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REPORT

PhD-Thesis:	Relational Approach to Universal Algebra
Author:	Jakub Opršal
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Scientific Content

The dissertation presents various ways how relations can be used to attack some long standing problems on algebraic structures (that is, sets with fixed sets of basic operations on them).

Chapter 1 of the thesis is on varieties. Part of it is based on the paper [2] by Jakub Opršal with his advisor Libor Barto and with Michael Pinsker. A variety is a class of algebras of the same type that is defined by equations. An equation is linear if it does not contain any nested terms but only basic operations and projections. Theorem 1.2.3 (Corollary 5.4 of [2]) is a linear version of Birkhoff's HSP-Theorem: a class of algebras is defined by linear identities iff it is closed under retractions and products. It is proved that several properties of varieties, like congruence regularity, uniformity, and singularity, cannot be described by linear equations (Theorems 1.4.1 and 1.4.2).

Varieties can be further ordered and classified by interpretability [3]. A variety $\mathcal{V}$ interprets into a variety $\mathcal{W}$ if basic operations of $\mathcal{V}$ are mapped to terms of $\mathcal{W}$ such that any algebra $\mathbf{A}$ in $\mathcal{W}$ induces an algebra in $\mathcal{V}$. Interpretability is a quasi-order on varieties. The equivalence classes of mutually interpretable varieties form the lattice of interpretability types. While interpretability types of linear varieties are closed under joins, they are not closed under meets in general. Via a bijection between the poset of linear varieties and homomorphism classes of clones, Jakub Opršal describes a linear meet of linear varieties (Proposition 1.3.3). Finally he proves that the meet of the (linear) Mal'cev conditions for congruence permutability and for congruence distributivity cannot be described by a linear Mal'cev condition (Theorem 1.5.2).

Chapter 2 of the thesis investigates which filters in the lattice of interpretability types are prime. The most famous open problem there is Taylor's modularity conjecture: whenever the join of two varieties is congruence modular, then one of them is congruence modular already. Jakub Opršal obtains the best partial result on this question so far by verifying it for idempotent varieties (Theorem 2.1.2). His proof uses the idea of "coloring terms by a relational structure" (Definition 2.2.1) that also appeared in [2]. This yields

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new descriptions of (idempotent) n-permutable varieties, varieties satisfying any non-trivial congruence identity, varieties with cube terms, and congruence modular varieties. In turn, the filters for these idempotent Mal'cev conditions are shown to be prime.

Chapter 3 is an extension of Jakub's Opršal's paper [4]. He provides a new relational description of Bulatov's higher commutators which generalize the binary term condition commutator in Universal Algebra. This allows for new short proofs of the basic properties of these commutators as already established by Aichinger and Mudrinski [1]. Moreover it is shown that every Mal'cev algebra has a unique maximal expansion with the same congruences and higher commutators (Corollary 3.1.4). In Theorem 3.6.1 identities are given that characterize supernilpotent Mal'cev algebras (algebras for which all higher commutators of some arity are trivial).

Conclusion

Unfortunately numerous typos, some inaccuracies, and terse proofs make the reading of the thesis more difficult than necessary. While it seems that everything can be easily rectified, I would have appreciated a more careful presentation. That being said, the results of the thesis are new, deep and interesting. They are a significant contribution to the field. Further applications to Constraint Satisfaction Problems and Computer Science are given in [2]. Most of the results in Chapter 3 are already accepted for publication (see [4]). Chapters 1 and 2 contain enough material for at least two more papers.

The scientific significance and the number of results of this work is certainly above the standard for a PhD-thesis in Europe or the USA. I recommend that Jakub Opršal is awarded a PhD in Mathematics.

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References

[1] E. Aichinger and Nebojša Mudrinski. Some applications of higher commutators in Mal'cev algebras. *Algebra Universalis*, 63(4):367 403, 2010.

[2] L. Barto, J. Opršal, and M. Pinsker. The wonderland of reflections. Unpublished.

[3] O. C. Garcia and W. Taylor. The lattice of interpretability types of varieties. Mem. Amer. Math. Soc. (50), 1984.

[4] J. Opršal. A relational description of higher commutators in Mal'cev varieties. To appear in *Algebra Universalis*.