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Relational Approach to Universal Algebra

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Abstract: We give some descriptions of certain algebraic properties using relations and relational structures. In the first part, we focus on Neumann's lattice of interpretability types of varieties. First, we prove a characterization of varieties defined by linear identities, and we prove that some conditions cannot be characterized by linear identities. Next, we provide a partial result on Taylor's modularity conjecture, and we discuss several related problems. Namely, we show that the interpretability join of two idempotent varieties that are not congruence modular is not congruence modular either, and the analogue for idempotent varieties with a cube term. In the second part, we give a relational description of higher commutator operators, which were introduced by Bulatov, in varieties with a Mal'cev term. Furthermore, we use this result to prove that for every algebra with a Mal'cev term there exists a largest clone containing the Mal'cev operation and having the same congruence lattice and the same higher commutator operators as the original algebra, and to describe explicit (though infinite) set of identities describing supernilpotence of a fixed degree in any congruence permutable variety.

Keywords: linear varieties, clone, interpretability, Mal'cev condition, Mal'cev algebra, commutator, supernilpotence

Název práce: Relační přístup k universální algebře

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Abstrakt: V této práci předkládáme popis některých algebraických vlastností pomocí relací a relačních struktur. V první části se zaměřujeme na Neumannův svaz interpretačních typů variet. Charakterizujeme variety definované lineárními rovnostmi a uvádíme příklad několika vlastností, které nejsou charakterizovatelné lineárními rovnostmi. Dále se věnujeme Taylorově domněnce o varietách s modulárními svazy kongruencí. Speciálně ukážeme, že interpretační spojení dvou idempotentních variet, které nemají modulární svazy kongruencí, samo nemá modulární svazy kongruencí. Uvádíme i obdobný výsledek pro variety s krychlovým termem. V druhé části práce uvádíme popis Bulatovových vyšších komutátorů ve varietách s mal'cevským termem. Dále použijeme tento výsledek na to, abychom ukázali, že pro každou algebru s mal'cevskou operací existuje největší klon, který obsahuje tu samou mal'cevskou operaci, má stejný svaz kongruencí a jehož komutátory se shodují s těmi v původní algebře. Nakonec uvádíme další aplikaci tohoto výsledku a to na explicitní popis (nekonečně mnoha) rovnic, které charakterizují supernilpotenci v každé varietě s přesmýkatelnými kongruencemi.

Klíčová slova: lineární varieta, klon, interpretabilita, mal'cevská podmínka, mal'cevská algebra, komutátor, supernilpotence

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Contents

Introduction

Many interesting properties of algebras are either directly defined (e.g. congruence modularity), or described by relations that are compatible with the basic operations of the algebra. We say that a relation $R \subseteq A^m$ is *compatible* with a function $f\colon A^n \to A$ (or f is a *polymorphism* of R) if for all tuples $(a_{11}, \dots, a_{1m}), \dots, (a_{n1}, \dots, a_{nm}) \in R$ we have

$$(f(a_{11}, \dots, a_{n1}), \dots, f(a_{1m}, \dots, a_{nm})) \in R,$$

equivalently, one can say that R is compatible with f if R is a subuniverse of the algebra $(A, f)^m$. Both subuniverses and congruences can be described as compatible relations of arity one and two, respectively—a congruence is exactly a binary compatible equivalence relation. The relation of compatibility between relations and functions on a fixed finite set gives rise to a well known Galois connection between sets of relations closed under taking primitive positive definitions and function clones; in universal algebra this Galois connection is often referred to as 'the Galois connection'. In this thesis we study some properties that are described by some relations. This is closely connected to Mal'cev conditions and clones, since the set of all polymorphisms of some relation (or a relational structure) is always a clone (note that this is true even if the underlying set is infinite), hence any property described by compatible relations does not depend on the particular choice of the signature of the algebra, or the variety.

This thesis consists of two main topics. First, we study Mal'cev conditions and the interpretability lattice of varieties that was introduced by Neumann [32]; second, we give a relational description of Bulatov's higher commutators in varieties with a Mal'cev term, and we show several applications thereof. Apart from some unpublished results, the first part is based on manuscripts (1) and (2). The whole second part, except its last section which is newer, is published in (3).

(1) Libor Barto, ______, and Michael Pinsker. The wonderland of reflections. manuscript, 2015.

(2) ______. Taylor's conjecture and related problems for idempotent varieties. manuscript, 2016.

(3) ______. A relational description of higher commutators in Mal'cev varieties. *Algebra Universalis*, accepted, 2015.

Mal’cev conditions

Vaguely speaking, a Mal’cev condition is a condition on a variety of the form ‘There exist some terms satisfying some identities.’ First such condition was Mal’cev’s description of congruence permutable varieties (i.e., varieties whose congruences satisfy $\alpha \circ \beta = \beta \circ \alpha$). As a formal definition of a Mal’cev condition we will use the following definition of Taylor [38].

Definition. A *strong Mal’cev condition* is a finite conjunction ϕ of identities using function (variable) symbols $f_1, \dots, f_n$ and symbols for individual variables. A variety $\mathcal{V}$ is said to satisfy such condition if there exist terms $\bar{f}_1, \dots \bar{f}_n$ in the language of $\mathcal{V}$ such that ϕ is satisfied in $\mathcal{V}$. Strong Mal’cev conditions are naturally ordered by syntactical consequence which allows us to define a *Mal’cev condition* as a disjunction of a countable chain of strong Mal’cev conditions.

There are many well-known properties of varieties that can be characterized by some Mal’cev condition. We focus on studying the conditions themselves. In particular, their natural ordering by implication. This ordering is very similar to the ordering of varieties by interpretation. Indeed, as seen from the definition below, if we view a strong Mal’cev condition as a finitely based variety of finite signature that is defined by the identities appearing in ϕ, then the ordering of strong Mal’cev conditions coincides with ordering of these varieties by interpretation.

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties, and let σ denote the signature of $\mathcal{V}$. An *interpretation* I of $\mathcal{V}$ in $\mathcal{W}$ is a mapping that maps basic operations of $\mathcal{V}$ to terms of $\mathcal{W}$ of the same arity such that for every algebra $\mathbf{A} \in \mathcal{W}$, $(A, (I(f)^{\mathbf{A}})_{f\in\sigma})$ is an algebra in $\mathcal{V}$. We say that a variety $\mathcal{V}$ is interpretable in a variety $\mathcal{W}$ if there exist an interpretation of $\mathcal{V}$ in $\mathcal{W}$. The lattice of interpretability types of varieties is then constructed by quasi-ordering all varieties by interpretability, and factoring out varieties that are interpretable in each other. This gives a partially ordered class such that every set has a join and a meet; we will give some constructions of joins and meets later, for complete proofs we refer to [17]. Now, given a Mal’cev condition, if we consider all varieties that satisfy this condition we will get a filter in the interpretability lattice (this filter is principal if the Mal’cev condition is equivalent to a strong Mal’cev condition). Hence studying Mal’cev conditions is studying some particular filters in the interpretability lattice.

The same lattice can also be introduced as the lattice of clones ordered by existence of a clone homomorphism. A mapping $h\colon \mathscr{A} \to \mathscr{B}$ between two clones $\mathscr{A}$ and $\mathscr{B}$ is a *clone homomorphism* if it preserves composition and projections. The lattice is then constructed by taking the class of all clones, quasi-ordering it by an existence of a clone homomorphism, and factoring out the homomorphically equivalent clones. The connection between the two lattices is given by the concepts of a clone of variety and a variety generated by a clone. Given a variety $\mathcal{V}$, we define its clone to be the clone of all term functions of the countably generated free algebra in $\mathcal{V}$. In the other direction, given a clone $\mathscr{C}$, its corresponding variety is the variety generated by the algebra $(C, (f)_{f\in\mathscr{C}})$. These two mappings are order-preserving and mutually inverse up to equi-interpretability of varieties and homomorphical equivalence of clones, hence they provide an isomorphism of the two lattices. We will often switch from the perspective of varieties to the perspective of clones and vice-versa.

In the first part of this thesis we focus on linear, and/or idempotent varieties and Mal'cev conditions. The first chapter is dedicated solely to linearity; we describe some properties of the subposet of the interpretability lattice consisting of all interpretability classes of linear varieties, and prove that some properties cannot be described by linear identities. In the second chapter, we study filters that correspond to Mal'cev conditions. Namely, we focus on some filters, and prove that their complements are closed under joins of linear, or idempotent varieties.

Commutator theory

Commutator theory generalizes commutator operator on normal subgroups of a group to a binary commutator operator on congruences of general algebras (the theory describing this commutator have been developed in the 80's, and is described in the book by Freese and McKenzie [16]). This commutator is defined by the centralizing relation.

Definition. Let $\mathbf{A}$ be an algebra, and α, β, γ three congruences of $\mathbf{A}$. We say that α *centralizes* β *modulo* γ if for all $x, y, u_1, \dots, u_n, v_1, \dots, v_n \in A$ such that $(u_i, v_i) \in \alpha$ for each i and $(x, y) \in \beta$ and all $(n+1)$-ary terms t we have

$$\big(t(u_1, \dots, u_n, x), t(v_1, \dots, v_n, x)\big) \in \gamma \implies \big(t(u_1, \dots, u_n, y), t(v_1, \dots, v_n, y)\big) \in \gamma.$$

The commutator $[\alpha, \beta]$ is then defined as the smallest congruence γ such that α centralizes β modulo γ.

This binary commutator has especially nice properties in varieties whose congruence lattices are modular. For example, in such varieties $[\alpha, \beta] = [\beta, \alpha]$ holds for any two congruences α, β of a single algebra. This binary commutator allows us to extend the notions of Abelianess, solvability, and nilpotence from groups to general algebras. However, nilpotent algebras do not share all the properties of nilpotent groups. In particular, unlike nilpotent groups, a finite nilpotent algebra does not need to be a product of prime-power order algebras. Nevertheless, there is another notion of supernilpotence that is defined from the generalization of the binary commutator to higher-arity commutators. This higher-arity commutators have been introduced by Bulatov [7] and they have nice properties in congruence permutable varieties (note that the variety of groups is congruence permutable). Supernilpotent finite algebras of finite type in congruence permutable varieties are exactly those algebras that can be factored as a product of prime-power order nilpotent algebras. This is a result of Aichinger and Mudrinski [2].

In the second part, we focus on these higher arity commutators and supernilpotent algebras. First, we give a description of the higher arity commutators in congruence permutable varieties by a certain relation that we will denote $\Delta(\alpha_1, \dots, \alpha_n)$. This is a generalization of the relation that is often denoted

$\Delta_{\alpha,\beta}$ and that encodes the binary commutator in congruence modular varieties (or more generally, in varieties with a difference term). This description allows us to prove that under some assumptions, there is a largest clone with given congruences and given higher arity commutator operators on its congruence lattice. In the last section, we provide explicit identities that describe supernilpotent algebras in congruence permutable varieties.

Preliminaries and notation

Throughout the thesis, we use standard notions from universal algebra, that can be found in any textbook on the topic. We denote algebras by $\mathbf{A}$, $\mathbf{B}$, etc.; relational structures by $\mathbb{A}$, $\mathbb{B}$, etc. In both cases, underlying sets are denoted by the same letter in italic (e.g. A is the underlying set of an algebra $\mathbf{A}$, and B is the underlying set of a relational structure $\mathbb{B}$). For an algebra $\mathbf{A}$ and a term t in the language of the algebra $\mathbf{A}$, we denote $t^{\mathbf{A}}$ the corresponding term operation. We will say that an operation f is *idempotent* if it satisfies $f(x, \dots, x) \approx x$; algebra is idempotent if all its operations are, a variety is idempotent if all its algebras are, and a Mal'cev condition is idempotent if it is satisfiable only by idempotent operations. Further, if $\mathbf{A}$ is an algebra and $X \subseteq A$ then $\operatorname{Sg}^{\mathbf{A}} X$ (or simply $\operatorname{Sg} X$, if the algebra is clear from context) denotes the subalgebra of $\mathbf{A}$ generated by X. Similarly, if $Y \subseteq A^2$, $\operatorname{Cg}^{\mathbf{A}} Y$ denotes the congruence of $\mathbf{A}$ generated by Y.

All clones in this thesis are function clones, i.e., a *clone* $\mathscr{C}$ is a set of operations on a set C, that contains all projections and is closed under composition. We will denote the n-ary projection on i-th coordinate by π_i^n, or just π_i, if the arity is clear from the context. Often, we will view a clone $\mathscr{C}$ as an algebra on C of the signature $\mathscr{C}$, i.e., the algebra $(C, (f)_{f \in \mathscr{C}})$. In particular, whenever we speak about subclones, congruences, or powers of a clone $\mathscr{C}$, we mean subalgebras, congruences, or powers of the algebra $(C, (f)_{f \in \mathscr{C}})$. Finally, if $\mathbf{A}$ is an algebra, $\operatorname{Clo} \mathbf{A}$ will denote the clone of all term operations of $\mathbf{A}$, and if $\mathbb{B}$ is a relational structure, $\operatorname{Pol} \mathbb{B}$ will denote the clone of all polymorphisms of $\mathbb{B}$. We will also use the operator Pol on a single relation R on B to mean the clone of polymorphisms of the relational structure (B, R).

Part I

Mal'cev conditions

Chapter 1

Linearity of Mal'cev conditions

1.1 Introduction

Given a signature τ and t, s τ-terms, an identity $t \approx s$ is said to be of *height 1* if both t and s are are terms of height 1, and it is *linear* (or *height at most 1*) if both t and s are terms of height at most 1. I.e., identities of the form

$$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$$

where f, g are functional symbols and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are not necessarily distinct variables are of height 1, and linear identities are either identities of height 1, or one of the forms

$$f(x_1, \ldots, x_n) \approx y, \text{ or } x \approx y.$$

We say that a variety is *linear*, if it can be defined by linear identities. First, we describe a variant of Birkhoff's HSP theorem for linear varieties, and for varieties defined by height 1 identities.

Mal'cev condition is linear if it involves only linear identities in the signature consisting of the term symbols used. Similarly, we may define Mal'cev conditions of height 1. Many important Mal'cev conditions are described using only linear identities. Such as the Mal'cev conditions describing congruence permutable varieties [28], congruence n-permutable varieties [35, 21], congruence modular varieties [13, 20], congruence distributive varieties [24], and many others. Nevertheless, there are some varietal properties that cannot be described by such condition. We present three such properties. Namely, we prove congruence regularity, congruence uniformity, and congruence singularity cannot be characterized by any linear identities.

Further, we focus on the interpretability lattice. If we take all interpretability classes of linear varieties we naturally get a subposet of the interpretability lattice of varieties. As easily seen from the description of interpretability join of two varieties (see [17], or the next chapter), this subposet is closed under joins. However, a meet of two incomparable linear varieties may not be linear. And in fact, this seems to be the usual case. We investigate the analogue for Mal'cev conditions, that is, meets of Mal'cev conditions (the strongest Mal'cev condition that is implied by both). From the description of interpretability meet of varieties in [17], one can see that a meet of Mal'cev conditions always exists, but the Mal'cev condition obtained directly from this description is not linear. In the last section of this chapter, we prove that the meet of Mal'cev conditions for congruence permutable varieties and for congruence distributive varieties cannot be characterized by a linear Mal'cev condition.

In the third section of this chapter, we also describe a clone counterpart of the poset of linear varieties; that is, the lattice of strong h1-classes of clones ordered by existence of a strong h1-clone homomorphism. This connection implies that meets in the interpretability poset of linear varieties always exist.

1.2 Linear Birkhoff

In this section, we proof an analogue for Birkhoff's HSP theorem for linear varieties. To recall, Birkhoff's theorem tells that a class of algebras is definable by some set of identities if and only if it is closed under taking products (the operator P), subalgebras (the operator S), and homomorphic images (the operator H). We will provide a similar characterization for classes of algebras definable by a set of linear identities (identities of height 1, respectively). For that we need the following construction that generalizes both subalgebras and homomorphic images.

Definition 1.2.1. Let $\mathbf{A}$ be an algebra in signature τ, and let $h_1 \colon A \to B$ and $h_2 \colon B \to A$. We define a τ-algebra $\mathbf{B}$ as follows: for every $f \in \tau$ of arity n and $b_1, \dots, b_n \in B$ let

$$f^{\mathbf{B}} = h_1(f^{\mathbf{A}}(h_2(b_1), \dots, h_2(b_n)))$$

We call $\mathbf{B}$ a *reflection* of $\mathbf{A}$. If $h_1 \circ h_2$ is the identity function on B, then we say that $\mathbf{B}$ is a *retraction*.

For a class of algebras $\mathcal{K}$, we write $\mathsf{R}\,\mathcal{K}$ for the class of all reflections of algebras in $\mathcal{K}$, and $\mathsf{R}_{\mathrm{ret}}\,\mathcal{K}$ for all retraction of algebras in $\mathcal{K}$. Also for a single algebra $\mathsf{R}\,\mathbf{A}$ is the class of all reflections of $\mathbf{A}$, and similarly for $\mathsf{R}_{\mathrm{ret}}\,\mathbf{A}$. Observe that a reflection of an algebra $\mathbf{A}$ satisfies all identities of height 1 that are satisfied in $\mathbf{A}$, and similarly, a retraction of $\mathbf{A}$ satisfies all linear identities that are satisfied in $\mathbf{A}$. Therefore, any class defined by identities of height 1 (linear identities, respectively) is closed under taking reflections (retractions, respectively).

Lemma 1.2.2. *Let $\mathcal{K}$ be a class of algebras of the same signature. Then*

(1) $\mathsf{R}_{\mathrm{ret}}\,\mathcal{K} \subseteq \mathsf{R}\,\mathcal{K}$;

(2) $\mathsf{R}\,\mathsf{R}\,\mathcal{K} = \mathsf{R}\,\mathcal{K}$ *and* $\mathsf{R}_{\mathrm{ret}}\,\mathsf{R}_{\mathrm{ret}}\,\mathcal{K} = \mathsf{R}_{\mathrm{ret}}\,\mathcal{K}$; *and*

(3) $\mathsf{HS}\,\mathcal{K} \subseteq \mathsf{R}_{\mathrm{ret}}\,\mathcal{K}$.

Proof. Item (1) is obvious from the definition. For item (2), observe that if $\mathbf{B}$ is a reflection of $\mathbf{A}$ by mappings h_1 and h_2, and $\mathbf{C}$ is a reflection of $\mathbf{B}$ by mappings g_1 and g_2, then $\mathbf{C}$ is a reflection of $\mathbf{A}$ by mappings $g_1 \circ h_1$ and $h_2 \circ g_2$. The proof of the second part is identical. Finally, for (3) we first prove that $\mathsf{H} \subseteq \mathsf{R}_{\text{ret}}$ and $\mathsf{S} \subseteq \mathsf{R}_{\text{ret}}$. Suppose that $\mathbf{B} \in \mathsf{H}\,\mathbf{A}$ and $h\colon A \to B$ is an epimorphism, and let $g\colon B \to A$ be arbitrary mapping that satisfies $h \circ g = 1_B$ (g is a choice of representatives of congruence classes of the kernel of h). Then $\mathbf{B}$ is a retraction of $\mathbf{A}$ by the mappings h and g. And similarly, if $\mathbf{B} \in \mathsf{S}\,\mathbf{A}$ and $g\colon B \to A$ is the inclusion, let $h\colon A \to B$ be an arbitrary mapping with $h \circ g = 1_B$, then $\mathbf{B}$ is a retraction of $\mathbf{A}$ by the mappings h and g. The rest is item (2). □

Finally, we get to the main theorem of this section. This theorem appeared in [4]; we present an alternative, but very similar proof.

Theorem 1.2.3. *Let $\mathcal{K}$ be a nonempty class of algebras of the same signature τ.*

- *$\mathcal{K}$ is closed under R_{ret} and P if and only if $\mathcal{K}$ is the class of models of some set of linear τ-identities.*
- *$\mathcal{K}$ is closed under R and P if and only if $\mathcal{K}$ is the class of models of some set of τ-identities of height 1.*

Proof. We will prove the first item, the proof of the second item is identical. The implication from right to left is trivial since R_{ret} preserves linear identities, and P preserves all identities. For the other implication, suppose that $\mathcal{K}$ is closed under R_{ret} and P, and let $\mathbf{B}$ be an algebra with signature τ satisfying all linear identities that hold in $\mathcal{K}$. Since $\mathsf{R}_{\text{ret}} \subseteq \mathsf{HS}$, we know that $\mathcal{K}$ is a variety; let $\mathbf{F}$ be its free algebra freely generated by the set B. Let $h_1\colon F \to B$ be a mapping such that $h_1(f^{\mathbf{F}}(b_1, \dots, b_n)) = f^{\mathbf{B}}(b_1, \dots, b_n)$ for all $b, b_1, \dots, b_n \in B$ and every $f \in \tau$, and $h_1(b) = b$ for all $b \in B$. Such mapping exists since $\mathbf{B}$ satisfies all linear τ-identities that hold in $\mathbf{F}$. Further, let $h_2\colon B \to F$ be the inclusion of B into F. Immediately from the definition of h_1, we get that $\mathbf{B}$ is the retraction of $\mathbf{F}$ by the mappings h_1 and h_2. This concludes that $\mathbf{B} \in \mathcal{K}$. □

The final observation of this chapter is that, similarly as $\mathsf{P}\,\mathsf{HS} \subseteq \mathsf{HS}\,\mathsf{P}$, we have $\mathsf{P}\,\mathsf{R}_{\text{ret}} \subseteq \mathsf{R}_{\text{ret}}\,\mathsf{P}$. This immediately follows from the above proof, since if $\mathbf{B} \in \mathsf{P}\,\mathsf{R}_{\text{ret}}\,\mathcal{K}$, then we know that $\mathbf{B} \in \mathsf{R}_{\text{ret}}\,\mathbf{F}$ for some $\mathbf{F} \in \mathsf{HSP}\,\mathcal{K}$, consequently $\mathbf{B} \in \mathsf{R}_{\text{ret}}\,\mathsf{HSP}\,\mathcal{K} = \mathsf{R}_{\text{ret}}\mathsf{P}\,\mathcal{K}$. We also get a similar result for the operator R.

1.3 The lattice

We introduced the poset of interpretability classes of linear varieties in the introduction. Here, we will define the clone counterpart of this poset, that is, the lattice of clones ordered by strong h1 homomorphisms. An h1 clone homomorphism is a mapping between two clones that preserves identities of height 1. Formally, let $\mathscr{A}$ and $\mathscr{B}$ be function clones and $\xi\colon \mathscr{A} \to \mathscr{B}$ a mapping preserving arities. Then ξ is an *h1 clone homomorphism* if

$$\xi(f(\pi^m_{i_1}, \dots, \pi^m_{i_n})) = \xi(f)(\pi^m_{i_1}, \dots, \pi^m_{i_n})$$

for all $m, n \geq 1$, all $i_1, \dots, i_n \in \{1, \dots, m\}$, and any n-ary operation $f \in \mathscr{A}$. An h1 clone homomorphism ξ is a *strong h1 clone homomorphism* if it in addition preserves all projections.

Since in the main focus of this chapter are linear identities, and linear (possibly idempotent) Mal'cev conditions, we will focus on strong h1 clone homomorphisms and linear identities. Nevertheless, analogous results can be also obtained for h1 clone homomorphism and identities of height 1.

If we consider a clone $\mathscr{A}$ as an algebra $\mathbf{A} = (A, (f)_{f\in\mathscr{A}})$, then we can naturally expand the operators H, S, P, R_{ret} and R to clones. Note that unlike H, S, and P, a retraction, or a reflection of a clone might not be a clone. In particular, operations of a retraction, or a reflection might not be closed under composition. We will need one more operator, taking expansions. A clone $\mathscr{B}$ is an *expansion* of an algebra $\mathbf{A}$ of signature τ if $B = A$ and for all $f \in \tau$, we have $f^{\mathbf{A}} \in \mathscr{B}$. If $\mathcal{K}$ is a class of algebras, the class of all expansions of algebras from $\mathcal{K}$ is denoted $\mathsf{E}\,\mathcal{K}$.

Finally, we have the following analogue of Theorem 1.2.3.

Theorem 1.3.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be clones. Then $\mathscr{B} \in \mathsf{E}\mathsf{R}_{\text{ret}}\mathsf{P}\,\mathscr{A}$ if and only if there exists a strong h1 homomorphism from $\mathscr{A}$ to $\mathscr{B}$.*

Proof. The implication from left to right follow from the fact that the operators P and R_{ret} preserve linear identities. For the converse, let $\xi\colon \mathscr{A} \to \mathscr{B}$ be an h1 clone homomorphism. From the definition of an h1 clone homomorphism we know that the algebra $\mathbf{B} = (B, (\xi(f))_{f\in\mathscr{A}})$ satisfies the linear identities in signature $\mathscr{A}$ that hold in $\mathbf{A} = (A, (f)_{f\in\mathscr{A}})$. Therefore, by Theorem 1.2.3, $\mathbf{B} \in \mathsf{R}_{\text{ret}}\mathsf{P}\,\mathbf{A}$, and consequently $\mathscr{B} \in \mathsf{E}\mathsf{R}_{\text{ret}}\mathsf{P}\,\mathbf{A}$. ☐

The following corollary of the previous theorem and Theorem 1.2.3 explains connection of strong h1 clone homomorphisms and linear varieties.

Corollary 1.3.2. *Let $\mathcal{V}$ be a linear variety, and $\mathscr{B}$ a clone. Then there is a strong h1 clone homomorphism from* $\operatorname{clo}(\mathcal{V})$ *to $\mathscr{B}$ if and only if there is a clone homomorphism from* $\operatorname{clo}(\mathcal{V})$ *to $\mathscr{B}$.*

Proof. If there is a strong h1 clone homomorphism from $\operatorname{clo}(\mathcal{V})$ to $\mathscr{B}$, we have that $\mathscr{B} \in \mathsf{E}\mathsf{R}_{\text{ret}}\mathsf{P}\operatorname{clo}(\mathcal{V})$. Therefore, $\mathscr{B} \in \mathsf{E}\mathsf{R}_{\text{ret}}\mathsf{P}\,\mathbf{F}$ where $\mathbf{F}$ is the countably-generated free algebra. But $\mathcal{V}$ is defined by linear identities, hence it is closed under R_{ret} and P, hence $\mathscr{B} \in \mathsf{E}\,\mathcal{V}$ which in turn implies that $\mathscr{B} \in \mathsf{EHSP}\operatorname{clo}(\mathcal{V})$, and there is a strong h1 clone homomorphism from $\mathscr{B}$ to $\operatorname{clo}(\mathcal{V})$.

The other implication is trivial, since every clone homomorphism is a strong h1 clone homomorphism. ☐

Finally, we can describe the connection between the lattice of strong h1-homomorphism classes of clones and the interpretability poset of linear varieties. To recall, the lattice of strong h1-homomorphism classes is obtained by quasi-ordering all clones by an existence of strong h1 clone homomorphism, and factoring the strong h1-homomorphically equivalent clones. Given a linear variety $\mathcal{V}$, its clone counterpart is its clone $\operatorname{clo}(\mathcal{V})$, and given a clone $\mathscr{C}$, its counterpart in the lattice of interpretability types of linear varieties is the variety $\mathsf{R}_{\text{ret}}\mathsf{P}\,\mathscr{C}$. To formulate the connection, let $[\mathscr{A}]_{\text{h1}}$ denote the class of all clones that are strongly h1-homomorphically equivalent to a clone $\mathscr{A}$, and let $[\mathcal{V}]$ denotes the interpretability class of a (linear) variety $\mathcal{V}$.

Proposition 1.3.3. *The mappings* $[\mathcal{V}] \mapsto [\mathrm{clo}(\mathcal{V})]_{h1}$ *and* $[\mathscr{A}]_{h1} \mapsto [\mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathscr{A}]$ *are mutually inverse order-preserving mappings between the poset of interpretability types of linear varieties and the lattice of strong h1-homomorphism classes of clones. Consequently, the poset of interpretability types of linear varieties is lattice-ordered.*

Proof. The previous theorem tells that if a linear variety $\mathcal{V}$ is interpretable in a variety $\mathcal{W}$ (i.e., there is a clone homomorphism from $\mathrm{clo}(\mathcal{V})$ to $\mathrm{clo}(\mathcal{W})$), then there is a strong h1 clone homomorphism from $\mathrm{clo}(\mathcal{V})$ to $\mathrm{clo}(\mathcal{W})$. Therefore, comparable varieties are mapped to comparable clones.

Starting with two clones $\mathscr{A}$ and $\mathscr{B}$ and a strong h1 clone homomorphism from $\mathscr{A}$ to $\mathscr{B}$, we know from Theorem 1.3.1 that $\mathscr{B} \in \mathsf{E}\mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathscr{A}$. Consequently, $\mathscr{B} \in \mathsf{E}\,\mathcal{V}$ where $\mathcal{V}$ is the variety corresponding to the clone $\mathscr{A}$, hence $\mathrm{clo}(\mathcal{V})$ has a clone homomorphism to $\mathscr{B}$. Further, if $\mathcal{W} = \mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathscr{B}$, then $\mathscr{B}$ has a strong h1 clone homomorphism to $\mathrm{clo}(\mathcal{W})$ which in turn implies that $\mathrm{clo}(\mathcal{V})$ has a strong h1 clone homomorphism to $\mathrm{clo}(\mathcal{W})$, and finally, by Corollary 1.3.2, $\mathrm{clo}(\mathcal{V})$ has a clone homomorphism to $\mathrm{clo}(\mathcal{W})$ which means that there is an interpretation from $\mathcal{V}$ to $\mathcal{W}$. Therefore, comparable clones are mapped to comparable varieties.

Altogether, we get that both mappings are well-defined and order-preserving. To prove that they are mutually inverse, observe that $\mathcal{V}$ is equi-interpretable with $\mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathrm{clo}(\mathcal{V})$ since $\mathcal{V}$ is linear, and $\mathscr{A}$ is strong h1-homomorphically equivalent to $\mathrm{clo}(\mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathscr{A})$. □

The above proposition gives that there are meets in the poset of interpretability types of linear varieties. The meets can be also directly described from the proof. Given two linear varieties $\mathcal{V}$ and $\mathcal{W}$, let $\mathcal{V} \wedge \mathcal{W}$ denote their (usual) interpretability meet. Then their linear meet is the variety $\mathsf{R}_{\mathrm{ret}}\mathsf{P}\,\mathrm{clo}(\mathcal{V} \wedge \mathcal{W})$, or in other words, the variety defined by all the linear identities satisfiable in $\mathcal{V} \wedge \mathcal{W}$.

Another observation on the proof gives the following fact: If $\mathscr{A}$, and $\mathscr{B}$ are two clones such that there is a strong h1 clone homomorphism from $\mathscr{A}$ to $\mathscr{B}$, then there exist two clones $\mathscr{A}' \in [\mathscr{A}]_{\mathrm{h1}}$ and $\mathscr{B}' \in [\mathscr{B}]_{\mathrm{h1}}$ such that there exists a clone homomorphism from $\mathscr{A}'$ to $\mathscr{B}'$. An example of two such clones are the clones of the corresponding linear varieties.

1.4 Non-linear properties

In this section we show some properties of varieties that can't be described by linear identities. Those will be congruence regularity, congruence uniformity, and congruence singularity. To recall, a variety is *congruence regular* if and only if any two congruences α, β of a single algebra from the variety that have a common congruence class are identical, and it is *congruence uniform* if any congruence of an algebra from the variety has all congruence classes of equal size. There are several Mal'cev conditions describing congruence regular varieties, see Csákány [12], Grätzer [19], and Wille [42]. For congruence uniformity, the situation is more complicated. Though it can be described by an infinite set of identities, it cannot be described by a Mal'cev condition (see [39]). Nevertheless, we will prove that no set of linear identities describe this property. The last discussed property is congruence singularity that was introduced by Bulatov in connection to complexity of the counting constraint satisfaction problem [9]. A variety is *congruence singular* if for every two congruences α, β of a finite algebra $\mathbf{A}$ from the variety and every $a \in A$, we have

$$|a/\alpha \wedge \beta| + |a/\alpha \vee \beta| = |a/\alpha| + |a/\beta|.$$

An example of variety that satisfies all these properties is the variety of groups.

Theorem 1.4.1. *Congruence regularity, and congruence uniformity cannot be characterized by linear identities.*

Proof. To prove this we will describe two varieties $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{W}$ satisfies the same linear identities as $\mathcal{V}$ does, but $\mathcal{V}$ is congruence regular and congruence uniform while $\mathcal{W}$ is neither. For $\mathcal{V}$ we can take the variety generated by the algebra $\mathbf{A}$ on the set $\{0,1\}$ whose basic operation are all term operations of the group $(\mathbb{Z}_2; +, -, 0)$. Obviously, $\mathcal{V}$ is congruence singular since it is equi-interpretable with a subvariety of groups. For the second variety take $\mathcal{W} = \mathsf{R}_{\mathrm{ret}}(\mathcal{V})$. This variety satisfies the same linear Mal'cev conditions as $\mathcal{V}$ by Theorem 1.2.3. Next we show that $\mathcal{W}$ is not congruence regular. For that consider the algebra $\mathbf{B} \in \mathcal{W}$ which is defined as a retraction of $\mathbf{A}^2$ with an underlying set $B = \{(0,0),(0,1),(1,0)\}$ define by the identity mapping and the mapping which is identical on B and maps $(1,1)$ to $(1,0)$. This algebra has a congruence with congruence classes $\{(0,0),(0,1)\}$ and $\{(1,0)\}$. This congruence is the retraction of the kernel of the projection to the first coordinate. This congruence has a one element class, but it is not trivial. Hence the variety $\mathcal{W}$ is not congruence regular, nor congruence uniform. □

Finally, we discuss congruence singularity. This property has a good meaning for finitely generated varieties, hence we will prove the following, stronger statement.

Theorem 1.4.2. *The subclass of finitely generated varieties that contains all congruence regular finitely generated varieties cannot be characterized by linear identities.*

Proof. The proof is very similar to the previous one. First observe that the variety $\mathcal{V}$ from the above proof is finitely generated and congruence regular. Next, we construct a second variety $\mathcal{W}'$ that is not congruence regular but is still finitely generated. For that define an algebra $\mathbf{C}$ as the retraction of $\mathbf{A}^3$ by the mappings

$$h_1 \colon 0 \mapsto (0,0,0), 1 \mapsto (0,0,1), 2 \mapsto (0,1,0), 3 \mapsto (1,0,0), 4 \mapsto (1,1,0)$$

and

$$h_2 \colon (0,0,0) \mapsto 0, (0,0,1) \mapsto 1, (0,1,a) \mapsto 2, (1,0,a) \mapsto 3, (1,1,a) \mapsto 4$$

for any $a \in \{0,1\}$. This algebra does not have singular congruences. Let, for example, α and β to be the retractions of the kernels of the projections to the first and the second coordinate. Then for $a = 0$ we have

$$|a/\alpha \wedge \beta| + |a/\alpha \vee \beta| = 2 + 5 \neq 3 + 3 = |a/\alpha| + |a/\beta|.$$

Finally, let $\mathcal{W}'$ to be the variety generated by the algebra $\mathbf{C}$. This variety is finitely generated by the definition, satisfies all linear identities that $\mathcal{V}$ does, since $\mathbf{C}$ is as retraction of $\mathbf{A}^3$, and is not congruence singular by the above argument. $\square$

Note that Theorem 1.4.1 can be also generalized to finitely generated varieties by simply taking $\mathcal{W}$ to be the variety generated by $\mathbf{B}$.

1.5 Meets of linear conditions

The meet of two strong Mal'cev conditions in the interpretability lattice is again a strong Mal'cev condition, and consequently meet of two Mal'cev conditions is again a Mal'cev condition. This is easily seen from the description of the meet of two varieties from [17].

Nevertheless, in this section we will work with clones rather than with varieties. The reason for that is the description of the meet in the clone lattice. The meet of homomorphism classes of two clones is the homomorphism class of the clone product of the two clones. A product of clones $\mathscr{A}$ and $\mathscr{B}$ is the clone $\mathscr{A} \times \mathscr{B}$ on the set $A \times B$ whose operations are all operation of the form $f \times g$ where $f \in \mathscr{A}$ and $g \in \mathscr{B}$ are of the same arity. The projections of this clone are products of the respective projections of the clones $\mathscr{A}$ and $\mathscr{B}$. This is exactly the product in the category of clones with clone homomorphisms. Therefore, it is easy to see that it is indeed the meet of the two clones in the lattice. From the definition, we also know that the product of clones $\mathscr{A}$ and $\mathscr{B}$ contains a binary operation $\cdot = \pi_1^{\mathscr{A}} \times \pi_2^{\mathscr{B}}$ which satisfies:

$$
\begin{aligned}
x \cdot x &\approx x \\
(x \cdot y) \cdot (z \cdot w) &\approx x \cdot w \\
h(x_1, \dots, x_n) \cdot h(y_1, \dots, y_n) &\approx h(x_1 \cdot y_1, \dots, x_n \cdot y_n) \\
(f \times g)(x_1, \dots, x_n) &\approx (f \times \pi_1^{\mathscr{B}})(x_1, \dots, x_n) \cdot (\pi_1^{\mathscr{A}} \times g)(x_1, \dots, x_n)
\end{aligned}
$$

for all $h \in \mathscr{A} \times \mathscr{B}$, $f \in \mathscr{A}$, and $g \in \mathscr{B}$.

The following lemma is a clone analogue of Proposition 3 of [17].

Lemma 1.5.1. *Let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be clones. There is a clone homomorphism from* $\mathscr{A} \times \mathscr{B}$ *to* $\mathscr{C}$ *if and only if* $\mathscr{C} \in \mathsf{E}(\mathscr{A}' \times \mathscr{B}')$ *where* $\mathscr{A}' \in \mathsf{HSP}\,\mathscr{A}$ *and* $\mathscr{B}' \in \mathsf{HSP}\,\mathscr{B}$.

Proof. The 'if' part is trivial. For the 'only if' part, let $\xi\colon \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ be a clone homomorphism. Now, consider the operation $t = \xi(\cdot)$ and a binary relation $\alpha = \{(a, t(a, b)) : a, b \in C\}$. This relation is reflexive, since $t(a, a) = a$, symmetric, since $t(t(a, b), t(a, a)) = t(a, a) = a$, and transitive since $t(t(a, b), c) = t(a, c)$. It is also compatible with all $\xi(f)$ for $f \in \mathscr{A} \times \mathscr{B}$, since

$$f(a_1 \cdot b_1, \dots, a_n \cdot b_n) = f(a_1, \dots, a_n) \cdot f(a_1, \dots, a_n)$$

is true in $\mathscr{A} \times \mathscr{B}$, hence α is a congruence of the algebra $\mathbf{C} = (C, (\xi(f))_{f \in \mathscr{A} \times \mathscr{B}})$. Similarly, if we define $\beta = \{(t(a, b), b) : a, b \in C\}$, we get a congruence of $\mathbf{C}$. Finally, we claim that $\alpha \wedge \beta = 0_{\mathbf{C}}$ and $\alpha \circ \beta = 1_{\mathbf{C}}$. The second claim is obvious from the definition. To prove the first one, suppose that $(a, b) \in \alpha \wedge \beta$; that is, there are c and d such that $a = t(c, b)$ and $b = t(a, d)$ which in turn gives that $a = t(c, t(a, d)) = t(c, d)$ and $b = t(t(c, b), d) = t(c, d) = a$. Therefore, $\mathbf{C} \simeq (\mathbf{C}/\alpha) \times (\mathbf{C}/\beta)$. Now, let $\mathscr{A}'$ be the set of operations of $\mathbf{C}/\alpha$ and $\mathscr{B}'$ be the set of operations of $\mathbf{C}/\beta$. Both $\mathscr{A}'$ and $\mathscr{B}'$ are clones since they are factors of the clone $\xi(\mathscr{A} \times \mathscr{B})$. The final claim is that $\mathscr{A}' \in \mathsf{HSP}\,\mathscr{A}$, this is due to the fact that $\xi(f \times g) = t(\xi(f \times \pi_1), \xi(\pi_1 \times g))$, hence if we define $\chi(f) = \xi(f \times \pi_1)$, we get a clone homomorphism from $\mathscr{A}$ onto $\mathscr{A}'$. Similarly, $\mathscr{B}' \in \mathsf{HSP}\,\mathscr{B}$ which completes the proof. □

Theorem 1.5.2. *Meet of Mal'cev conditions for congruence permutability and for congruence distributivity cannot be described by a linear Mal'cev condition.*

Proof. We will construct two clones such that the first one satisfies the Mal'cev condition defined as the meet of the Mal'cev conditions for congruence permutability and for congruence distributivity, and the second one does not, but there is a strong h1 clone homomorphism from the first to the second, hence the meet could not be possibly described by linear identities, or a linear Mal'cev condition.

First, we describe the clone description of the meet of the two conditions. Let $\mathscr{M}$ be the clone of the variety with single Mal'cev operation, and let $\mathscr{J}_n$ be the clone of the variety with chain of Jónsson operations of length n. Then the meet is described as all the clones $\mathscr{K}$ such that there exists n and a clone homomorphism from $\mathscr{M} \times \mathscr{J}_n$ to $\mathscr{K}$. The first clone will be product of clones $\mathscr{A}$ and $\mathscr{B}$ where $\mathscr{A}$ is the clone on $\{0, 1\}$ generated by the minority function and $\mathscr{B}$ is the clone of $\{0, 1\}$ generated by the majority operation. The clone $\mathscr{A} \times \mathscr{B}$ clearly satisfy the Mal'cev condition for the meet, since there is a clone homomorphism from $\mathscr{M} \times \mathscr{J}_1$ to $\mathscr{A} \times \mathscr{B}$. Next, we construct the second clone by a retraction of $\mathscr{A} \times \mathscr{B}$. Let $C = \{(0, 0), (1, 0), (0, 1)\}$, and define mappings $h_1 \colon \{0, 1\}^2 \to C$ and $h_2 \colon C \to \{0, 1\}^2$ by $h_1(c) = c$ for $c \in C$, $h_1((1, 1)) = (1, 0)$, and $h_2(c) = c$. Then let $\mathscr{C}$ be the clone of the retraction of $\mathbf{A} \times \mathbf{B}$ by h_1 and h_2. Clearly, $\mathscr{C} \in \mathsf{E}\,\mathsf{R}_{\mathrm{ret}}(\mathscr{A} \times \mathscr{B})$, hence there is a strong h1 clone homomorphism from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$.

The rest to prove is that $\mathscr{C}$ does not satisfy the Mal'cev condition for the meet. Suppose the contrary, i.e., there is a clone homomorphism from $\mathscr{M} \times \mathscr{J}_n$ to $\mathscr{C}$ for some n. Therefore, by the previous lemma, $\mathscr{C}$ is an expansion of a product of $\mathscr{A}'$ and $\mathscr{B}'$ where $\mathscr{A}'$ contains a Mal'cev operation and $\mathscr{B}'$ contains Jónsson chain of length n. But since $|C|$ is a prime, either $\mathscr{A}'$, or $\mathscr{B}'$ is trivial (a clone on one element set), hence $\mathscr{C}$ contains either a Mal'cev operation, or Jónsson operations. To show it does not contain a Mal'cev operation observe that $\{(0, 0), (0, 1)\}$ is a subuniverse of $\mathscr{C}$. This follows from the fact that $\mathscr{A}$ is idempotent, and consequently $\{(0, 0), (0, 1)\}$ is a subuniverse of $\mathscr{A} \times \mathscr{B}$. Also, the corresponding subclone is isomorphic to $\mathscr{B}$ which does not contain a Mal'cev operation. Similarly, $\mathscr{C}$ does not contain Jónsson operations, since $\{(0, 0), (1, 0)\}$ is a subuniverse of $\mathscr{C}$ with the corresponding subclone isomorphic to $\mathscr{A}$. Therefore, we obtain the contradiction. □

The same argumentation can be used to any other two idempotent linear

Mal'cev conditions given that we have two finite algebras **A**, **B** which satisfy one but not the other. These algebras are usually easy to find for any two particular Mal'cev conditions that are not comparable. For example, for all the Mal'cev conditions mentioned in the next chapter, we can take the polymorphism clone of the relational structure used in the characterization of the condition using a coloring. In the case that two such algebras can be found, we have the following.

Theorem 1.5.3. *Suppose that ϕ_1 and ϕ_2 are two idempotent Mal'cev conditions,* **A** *is a finite algebra that satisfies ϕ_1 but not ϕ_2, and* **B** *is a finite algebra that satisfies ϕ_2 but not ϕ_1. Then the meet of ϕ_1 and ϕ_2 cannot be characterized by a linear Mal'cev condition.*

Proof. The proof is identical to the proof of the previous theorem with the only difference in the construction of the clones $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$. For $\mathscr{A}$ and $\mathscr{B}$ we take the clones of **A** and **B**, respectively. Further, suppose that $|A| = m$, $|B| = n$. We claim that there is a prime p, $m + n - 1 \leq p \leq mn$. Without loss of generality assume that $m \geq n > 1$. From Bertrand's postulate there exists a prime p with $m + n - 1 \leq p \leq 2m + 2n - 4$ and

$$2m + 2n - 4 = 2m + 2(n - 2) \leq 2m + m(n - 2) = mn$$

which completes the claim.

We can get back to the construction of $\mathscr{C}$. Fix two elements $a \in A$ and $b \in B$, and let C be a subset of $A \times B$ of size exactly p containing the set $(\{a\} \times B) \cup (A \times \{b\})$. Further, define mappings $h_1 \colon A \times B \colon C$ and $h_2 \colon C \to A \times B$ by $h_1((x, y)) = (x, y)$ if $(x, y) \in C$ and $h_1(x, y) = (x, b)$, otherwise, and $h_2(x, y) = (x, y)$ for any $(x, y) \in C$. Finally, we define $\mathscr{C}$ as the clone of the retraction of $\mathscr{A} \times \mathscr{B}$ by the mappings h_1 and h_2. The rest of the proof is identical to the previous one with $\{a\} \times B$ and $A \times \{b\}$ being the two distinctive subuniverses of $\mathscr{C}$. □

This proof can be also used to prove that the meet of two idempotent varieties is not linear. For example, using the clones $\mathscr{A}$ and $\mathscr{B}$ form the proof of Theorem 1.5.2, we can show that the meet of the variety with a single Mal'cev operation and the variety with a single majority operation is not equi-interpretable with any linear variety.

Chapter 2

Taylor's modularity conjecture and related problems

2.1 Introduction

As noted in the introduction, Mal'cev conditions describe filters in the lattice of interpretability types; the class of all interpretability types of varieties that satisfy some given Mal'cev condition is a filter in the lattice. A general question that we will study in this chapter is which of the important Mal'cev conditions give rise to a prime filter. Some of the Mal'cev conditions with this property have been described in the monograph by Garcia and Taylor [17], e.g. having group terms, or having cyclic terms of given prime arity. Further, Taylor and Garcia formulated two conjectures about the filters of congruence permutable varieties and the filter of congruence modular varieties. The first one has been resolved by Tschantz, who proved that the filter of congruence permutable varieties is prime [40]. However, this proof has never been published due to its length and technicality. The second conjecture still remains open. This conjecture is usually referred to as the Taylor's modularity conjecture.

Conjecture 2.1.1 (Taylor's modularity conjecture)**.** *The filter of congruence modular varieties is prime.*

Some partial results have been discovered by Sequeira and Bentz [36, 5]. The conjecture can be reformulated as 'Whenever we have two varieties such that their join is congruence modular, then one of the original varieties is congruence modular.' The result by Bentz and Sequeira is that this is true for varieties that are both idempotent and linear, we present two generalizations of this result, namely that either idempotence, or linearity can be dropped. The first generalization is given by the following theorem.

Theorem 2.1.2. *If $\mathcal{V}$, $\mathcal{W}$ are two idempotent varieties such that $\mathcal{V} \vee \mathcal{W}$ is congruence modular then either $\mathcal{V}$, or $\mathcal{W}$ is congruence modular.*

The idea of the proof is to study the varieties that do not satisfy the condition. In every idempotent variety that is not congruence modular, we find an algebra with three special congruences that do not satisfy the modularity law.

This construction generalizes one of McGarry [30] for locally finite idempotent varieties.

A similar method can be also used for other filters. Sequeira [36] observed that for the filter of k-permutable varieties for a fixed k, and for the filter of congruence modular varieties, the complements can be described as the varieties whose terms are 'compatible with some projections'. We use a different notion of coloring of terms by a relational structure (see Definition 2.2.1). If all the relations of the relational structures are equivalences then strong coloring of terms is equivalent to compatibility with projections. The second result can be then formulated as follows.

Theorem 2.1.3. *Let $\mathbb{B}$ be a relational structure, and $\mathcal{V}$, $\mathcal{W}$ two linear varieties that have strongly $\mathbb{B}$-colorable terms. Then $\mathcal{V} \vee \mathcal{W}$ also has strongly $\mathbb{B}$-colorable terms.*

Other filters whose complements can be characterized as the classes of varieties that are strongly colorable by some relational structures are the filter of varieties that are congruence n-permutable for some n, the filter of varieties satisfying a non-trivial congruence identity, and the filters of varieties with a cube term either of fixed arity, or some arity. These characterizations appear in Section 2.3. This is also an evidence of primeness of the above listed filters. This seems especially plausible because all of the above conditions have a nice characterization for idempotent varieties. Valeriote and Willard proved that a linear variety that is not congruence n-permutable for any n is interpretable in the variety of distributive lattices [41], and Kearnes and Kiss proved that an idempotent variety that does not satisfy any non-trivial congruence identity is interpretable in the variety of semilattices [25]. From this immediately follows that the complements of these filters are closed under joins of idempotent varieties. For cube terms we will use a known characterization of idempotent varieties without a cube term by so-called cube term blockers. This has been discovered for finitely generated idempotent varieties by Marković, Maróti, and McKenzie [29], and recently generalized to all idempotent varieties by Kearnes and Szendrei [27]. Using this description we obtain the following theorem.

Theorem 2.1.4. *Suppose that n is a positive integer, and $\mathcal{V}$ and $\mathcal{W}$ are two idempotent varieties such that $\mathcal{V} \vee \mathcal{W}$ has an n-cube term. Then so does either $\mathcal{V}$, or $\mathcal{W}$.*

Altogether, we can prove that all the complements of the mentioned filters, except for the filter of varieties that are congruence n-permutable for a fixed n, are closed under taking joins of idempotent varieties, and taking joins of linear varieties. For the filter of varieties that are congruence n-permutable for a fixed n, we can prove only that its complement is closed under taking joins of linear varieties.

2.2 Preliminaries

Though we will often speak about polymorphism clones of relational structures, the primary perspective for us will be the lattice of interpretability types of varieties. One important observation is that a variety $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of a relational structure $\mathbb{B}$ if and only if there is an algebra $\mathbf{B}$ with the underlying set identical with the one of the relational structure such that all relations of $\mathbb{B}$ are compatible with the operations of $\mathbf{B}$.

To recall, an interpretability join of two varieties $\mathcal{V}$ and $\mathcal{W}$ with signatures τ and σ, respectively, is the variety $\mathcal{V} \vee \mathcal{W}$ whose signature is the discreet union of τ and σ, and is defined by all τ-identities that are satisfied in $\mathcal{V}$ together with all σ-identities that are satisfied in $\mathcal{W}$. This gives another nice reformulation of our problem: Given a Mal'cev condition. Does the union of any two sets of (linear, idempotent) identities in disjoint signatures, such that none of these sets imply the Mal'cev condition, imply the Mal'cev condition?

We include the following reformulation of the definition of coloring of a clone by a relational structure from [4] from the perspective of varieties.

Definition 2.2.1. Given a variety $\mathcal{V}$ and a relational structure $\mathbb{B}$ on a set B. Let $\mathbf{F}$ denote the free algebra freely generated by the set $\{x_b : b \in B\}$, and for a relation R of $\mathbb{B}$ let $R^{\mathcal{V}}$ denote the smallest compatible relation relation of $\mathbf{F}$ containing all tuples $(x_{b_1}, \dots, x_{b_n})$ for $(b_1, \dots, b_n) \in R$. A mapping $c\colon \mathbf{F} \to B$ is called a *coloring of terms* of $\mathcal{V}$ by $\mathbb{B}$ if for any $(f_1, \dots, f_n) \in R^{\mathcal{V}}$ we get $(c(f_1), \dots, c(f_n)) \in R$. A coloring is strong if it in addition satisfies $c(x_b) = b$ for every $b \in B$. We say that a variety has *(strongly)* $\mathbb{B}$*-colorable terms* if and only if there is a (strong) coloring of its terms by $\mathbb{B}$.

Note that the elements of $\mathbf{F}$ from the above definition can be viewed as terms of $\mathcal{V}$ with variables x_b, $b \in B$. This justifies the name. Throughout this chapter we will use the symbol $\mathbf{F}_{\mathcal{V}}(x_1, \dots, x_n)$ to denote the free algebra in a variety $\mathcal{V}$ that is freely generated by the set $\{x_1, \dots, x_n\}$.

2.3 Related problems

As noted in the introduction, our method is to look at the complements of the filters. In all the cases below the complement can be described as the class of varieties that have strongly $\mathbb{B}$-colorable terms for some relational structure $\mathbb{B}$. The key step in the proof of Theorem 2.1.3 is the following.

Theorem 2.3.1. *A linear variety has strongly* $\mathbb{B}$*-colorable terms if and only if it is interpretable in the variety generated by the polymorphism clone of* $\mathbb{B}$*.*

Proof. Suppose that $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{B}$. This happens if there is an algebra $\mathbf{B} \in \mathcal{V}$ such that any relation of $\mathbb{B}$ is compatible with all the operations of $\mathbf{B}$. Now, since $\mathbf{B} \in \mathcal{V}$ it is a factor of the free algebra $\mathbf{F}$ generated by the set $\{x_b : b \in B\}$. Let $c\colon \mathbf{F} \to \mathbf{B}$ denote the homomorphism defined on generators by $c(x_b) = b$. We claim that this is a strong coloring of terms of $\mathcal{V}$. Indeed, if $(f_1, \dots, f_n) \in R^{\mathcal{V}}$ then there is a $\mathcal{V}$-term t of arity m and tuples $(b_{i1}, \dots, b_{in}) \in R$, $i = 1, \dots, m$ such that

$t^{\mathbf{F}}(x_{b_{1j}}, \ldots, x_{b_{mj}}) = f_j$ for all $j \in \{1, \ldots, m\}$. Finally, $t^{\mathbf{B}}$ is compatible with R, hence

$$(c(f_1), \ldots, c(f_n)) = (t^{\mathbf{B}}(b_{11}, \ldots, b_{1n}), \ldots, t^{\mathbf{B}}(b_{m1}, \ldots, b_{mn})) \in R.$$

For the converse, suppose that $c \colon \mathbf{F} \to B$ is a strong coloring of terms of $\mathcal{V}$. Let $\mathbf{B}$ be the retraction of $\mathbf{F}$ on the set B given by the mappings c and the mapping $b \mapsto x_b$ (note that $c(x_b) = b$). Since $\mathcal{V}$ is linear, we have $\mathbf{B} \in \mathcal{V}$. We claim that each operation $t^{\mathbf{B}}$ of $\mathbf{B}$ is a polymorphism of $\mathbb{B}$. To verify this, consider a relation R of $\mathbb{B}$ and tuples $(b_{i1}, \ldots, b_{in}) \in R$, $i = 1, \ldots, m$. We have that $(t^{\mathbf{F}}(x_{b_{11}}, \ldots, x_{b_{1n}}), \ldots, t^{\mathbf{F}}(x_{b_{m1}}, \ldots, x_{b_{mn}})) \in R^{\mathcal{V}}$ which, together with the definition of coloring, implies that

$$\begin{aligned}(t^{\mathbf{B}}(b_{11}, \ldots, b_{1n}), \ldots, t^{\mathbf{B}}(b_{m1}, \ldots, b_{mn})) = \\ (c(t^{\mathbf{F}}(x_{b_{11}}, \ldots, x_{b_{1n}})), \ldots, c(t^{\mathbf{F}}(x_{b_{m1}}, \ldots, x_{b_{mn}}))) \in R.\end{aligned}$$

This shows that the polymorphism clone of $\mathbb{B}$ is an expansion of $\mathbf{B} \in \mathcal{V}$, hence there is an interpretation from $\mathcal{V}$ to the variety generated by the polymorphism clone of $\mathbb{B}$. □

Proof of Theorem 2.1.3. Given $\mathcal{V}$ and $\mathcal{W}$ are both linear and they both have strongly $\mathbb{B}$-colorable terms, we know from the previous theorem that they are both interpretable in the variety generated by the polymorphism clone of $\mathbb{B}$. Therefore, their interpretability join $\mathcal{V} \vee \mathcal{W}$ is interpretable in the variety generated by polymorphism clone of $\mathbb{B}$. This gives us the claim by using the previous theorem again. □

A general aim is to prove that for any relational structure $\mathbb{B}$ there is a chain of varieties such that every idempotent variety that has strongly $\mathbb{B}$-colorable terms is interpretable in some variety from the chain. If this is the case, we will immediately get that the down-set of strongly $\mathbb{B}$-colorable idempotent varieties is an ideal, since any two varieties will be interpretable to a common variety from the chain. Of course, it would be especially nice if the chain would have only one element, and this is actually true for varieties that are not n-permutable for any n and varieties that do not satisfy any non-trivial congruence identity. Nevertheless, in other cases (varieties without a cube term and varieties that are not congruence modular) the chain is of class-size and there is no suitable chain that is a set. Note that if the chain would be small (a set) then it would necessarily have a largest element, hence one variety would suffice. The varieties in the chains that we will construct will be generated by polymorphism clones of some relational structures. These relational structures will be similar to the relational structures used for the coloring. This fact will allow us to connect the results about idempotent and linear varieties.

2.3.1 Having n-permutable congruences

Recall that a variety is said to have n-permutable congruences if and only if every two congruences α, β of a single algebra satisfy $\alpha \circ_n \beta = \beta \circ_n \alpha$. This has two well-known Mal'cev characterizations, the original $(n+1)$-ary terms by Schmidt [35] and ternary terms by Hagemann and Mitschke [21].

Theorem 2.3.2. *The following is equivalent for any variety $\mathcal{V}$ and every positive integer n.*

(1) *$\mathcal{V}$ is n-permutable;*

(2) *there are $(n+1)$-ary $\mathcal{V}$-terms $s_0, \dots, s_n$ such that the identities*

$$s_0(x_0, \dots, x_n) \approx x_0, \ \ s_n(x_0, \dots, x_n) \approx x_n,$$
$$s_i(x_0, x_0, x_2, x_2, \dots) \approx s_{i+1}(x_0, x_0, x_2, x_2, \dots) \text{ for odd } i, \text{ and}$$
$$s_i(x_0, x_1, x_1, x_3, \dots) \approx s_{i+1}(x_0, x_1, x_1, x_3, \dots) \text{ for even } i$$

are satisfied in $\mathcal{V}$;

(3) *there are ternary $\mathcal{V}$-terms $p_0, \dots, p_n$ such that the identities*

$$p_0(x, y, z) \approx x, \ \ p_n(x, y, z) \approx z, \text{ and}$$
$$p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for every } i < n$$

are satisfied in $\mathcal{V}$. □

The primeness of these strong Mal'cev conditions was studied by Sequeira [36]. One of his results is a characterization by means of compatibility with projections. This is translated to the language of colorings as the strong coloring by the relational structure $\mathbb{Z}_n = (\{0, 1, \dots, n\}; \alpha, \beta)$ where α and β are equivalences defined by partitions $01|23|\dots$ and $0|12|3\dots$, respectively. This result is based on Schmidt terms.

Proposition 2.3.3. *Let $n \geq 2$. A variety is not congruence n-permutable if and only if has strongly $\mathbb{Z}_n$-colorable terms.* □

As a corollary of the previous proposition and Theorem 2.1.3 we get the following.

Corollary 2.3.4. *Let $n > 2$ be arbitrary. If $\mathcal{V}$ and $\mathcal{W}$ are two linear varieties such that $\mathcal{V} \vee \mathcal{W}$ is n-permutable, then either $\mathcal{V}$, or $\mathcal{W}$ is n-permutable.* □

Let us focus on the filter of varieties that are n permutable for some n. In the introduction we mentioned the following result of Valeriote and Willard [41].

Theorem 2.3.5. *An idempotent variety is not n-permutable for any n if and only if it is interpretable in the variety of distributive lattices.* □

The following characterization of the class of varieties that are not congruence n-permutable for any n using a strong coloring appeared in [4].

Proposition 2.3.6. *A variety is not n-permutable for any n if and only if it has strongly $(\{0, 1\}, \leq)$-colorable terms.*

Proof. let **F** denote the free algebra freely generated by the set $\{x_0, x_1\}$. The relation $\leq^{\mathcal{V}}$ is the subpower of **F** generated by the set $\{(x_0, x_0), (x_0, x_1), (x_1, x_1)\}$, therefore it is equal to

$$\{(t^{\mathbf{F}}(x_0, x_0, x_1), t^{\mathbf{F}}(x_0, x_1, x_1)) : t \text{ is a ternary term}\}.$$

In other words, for two binary terms f, g we have $f^{\mathbf{F}}(x_0, x_1) \leq^{\mathcal{V}} g^{\mathbf{F}}(x_0, x_1)$ if and only if there exists a ternary term t satisfying $t(x, x, y) \approx f(x, y)$ and $t(x, y, y) \approx g(x, y)$. It follows, that if a variety has Hagemann-Mitschke operations, then it does not have strongly $(\{0, 1\}; \leq)$-colorable terms. Since such terms force $c(x_1) \leq c(x_0)$, a contradiction with $c(x_i) = i$. For the other implication, we define a strong coloring c by $c(h^{\mathbf{F}}(x_0, x_1)) = 0$ if and only if there exists a Hagemann-Mitschke chain connecting x and $h(x, y)$, i.e., there is n, and ternary terms $p_1, \dots, p_n$ such that $p_1(x, y, y) \approx x$, $p_n(x, y, y) \approx h(x, y)$, and $p_i(x, x, y) \approx p_{i+1}(x, y, y)$ for every $i = 1, \dots, n - 1$. Since $\mathcal{V}$ does not have Hagemann-Mitschke terms, we get that $c(x_1) = 1$. The rest is an easy exercise. □

Since the idempotent reduct of the polymorphism clone of $(\{0, 1\}, \leq)$ generates a variety that is equi-interpretable with the variety of distributive lattices, we can get the following by combination of the above proposition and Theorem 2.3.5.

Theorem 2.3.7. *If $\mathcal{V}$ is not n-permutable for any n, and it is either linear, or idempotent then $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $(\{0, 1\}, \leq)$.* □

This gives a relative primeness of this filter as a corollary.

Corollary 2.3.8. *Suppose that $\mathcal{V}$ and $\mathcal{W}$ are two varieties such that each of them is either linear, or idempotent, and $\mathcal{V} \vee \mathcal{W}$ is n-permutable for some n. Then either $\mathcal{V}$, or $\mathcal{W}$ is n-permutable for some n.*

Proof. Suppose that neither of the two varieties is n-permutable. By the previous theorem, we get that both $\mathcal{V}$ and $\mathcal{W}$ are interpretable in the variety generated by the polymorphism clone of $(\{0, 1\}, \leq)$ not depending whether the varieties are idempotent, linear, or one idempotent and the other linear. Consequently, we get that their join is interpretable in the variety generated by the polymorphism clone of $(\{0, 1\}, \leq)$ which is not n-permutable for any n. □

2.3.2 Satisfying a non-trivial congruence identity

In [5], Bentz and Sequeira proved that if a join of two linear idempotent varieties satisfy a non-trivial congruence identity, then either of the varieties does. This result was build on a description by so-called derivatives of sets of identities [14] and a Mal'cev condition very similar to Day terms that have been described by Kearnes and Kiss in [25]. There is also an older Mal'cev condition equivalent to the mentioned one, a Hobby-McKenzie term (for the definition we refer to [22], or [25]). The equivalence of these two conditions follows from the following corollary of Theorems 5.25, 5.28 and 7.15 of [25].

Theorem 2.3.9. *The following is equivalent for a variety $\mathcal{V}$.*

(1) *$\mathcal{V}$ satisfies a non-trivial congruence identity,*

(2) *$\mathcal{V}$ has a Hobby-McKenzie term,*

(3) *$\mathcal{V}$ satisfies an idempotent Mal'cev condition that fails in the variety of semilattices,*

(4) *there exists 4-ary terms $t_0, \dots, t_n$ such that the identities*

$$t_0(x,y,z,w) \approx x \text{ and } t_n(x,y,z,w) \approx w,$$
$$t_i(x,y,y,y) \approx t_{i+1}(x,y,y,y) \text{ for even } i,$$
$$t_i(x,x,y,y) \approx t_{i+1}(x,x,y,y) \text{ and } t_i(x,y,x,y) \approx t_{i+1}(x,y,x,y) \text{ for odd } i$$

are satisfied in $\mathcal{V}$. □

We will refer to the terms in item (4) as to Kearnes-Kiss terms. In the proof of the above theorem, authors first prove that item (1) implies that the variety contains an algebra $\mathbf{A}$ with a compatible semilattice operation, i.e., there is a semilattice operation $\vee$ on A such that its graph—the relation $G(\vee) = \{(a, b, a \vee b) : a, b \in A\}$ is a compatible relation. This is equivalent to item (3) because of the following lemma which appeared in [25] as Lemma 5.24.

Lemma 2.3.10. *If $\mathbf{C}$ is a nontrivial idempotent algebra with a compatible semilattice operation, then there is an algebra $\mathbf{D}$ that is a homomorphic image of $\mathbf{C}$ and is term equivalent to the 2-element semilattice.* □

This suggests a relational structure to be used in the coloring description of this Mal'cev condition: the graph of a semilattice operation on a two element set. Therefore, let $\mathbb{S}$ denote the relational structure on $\{0, 1\}$ with one ternary relation $G(\vee) = \{(x, y, x \vee y) : x, y \in \{0, 1\}\}$.

Lemma 2.3.11. *A variety does not satisfy a non-trivial congruence identity if and only if it has strongly $\mathbb{S}$-colorable terms.*

Proof. The fact that Kearnes-Kiss terms are not colorable by $\mathbb{S}$ can be argued using the arguments of [25], since it is an idempotent linear Mal'cev condition that is not satisfiable in the variety of semilattices. The rest is Lemma 2.3.10. Alternatively, one can show this directly by observing that, for any variety $\mathcal{V}$, the relation $G(\vee)^{\mathcal{V}}$ consists of triples $(s(x_0, x_1), t(x_0, x_1), r(x_0, x_1))$ such that there exists a 4-ary term f satisfying $f(x, x, y, y) \approx s(x, y)$, $f(x, y, x, y) \approx t(x, y)$, and $f(x, y, y, y) \approx r(x, y)$. By the definition of coloring and the relation $G(\vee)$ we can deduce that for every triple $(s, t, r) \in G(\vee)^{\mathscr{A}}$ if $c(s) = 0$ and $c(t) = 0$ then also $c(r) = 0$, and similarly if $c(r) = 0$ then both $c(s)$ and $c(t)$ are also 0. By combining these two observations, one can prove by induction on i that for an operation t_i from Kearnes-Kiss chain we have that $c(t_i(x_0, x_0, x_1, x_1)) = 0$, $c(t_i(x_0, x_1, x_0, x_1)) = 0$, and $c(t_i(x_0, x_1, x_1, x_1)) = 0$. This shows that any coloring c has to satisfy $c(x_1) = 0$, hence the contradiction.

For the other implication suppose that a variety does not have Kearnes-Kiss terms, and define a coloring c in such a way that $c(t) = 0$ if and only if this fact is forced by the argument in the previous paragraph. That is, there exists tuples $(s_i, t_i, r_i) \in G(\vee)^{\mathscr{A}}$, $i = 1, \dots, n$ such that $s_0 = t_0 = x_0$, $r_i = r_{i+1}$ for even i, $s_i = s_{i+1}$ and $t_i = t_{i+1}$ for odd i, and $t = r_n$ for n odd, or $t \in \{s_n, t_n\}$ for n even. Now, since the clone does not contain Kearnes-Kiss operations we get that $c(x_1) = 1$. The rest is to check that c is a valid coloring which is an easy exercise. □

As a corollary of the previous lemma and Theorem 2.1.3 we get the linear case of the following theorem. The idempotent case is a result of Kearnes and Kiss. Note that the idempotent reduct of the polymorphism clone $\mathbb{S}$ is isomorphic to the clone of semilattices (see Lemma 2.3.10).

Theorem 2.3.12. *If $\mathcal{V}$ does not satisfy any non-trivial congruence identity and it is either linear, or idempotent, then $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{S}$.* □

Again, as in the previous section we get this straightforward corollary.

Corollary 2.3.13. *Suppose that $\mathcal{V}$ and $\mathcal{W}$ are two varieties that are each either idempotent, or linear, and $\mathcal{V} \vee \mathcal{W}$ satisfies a non-trivial congruence identity. Then so does either $\mathcal{V}$, or $\mathcal{W}$.* □

2.3.3 Having a cube term

Cube terms describe finite algebras with few subpowers (i.e., with a polynomial bound in n on the number of generators of subalgebras of n-th power). This result and many more interesting properties of algebras with cube terms can be found in [6], [1], and [26]. There are several Mal'cev conditions equivalent to having a cube term, e.g. having an edge term, or having a parallelogram term. For our purpose, the most useful of these equivalent conditions is the cube term itself.

Fix a variety, an let $\mathbf{F}$ be an algebra that is freely generated by the set $\{x, y\}$. An n-cube term is a $(2^n - 1)$-ary term c such that

$$c^{\mathbf{F}^n}(\mathbf{x}_1, \dots, \mathbf{x}_{2^n-1}) = (x, \dots, x)$$

where $\mathbf{x}_1, \dots, \mathbf{x}_{2^n-1}$ are all the n-tuples of x's and y's containing at least one y in some order. For example, a Mal'cev term q is a 2-cube term since it satisfies $q^{\mathbf{F}^2}((x, y), (y, y), (y, x)) = (x, x)$. The order of variables in cube terms will not play any role for us.

Our result is based on the description of idempotent algebras without a cube term using cube term blockers. We say that U is a *cube term blocker* in an algebra $\mathbf{A}$ if U is a proper subset, i.e., $\emptyset \neq U \subseteq A$, and $A^n \setminus (A \setminus U)^n$ is a subuniverse of $\mathbf{A}^n$ for every $n \in \mathbb{N}$. Cube term blockers have been introduced by Marković, Maróti, and McKenzie [29] who proved that a finite idempotent algebra has a cube term if and only if none of its subalgebras has a cube term blocker. This theorem was recently generalized to infinite algebras by Kearnes and Szendrei [27]. They also proved a similar characterization for cube terms of fixed arity using crosses, that is, relations of the form

$$\mathrm{Cross}(U_1, \dots, U_n) = \{(x_1, \dots, x_n) : x_i \in U_i \text{ for some } i\}.$$

Note that U is a cube term blocker if and only if U is a proper subset of A and $\mathrm{Cross}(U, \dots, U)$ is a subuniverse of $\mathbf{A}^n$ for any n. The theorem of Kearnes and Szendrei can be then formulated as follows.

Theorem 2.3.14. *Suppose that $\mathcal{V}$ is an idempotent variety, $\mathbf{F}$ denotes the two generated free algebra $\mathbf{F}_{\mathcal{V}}(x, y)$, and $n \in \mathbb{N}$.*

- *$\mathcal{V}$ has no n-cube term if and only if there exist U_1, ..., $U_n \subset F$ such that $x \in U_i$, $y \notin U_i$ for all i, and $\mathrm{Cross}(U_1, \dots, U_n)$ is a subuniverse of $\mathbf{F}^n$.*
- *$\mathcal{V}$ has no cube term if and only if there exist a cube term blocker U in $\mathbf{F}$ such that $x \in U$, $y \notin U$.* □

The next step is to construct algebras of equal sizes and common cube term blocker U in two distinct idempotent varieties without a cube term.

Proposition 2.3.15. *Let $\mathbf{A}$ be an algebra with a cube term blocker U. Then for every cardinal $\kappa \geq \min\{|A|, \aleph_0\}$ there is an algebra $\mathbf{B}$ of size κ in the variety generated by $\mathbf{A}$ with a cube term blocker V such that $|V| = |B \setminus V| = \kappa$.* □

Proof. This proof is based on a transfinite construction by Tarski that appeared in [11]. Put $\mathbf{A}_0 = \mathbf{A}$ and $U_0 = U$. Then for an ordinal λ define $\mathbf{A}_{\lambda+1}$ to be the algebra isomorphic to $\mathbf{A}_\lambda^2$ via an isomorphism $f\colon A_\lambda^2 \to A_{\lambda+1}$ such that $f(a, a) = a$, in particular $A_\lambda \subseteq A_{\lambda+1}$, also let $U_{\lambda+1} = f(A_\lambda \times U_\lambda)$. For a limit ordinal λ put $\mathbf{A}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{A}_\alpha$ and $U_\lambda = \bigcup_{\alpha<\lambda} U_\alpha$. Finally, observe that by induction on λ, U_λ is a cube term blocker in $\mathbf{A}_\lambda$. For successor ordinal we need to prove that $U_\lambda \times A_\lambda$ is a cube term blocker in $\mathbf{A}_\lambda^2$, but this is true since everything interesting happens on the first coordinate and U_λ is a cube term blocker in $\mathbf{A}$. For limit ordinal λ, we have that $A_\lambda^n \setminus (A_\lambda^n \setminus U_\lambda)^n$ is a subuniverse of $\mathbf{A}_\lambda^n$, since it is a union of the chain of subuniverses $A_\alpha^n \setminus (A_\alpha^n \setminus U_\alpha)^n$. Further, for every $\lambda \geq \omega_0$ we have $|A_\lambda| = \min\{|A|, |\lambda|\}$, and the analogues for U_λ and its complement. Therefore, $\mathbf{B} = \mathbf{A}_\kappa$ and $V = U_\kappa$ have the requested properties. □

For the strong Mal'cev conditions of having a n-cube term, we will use a multi-sorted variation on the cross subalgebras, hence taking a product of n different algebras rather then the n-th power of $\mathbf{A}$. We will say that an n-tuple $U_1, \dots, U_n$ of proper subsets of respective A_i's is a *cube term blocking tuple* in $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ if

$$(A_1 \times \dots \times A_n) \setminus \big((A_1 \setminus U_1) \times \dots \times (A_n \setminus U_n)\big)$$

is a subuniverse of $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$. Observe that whenever there are algebras $\mathbf{A}_1$, ..., $\mathbf{A}_n$ in some variety with a cube term blocking tuple, then the variety does not have an n-cube term. The other implication in the case that the variety is idempotent is given by Theorem 2.3.14. Again, as in the previous proposition we can increase the size of algebras with a cube term blocking tuple.

Proposition 2.3.16. *Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be algebras with common signature such that their product has a cube term blocking tuple $U_1, \dots, U_n$. Then for every cardinal $\kappa \geq \min\{|A_1|, \dots, |A_n|, \aleph_0\}$ there are algebras $\mathbf{B}_1,\dots,\mathbf{B}_n$ of size κ in the varieties generated by the respective $\mathbf{A}_i$'s such that their product has a cube term blocking tuple $V_1, \dots, V_n$ with $|V_i| = |B_i \setminus V_i| = \kappa$ for every i.*

Proof. We repeat the same transfinite construction as in the previous proof for all algebras $\mathbf{A}_i$ with subsets U_i. Let $\mathbf{A}_{i,\lambda}$ and $U_{i,\lambda}$ be the algebra, and the set constructed from $\mathbf{A}_i$ and U_i in λ steps. We can repeat the argument from the previous proof to obtain that $|A_{i,\lambda}| = \min\{|A_i|, |\lambda|\}$ for every i and $\lambda \geq \omega_0$, and the analogue for $U_{i,\lambda}$ and its complement. Finally, observe that $U_{1,\lambda}, \dots, U_{n,\lambda}$ is a cube blocking tuple in $\mathbf{A}_{1,\lambda} \times \dots \times \mathbf{A}_{n,\lambda}$. This follows from a similar induction argument as before. Therefore, algebras $\mathbf{B}_i = \mathbf{A}_{i,\kappa}$ and sets $V_i = U_{i,\kappa}$ have the requested properties. □

These two propositions motivate the following definition. To encode the product of several algebras we will add compatible equivalence relation that correspond to the kernels of projections.

Definition 2.3.17. For a cardinal κ fix $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$ and define $\mathbb{C}_\kappa$ as the relational structure $(\kappa, (R_n)_{n\in\mathbb{N}})$ where $R_n = \kappa^n \setminus (\kappa \setminus U_\kappa)^n$. Moreover for a positive integer n define $\mathbb{C}_{n,\kappa}$ to be the relational structure $(\kappa^n; \alpha_1, \dots, \alpha_n, R'_n)$ where α_i is the kernel of the i-th projection, and $R'_n = \kappa^n \setminus (\kappa \setminus U_\kappa)^n$.

Directly from Propositions 2.3.15 and 2.3.16 we get the following.

Corollary 2.3.18. *Let $\mathcal{V}$ be a variety.*

- *If $\mathcal{V}$ contains an algebra $\mathbf{A}$ with a cube term blocker, then it is interpretable in the variety generated by the polymorphism clone of $\mathbb{C}_\kappa$ for all $\kappa \geq \min\{|A|, \aleph_0\}$.*
- *Similarly, if $\mathcal{V}$ contains an algebra $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ with a cube term blocking tuple then it is interpretable in the variety generate by the polymorphism clone of $\mathbb{C}_{n,\kappa}$ for all $\kappa \geq \min\{|A_1|, \dots, |A_n|, \aleph_0\}$.* □

The above corollary claims that there is a clone homomorphism from the polymorphism clone of $\mathbb{C}_\kappa$ to the polymorphism clone of $\mathbb{C}_\lambda$ for any $\kappa > \lambda \geq \aleph_0$. The next example shows that there is no clone homomorphism in the other direction, and as the consequence of this fact, that there is no largest idempotent variety without a cube term.

Example 2.3.19. We will describe identities that are satisfiable by polymorphisms of $\mathbb{C}_\kappa$ but not by polymorphisms of $\mathbb{C}_\lambda$ for $\lambda < \kappa$. They use κ binary symbols f_i, $i \in \kappa$ and ternary symbols $p_{i,j}$, $q_{i,j}$, $i, j \in \kappa$:

$$\begin{aligned} x &\approx p_{i,j}(x, f_j(x,y), y), \\ p_{i,j}(x, f_i(x,y), y) &\approx q_{i,j}(x, f_j(x,y), y), \\ q_{i,j}(x, f_i(x,y), y) &\approx y \end{aligned}$$

for all $i \neq j$, and $f_i(x,x) \approx x$ for all i.

First, observe that these identities are not satisfiable on any set of size smaller than κ except a one-element set. Indeed, if $a \neq b$ then $f_i(a,b) \neq f_j(a,b)$ for $i \neq j$, otherwise

$$a = p_{i,j}(a, f_j(a,b), b) = p_{i,j}(a, f_i(a,b), b) = \\ q_{i,j}(a, f_j(a,b), b) = q_{i,j}(a, f_i(a,b), b) = b.$$

Therefore, the set $\{f_i(a,b) : i \in \kappa\}$ has cardinality κ. On the other hand, $\mathbb{C}_\kappa$ has polymorphisms satisfying these equations, for example fix $c \in U_\kappa$ and put $f_i(x,y) = i$ for $x \neq y$, $f_i(x,x) = x$, and $p(x,y,z) = x$ if $y = f_j(x,z)$ and $p_{i,j}(x,y,z) = c$ otherwise. Similarly, $q(x,y,z) = z$ if $y = f_i(x,z)$ and $q_{i,j}(x,y,z) = c$ otherwise. It is easy to check that these operations are well-defined and satisfy the identities. Also they are compatible with $\mathbb{C}_\kappa$. To prove that, observe that any operation t which has a coordinate i such that $t(x_1,\dots,x_n) \in U_\kappa$ whenever $x_i \in U_\kappa$ is a polymorphism of $\mathbb{C}_\kappa$. This is a simple generalization of the well-known description of polymorphisms of $\mathbb{C}$. The corresponding coordinates for our functions are: the first for p's, the last for q's, and arbitrary for f's.

Finally, we get to the linear case. The relational structure we will use is the cube term blocker of the minimal size. We define $\mathbb{C}$ to be the relational structure $(\{0,1\}, (R_n)_{n\in\mathbb{N}})$ with R_n being the n-ary relations containing all tuples except the tuple $(0,\dots,0)$. For the corresponding strong Mal'cev conditions we use the structures $\mathbb{C}_n = (\{0,1\}, R_n)$ for $n \geq 2$.

Proposition 2.3.20. *The following is true for every variety $\mathcal{V}$ and all $n \geq 2$.*

- *$\mathcal{V}$ does not have an n-cube term if and only if it has strongly $\mathbb{C}_n$-colorable terms.*
- *$\mathcal{V}$ does not have a cube term if and only if it has strongly $\mathbb{C}$-colorable terms.*

Proof. For the first item, observe that tuples in $R_n^{\mathcal{V}}$ are exactly those tuples that can be the result of applying some term f coordinatewise to all tuples of x_0's and x_1's except the tuple $(x_0,\dots,x_0)$. Therefore, whenever we have an n-cube term t, the terms of $\mathcal{V}$ are not strongly $\mathbb{C}_n$-colorable since $(x_0,\dots,x_0) = t(\mathbf{x}_1,\dots,\mathbf{x}_{2^n-1}) \in R_n^{\mathcal{V}}$ where $\mathbf{x}_i$'s are all tuples of x_0's and x_1's except the tuple $(x_0,\dots,x_0)$. For the other direction, suppose that $\mathcal{V}$ does not have an n-cube term and define a coloring c by $c(x_0) = 0$ and $c(t(x_0,x_1)) = 1$ if $t(x,y) \approx x$ is not satisfied in $\mathcal{V}$. To check that this is a strong coloring is an easy exercise. The proof of the second item is analogous. □

In conclusion, we can obtain the following stronger version of Theorem 2.1.4.

Theorem 2.3.21. *Let $n \geq 2$. If $\mathcal{V}$, $\mathcal{W}$ are two varieties such that each of them is either linear, or idempotent, and $\mathcal{V} \vee \mathcal{W}$ has an n-cube term then either $\mathcal{V}$, or $\mathcal{W}$ has an n-cube term.*

Proof. We will show that for every variety $\mathcal{V}$ that does not have an n-cube term, and is either linear, or idempotent, there is κ_0 such that for all $\kappa \geq \kappa_0$, $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{C}_{n,\kappa}$. This will immediately give the theorem.

In the case $\mathcal{V}$ is linear, we get that the variety is interpretable in the polymorphism clone of $\mathbb{C}_n$ by Proposition 2.3.20 and Theorem 2.3.1. Consequently, there is an algebra $\mathbf{A} \in \mathcal{V}$ with $A = \{0, 1\}$ that have R_n as a subuniverse. But this means that the n-tuple $\{1\}, \ldots, \{1\}$ is a cube term blocking tuple in $\mathbf{A}^n$. Finally, by Proposition 2.3.18 we get that it is interpretable in the variety generated by the polymorphism clone of $\mathbb{C}_{n,\kappa}$ for all $\kappa \geq \aleph_0$. In the case $\mathcal{V}$ is idempotent, we know from Theorem 2.3.14 that there is an algebra $\mathbf{A} \in \mathcal{V}$ and proper subsets $U_1, \ldots, U_n$ of A which form a cube term blocking n-tuple of $\mathbf{A}^n$. Again, we get that $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{C}_{n,\kappa}$ for all $\kappa \geq \min\{|A|, \aleph_0\}$ by Proposition 2.3.18. □

2.4 Taylor's modularity conjecture

Finally, we will discuss Taylor's modularity conjecture itself. There are several notable partial results. In [5], authors proved that the conjecture is true if we restrict to linear idempotent varieties. We will prove a generalization of this result to linear varieties that do have to be idempotent, and also provide another partial result. That is, that the conjecture is true if we restrict to idempotent varieties. The second result uses a construction that generalizes one of McGarry [30], who constructed a special congruence in a product of two algebras in any locally finite idempotent variety (we will call such situation 'a modularity blocker'). One of McGarry's question was also whether this construction can be generalized to idempotent varieties with infinite two-generated free algebra. We give an affirmative answer to this question.

Let us first start with the linear case. We will use the well-known Mal'cev characterization of modular varieties by A. Day [13]. The terms appearing in the theorem are commonly known as *Day terms.*

Theorem 2.4.1. *Every algebra in a variety $\mathcal{V}$ has a modular congruence lattice if and only if there exist a positive integer n and 4-ary $\mathcal{V}$ terms $d_0, \ldots, d_n$ such that the identities*

$$d_0(x,y,z,w) \approx x \text{ and } d_n(x,y,z,w) \approx w,$$
$$d_i(x,y,y,z) \approx d_{i+1}(x,y,y,z) \text{ for even } i,$$
$$d_i(x,x,y,y) \approx d_{i+1}(x,x,y,y) \text{ and } d_i(x,y,x,y) \approx d_{i+1}(x,y,x,y) \text{ for odd } i$$

are satisfied in $\mathcal{V}$. □

The following structure is based on Sequeira's description of varieties without Day terms using 'compatibility with projections' [36]. Let $\mathbb{D}$ be a four-element relational structure with three relations that are equivalences that do not satisfying the modular law. In detail, $\mathbb{D} = (D; \alpha, \beta, \gamma)$ where $D = \{0, 1, 2, 3\}$, and

α, β, and γ are equivalences defined by partitions 01|23, 03|12, and 0|12|3, respectively. It is easy to see that Day terms cannot be strongly $\mathbb{D}$-colorable. The reverse implication is an easy exercise using the same methods as in the previous sections.

Proposition 2.4.2. *A linear variety $\mathcal{V}$ is not congruence modular if and only if it has strongly $\mathbb{D}$-colorable terms.* □

Observe that if all relations of $\mathbb{D}$ are compatible with all the operations of some algebra $\mathbf{D}$ with the same underlying set, then since α and β are congruences of $\mathbf{D}$, we get that $\mathbf{D} = \mathbf{A} \times \mathbf{B}$ for some two element algebras $\mathbf{A}$ and $\mathbf{B}$. Further, $\mathbf{D}$ has γ as a congruence. This congruence is a nontrivial congruence strictly smaller then one of the projection kernels. In fact, it is a minimal case of the situation described in the following definition.

Definition 2.4.3. Let γ be a congruence of a product $\mathbf{A} \times \mathbf{B}$ of two algebras, and let γ^a denote the equivalence $\{(b_1, b_2) : ((a, b_1), (a, b_2)) \in \gamma\}$. We say that γ is a *modularity blocker* in $\mathbf{A} \times \mathbf{B}$ if γ is smaller then the kernel of the first projection, and there is an equivalence $\eta \neq 1_{\mathbf{B}}$ on B such that for all $a \in A$ either $\gamma^a = 1_{\mathbf{B}}$, or $\gamma^a = \eta$ where each of the cases apply to at least one $a \in A$. A modularity blocker is *special* if the above is satisfied for $\eta = 0_{\mathbf{B}}$.

Note that a special modularity blocker γ is uniquely determined by the set $U_\gamma = \{a \in A : \gamma^a = 1_{\mathbf{B}}\}$. This allows us to define an equivalence on $A \times B$ which, if compatible with the operations, would be a special modularity blocker. We will abuse the notation and call this equivalence a *special modularity blocker defined by U*. The motivation of the above definition is a theorem of McGarry [30] which can be rephrased as follows.

Theorem 2.4.4. *A locally finite idempotent variety $\mathcal{V}$ is not congruence modular if and only if it contains two algebras $\mathbf{A}$, $\mathbf{B}$ such that $\mathbf{A} \times \mathbf{B}$ has a modularity blocker.* □

We will prove a generalization of the above theorem for idempotent varieties that do not need to be locally finite. The first step of the proof coincides with McGarry's proof. Nevertheless, we present an alternative proof.

Lemma 2.4.5. *Let $\mathcal{V}$ be an idempotent variety, and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. Then $\mathcal{V}$ is congruence modular if and only if $((y, x), (y, y)) \in \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{((x, x), (x, y))\}$.*

Proof. It is easy to show that in any congruence modular variety the lower is true. To show that it also implies modularity suppose that $\mathcal{V}$ is idempotent and not congruence modular. From the standard proof of Day's result we know that in the four-generated free algebra $\mathbf{F}_{\mathcal{V}}(x, y, z, w)$ the congruences $\alpha = \mathrm{Cg}\{(x, y), (z, w)\}$, $\beta = \mathrm{Cg}\{(x, w), (y, z)\}$, and $\gamma = \mathrm{Cg}\{(y, z)\}$ do not satisfy the modularity law, in particular $(x, w) \notin \gamma \vee (\beta \wedge \alpha)$. The idea of the proof is to shift this property to the second power of the two-generated free algebra. To do that, consider the mapping $h \colon \mathbf{F}_{\mathcal{V}}(x, y, z, w) \to \mathbf{F} \times \mathbf{F}$ defined on the generators by $h(x) = (x, x)$, $h(y) = (y, x)$, $h(z) = (y, y)$, and $h(w) = (x, y)$. Observe that if $\mathcal{V}$ is idempotent then h is onto, since for every two binary idempotent terms t, s we have $h(t(s(x,w), s(y,z))) = (t(x,y), s(x,y))$. Finally, since the kernel of h is $\alpha \wedge \beta$ we get that

$$(x,w) \notin h^{-1}(\mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{((y,x),(y,y))\}) = \gamma \vee (\alpha \wedge \beta),$$

and consequently $((x,x),(x,y)) = (h(x), h(w)) \notin \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{((y,x),(y,y))\}$. □

Proposition 2.4.6. *Let $\mathcal{V}$ be an idempotent variety, and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$ then there is a modularity blocker γ in $\mathbf{F} \times \mathbf{F}$ such that $(xx, xy) \in \gamma$ and $(yx, yy) \notin \gamma$,*

Proof. We start with $\gamma_0 = \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{(xx, xy)\}$. From the previous lemma, we know that $(yx, yy) \notin \gamma_0$. We will expand γ_0 to get a modularity blocker. For that take η to be a maximal equivalence on F such that

$$(yx, yy) \notin \gamma = \gamma_0 \vee \mathrm{Cg}\{(ya, yb) : (a,b) \in \eta\}.$$

Such equivalence exists from Zorn's lemma. First, observe that $\gamma^p \geq \eta$ for each $p \in \mathbf{F}$. This is true since it is satisfied for $p = x$ and $p = y$, and $\{x, y\}$ generates $\mathbf{F}$. Second, suppose that $\gamma^p > \gamma^y$ for some $p \in F$. Let $e' \colon \mathbf{F} \to \mathbf{F}$ be the homomorphism defined by $x \mapsto x$ and $y \mapsto p$, and let $e = e' \times 1_{\mathbf{F}}$. Now, consider the congruence $\gamma_1 = e^{-1}(\gamma)$. From the fact that $\gamma^p > \gamma^y$ we get that $\gamma_1^y > \gamma^y$ since for any $(a,b) \in \gamma^p$ we have that $(e(y,a), e(y,b)) = ((p,a),(p,b)) \in \gamma$, hence $((y,a),(y,b)) \in \gamma_1$. From this and the maximality of η we get that $(x,y) \in \gamma_1^y$, and consequently $((p,x),(p,y)) = (e(y,x), e(y,y)) \in \gamma$ which shows that $\gamma^p = 1_{\mathbf{F}}$ and concludes the proof. □

In order to prove the modularity conjecture for idempotent varieties, we will construct modularity blockers of equal size in two different varieties. More precisely, we will construct in any idempotent variety a special modularity blocker γ in $\mathbf{A} \times \mathbf{B}$ where $|A| = |B| = \kappa$ and also $|U_\gamma| = |A \setminus U_\gamma| = \kappa$ for some big enough cardinal κ. To construct a special modularity blocker from a modularity blocker γ in an idempotent algebra $\mathbf{A} \times \mathbf{B}$, it is enough to take $\mathbf{A} \times (\mathbf{B}/\gamma^a)$ for $a \in A$ with $\gamma^a < 1_{\mathbf{B}}$ with the corresponding factor of the modularity blocker γ. This is possible, since for any idempotent $\mathbf{B}$, γ^a is a congruence of $\mathbf{B}$. The next step is to show that we can increase the size of any special modularity blocker. First, we need the following observation.

Lemma 2.4.7. *Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same signature τ, let U be a proper subset of $\mathbf{A}$, and let γ be the modularity blocker defined by U. Then γ is a congruence of $\mathbf{A} \times \mathbf{B}$ if and only if for every symbol $f \in \tau$ we have that if $f^{\mathbf{A}}(a_1, \dots, a_n) \notin U$ for some $a_1, \dots, a_n \in A$ then $f^{\mathbf{B}}$ does not depend on any coordinate i such that $a_i \in U$.*

Proof. It is enough to show that an operation $f^{\mathbf{A}\times\mathbf{B}}$ is compatible with γ if and only if it has the above property. First observe that $((a,b),(a',b')) \in \gamma$ if and only if $a = a'$ and either $a \in U$, or $b = b'$. Therefore, if $f^{\mathbf{A}\times\mathbf{B}}$ is compatible with γ and $f^{\mathbf{A}}(a_1, \dots, a_n) \notin U$ then

$$f^{\mathbf{B}}(b_1, \dots, b_n) = f^{\mathbf{B}}(b'_1, \dots, b'_n)$$

for all b_i, b'_i where $b_i = b'_i$ whenever $a_i \notin U$. This exactly means that $f^{\mathbf{B}}$ does not depend on any coordinate i with $a_i \in U$. The other implication is given by reversing this argument. $\square$

Proposition 2.4.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two algebras of the same signature, and let* γ *be a special modularity blocker in* $\mathbf{A} \times \mathbf{B}$*. Then for every cardinal* $\kappa \geq \min\{|A|, |B|, \aleph_0\}$ *there exist algebras* $\mathbf{A}'$*,* $\mathbf{B}'$ *with a special modularity blocker* γ' *such that* $|A'| = |B'| = |U'| = |A' \setminus U'| = \kappa$*.*

Proof. We will use the identical construction as in the proof of Proposition 2.3.15 to increase the size of B. Let $\mathbf{B}_\lambda$ denote the algebra obtained from $\mathbf{B}$ by Tarski's construction after λ steps. From the previous lemma, it is easy to see that U is modularity blocker in $\mathbf{A} \times \mathbf{B}_\lambda$ for every lambda. Finally, we take $\mathbf{B}' = \mathbf{B}_\kappa$, $\mathbf{A}' = \mathbf{A} \times \mathbf{B}'$ and $U' = U \times B$. Again, this gives a compatible special modularity blocker defined by U' in $\mathbf{A}' \times \mathbf{B}'$ of the required size. $\square$

For every cardinal κ, fix a subset $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$, and define $\mathbb{D}_\kappa$ to be the relational structure with $D_\kappa = \kappa \times \kappa$, and three relations, α, β, and γ, where the first two are kernels of the projections to first and second coordinate, and γ the special modularity blocker defined by U_κ. From the last proposition, we immediately get that the polymorphism clones of these structure form an increasing chain in the clone lattice.

Corollary 2.4.9. *Suppose that* $\mathcal{V}$ *is not congruence modular and it is either linear, or idempotent. Then there exists a cardinal* κ_0 *such that* $\mathcal{V}$ *is interpretable in the variety generated by the polymorphism clone of* $\mathbb{D}_\kappa$ *for all* $\kappa \geq \kappa_0$*.*

Proof. In the case $\mathcal{V}$ is linear, we know that it is interpretable in the variety generated by the polymorphism clone of $\mathbb{D}$ which implies that it contains an algebra $\mathbf{D}$ with the congruences α, β, and γ. Therefore, $\mathbf{D}$ is a product of algebras $\mathbf{D}/\alpha$ and $\mathbf{D}/\beta$ with a special modularity blocker γ. From the previous proposition, we get that $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{D}_\kappa$ for all $\kappa \geq \aleph_0$. For an idempotent variety, we know that it has a special modularity blocker in some product $\mathbf{A} \times \mathbf{B}$ from Proposition 2.4.6 and the comment under it. Again, by the previous proposition, we get that $\mathcal{V}$ is interpretable in the variety generated by the polymorphism clone of $\mathbb{D}_\kappa$ for all $\kappa \geq \min\{|A|, |B|, \aleph_0\}$. $\square$

As a direct corollary of the above, we are given the following stronger version of Theorem 2.1.2.

Corollary 2.4.10. *If* $\mathcal{V}$*,* $\mathcal{W}$ *are two varieties such that each of them is either linear, or idempotent, and* $\mathcal{V} \vee \mathcal{W}$ *is congruence modular then either* $\mathcal{V}$*, or* $\mathcal{W}$ *is congruence modular.* $\square$

Similarly as for the cube term blockers, we can show that there is no largest idempotent non-modular variety. This is achieved by the following construction of varieties without small modularity blockers.

Example 2.4.11. We will show that the variety generated by the polymorphism clone of $\mathbb{D}_\kappa$ does not include non-trivial algebras of size smaller then κ. For that we will use similar identities as in Example 2.3.19. They use κ-binary symbols f_i, $i \in \kappa$ and ternary symbols $p_{i,j}$, $q_{i,j}$, $r_{i,j}$, $i, j \in \kappa$:

$$\begin{aligned} x &\approx p_{i,j}(x, f_j(x,y), y), \\ p_{i,j}(x, f_i(x,y), y) &\approx q_{i,j}(x, f_j(x,y), y), \\ q_{i,j}(x, f_i(x,y), y) &\approx r_{i,j}(x, f_j(x,y), y), \\ r_{i,j}(x, f_i(x,y), y) &\approx y \end{aligned}$$

for all $i \neq j$, and $f_i(x,x) \approx x$ for all i.

Similarly as in Example 2.3.19 we get that these identities are not satisfiable in any algebra of size strictly smaller than κ. We also claim that they are satisfiable in the polymorphism clone $\mathbb{D}_\kappa$. Let $f_i(x,y)$ be idempotent and constantly $c_i \in U_\kappa \times B$ for $x \neq y$ with choosing different constants for different i's. Now fix $c \in U_\kappa$ and $c' \in B$, and define $p^{\mathbf{A}}_{i,j}(x,y,z)$ to be x if $y = f_j(x,z)$ and c otherwise, similarly $q^{\mathbf{A}}_{i,j}(x,y,z) = x$ if $x = y = z$ and it is c otherwise, finally $p^{\mathbf{A}}_{i,j}(x,y,z) = z$ if $y = f_i(x,z)$ and it is constantly c otherwise. For the $\mathbf{B}$ parts of the terms we take $p^{\mathbf{B}}_{i,j}$ to be the first projection, $r^{\mathbf{B}}_{i,j}$ is the third projection, and we put $q^{\mathbf{B}}_{i,j}(x,y,z)$ to be x if $y = f_j(x,z)$, z if $y = f_i(x,z)$, and constantly c' otherwise. It is easy to check that these functions satisfy the identities. They are also compatible with α and β since they are defined as operations of the product algebra. To check that they are also compatible with γ is an easy application of Lemma 2.4.7.

The reason that we did not use the same identities as in Example 2.3.19 is that we believe that those identities imply that the variety has 3-permutable congruences, and hence it is congruence modular. Similarly, we believe that the identities from the above example imply that the variety has 4-permutable congruences. From the result of Valeriote and Willard, we know that both sets of identities have to imply n-permutability for some n, since otherwise they would have to have a two-element model (some restriction of the polymorphism clone of $(\{0,1\}, \leq)$.

2.5 Conclusions

We showed that the complements of almost all the filters corresponding to mentioned Mal'cev conditions are under taking joins of varieties that are either linear, or idempotent. The only exception are the filters of n-permutable varieties for a fixed n. The author believes that the same method can be applied. The key is to find suitable chain of varieties such that any variety that does not have n-permutable congruences is interpretable in one of the varieties from the chain. These varieties could probably be the varieties generated by the polymorphism clones of some suitable relational structures.

We also provided some evidence for the analogues of Taylor's modularity conjecture for all of the discussed Mal'cev conditions, hence expanding the list of conjectures with the following two.

Conjecture 2.5.1. *The filter of varieties satisfying a non-trivial congruence identity is prime.*

Conjecture 2.5.2. *The filter of varieties with an n-cube term is prime.*

Part II

Commutator theory

Chapter 3

A relational description of higher commutators in Mal'cev varieties

3.1 Introduction

Two algebras are called *polynomially equivalent* if they have the same underlying set and the same clone of all polynomial operations. One of the invariants to distinguish polynomially inequivalent algebras is the congruence lattice of the corresponding algebra, and the binary commutator operation $[\cdot,\cdot]$ on this lattice. In fact, from the results of Idziak [23] and Bulatov [8], one can see that on the three-element set, every Mal'cev algebra is up to polynomial equivalence described by its congruence lattice, and the binary commutator operation. This is no longer true for sets with at least four elements. But one can generalize the binary commutator operator to higher arities. These higher arity commutators have been introduced by Bulatov [7]. From the description of polynomial clones with a Mal'cev operation on the four-element set [10], one can obtain that every four-element Mal'cev algebra is determined up to polynomial equivalence by its unary polynomials, congruence lattice, and higher commutator operators on this lattice. The higher commutators are defined by the following 'term-condition'.

Definition 3.1.1 (Bulatov's higher commutator operators)**.** Let $\alpha_0, \dots, \alpha_{n-1}$, and γ be congruences of some algebra $\mathbf{A}$. We say that $\alpha_0, \dots, \alpha_{n-2}$ *centralize* α_{n-1} *modulo* γ if for all tuples $\mathbf{a}_i$, $\mathbf{b}_i$, $i = 0, \dots, n-1$, and all terms t of $\mathbf{A}$ such that

(1) $\mathbf{a}_i \neq \mathbf{b}_i$, but the corresponding entries are congruent modulo α_i for all $i \in \{0, \dots, n-1\}$, and

(2) $(t(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}, \mathbf{a}_{n-1}), t(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}, \mathbf{b}_{n-1})) \in \gamma$ for all

$$(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}) \in (\{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{n-2}, \mathbf{b}_{n-2}\}) \setminus \{(\mathbf{b}_0, \dots, \mathbf{b}_{n-2})\},$$

we have $(t(\mathbf{b}_0, \dots, \mathbf{b}_{n-2}, \mathbf{a}_{n-1}), t(\mathbf{b}_0, \dots, \mathbf{b}_{n-2}, \mathbf{b}_{n-1})) \in \gamma$.

The n-ary *commutator* $[\alpha_0, \dots, \alpha_{n-1}]$ is then defined as the smallest congruence γ such that $\alpha_0, \dots, \alpha_{n-2}$ centralize α_{n-1} modulo γ. We define the nulary commutator to be trivially the full congruence on $\mathbf{A}$, and for the unary commutator of α we put $[\alpha] = \alpha$.

One of important notions that came from higher commutators is a notion of supernilpotence: an algebra is *k-supernilpotent* (or supernilpotent of degree k) if it satisfies the commutator identity

$$\underbrace{[1, 1, \ldots, 1]}_{k+1} = 0.$$

If an algebra is k-supernilpotent for some k we say that is it supernilpotent. For general algebras supernilpotence is a strictly stronger notion then nilpotence; i.e., there is a nilpotent algebra which is not supernilpotent. However, this is not the case in the variety of groups where both notions coincide. Therefore supernilpotent algebras can be viewed as natural generalization of nilpotent groups. They also share several properties with nilpotent groups, in particular a Mal'cev algebra of finite type is supernilpotent if and only if it is a product of prime power order supernilpotent algebras [2]. It has been shown in [3] that there are two expansions of the same group that are both 2-supernilpotent, but the clone given as the join of their clones is not. In this chapter we establish additional properties to ensure that the join of two k-supernilpotent clones sharing a Mal'cev operation is k-supernilpotent.

To achieve that goal we give a description of higher commutators using a certain 2^n-ary relation denoted $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ (see Definition 3.3.1). A similar relation have been also defined in [37]. The relation $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ encodes the value of $[\alpha_0, \ldots, \alpha_{n-1}]$ as its *forks* at the last coordinate—by a *fork* of a relation R at a coordinate i we mean a pair (a, b) such that there exists $\mathbf{c}, \mathbf{d} \in R$ with $c_i = a$, $d_i = b$, and $c_j = d_j$ for all $j \neq i$; and we denote $\psi_i(R)$, the set of all forks of R at i. A similar notion has been used to investigate some properties of algebras with a cube term [6, 1]. The description of higher commutators is then given by the following theorem.

Theorem 3.1.2. *If* $\mathbf{A}$ *is an algebra with a Mal'cev term, and* $\alpha_0, \ldots, \alpha_{n-1}$ *are congruences of* $\mathbf{A}$ *then*

$$[\alpha_0, \ldots, \alpha_{n-1}] = \psi_{2^n-1}(\Delta(\alpha_0, \ldots, \alpha_{n-1})).$$

Further, we show that $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ encodes not only the n-ary commutator $[\alpha_0, \ldots, \alpha_{n-1}]$ but also all smaller-arity commutators that can be obtained by omitting one, or more of the congruences α_i. We show that if we take the clone of all polymorphisms of the relation $\Delta(\alpha_0, \ldots, \alpha_{n-1})$, we get exactly the clone $\mathscr{C}(\alpha_0, \ldots, \alpha_{n-1})$ with the properties described in the following theorem, and consequently one can construct a largest clone with the same commutator operators as the original Mal'cev algebra.

Theorem 3.1.3. *Let* $\mathbf{A}$ *be an algebra with Mal'cev term* q*, and let* $\alpha_0, \ldots, \alpha_{n-1}$ *be congruences of* $\mathbf{A}$*. Then there exists a largest clone* $\mathscr{C}(\alpha_0, \ldots, \alpha_{n-1})$ *on* A *containing* q *such that it preserves congruences* $\alpha_0, \ldots, \alpha_{n-1}$*, and all commutators of the form* $[\alpha_{i_0}, \ldots, \alpha_{i_{k-1}}]$ *(where* $k \leq n$ *and* $0 \leq i_0 < \cdots < i_{k-1} < n$*) agree in* $\mathbf{A}$ *and* $\mathscr{C}(\alpha_0, \ldots, \alpha_{n-1})$*.*

Corollary 3.1.4. *Let $\mathbf{A}$ be an algebra with a Mal'cev term q, then there exists a largest clone on A containing q such that the algebra corresponding to this clone has the same congruence lattice as $\mathbf{A}$ and the same higher commutator operators as $\mathbf{A}$.*

Proof of Corollary 3.1.4 given Theorem 3.1.3. The largest such clone is the intersection of all clones $\mathscr{C}(\alpha_1, \dots, \alpha_n)$ from Theorem 3.1.3 for all n and all tuples $\alpha_1, \dots, \alpha_n$ of congruences of $\mathbf{A}$. □

Although our main motivation of developing this theory lies in the application to Mal'cev algebras on a finite domain, the same results are valid even for algebras with infinite domains. Moreover, since the largest clone in the previous theorem is described as a polymorphism clone, we know that such clone is closed in the natural topology given by pointwise convergence by a result of Romov [34]. More on clones on infinite sets can be found in [18].

The theory developed to prove Theorem 3.1.3 is strong enough to give relatively short proofs of several basic properties of higher commutators (usually referred as (HC1)–(HC8)) that have been established in [7]; their proofs have been published in [2]. Our alternative proofs of some of these properties are given in the fifth section. In the last section, we provide explicit set of identities that define supernilpotence of a fixed degree in a Mal'cev variety. This is another application of the relational description.

3.2 Preliminaries and notation

We will use the same notation as in the rest of the thesis with a few alternations. If α is a congruence we will write $a \equiv_\alpha b$ instead of $(a, b) \in \alpha$. Also, the symbol 2 will denote both the natural number 2 and the two-element set $\{0, 1\}$. We also fix some notations concerning tuples. The i-th coordinate of tuple $\mathbf{a}$ is denoted by either a_i, or $\mathbf{a}(i)$. So, $\mathbf{a} = (a_0, \dots, a_{n-1})$ and $(a_0, \dots, a_{n-1})(i) = a_i$. Tuples will be usually numbered by an increasing sequence of consecutive integers starting at 0. So every n-ary relation is a subset of $A^{\{0,\dots,n-1\}} = A^n$. The only exception will be elements of the relation $\Delta(\alpha_0, \dots, \alpha_{n-1})$. In the theory of binary commutator described in [16], it is usual to denote the elements of 4-ary relation $\Delta_{\alpha,\beta}$ (we will denote the same relation $\Delta(\alpha, \beta)$) as 2×2 matrices. Similarly, when generalizing this concept to $\Delta(\alpha_0, \dots, \alpha_{n-1})$ one should write elements of this relation as $2 \times \cdots \times 2$ n-dimensional matrices. We will denote those elements by tuples whose coordinates will be labeled by the set $2^n = \{0, 1\}^n$, we will write these coordinates as binary sequences omitting brackets and commas, and if needed we will view them as reverse binary expansions of natural numbers $0, \dots, 2^n - 1$; i.e., the tuple $\mathbf{k} = k_0 \dots k_{n-1}$ represents the number $\sum k_i 2^i$. This gives us a natural linear ordering of the set 2^n that we will use to write the elements of A^{2^n} as linear 2^n-tuples. So, the tuple $\mathbf{a} \in A^{2^n}$ will be written as $(a_{00\dots0}, a_{10\dots0}, a_{010\dots0}, a_{110\dots0}, \dots, a_{11\dots1})$. For $d \in \{0, 1\}$ we will use symbol $\overline{d}$ for the negation of d, i.e., $\overline{0} = 1$ and $\overline{1} = 0$. We will also refer to forks of these relations at some coordinate $\mathbf{k}$ the same way as if all the coordinates would be integer.

The last piece of notation has a close connection to a simple lemma about forks of a relation. For any map $e\colon J \to I$ and $\mathbf{a} \in A^I$ the symbol $\mathbf{a}^e$ denotes the J-tuple defined by $\mathbf{a}^e(j) = a_{e(j)}$. Similarly, for a relation $R \leq \mathbf{A}^I$, R^e denotes the relation $\{\mathbf{a}^e \mid \mathbf{a} \in R\}$.

Lemma 3.2.1. *Let $\mathbf{A}$ be an algebra, $R \leq \mathbf{A}^I$, $S \leq \mathbf{A}^J$, $e\colon J \to I$, and $R^e \subseteq S$. If $i \in I$ and there is a unique $j \in J$ such that $e(j) = i$ then $\psi_i(R) \subseteq \psi_j(S)$. In particular,*

(i) if $R \subseteq S$ then $\psi_i(R) \subseteq \psi_i(S)$ for every $i \in I$;

(ii) if $e\colon I \to I$ is bijective then $\psi_{e(i)}(R) = \psi_i(R^e)$ for every $i \in I$.

Proof. Suppose that $(a, b) \in \psi_i(R)$; i.e., there are tuples $\mathbf{a}$, $\mathbf{b} \in R$ such that $a_i = a$, $b_i = b$, and $a_k = b_k$ for all $k \neq i$. Then from $R^e \subseteq S$ we know that $\mathbf{a}^e$, $\mathbf{b}^e \in S$. These tuples witness that $(a, b) \in \psi_j(S)$, because $\mathbf{a}^e(j) = a_i = a$, $\mathbf{b}^e(j) = b_i = b$, and $\mathbf{a}^e(k) = \mathbf{b}^e(k)$ for $k \neq j$.

The statement (i) is a special case of the former for $I = J$, and e being the identity mapping. To prove (ii), suppose that e is a bijection on the set I. Then from the statement for $S = R^e$ we get that $\psi_{e(i)}(R) \subseteq \psi_i(R^e)$. For the other inclusion substitute e with e^{-1}, R with R^e, and i with $e(i)$. ☐

We recall two simple well-known lemmata for Mal'cev algebras.

Lemma 3.2.2. *Let $\mathbf{A}$ be a Mal'cev algebra. Then any binary reflexive compatible relation on $\mathbf{A}$ is a congruence.* ☐

Lemma 3.2.3. *Let $\mathbf{A}$ be a Mal'cev algebra, and let R be n-ary compatible relation on A. If $(a, b) \in \psi_i(R)$, and $(c_0, \dots, c_{i-1}, a, c_{i+1}, \dots, c_{n-1}) \in R$ then*

$$(c_0, \dots, c_{i-1}, b, c_{i+1}, \dots, c_{n-1}) \in R.$$

Proof. Without loss of generality suppose that $i = 0$. Let q be a Mal'cev term of $\mathbf{A}$, and let $(a, u_1, \dots, u_{n-1})$ and $(b, u_1, \dots, u_{n-1})$ be witnesses for $(a, b) \in \psi_0(R)$. Then

$$q\begin{pmatrix} b & a & a \\ u_1 & u_1 & c_1 \\ \vdots & & \vdots \\ u_{n-1} & u_{n-1} & c_{n-1} \end{pmatrix} = \begin{pmatrix} b \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in R,$$

since we know that R is compatible with q. ☐

3.3 Description of higher commutators

Definition 3.3.1. Let $\mathbf{A}$ be an algebra, and $\alpha_0, \dots, \alpha_{n-1} \in \operatorname{Con} \mathbf{A}$. First, for each congruence α_i choose one dimension in the n-dimensional space. We define the relation $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ as the 2^n-ary relation indexed by the set 2^n generated by tuples that are constant on two opposing $(n-1)$-dimensional hyperfaces of the hypercube orthogonal to the dimension corresponding to α_i and these constants are α_i congruent.

We will use $\mathrm{face}_i^d(\mathbf{a})$ to denote the $(d+1)$-th hyperface orthogonal to dimension i, i.e., $\mathrm{face}_i^d(\mathbf{a}) = \mathbf{a}^{f_{i,d}}$ where $f_{i,d}(\mathbf{k}) = k_0 \dots k_{i-1} d k_i \dots k_{n-2}$. The generating tuples of the relation $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ will be denoted $\mathrm{cube}_i^n(a,b)$. By definition, $\mathrm{face}_i^0 \,\mathrm{cube}_i^n(a,b) = (a, \dots, a)$, and $\mathrm{face}_i^1 \,\mathrm{cube}_i^n(a,b) = (b, \dots, b)$; or equivalently, $\mathrm{cube}_i^n(a,b) = (a,b)^{d_i}$ where $d_i \colon 2^n \to 2$ is defined by $\mathbf{k} \mapsto k_{(i)}$. Finally, $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ is defined as

$$\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) := \mathrm{Sg}\left\{ \mathrm{cube}_i^n(a,b) \mid i < n, a \equiv_{\alpha_i} b \right\} = \bigvee_{i<n} \mathrm{cube}_i^n(\alpha_i).$$

For the trivial case when $n = 0$, we put $\Delta_{\mathbf{A}}() := A$. If the algebra is clear from the context, we will write just $\Delta(\alpha_0, \dots, \alpha_{n-1})$ instead of $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$.

Example 3.3.2. We will describe generators of $\Delta(\alpha_0, \alpha_1, \alpha_2)$ for three congruences α_0, α_1, α_2 of an algebra $\mathbf{A}$. The elements of $\Delta(\alpha_0, \alpha_1, \alpha_2)$ are indexed by vertices of a three-dimensional hypercube. The generators are tuples of one of the following forms (a,b,a,b,a,b,a,b), where $a \equiv_{\alpha_0} b$, (a,a,b,b,a,a,b,b) where $a \equiv_{\alpha_1} b$, and (a,a,a,a,b,b,b,b) where $a \equiv_{\alpha_2} b$. Their graphical representation is given in Figure 3.1.

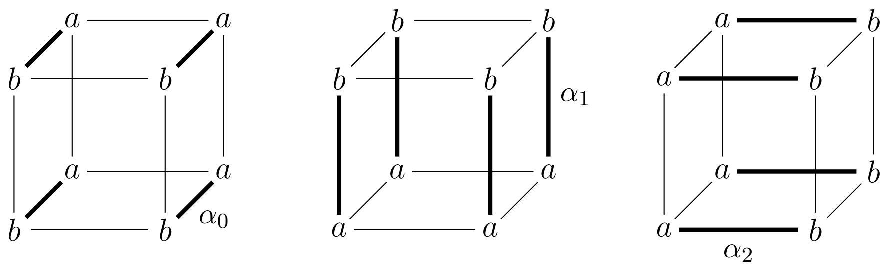

Figure 3.1: Generators of $\Delta(\alpha_0, \alpha_1, \alpha_2)$

Before we get to the proof of Theorem 3.1.2, we will describe some basic properties of the relation $\Delta(\alpha_0, \dots, \alpha_{n-1})$. The first lemma gives a term description of $\Delta(\alpha_0, \dots, \alpha_{n-1})$. This description gives a clear connection of $\Delta(\alpha_0, \dots, \alpha_{n-1})$ and the term condition.

Lemma 3.3.3. *For every algebra* $\mathbf{A}$, *and congruences* $\alpha_i \in \mathrm{Con}\, A$, $i < n$,

$$\begin{aligned}
&\Delta(\alpha_0, \dots, \alpha_{n-1}) \\
&\quad = \big\{ \big(t(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}), t(\mathbf{b}_0, \mathbf{a}_1 \dots, \mathbf{a}_{n-1}), \dots, t(\mathbf{b}_0, \mathbf{b}_1 \dots, \mathbf{b}_{n-1})\big) \mid \\
&\qquad\qquad \forall i < n : m_i \in \mathbb{N}_0, \mathbf{a}_i, \mathbf{b}_i \in A^{m_i}, \mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i; t \in \mathrm{Clo}_{\sum m_i} \mathbf{A} \big\}
\end{aligned}$$

where the elements of the 2^n*-tuple include the term* t *applied to all the combinations of corresponding* $\mathbf{a}_i$*'s and* $\mathbf{b}_i$*'s.*

Proof. The relation $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ is generated by tuples $\mathrm{cube}_i^n(a, b)$ for $a \equiv_{\alpha_i} b$. So, $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ is the set of all tuples of the form

$$t(\mathrm{cube}_{i_0}^n(a_0, b_0), \ldots, \mathrm{cube}_{i_{k-1}}^n(a_{k-1}, b_{k-1}))$$

where $t \in \mathrm{Clo}_k \mathbf{A}$, and for all $j < k$ we have $i_j < n$, $a_j \equiv b_j$ modulo α_{i_j}. The description in the statement of the lemma can be obtained from this by grouping together $\mathrm{cube}_{i_j}^n(a_j, b_j)$'s with the same index i_j, and applying the term t coordinatewise. □

Example 3.3.4. In the ternary commutator case, the lemma tells that

$$\begin{aligned}\Delta(\alpha_0, \alpha_1, \alpha_2) = \big\{ \big(& t(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2), t(\mathbf{b}_0, \mathbf{a}_1, \mathbf{a}_2), t(\mathbf{a}_0, \mathbf{b}_1, \mathbf{a}_2), t(\mathbf{b}_0, \mathbf{b}_1, \mathbf{a}_2), \\ & t(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_2), t(\mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_2), t(\mathbf{a}_0, \mathbf{b}_1, \mathbf{b}_2), t(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) \big) \mid \\ & m_0, m_1, m_2 \in \mathbb{N}_0, t \in \mathrm{Clo}_{m_0+m_1+m_2} \mathbf{A}, \forall i < 3 : \mathbf{a}_i, \mathbf{b}_i \in A^{m_i}, \mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i \big\}.\end{aligned}$$

The graphical representation of a typical element of this relation is given in Figure 3.2.

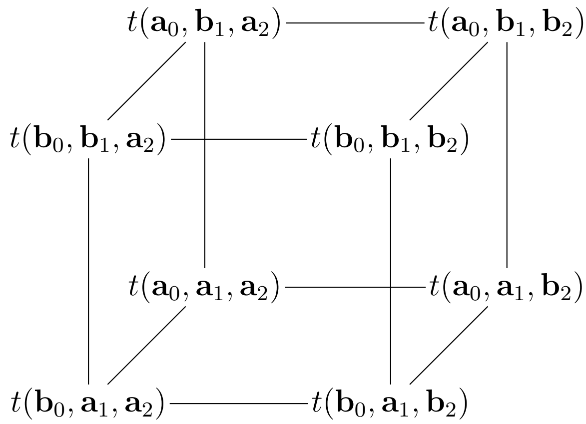

Figure 3.2: Graphical representation of an element of $\Delta(\alpha_0, \alpha_1, \alpha_2)$

Lemma 3.3.5. *Let* $\mathbf{A}$ *be an algebra, and* $\alpha_0, \ldots, \alpha_{n-1}, \gamma \in \mathrm{Con}\,\mathbf{A}$. *Then* $\alpha_0, \ldots, \alpha_{n-2}$ *centralize* α_{n-1} *modulo* γ *if for every tuple*

$$(a_{0\ldots0}, \ldots, a_{1\ldots1}, b_{0\ldots0}, \ldots, b_{1\ldots1}) \in \Delta(\alpha_0, \ldots, \alpha_{n-1})$$

such that $a_{\mathbf{k}} \equiv_\gamma b_{\mathbf{k}}$ *for all* $\mathbf{k} \in 2^{n-1} \setminus \{1\ldots1\}$ *we have also* $a_{1\ldots1} \equiv_\gamma b_{1\ldots1}$. □

Lemma 3.3.6. *Let* $\mathbf{A}$ *be an algebra with congruences* $\alpha_0, \ldots, \alpha_n$, $i < n$, *and* $s_i \colon k \mapsto k_{(0)} \ldots k_{(i-1)} \overline{k_{(i)}} k_{(i+1)} \ldots k_{(n-1)}$. *Then*

(i) if $d \in \{0, 1\}$ *and* $\mathbf{a} \in \Delta(\alpha_0, \ldots, \alpha_{n-1})$ *then*

$$\mathrm{face}_i^d\, \mathbf{a} \in \Delta(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-1});$$

(ii) *if* $\mathbf{a} \in \mathbf{A}^{2^{n-1}}$ *such that*

$$\operatorname{face}_i^0 \mathbf{a} = \operatorname{face}_i^1 \mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1}),$$

then $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$*;*

(iii) $\Delta(\alpha_0, \dots, \alpha_{n-1})^{s_i} = \Delta(\alpha_0, \dots, \alpha_{n-1})$.

Furthermore, if $\mathbf{A}$ *is a Mal'cev algebra then the binary relation*

$$\delta = \big\{ \big(\operatorname{face}_i^0 \mathbf{a}, \operatorname{face}_i^1 \mathbf{a}\big) \mid \mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1}) \big\}$$

is a congruence on $\Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)$*.*

Proof. Since all the considered relations are compatible, we can check the validity of particular inclusions on the generators of the relations, i.e., tuples $\operatorname{cube}_{i,n}(a, b)$. In detail, to prove (i) one has to observe that

$$\operatorname{face}_i^d \operatorname{cube}_j^n(a,b) = \begin{cases} \operatorname{cube}_j^{n-1}(a,b) & \text{if } j < i, \\ (c, \dots, c) & \text{where } c \in \{a, b\} \text{ if } j = i, \\ \operatorname{cube}_{j-1}^{n-1}(a,b) & \text{if } j > i, \end{cases}$$

and consequently

$$\begin{aligned} &\operatorname{face}_i^d \{\operatorname{cube}_j^n(a,b) \mid j < n, a \equiv_{\alpha_j} b\} \\ &\quad = \{\operatorname{cube}_j^{n-1}(a,b) \mid j < i, a \equiv_{\alpha_j} b\} \cup \{\operatorname{cube}_{j-1}^{n-1}(a,b) \mid i < j < n, a \equiv_{\alpha_j} b\}, \end{aligned}$$

hence the relation generated by the left hand side is the same as the relation generated by the right hand side which gives the desired equality.

For (ii), first observe that

$$\{\mathbf{a} \mid \operatorname{face}_i^0 \mathbf{a} = \operatorname{face}_i^1 \mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})\}$$

is the 2^n-ary relation generated by tuples $\mathbf{a}$ such that

$$\operatorname{face}_i^0 \mathbf{a} = \operatorname{face}_i^1 \mathbf{a} = \operatorname{cube}_{j,n-1}(a,b)$$

where $a \equiv b$ modulo α_j, or α_{j+1} when $j < i$, or $j \geq i$ respectively. Second, if $\mathbf{a}$ is such a tuple then $\mathbf{a} = \operatorname{cube}_j^n(a,b)$ if $j < i$, or $\mathbf{a} = \operatorname{cube}_{j+1}^n(a,b)$ if $j \geq i$. In either case, $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ for a, b that are congruent modulo the corresponding congruence.

The item (iii) is a consequence of the fact that

$$(\operatorname{cube}_j^n(a,b))^{s_i} = \begin{cases} \operatorname{cube}_j^n(b,a) & \text{if } j = i, \text{ or} \\ \operatorname{cube}_j^n(a,b) & \text{otherwise,} \end{cases}$$

hence $(\operatorname{cube}_j^n \alpha_j)^{s_i} = \operatorname{cube}_j^n \alpha_j$ from the symmetry of congruences. From the definition of $\Delta(\alpha_0, \dots, \alpha_{n-1})$, we get that

$$\Delta(\alpha_0, \dots, \alpha_{n-1})^{s_i} = \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

From items (i)–(iii) we already know that the binary relation δ is a reflexive symmetric binary relation on $\Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})$. The rest follows from Lemma 3.2.2. □

Lemma 3.3.7. *Let* $\mathbf{A}$ *be a Mal'cev algebra with congruences* $\alpha_0, \dots, \alpha_{n-1}$*. Then* $\psi_{\mathbf{j}}(\Delta(\alpha_0, \dots, \alpha_{n-1})) = \psi_{\mathbf{k}}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$ *for all* $\mathbf{j}, \mathbf{k} \in 2^n$*.*

Proof. For simplicity, let $\psi_{\mathbf{k}} = \psi_{\mathbf{k}}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$. If s_i is the permutation on 2^n defined by $s_i(\mathbf{k}) = k_0 \dots k_{i-1}\overline{k_i}k_{i+1} \dots k_{n-1}$ then from Lemma 3.3.6(iii), we know that $\Delta(\alpha_0, \dots, \alpha_{n-1}) = \Delta(\alpha_0, \dots, \alpha_{n-1})^{s_i}$. This gives, by Lemma 3.2.1, that $\psi_{\mathbf{k}} = \psi_{s_i(\mathbf{k})}$ for all $i < n$. But, if $i_0, \dots, i_{m-1}$ are exactly those indices i such that $k_i \neq j_i$ then $\mathbf{j} = s_{i_0} \circ \dots \circ s_{i_{m-1}}(\mathbf{k})$, and consequently $\psi_{\mathbf{k}} = \psi_{\mathbf{j}}$ for all $\mathbf{j}, \mathbf{k}$. □

Instead of proving Theorem 3.1.2 directly, we will prove the following refinement. The theorem is then given by equivalence of (1) and (4).

Proposition 3.3.8. *If* $\mathbf{A}$ *is a Mal'cev algebra,* $\alpha_0, \dots, \alpha_{n-1} \in \operatorname{Con} \mathbf{A}$*, and* $a, b \in \mathbf{A}$*; then the following is equivalent*

(1) $(a, b) \in \psi_{1\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$*;*

(2) $(a, \dots, a, b) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$*;*

(3) *there exists* $c_0, \dots, c_{2^{n-1}-2}$ *such that*

$$(c_0, \dots, c_{2^{n-1}-2}, a, c_0, \dots, c_{2^{n-1}-2}, b) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

(4) $a \equiv_{[\alpha_0, \dots, \alpha_{n-1}]} b$*;*

Proof. The implication (1) → (2) is direct corollary of Lemma 3.2.3, given that $\Delta(\alpha_0, \dots, \alpha_{n-1})$ contains all constant tuples, in particular $(a, \dots, a)$. (2) → (3) is trivial. For (3) → (4) suppose that (3) is satisfied for a pair (a, b) then since $c_i \equiv c_i \bmod [\alpha_0, \dots, \alpha_{n-1}]$ for all $i < 2^{n-1} - 1$ and $\alpha_0, \dots, \alpha_{n-2}$ centralize α_{n-1} modulo $[\alpha_0, \dots, \alpha_{n-1}]$, we have $a \equiv b \bmod [\alpha_0, \dots, \alpha_{n-1}]$ from Lemma 3.3.5.

The last to prove is (4) → (1), in other words that

$$[\alpha_0, \dots, \alpha_{n-1}] \leq \psi_{1\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1})). \tag{3.3.1}$$

Let $\psi = \psi_{1\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$; from Lemma 3.3.7 we know that

$$\psi = \psi_{1\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1})) = \psi_{\mathbf{k}}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$$

for all $\mathbf{k} \in 2^n$, so we do not have to distinguish between forks at different coordinates. To prove (3.3.1) by the definition of the commutator, it is enough to prove that $\alpha_0, \dots, \alpha_{n-2}$ centralize α_{n-1} modulo ψ. For that we will use Lemma 3.3.5. Suppose that

$$(a_{0\dots0}, \dots, a_{1\dots1}, b_{0\dots0}, \dots, b_{1\dots1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}),$$

and $a_{\mathbf{i}} \equiv_\psi b_{\mathbf{i}}$ for all $\mathbf{i} \in 2^{n-1} \setminus \{1 \dots 1\}$. By repeated use of Lemma 3.2.3 we can replace $b_{00\dots0}, \dots, b_{01\dots1}$ by the respective $a_{\mathbf{i}}$'s. Therefore,

$$(a_{00\dots0}, \dots, a_{01\dots1}, a_{11\dots1}, a_{00\dots0}, \dots, a_{01\dots1}, b_{11\dots1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

In addition, we know from Lemma 3.3.6(i) and 3.3.6(ii) that

$$(a_{00\dots0}, \dots, a_{01\dots1}, a_{11\dots1}, a_{00\dots0}, \dots, a_{01\dots1}, a_{11\dots1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

So, $a_{11\dots1} \equiv_\psi b_{11\dots1}$ which concludes the proof that $\alpha_0, \dots, \alpha_{n-2}$ centralize α_{n-1} modulo ψ. □

In the last proposition, some parts have been already known. The proposition (in the case of Mal'cev algebras) generalize Theorem 4.9 of [16] which gives equivalence of (2), (3), and (4) for binary commutators in congruence modular varieties. The omitted equivalence of (1) and (3) in the binary case can be easily derived from the known fact that $\Delta(\alpha_0, \alpha_1)$ is a congruence on rows. Furthermore, for the higher commutators, the implication (3) $\to$ (4) for the variety of groups has appeared in [37]; and if all α_i's are principal congruences, the equivalence of (2) and (4) is given by [2, Lemma 6.13] (via an easy translation similar to Lemma 3.3.3).

3.4 Strong cube terms, and clones of operations preserving commutators

We will use terms that generalize Mal'cev terms. These terms will play similar role as a difference term in the case of binary commutator. A $(2^n - 1)$-ary term q_n is a *strong n-cube term* if it satisfies

$$q_n(x_{00\dots0}, x_{10\dots0}, \dots, x_{01\dots1}) \approx x_{11\dots1}$$

whenever there is $i < n$ such that for all $\mathbf{k}$ we have $x_{\mathbf{k}} = x_{k_0\dots k_{i-1}\overline{k_i}k_{i+1}\dots k_{n-1}}$. This gives a set of n identities, each with 2^{n-1} variables. The two identities for strong 2-cube term are

$$q_2(x, y, x) \approx y \quad \text{and} \quad q_2(x, x, y) \approx y.$$

So, the term $q_2(y, x, z)$ is a Mal'cev term, and if q is a Mal'cev term then $q(y, x, z)$ is a strong 2-cube term. The three identities for strong 3-cube term are

$$\begin{aligned} q_3(x, y, z, w, x, y, z) &\approx w \\ q_3(x, y, x, y, z, w, z) &\approx w \\ q_3(x, x, y, y, z, z, w) &\approx w. \end{aligned}$$

Strong cube terms are stricter version of cube terms introduced in [6]; a $(2^n - 1)$-ary term is an *n-cube term* if it satisfies

$$q_n(x_{00\dots0}, x_{10\dots0}, \dots, x_{01\dots1}) \approx x_{11\dots1}$$

whenever there is $i < n$ such that $x_{\mathbf{k}} = x_{\mathbf{l}}$ for all $\mathbf{k}, \mathbf{l}$ such that $k_i = l_i$. This gives n two-variable identities compared to the 2^{n-1} variables used in the identities for strong cube terms. One can see that for $n \geq 2$ every strong n-cube term satisfies the identities of an n-cube term; the i-th identity of a cube term is implied by almost any identity for a strong cube term except for the i-th one.

Lemma 3.4.1. *The following is equivalent for any algebra* $\mathbf{A}$.

(1) $\mathbf{A}$ *has a strong n-cube term for all $n \geq 2$,*

(2) $\mathbf{A}$ *has a strong n-cube term for some $n \geq 2$,*

(3) $\mathbf{A}$ *has a Mal'cev term.*

Proof. From the observation in the above paragraphs, we know that the condition (3) is equivalent to

(3') $\mathbf{A}$ has a strong 2-cube term.

We will prove equivalence of (1), (2), and (3'). The implication (1) $\to$ (2) is trivial. For (2) $\to$ (3$'$) observe that if q_n is a strong n-cube term then $q_2(x,y,z) = q_n(x,\dots,x,x,y,z)$ is a strong 2-cube term. For the last implication (3$'$) $\to$ (1) we can construct strong cube terms by the recursion:

$$q_{n+1}(x_0,\dots,x_{2^{n+1}-1}) = q_2(q_n(x_0,\dots,x_{2^n-2}), x_{2^n-1}, q_n(x_{2^n},\dots,x_{2^{n+1}-2}))$$

It is easy to check that if q_n is a strong n-cube term and q_2 is a strong 2-cube term then q_{n+1} is a strong $(n+1)$-cube term. □

The following lemma is the key for proving Theorem 3.1.3 and a lot of properties of higher commutators in Mal'cev varieties.

Lemma 3.4.2. *Let* $\mathbf{A}$ *be an algebra with a strong* n*-cube term* q_n*,* $\alpha_1,\dots,\alpha_n \in \operatorname{Con}\mathbf{A}$*. Then* $\mathbf{a} \in \Delta(\alpha_1,\dots,\alpha_n)$ *if and only if*

$$\operatorname{face}_i^0 \mathbf{a} \in \Delta(\alpha_0,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_{n-1})$$

for each i *and* $q_n(a_{00\dots0}, a_{10\dots0},\dots,a_{01\dots1}) \equiv_{[\alpha_0,\dots,\alpha_{n-1}]} a_{11\dots1}$.

Proof. We will prove the lemma in two steps.
Claim 1. If $\operatorname{face}_i^0 \mathbf{a} \in \Delta(\alpha_0,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_{n-1})$ for each $i < n$, and

$$q_n(a_{00\dots0}, a_{10\dots0},\dots,a_{01\dots1}) = a_{11\dots1}$$

then $\mathbf{a} \in \Delta(\alpha_1,\dots,\alpha_{n-1})$;

For $\mathbf{k} \in 2^n$ and $\mathbf{b} \in A^{2^n}$ let $\mathbf{a}^{\mathbf{k}}$ denotes the 2^n-tuple

$$\mathbf{a}^{\mathbf{k}} = (a_{00\dots0}, a_{k_00\dots0}, a_{0k_1\dots0},\dots,a_{k_0k_1\dots k_{n-1}}).$$

Note that if $k_i = 0$ then $\operatorname{face}_i^0 \mathbf{a}^{\mathbf{k}} = \operatorname{face}_i^1 \mathbf{a}^{\mathbf{k}} = \operatorname{face}_i^0 \mathbf{a}^{k_0\dots k_{i-1}1k_{i+1}\dots k_{n-1}}$. Suppose that the tuple $\mathbf{a}$ satisfies the premise of the claim. The fact that $\mathbf{a}^{\mathbf{k}} \in \Delta(\alpha_0,\dots,\alpha_{n-1})$ for all $\mathbf{k} \neq 11\dots1$ follows from induction on the number of 0's in $\mathbf{k}$—the base step is given by Lemma 3.3.6(ii) and the assumption; the induction step is given by Lemma 3.3.6(i) and (ii) and the above observation. Next we claim that

$$q_n^{\mathbf{A}^{2^n}}(\mathbf{a}^{00\dots0},\dots,\mathbf{a}^{01\dots1}) = (a_{00\dots0},\dots,a_{01\dots1}, q_n^{\mathbf{A}}(a_{00\dots0},\dots,a_{01\dots1})); \qquad (3.4.1)$$

i.e., we have to show that

$$q_n(a_{00\dots0}, a_{j_00\dots0},\dots,a_{0j_1\dots j_{n-1}}) = a_{\mathbf{j}}$$

for every coordinate $\mathbf{j} \neq 11\dots1$. The above is trivial for the coordinate $\mathbf{j} = 11\dots1$; for $\mathbf{j}$ with $j_i = 0$ the identity follows from the i-th identity for a strong cube term. Finally, since the relation $\Delta(\alpha_0,\dots,\alpha_{n-1})$ is compatible with q_n we know that the right hand side of (3.4.1) is in $\Delta(\alpha_0,\dots,\alpha_{n-1})$.

Claim 2. If $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ then $q_n(a_{00\dots0}, \dots, a_{01\dots1}) \equiv_{[\alpha_0,\dots,\alpha_{n-1}]} a_{11\dots1}$.

If $(a_{00\dots0}, \dots, a_{11\dots1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ then from Lemma 3.3.6(i) we know that the tuple $(a_{00\dots0}, \dots, a_{01\dots1}, q_n(a_{00\dots0}, \dots, a_{01\dots1}))$ satisfies the prerequisites of the first claim. Hence, we know that

$$(a_{00\dots0}, \dots, a_{01\dots1}, q_n(a_{00\dots0}, \dots, a_{01\dots1})) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$$

which shows that

$$(a_{11\dots1}, q_n(a_{00\dots0}, \dots, a_{01\dots1})) \in \psi_{11\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1})) = [\alpha_0, \dots, \alpha_{n-1}].$$

Finally, we get to the statement of this lemma. The 'only if' part is Claim 2 together with Lemma 3.3.6(i). For the 'if' part, if

$$\mathrm{face}_i^0\, \mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})$$

for all $i < n$, we know from Claim 1 that

$$(a_{00\dots0}, \dots, a_{01\dots1}, q_n(a_{00\dots0}, \dots, a_{01\dots1})) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

From the last condition and Theorem 3.1.2, we know that $q_n(a_{00\dots0}, \dots, a_{01\dots1})$ and $a_{11\dots1}$ are congruent modulo $\psi_{11\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$. Hence, $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ from Lemma 3.2.3. □

In the rest of this chapter we use symbol $[\alpha_0, \dots, \alpha_{n-1}]_{\mathbf{X}}$ to denote the commutator $[\alpha_0, \dots, \alpha_{n-1}]$ computed in the algebra $\mathbf{X}$.

Corollary 3.4.3. *Let $\mathbf{A}$, $\mathbf{B}$ are algebras that share a universe, a Mal'cev operation, and congruences $\alpha_0, \dots, \alpha_{n-1}$. Then*

$$[\alpha_{i_0}, \dots, \alpha_{i_{k-1}}]_{\mathbf{A}} = [\alpha_{i_0}, \dots, \alpha_{i_{k-1}}]_{\mathbf{B}}$$

for all $k \le n$ and $0 \le i_0 < \dots < i_{k-1} < n$ if and only if $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \dots, \alpha_{n-1})$.

Proof. We will prove the corollary by induction on n. The case $n = 1$ is trivial; for the induction step suppose that for all congruences $\beta_0, \dots, \beta_{n-1} \in \operatorname{Con} \mathbf{A} \cap \operatorname{Con} \mathbf{B}$ such that the commutators $[\beta_{i_0}, \dots, \beta_{i_{k-1}}]$ agree in $\mathbf{A}$ and $\mathbf{B}$ for all $k < n$ and $i_0, \dots, i_{k-1}$ pairwise distinct elements from $\{0, \dots, n-1\}$, we have $\Delta_{\mathbf{A}}(\beta_{i_0}, \dots, \beta_{i_{k-1}}) = \Delta_{\mathbf{B}}(\beta_{i_0}, \dots, \beta_{i_{k-1}})$. In particular, we have

$$\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})$$

for all $i < n$. Let q_n be a common strong n-cube operation of $\mathbf{A}$ and $\mathbf{B}$ (it can be derived from the common Mal'cev operation). From Lemma 3.4.2 we know that whenever $\mathbf{X}$ is a Mal'cev algebra then $\mathbf{a} \in \Delta_{\mathbf{X}}(\alpha_0, \dots, \alpha_{n-1})$ if and only if

$$\mathrm{face}_j^0\, \mathbf{a} \in \Delta_{\mathbf{X}}(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1}) \text{ for all } j < n, \text{ and}$$
$$a_{2^n-1} \equiv_{[\alpha_0,\dots,\alpha_{n-1}]_{\mathbf{X}}} q_n(a_0, \dots, a_{2^n-2}). \quad (3.4.2)$$

Now, suppose that $\mathbf{a} \in \Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})$ hence (3.4.2) is valid for $\mathbf{X} = \mathbf{A}$. But since the operation q_n, the commutators $[\alpha_0, \ldots, \alpha_{n-1}]$, and the relations $\Delta(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-1})$ agree in $\mathbf{A}$ and $\mathbf{B}$, we get that (3.4.2) is also true for $\mathbf{X} = \mathbf{B}$, hence $a \in \Delta_{\mathbf{B}}(\alpha_0, \ldots, \alpha_{n-1})$. This shows that $\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1}) \subseteq \Delta_{\mathbf{B}}(\alpha_0, \ldots, \alpha_{n-1})$. The other inclusion is analogous.

For the 'only if' part, suppose that $\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \ldots, \alpha_{n-1})$. From Lemma 3.3.6(i) it follows that $\Delta_{\mathbf{A}}(\alpha_{i_0}, \ldots, \alpha_{i_{k-1}}) = \Delta_{\mathbf{B}}(\alpha_{i_0}, \ldots, \alpha_{i_{k-1}})$ for all $\{i_0, \ldots, i_{k-1}\} \subseteq \{0, \ldots, n-1\}$ with i's pairwise distinct; and consequently from Theorem 3.1.2, $[\alpha_{i_0}, \ldots, \alpha_{i_{k-1}}]_{\mathbf{A}} = [\alpha_{i_0}, \ldots, \alpha_{i_{k-1}}]_{\mathbf{B}}$. □

Finally, we get to the proof of the Theorem 3.1.3. We restate the theorem once again.

Theorem 3.1.3. *Let* $\mathbf{A}$ *be an algebra with Mal'cev term* q, *and* $\alpha_0, \ldots, \alpha_{n-1}$ *be congruences of* $\mathbf{A}$. *Then there exists a largest clone* $\mathscr{C}$ *on* A *containing* q *such that it preserves congruences* $\alpha_0, \ldots, \alpha_{n-1}$, *and all commutators of the form* $[\alpha_{i_0}, \ldots, \alpha_{i_k}]$ *where* $0 \leq i_0 < \cdots < i_{k-1} < n$ *agree in* $\mathbf{A}$ *and* $\mathscr{C}$.

Proof. We will show that the largest clone satisfying the required properties is the clone $\mathscr{C}$ of all polymorphisms of the relation $\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})$. Obviously $\mathscr{C} \supseteq \operatorname{Clo} \mathbf{A}$ which implies that $q \in \mathscr{C}$ and $\Delta_{\mathscr{C}}(\alpha_0, \ldots, \alpha_{n-1}) \supseteq \Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})$. But since the relation $\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})$ is compatible with $\mathscr{C}$, and the relation $\Delta_{\mathscr{C}}(\alpha_0, \ldots, \alpha_{n-1})$ is the smallest compatible relation containing $\operatorname{cube}_i^n(a, b)$ for all $a \equiv_{\alpha_i} b$ and $i < n$, we have $\Delta_{\mathscr{C}}(\alpha_0, \ldots, \alpha_{n-1}) = \Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})$.

From Corollary 3.4.3, we know that $\mathscr{C}$ satisfies the specified property. The rest is to prove that $\mathscr{C}$ is the largest such clone. Let $\mathscr{B}$ be another clone satisfying the property. Then from the same corollary we get $\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1}) = \Delta_{\mathscr{B}}(\alpha_0, \ldots, \alpha_{n-1})$, and consequently $\mathscr{B} \subseteq \operatorname{Pol}(\Delta_{\mathbf{A}}(\alpha_0, \ldots, \alpha_{n-1})) = \mathscr{C}$. □

3.5 Proofs of basic properties of higher commutators

In this chapter we will present alternative proofs of basic properties of higher commutators formulated in [7, 2]. For an arbitrary algebra $\mathbf{A}$ and its congruences $\alpha_0, \ldots, \alpha_{n-1}$, $\beta_0, \ldots, \beta_{n-1}$, γ, and η the following is satisfied

(HC1) $[\alpha_0, \ldots, \alpha_{n-1}] \leq \alpha_0 \wedge \cdots \wedge \alpha_{n-1}$;

(HC2) if $\alpha_i \leq \beta_i$ for all i then $[\alpha_0, \ldots, \alpha_{n-1}] \leq [\beta_0, \ldots, \beta_{n-1}]$;

(HC3) $[\alpha_0, \ldots, \alpha_{n-1}] \leq [\alpha_1, \ldots, \alpha_{n-1}]$.

Furthermore, if $\mathbf{A}$ is a Mal'cev algebra then

(HC4) if σ is a permutation on the set $\{0, \ldots, n-1\}$ then

$$[\alpha_0, \ldots, \alpha_{n-1}] = [\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(n-1)}];$$

(HC5) congruences $\alpha_0, \ldots, \alpha_{n-2}$ centralize α_{n-1} modulo γ if and only if

$$[\alpha_0, \ldots, \alpha_{n-1}] \leq \gamma;$$

(HC6) if $\eta \leq \alpha_0, \ldots, \alpha_{n-1}$ then

$$[\alpha_0/\eta, \ldots, \alpha_{n-1}/\eta]_{\mathbf{A}/\eta} = ([\alpha_0, \ldots, \alpha_{n-1}]_{\mathbf{A}} \vee \eta)/\eta;$$

(HC7) if I is a non-empty set, and ρ_i are congruences of $\mathbf{A}$ for all $i \in I$ then

$$\bigvee_{i \in I} [\alpha_0, \ldots, \alpha_{n-2}, \rho_i] = [\alpha_0, \ldots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i];$$

(HC8) if $i = 1, \ldots, n-1$ then $[[\alpha_0, \ldots, \alpha_{i-1}], \alpha_i, \ldots, \alpha_{n-1}] \leq [\alpha_0, \ldots, \alpha_{n-1}]$.

Although properties (HC1–3) can be derived directly from Theorem 3.1.2 and the definition of $\Delta(\alpha_0, \ldots, \alpha_{n-1})$, we will omit these proofs because methods would work only for Mal'cev algebras; the general case have been proved in [31, Proposition 1.3]. We will prove properties (HC4), (HC5), (HC7), and (HC8)—the last property (HC6) is a corollary of (HC5).

Proposition 3.5.1 (HC4, [2, Proposition 6.1])**.** *Let* $\mathbf{A}$ *be a Mal'cev algebra and let* $\alpha_0, \ldots, \alpha_{n-1} \in \operatorname{Con} \mathbf{A}$. *Then* $[\alpha_0, \ldots, \alpha_{n-1}] = [\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(n-1)}]$ *for each permutation* σ *of* $\{0, \ldots, n-1\}$.

Proof. We claim that the relations $\Delta(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(n-1)})$ and $\Delta(\alpha_0, \ldots, \alpha_{n-1})$ are identical up to permuting coordinates; precisely

$$\Delta(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(n-1)}) = \Delta(\alpha_0, \ldots, \alpha_{n-1})^{\sigma'}$$

where σ' is defined by $\sigma'(\mathbf{k}) = k_{\sigma^{-1}(0)} \ldots k_{\sigma^{-1}(n-1)}$. One can check this fact by observing that $\sigma'(\mathbf{k})(\sigma(i)) = k_i$, and consequently

$$\Delta(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(n-1)}) = \bigvee_{i<n} \operatorname{cube}_i^n(\alpha_{\sigma(i)}) = \bigvee_{i<n} \left(\operatorname{cube}_{\sigma(i)}^n(\alpha_{\sigma(i)})\right)^{\sigma'}$$
$$= \left(\bigvee_{i<n} \operatorname{cube}_{i,n}(\alpha_i)\right)^{\sigma'} = \Delta(\alpha_0, \ldots, \alpha_{n-1})^{\sigma'}.$$

Finally, $\sigma'(11\ldots1) = 11\ldots1$, so the statement is true from Theorem 3.1.2 and Lemma 3.2.1(ii). □

Proposition 3.5.2 (HC5, [2, Lemma 6.2])**.** *Let* $\mathbf{A}$ *be a Mal'cev algebra and* $\alpha_0, \ldots, \alpha_{n-1}, \gamma$ *be congruences of* $\mathbf{A}$. *Then* $\alpha_0, \ldots, \alpha_{n-2}$ *centralizes* α_{n-1} *modulo* γ *if and only if* $\gamma \geq [\alpha_0, \ldots, \alpha_{n-1}]$.

Proof. The 'only if' part is given by the definition of the commutator, to prove the 'if' part suppose that $\mathbf{a} \in \Delta(\alpha_0, \ldots, \alpha_{n-1})$ such that $a_{k_0 \ldots k_{n-2} 0} \equiv_\gamma a_{k_0 \ldots k_{n-2} 1}$ for all $k \in 2^{n-1} \setminus \{1 \ldots 1\}$. We want to prove that $a_{11\ldots10} \equiv_\gamma a_{11\ldots1}$. By Lemma 3.4.2 we know that

$$a_{11\ldots1} \equiv_{[\alpha_0, \ldots, \alpha_{n-1}]} q_n(a_{00\ldots0}, \ldots, a_{01\ldots1})$$

but the right hand side is modulo γ congruent to

$$q_n(a_{00\dots0},\dots,a_{01\dots10},a_{1\dots10},a_{00\dots0},\dots,a_{01\dots10})=a_{1\dots10}.$$

So, we know that $a_{1\dots11}\equiv a_{1\dots10}$ modulo γ since $\gamma\geq[\alpha_0,\dots,\alpha_{n-1}]$. And finally, $\alpha_0,\dots,\alpha_{n-2}$ centralizes α_{n-1} modulo γ from Lemma 3.3.5. □

The condition (HC6) is a direct corollary of condition (HC5); for a proof see [2, Corollary 6.3]. The following two lemmata prepare for the proof of (HC7).

Lemma 3.5.3. *Let* $\mathbf{A}$ *be an algebra,* I *a non-empty set,* $\rho_i\in\operatorname{Con}\mathbf{A}$ *for all* $i\in I$, *and* $\alpha_0,\dots,\alpha_{n-2}\in\operatorname{Con}\mathbf{A}$. *Then*

$$\Delta(\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in I}\rho_i)=\bigvee_{i\in I}\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_i).$$

Proof. The statement can be derived directly from the definition of the relation $\Delta(\alpha_0,\dots,\alpha_{n-1})$ by a simple calculation:

$$\begin{aligned}\Delta(\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in I}\rho_i)&=\Big(\operatorname{cube}_0^n\alpha_0\vee\dots\vee\operatorname{cube}_{n-2}^n\alpha_{n-2}\vee\operatorname{cube}_{n-1}^n\bigvee_{i\in I}\rho_i\Big)\\&=\bigvee_{i\in I}\big(\operatorname{cube}_0^n\alpha_0\vee\dots\vee\operatorname{cube}_{n-2}^n\alpha_{n-2}\vee\operatorname{cube}_{n-1}^n\rho_i\big)\\&=\bigvee_{i\in I}\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_i).\end{aligned}$$

□

The following lemma was in fact proved during the proof of [2, Lemma 6.4].

Lemma 3.5.4. *Let* $\mathbf{A}$ *be an algebra,* I *a non-empty set,* $\rho_i\in\operatorname{Con}\mathbf{A}$ *for all* $i\in I$, *and* $\alpha_0,\dots,\alpha_{n-2}\in\operatorname{Con}\mathbf{A}$. *Then*

$$[\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in I}\rho_i]=\bigvee_{\substack{\{i_0,\dots,i_{k-1}\}\subseteq I,\\k<\infty}}[\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in\{i_0,\dots,i_{k-1}\}}\rho_i].$$

Proof. Throughout the proof we will extensively use compactness of subuniverses of some fixed algebra that is if $\mathbf{A}_i$ for $i\in J$ are subalgebras of some algebra $\mathbf{A}$, and $a\in\bigvee_{i\in J}\mathbf{A}_i$ then there exists a finite set $K\subseteq J$ such that $a\in\bigvee_{i\in K}\mathbf{A}_i$. To shorten the notation let η denotes the right hand side of the statement. Hence

$$\eta=\bigvee_{\substack{\{i_0,\dots,i_{k-1}\}\subseteq I,\\k<\infty}}[\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in\{i_0,\dots,i_{k-1}\}}\rho_i].$$

First, we prove that α_0, ..., α_{n-2} centralize $\bigvee_{i\in I}\rho_i$ modulo η.
Claim 1. If $\mathbf{a}\in\Delta(\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in I}\rho_i)$ then there is a finite set $S\subseteq I$ such that $\mathbf{a}\in\Delta(\alpha_0,\dots,\alpha_{n-2},\bigvee_{i\in S}\rho_i)$.

The claim follows from Lemma 3.5.3 and the note at the beginning of this proof.

Claim 2. If $a \equiv_\eta b$ then there is a finite set $T \subseteq I$ such that a and b are congruent modulo $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in T} \rho_i]$.

Again, there are finite sets $T_0, \dots, T_{k-1}$ such that a and b are congruent modulo $\bigvee_{j=0}^{k-1}[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in T_j} \rho_i]$. And,

$$\bigvee_{j=0}^{k-1}[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in T_j} \rho_i] \le [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{\rho\in T} \rho]$$

where $T = \bigcup_{j=0}^{k-1} T_j$. Which completes the proof of the second claim.

Suppose that $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in I} \rho_i)$ and $a_{k_0\dots k_{n-2}0} \equiv_\eta a_{k_0\dots k_{n-2}1}$ for all $\mathbf{k} \in 2^{n-1} \setminus \{1\dots1\}$. Let S be a finite set from Claim 1, and let $T_\mathbf{k}$ be finite sets such that $a_{k_0\dots k_{n-2}0}$ and $a_{k_0\dots k_{n-2}1}$ are congruent modulo $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{\rho\in T_\mathbf{k}} \rho]$; such sets exist by Claim 2. Let $U = S \cup T_{0\dots0} \cup \dots \cup T_{1\dots10}$ (note that U is a finite set) and $\eta' = \bigvee_{i\in U} \rho_i$. Then

(1) $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \eta')$, and

(2) $a_{k_0\dots k_{n-2}0} \equiv_{[\alpha_0,\dots,\alpha_{n-1},\eta']} a_{k_0\dots k_{n-2}1}$ for all $k \in 2^{n-1} \setminus \{1\dots1\}$.

So, from the Lemma 3.3.5 we know that $a_{1\dots10} \equiv_{[\alpha_0,\dots,\alpha_{n-1},\eta']} a_{1\dots11}$. Finally, $[\alpha_0, \dots, \alpha_{n-1}, \eta'] \le [\alpha_0, \dots, \alpha_{n-1}, \eta]$ because $\eta' \le \eta$. Hence $\alpha_0, \dots, \alpha_{n-2}$ centralize $\bigvee_{i\in I} \rho_i$ modulo η, and consequently $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in I} \rho_i] \le \eta$.

The other inclusion is obvious from the fact that

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in I} \rho_i] \ge [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i\in J} \rho_i]$$

for every finite set $J \subseteq I$. □

Lemma 3.5.5 ([2, Corollary 6.6]). *Let* $\mathbf{A}$ *be a Mal'cev algebra and* $\alpha_1, \dots, \alpha_{n-1}$, $\rho_1, \dots, \rho_k$ *congruences of* $\mathbf{A}$. *Then*

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i=1}^{k} \rho_i] = \bigvee_{i=1}^{k}[\alpha_0, \dots, \alpha_{n-2}, \rho_i].$$

Proof. It suffices to prove the statement just for $k = 2$. We will write

$$(a_0, \dots, a_{2^{n-1}-1}) \equiv_{\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_i)} (b_0, \dots, b_{2^{n-1}-1})$$

if $(a_0, \dots, a_{2^{n-1}-1}, b_0, \dots, b_{2^{n-1}-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i)$. Note that from Lemma 3.3.6, we know that the binary relation $\{(\mathbf{a}, \mathbf{b}) : \mathbf{a} \equiv_{\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_i)} \mathbf{b}\}$ is a congruence on $\Delta(\alpha_0, \dots, \alpha_{n-2})$. From Lemma 3.5.3, we know that

$$\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2) = \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1) \vee \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_2).$$

Since in Mal'cev algebras $\alpha \circ \beta = \mathrm{Sg}(\alpha \cup \beta)$ for any pair of congruences α, β, we have that for all $\mathbf{a}, \mathbf{b} \in \Delta(\alpha_0, \dots, \alpha_{n-2})$, $\mathbf{a} \equiv_{\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_1\vee\rho_2)} \mathbf{b}$ if and only if there exists $\mathbf{c}$ such that $\mathbf{a} \equiv_{\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_1)} \mathbf{c}$ and $\mathbf{b} \equiv_{\Delta(\alpha_0,\dots,\alpha_{n-2},\rho_2)} \mathbf{c}$.

Now, we prove that

$$[\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2] \leq [\alpha_0, \dots, \alpha_{n-2}, \rho_1] \vee [\alpha_0, \dots, \alpha_{n-2}, \rho_2]. \tag{3.5.1}$$

Suppose that a and b are congruent modulo $[\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2]$, hence from Proposition 3.3.8 there are $e_0, \dots, e_{2^{n-1}-2}$ such that

$$(e_0, \dots, e_{2^{n-1}-2}, a) \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2)} (e_0, \dots, e_{2^{n-1}-2}, b).$$

From the above observation, we know that there is a tuple $\mathbf{c} \in \Delta(\alpha_0, \dots, \alpha_{n-2})$ such that

$$\mathbf{c} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1)} (e_0, \dots, e_{2^{n-1}-2}, a) \tag{3.5.2}$$

and

$$\mathbf{c} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_2)} (e_0, \dots, e_{2^{n-1}-2}, b) \tag{3.5.3}$$

If we use Lemma 3.4.2 for (3.5.2), we get that

$$a \equiv_{[\alpha_0, \dots, \alpha_{n-2}, \rho_1]} q_n(c_0, \dots, c_{2^{n-1}-1}, e_0, \dots, e_{2^{n-1}-2});$$

similarly for (3.5.3), we get that

$$b \equiv_{[\alpha_0, \dots, \alpha_{n-2}, \rho_2]} q_n(c_0, \dots, c_{2^{n-1}-1}, e_0, \dots, e_{2^{n-1}-2}).$$

Altogether, a and b are congruent modulo $[\alpha_0, \dots, \alpha_{n-2}, \rho_1] \vee [\alpha_0, \dots, \alpha_{n-2}, \rho_2]$. Which completes the proof of (3.5.1). The other inclusion is given by (HC2). □

Proposition 3.5.6 (HC7, [2, Lemma 6.7])**.** *Let* **A** *be a Mal'cev algebra with congruences* $\alpha_0, \dots, \alpha_{n-2}$*, and* ρ_i*,* $i \in I$ *for* I *non-empty set. Then*

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] = \bigvee_{i \in I} [\alpha_0, \dots, \alpha_{n-2}, \rho_i].$$

Proof. If I is a finite set then the proposition is given by Lemma 3.5.5. If I is infinite, we can first use Lemma 3.5.4 to get

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] = \bigvee_{\{i_0, \dots, i_{k-1}\} \subseteq I} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in \{i_0, \dots, i_{k-1}\}} \rho_i].$$

Then by using the finite case, the right hand side is equal to

$$\bigvee_{\substack{\{i_0, \dots, i_{k-1}\} \subseteq I, \\ i \in \{i_0, \dots, i_{k-1}\}}} [\alpha_0, \dots, \alpha_{n-2}, \rho_i] = \bigvee_{i \in I} [\alpha_0, \dots, \alpha_{n-2}, \rho_i].$$ □

Proposition 3.5.7 (HC8, [2, Proposition 6.14])**.** *Let* **A** *be a Mal'cev algebra with congruences* $\alpha_0, \dots, \alpha_{n-1}$*, and* $i \in \{1, \dots, n-1\}$*. Then*

$$[[\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{n-1}] \leq [\alpha_0, \dots, \alpha_{n-1}].$$

Proof. For $m \geq i$, define the map $e_m\colon 2^m \to 2^{m-i+1}$ by $\mathbf{k} \mapsto k' k_i \dots k_{m-1}$ where $k' = k_0 \cdot k_1 \cdot \dots \cdot k_{i-1}$. We claim that

$$\Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{m-1})^{e_m} \leq \Delta(\alpha_0, \dots, \alpha_{m-1}). \tag{3.5.4}$$

Because $\Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{m-1})^{e_m}$ is clearly a subuniverse of $\mathbf{A}^{2^m}$ generated by the set

$$(\mathrm{cube}_0^{m-i+1}[\alpha_0, \dots, \alpha_{i-1}])^{e_m} \cup \bigcup_{j<m-i} (\mathrm{cube}_{j+1}^{m-i+1}\, \alpha_{i+j})^{e_m},$$

it suffices to prove that this is a subset of $\Delta(\alpha_0, \dots, \alpha_{m-1})$. The inclusions

$$(\mathrm{cube}_{j+1}^{m-i+1}\, \alpha_{i+j})^{e_m} \subseteq \Delta(\alpha_0, \dots, \alpha_{n-1})$$

are consequences of the fact that $e_m(\mathbf{k})(j+1) = k_{j+i}$ for all $j < m - i$. The other inclusion, $(\mathrm{cube}_0^{m-i+1}[\alpha_0, \dots, \alpha_{i-1}])^{e_m} \subseteq \Delta(\alpha_0, \dots, \alpha_{m-1})$, can be proved by induction on m. If $m = i$ then

$$(\mathrm{cube}_0^1[\alpha_0, \dots, \alpha_{i-1}])^{e_m} = \{(a, \dots, a, b) \mid a \equiv_{[\alpha_0, \dots, \alpha_{i-1}]} b\},$$

and all the elements of this set are in $\Delta(\alpha_0, \dots, \alpha_{i-1})$ from Proposition 3.3.8. For the induction step, observe that

$$\begin{aligned} \mathrm{face}_m^0\big((\mathrm{cube}_0^{(m+1)-i+1}(a,b))^{e_{m+1}}\big) = \mathrm{face}_m^1\big((\mathrm{cube}_0^{(m+1)-i+1}(a,b))^{e_{m+1}}\big) \\ = \big(\mathrm{cube}_0^{m-i+1}(a,b)\big)^{e_m}. \end{aligned}$$

So the inclusion follows from Lemma 3.3.6(ii). Finally, from (3.5.4) for $m = n$, the fact that $e_n(\mathbf{k}) = 1 \dots 1$ if and only if $\mathbf{k} = 1 \dots 1$, and Lemma 3.2.1 we get

$$\psi_{1\dots1}(\Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{n-1})) \leq \psi_{1\dots1}(\Delta(\alpha_0, \dots, \alpha_{n-1})).$$

The rest is Theorem 3.1.2. □

3.6 Identities for supernilpotent algebras

To recall, we say an algebra $\mathbf{A}$ is supernilpotent of degree n if the $n+1$-ary commutator satisfies the identity $[1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$. In this section, we are interested in the identities satisfied by supernilpotent algebras. This is motivated by a finite basis theorem of Freese and McKenzie [16], that has been recently generalized by Faulkner [15].

Theorem. *Let $n \geq 2$, and let $\mathcal{V}$ be a locally finite, Mal'cev variety of finite signature consisting solely of algebras that are supernilpotent of degree n. Then $\mathcal{V}$ has a finitely based equational theory.*

In the scope of this result, we give explicit identities that define supernilpotence in a fixed Mal'cev variety. The following theorem provides such (infinite) set of identities defining supernilpotence of a fixed degree.

Theorem 3.6.1. *Let $n \geq 2$, and $\mathcal{V}$ be a variety with a strong n-cube term q_n. Then an algebra $\mathbf{A} \in \mathcal{V}$ is supernilpotent of degree $n-1$ if and only if*

$$q_n^{\mathbf{A}}(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{a}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1})) = t^{\mathbf{A}}(\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1})$$

for every $m_1, \ldots, m_n \in \mathbb{N}$, $\mathbf{a}_i, \mathbf{b}_i \in A^{m_i}$ for $i = 1, \ldots, n$, and term t of arity $m_1 + \cdots + m_n$.

Proof. For simplicity, let 1 denote the full congruence on $\mathbf{A}$. First, from Lemma 3.3.3 we get that a typical element of $\Delta(1, \ldots, 1)$ (with 1 appearing n times) is of the form

$$\big(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), t^{\mathbf{A}}(\mathbf{b}_0, \mathbf{a}_1 \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{b}_0, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1})\big)$$

where $\mathbf{a}_i, \mathbf{b}_i \in A^{m_i}$, $i = 1, \ldots, n$, and t a term of arity $m_1 + \cdots + m_n$. Also, from this and Lemma 3.4.2, we get that

$$\big(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{a}_0, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}),\\ q_n^{\mathbf{A}}(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{a}_0, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}))\big) \in \Delta(1, \ldots, 1).$$

This immediately gives that every $(n-1)$-supernilpotent algebra satisfies the property by Theorem 3.1.2. For the converse, suppose that the identity is satisfied for all $\mathbf{a}_i, \mathbf{b}_i$ and t. Then from Lemma 3.3.3 and the identity, we get that any tuple in $\Delta(1, \ldots, 1)$ is of the form

$$\big(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{a}_0, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}),\\ q_n^{\mathbf{A}}(t^{\mathbf{A}}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \ldots, t^{\mathbf{A}}(\mathbf{a}_0, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}))\big)$$

which means that the last coordinate is uniquely determined by the previous ones, i.e., $\psi_{1\ldots1}\Delta(1, \ldots, 1) = 0_{\mathbf{A}}$. Therefore, $\mathbf{A}$ is supernilpotent of degree $n-1$ again by Theorem 3.1.2. □

Although, it may be also proven directly from the basic properties of higher commutators, this implies that the class of all algebras of a fixed supernilpotence degree from a given Mal'cev variety form a subvariety.

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