30	0%	.10	.10	.08	-.10	.30	0%
.5	.50	.49	.42	-.50	1.46	0%	.50	.50	.17	.13	.83	0%
1	.99	.98	.63	-.62	2.70	0%	1.00	1.00	.24	.58	1.37	3%
3	3.02	3.08	1.24	-.20	6.18	0%	3.00	3.03	.38	2.65	3.59	15%
5	4.99	4.89	1.67	.58	8.08	1%	5.00	4.96	.88	4.27	5.51	28%
10	10.00	9.96	2.85	3.87	14.45	4%	10.00	9.89	1.26	8.84	10.74	46%
	(c) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.3^2)$						(d) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.4^2)$					
0	-.00	-.00	.07	-.15	.11	1%	.00	.00	.04	-.06	.06	10%
.1	.10	.10	.05	-.05	.23	0%	.10	.10	.03	.04	.17	2%
.5	.50	.50	.11	.35	.68	7%	.50	.50	.06	.43	.57	24%
1	1.00	.99	.17	.77	1.19	18%	1.00	1.00	.13	.84	1.08	40%
3	3.00	3.00	.12	2.79	3.18	46%	3.00	2.99	.33	2.79	3.21	64%
5	5.00	5.03	.39	4.77	5.32	56%	5.00	5.01	.13	4.82	5.20	74%
10	10.00	9.94	1.21	9.63	10.29	74%	10.00	10.00	.29	9.42	10.31	85%
	(e) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.6^2)$						(f) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.8^2)$					
0	-.00	-.00	.02	-.02	.02	36%	-.00	-.00	.02	-.03	.02	54%
.1	.10	.10	.01	.08	.12	15%	.10	.10	.01	.08	.11	35%
.5	.50	.50	.05	.46	.55	51%	.50	.50	.02	.47	.52	68%
1	1.00	1.00	.05	.92	1.07	67%	1.00	1.00	.01	.98	1.02	83%
3	3.00	3.01	.09	2.93	3.09	86%	3.00	3.02	.18	2.94	3.09	94%
5	5.00	5.05	.32	4.90	5.42	91%	5.00	5.00	.03	4.92	5.07	97%
10	10.00	9.97	.60	9.76	10.38	96%	9.99	10.28	4.14	.80	25.02	98%
	(g) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1)$						(h) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1.2^2)$					
0	-.00	-.00	.01	-.02	.01	72%	.00	.00	.03	-.01	.01	79%
.1	.10	.10	.01	.09	.11	53%	.10	.10	.00	.10	.10	67%
.5	.50	.50	.01	.48	.51	78%	.50	.50	.02	.47	.51	88%
1	1.00	1.00	.03	.97	1.02	89%	1.00	1.00	.02	.98	1.11	94%
3	3.00	3.30	1.16	2.96	8.29	97%	3.00	3.16	.49	2.98	4.82	98%
5	5.00	5.00	.05	4.91	5.15	99%	5.00	5.59	1.68	4.93	9.76	99%
10	10.02	11.61	4.69	5.79	20.28	99%	10.01	14.29	7.09	9.62	29.62	99%

Note: Belief parameters g_h and b_h drawn from various normal distributions of given parameter, stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0, 10^{-14})$, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table B.2: Results for β estim. w.r.t. various dist. of g_h and b_h IV.

β	(a) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.1^2)$						(b) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.2^2)$					
	Med.	Mean	SD	LQ	HQ	NN	Med.	Mean	SD	LQ	HQ	NN
0	.01	.00	.32	-.50	.50	0%	.01	.01	.26	-.50	.50	0%
.1	.10	.10	.13	-.10	.30	0%	.09	.09	.14	-.10	.30	0%
.5	.49	.50	.60	-.50	1.50	0%	.50	.49	.39	-.43	1.39	0%
1	.99	.98	1.06	-.96	2.93	0%	.99	1.01	.61	-.31	2.56	0%
3	3.03	3.05	2.59	-2.46	8.24	0%	3.00	3.06	1.11	.73	5.72	0%
5	5.06	5.20	3.77	-3.64	13.56	0%	5.00	4.90	1.73	-.77	7.65	0%
10	9.94	9.59	6.60	-7.20	25.75	0%	10.00	9.89	2.03	6.75	11.93	4%
	(c) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.3^2)$						(d) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.4^2)$					
0	-.00	-.00	.18	-.49	.46	0%	-.00	-.00	.12	-.34	.25	0%
.1	.10	.10	.11	-.10	.30	0%	.10	.10	.08	-.10	.30	0%
.5	.50	.51	.25	-.14	1.15	0%	.50	.50	.14	.19	.76	0%
1	1.00	1.00	.35	.27	1.74	0%	1.00	.99	.23	.58	1.35	3%
3	3.00	3.01	.51	2.43	3.73	4%	3.00	3.00	.47	2.53	3.45	15%
5	5.00	5.04	.59	4.34	5.83	10%	5.00	5.00	.31	4.51	5.44	29%
10	10.01	9.98	1.27	9.23	11.22	29%	10.01	9.99	.48	9.31	10.59	60%
	(e) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.6^2)$						(f) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.8^2)$					
0	-.00	-.00	.07	-.11	.11	2%	-.00	.00	.05	-.06	.08	11%
.1	.10	.10	.05	-.01	.22	0%	.10	.10	.03	.05	.14	3%
.5	.50	.50	.08	.40	.61	7%	.50	.50	.04	.43	.58	23%
1	1.00	1.00	.09	.86	1.15	18%	1.00	1.00	.04	.94	1.06	41%
3	3.00	3.00	.10	2.83	3.19	51%	3.00	3.00	.09	2.75	3.16	75%
5	5.00	5.00	.19	4.71	5.33	66%	5.00	5.01	.06	4.89	5.14	87%
10	10.00	10.12	.94	9.56	10.71	86%	10.00	9.95	.34	9.38	10.45	95%
	(g) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1)$						(h) $\widehat{\beta}$, g_h & $b_h \sim N(0, 1.2^2)$					
0	-.00	-.00	.04	-.03	.04	27%	-.00	-.00	.02	-.02	.02	42%
.1	.10	.10	.02	.07	.13	12%	.10	.10	.02	.08	.12	25%
.5	.50	.50	.04	.46	.55	43%	.50	.50	.02	.47	.53	58%
1	1.00	1.00	.04	.94	1.04	62%	1.00	1.00	.05	.93	1.07	73%
3	3.00	3.00	.12	2.89	3.11	86%	3.00	3.00	.05	2.87	3.13	94%
5	5.00	5.25	1.16	4.85	10.13	94%	5.00	5.59	2.29	4.72	14.47	97%
10	10.00	10.76	3.36	5.79	22.97	97%	10.01	14.99	8.23	9.65	30.00	98%

Note: Belief parameters g_h and b_h drawn from various normal distributions of given parameter, stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^{N}$ drawn from normal distribution $N(0, 10^{-12})$, $R = 1.0001$. Each sample is based on 1000 random runs, $H = 5$ possible trading strategies, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table B.3: Results of 5-parameter estimation of a 3-type model I.

True β, g_2, b_2, g_3, b_3	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{g_3}$			(e) $\widehat{b_3}$			
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	NN
.5, .4, .3, .2, .15	.39	.45	.53	.37	.38	.27	.25	.28	.19	.24	.22	.27	.20	.17	.19	0%
3, .4, .3, .2, .15	2.53	2.77	1.66	.37	.38	.30	.25	.28	.19	.23	.21	.30	.20	.17	.19	0%
10, .4, .3, .2, .15	9.23	8.70	4.23	.34	.35	.25	.24	.26	.16	.25	.24	.25	.22	.19	.16	1%
.5, -.4, -.3, .2, .15	.46	.53	.50	-.53	-.53	.17	-.26	-.28	.11	.34	.33	.18	.11	.13	.11	0%
3, -.4, -.3, .2, .15	2.87	2.92	3.22	-.39	-.40	.16	-.24	-.27	.12	.23	.26	.19	.09	.12	.11	0%
10, -.4, -.3, .2, .15	11.16	11.10	10.62	-.35	-.35	.17	-.25	-.27	.12	.21	.28	.19	.10	.13	.11	0%
.5, -.4, .3, .2, -.15	.50	.55	.50	-.53	-.53	.17	.26	.28	.11	.34	.34	.17	-.11	-.13	.11	0%
3, -.4, .3, .2, -.15	2.76	2.84	3.32	-.38	-.40	.17	.24	.26	.11	.24	.27	.20	-.09	-.11	.11	0%
10, -.4, .3, .2, -.15	12.14	12.27	10.45	-.35	-.36	.17	.25	.28	.12	.20	.26	.18	-.10	-.14	.11	0%
.5, .8, .6, .4, .3	.51	.48	.19	.75	.78	.45	.45	.50	.33	.45	.42	.45	.45	.40	.33	25%
3, .8, .6, .4, .3	2.47	2.32	.90	.59	.65	.41	.44	.48	.29	.62	.55	.41	.45	.41	.29	45%
10, .8, .6, .4, .3	9.97	9.89	.75	.79	.63	.20	.59	.47	.15	.42	.58	.21	.31	.43	.15	89%
.5, -.8, -.6, .4, .3	.33	.43	.34	-1.00	-1.01	.30	-.69	-.71	.22	.61	.62	.31	.39	.41	.22	0%
3, -.8, -.6, .4, .3	2.29	3.15	2.49	-.85	-.91	.26	-.64	-.66	.22	.49	.54	.30	.34	.37	.21	9%
10, -.8, -.6, .4, .3	10.06	10.12	2.09	-.80	-.80	.06	-.60	-.60	.05	.40	.41	.09	.30	.30	.04	37%
.5, -.8, .6, .4, -.3	.34	.43	.34	-1.02	-1.01	.32	.69	.73	.23	.62	.61	.33	-.39	-.43	.23	0%
3, -.8, .6, .4, -.3	2.49	3.32	2.51	-.83	-.89	.26	.62	.66	.21	.45	.51	.28	-.33	-.36	.21	8%
10, -.8, .6, .4, -.3	10.03	10.05	2.67	-.80	-.79	.07	.60	.59	.06	.40	.41	.12	-.30	-.29	.05	37%
.5, 1.2, .9, .8, .6	.50	.43	.15	1.04	1.08	.60	.77	.82	.58	.96	.92	.60	.71	.68	.58	85%
3, 1.2, .9, .8, .6																100%
10, 1.2, .9, .8, .6																100%
.5, -1.2, -.9, .8, .6	.26	.32	.30	-1.58	-1.60	.46	-1.12	-1.14	.43	1.22	1.23	.49	.82	.84	.42	5%
3, -1.2, -.9, .8, .6	2.79	2.49	1.15	-1.24	-1.32	.24	-.91	-.88	.25	.84	1.02	.49	.61	.59	.23	62%
10, -1.2, -.9, .8, .6																100%
.5, -1.2, .9, .9, -.6	.24	.32	.31	-1.60	-1.60	.45	1.15	1.14	.43	1.22	1.23	.49	-.85	-.85	.43	5%
3, -1.2, .9, .9, -.6	2.74	2.38	1.18	-1.24	-1.34	.25	.91	.88	.26	.85	1.05	.51	-.61	-.59	.25	62%
10, -1.2, .9, .8, -.6																100%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0, 0.1^2)$, $R = 1.0001$. Each sample is based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD) are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

Table B.4: Results of 5-parameter estimation of a 3-type model II.

True β, g_2, b_2, g_3, b_3	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{g_3}$			(e) $\widehat{b_3}$			
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	NN
.5, .4, .3, .2, .15	.39	.48	.35	.30	.33	.23	.25	.25	.19	.27	.25	.22	.22	.20	.18	0%
3, .4, .3, .2, .15	3.33	3.86	2.43	.35	.33	.16	.26	.26	.14	.23	.24	.16	.21	.18	.14	0%
10, .4, .3, .2, .15	10.61	12.69	8.15	.37	.31	.13	.27	.24	.12	.25	.25	.15	.20	.18	.12	3%
.5, -.4, -.3, .2, .15	.42	.57	.40	-.42	-.45	.14	-.32	-.34	.13	.21	.24	.15	.16	.18	.12	0%
3, -.4, -.3, .2, .15	3.56	3.53	3.17	-.35	-.31	.14	-.25	-.23	.12	.17	.15	.08	.12	.12	.07	0%
10, -.4, -.3, .2, .15	10.33	9.05	10.12	-.37	-.29	.17	-.27	-.22	.13	.18	.14	.09	.13	.12	.06	11%
.5, -.4, .3, .2, -.15	.41	.56	.40	-.43	-.45	.14	.32	.33	.13	.21	.24	.14	-.17	-.19	.12	0%
3, -.4, .3, .2, -.15	3.80	3.51	3.31	-.35	-.30	.14	.25	.22	.12	.17	.14	.08	-.12	-.12	.06	0%
10, -.4, .3, .2, -.15	10.28	9.37	10.02	-.37	-.30	.16	.27	.22	.13	.18	.15	.09	-.13	-.12	.06	12%
.5, .8, .6, .4, .3	.47	.45	.18	.72	.68	.34	.52	.52	.29	.49	.51	.34	.42	.38	.28	1%
3, .8, .6, .4, .3	2.94	2.99	.41	.46	.57	.21	.37	.42	.17	.77	.63	.21	.56	.47	.17	21%
10, .8, .6, .4, .3	9.41	9.68	2.83	.51	.56	.23	.37	.42	.20	.79	.64	.22	.58	.46	.19	70%
.5, -.8, -.6, .4, .3	.49	.52	.19	-.81	-.81	.09	-.60	-.61	.10	.40	.41	.08	.30	.31	.08	0%
3, -.8, -.6, .4, .3	2.94	2.43	1.76	-.79	-.70	.27	-.59	-.53	.19	.39	.35	.13	.30	.30	.05	7%
10, -.8, -.6, .4, .3	8.26	3.09	7.90	-.78	-.46	.40	-.56	-.35	.29	.38	.23	.20	.28	.27	.09	67%
.5, -.8, .6, .4, -.3	.49	.52	.18	-.81	-.81	.09	.60	.61	.10	.40	.41	.07	-.30	-.31	.08	0%
3, -.8, .6, .4, -.3	2.96	2.52	1.68	-.80	-.71	.25	.59	.54	.18	.39	.35	.13	-.29	-.29	.05	5%
10, -.8, .6, .4, -.3	8.56	4.07	7.65	-.79	-.50	.39	.58	.38	.28	.39	.25	.19	-.28	-.27	.09	69%
.5., 1.2, .9, .8, .6																100%
3, 1.2, .9, .8, .6																100%
10, 1.2, .9, .8, .6																100%
.5, -1.2, -.9, .8, .6	.51	.49	.14	-1.19	-1.16	.17	-.89	-.87	.14	.79	.77	.12	.59	.60	.07	86%
3, -1.2, -.9, .8, .6																100%
10, -1.2, -.9, .8, .6																100%
.5, -1.2, .9, .8, -.6	.51	.49	.12	-1.19	-1.16	.18	.89	.87	.14	.79	.77	.12	-.60	-.62	.13	87%
3, -1.2, .9, .8, -.6																100%
10, -1.2, .9, .8, -.6																100%

Note: Stochastic noise ϵ_t and $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distribution $N(0,1)$, $R = 1.0001$. Each sample is based on 1000 random runs, number of observations $t = 5000$, and the kernel estimation precision $N = 1000$. Sample medians, means, standard deviations (SD) are reported. Figures are rounded to 2 decimal digits. 'NN' column reports the percentage of runs with 'NaN' outcome rounded to integer numbers.

Source: Author's own computations in *MATLAB*.

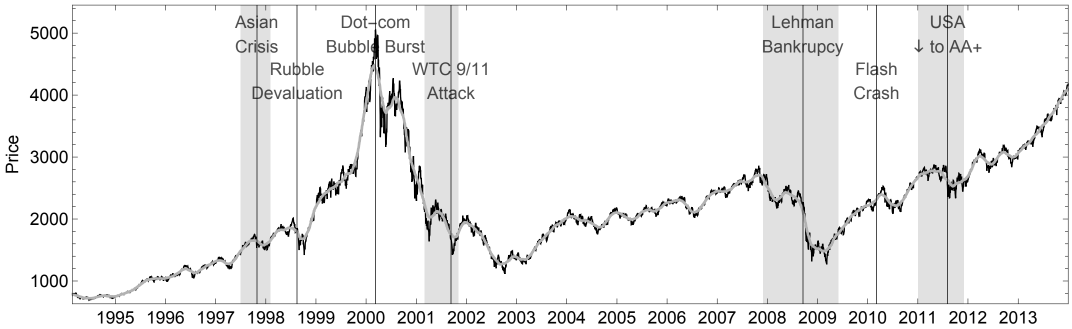
Appendix C

Supplementary Figures

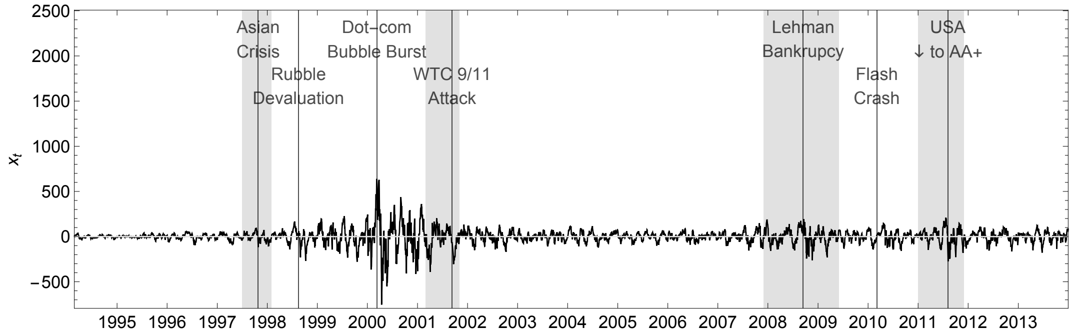
On the following pages, a few supplementary figures are provided.

Figure C.1: NASDAQ fundamental price MA61 approximation

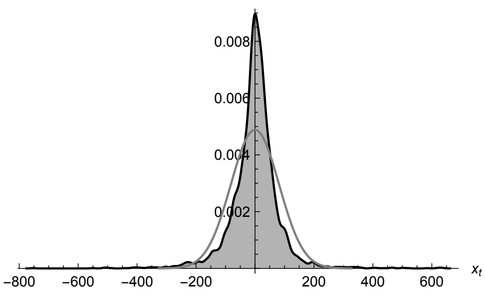
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

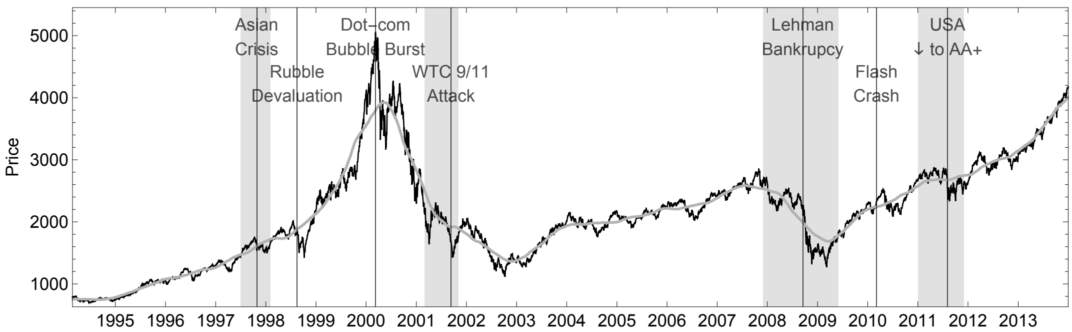


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

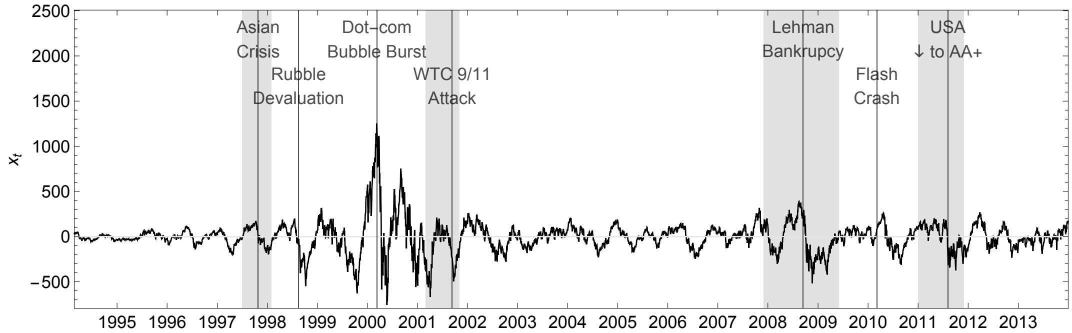
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.2: NASDAQ fundamental price MA241 approximation

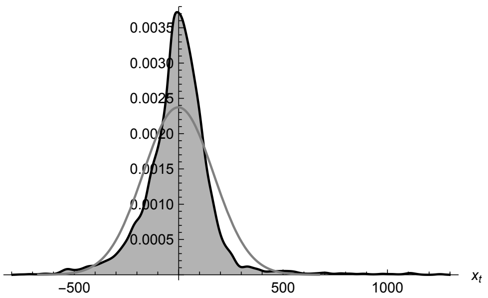
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram

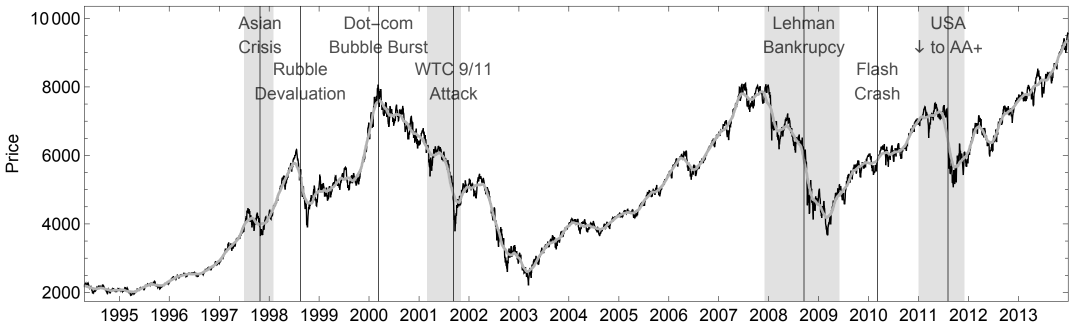


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

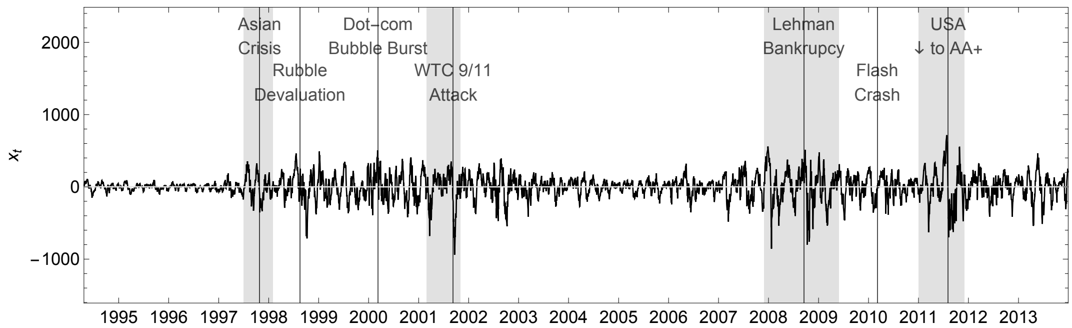
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.3: DAX fundamental price MA61 approximation

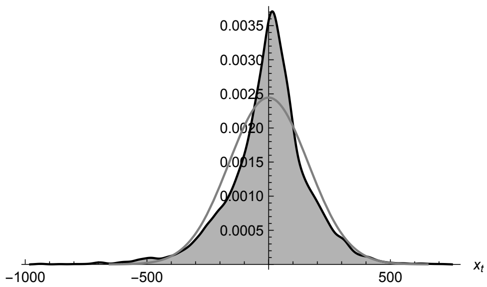
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

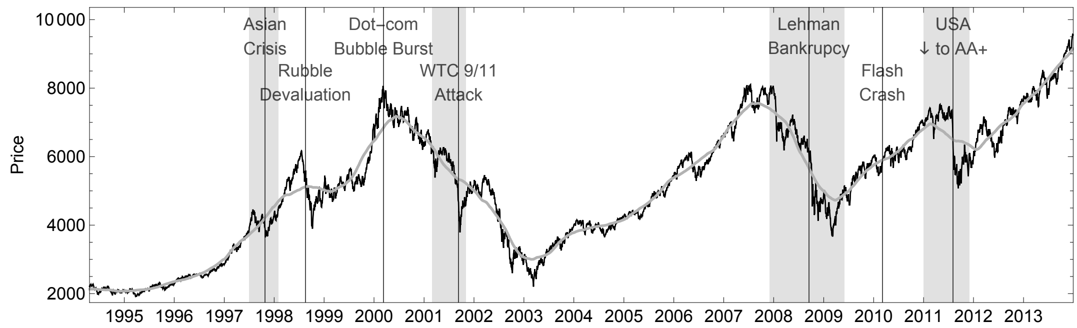


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

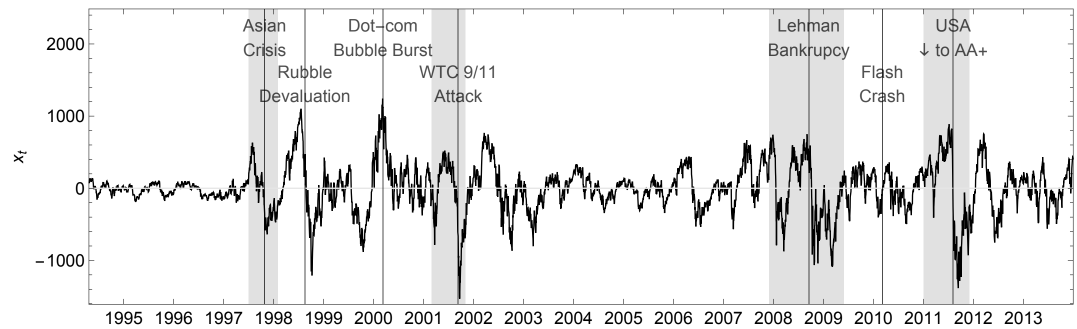
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.4: DAX fundamental price MA241 approximation

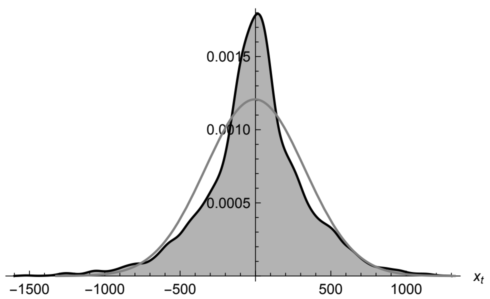
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram

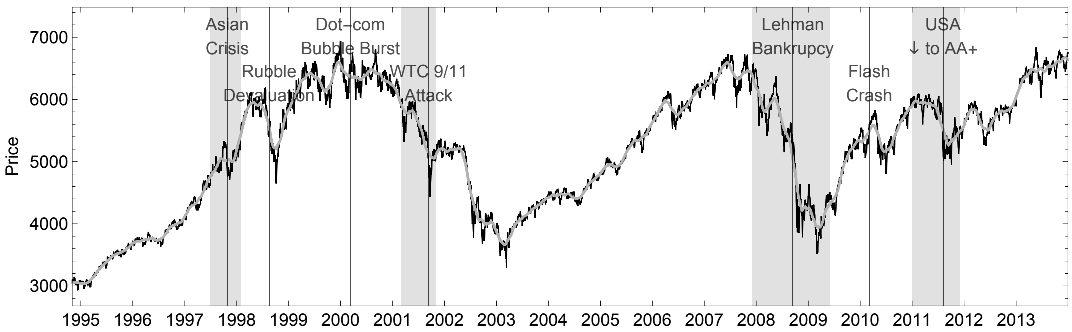


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

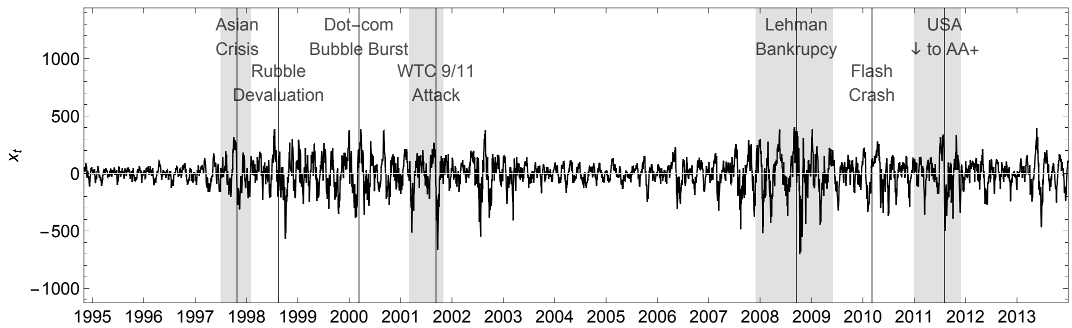
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.5: FTSE fundamental price MA61 approximation

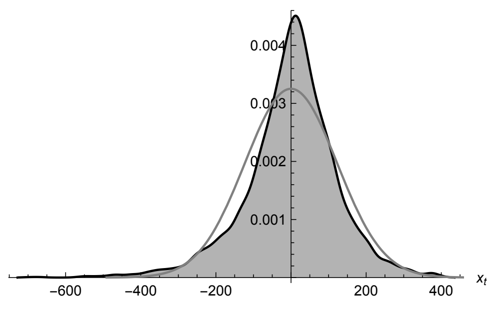
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

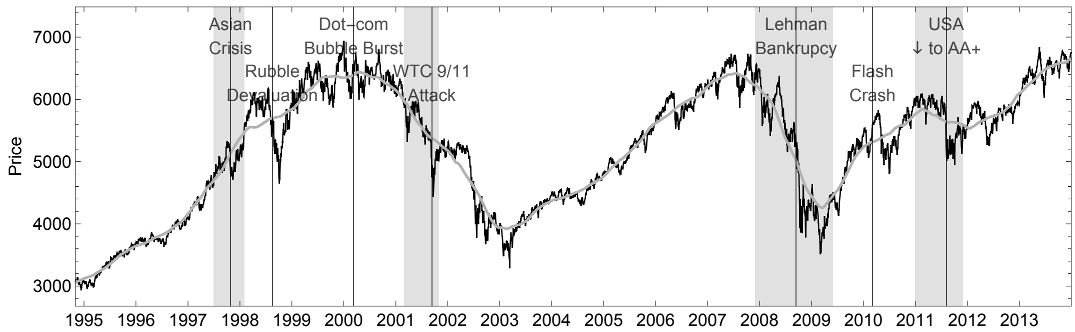


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

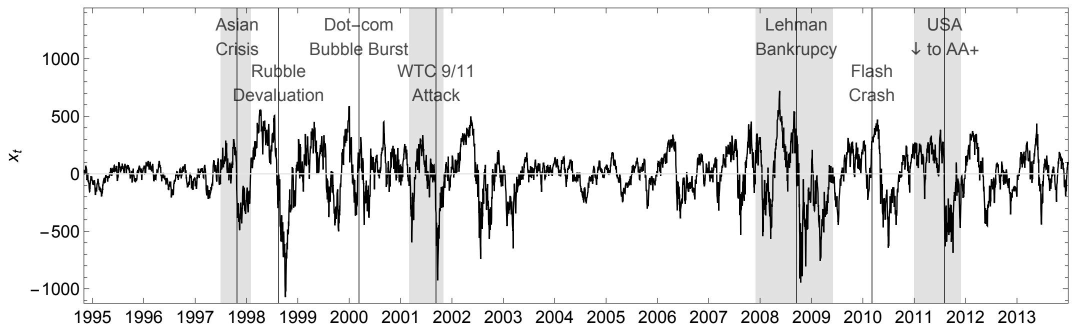
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.6: FTSE fundamental price MA241 approximation

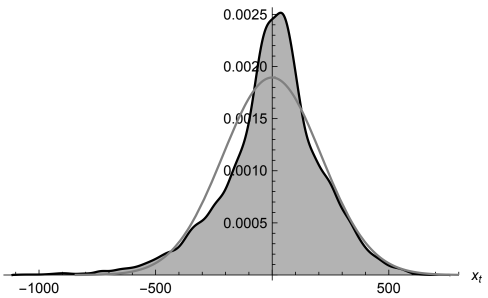
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram

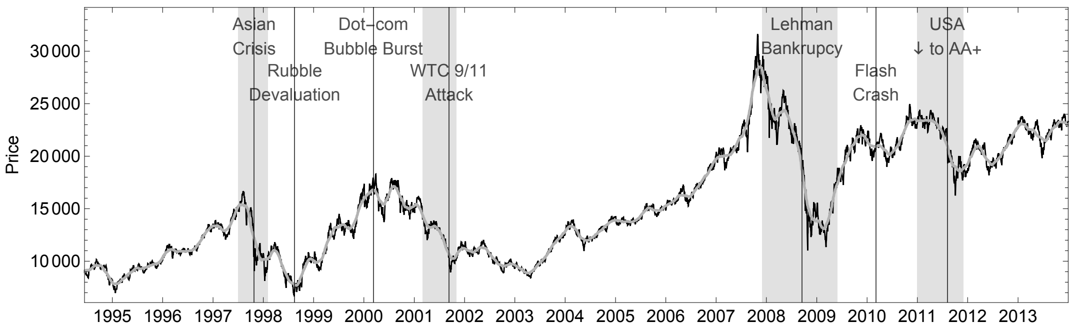


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

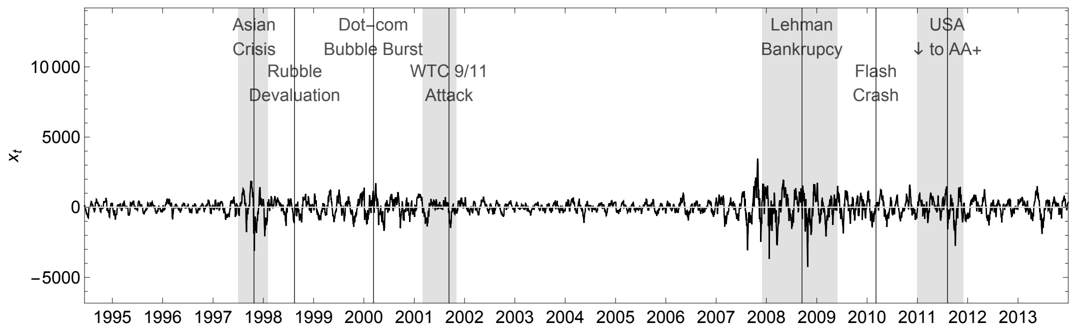
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.7: HSI fundamental price MA61 approximation

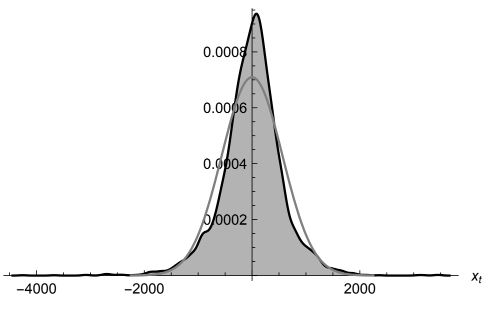
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

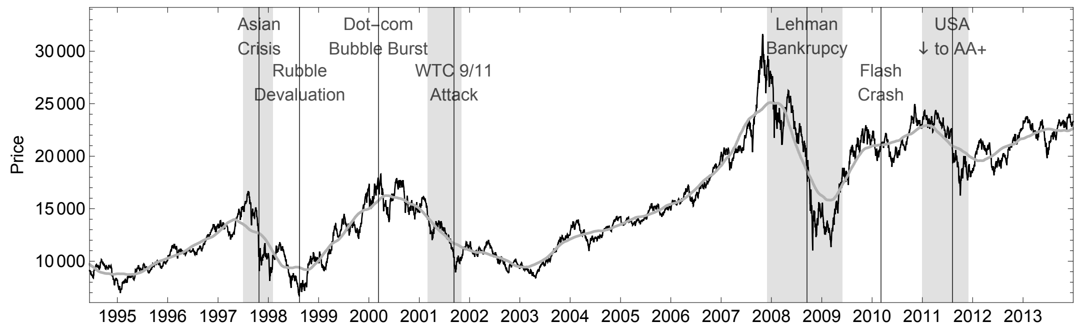


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

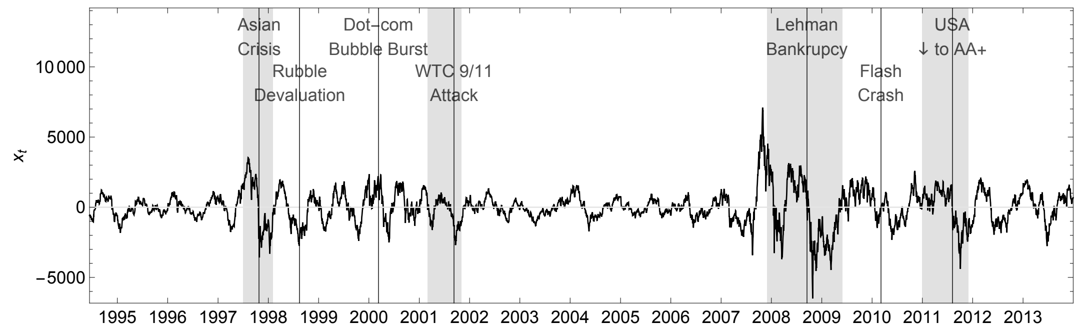
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.8: HSI fundamental price MA241 approximation

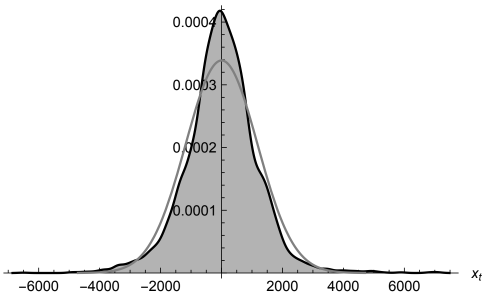
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram

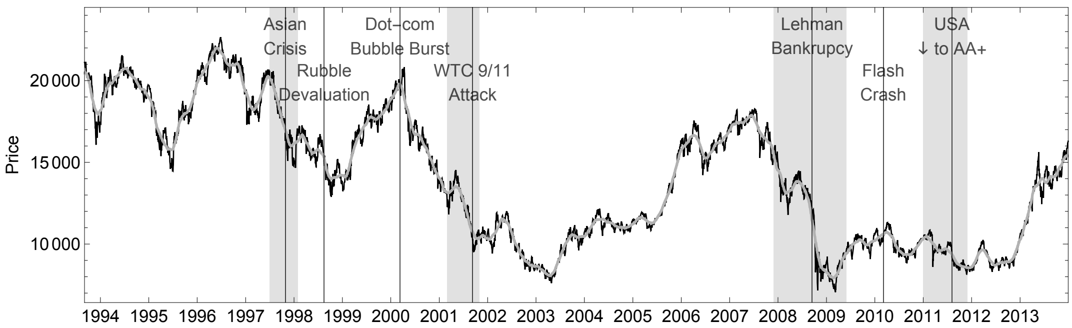


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

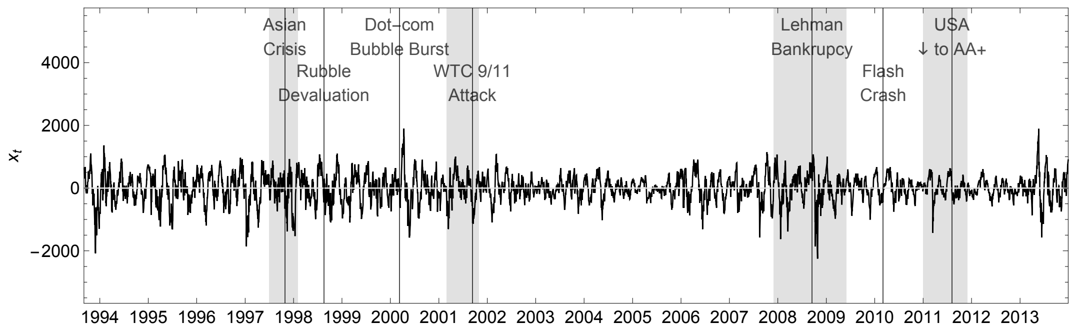
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.9: NIKKEI 225 fundamental price MA61 approximation

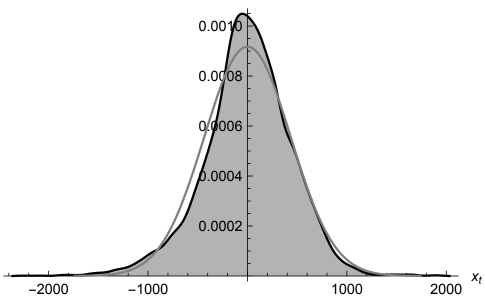
(a) MA window 61 days



(b) x_t time series



(c) x_t histogram

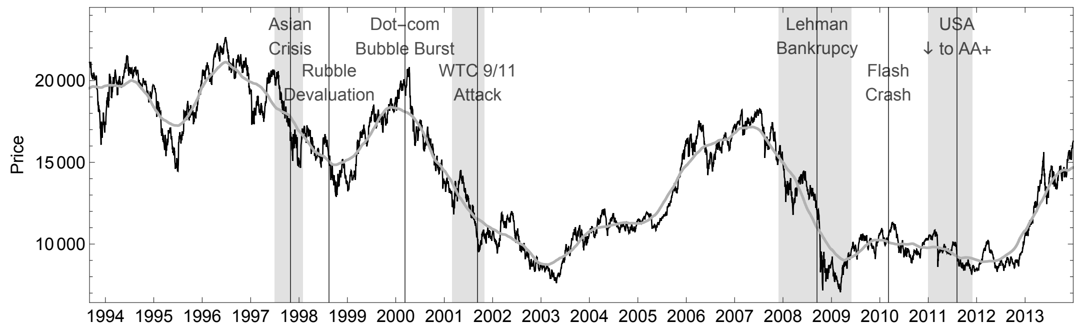


Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 61 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

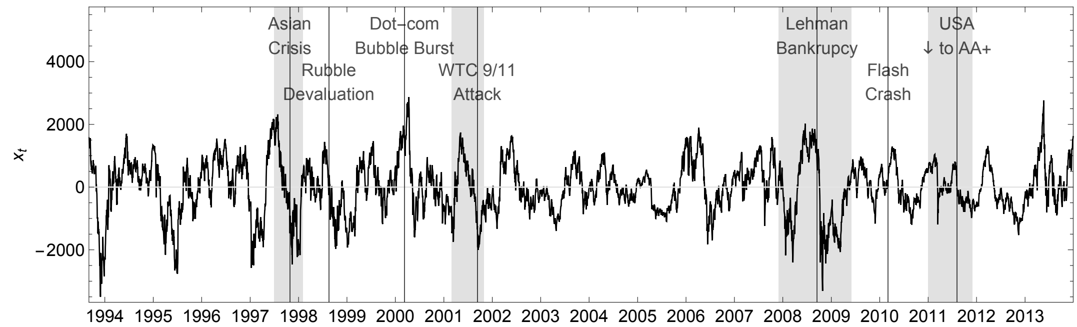
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure C.10: NIKKEI 225 fundamental price MA241 approximation

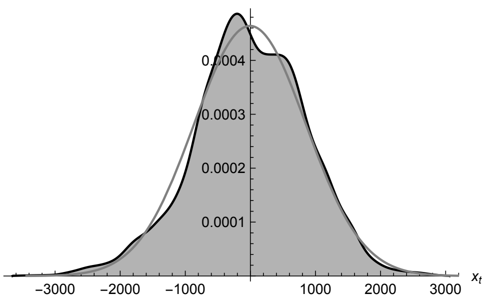
(a) MA window 241 days



(b) x_t time series



(c) x_t histogram



Note: (a) depicts the original price p_t in blacks and the fundamental price p_t^* approximation via 241 days centred MA in light grey. (b) plots the implied $x_t = p_t - p_t^*$. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey). (c) shows the same data as (b) in a smooth histogram kernel approximation format in black together with the fit of $N(\mu, \sigma^2)$ in grey.

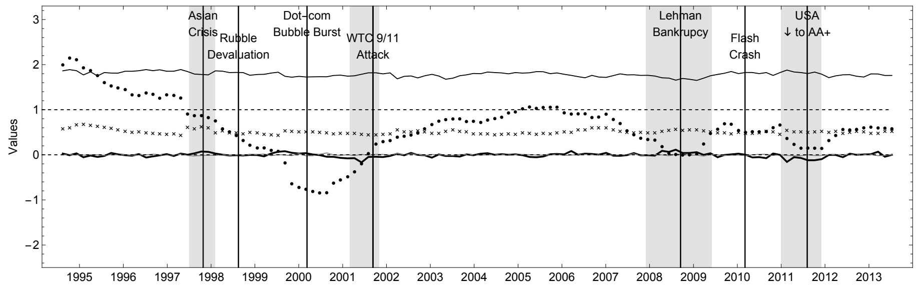
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Appendix D

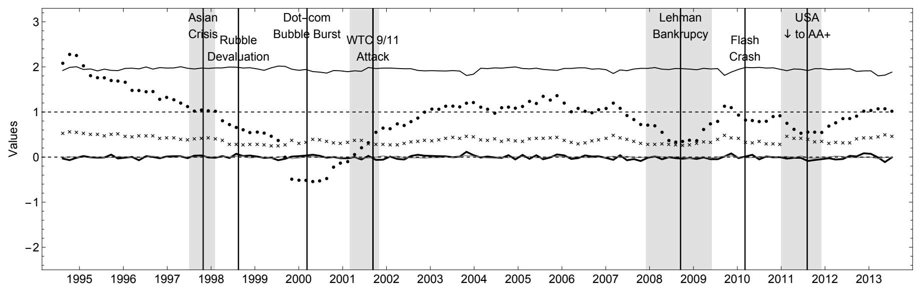
Rolling HAM estimates

Figure D.1: Rolling estimates of the 2-type β model for NASDAQ

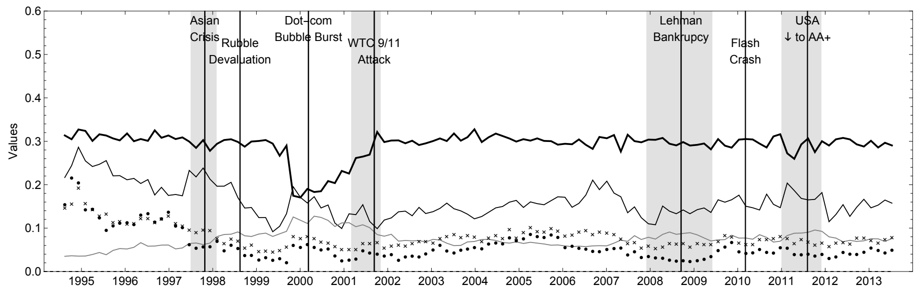
(a) MA61 fundamental price approximation



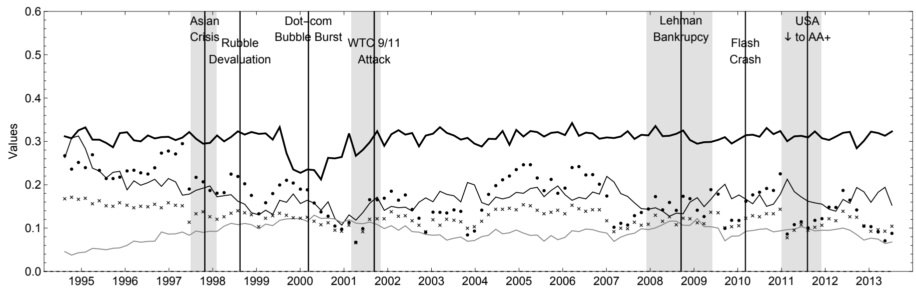
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

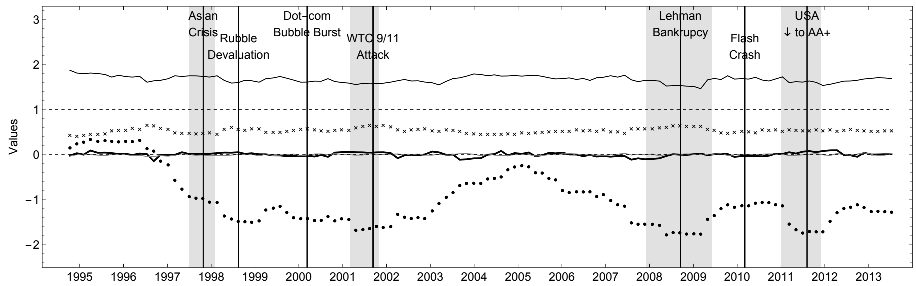


Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

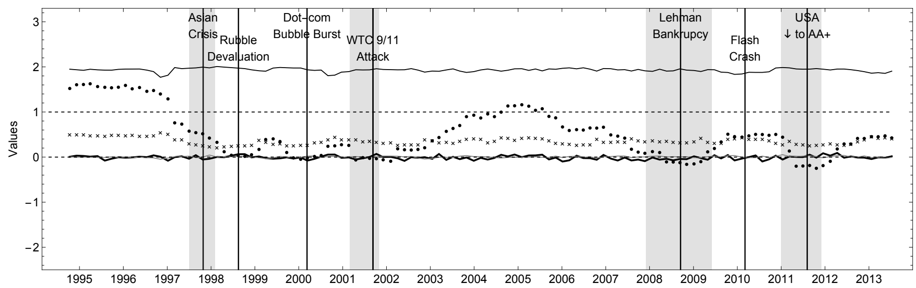
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.2: Rolling estimates of the 2-type β model for DAX

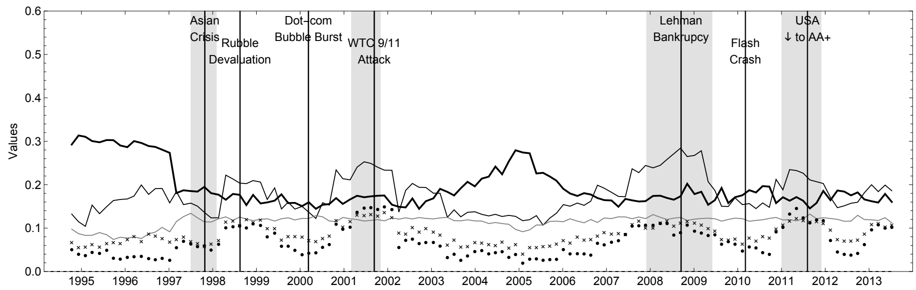
(a) MA61 fundamental price approximation



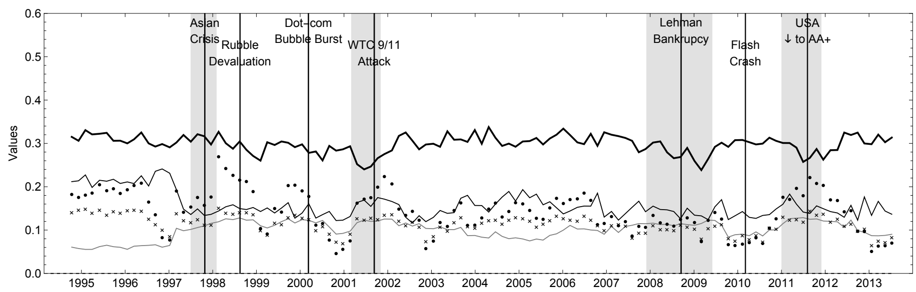
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

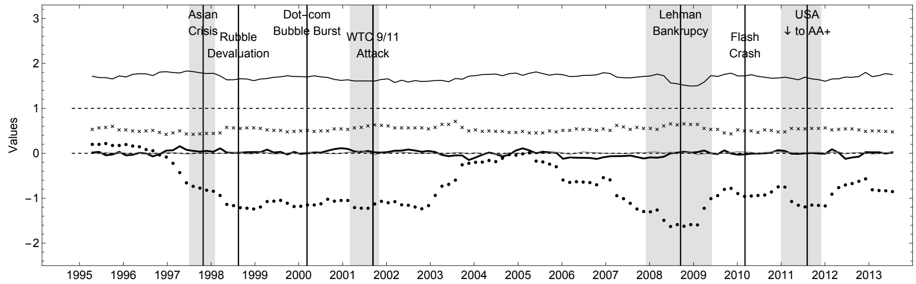


Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

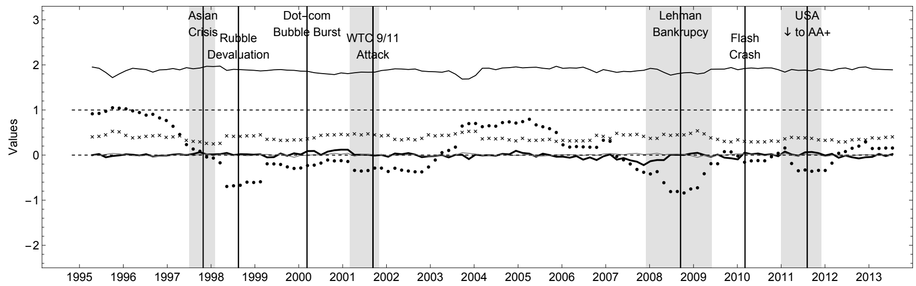
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.3: Rolling estimates of the 2-type β model for FTSE

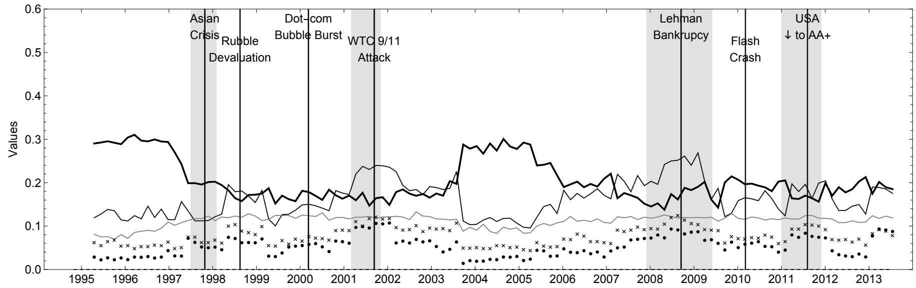
(a) MA61 fundamental price approximation



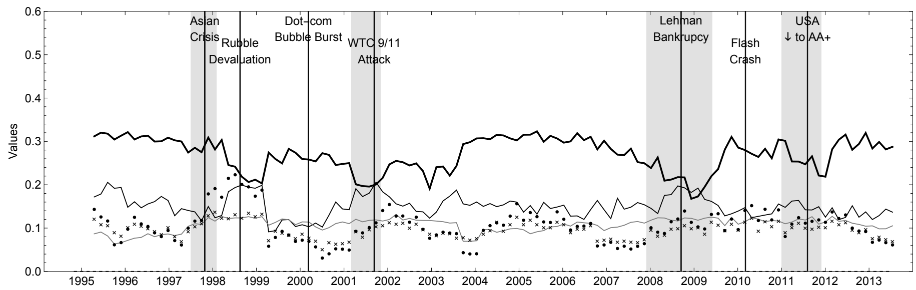
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

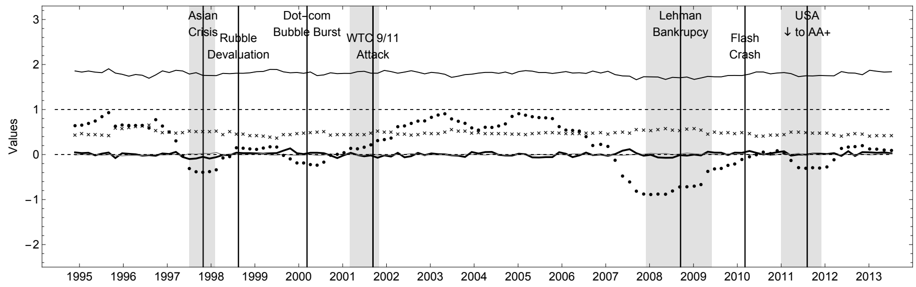


Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

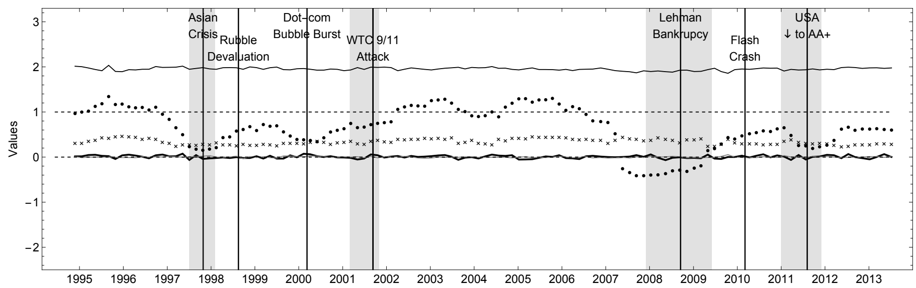
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.4: Rolling estimates of the 2-type β model for HSI

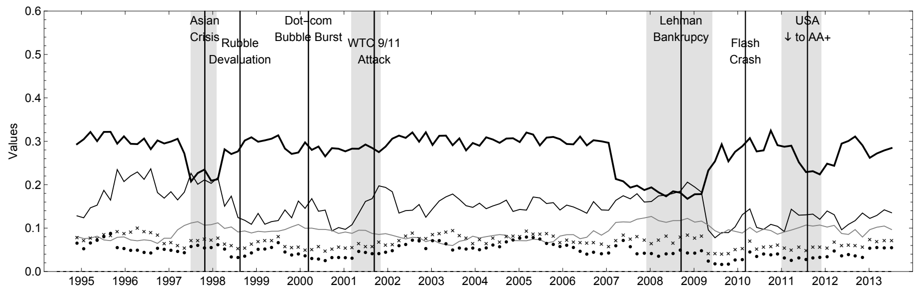
(a) MA61 fundamental price approximation



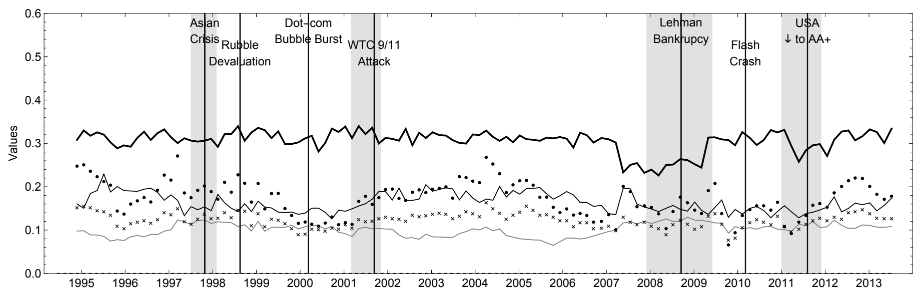
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

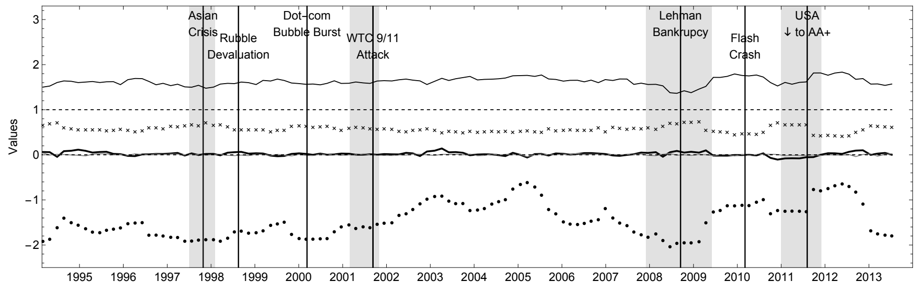


Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

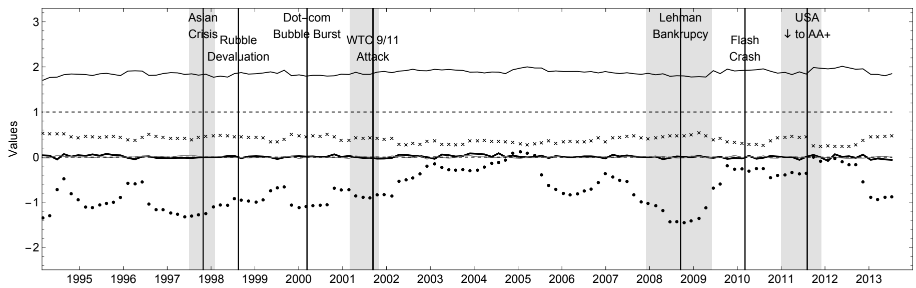
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.5: Rolling estimates of the 2-type β model for NIKKEI 225

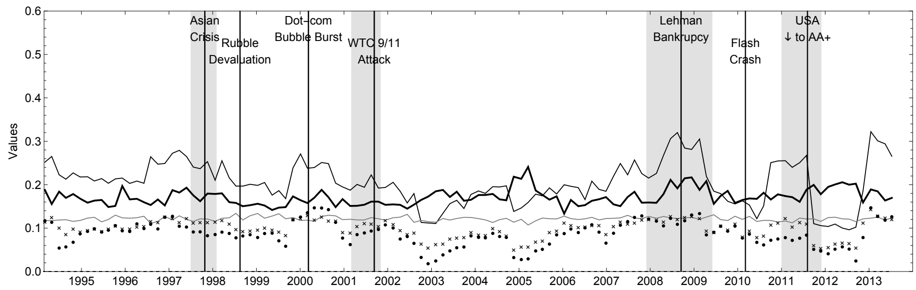
(a) MA61 fundamental price approximation



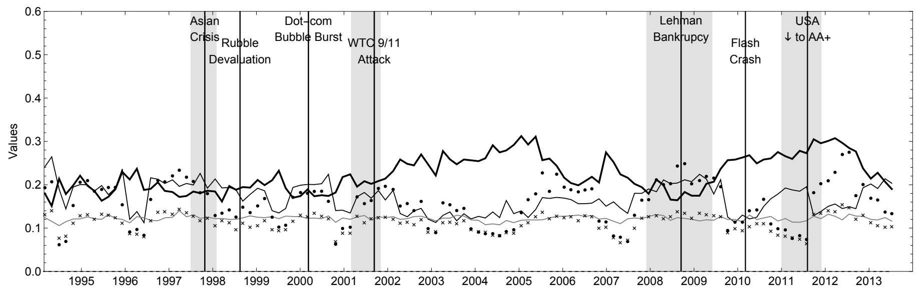
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

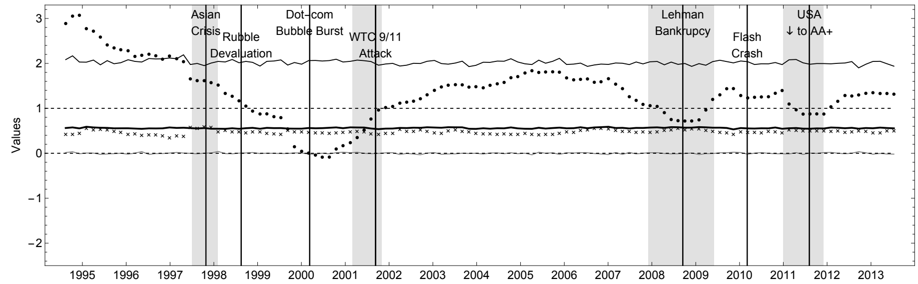


Note: Bold black full line depicts $\widehat{\beta}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and LL are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

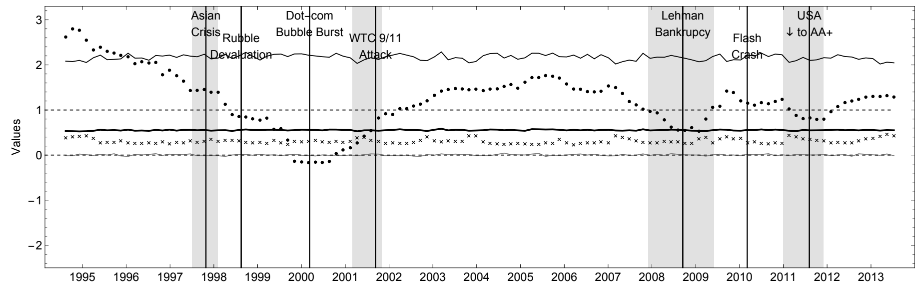
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.6: Rolling est. of the 2-type *fraction* model for NASDAQ

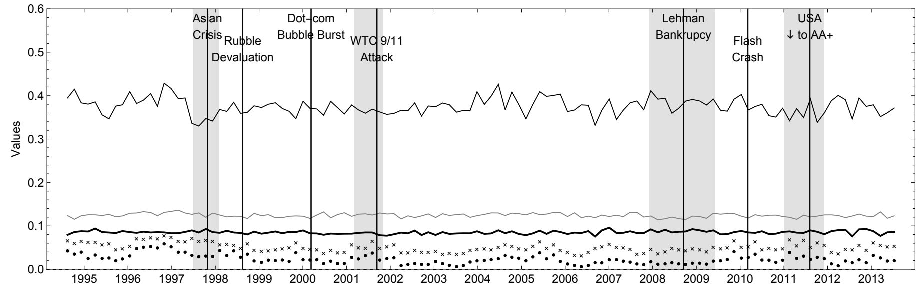
(a) MA61 fundamental price approximation



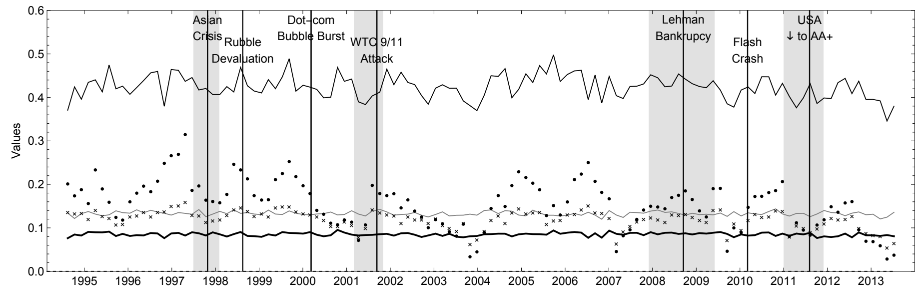
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

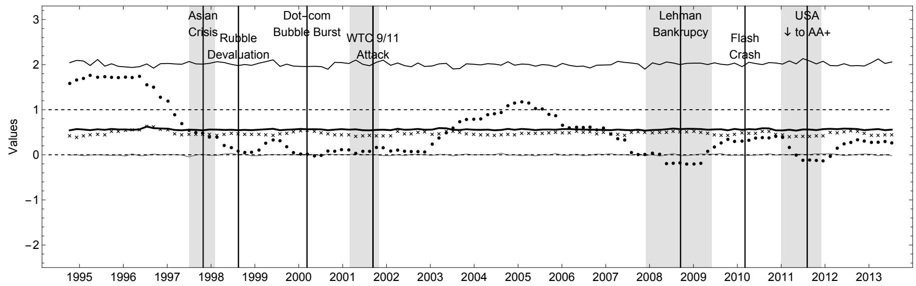


Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $L+6L$ are represented by × and •, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

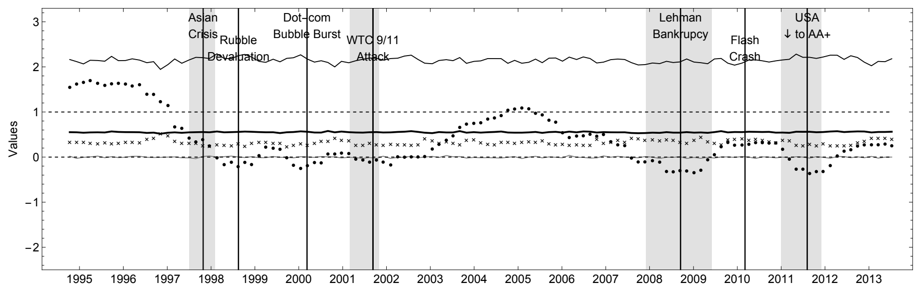
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.7: Rolling estimates of the 2-type *fraction* model for DAX

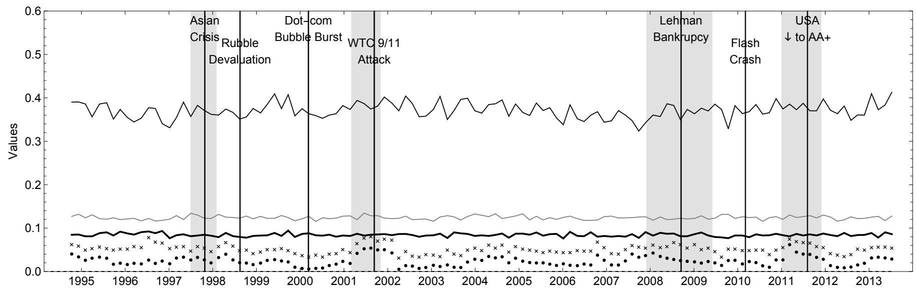
(a) MA61 fundamental price approximation



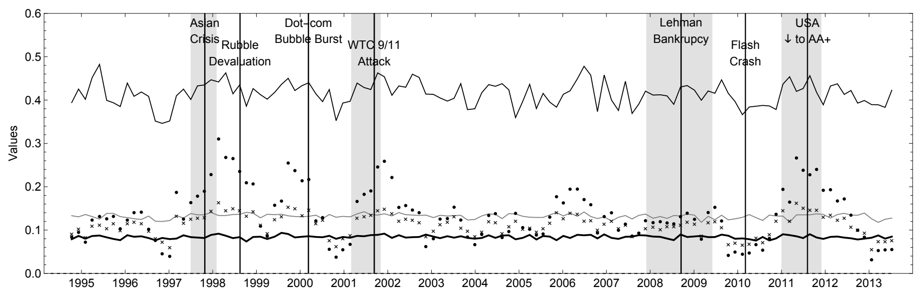
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

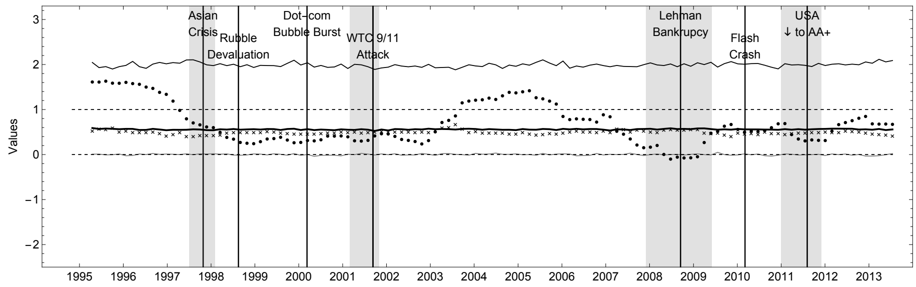


Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $LL+6$ are represented by × and •, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

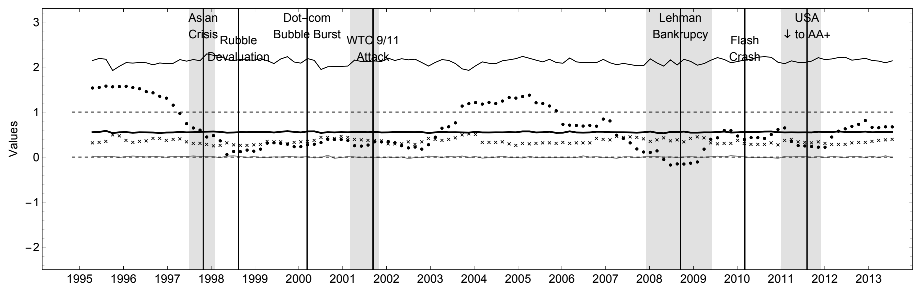
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.8: Rolling estimates of the 2-type *fraction* model for FTSE

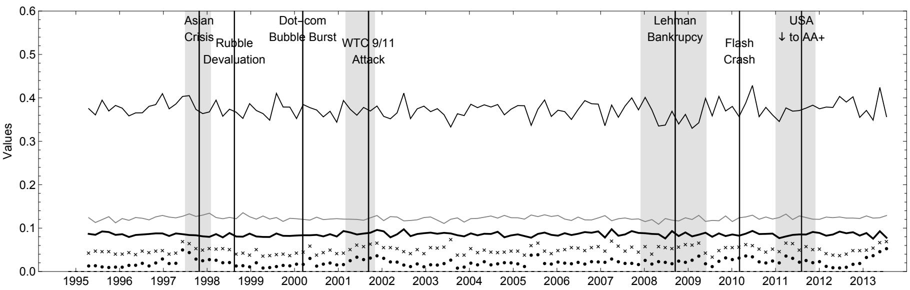
(a) MA61 fundamental price approximation



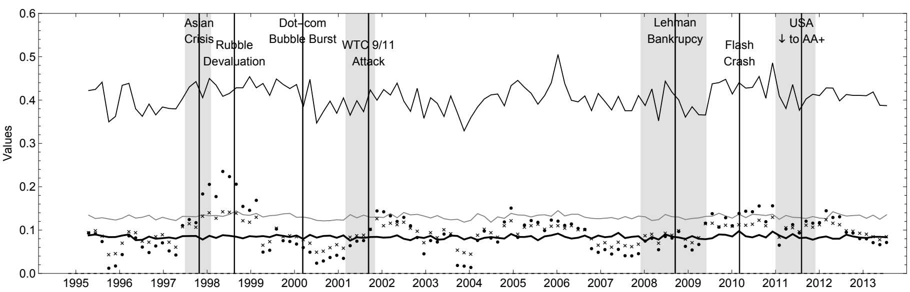
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

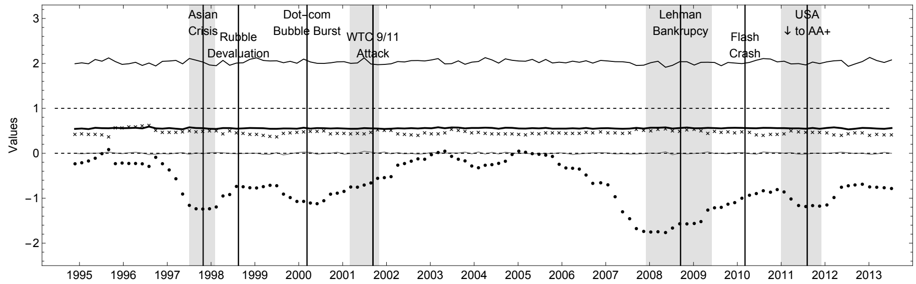


Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $LL+6$ are represented by × and •, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

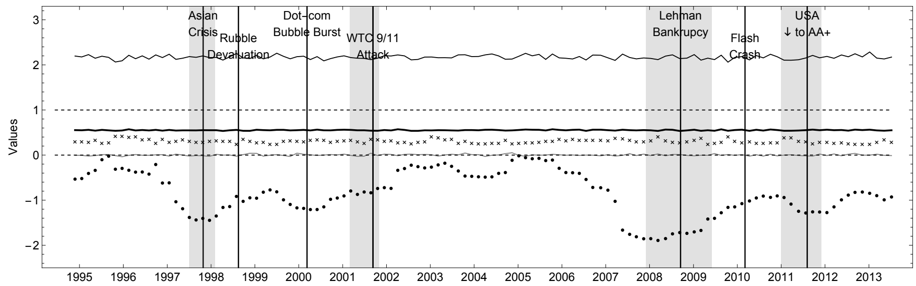
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.9: Rolling estimates of the 2-type *fraction* model for HSI

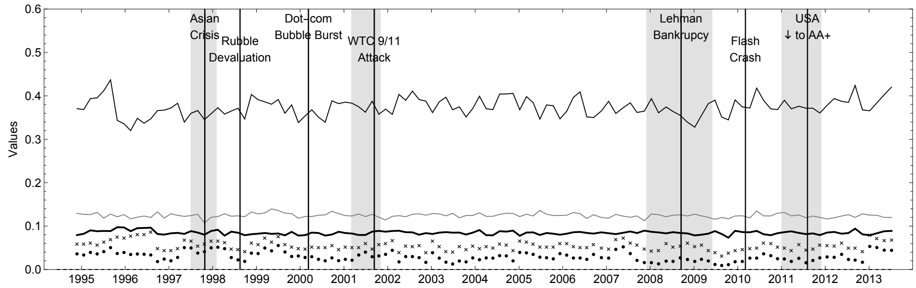
(a) MA61 fundamental price approximation



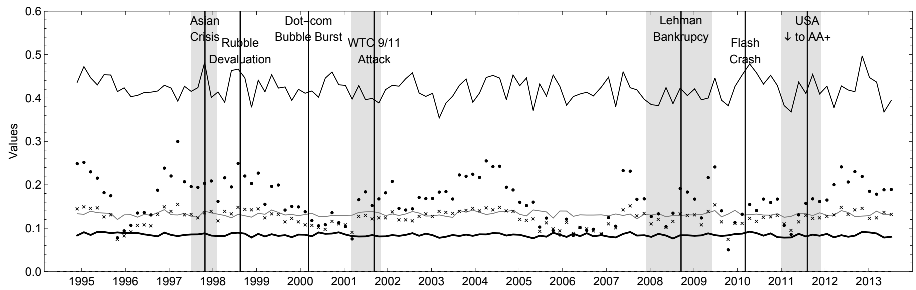
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations

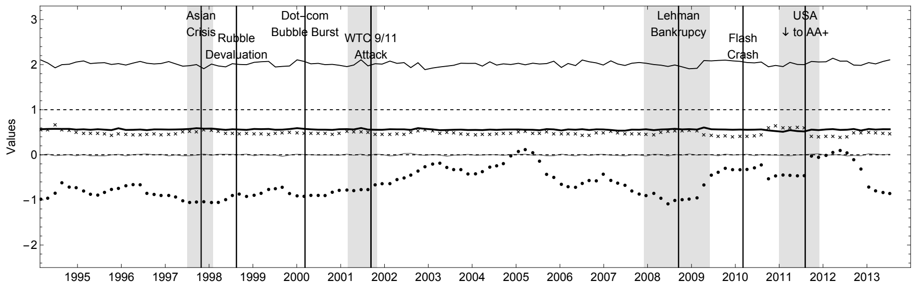


Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $LL+6$ are represented by × and •, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

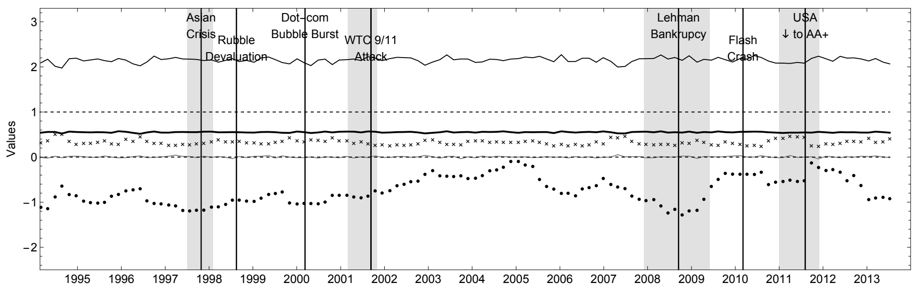
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure D.10: Rolling est. of the 2-type *fraction* model for NIKKEI 225

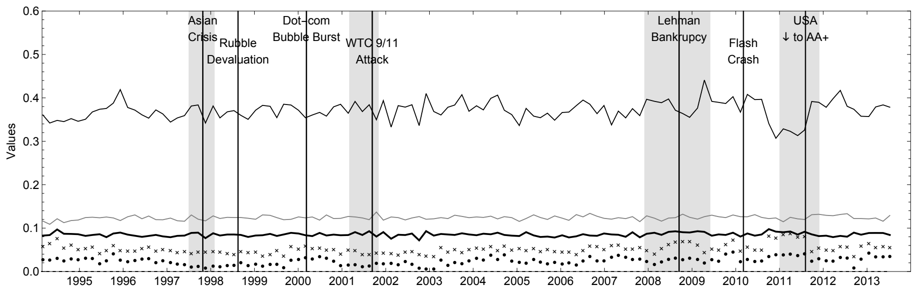
(a) MA61 fundamental price approximation



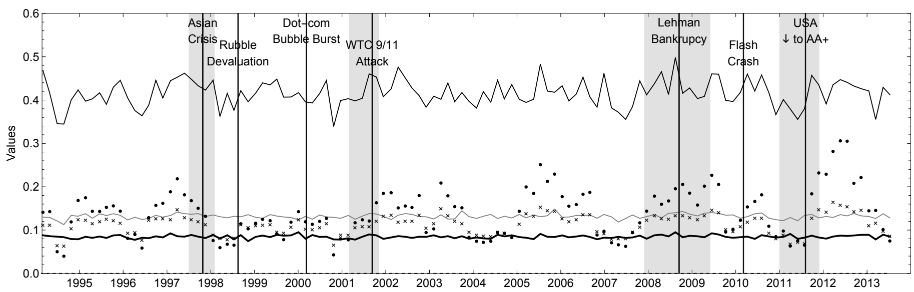
(b) MA241 fundamental price approximation



(c) MA61—related standard deviations



(d) MA241—related standard deviations



Note: Bold black full line depicts $\widehat{fraction}$, black full line depicts $\widehat{g_2}$, and grey full line depicts $\widehat{b_2}$. $\widehat{noise\ intensity}$ and $LL+6$ are represented by $\times$ and $\bullet$, respectively. Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$ i.i.d. draws from normal distribution. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

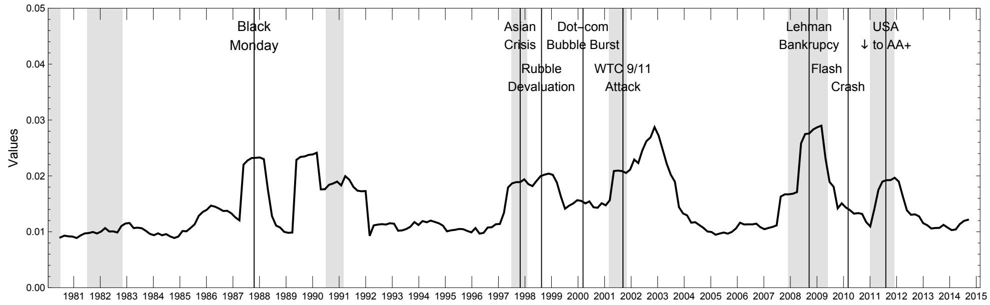
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Appendix E

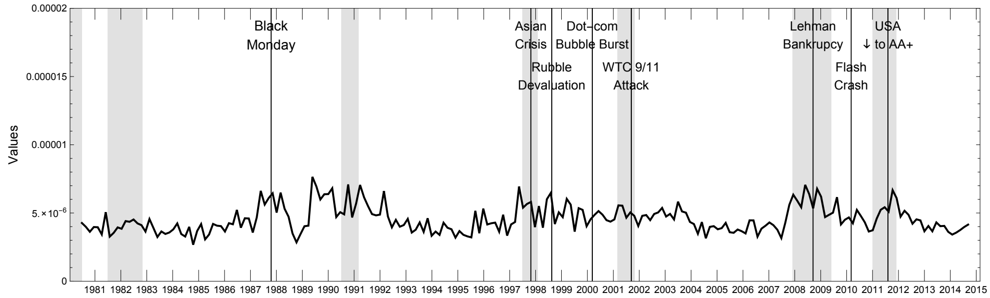
Rolling Alfarano et al. model estimates

Figure E.1: Rolling estimates for DAX

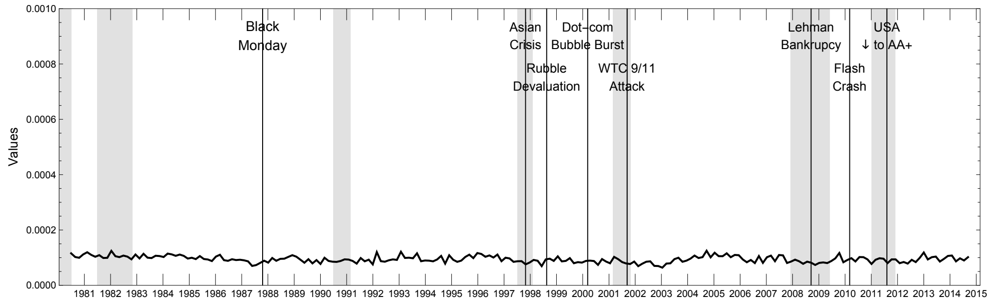
(a) $\widehat{\sigma_f},\ meanSD = .0026$



(b) $\widehat{b},\ meanSD = .0000128$



(c) $\widehat{a},\ meanSD = .000089$

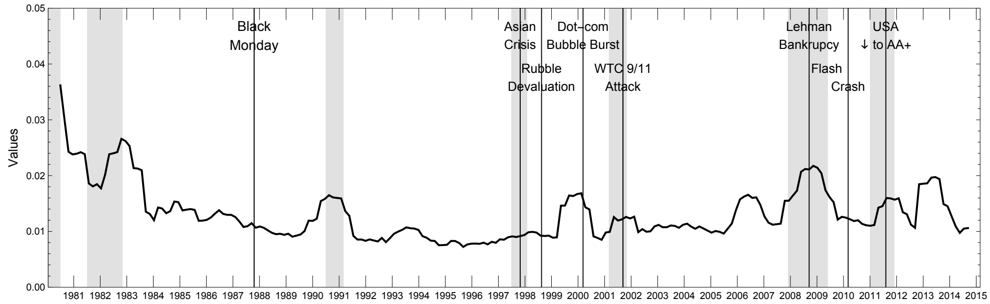


Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

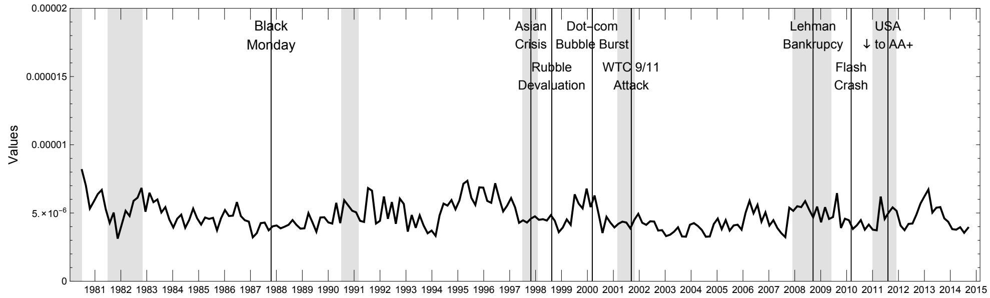
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure E.2: Rolling estimates for GOLD

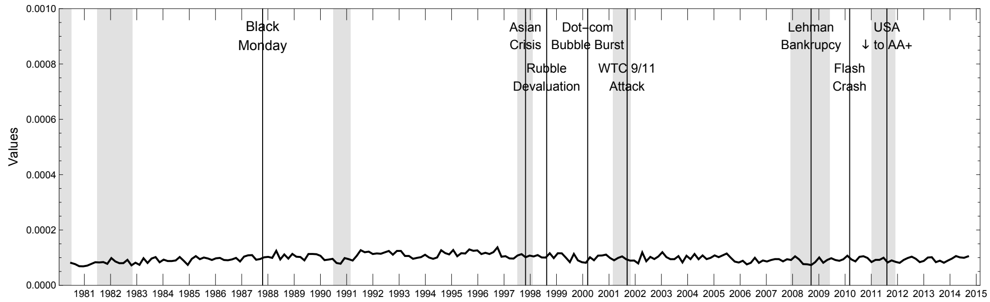
(a) $\widehat{\sigma_f},\ meanSD = .0026$



(b) $\widehat{b},\ meanSD = .0000133$



(c) $\widehat{a},\ meanSD = .000088$

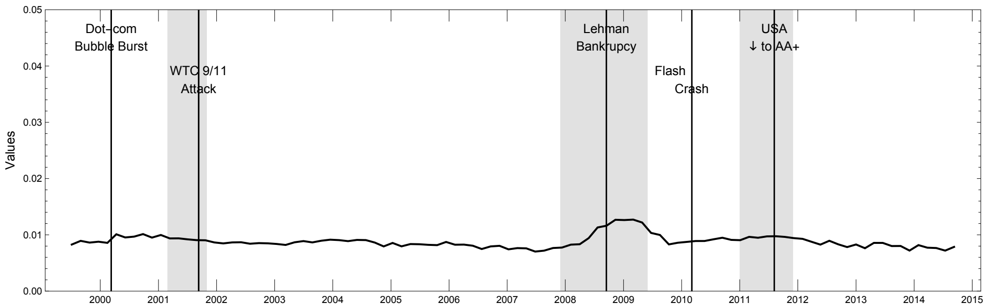


Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

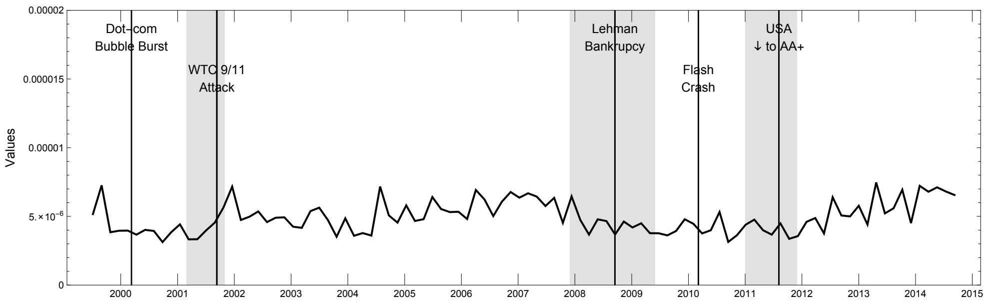
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure E.3: Rolling estimates for U.S./EUR

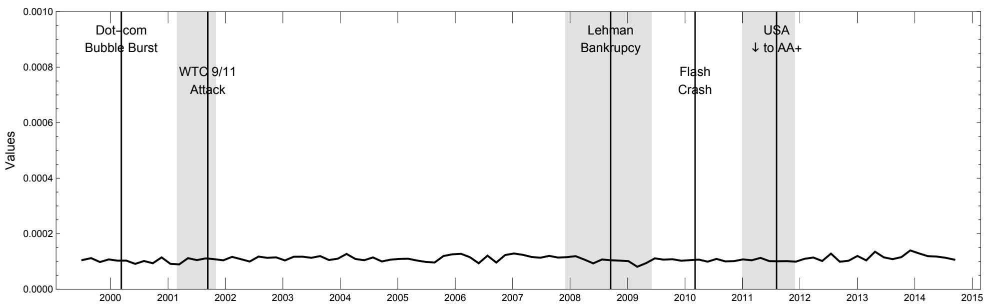
(a) $\widehat{\sigma_f},\ meanSD = .0029$



(b) $\widehat{b},\ meanSD = .0000132$



(c) $\widehat{a},\ meanSD = .000087$

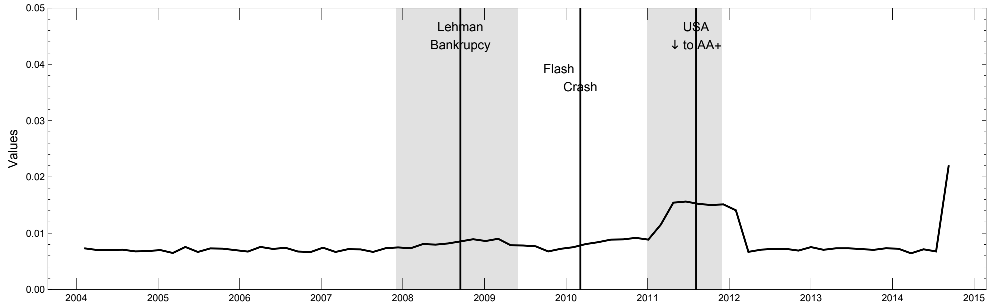


Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

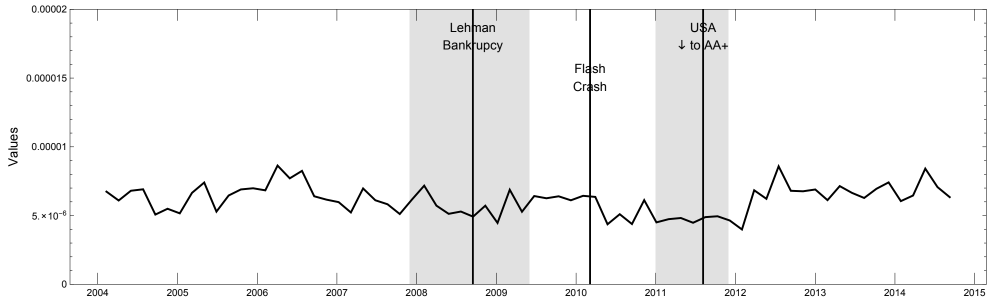
Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Figure E.4: Rolling estimates for EUR/CHF

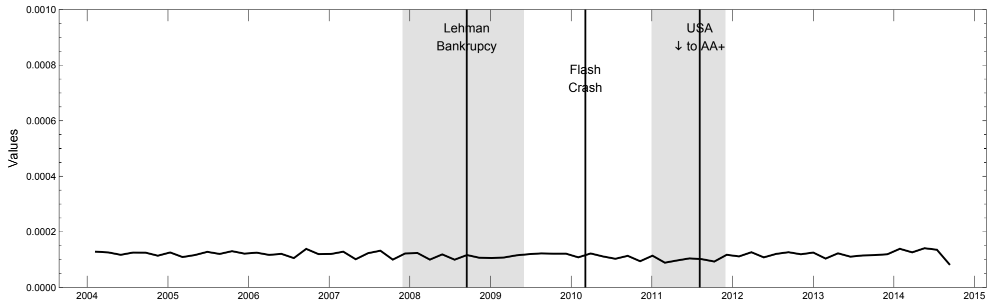
(a) $\widehat{\sigma_f}$, $meanSD = .0033$



(b) $\widehat{b}$, $meanSD = .0000150$



(c) $\widehat{a}$, $meanSD = .000086$



Note: Results are based on 200 random runs, length of the rolling window is 240 days with 40 days steps, and the kernel estimation precision $N = 500$. Sample medians are reported. The figure also depicts several important stock market crashes (marked as vertical black lines) and recession periods (depicted in grey).

Source: Author's own computations in *MATLAB* and *Wolfram Mathematica*.

Appendix F

Response to opponents

for the Dissertation Defense
February 26, 2016

on manuscript
Estimation of Financial Agent-Based Models
by Jiri Kukacka

Dear Dr. Gerba, Dr. Vacha, and Dr. Zwinkels,

at the beginning, allow me to greatly thank you for your detailed discussion and thoughtful suggestions on the pre-defense version of my dissertation. I honestly appreciate your careful reading of the thesis and many insightful comments. I have taken into account all your proposals and I believe that the thesis has gained greater scientific value after all changes I have made. This report provides a response to your Opponent's Reports with detailed discussion and list of revisions.

Note: text in *italics* is copied from the Opponent's Reports to remind the context for particular answers. I also specify respective pages for each comment where the text of the dissertation has been modified or amended.

Response to comments from Dr. Gerba

Q: ***Introduction:*** *The introduction to the FABM is extremely vague and excessively broad. The student does not show he understands the reasons behind the rational expectations (RE) departure of recent models, and what these new models are trying to achieve. Relevant references are missing, as well as a conceptual comparison of the standard RE models with their heterogeneous AB counterpart. Also there is no discussion on the 'endogeneity problem of heterogeneity'. The student should attempt to answer how much heterogeneity truly exists in financial markets, and how much is 'supra-imposed' by the scientist. This would help to focus much more the introduction and literature review of the thesis. At the moment, the student simply lists a few references, with no clear focus or conclusion on where the literature (currently) stands. Therefore, synthesis is required. Some references that can assist the student in structuring the ideas around heterogeneous agent-models is Branch and Evans (multiple publications), or a recent work that I performed on RE versus AB modelling (De Grauwe and Gerba (2015), "Stock Market Cycles and Supply Side Dynamics: Two Worlds, One Vision", Banco de Espana Working Paper Series (forthcoming)). Paper is attached.*

A: Thank you very much for this guidance how to restructure and complete the Introduction, often the most 'tricky' part of the entire work. According to above mentioned suggestions, I have added several pages to cover all proposed areas. In consideration of a reasonable length of the introductory chapter, I have elaborated an essential part of the literature synthesis within Chapter 2 in new sections '2.1 Estimation vs. calibration', '2.4 Categorisation of findings', and '2.5 Best practices from DSGE estimation' that I comment on in more details in my further answers below. In the Introduction, I have completed the text by a conceptual comparison between 'traditional' RE models and 'modern' ABMs and further via a discussion of the importance of the ABMs development for Economics. I have also outlined issues related to but also advantages of the RE Hypothesis and motivation for the 'AB paradigm shift' in reaction to unrealistic assumptions of the RE framework, the 'Aggregation Problem', criticism of the EMH, and others. Regarding the 'endogeneity problem of heterogeneity', based on preceding introduction of the term and a discussion on the design of FABMs that is largely motivated by empirical evidence, I conclude that for the 2-type, 3-type, or generally 'N-type' models', the phenomenon of supra-imposed artificial heterogeneity does not seem to play an important role, actually, researcher rather balance on the other side of the problem when designing relatively trivial models that account only for the most robust heterogeneous features of real markets. [pg. 2–8, 12–17, 28–38]

Q: ***Literature Review:*** *What are the advantages (or marginal benefits) of estimating AB-models compared to simply calibrating them? Are there any advantages/benefits that are different from the ones in standard financial/ macroeconomic models? Much of the recent DSGE/APT literature has shown that estimations of those have serious limitations, such as identification issues, limited number of parameters allowed to be estimated, or the lack of structural interpretation. This has led some to argue for calibration as the more 'trustworthy approach. Is that the same in (non RE) AB-models, or do other principles apply?*

A: This is a very important methodological question and topic not solely related to AB-models. A new section '2.1 Estimation vs. calibration' summarising, comparing, and discussing advantages and shortcomings of both approaches has been added to Chapter 2 'Literature Review'. It is also evident that the evolution of the DSGE methodology in the last decade constitutes a crucial source of knowledge, experience, and hints for current development of validation methods for ABMs as many challenges clearly overlap. Issues discussed in the new section are thus shared also by the ABM field to the similar extent as the subject matter.

Compared to DSGE models and Macro ABMs, a distinguishing feature of FABMs is the structure of the output which almost always is a single time series. Thus, there are no cross-correlations observed and usual autocorrelations are employed in a standard set together with other financial stylised facts. The modellers are thus likely to avoid possibly arbitrary decisions what information should be used for calibration and what reserved for model testing. On the other hand, FABMs often share with DSGE models problems related to flat likelihood function—most frequently likelihood encompasses little information in a direction of a particular parameter. In such cases, according to Fagiolo & Roventini (2012, pg. 81), informal calibration might be "a more honest and internally consistent strategy to set up a model". When it comes to estimation, opposed to DSGE models and Macro ABMs with a relative large number of parameters, where empirical estimation might not be feasible or advisable, simple FABMs might contain only few parameters that often do not have any obvious empirically measurable counterparts. It might even seem that some researchers 'deliberately' design simple stylised or highly aggregated models to facilitate computations and estimation of a small set of crucial parameters, but such an approach is fully scientifically legitimate and follows so called KISS modelling principle.

This thesis focuses on three well-known FABMs. In the Cusp model (Zeeman 1974,Thom 1975, see Chapter 3), the parameters are to a large extent artificial without any apparent economic interpretation. They rather represent weights of control variables that need to be optimised to fit the data. I can hardly imagine a reasonable calibration procedure in this case. For the Brock & Hommes (1998) model (see Chapters 4, 5, 6), the crucial switching coefficient—the intensity of choice β—needs to be retrieved from data also using some optimisation technique. Since the literature lacks a general consensus either on existence of behavioural switching on various markets or its intensity, the calibration approach would not be of much help in this situation. As aptly summarised by Chen *et al.* (2012, pg. 202), "supposing that we are given the significance of the intensity of choice in generating some stylized facts, then the next legitimate question will be: can this intensity be empirically determined, and if so, how big or how small is it?". In some ways similar uncertainty hinders calibration also for the market noise intensity that we estimate in Chapter 6. Conversely, some of model parameters allow for calibration based on micro-studies or literature surveys, e.g. the overall market risk aversion or specifications of trading strategies on specific markets. Finally, in the Alfarano *et al.* (2008) model (see Chapter 7), the autonomous and herding switching intensities also do not have reasonable empirical proxies. Clearly, the model allows for calibration using the 'natural sciences' approach to replicate a set of financial stylised facts—that was perhaps done by the original authors as well as by Chen & Lux (2015) and Ghonghadze & Lux (2015) resulting in the proposed simulation setting—but then an interesting scientific question appears whether one can estimate comparable values from the market data. [pg. 12–17]

Q: Table 2.1 and 2.2 are incomplete. To give the reader a better overview of the literature, I would also include information such as: Total number of parameters in the model, number of estimated parameters, number of endogenous variables in the model, number of simulation periods, the statistical fit of the estimations (i.e. R-squared, or analogues), the estimated values of the beta parameter (where relevant).

A: Thank you for the suggestion, Tables 2.1 and 2.2 have been supplemented by the following information (resulting in creation of a new Table 2.3) that I believe is the most beneficial for the subject matter of the thesis: total number of estimated parameters; data frequency coded by 'd/w/m/q/a' for daily/weekly/monthly/quarterly/annual; number of observations/periods; type of data coded by 's/fx/c/g/re' for stock markets/FX/commodity markets/gold/real estate; statistical fit of the estimation (R^2, its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and the absolute estimated value of the intensity of choice—the switching parameter from the multinomial logit model—together with its statistical significance/insignificance at 5% level coded by 's'/'i'. [pg. 19–21]

Q: To iterate on my point above, it is not clear what the 'take home' message of the literature review section is, apart from studies not agreeing on the parameter values. More analytical work on the review is needed. In particular you should answer questions such as: Are there any patterns in the studies that you mention which can assist you in categorizing them according to methods, structures, dimensions, etc? In what cases do estimations perform better (and with what methodologies)? When do estimations fail? Are the only issues with estimations of AB model the dimensionality curse and non-linearity complication, or do these studies point to some additional problems?

A: I have elaborated a new section '2.4 Categorisation of findings' where I especially focus on aspects mentioned in this comment. I aim at providing a less robust and possibly incomplete or ad hoc, but deeper and more quantitatively based alternative categorisation based on findings in Chapter 2 and focused on empirical estimation aspects of the analysed models. Given the fact that "a strongly heterogeneous set of approaches to empirical validation is to be found in the AB literature" (Fagiolo *et al.* 2007, pg. 199), categorisation attempts are far from being a simple and clear-cut task. A special attention is devoted to possible connections between estimation methods and types or frequencies of data, to the performance of estimation methods based on model fits and its relation to analysed categories, and to various aspects of the statistical significance of behavioural switching. [pg. 28–34]

Because many analogies can be found between ABMs and DSGE models and several econometric issues evidently overlap, I have further added a new section '2.5 Best practices from DSGE estimation' where I outline some important sources of guidance, experience, hints, but also caution, that may be potentially utilised or adapted within the field of ABMs estimation. [pg. 35–38]

Q: ***Chapter 3 and onwards:*** *You do not provide a motivation on why you have chosen to estimate those parameters in the model over others. Did you choose them because they are elementary for driving the full model dynamics, or did you use any other criteria when selecting specifically those? (For instance, why is the beta parameter so important for the micro-dynamics of AB models?)*

A: In accordance with the opponents' suggestion, I have now added further motivation about the choice of estimated parameters to relevant parts of the text, i.e. pg. 55, 89, 112, and 161. I also added a (last) paragraph [pg. 16–17] of the new section '2.1 Estimation vs. calibration' summarizing motivation for estimation of particular parameters and discussing why calibration would not be of much help in these specific situations. Conversely, I also suggest there for which parameters a calibration based on micro-studies or literature surveys might be an advisable strategy. Finally, I have amended the sections explaining the model and estimation setups [pg. 83–84, 87–90, and 89] by sources of rationale and inspiration for the specific setting, what previous literature it follows, where has it already been successfully applied, etc.

An important advantage of simple FABMs is that their dynamics is mostly driven by a few crucial parameters. As a result, we might promisingly attempt to estimate all essential coefficients simultaneously and thus we do not need any rigorous criteria for selection. Moreover, these parameters often do not have obvious empirically measurable counterparts that would allow for alternative calibration. Some other parameters, on the other hand, can be interpreted as scale factors. For those coefficients, a calibration is an advisable strategy because adding too much parameters into the estimation problem may deteriorate outcomes and bring more disadvantage then new knowledge. I believe that the extended discussion and also findings from relevant literature offer a good justification of my choice, i.e. that I for each analysed model managed to select an optimal set of key parameters that can on one hand be reasonably well estimated and on the other hand brings sufficient information about the model dynamics.

In the cusp I simply estimate all parameters of the cusp equation, i.e. ω_0 and ω_1, defining the first order approximation of a smooth transformation of the actual state variable, together with six parameters $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ determining two independent variables. At first, I estimate all parameters of the cusp equation and contrast these result to estimation results of the restricted model with $\alpha_2 = \beta_1 = 0$ according to our hypothesis about primary driving forces of the asymmetry and bifurcation sides of the model.

In the Brock & Hommes (1998) setting I select estimated parameters consistently with the current literature (see Tables 2.1 and 2.2), i.e. the key switching parameter β and the behavioural belief coefficients. The intensity of choice β is the most important parameter influencing the dynamics of the system through the multinomial logit model of a continuous adaptive evolution of market fractions. Not only its magnitude between two extreme cases $\beta = 0$ and $\beta = \infty$ is important, but β also determines the type of the model equilibrium that can generally take the form of a (multiple) steady state(s), cycles, or even chaotic behaviour. The intensity of choice β is also crucial for its conceptual importance—it represents the dominant approach how the boundedly rational choices of agents are mathematically modelled in the current literature indicated by the ABS origin in Tables 2.1 and 2.2. The other coefficients, e.g. the risk aversion a, the conditional variance of excess returns σ^2, or the risk free rate R are simplified already in the original model as constants and shared by all investor types. The model is then theoretically derived based on those assumptions. These parameters only influence the absolute values of the profitability measures U_h but not their relative proportions (R additionally a little bit adjusts the model output x_t). Thus I can naturally consider them not influencing dynamics of the model. Number of strategies is also taken from literature where almost only the simplest 2-type and 3-type models are estimated.

In the Alfarano *et al.* (2008) model I estimate a natural parameter-triplet essential for the model dynamics—the switching rates a and b, and the fundamental volatility σ_f. As argued by Chen & Lux (2015, pg. 11), "adding $\frac{N_c V_c}{N_f V_f}$ as a fourth parameter would deteriorate results by so much that the outcomes of our estimation would become almost useless" due to high autocorrelation between parameters and implied problems with their identification. Therefore the $\frac{N_c V_c}{N_f V_f}$ term, which can, moreover, be interpreted as only a scale factor and thus omitted, is set to 1. There are no other parameters in the Alfarano *et al.* (2008) model.

*Q: **Chapter 3:** page 20: You reference Creedy et al (1993, 1996) and Koh et al (2007) as the closest studies to yours, but they estimate exchange rates. What are the core differences with respect to modelling equity markets (as in your case), and how will this influence the assumptions that you impose on the underlying process? In other words, how do you expect your results to differ from theirs based on the different processes governing Forex and equity markets?*

A: Thank you for an interesting comment. Although FOREX and equity markets might be based on (partially) different underlying processes, in both cases the dominant part of the process would be the assumed cusp model. It is a very general process which principal intriguing part is the possible presence of bifurcation (bistability). The presence of bifurcation is according to current financial literature a reasonable assumption also for FOREX markets (De Grauwe & Grimaldi 2006b;a; De Grauwe & Kaltwasser 2012). Both markets are to a large extent similar and exhibit same important empirical stylised facts of financial time series, so I would expect a different intensity and frequency of crashes, but generally the process should be to a large extent the same. On the other hand, with reference to Alfarano *et al.* (2007, pg. 183), "noise traders seem to dominate in stock markets, while fundamentalists dominate in foreign exchange markets". Thus one of possible hypotheses when utilising FOREX data might be a stronger impact of variables that proxy activities of fundamental investors (compared to the stock market data application) and vice versa for the chartistic side proxy.

However, I would expect that differences in estimated parameters are likely to be more influenced by selected proxy data for the fundamental and chartist side of the FOREX market then by differences in returns between these two markets. The main difficulty in application is the availability of empirical proxies for control variables. For modelling of a single currency pair, any FOREX counterpart of the ADV/DEC ratio does not exist. On the other hand, some fundamental or news-driven indicators might appear useful. For FOREX, the Purchasing Power Parity is often used as a good proxy variable of the fundamental value, but it describes the long-term behaviour of exchange rates and does not affect daily exchange rate changes. The main issue for empirical application thus remains in the availability of data in a reasonable (daily) frequency. Such data are (although paid) available for the U.S. stock market, but I am not aware of any advisable data source in case of FOREX markets. [pg. 44]

Q: Page 28: Could you compare the cusp catastrophe model results to the ones obtained in standard GARCH-type models, or just a plain random walk? (The second is frequently used as a comparison benchmark for model performance in the finance industry).

A: I largely share the opponent's concern in comparison of the cusp results with traditional models and I always report estimation performance comparison with the linear and logistic models in Tables 3.1, 3.3, and 3.4. On the other hand, contrasting the cusp catastrophe model results to GARCH or random walk seems methodologically disputable. The cusp catastrophe framework is based on modelling the endogenous structure of the system based on the interactions of control variables determining the state variable at time t, but GARCH and RW represent temporal dependence modelling based on a single variable an its lags. These are two relatively distinct modelling approaches and their comparison is problematic also because they do not share the same dataset (the cusp model needs market returns and a set of control variables, however, for GARCH or RW market returns are sufficient). Additionally, the ARIMA-GARCH error term does not have any counterpart in the cusp model. The linear and logistic model are natural candidates for estimation performance comparison as they share the similar empirical dataset. A comparison of forecasting performance is another issue for the cusp models as for determining the state variable (market return) in time t the control variables from time t are used—i.e. in contrast to the GARCH family of models or RW, the cusp catastrophe model is not designed for forecasting. [pg. 52]

Another important point connected to this comment is the fact that our two-step estimation procedure could have been applied using any other model for market volatility. A related footnote was added to the text [pg. 43]. However, there is a general consensus in the literature that realised volatility is a supreme model of volatility to use and therefore I adhere to the realised volatility approach.

Q: Page 29: You assume the parameters in the cusp model to be Gaussian. Could you please provide some motivation as many would disagree with that (strong) assumption?

A: This is an important question and I have extended the study by smooth histogram kernel approximations of the probability density of individual parameters depicted in Appendix A, Figures A.1 and A.2 compared to normal distribution. I have further statistically tested the normality via the Jarque-Bera ALM test at 5% level. I have additionally edited the text to state that the distribution of the parameters of our main interest using $y_t = r_t/\sigma_t$ is Gaussian. Based on the Jarque-Bera ALM test at the 5% level, the null hypothesis of normality is only rejected for ω_0 and α_2 of the unrestricted model. Moreover, ω_0 is a constant term and α_2 is left out in the restricted model which further testifies the Gaussianity of key parameters. For the restricted model normality of parameters is not rejected in any case. [pg. 54]

Q: Page 32: Could you provide some descriptive statistics for the time period that you use in your empirical modelling section so that the reader can get a rough idea on how volatile (and asymmetric) those periods were on the US equity market.

A: In accordance with the opponent's suggestion, I have added Table 3.2. [pg. 61] containing eleven descriptive statistics of all important datasets I use for empirical modelling: the S&P500 stock market returns r_t, realized volatility RV_t, daily returns normalized by the realized volatility $r_t RV_t^{-1/2}$ and data for the independent variables—the ratio of advancing and declining stock volume, the OEX put/call options and the change in total volume. I have displayed this set of descriptive statistics for the full sample period 1984-2010 as well as for two crash periods to allow for direct comparison: 1987 crash and 2008 crash analysed in Section '3.4.3 Examples of the 1987 and 2008 crashes'.

Q: ***Chapters 4 and 5:*** *Very nice discussion and the chapters are well-structured and clear.*

A: I am really happy for this appreciation. Thank you!

Q: ***Chapter 6: page 93:*** *Can you run more than 500 runs in order to assure that a convergence in the estimation is achieved? In standard time-series methods between 1000 and 10.000 runs are needed to achieve convergence.*

A: I agree with the opponent that the convergence issue is one of the crucial aspects in the estimation and 500 runs might not be sufficient. Unfortunately, I need to take into account also the considerable computational burden of the estimation procedure (obtaining pre-defense results e.g. for Table 6.2 used to full capacity of a high-speed multi-core server at the Czech Academy of Science for circa ten days). I have doubled number of runs to 1000 and after a detailed check of the new results, I neither observed any considerable difference in absolute values of parameters, nor any pattern in tiny decimal differences (e.g. expected prevailing decrease on standard deviations). I therefore did not continue in increasing number of runs and I finally report final results based 1000 runs in Tables 6.2 and 6.4. (Due to a very problematic computational stability of the 3-type model under S&P500 MA241 fundamental value approximation, I have re-computed results for 1000 random runs only for the S&P500 MA061 case. Again, no considerable difference was observed and therefore in Table 6.3 I consistently report original results based on 500 random runs for both MA061 and MA241 FV approximations.)

W.r.t. to the number of runs, it is important to highlight that in each run the number of starting points for the numerical optimisation is 8 and I choose the best result according to maximal log-likelihood. Technically, the set of 1000 estimates is therefore based on 8000 optimisation calculations to solve possible problematic numerical stability of the model when real data analysis is introduced. [pg. 133]

Q: ***Chapter 7:*** *I am fully aware that this chapter is still work in progress, and that more time will be spent in drafting it. However, I wish to provide some comments/questions that will be useful for the student to take into account when drafting the final version of this piece.*

While the application to the Alfarano et al model is interesting, it is not clear what the contribution of that chapter is (apart from just another application of the same method). Are there are any deeper insights from this estimation that you can bring forward (which were not discussed or hinted in the previously)? Why exactly did you choose the Alfarano et al (2008) model, and not any other? Also 300 runs seem to be a small number in order assure convergence. Could you maybe increase it?

A: The opponent is absolutely correct to point out that motivation for the Alfarano *et al.* (2008) model definitely needs to be extended and more properly discussed in the text. The following sections were added to the text [pg. 154–155]:

"Unlike the most widely used discrete-choice multinomial logit switching rule approach (Brock & Hommes 1998, Equation 4.16) studied in previous chapters, the Alfarano *et al.* (2008) model is based on the other typical ingredient of FABMs—the herding behaviour. The concept of herding represents the second widely accepted principle of possible evolution of market fractions applied in FABMs that can trigger interesting nonlinear endogenous dynamics resulting in large aggregate price fluctuations. Therefore estimation of two models based on these two leading principles—switching and herding—is introduced in the thesis. A further motivation for the analysis and repetitious empirical testing of the Alfarano *et al.* (2008) model comes from somewhat puzzling conclusions of previous estimation attempts of similar concepts via MSM. As concluded by Chen *et al.* (2012, pg. 207), "the Lux model was rejected, similar to the rejection of the ANT model" based on its empirical validation (Winker *et al.* 2007) in favour of ABS. However, the most recent studies (Franke & Westerhoff 2012; Ghonghadze & Lux 2015) accept the model or systems based on the same origin as possible data generating processes with high p-values of the J-test (see Subsection 2.4.4)."

"The model generically replicates the leptokurtic distributions of returns and volatility clustering that are directly linked to the herding component of the model which puts emphasis on "the importance of bounded rational behaviour as a potential explanation of the stylised facts" (Alfarano *et al.* 2008, pg. 125)."

Moreover, the bimodality of the sentiment variable surprisingly brings crucial difficulties to the estimation procedure—a pattern not observed for any setting in the switching Brock & Hommes (1998) model.

Another practical reasoning is that that the same model has recently been investigated by FinMaP colleagues from the University of Kiel, Germany, via MSM and GMM, bringing merits of possible future comparisons.

Regarding number of runs, 300 repeats were applied for preliminary results only. To report final results, I have increased the number of runs to 1000 (taking the advantage of our knowledge from previous chapters).

Q: ***General for the entire thesis:*** *A spelling-check needs to be run to remove all typos. Also, the LaTex PDF-compiler did not compile correctly some of the words. Please, re-run it.*

A: I apologise for these imperfections. I re-read the thesis and ran a spell-check once again to get rid of remaining typos. I also multiply checked the final version of the thesis for possible LaTeXcompiling errors.

Q: At the beginning of the thesis, state how much of the thesis is your own work, and how much is of supervisor(s), co-authors, or others (in percentages). That is normal practice at UK institutions.

A: Thank you for this suggestion. A new page 'Declaration of Authorship' defining how much of the thesis text is the work of the author, and how much is of supervisor has been added. [pg. v]

Chapter 3 has already been published under the full title "Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility" in *Quantitative Finance*, 2015, 15 (6), pp. 959-973. It is a joint work with the thesis supervisor Jozef Barunik and both authors contributed equally to this work. The rest of the (yet unpublished) dissertation thesis text has been composed solely by Jiri Kukacka under the standard academic guidance of the thesis supervisor. The approximate contribution to the text of the dissertation thesis can be divided to 90% by Jiri Kukacka and 10%

by Jozef Barunik. There are no other co-authors, collaborators, or students involved.

Response to comments from Dr. Vacha

I am grateful to Dr. Vacha for his kind assessment of my dissertation. Dr. Vacha recommended the thesis for defense without substantial changes and did not suggest any major comments. I would like to thank to Dr. Vacha for many useful discussions with during doctoral seminars and collectively attended conferences.

Q: However, Dr. Vacha raised an interesting question during the pre-defense regarding possible datasets in which the intensity of choice might be potentially expected statistically significant using NPSMLE. I would be happy to summarise my answer and comment on this question here in more detail than discussed in September 2015.

A: I would suggest four potential categories of data where I personally believe the probability of revealing statistically significant behavioural switching is relatively high. The first candidate are single stocks of specific industries vulnerable to speculative mispricing and subsequent corrections, typically the IT sector, especially in specific periods as the Dot-com financial bubble. However, a related research would necessarily require arbitrary decisions regarding the selected data, sample coverages, etc., and embody signs of data-mining. Moreover, the fundamental value approximation would be largely problematic and unrepresentative for single stocks compared to aggregate market indices.

The second candidate are low-frequency data, that is, quarterly or annual observations for which, on the other hand, the fundamental value approximation is much simpler, e.g. via the dynamic Gordon growth model following Boswijk *et al.* (2007) or using the Purchasing Power Parities as the fundamental rate for FX data. However, as analysed in Section '2.4.6 Switching', so far no conjunction can be observed between the intensity of choice and the frequency and length of the data as statistically significant as well as insignificant findings are reported across these categories without any clear pattern. Combining these two aspects, we might attempt to approximate the fundamental value of single stocks using valuation techniques based on information from annual reports of selected companies.

Third, as interestingly concluded in Section 2.4.6 'Switching', statistically significant estimates largely dominate for commodities. Finally, important insights into behavioural switching might be gained via data from highly speculative markets such as Bitcoin exchanges, where a ‘laymen’ trend extrapolation seems to be a strongly dominating belief principle in several historical periods.

Response to comments from Dr. Zwinkels

In what follows, I will present a list of my comments, in order as they appear in the thesis. Not all of them need to be addressed before the defense, but the author can use them to his advantage. I would like to see the ***highlighted*** *points addressed by the candidate before the final defense.*

Chapter 1:

Q: 1. Pg 1: 'beliefs and expectations'. What is the difference?

A: This is an interesting point. Technically (in the terminology of Chapters 4-6) there is no difference. I personally can feel some narrative distinction, but the comment is absolutely correct, this might be confusing for the reader. I deleted 'beliefs' in the final text as the other term 'expectations' is more standard and clear in Economics. [pg. 1]

Chapter 2:

Q: ***1. Pg 10 , . . . estimate a HAM based oN Kirman's ant. . .***

A: Corrected, thank you for noticing! [pg. 23]

Q: ***2. Page 13, middle of the page: In fact, Frijns et al. (2010) do not use QML to estimate their model, but apply a simulation approach.***

A: I apologise, the entry was corrected in Table 2.2 as well as in the text, thank you! [pg. 20, 26]

Q: ***3. Pg 14, middle of the page: Here, the author is comparing intensity of choice (IOC) parameters that different authors have found. The IOC parameter, however, can NOT be compared across assets or time periods because it is unit free. The magnitude of the estimated IOC parameter is conditional on the exact definition of performance of the rules as well as the exact market characteristics in that particular sample period. Ter Ellen and Zwinkels (2010) propose an alternative switching functions that DOES allow for comparison of the IOC across time/markets. This issue repeats itself on several occasions throughout the thesis.***

A: This is an important methodological comment. The opponent is absolutely correct that a very same values of the intensity of choice are likely to have somewhat different effect across various specification of models/time periods/assets in the same family of switching HAMs. On the other hand, the intensity of choice is a crucial and robust driver of the data generating process behind switching HAMs and to a large extent determines the behaviour of the system in a consistent manner: zero intensity of choice fixes market fractions and does not allow for any evolutionary switching, high values implicate wild switching for vast majority of model specifications, assets, or periods. Relatively small positive intensity of choice is associated with a presence of some detectable behavioural switching. A rigorous comparison is truly not possible and it was not my aim. I mainly intended to avail the general knowledge of previous estimation results from literature for setting meaningful simulation grids in Chapter 5 or to constrain random generation of initial points in Chapter 6. I honestly believe that taking such advantage from knowledge of recent literature makes sense and brings important benefits.

A paragraph with related discussion has been added. [pg. 18, 22, 84]

Chapter 3:

Q: ***1. Entire chapter: Throughout the chapter, the authors uses the "we" form, and he refers to Barunik & Vosvrada (2009) as "we". First of all, this is inconsistent with the rest of the thesis. Second, and more importantly, for the cynical reader (which I am not), this could raise the impression that the author's contribution to this particular chapter are not as much as for the other chapters.***

A: Thank you very much for noticing, the 'we' form was corrected throughout the entire text. Chapter 3 has already been published under the full title "Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility" in Quantitative Finance, 2015, 15 (6), pp. 959-973. It is a joint work with the thesis supervisor Jozef Barunik and both authors contributed equally to this work. However, as Jozef is the co-author of both Cusp research projects (Barunik & Vosvrda 2009; Barunik & Kukacka 2015), this appeared in the text of the paper as well as Chapter 3.

A new page 'Declaration of Authorship' defining how much of the thesis text is the work of the author, and how much is of supervisor has been added. [pg. v]

Q: 2. The model used in this chapter, the CUSP model, originates from the natural sciences. I would like to see a more thorough motivation of why and how it is possible to apply this particular model (without modification) into a social science, what economics is.

A: There are many applications of the cups model in psychology, political science, sociology, marketing, and also economics. I have elaborated an extended literature review comprising and motivating both applications in other natural sciences as well as in social sciences. [pg. 40–42]

Q: 3. Pg. 19: The author argues that the availability of high-frequency data and the Realized-Volatility (RV) methodology allows him to use the two-step procedure as presented in the chapter. In fact, this is NOT the case, because the two-step procedure can be applied using ANY model for market volatility. I agree that using RV is the BEST model to use, however, it is not the ONLY possible model (think of GARCH type models or Heston type stochastic volatility models).

A: I do agree and a related footnote has been added to the text. In any case, using the supreme model of volatility is likely to produce the best results, therefore I adhere to the realised volatility approach. [pg. 43]

Q: 4. The two step method in this chapter is born from the restriction that the CUSP model requires data with stationary volatility. As such, the authors proposes to first normalize the input data by dividing returns by their realized volatility before estimating the CUSP model. This is an intuitive way to address to issue. This method, however, is somewhat counter intuitive to me because FABMs are especially designed to be able to generate time-varying volatility! Hence, the CUSP model itself should, theoretically, be able to generate time varying volatility. As such, estimating the model itself in the raw data could, theoretically, resolve the problem (as long as the residuals have stationary variance, this should be consistent with the model's assumptions). As such, I would like to see a comparison in the chapter in which the author first estimates the model in the raw data, then on the normalized data, and compare the results.

A: Thank you for an insightful comment. The constant difusion function is an assumption of the catastrophe theory that was developed in the framework of natural sciences (biology). As such, it was also adopted by the estimation procedure (Grasman *et al.* 2009). The cusp model can be understood as one of the first HAMs not because it replicates the stylised facts of financial time series but because of its economic interpretation by Zeeman (1974). Following Zeeman (1974) the application of the cusp model in Finance utilises the idea of interaction between evolving populations of fundamentalists and chartists as all later HAMs. In fact, the model itself produces time series with constant volatility (see Fig. 3.1 b), volatility clustering appears only when time-varying volatility is artificially added (see Fig. 3.1 c). A comparison of estimation using raw and normalised data is always offered in all important cases both for the Monte Carlo study as well as for the empirical estimation: right (b) parts of Tab. 3.1, 3.2. Naturally, normalised data always perform significantly better.

In a sense, our approach can be viewed as generalisation of the original cusp model, that in the two equation setting is able to produce time-varying volatility and reproduce the stylised fact about volatility of returns.

Q: ***5. Pg 34: in the data description, exogenous data is introduced. At that point, it was not clear to me why the exogenous data was needed.***

A: This was not very clear from the text, a further extended explanation regarding the empirical datasets has been added. Exogenous data is needed to drive the asymmetry (fundamental) side and bifurcation (chartistic) side of the market/model. A detailed description of the data appears in the text. The choice of CME data was also determined by their availability as I was able to obtain high-frequency data related to S&P500 from there by Tick Data, Inc. [pg. 59]

Q: ***6. Following up on the previous point, the choice of exogenous variables appears to be a rather ad-hoc choice. Why these variables? Why not others?***

A: The choice of variables partially follows a successfull aplication in Barunik & Vosvrda (2009), where more options are compared: the daily change of total trading volume, ratio of advancing stocks volume and declining stocks volume, OEX put/call ratio, Dow Jones Composite Bond Index, and one-day lag of SP 500 returns. Authors show that fundamentalists are best described by the ratio of advancing and declining stock volume, and chartists are best described by the OEX put/call ratio. We have used the same proxy variables as in the original Barunik & Vosvrda (2009) work to maintain reproducibility, and to motivate the importance of our generalisation of the model. Moreover, nobody else so far estimated cusp on long-span stock market data and we are not aware of any better dataset of exogenous variables for this purpose.

I provide further motivation for our choice directly in the text: e.g. "The variables related to the trading volume generally correlate with the volatility and therefore are considered good measures of the trading activity of large funds and other institutional investors" or "Financial options are widely used and are the most popular instruments for speculative purposes. Therefore, they serve as a good measure of speculative money in capital markets (see e.g. Bates (1991), Finucane (1991), or Wang *et al.* (2006)) because they represent the data about extraordinary premiums and excessive greed or fear on the market."

A further explanation about a relation between market volume, liquidity, and fundamental money has been added to the text. [pg. 60]

Q: ***7. Pg 38: Looking at the LL values of the restricted and unrestricted models, it becomes clear that the fit of the RESTRICTED model is HIGHER than the fit of the UNRESTRICTED model. This is impossible; the LL of the restricted model can only be smaller or equal than the LL of the unrestricted model. If the restricted model was best, the alpha 2 and beta 1 coefficients in the unrestricted version of the model should have been put equal to zero. Therefore, it appears that one of the two models has not converged to its global optimum. As such, the interpretation of the coefficient cannot be trusted.***

A: Many thanks for the detailed inspection, I checked the computational results and found that figures from a slightly different estimation based on a shorter dataset were copied in case of the (a) part, restricted model (second column). I have corrected the coefficients (only minor changes) as well as R^2, LL, AIC, BIC (natural significant changes). I have adapted the interpretation of results with this respect. [pg. 63–64]

Q: 8. Pg 42, Figure 3.5: BIC values cannot be compared over different sample periods. It is a unit-free metric.

A: I of course agree with the opponent that BIC cannot be directly compared across various time periods. However, I did not intend to track their dynamic evolution in time but to contrast the criteria of the cusp model and the logistic model in every single rolling one-half year period. I have clarified this in the text. [pg. 69]

Q: 9. Pg 43: volume is used as a proxy for fundamentalist activity. This is a rather loose interpretation of the model. Giving a fundamentalist/chartist type interpretation to the CUSP model is hard to sell.

A: Fundamentalist-chartistic interpretation of the Catastrophe theory comes from Zeeman (1974), this Chapter is in fact only an empirical verification of his hypotheses. Regarding the ratio of advancing and declining stock volume, a further possible motivation than already provided in the text (Section '3.4.1. Data description') might be related e.g. to the ARMS INDEX used in technical analysis that is based exactly on (not only) the volume of advancing and declining stocks and is generally used as an indicator of overall market sentiment. I have already provided motivation for our choice of this proxy variable in Section '3.4.1. Data description': "The variables related to the trading volume generally correlate with the volatility and therefore are considered good measures of the trading activity of large funds and other institutional investors. Trading volume indicators thus represent the fundamental side of the market and can be used as a good proxy for fundamental investors. Therefore, the ratio of advancing and declining stock volume should mainly contribute to the asymmetry side of the model." [pg. 68–69]

Q: ***10. Pg 44: The high frequency data is said to be unique. This is not the case.***

A: Thank you for the point, 'unique' was deleted from the text. [pg. 70]

Chapter 4:

Q: 1. Chapter 4 merely contains the background for the following chapters 5 to 7. As such, it doesn't contain original work. Therefore, my suggestion would be to integrate Chapters 4 and 5.

A: This is a meaningful suggestion. Unfortunately, Chapter 5 is already relatively extensive. More importantly, the NPSMLE method is a general framework not solely related to the Brock & Hommes (1998) model. Thus I would like to keep it as a separate part of the work and I believe such division helps to make the structure of the thesis more clear. In the preliminary draft of the thesis, however, the NPSMLE was presented in a preceding detached chapter, but then I have changed the order to the more standard one: model—estimation methodology—empirical estimation.

Chapter 5:

Q: 1. Pg 56: under equation 5.3: it would be good to say something about the properties of ϵ?

A: I have provided the reader with an extended description of the properties of ϵ in the text and a related footnote. [pg. 83]

Q: 2. Pg 56: Why is the risk free rate r kept constant? A stochastic process would be closer to reality.

A: I agree with the opponent that a stochastic process would more closely resemble reality, but I keep to the constant value for the reason of simplicity. Our 'daily' value translates into approximately 2.5% risk free rate annually. We can see that the estimation performance is very robust w.r.t. absolute value of r in Table 5.2 as commented on [pg. 101]: *"The robustness of the method w.r.t. assumption of the real market risk free rate therefore relaxes the need of a very precise derivation of this parameter for various countries and historical periods and the reasonable approximation* $R = 1 + r = 1.0001$ *representing circa 2.5% annual risk free interest rate can be generally used in Chapter 6."* Furthermore, r only influences the absolute value of x_t (see Equation 5.1), not the dynamics of the model. [pg. 83]

Q: 3. Pg 56: The author makes the assumption that variance is constant. As also pointed out above, this is a peculiar assumption because these types of models are specifically designed to generate time-varying variance. I realize this is a common assumption to make, but it might be good to point this out.

A: I agree this is an important point and as suggested, I have pointed this out in the text: I have added a further discussion about the computational setting of the model and clarified advisability of related assumptions. From the theoretical derivation of the model, constant σ^2 denotes traders' beliefs about the conditional variance of excess returns. The model output is, on the other hand, usually characterised by time-varying variance as a resut of interaction of fundamental and chartistic strategies. For analytical tractability the assumption of constatnt σ^2 has already been made in the original Brock & Hommes (1998) model that I aim to estimate, so I stick to their design. It is important to note that a and σ^2 are in fact only scale factors for the profitability measure U. Their magnitudes do not affect relative proportions of $U_{h,t}$ and thus do not influence the dynamics of the model output. In other words, although I assume constant σ^2, the output time series generated by the model does not have constant variance. Strategy-specific a_h or time-varying $\sigma^2_{h;t}$ are appealing concepts mainly for simulation analyses of HAMs. [pg. 83]

Q: 4. Pg 57: The author argues that the IOC is typically a single-digit positive number. As pointed out above, this is hard to rationalize because the IOC parameter cannot be compared across models/time/assets.

A: I have clarified in the text that although the intensity of choice β cannot be directly rigorously compared across various models, assets, or time periods, I utilise the general knowledge of previous estimation efforts for models sharing similar framework to set meaningful simulation grids in this chapter. [pg. 84]

Q: 5. Pg 57: The author states that it is out of the scope of the model to study the dynamics of the model for different values of β. I would agree with this statement, although it IS important to check what type of equilibrium the model is in with the stated values of β (i.e., a stable fixed point equilibrium, limit cycle, or chaotic). This could have a major influence on the estimation procedure.

A: The suggested analysis constitutes a considerable enhancement of the estimation performance investigation and I would like to thank the opponent for this notable remark. This can be generally achieved using the tools of the bifurcation analysis: depicting the bifurcation diagram or computing evolution of the Lyapunov exponent for the stated values of β. However, for the bifurcation analysis to provide us with meaningful and unambiguous results, we need to define a very specific model setting. Moreover, the bifurcation analysis is an appropriate tool mainly for the deterministic part of the model (i.e. without the i.i.d. noise term sequence ϵ_t). In our setting of a robust Monte Carlo analysis based on random generation of 4 sets of belief coefficients combined with 10 intensities of the stochastic noise ϵ_t, a simple analysis of the stability of the model is unfortunately not likely to bring credible and unambiguous conclusions. [pg. 85]

Q: ***6. Pg 58: The author states that he is using different types of distributions for the noise process. 1) Give a full list of the different specification. 2) What I miss, is a heavy-tailed distribution, such as the Student-t. Again, the FABMs are specifically designed to generate time-varying volatility and heavy tails. This should come back in the simulation study of this chapter. In addition, the uniform distribution that the author uses does not make much economic sense to me; I do not know of any processes that generate this type of distribution.***

A: I largely agree this part was not clearly written and I have enriched the text regarding the description of different noise specifications. I also have made clearer in the text that a detailed description of all 30 stochastic noise specifications can be found in Table 5.1, Table 5.4, and Table 5.5. My original intention was to not double the information that is already included.

The normality of market noise is a usual assumption in the literature: "the nonlinear models are fed with an exogenous stochastic process, but the noise process is 'nice', which in this case means that it is normally distributed", as pointed out by Amilon (2008, pg. 344). It is also crucial that using normal distribution we can take the advantage of the favourable theoretical properties of the Gaussian kernel (Kristensen & Shin 2012, pg. 81) in Equation 4.23 for the NPSMLE. The same kernel distribution as is the distribution of stochastic noise is an important assumption of the estimation procedure. Normality of the stock market noise seems as reasonable assumption, at least I am not aware of any more resonable assumption for nonobservable noise. The uniform noise concurs the previous research in Barunik *et al.* (2009); Vacha *et al.* (2012); Kukacka & Barunik (2013). As for the NPSML estimation (in a simulation study) the theoretical distribution of the stochastic noise can be any, the uniform distribution represents an extreme not-very-realistic case intended for comparison with and contrasting the results based on normally distributed noise.

Whilst I agree with the opponent that heavy tails are one of the key stylised facts FABM are specifically designed to generate, I intentionally do not consider any at first sight soliciting heavy-tailed noise distribution. The fact that financial data are heavy-tailed does not suggest any specific distribution of the market noise. In fact the situation is opposite. The attractiveness of the HAM is based on its ability to produce a heavy-tailed distribution of model output although we input normally distributed stochastic noise. Thus the HAM explains one of the most important stylised facts of financial time series via endogenous interactions of fundamentalists and boundedly rational chartists, not as an effect of a specific distribution of noise input.

This discussion has been added to the text. [pg. 85–86]

Q: 7. Pg 62 + 63: My conclusion from these graphs would be that the mean/ median estimate of the coefficients is fine, but that the confidence bands are completely uninformative. This is directly comparable to the results of Terasvirta (1997) when using NLLS/(Q)ML. This calls for a proper comparison of model with/without switching.

A: The analysis of the confidence bands primarily shows that theoretical properties of the estimator, the consistency and asymptotic efficiency, also hold in small samples for the model. We can clearly observe the consistency of the estimator and how the efficiency of the mean estimate increases simultaneously with increasing length of generated time series t as well as the precision of the kernel estimation N. Therefore, the confidence bands were not expected to be completely informative for all combinations of N and t. Nonetheless, the crucial result is that they are informative for larger lengths of generated time series, i.e. 5000 and 10000 for small βs, but only 100 is sufficient for $\beta = 10$. These features have important favourable consequences for application of the method to datasets of various lengths—we should be able to detect even weak signs of behavioural switching in long-span daily financial data, but also stronger signs of switching in macroeconomic data where typically lower-frequency time series of shorter lengths are available.

The opponent is correct that the precision of confidence bands in the Monte Carlo analysis might have been somewhat better, but w.r.t. the complexity of the estimation issue in the nonlinear HAM setting with five repeatedly randomly generated strategies (as well as to many other estimation attempts from literature that have found the switching coefficient insignificant) I consider the results very promising. The most important property of the estimation method in the current setting is the ability to distinguish between various βs and assess statistical significance and this objective is well achieved.

It is important to stress that the confidence bands provide only a part of the information about estimates of the coefficients. Another important information is reported by Figure 5.4. that depicts smooth histograms of selected estimated βs. Although we can observe a nontrivial interplay between the magnitude of βs, noise intensity, and the estimation precision, the mass of the distribution is generally concentrated around the true value and we observe considerably positive excess of kurtosis, especially in cases of noise specification for which the estimation performs best.

Figures 5.2 and 5.3 also allow for comparison between estimation of models with and without switching. The left column of Figure 5.2 represents the model without switching ($\beta = 0$), the right column and Figure 5.3 illustrate estimation performance for models with switching ($\beta > 0$). The model without switching is then always included as the first row of all panels in all tables in Subsection 5.3.1. However, β is always being estimated as it is the only parameter of interest in these sections.

To make the situation more clear, this discussion has been added on [pg. 92–93]. [Tables can be now found on pg. 90–91]

A rigorous statistical comparison of the estimation method performance for model with/without switching is possible for the 2-type model. Please, see extended Section 5.3.2. [pg. 121–122] and related discussion to point 9 below.

Q: 8. Pg 66 Table 5.1: The results in the bottom row of panel j are different thus interesting. They require a more in-depth discussion.

A: I agree that these figures are interesting, however, at the same moment very problematic as they go along with an extreme number of 'NaN' outcomes (96%) as a result of a very strong stochastic noise $N(0, 2^2)$ added to the system. This simply means that only 40 out of 1000 random runs converged and produced point estimate for β. I have addressed this issue further on [page 96]: "We do not consider results with high number of 'NaN' outcomes relevant within this analysis, however, we keep displaying them to retain the completeness of provided information as well as an optimal warning signal of an improper behaviour of the system under scrutiny."

Q: 9. The author focuses fully on the proper estimation of the IOC parameter β in this chapter. This makes sense because it is an important question to address. Another important question, though, is the issue surrounding model fit. Specifically, the author does different checks with two and three type models. I would be interested in seeing an extension in which the author tests whether the method is capable of 1) distinguishing static from switching models, and 2) distinguishing two from three (or more) type models.

A: This is a very important suggestion and I share with the opponent the notion of importance of a rigorous statistical assessment of the NPSMLE capability to distinguish between 'competing' models. With regard to 1), i.e. the comparison between static and switching model, as these models are nested, we can apply the standard likelihood-ratio goodness of fit test. For this purpose, Tables 5.11 and 5.12 [pg.114–115] summarising result for the 2-type model have been extended by information about likelihood ratios, log-likelihood ratio test statistics, and related p-values to reject the null of the static model. I have also added a new subsection 'Likelihood-ratio test' discussing test results [pg. 121–122]. The situation with 2), i.e. distinguishing the 2-type from the 3-type model is more challenging. As a result of nonlinear nature of the model governed by discrete choice probability multinomial logit model (Equation 5.2), we are not able to apply simple restrictions so that we compare nested models again. Technically, two options suggest itself to design the restricted (null) 2-type model. First, we can restrict the parameters $g_3 = 0$ and $b_3 = 0$ to displace the third strategy type, but then the displaced strategy in fact follows the role of the fundamental strategy and for small βs the market fractions fluctuate around 66.6% for fundamentalists and 33.3% for chartists (instead of 50%/50% in the 2-type model). Second, we can decrease the number of strategies to 2 and use the 'usual' 2-type model from Section 5.3.2, but then the structure of compared models is not a result of parameter restrictions but of a different structure of the model (because number of strategies is not an estimated parameter) where for small βs market fractions fluctuate around 50%/50% (instead of 33.3%/33.3%/33.3% in the 3-type model). In both cases, I apprehend the non-nested character of the models prohibiting application of the simple likelihood-ratio test. These concerns were further confirmed by preliminary trial computations bringing absurd results. For the future research I plan to consider a version of the Vuong's closeness test (Vuong 1989), which is an advanced likelihood-ratio test for model selection that allows also for overlapping or non-nested models.

I have reflected upon this important methodological issue in the text as one of the future extension of the NPSMLE of HAM related research in a new section '5.3.4 Suggestions for future research' on [pg. 123].

Q: 10. The added value of this chapter is the empirical methodology to estimate the FABM. The author shows that the model does a good job in identifying the coefficients. What I would be interested in seeing, though, is a comparison of this new method with the traditional methods used to estimate this model (i.e., NLLS as in Boswijk et al. 2006). Does the method the author proposes also yield BETTER results than the traditional methods? This important also given the level of computational demands of the new method.

A: I definitely agree that this is an important area that requires further research. And also a very difficult task to tackle as traditional methods are largely infeasible for the HAM. The Boswijk *et al.* (2007) serves as a good example: it is in fact philosophically close to the model under our scrutiny, but unfortunatelly technically it has markedly different specification: 1) the model is considerably redesigned compared to the original (and thus to our) approach; 2) different assumptions about fundamentalists are used (the trend coefficient g_1 is expected to be lower than 1 to reflect gradual mean-reverting tendency instead of strict $g_1 = 0$ in the original Brock & Hommes (1998) model design); 3) the identification issue of the fundamental and trend following AR(1) coefficients is problematic from definition in Boswijk *et al.* (2007) as both coefficients are based on the same value x_{t-1}; 4) it uses annual empirical data (those are difficult to proceed in our framework of MA fundamental value as the jumps in annual data are huge) instead of daily data I use in the analysis.

I have reflected upon this important methodological issue in the text as one of the future extension of the NPSMLE of HAM related research in a new section '5.3.4 Suggestions for future research' on [page 123].

Chapter 6:

Q: ***1. Pg 95: Important in the discussion about the fundamental value is not necessarily whether the fundamental value estimate reflects the TRUE fundamental value, but rather whether it reflects a value that could reasonably proxy for the fundamental value as seen by market practitioners.***

A: I largely support the opponent's point. Long-term and short-term MAs are commonly used by practitioners in trading to extrapolate divergence from the fundamental value in technical analysis. Since the fundamental value of stocks is essentially unknown, market practitioners often tend to at least estimate whether the stock is over or under-valued, whether the possible mispricing is small or large, and whether the gap is going to increase or whether a soon correction is more likely. As the Brock & Hommes (1998) model is also formulated in deviations from the fundamental price, the MA approach seems to be one of reasonable guidances. Moreover, I decided to keep to current literature and follow some other works using MA as an approximation for the fundamental value, e.g. Winker *et al.* (2007); ter Ellen & Zwinkels (2010); Huisman *et al.* (2010). To reduce the arbitrariness, I test the effect of 5 window lengths, namely: 21, 61, 121, 241, and 481 days. As all specifications lead to comparable results, I report results for 61 and 121 days in the thesis.

I have added a new paragraph discussing this issue and a short literature review summarising the use of MA filtering by practitioners and active traders [pg. 126–127]

Q: 2. Pg 95: Using a CENTERED moving average for the fundamental value leads to a look-ahead bias. Why not a simple moving average?

A: This question is closely related to the previous one and I agree with the opponent that the look-ahead bias is likely to play some role in our proxy variable. Both MA versions were analysed and found to produce to a large extent comparable results. The centred MA is therefore suggested to reduce the delay of the information flow and for several additional related reasons stated in the paragraph below. Moreover, the centred MA incorporates a convenient property that the price converges to it by definition that is exactly a feature one would expect from the fundamental value. These justifications have been added to the text. [pg. 127]

First, the simple MA is associated with problematic numerical stability of the model/estimation method because resulting deviations based only on past information are significantly larger. Second, I might argue that approximation based on a simple (historical) MA struggles from something what might be called a 'historical bias': the simple MA might systematically poorly-approximate. Imagine e.g. the situation that the price is growing for some period (this might not necessarily be an effect of growing mispricing but also of rising fundamental value or likely of both going hand in hand). The fundamental value proxy based on historical MA will constantly underestimate. Third, the future expectations (which might have been part of the information set of market participants at given time) should also be reflected in the rational fundamental value, but based on historical data in hand, I am not able to extract possible expectations about future price from anywhere else than somewhat implicitly from the future price evolution itself.

Q: ***3. Table 6.1: Include the autocorrelation in x as well as the autocorrelation in x^2.***

A: Thank you for the suggestion, this is an important aggregate information that can be simply checked. As the autocorrelation in x as well as in x^2 is closely related to important financial stylised facts, together with Table 6.1 [pg. 131]. I have also added this informmation to all other tables of descriptive statistics: Tab. 3.2. on [pg. 61] for Cusp, Tab. 7.1. on [pg. 161] and 7.3. on [pg. 172] for the Alfarano *et al.* (2008) model.

Q: ***4. Pg 99: Please repeat the exact model that you are estimating. This would help the reader.***

A: In accordance with the opponent's suggestion, I have repeated the structure of the model for reader's convenience. Thank you. [pg. 132]

Q: ***5. Pg 99: The author concludes that the estimated β is insignificant. In the previous chapter, however, we have seen that the confidence bands are largely uninformative (see point 7 of chap-***

ter 5). Hence, how would the results from Chapter 5 apply to the findings of Chapter 6?

A: This comment is likely to stem from possible misunderstanding which is clarified in the discussion to point 7. of Chapter 5. In fact results from Chapter 5 were directly used for the computational setting in Chapter 6. The crucial result of Chapter 5 is that the confidence bands are informative for larger lengths of generated time series, i.e. 5000 and 10000 even for small βs and a reasonable kernel estimation precision $N = 500$. So we should be able to detect even very weak signs of behavioural switching in long-span daily financial data. This is exactly the reason why the length of 5000 observations is used for the empirical datasets as I mention in Section '6.1 The estimation setting' on [pg. 124].

In order to clarify this in the text as well, a paragraph with this discussion has been added on [pg. 134].

Q: 6. Related to the former: If there is no switching, the different types are actually not identified from each other; there will be nuisance parameters.

A: It is a good point that if we estimate statistically insignificant intensity of choice parameter β, the model in fact boils down to a simple weighted AR(1) process because the population weights are fixed to constant $1/H$. Different types of traders then cannot be identified (because they do not switch over time) and the model actually is not heterogeneous agent model any more. In such a case the trend and bias parameters $\widehat{g_2}$ and $\widehat{b_2}$ (or $\widehat{g_3}$ in the 3-type model) can be viewed as nuisance parameters—they to a large extent lose the original model interpretation and we cannot fully trust the estimated magnitudes of these parameters. A stable population ratio of trading strategies $n_{1,t}/n_{2,t} \doteq 0.5/0.5$ technically means that the population of fundamentalists is forced to be of the same magnitude as the population of chartists. As we cannot agree with such a strong assumption of similar population magnitudes for both strategies (that are likely not capturing the real market population proportions), in Section '6.6 Estimation of market fractions' I trivialise the simulated model via disabling the evolutionary switching behaviour and fixing the population ratio of trading strategies to $n_{1,t}/n_{2,t} = const.$ The population ratio of trading strategies n_1/n_2 and implied percentage *fraction* of fundamentalists on the market is then a direct subject of the estimation interest.

A further explanation regarding nuisance parameters has been added to the text. [pg. 146]

It is, however, important to note that zero intensity of choice β does not disrupt identification of estimated parameters. This is confirmed by the analysis of the empirical log-likelihood function in Figure 6.3. [pg. 136] where β is estimated close to 0, but g_2, b_2 as well as *noise intensity* are generally well identified via unique maxima of transversal cuts of the log-likelihood function. Also for the trivialised *fraction* model the empirical log-likelihood function in Figure 6.9. [pg. 150] is very informative for all estimated parameters except the bias parameter b_2.

Q: 7. Pg 104: Why not use actual risk-free data?

A: I agree with the opponent that using actual risk-free data represents a more proper way of approximating the risk free rate parameter, but I decided to follow the same setting as I analyse in the Monte Carlo sections to be consistent. It would be relatively difficult to precisely proxy the r for all analysed stock markets from 1994 and finally it would only make things more complicated without bringing any difference in results. Negligible (because daily) differences caused by deviations of the actual risk-free figure from our constant r are likely to be to a large extent nullified by the fundamental value approximation. [pg. 138]

Q: 8. Pg 106: Can you statistically distinguish the two-type model from the three-type model? The thesis needs proper tests of model fit.

A: This is another important suggestion closely related to the point 9. of the previous part. Although we now only have one empirical dataset, the problem with non-nested character of the models persists also for the empirical application. Primarily, however, the 'message' from the estimation is essential and clear: zero β implicates that the goodness of fit test does not make much sense, because the model boils down into a simple model without switching. Thus neither the 2-type nor the 3-type model is likely to be optimal. For my interest, I have preliminary checked results of the simple likelihood ratio test (perhaps incorrectly applied due to the non-nested character of the models) and it is not able to distinguish between the two types of the model. I do not report results of this preliminary testing in the thesis.

Q: 9. Pg 108: The author compares LL values across different sub-samples. This is not possible; only nested models are comparable.

A: The opponent is again absolutely correct. I have added a note to the text [pg. 143] that a direct comparison of rolling log-likelihoods is methodologically disputable because it is based on different rolling sub-samples. However, I argue that since rolling datasets keep the same length and overlap by circa 83% between adjacent steps, the overall evolution of the LL pattern provides us with a valuable information.

Q: 10. As for the previous chapter, it would be interesting to see the performance of the new estimation method in comparison to the performance of the traditional method.

A: As already discussed in point 10. of Chapter 5, I completely agree with the opponent that this is an important area for further research. At the same moment, I would like to highlight the fact that the main motivation of this work is that traditional methods are largely infeasible for many FABMs and generally require strong compromises in theoretical assumptions. As I see the future of the field primarily in application of simulation-based methods rather than compromising ABMs' design in order to apply traditional method (i.e. NLS in the specific case of Brock & Hommes (1998) model), I consider these performance comparison efforts mostly as a complement or a verification tool for the development and application of simulation-based techniques.

Chapter 7:

Q: 1. I realize that this chapter is work in progress. It is interesting work, though, and the results are promising. Currently, it is still a rather dry summation of empirical results. What about economic intuition? To give an example: Figure 7.10 panel (b) shows that the fundamental variance shoots up at Black Monday. Does this imply that Black Monday was a fundamental event?

A: To interpret these results correctly, we need to discuss the nature of volatility in the model. The total volatility of the model output p_t is derived from so called fundamental value F_t and the effect of market sentiment x_t. However, the fundamental volatility in this highly stylised simple model cannot be fully interpreted as the real world fundamental risk. When bringing the model to empirical data, the fundamental volatility term to a large extent represents all the remaining volatility that is not caused as the effect of noise traders' switching between the optimistic and pessimistic mood. E.g. based on rolling estimation results, the estimation method seems to predominantly assign the cause of the elevated market volatility to the fundamental value term, although e.g. Black Monday can be hardly denoted as a fundamental event. However, we also observe jumps in the estimates of the herding intensity b that definitely have a good economic interpretation for selected historical events. The NPSMLE in combination with a very simple stylised model are thus perhaps weak in distinguishing well between these two theoretical sources of volatility. Comparing the shapes of the simulated sub-log-likelihood-functions in Figure 7.8. and Figure 7.9. in dimensions of individual model parameters and their combinations, we clearly observe relative flatness of the resulting log-likelihood-function in dimensions of a and b w.r.t. the high-pitched shape in the dimension of σ_f. Thus, the market volatility amplification is likely to be assigned based on rather technical optimisation criteria mainly to the fundamental volatility, in which dimension the optimisation algorithm search can work well better and which is likely to overshadow the effect of switching parameters a and b during the estimation procedure.

This discussion has been added to the text [pg. 177 and 181]. I have also completed the chapter by an extended model description [pg. 155–156], via further discussion of new results, and a short section of concluding remarks. [pg. 176–182]

Chapter 8:

Q: ***1. The chapter gives a good overview of the work the author has done in the thesis. I would like to challenge the author, though, to also include his thoughts about the bigger picture: What is the current stance of the literature, where is it going, or where should it be going? What are the current strengths, weaknesses, threats, and opportunities.***

A: I would like to thank the opponent for this challenge. It motivated me to take a deep think about the field and its future. As an outcome, I have added an attempt at a largely subjective SWOT analysis of the field as the very last part of the Conclusion. [pg. 186–188]