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Dissertation

Essays in Financial Econometrics

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Contents

Abstract

Proper understanding of the dependence between assets is a crucial ingredient for a number of portfolio and risk management tasks. While the research in this area has been lively for decades, the recent financial crisis of 2007-2008 reminded us that we might not understand the dependence properly. This crisis served as catalyst for boosting the demand for models capturing the dependence structures. Reminded by this urgent call, literature is responding by moving to nonlinear dependence models resembling the dependence structures observed in the data. In my dissertation, I contribute to this surge with three papers in financial econometrics, focusing on nonlinear dependence in financial time series from different perspectives.

I propose a new empirical model which allows capturing and forecasting the conditional time-varying joint distribution of the oil – stocks pair accurately. Employing a recently proposed conditional diversification benefits measure that considers higher-order moments and nonlinear dependence from tail events, I document decreasing benefits from diversification over the past ten years. The diversification benefits implied by my empirical model are, moreover, strongly varied over time. These findings have important implications for asset allocation, as the benefits of including oil in stock portfolios may not be as large as perceived.

Further, I investigate the dependence structure in financial time series using quantile regression framework. I model conditional quantiles of returns using nonlinear quantile regression models based on copula functions. I explore further non-linearities in the data, and propose to use realized measures in the nonlinear quantile regression framework to explain and forecast conditional quantiles of financial returns. The nonlinear quantile regression models are implied by copula specifications and allow us to capture possible nonlinearities, and asymmetries in conditional quantiles of financial returns. Using high frequency data covering most liquid U.S. stocks in seven sectors, I provide ample evidence of asymmetric conditional dependence and different level of dependence characteristic for each industry.

Finally, I consider conditional Value-at-Risk estimation under copula quantile regression models. I follow a slightly different approach compared to the current literature, where in the focus is systemic risk, and estimate the risk contribution that an asset has on some other individual asset. This approach allows the study of risk spillovers among assets. The dataset which I use for the model is the same as the one in Chapter 3. I find different risk spillover levels for distinctive industries. Furthermore, in some cases the risk spillover levels within the assets of the same industry are very different. These findings have great potential on portfolio re-balancing policies under stress events.

List of Tables

List of Figures

Acronyms

AIC Akaike Information Criterion
BIC Bayesian Information Criterion
CDB conditional diversification benefit
CAViaR conditional autoregressive Value-at-Risk
CoVaR conditional Value-at-Risk
CPA conditional predictive ability
CQ copula-quantile
CvM Cramer von-Mises
GARCH Generalized Autoregressive Conditional Heteroskedasticity
GAS Generalized Autoregressive Score
KS Kolmogorov-Smirnov
LAD least absolute deviation
LQR linear quantile regression
NQR nonlinear quantile regression
OLS ordinary least squares
OOS out-of-sample
QAR quantile autoregression
VaR Value-at-Risk

Chapter 1

Introduction

Modelling the dependence of financial time series has received high attention in recent years. This not only because it is a crucial ingredient for a number of portfolio and risk management tasks, but also because of the increased complexity of the financial systems. Financial systems of today operate in global levels and require far more complex models to capture the profound dependence structures. Thus, quantifying the dependence is a much more difficult task that what it might seem.

Linear correlation is the most common type of dependence used in financial theory. This type of dependence is limited as it is a natural one only in the case of multivariate normal distributions, or more broadly in elliptical distributions family. Use of correlation coefficient outside the world of elliptical distributions produces misleading conclusions. As a testimony, the financial crises of 2007-2008 revealed that we might not understand the dependence properly. Typically, the *linear correlation* was improperly used (mainly by practitioners) to model situations far from elliptical. Thus, appropriate dependence measures such as *rank correlations* and *coefficients of tail dependence* are attractive candidates to model and estimate the nonlinear dependence which is commonly found in financial time series.

Both, *rank correlations* and *coefficients of tail dependence*, are copula based dependence measures. They depend on copula function only and are independent of the margins. This property comes from the fact that copulas allow a *bottom-up* approach to multivariate modelling *i.e.* model the marginal distributions first and the dependence structure in a second step. Another useful property of copula models is their help in understanding the dependence at a deeper level as they express the dependence at a *quantile scale*.

Considering the advantages of copulas in multivariate modelling I turn them into the main tool for this work. In my dissertation thesis I include three papers in financial econometrics focusing on nonlinear dependence in financial time series from different perspectives. Before continuing with the introduction of the papers let me first recall some stylized facts on financial time series. These facts motivated my research in nonlinear dependence and increased further the support in favour of using copula models as the main tool for co-dependence estimation.

Stylized facts about financial time series do not support the assumption of normality, which for many decades prevailed many econometric models. Some stylized facts common to a wide set of financial assets include heavy tails, volatility clustering, gain/loss asymmetry, slow decay of autocorrelations of absolute returns etc. (for more stylized facts see Cont (2001)). Additionally, the correlations among assets are not constant over time as Erb *et al.* (1994); Longin & Solnik (1995), and Engle (2002) have documented. Furthermore, when estimating correlations the asymmetry is a serious issue to deal with. Harlow (1991) shows that portfolios which take into consideration an asymmetric measure of risk outperform the classical mean-variance portfolios. The same principle should be applied when considering correlation. Erb *et al.* (1994) emphasize the importance of future correlation structure when making asset allocation decisions. Moreover, they examine cross-correlation stock returns in G-7 countries and find that the correlations are higher during recessions than during growth periods.

In the last three decades several models for measuring correlation in financial time series have been proposed, simple ones like rolling historical correlations and exponential smoothing or more sophisticated based on multivariate GARCH. The multivariate GARCH models are the most popular multivariate models for correlation estimation. We can mention here the Dynamic Conditional Correlation (DCC)-GARCH model of Engle (2002) or the MGARCH model of Tse & Tsui (2002).

In addition to these approaches financial econometrics attention is turning towards dependence models based on copula theory. Copula models are a general way of describing dependence in probability theory and statistics and in the fundamentals stands the theorem of Sklar (1959). This theorem shows that any joint distribution can be written as a function of marginal distributions. Due to this theorem copulas allow to specify the models for the marginal distributions separately from the dependence structure that links these distributions. In other words they join multivariate distribution functions to their one-dimensional margins. This property transforms the copulas into a bridge between univariate and multivariate distribution functions. We can combine marginal distributions with copula functions and get a valid multivariate distribution. Theoretically, the multivariate distributions which we are able to model or generate are limitless.

As mentioned earlier in the text, tail dependence notion is closely related to copula theory. Tail dependence is very useful in studying the joint tail behaviour *i.e.* the probability that two or more assets will have large (often negative) returns at the same time. Thus, to model a multivariate distribution we should select from the pool of copula functions the ones that exhibit the dependence found or observed in that distribution because for different situations different copulas are appropriate *e.g.* we should not use Gumbel copula to model lower tail dependence, Clayton or t copula would be appropriate in such situation. For an introduction to copulas Joe (1997) and Nelsen (2006) are highly recommended.

Although copulas in financial theory are relatively new, they constitute a fast-growing field of research. To my best knowledge Embrechts *et al.* (1999) and Bouyé *et al.* (2000) were among the first to use copula theory in financial applications. Since then many papers have followed with applications in different fields *e.g.* Li (2000) uses copulas in pricing Collaterized Debt Obligations; Patton (2006) studies the dependence between foreign exchange rates (and introduces the first dynamic copula models); Salmon & Schleicher (2006) uses copula models in currency option pricing; Giacomini *et al.* (2009) estimate portfolio VaR; McNeil *et al.* (2005; 2015) use copula models in the risk management context. For a survey on copula methods please refer to Manner & Reznikova (2012) or the review of Patton (2012).

Having introduced the copula models let me continue with the introduction of the chapters of this dissertation. I want to emphasize that all chapters in this dissertation have resulted from cooperation with my PhD studies' Supervisor, Jozef Baruník.

In Chapter 2 I study oil as a diversification tool for stock markets. We document decreasing diversification benefits over the past ten years. This paper is published recently in Energy Economics (Avdulaj & Barunik (2015)). A previous version of this paper received the award "Best research paper in Energy Economics 2013", organized by Institute of Energy Economics of the Faculty of Finance and Accounting at the University Economics in Prague.

The risk reduction from diversification has been a major subject of literature for many decades. The literature studying the role of oil prices in equity returns is scarce, but the idea of utilizing oil as a diversification tool has attracted a number of publications *e.g.* Gorton & Rouwenhorst (2006); Buyuksahin *et al.* (2010) and Fratzscher *et al.* (2014). They find that oil is a perfect diversification tool for stocks because of either zero, or (even) negative correlation. This feature was reflected in investor's behaviour whose demand for products to diversify risk increased. However, after the financial turmoil of 2008 literature identified an increase in dependence due financialization of commodities Tang & Xiong (2012); Büyükşahin & Robe (2014). Thus, given this increase in dependence we asked the question whether oil is still a perfect diversification tool for stocks.

To answer the question two things are worth considering. First, the time-varying nature of the correlations must be addressed properly. Second, linear correlation may not be a satisfactory measure of dependence, as it does not account for dependence between tail events. To cope with these considerations we propose a flexible empirical model for oil-stock dependence, which couples time-varying copula models with high frequency data. We employ the increasingly popular Generalized Autoregressive Score (GAS) framework (Creal *et al.* 2013) and model the joint distribution of oil-stock returns utilizing time-varying copula functions to capture nonlinear dependence. The GAS framework, is an observation driven one that uses the score of the conditional density function to drive the dynamics of the time-varying parameters. In addition, we employ the recently developed realized GARCH model (Hansen *et al.* 2012), which uses high frequency-based measures of volatility to better capture the volatility process in the margins of the oil-stocks return distribution. Our newly proposed empirical model, the realized GARCH time-varying GAS copula, is thus a very flexible approach. Furthermore, we study conditional diversification benefits, which are implied by our model using the appealing framework of Christoffersen *et al.* (2012). This framework considers higher-order moments and non-linear dependence, which is an important step to take, as diversification benefits implied by simple linear correlations will likely be under- or over-estimated, depending on the degree of dependence coming from the tails.

Our realized GARCH (GAS) copula model identifies strongly varying correlations over the data span, from January 2003 to December 2011. It fits the data very well and the out-of-sample forecasts have good performance too. To translate these results in economic implications we quantify the risk of an equally weighted portfolio composed from oil and stocks, and study the benefits from diversification to see how the strongly varying correlation affects the diversification benefits. We evaluate model performance with respect to quantile forecasts, which represents the Value-at-Risk, and find that the model is accurate. Most important, we translate the results into the conditional diversification benefits measure recently proposed by Christoffersen *et al.* (2012). We find that the diversification benefits greatly vary over time. Initially, the benefits from diversification were relatively high, in the second period between 2006 and 2008, they decreased corresponding to increasing correlation. In the several years after the 2008 crisis, benefits from diversification between oil and stocks were decreasing rapidly, while we can see some rebound in the last few years.

We claim that for the period under research, oil and stocks could be used in a well-diversified portfolio less often than common perception would imply. This because we find substantial evidence of dynamics in tail dependence, which translates into dynamically decreasing diversification benefits from employing oil as a hedging tool for stocks.

In Chapter 3 I investigate the VaR estimation under a nonlinear quantile regression framework. This paper is submitted to *Studies in Nonlinear Dynamics and Econometric* and is under revision. Financial institutions use VaR as the standard measure of market risk. Despite its simplicity, measuring and forecasting it accurately is a challenging task. Recently, quantile regression models have been used successfully to capture the conditional quantiles of returns.

Quantile regression models have important applications in risk management, portfolio optimization, and asset pricing. Koenker & Bassett (1978) introduced the quantiles regression more than three decades ago, but only in the last one the financial literature paid more interest to it. Engle & Manganelli (2004) introduce the conditional autoregressive value at risk, which is also known as the CAViaR model. Instead of modelling the whole distribution, the authors model the quantiles directly. In CAViaR model the quantile of the distribution is regressed on its lagged values and a term which plays the role of the news impact curve for GARCH models. The former ensures a smooth change of quantile, while the latter links the quantile with the observable variables that belong to the information set. Under a semi-parametric quantile regression framework Žikeš & Baruník (2014) utilize nonparametric measures of the various components of *ex post* variation in asset prices to study the properties of conditional quantiles of daily asset returns and realized volatility, and forecast their future values. We exploit the ideas discussed in this paper in a nonlinear semiparametric conditional quantile regression framework to estimate the dependency between returns and realized volatility at quantiles of interest.

In our model the quantile dependence is implied by copula function alone. We use Normal and t copula for quantile curves. Both copulas belong to the elliptical families, and only the second one allows for tail dependence. For comparison we also use the linear quantile regression. In all cases, nonlinear and linear models, we quantile regress the returns at time *t+1* (r_{t+1}) on lagged realized volatility at time t (ϑ_t). We apply the proposed model on 21 most liquid U.S. stocks from seven main market sectors defined in accordance with the Global Industry Classification Standard. We use three stocks with the highest market capitalization in a sector as representative of the analyzed sector. The selected stocks account for approximately half of the total capitalization of the sector. The data spans from August 2004 to December 2011. The period under study is very informative because it covers the recent U.S. recession of Dec. 2007 - June 2009 and three years before and after the crisis.

We run our model on full sample and find nonlinear dependencies especially for lower quantiles. However, most of the times we are interested in utilizing the models to conduct predictions and not just to fit the data. For this reason we split the data in the in-sample part where we estimate our model, and the out-of-sample part where we forecast. We obtain the one-step-ahead and five-step-ahead forecast for the quantiles of returns (or the VaR) for a forecasting window of around two years. We asses absolute and relative performance of the conditional quantile models. In absolute performance nonlinear and linear models perform similarly well. We also find that the relative performance of nonlinear quantile regression models improves significantly for longer forecasting periods, especially for the t copula. We conclude that using the realized volatility under a copula quantile framework is useful, especially in the cases where the quantile dependence is nonlinear.

In Chapter 4, which is also the last one, we focus on conditional Value-at-Risk (CoVaR) estimation under a copula quantile framework. The paper is work in progress and is mostly based in methodology developed in Chapter 3. In this work we analyze the risk contribution of institution i on institution j, or risk spillover of i on j, where both institutions belong to the same industry. For VaR estimation we use the nonlinear quantile copula regression models which we introduced in Chapter 3 of this dissertation. In contrast to the model introduced in 3.2 where we use *own* lagged realized volatility as state variable, here we estimate the VaR using *inter* lagged realized volatility as state variable[1]. We use the same data as in Chapter 3, which consider 21 assets from 7 different industries. Our analysis identifies the risk transmission differences that exists between companies and industries. We compare our results with a benchmark model for VaR based on rescaled realized volatility and also compare with linear quantile regression model.

We propose to use a *two-step* procedure for CoVaR estimation. First step consists in using semiparametric copula-quantile (CQ) regression models to obtain the VaR. We quantile-regress the returns of institution j on realized volatility of institution i. Using this nonlinear framework we obtain the $VaR^{j|i}$. The second step consists in CoVaR estimation, which is done by using the linear quantile regression as in Adrian & Brunnermeier (2011). We apply this methodology on the same dataset as in Chapter 3.

Overall we have four models for CoVaR estimation. Two of them are non-linear, Normal and t copula quantile models, then the linear quantile regression model and finally the benchmark model which estimates VaR using the rescaled

[1]By *own* we mean that the returns and volatility come from the same asset, while by *inter* we mean that the return and volatility come from different assets (within the same industry).

realized volatility. Based on t copula degree of freedom parameter we confirm the stylized fact that the distribution of returns is heavy tailed. For identifying the assets with high risk spillovers and for model comparison we use the average of CoVaR ($\Delta CoVaR$) risk measure. We compare the average $\Delta CoVaR$ estimated from four different models. The ranking of the risk contributors differs significantly among the models used for CoVaR estimation but is more consistent if we compare ranking of industries. We find that highest average risk contributors are among Financial sector. In fact, most of top five or all top contributors belong to assets from this sector. For example, for the benchmark model all the Financials make the first six positions. Normal copula, t copula and the benchmark model identify the contribution of Wells Fargo & Company on Citigroup as the highest risk contributor, with very similar values for $\Delta CoVaR$ estimates. While for Financial sector it makes sense to see their assets to have higher risk spillovers among each-other, it is interesting to see that Information Technology assets are ranked the highest risk contributors based on Linear quantile regression model. Based on this model, if Apple is under distress, Microsoft (at a higher extent) and Intel, are both highly affected from its risk contribution. Another interesting result is that Consumer Staples industry and Health Care have the lowest risk spreading among seven industries considered. This result is supported by all models, even though their estimated absolute $\Delta CoVaR$ value differs.

Finally, in Chapter 5, I make a short summary of the main results and conclude the dissertation.

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Chapter 2

Are benefits from oil – stocks diversification gone? New evidence from a dynamic copula and high frequency data

Abstract

Oil is perceived as a good diversification tool for stock markets. To fully understand this potential, we propose a new empirical methodology that combines generalized autoregressive score copula functions with high frequency data and allows us to capture and forecast the conditional time-varying joint distribution of the oil – stocks pair accurately. Our realized GARCH with time-varying copula yields statistically better forecasts of the dependence and quantiles of the distribution relative to competing models. Employing a recently proposed conditional diversification benefits measure that considers higher-order moments and nonlinear dependence from tail events, we document decreasing benefits from diversification over the past ten years. The diversification benefits implied by our empirical model are, moreover, strongly varied over time. These findings have important implications for asset allocation, as the benefits of including oil in stock portfolios may not be as large as perceived.

Keywords: portfolio diversification, dynamic correlations, high frequency data time-varying copulas, commodities
JEL: C14, C32, C51, F37, G11

2.1 Introduction

The risk reduction benefit from diversification has been a major subject in the finance literature for decades. The number of studies exploring the role of oil

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prices in equity returns remains limited, with no consensus regarding the nature and number of factors that play a role in determining equity returns. Despite the rather scarce literature, the idea of utilizing crude oil as a diversification tool for financial assets attracted number of publications (Gorton & Rouwenhorst 2006; Buyuksahin *et al.* 2010; Fratzscher *et al.* 2014), which conclude that oil is a nearly perfect diversification tool for stocks due to their null, or even negative, correlation. This feature is also reflected in investors' demand for products to diversify risk. For example Morgan Stanley offers a product composed of oil and S&P 500 prices with equal weights. After the recent financial turmoil in 2008, the literature began to document rising dependence due to financialization of commodities (Tang & Xiong 2012; Büyükşahin & Robe 2014). However, a majority of empirical studies use linear (time-varying) correlations, ignoring the possible non-linear dependence of tail events. Hence, it is natural to ask the question posed in the title of this paper: "Are the benefits from oil-stocks diversification gone?" To answer this question, a proper measure of the benefits, which will account for the following two key issues, must be employed.

First, the time-varying nature of the correlations must be addressed properly. The recent turmoil in financial markets, which began in September 2008, further strengthened the focus on models that are able to capture dynamic dependencies in data. Miller & Ratti (2009) analyze the long-run relationship between oil and international stock markets utilizing cointegration techniques, and they find that stock markets responded negatively to increases in oil prices in the long run before 1999, but after this point, the relationship collapsed. This finding is in line with a number of studies reporting the negative influence of rising oil prices on stock markets (Sadorsky 1999; Ciner 2001; Nandha & Faff 2008; O'Neill *et al.* 2008; Park & Ratti 2008; Chen 2010). Generally, these results are consistent with economic theory, as rising oil prices are expected to have an adverse effect on real output and, hence, an adverse effect on corporate profits if oil is employed as a key input. This phenomenon suggests that oil could be an important factor for equity returns. In a large study of the relationship among oil prices, exchange rates and emerging stock markets, Basher *et al.* (2012) confirm previous research by finding that a positive shock in oil prices tends to depress stock markets and U.S. dollar exchange rates in the short run. Wu *et al.* (2012) further study the depreciation in the U.S. dollar causing an increase in crude oil prices. Geman & Kharoubi (2008) study the maturity effect in the choice of oil futures with respect to diversification benefits and find that futures with more distant maturities are more negatively correlated with the S&P 500. More recently, new evidence from data during and after the 2008 financial crisis has emerged. Mollick & Assefa (2013) find that, while stock returns were negatively affected by oil prices prior to the crisis, this relationship became positive after 2009. The authors interpret this reversal as stocks positively responding to expectations of recovery. Aloui *et al.* (2013) study the dynamics of oil and Central and Eastern European (CEE) stock markets, and they find a positive time-varying relationship utilizing recent data. Wen *et al.* (2012) finds contagion between energy and stock markets that arose during the 2008 financial crisis. Finally, Broadstock *et al.* (2012); Sadorsky (2012) document price and volatility spillovers in oil and stocks.

Second, linear correlation may not be a satisfactory measure of dependence, as it does not account for dependence between tail events. Correlation asymmetries and changes in correlation due to business cycle conditions are crucial, as these dependences will impact to a large degree the benefits from diversification.

In this paper, we provide a comprehensive empirical study of the dynamics in dependence and tail dependence and translate the results directly to the diversification benefits. We offer two contributions. First, we propose a flexible empirical model for oil-stock dependence, which couples time-varying copula models with high frequency data. Employing the increasingly popular Generalized Autoregressive Score (GAS) framework (Creal *et al.* 2013), we model the joint distribution of oil-stock returns utilizing time-varying copula functions and capture nonlinear dependence. This is an important step to take, as the dynamics of correlations may depart from the one imposed by the assumption of multivariate normality used in many different approaches. In addition, we employ the recently developed realized GARCH model (Hansen *et al.* 2012), which uses high frequency-based measures of volatility to better capture the volatility process in the margins of the oil-stocks return distribution. Our newly proposed empirical model, the realized GARCH time-varying GAS copula, is thus a very flexible approach. Moreover, we employ a semiparametric alternative to modeling strategy, which combines a nonparametric estimate of a margins distribution and parametric copula function. While this approach is empirically attractive, it is not often employed in the literature.

Second, we study conditional diversification benefits, which are implied by our model using the appealing framework of Christoffersen *et al.* (2012). This framework considers higher-order moments and non-linear dependence, which is an important step to take, as diversification benefits implied by simple linear correlations will likely be under- or over-estimated, depending on the degree of dependence coming from the tails. In addition, we evaluate our empirical model utilizing Value-at-Risk (VaR) forecasts.

We demonstrate that our newly proposed empirical model is able to capture and forecast the time-varying dynamics in a joint distribution accurately. We revisit the oil-stock relationship utilizing the large span of data covering the periods for which the literature finds a negative relationship, as well as the recent stock market turmoil. The main finding is that we document decreasing benefits from the usage of oil as a diversification tool for stocks until the year 2011, which can be attributed to the changing expectations of investors after the recent market turmoil. After the year 2011, diversification benefits started to increase slowly and showed promise to investors who wish to use oil as a hedge for their stock portfolio. Moreover, in an empirical section, we test the economic significance of the model and demonstrate that our method yields more accurate quantile forecasts, which are central to risk management due to the popular VaR measure.

The work is organized as follows. The second section introduces our empirical model based on a dynamic copula realized GARCH modeling framework in detail. The third section introduces the data, and the fourth section offers empirical results documenting the time-varying nature of dependence between oil and stocks and good out-of-sample performance of models. The fifth section then elaborates on the economic implications of our modeling strategy, employing quantile forecasts, quantifying the risk of an equally weighted portfolio composed of oil and stocks and, finally, documents the time variation in the benefits of utilizing oil as a diversification tool for stocks. The last section concludes.

2.2 Dynamic copula realized GARCH modeling framework

Our modeling strategy utilizes high frequency data to capture the dependence in the margins and recently proposed dynamic copulas to model the dynamic dependence. The final model is thus able to describe the conditional time-varying joint distribution of oil and stocks, which will be very useful in the economic application.

The methodology employed in this work is based on Sklar's (1959) theorem extended to conditional distributions by Patton (2006b). Sklar's extended theorem allows us to decompose a conditional joint distribution into marginal distributions and a copula. Consider the bivariate stochastic process $\{\mathbf{X}_t\}_{t=1}^{T}$ with $\mathbf{X_t} = (X_{1t}, X_{2t})'$, which has a conditional joint distribution $\mathbf{F_t}$ and conditional marginal distributions F_{1t} and F_{2t}. Then

$$\mathbf{X_t}|\mathcal{F}_{t-1} \sim \mathbf{F_t} = \mathbf{C_t}\left(F_{1t}, F_{2t}\right), \tag{2.1}$$

where $\mathbf{C_t}$ is the time-varying conditional copula of $\mathbf{X_t}$ containing all information about the dependence between X_{1t} and X_{2t}, and $\mathcal{F}_{t-1}$ the information set. Due to Sklar's theorem, we are thus able to construct a dynamic joint distribution

$\mathbf{F_t}$ by linking together any two marginal distributions F_{1t} and F_{2t} with any copula function providing very flexible approach for modeling joint dynamic distributions.[1]

2.2.1 Time-varying conditional marginal distribution with realized measures

The first step in building an empirical model based on copulas is to find a proper model for the marginal distributions. As the most pronounced dependence that can be found in the returns time series is the one in variance, the vast majority of the literature utilizes the conventional generalized autoregressive Generalized Autoregressive Conditional Heteroskedasticity (GARCH) approach of Bollerslev (1986) in this step.

With the increasing availability of high frequency data, the literature moved to a different concept of volatility modeling called realized volatility. This very simple and intuitive approach for computing daily volatility employing the sum of squared high-frequency returns was formalized by Andersen *et al.* (2003) and Barndorff-Nielsen & Shephard (2004). While realized volatility can be measured simply from the high frequency data, one must specify a correct model to be able to use this parameter for forecasting. In past years, researchers found ways to include a realized volatility measure to assist GARCH-type parametric models in capturing volatility.

As noted previously, the key object of interest in financial econometrics, the conditional variance of returns, $h_{it} = var(X_{it}|\mathcal{F}_{t-1})$, is usually modeled by GARCH. While in a standard GARCH(1,1) model the conditional variance, h_{it}, is dependent on its past values h_{it-1} and the values of X^2_{it-1}, Hansen *et al.* (2012) propose to utilize a realized volatility measure and make h_{it} dependent on the realized variance. The authors propose a so-called measurement equation that ties the realized measure to latent volatility. The general framework of realized GARCH(p,q) models is well connected to the existing literature in Hansen *et al.* (2012). Here, we restrict ourselves to the simple log-linear specification of the realized GARCH(1,1). While it is important to model the conditional time-varying mean $E(X_{it}|\mathcal{F}_{t-1})$, we also include the standard AR model in the final modeling strategy. As we will discuss later, the autoregressive term of order no larger than two is appropriate for the oil and stocks data in the study; thus, we restrict ourselves to specifying the AR(2) with log-linear realized GARCH(1,1) model as in Hansen *et al.* (2012)

$$X_{it} = \mu_i + \alpha_1 X_{it-1} + \alpha_2 X_{it-2} + \sqrt{h_{it}} z_{it}, \qquad \text{for } i = 1, 2 \quad (2.2)$$

$$\log h_{it} = \omega_i + \beta_i \log h_{it-1} + \gamma_i \log RV_{it-1}, \quad (2.3)$$

[1]Please note that the information set for the margins and the copula conditional density is the same.

$$\log RV_{it} = \psi_i + \phi_i \log h_{it} + \tau_i(z_{it}) + u_{it}, \tag{2.4}$$

where μ_i is the constant mean, h_{it} conditional variance, which is latent, RV_{it} realized volatility measure, $u_{it} \sim N(0, \sigma_{iu}^2)$, and $\tau_i(z_{it}) = \tau_{i1} z_{it} + \tau_{i2}(z_{it}^2 - 1)$ leverage function. For the RV_{it}, we employ the high frequency data and compute it as a sum of squared intraday returns (Andersen *et al.* 2003; Barndorff-Nielsen & Shephard 2004). We will provide more detail on how we compute the realized volatility measure in the empirical section. Hansen *et al.* (2012) suggests estimating the parameters utilizing a quasi-maximum likelihood estimator (QMLE), which is very similar to the standard GARCH. While we have realized measures in the estimation yielding additional measurement error u_{it}, we need to factorize the joint conditional density[2] $f(X_{it}, RV_{it}|\mathcal{F}_{t-1}) = f(X_{it}|\mathcal{F}_{t-1}) f(RV_{it}|X_{it}, \mathcal{F}_{t-1})$ which results in a sum after logarithmic transform and thus, is readily available for finding parameters. In our model, we allow the innovations z_{it} to follow skewed-t distribution of Hansen (1994), having two shape parameters, a skewness parameter $\lambda \in (-1, 1)$ controlling the degree of asymmetry, and a degree of freedom parameter $\nu \in (2, \infty]$ controlling the thickness of tails. When $\lambda = 0$, the distribution becomes the standard Student's t distribution, when $\nu \to \infty$, it becomes skewed Normal distribution, while for $\nu \to \infty$ and $\lambda = 0$, it becomes $N(0, 1)$. Thus, the choice of the skewed-t distribution gives us flexibility to capture the potential measurement errors from realized volatility and hence, possible departures from the normality of residuals.

Thus, after the time-varying dependence in the mean and volatility is modeled, we are left with residuals

$$\hat{z}_{it} = \frac{X_{it} - \hat{\mu}_i - \hat{\alpha}_1 X_{it-1} - \hat{\alpha}_2 X_{it-2}}{\sqrt{\hat{h}_{it}}} \tag{2.5}$$

$$\hat{z}_{it}|\mathcal{F}_{t-1} \sim F_i(0, 1), \qquad \text{for } i = 1, 2. \tag{2.6}$$

which have a constant conditional distribution with zero mean and variance one. Then, the conditional copula of $\mathbf{X_t}|\mathcal{F}_{t-1}$ is equal to the conditional distribution of $\mathbf{U_t}|\mathcal{F}_{t-1}$:

$$\mathbf{U_t}|\mathcal{F}_{t-1} \sim \mathbf{C_t}(\gamma_\mathbf{0}), \tag{2.7}$$

with γ being copula parameters, and $\mathbf{U_t} = [U_{1t}, U_{2t}]'$ conditional probability integral transform

$$U_{it} = F_i\left(\hat{z}_{it}; \phi_{i,0}\right), \qquad \text{for } i = 1, 2. \tag{2.8}$$

[2]Please note that information set $\mathcal{F}_{t-1}$ contains the lagged values of RV_{it} as well.

2.2.2 Dynamic copulas: A "GAS" dynamics in parameters

After finding a model for the marginal distribution, we proceed to the copula functions. An important feature that is required for our work is the specification that parameters are allowed vary over time. Recently, Hafner & Manner (2012); Manner & Segers (2011) proposed a stochastic copula model that allows the parameters to evolve as a latent time series. Another possibility is offered by ARCH-type models for volatility (Engle 2002) and related models for copulas (Patton 2006b; Creal *et al.* 2013), which allow the parameters to be some function of lagged observables. An advantage of the second approach is that it avoids the need to "integrate out" the innovation terms driving the latent time series processes.

For our empirical model, we adopt the Generalized Autoregressive Score (GAS) model of Creal *et al.* (2013), which specifies the time-varying copula parameter (δ_t) as a function of the lagged copula parameter and a forcing variable that is related to the standardized score of the copula log-likelihood[3]. Consider a copula with time-varying parameters:

$$\mathbf{U_t}|\mathcal{F}_{t-1} \sim \mathbf{C_t}(\delta_\mathbf{t}(\gamma)). \tag{2.9}$$

Often, a copula parameter is required to fall within a specific range, e.g., the correlation for Normal or t copula is required to fall in between values of -1 and 1. To ensure this, Creal *et al.* (2013) suggest transforming the copula parameter by an increasing invertible function[4] (e.g., logarithmic, logistic, etc.) to the parameter:

$$\kappa_t = h(\delta_t) \Longleftrightarrow \delta_t = h^{-1}(\kappa_t) \tag{2.10}$$

For a copula with transformed time-varying parameter κ_t, a GAS(1,1) model is specified as

$$\kappa_{t+1} = w + \beta\kappa_t + \alpha I_t^{-1/2}\mathbf{s}_t \tag{2.11}$$

$$\mathbf{s}_t \equiv \frac{\partial \log \mathbf{c}(\mathbf{u_t}; \delta_t)}{\partial \delta_t} \tag{2.12}$$

$$I_t \equiv E_{t-1}[\mathbf{s}_t\mathbf{s}_t'] = I(\delta_t). \tag{2.13}$$

While this specification for the time-varying parameters is arbitrary, Creal *et al.* (2013) motivates it in a way that the model nests a variety of popular approaches

[3] Harvey (2013); Harvey & Sucarrat (2014) propose a similar method for modeling time-varying parameters, which they call a dynamic conditional score model.

[4] For example, for the Normal and t copula, the transformation is $(1 - e^{-\kappa_t})/(1 + e^{-\kappa_t})$. Concrete functions for other copulas employed in our work are given in Appendix 2.A, which introduces copula functions.

from conditional variance models to trade durations and counts models. Additionally, the recursion is similar to numerical optimization algorithms such as the Gauss-Newton algorithm. In comparison to the approach of Patton (2006b), the GAS specification implies more sensitivity to correlation shocks. Hence, reactions from returns of an opposite sign in the situation of a positive correlation estimate will be captured. For more details, and empirical comparison of the two approached, see Section 3.1. in Creal *et al.* (2013).

Until now, we have focused attention on the specification of the dynamics of the models. What remains to be specified is the shape of the copula. In our modeling strategy, we will compare several of the most often utilized shapes of copula functions, while the rest of the model will be fixed. For the dynamic parameter models, we will employ the rotated Gumbel, Normal and Student's t functional forms described briefly in the 2.A. In our empirical application, we also employ constant copula functions as a benchmark. These are described in the 2.A as well.

2.2.3 Estimation strategy

The final dynamic copula realized GARCH model defines a dynamic parametric model for the joint distribution. The joint likelihood is

$$\mathcal{L}(\theta) \equiv \sum_{t=1}^{T} \log \mathbf{f}_t(\mathbf{X_t}; \theta) = \sum_{t=1}^{T} \log f_{1t}(X_{1t}; \theta_1) + \sum_{t=1}^{T} \log f_{2t}(X_{2t}; \theta_2) \tag{2.14}$$

$$+ \sum_{t=1}^{T} \log \mathbf{c}_t(F_{1t}(X_{1t}; \theta_1), F_{2t}(X_{2t}; \theta_2); \theta_c), \tag{2.15}$$

where $\theta = (\phi', \gamma')'$ is vector of all parameters to be estimated, including parameters of the marginal distributions ϕ and parameters of the copula, γ. The parameters are estimated utilizing a two-step estimation procedure, generally known as multi-stage maximum likelihood (MSML) estimation, first estimating the marginal distributions and then estimating the copula model conditioning on the estimated marginal distribution parameters. While this greatly simplifies the estimation, inference on the resulting copula parameter estimates is more difficult than usual as the estimation error from the marginal distribution must be considered. As a result, MSMLE is asymptotically less efficient than one-stage MLE; however, as discussed by Patton (2006a), this loss is small in many cases. Moreover, the bootstrap methodology can be utilized, as discussed in following sections.

Semiparametric models

One of the appealing alternatives to a fully parametric model is to estimate univariate distribution non-parametrically, for example, by utilizing the empirical distribution function. Combination of a nonparametric model for marginal distribution and parametric model for the copula results in a semiparametric copula model, which we use for comparison to its fully parametric counterpart. In our modeling strategy, we concentrate on a full parametric model combining fully parametric marginal distribution F_i with a copula function, while the theory is developed for the inference. Still, a nonparametric distribution F_i has great empirical appeal; thus, we utilize it for comparison and rely on bootstrap-based inference for parameter estimates, as discussed later in the text. Forecasts based on a semiparametric estimation where nonparametric marginal distribution is combined with parametric copula function are not common in economic literature, thus, it is interesting to compare it in our modeling strategy. For the margins of the semi-parametric models, we employ the non-parametric empirical distribution F_i introduced by Genest *et al.* (1995)[5], which consists of modeling the marginal distributions by the (rescaled) empirical distribution.

$$\hat{F}_i(z) = \frac{1}{T+1}\sum_{t=1}^{T} \mathbf{1}\{\hat{z}_{it} \leq z\} \tag{2.16}$$

In this case, the parameter estimation is conducted by maximizing likelihood

$$\mathcal{L}(\gamma) \equiv \sum_{t=1}^{T} \log \mathbf{c}_t(\hat{U}_{1t}, \hat{U}_{2t}; \gamma), \tag{2.17}$$

Again, the inference of parameters is more difficult than usual. We discuss the inference in the following section.

2.2.4 Inference for parameter estimates

For the statistical inference of parameters, we utilize the bootstrapping methodology as suggested by Patton (2006b). More specifically, for constant parametric copulas, we employ the stationary bootstrap of Politis & Romano (1994), while for the constant semi-parametric *i.i.d.* bootstrapping. The use of these bootstrapping methods is justified by the work of Gonçalves & White (2004); Chen & Fan (2006a) and Rémillard (2010). The algorithm utilized to obtain the statistical inference for parametric model (both constant and time-varying) follows these steps:

1: Use a block bootstrap to generate a bootstrap sample of the data of length T.
2: Estimate the model using the same multi-stage approach as applied for the real data.

[5] The asymptotic properties of this estimator can be found in Chen & Fan (2006b).

3: Repeat steps 1-2 S times[6].
4: Use the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of estimated parameters to obtain a $1-\alpha$ confidence interval for these parameters.

For the constant semi-parametric copulas the algorithm follows:

1: Use an *i.i.d.* bootstrap to generate a bootstrap sample of the estimated standardized residuals of length T.
2: Transform each time series of bootstrap data using its empirical distribution function.
3: Estimate the copula model on the transformed data.
4: Repeat steps 1-3 S times.
5: Use the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of estimated parameters to obtain a $1-\alpha$ confidence interval for these parameters.

When we consider semi-parametric time-varying copulas, we cannot utilize the *iid* bootstrap because the true standardized residuals are not jointly *iid.* Inference methods for these models are not yet available. However, Patton (2013) suggests employing the block bootstrap technique (*e.g.* stationary bootstrap of Politis & Romano (1994)), stressing the need for formal justification.

2.2.5 Goodness-of-fit and copula selection

A crucial issue in empirical copula applications is related to the goodness-of-fit. While copula models allow great flexibility, it is crucial to find the model that is well specified for the data as more harm then help can be done when one relies on a misspecified model. Genest *et al.* (2009) make a review on available goodness-of-fit tests for copulas. Two tests that are widely used for goodness-of-fit tests of copula models and that we utilize are the standard Kolmogorov-Smirnov (KS) and Cramer von-Mises (CvM) tests. These approaches work only for constant copula models. When dealing with time-varying copulas we should modify the testing procedure. Thus, we utilize the fitted copula model to obtain the Rosenblatt transform of the data, which is a multivariate version of the probability integral transformation. In the multivariate version, these tests then measure the distance between the empirical copula estimated on Rosenblatt's transformed data denoted by $\hat{\mathbf{C}}_T$ and the independence copula denoted by $\mathbf{C}_{\perp}$.

Rosenblatt's probability integral transform of a copula $\mathbf{C}$ is the mapping $\mathcal{R}$: $(0,1)^n \rightarrow (0,1)^n$. To every $\mathbf{U}_t = (U_{1t}, \ldots, U_{nt}) \in (0,1)^n$, this mapping assigns

[6] We use S=100 due to the high computational power needed for time-varying t copula and because larger S in fact does not substantially improve the results (these "testing" results with S=1000 are available upon request from authors).

another vector $\mathcal{R}(\mathbf{U}_t) = (V_{1t}, \ldots, V_{nt})$ with $V_{1t} = U_{1t}$ and for each $i \in \{2, \ldots, n\}$,

$$V_{it} = \frac{\partial^{i-1} C(U_{1t}, \ldots, U_{it}, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C(U_{1t}, \ldots, U_{i-1,t}, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}} \tag{2.18}$$

For $i = 2$, Equation (2.18) reduces to $V_{1t} = U_{1t}$, $V_{2t} = \partial C(U_{1t}, U_{2t})/\partial u_1$ because the denominator $\partial C(U_{1t}, 1)/\partial u_1 = 1$. Rosenblatt's transformation has the very convenient property that $\mathbf{U}$ is distributed as copula $\mathbf{C}$ if and only if $\mathcal{R}(\mathbf{U})$ is the n-dimensional independent copula

$$C_{\perp}(\mathbf{V}_t; \hat{\theta}_t) = \prod_{i=1}^{n} V_{it} \tag{2.19}$$

Thus, the Rosenblatt transformation of the original data gives us a vector of *i.i.d.* and mutually independent $Unif(0,1)$ variables, and we can utilize this vector to compare the empirical copula on the transformed data with the independence copula. The KS and CvM tests follow in Equations 2.21 and 2.22, respectively.

$$\hat{\mathbf{C}}_T(\mathbf{v}) \equiv \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{n} \mathbf{1}\{V_{it} \leq v_{it}\} \tag{2.20}$$

$$KS_R = \max_t \left| \mathbf{C}_{\perp}(\mathbf{V}_t; \hat{\theta}_t) - \hat{\mathbf{C}}_T(\mathbf{V}_t) \right| \tag{2.21}$$

$$CvM_R = \sum_{t=1}^{T} \left\{ \mathbf{C}_{\perp}(\mathbf{V}_t; \hat{\theta}_t) - \hat{\mathbf{C}}_T(\mathbf{V}_t) \right\}^2 \tag{2.22}$$

Critical values of the goodness-of-fit tests are obtained with simulations, as in the Genest *et al.* (2009) algorithm, as asymptotic distributions are not applicable in the presence of parameter estimation error. In the case of the full-parametric model, the simulations involve generation and estimation of the data from both the model for the margins and for the copula. For the semi-parametric model, the data are generated and estimated only for the copula model. However, as Patton (2013) notes, the approach of combining the non-parametric margins with dynamic copulas does not yet have theoretical support.

Another important issue when working with copulas is the selection of the best copula from the pool. Several methods and tests have been proposed for the selection procedure. The methods proposed by Durrleman *et al.* (2000) are based on the distance from the empirical copula. The authors show how to choose among Archimedian copulas and among a finite subset of copulas. Chen & Fan (2005) propose the use of pseudo-likelihood ratio test for selecting semiparametric multivariate copula models.[7] A test on conditional predictive ability (CPA) is proposed by Giacomini & White (2006). This is a robust test that allows one to accommodate both unconditional and conditional objectives. Recently, Diks *et al.* (2010) have proposed a test for comparing the predictive ability of competing copulas. The test is based on the Kullback-Leibler information criterion (KLIC), and its statistics is a special case of the unconditional version of Giacomini & White (2006).

As our main aim is to employ the model for forecasting, out-of-sample performance of models will be tested by CPA, which consider the forecast performance of two competing models conditional on their estimated parameters to be equal under the null hypothesis

$$H_0 : E[\hat{\mathbf{L}}] = 0 \tag{2.23}$$

$$H_{A1} : E[\hat{\mathbf{L}}] > 0 \text{ and } H_{A2} : E[\hat{\mathbf{L}}] < 0, \tag{2.24}$$

where $\hat{\mathbf{L}} = \log \mathbf{c}_1(\hat{\mathbf{U}}, \hat{\gamma}_{1t}) - \log \mathbf{c}_2(\hat{\mathbf{U}}, \hat{\gamma}_{2t})$. This test can be used for both nested and non-nested models, and we can utilize it for comparison of parametric and semiparametric models as well. The asymptotic distribution of the test statistic is $N(0, 1)$, and we compute the asymptotic variance utilizing HAC estimates to correct for possible serial correlation and heteroskedasticity in the differences in log-likelihoods.

2.3 The Data description

The data set consists of tick prices of crude oil and S&P 500 futures traded on the platforms of Chicago Mercantile Exchange (CME)[8]. More specifically, oil (Light Crude) is traded on the New York Mercantile Exchange (NYMEX) platform, and the S&P 500 is traded on the CME in Chicago. We use the most active rolling contracts from the pit (floor traded) session. Prices of all futures are expressed in U.S. dollars.

The sample period spans from January 3, 2003 to December 11, 2012, covering the recent U.S. recession of Dec. 2007 - June 2009. We acknowledge the fact that the CME introduced the Globex electronic trading platform on Monday, December

[7] Although some authors use Akaike Information Criterion (AIC) (or Bayesian Information Criterion (BIC)) for choosing among two copula models, selection based on these indicators may hold only for the particular sample in consideration (due to their randomness) and not in general. Thus, proper statistical testing procedures are required [see Chen & Fan (2005)].

[8] The data were obtained from the Tick Data, Inc.

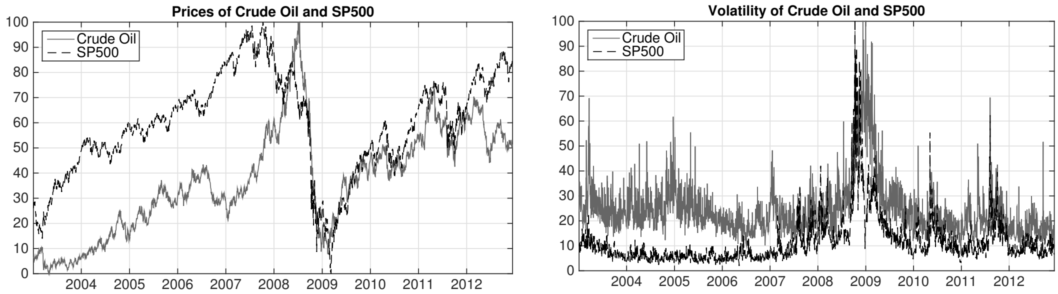

Figure 2.1: Normalized prices and annualized realized volatilities of oil and stocks (S&P 500), over the sample period extending from January 3, 2003 to December 11, 2012.

18, 2006, and begun to offer nearly continuous trading. However, we restrict the analysis on the intraday 5-minutes returns within the business hours of the New York Stock Exchange (NYSE) as most of their liquidity of S&P 500 futures comes from the period when U.S. markets are open. Time synchronization of our data is achieved in a way that oil prices are paired with the S&P 500 by matching the identical Greenwich Mean Time (GMT) stamps. We eliminate transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2 because of the low activity on these days, which could lead to estimation bias. Hence, in our analysis we work with data from 2,436 trading days.

For our empirical model, we need two time series, namely daily returns and realized variance to be able to estimate the realized GARCH model in margins. We consider open-close returns; thus, daily returns are simply obtained as a sum of logarithmic intraday returns. Realized variance is computed as a sum of squared 5-minute intraday returns

$$RV_t = \sum_{i=1}^{M} r_i^2, \tag{2.25}$$

Figure 2.1 plots the development of prices of the oil and stocks together with its realized volatility. Please note that plot of prices is normalized to make them comparable and for the plot of realized volatility, we utilize daily volatility annualized according to the following convention: $100 \times \sqrt{250 \times RV_t}$. Strong time-varying nature of the volatility can be noticed immediately for both oil and stocks. In Table 2.5, we present descriptive statistics of the returns of the data that constitute our sample. Distributions of the daily returns are showing excess kurtosis. It is interesting that the volatility of oil is on average more than twice larger in comparison to the volatility of stocks. In addition, the dynamics of volatilities differ, primarily in the first part of the period. This, in fact, motivates a need for the flexible model, which will capture the different dynamics in the marginal distributions for oil and stocks.

2.4 Empirical Results

Before modeling the dependence structures between oil and stocks, we must model their conditional marginal distributions first. Utilizing the Bayesian Information Criterion (BIC) and considering general ARMA models up to five AR as well as MA lags, we find an AR(2) model to best capture time-varying dependence in the mean of S&P 500 stock market returns, while no significant dependence in the mean was found for oil.

Table 2.1 summarizes the in-sample realized GARCH(1,1) fit for both oil and stocks represented by the S&P 500 in our study. In addition, the benchmark volatility model from the GARCH family, namely, GJR model (Glosten *et al.* 1993), is used for comparison. All the estimated parameters are significantly different from zero and are similar to those obtained by Hansen *et al.* (2012). We can see that realized volatility plays its role in the model as it helps to model volatility significantly. By observing partial log-likelihood $\mathcal{LL}_r$ as well as information criteria, we can see that the realized GARCH brings significant improvement over the GJR GARCH model in both oil and stocks. This is a crucial result for copulas, as we need to specify the best possible model in the margins to make sure there is no univariate dependence left. If a misspecified model is utilized for the marginal distributions, then the probability integral transforms will not be $Unif(0,1)$ distributed, and this will result in copula misspecification.

For the estimated standardized residuals from the realized GARCH(1,1), we consider both parametric and nonparametric distributions, as noted previously. The figure in 2.4 plots the histogram of the standardized residuals together with quantile plots against skewed t distribution. We can see a reasonable fit of skewed t distribution with the data, although a very small departure from tails can be observed for the S&P 500 data. This also motivates us to choose to estimate a full battery of copula models including those combining a nonparametric empirical distribution for margins and a parametric copula function; although from the density fits, we can see that the gains will probably not be large.

To study the goodness of fit for the skewed t distribution, we compute the Kolmogorov-Smirnov (KS) and Cramer von-Mises (CvM) test statistics with p-values from 1,000 simulations, and we find KS (CvM) p-values of 0.452 (0.577) and

	Crude Oil		S&P 500			Crude Oil		S&P 500	
	AR(2)					AR(2)			
c	0.0001	(0.29)	0.0000	(0.21)	c	0.0001	(0.29)	0.0000	(0.21)
α_1	-	-	-0.1095	(-5.42)	α_1	-	-	-0.1095	(-5.42)
α_2	-	-	-0.0744	(-3.68)	α_2	-	-	-0.0744	(-3.68)
	realized GARCH(1,1)					GJR GARCH(1,1)			
ω	0.0626	(6.14)	0.2000	(14.07)	κ	0.0143	(2.69)	0.0028	(2.79)
β	0.7622	(46.41)	0.7176	(45.11)	ϕ	0.0270	(2.57)	0.0187	(1.71)
γ	0.2081	(12.59)	0.2413	(19.04)	ι	0.0390	(2.61)	0.0883	(5.72)
ξ	-0.3173	(-9.26)	-0.9018	(-21.84)	ψ	0.9363	(72.58)	0.9321	(85.86)
ϕ	1.0758	(23.36)	1.1130	(40.87)		-	-	-	-
τ_1	-0.0627	(-7.18)	-0.0772	(-8.15)		-	-	-	-
τ_2	0.1053	(16.62)	0.0999	(16.56)		-	-	-	-
ν	13.4633	(4.07)	12.2552	(5.49)	ν	12.7026	(4.29)	7.9716	(6.48)
λ	-0.0885	(-3.21)	-0.1544	(-6.37)		-	-	-	-
$\mathcal{LL}_{r,x}$	-4558.16		-4167.49			-		-	
$\mathcal{LL}_r$	-3189.22		-2473.56		$\mathcal{LL}$	-3207.74		-2501.89	
AIC_r	6396.43		4965.11		AIC	6425.49		5013.78	
BIC_r	6448.61		5017.30		BIC	6454.48		5042.77	

Table 2.1: Parameter estimates from AR(2) *log-linear* realized GARCH(1,1) and benchmark GJR GARCH(1,1), the former with *skew-t* innovations and the latter with standard Student's t. t-statistics are reported in parentheses.

0.254 (0.356) for the oil and S&P 500 standardized residuals, respectively. Thus, we are not able to reject the null hypothesis that these distributions come from the skewed t, which provides support for these models of the marginal distribution. The estimated parameters ν (λ) for the oil and stocks are 13.462 (-0.088) and 12.255 (-0.154), respectively. This allows us to continue with modeling time-varying dependence.

2.4.1 Time-varying dependence between oil and stocks

By studying simple correlation measures of original returns, we find the linear correlation and rank correlation for oil and stocks to be 0.29 and 0.224, respectively, both significantly different from zero. Before specifying a functional form for a time-varying copula function, we test for the presence of time-varying dependence utilizing the simple approach based on the ARCH LM test. The test statistics are computed from the OLS estimate of the covariance matrix, and critical values are obtained employing *i.i.d.* bootstrap (for detailed information, consult Patton

(2013)). Computing the test for the time-varying dependence between oil and stocks up to $p = 10$ lags, we find the joint significance of all coefficients. Thus, we can conclude that there is evidence against constant conditional correlation for oil and stocks.

		Parametric			Semiparametric		
Constant copula							
		Est.	Param	$\log \mathcal{L}$	Est.	Param	$\log \mathcal{L}$
Normal	ρ	0.2060	(0.0290)	**52.56**	0.2053	(0.0231)	**52.43**
Clayton	κ	0.2392	(0.0353)	**56.90**	0.2738	(0.0322)	**58.69**
RGumb	κ	1.1403	(0.0213)	**66.40**	1.1588	(0.0176)	**69.13**
Student's t	ρ	0.2051	(0.0261)		0.2183	(0.0214)	
	ν^{-1}	0.1376	(0.0244)	**79.74**	0.1660	(0.0252)	**81.78**
Sym. Joe-Clayton	τ^L	0.0941	(0.0268)		0.1209	(0.0254)	
	τ^U	0.0208	(0.0226)	**66.14**	0.0242	(0.0193)	**67.38**
"GAS" time-varying copula							
		Est.	Param	$\log \mathcal{L}$	Est.	Param	$\log \mathcal{L}$
$RGumb_{GAS}$	$\hat{\omega}$	-0.0074	(0.2071)		-0.0097	(0.4219)	
	$\hat{\alpha}$	0.1038	(0.3131)		0.1184	(0.3575)	
	$\hat{\beta}$	0.9972	(0.0122)	**135.06**	0.9960	(0.0447)	**139.15**
N_{GAS}	$\hat{\omega}$	0.0017	(0.0037)		0.0019	(0.0041)	
	$\hat{\alpha}$	0.0474	(0.0109)		0.0553	(0.0124)	
	$\hat{\beta}$	0.9952	(0.0070)	**152.47**	0.9947	(0.0075)	**153.59**
t_{GAS}	$\hat{\omega}$	0.0016	(0.0040)		0.0018	(0.0073)	
	$\hat{\alpha}$	0.0493	(0.0128)		0.0579	(0.0193)	
	$\hat{\beta}$	0.9957	(0.0076)		0.9952	(0.0205)	
	$\hat{\nu}^{-1}$	0.0775	(0.0259)	**162.82**	0.0940	(0.0315)	**165.39**

Table 2.2: Constant and time-varying copula model parameter estimates with AR(2) realized GARCH(1,1) model for both fully parametric and semiparametric cases. Bootstrapped standard errors are reported in parentheses.

Motivated by the possible time-varying dependence in oil and stocks, we can specify the copula functions. We estimate three time-varying copula functions, namely, Normal, rotated Gumbel and Student's t using the GAS framework described in the methodology part. As a benchmark, we also estimate the constant copulas to be able to compare the time-varying models against the constant ones. While semiparametric approach is empirically interesting and not often used in literature, we employ it for all the estimated models as well.

Table 2.2 presents the fit from all estimated models. Starting with constant copulas, all the parameters are significantly different from zero, and Student's t copula appears to describe the oil and stock pair best according to highest log-likelihood. Semiparametric specifications combining nonparametric distribution in margins with parametric copula function bring further improvement in the log-likelihoods. Importantly, time-varying specifications bring large improvement in log-likelihoods and confirm strong time-varying dependence between oil and stocks.

To study the goodness of fit for all the specified models, we utilize[9] Kolmogorov-Smirnov (KS) and Cramer von-Mises (CvM) test statistics with p-values obtained from 1,000 simulations. The methodology is described in detail in previous sections. None of the fully parametric models is rejected, while most of the semiparametric models are rejected with exception of constant Student's t, Sym. Joe-Clayton and time-varying Student's t. This result suggests that fully parametric models with realized GARCH and parametric distribution in margins are all well specified. Thus, the realized GARCH appears to model very well all the dependence in margins, which is crucial for the good specification of the model in the copula-based approach. Semiparametric models are interestingly rejected and are not specified well, except for a few mentioned cases. This is in line with results of Patton (2013), who finds rejections in semiparametric specifications in the U.S. stock indices data. Still, both tests strongly support the realized GARCH time-varying GAS copulas for the oil and stock pair.

2.4.2 Out-of-sample comparison of the proposed models

While it is important to have a well-specified model that describes the data, most of the times we are interested in utilizing the model in predictions. Thus, we conduct an out-of-sample evaluation of the proposed models. For this, the sample is divided into two periods. The first, the in-sample period, is used to obtain parameter estimates from all models and spans from January 3, 2003 to July 6, 2010. The second, the out-of-sample period, is then used for evaluation of forecasts. Due to highly computationally intensive estimations of the models, we restrict ourselves to a fixed window evaluation, where the models are estimated only once, and all the forecasts are performed using the recovered parameters from this fixed in-sample period. This makes it even harder for the models to perform

[9] The results of the in-sample goodness of fit tests are available on request from the authors. We do not include them in the text to save space.

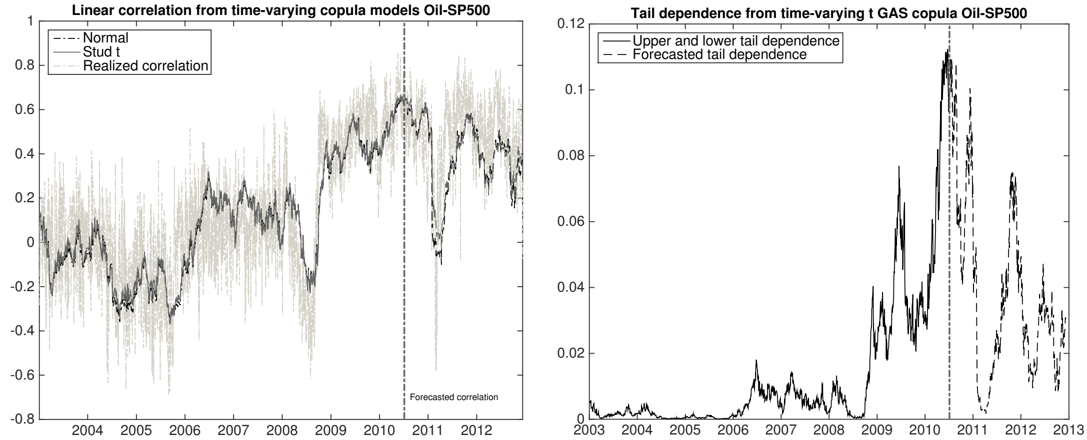

Figure 2.2: Left: linear correlation plotted against realized correlation. Right: tail dependence from time-varying Student's t-copula. The vertical dashed line separates the in-sample from the out-of-sample (forecasted) part.

well in the highly dynamic data. The procedure for the out-of-sample estimations is given in Algorithm 1, 2.C.

For the out-of-sample forecast evaluation, we employ the conditional predictive ability (CPA) test of Giacomini & White (2006). Table 2.3 presents the results from this test. The time-varying copula models significantly outperform the constant copula models in the out-of-sample evaluation. This holds both for the parametric and semiparametric cases. Thus, time-varying copulas have much stronger support for forecasting the dynamic distribution of oil and stocks. When comparing the different time-varying copula functions, the test is not so conclusive. While Student's t statistically outperform rotated Gumbel, the forecasts from Student's t can not be statistically distinguished from the normal copula. When looking at the out-of-sample log-likelihoods, Student's t copula is the most preferred. Finally, the bottom row shows that forecasts from parametric models and semiparametric ones cannot be statistically distinguished.

Thus, we find strong statistical support that the realized GARCH time-varying copula methodology well describes the dynamic joint distribution of the oil and stocks in both the in-sample and out-of-sample.

2.4.3 Time-varying correlations and tails

Having correctly specified the empirical model capturing the dynamic joint distribution between oil and stocks, we can proceed to studying the pair. Figure

	Parametric margins							
	Normal	Clayton	R. Gum.	Stud. t	SJC	$RGum_{GAS}$	N_{GAS}	t_{GAS}
Normal								
Clayton	0.36							
R. Gum.	2.00**	4.37***						
Stud. t	1.78*	2.09**	-0.15					
SJC	1.91*	3.80***	-1.94	-0.68				
$RGum_{GAS}$	2.75***	3.09***	2.34***	1.99**	2.53***			
N_{GAS}	2.49***	2.31**	1.65*	1.48	1.94*	0.33		
t_{GAS}	3.94***	4.02***	3.42***	3.27***	3.73***	2.37***	1.09	
$\mathcal{LL}^{OOS}$	33.14	34.62	43.74	43.17	40.74	60.10	62.65	69.79
Rank	8.00	7.00	4.00	5.00	6.00	3.00	2.00	1.00
	Nonparametric margins							
	Normal	Clayton	R. Gum.	Stud. t	SJC	$RGum_{GAS}$	N_{GAS}	t_{GAS}
Normal								
Clayton	1.12							
R. Gum.	2.23**	3.88***						
Stud. t	2.17**	1.93*	0.43					
SJC	2.39***	3.77***	-1.18	-0.74				
$RGum_{GAS}$	3.24***	3.34***	2.90***	2.35***	2.95***			
N_{GAS}	2.64***	2.32***	1.90*	1.57	2.03**	0.13		
t_{GAS}	4.05***	3.92***	3.55***	3.20***	3.69***	1.97**	1.06	
$\mathcal{LL}^{OOS}$	28.74	32.73	38.51	39.96	37.42	59.11	60.04	66.49
Rank	8.00	7.00	5.00	4.00	6.00	3.00	2.00	1.00
	Parametric vs. nonparametric margins							
	Normal	Clayton	R. Gum.	Stud t	SJC	$RGum_{GAS}$	N_{GAS}	t_{GAS}
t-stat	0.85	0.77	0.90	0.83	0.82	0.73	0.78	0.83

$*$, $**$ and $***$ denote significantly better performance at the 90%, 95% and 99% significance levels, respectively.

Table 2.3: The t-statistics from the out-of-sample pair-wise comparisons of log-likelihood values for five constant copula models and three time-varying copula models, with fully parametric or semiparametric marginal distribution models. Positive (negative) values indicate better performance of copula in the row (column) to a copula in the column (row). $\mathcal{LL}^{OOS}$ is the out-of-sample log likelihood, and "Rank" simply ranks all the models with respect to the log likelihood. In the bottom row, we compare the performance of the same copula with different margins *i.e.* parametric vs. nonparametric ones. The out-of-sample period is from July 6, 2010 to December 11, 2012 and includes 609 observations.

2.2 plots the time-varying correlations implied by our model with normal and Student's t GAS copulas. In the first period, correlations are obtained from an in-sample fit of our model. In the second period, the model is used to forecast the correlations. These dynamics are very close to the one reported by a recent study of Wen *et al.* (2012), although they are more accurate due to the help of realized measures utilized in our modeling strategy and GAS structure as well. To highlight this point, we compare the in-sample and forecasted correlations from the model to the actual correlation measured with the help of high-frequency data. The realized correlation between oil and stocks for a given day t is measured using $k = 1, \ldots, M$ five-minute returns non-parametrically using realized volatility (Andersen *et al.* 2003; Barndorff-Nielsen & Shephard 2004) as

$$RCorr_t = \frac{\sum_{k=1}^{M} r_{(1)k,t} r_{(2)k,t}}{\sqrt{\sum_{k=1}^{M} r_{(1)k,t}^2} \sqrt{\sum_{i=1}^{M} r_{(2)k,t}^2}},$$

and is depicted by the black dotted line in the Figure 2.2. We note that realized correlations provide noisy estimates, and we can clearly see how correlations implied by our model fit the actual correlations, as they also capture abrupt change in the year 2008. The out-of-sample forecast of the correlations lags the actual correlations and is slightly downward biased. The reason for this departure is that we use a static forecast utilizing parameters estimated on the in-sample data set as explained earlier in the text. Rolling sample forecasts would recover the actual correlations with much better precision. Hence, the proposed approach can capture the time-varying dynamics very well and is also able to forecast the dependence.

As we can see from the Figure 2.2, the dependence between oil and stocks varies strongly over time. Wen *et al.* (2012) suggests that the correlations changed dramatically during the 2008 crisis, but employing a larger data span, we suspect the correlation to have more regimes. To find the presence of structural breaks statistically, we employ the *supF* test (Hansen 1992; Andrews 1993), with p-values computed based on Hansen (1997) and apply it to the correlations.[10] Employing this approach, we confirm two endogenous changes in the dependence, with March 14, 2006 and October 9, 2008 dividing the data into three distinct periods.

[10]To conserve space, we do not report the test statistics for the detection of structural breaks. The results are available upon request. The test is used to evaluate the null hypothesis of no structural change, utilizing an extension of the F test statistics. In the first step, the error sum of squares is computed together with the restricted sum of the squares for every potential change point utilizing least squares fits. Second, F test statistics are computed for every potential change point, and third, a supremum is found from all the F test statistics constituting the structural break. The p-values are computed based on Hansen (1997).

During the first period, the correlation was decreasing from zero to negative values. An economic reasoning for this finding comes from the fact that the response of the stock markets to oil price shocks differs according to the origin of such shocks (Hamilton 1996). Specifically, a supply-side shock negatively impacts stock market returns and leads to negative correlation in the oil-stocks pair. An increase in oil prices – a supply-side shock – might result from an abrupt reduction of output by major producers (e.g., OPEC countries) or due to a major political event, such as the 1990-1991 Gulf War.

During the second period, beginning March 14, 2006, the correlation increased to positive values, while after the turmoil of October 2008, identified by the test very precisely with the date of October 9, 2008, the correlation became significantly positive, suggesting that diversification opportunities are disappearing. In the following years, correlations remained high, while in the last years of the sample, they began to decrease but remained in positive territory. This may be attributed to the changing expectations of market participants. After the financial crisis, the oil market became very strongly financialized (Büyükşahin & Robe 2014), and moves in stock prices also appeared to carry over to oil prices as well. After 2008, stock market participants were much more uncertain about future behavior, which translated into high volatility during this period.

In addition, the second part of the Figure 2.2 plots the dynamic tail dependence from the Student's t GAS model. In the first period, we document tail independence. The 2008 turmoil brought a large increase in tail dependence, which remained highly dynamic. While one of the additional advantages time-varying copula functions bring is allowing for asymmetric tails; in the final empirical model, we also study tail asymmetry. Interestingly, we find no evidence for asymmetry in the tails.

Our results have serious implications for investors as they suggest that diversification possibilities may be even larger than commonly perceived from the mere dynamics of the correlations. In addition, correlations as well as tail dependence have been rapidly changing over the past few years. We will utilize the results and study the possible economic benefits of our analysis, with the main focus on translating the dynamic correlation and tail dependence to a proper quantification of the diversification benefits.

2.5 Economic implications: Time-varying diversification benefits and VaR

Statistically significant improvement in the fit, or even out-of-sample forecasts does not necessarily need to translate into economic benefits. Thus, we test the proposed methodology in economic implications. First, we quantify the risk of an equally weighted portfolio composed from oil and stocks, and second, we study the benefits from diversification to see how the strongly varying correlation affects the diversification benefits.

2.5.1 Quantile forecasts

Quantile forecasts are central to risk management decisions due to a widespread Value-at-Risk (VaR) measurement. VaR is defined as the maximum expected loss that may be incurred by a portfolio over some horizon with a given probability. Let q_t^α denote a α quantile of a continuous and increasing distribution. VaR[11] of a given portfolio at time t is then simply

$$q_t^\alpha \equiv F_t^{-1}(\alpha), \text{for } \alpha \in (0,1). \tag{2.26}$$

Thus, the choice of the distribution is crucial to the VaR calculation. For example, assuming normal distribution may lead to underestimation of the VaR. Our objective is to estimate one-day-ahead[12] VaR of an equally weighted portfolio composed from oil and stock returns $Y_t = 0.5X_{1t} + 0.5X_{2t}$, which have conditional time-varying joint distribution F_t. In the previous analysis, we found that the realized GARCH model with time-varying GAS copulas well fits and forecasts the data; thus, we utilize it in VaR forecasts to see if it also correctly forecasts the joint distribution. As there is no analytical formula that can be utilized, the future conditional joint distribution is simulated from the estimated models. Once we obtain the future distribution of the portfolio, the VaR is computed from the corresponding quantile.

While quantile forecasts can be readily evaluated by comparing their actual (estimated) coverage $\hat{C}_\alpha = 1/n \sum_{n=1}^{T} 1(y_{t,t+1} < \hat{q}_{t,t+1}^\alpha)$, against their nominal coverage rate, $C_\alpha = E[1(y_{t,t+1} < q_{t,t+1}^\alpha)]$, this approach is unconditional and does not capture the possible dependence in the coverage rates. The number of approaches has been proposed for testing the appropriateness of quantiles conditionally; for

[11]VaR is typically a negative number, but in literature it is common to report it as positive value. In my dissertation I do not follow this sign convention and report all downside risks in negative numbers.

[12]It is possible to estimate h-step-ahead forecasts as well, but these are interesting when a rolling scheme is utilized for forecasting. As explained previously, due to the computational burden of the estimation methodology, we employ static forecasts. In addition, h-step ahead forecast requires simulation of the conditional distribution from the model; hence, computational intensity would increase with the horizon employed. The procedure for one-step-ahead VaR is given in Algorithm 2, 2.C.

Table 2.4: Out-of-sample VaR evaluation for GJR GARCH and realized GARCH models in margins. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with the Newey-West estimator for variance. All models are compared to t_{GAS} with realized GARCH in margins, while models with significantly less accurate forecasts at 95% level are reported in bold.

		Parametric						Semiparametric					
		0.01	0.05	0.1	0.9	0.95	0.99	0.01	0.05	0.1	0.9	0.95	0.99
GJR GARCH	**Normal**												
	$\hat{C}_\alpha$	0.015	0.069	0.110	0.893	0.946	0.995	0.015	0.071	0.110	0.893	0.946	0.993
	$\hat{L}$	0.025	**0.083**	0.132	**0.108**	**0.062**	0.015	0.024	0.082	0.131	**0.108**	**0.061**	0.015
	DQ	8.392	8.031	4.323	3.740	8.755	3.572	8.392	8.468	3.126	7.352	4.962	1.674
	p-val	0.211	0.236	0.633	0.712	0.188	0.734	0.211	0.206	0.793	0.290	0.549	0.947
	$\mathbf{RGumb_{GAS}}$												
	$\hat{C}_\alpha$	0.008	0.043	0.090	0.901	0.957	0.998	0.008	0.039	0.087	0.905	0.959	0.998
	$\hat{L}$	0.024	0.082	0.132	**0.107**	**0.061**	0.016	0.024	0.082	0.132	**0.107**	**0.061**	0.016
	DQ	5.779	6.305	3.911	10.334	5.496	6.494	5.779	6.831	3.260	9.267	6.232	6.494
	p-val	0.448	0.390	0.689	0.111	0.482	0.370	0.448	0.337	0.776	0.159	0.398	0.370
	$\mathbf{N_{GAS}}$												
	$\hat{C}_\alpha$	0.013	0.043	0.090	0.911	0.967	0.998	0.010	0.041	0.087	0.913	0.970	0.998
	$\hat{L}$	0.024	0.083	0.133	0.106	**0.061**	**0.016**	0.024	**0.083**	**0.133**	0.107	**0.062**	**0.017**
	DQ	4.486	4.284	3.869	7.479	9.155	6.494	4.838	4.667	5.314	10.466	13.575	6.494
	p-val	0.611	0.638	0.694	0.279	0.165	0.370	0.565	0.587	0.504	0.106	0.035	0.370
	$\mathbf{t_{GAS}}$												
	$\hat{C}_\alpha$	0.010	0.046	0.085	0.908	0.966	0.998	0.011	0.044	0.089	0.905	0.969	0.998
	$\hat{L}$	0.024	0.082	0.133	0.107	**0.061**	**0.017**	0.024	0.082	0.133	0.107	**0.062**	**0.017**
	DQ	4.838	3.072	3.610	8.370	9.413	6.494	9.276	5.442	3.167	11.390	11.982	6.494
	p-val	0.565	0.800	0.729	0.212	0.152	0.370	0.159	0.488	0.788	0.077	0.062	0.370
realized GARCH	**Normal**												
	$\hat{C}_\alpha$	0.023	0.082	0.130	0.877	0.931	0.987	0.021	0.085	0.126	0.878	0.931	0.985
	$\hat{L}$	**0.026**	**0.083**	0.132	**0.107**	**0.062**	0.015	0.025	**0.083**	0.132	**0.107**	**0.062**	0.015
	DQ	11.782	13.015	7.838	8.558	10.076	3.421	10.578	14.582	8.282	8.101	10.076	4.249
	p-val	0.067	0.043	0.250	0.200	0.121	0.755	0.102	0.024	0.218	0.231	0.121	0.643
	$\mathbf{RGumb_{GAS}}$												
	$\hat{C}_\alpha$	0.016	0.064	0.117	0.890	0.934	0.990	0.016	0.062	0.112	0.890	0.938	0.990
	$\hat{L}$	**0.025**	**0.082**	0.131	**0.106**	**0.060**	0.015	0.024	0.081	0.131	**0.106**	**0.060**	0.015
	DQ	5.654	4.939	5.976	12.658	9.639	3.969	5.654	3.555	5.374	12.658	5.306	3.969
	p-val	0.463	0.552	0.426	0.049	0.141	0.681	0.463	0.737	0.497	0.049	0.505	0.681
	$\mathbf{N_{GAS}}$												
	$\hat{C}_\alpha$	0.018	0.062	0.115	0.893	0.944	0.990	0.018	0.061	0.115	0.895	0.947	0.992
	$\hat{L}$	0.025	**0.081**	**0.131**	0.105	**0.059**	0.015	0.025	0.081	0.131	**0.106**	0.059	0.015
	DQ	6.695	2.735	5.886	14.911	5.558	3.969	6.695	2.640	5.623	15.042	4.470	0.617
	p-val	0.350	0.841	0.436	0.021	0.474	0.681	0.350	0.853	0.467	0.020	0.613	0.996
	$\mathbf{t_{GAS}}$												
	$\hat{C}_\alpha$	0.016	0.066	0.113	0.893	0.946	0.993	0.015	0.061	0.112	0.895	0.949	0.997
	$\hat{L}$	0.025	0.081	0.131	0.106	0.060	0.015	0.024	0.081	0.131	0.105	0.059	0.015
	DQ	5.654	3.298	3.670	14.911	4.723	0.450	4.906	2.336	3.343	14.616	4.060	3.582
	p-val	0.463	0.771	0.721	0.021	0.580	0.998	0.556	0.886	0.765	0.023	0.669	0.733

the best discussion, see Berkowitz *et al.* (2011). In our out-of-sample VaR testing, we employ an approach originally proposed by Engle & Manganelli (2004), who use the n-th order autoregression

$$I_t = \omega + \sum_{k=1}^{n} \beta_{1k} I_{t-k} + \sum_{k=1}^{n} \beta_{2k} q_{t-k+1}^{\alpha} + u_t, \tag{2.27}$$

where I_{t+1} is 1 if $y_{t+1} < q_t^{\alpha}$, and zero otherwise. While the hit sequence I_t is a binary sequence, u_t is assumed to follow a logistic distribution, and we can estimate it as a simple logit model and test whether $Pr(I_t = 1) = q_t^{\alpha}$. To obtain the p-values, we rely on simulations as suggested by Berkowitz *et al.* (2011), and we refer to this test as a DQ test in the results.

The main motivation of the DQ test is to determine whether the conditional quantiles are correctly dynamically specified; hence, it evaluates the absolute performance of the various models. To assess the relative performance of the models, we evaluate the accuracy of the VaR forecasts statistically by defining the expected loss of the VaR forecast made by a forecaster m as

$$L_{\alpha,m} = E\left[\left(\alpha - 1\left\{y_{t,t+1} < q_{t,t+1}^{\alpha,m}\right\}\right)\left(y_{t,t+1} - q_{t,t+1}^{\alpha,m}\right)\right], \tag{2.28}$$

which was proposed by Giacomini & Komunjer (2005). The tick loss function penalizes quantile violations more heavily, and the penalization increases with the magnitude of the violation. As argued by Giacomini & Komunjer (2005), the tick loss is a natural loss function when evaluating conditional quantile forecasts. To compare the forecast accuracy of the two models, we test the null hypothesis that the expected losses for the models are equal, $H_0 : d = L_{\alpha,1} - L_{\alpha,2} = 0$, against a general alternative. The differences can be tested using Diebold & Mariano (2002) test statistics, $S = \overline{d}/\sqrt{\widehat{LRV}/T}$, where $\overline{d}$ is the unconditional average of loss difference d, and $\widehat{LRV}$ a consistent estimate of the long-run variance of $\sqrt{T}d$. Under the null of equal predictive accuracy, $S \sim N(0, 1)$

Table 3.10 reports the out-of-sample VaR evaluation of all models. As standard normal distribution is the most common choice for VaR computations, we also report the results for the constant normal copula. In addition, we benchmark all the models to versions with the GJR GARCH, which does not employ high frequency data. We can see that all the time-varying models are well specified, and the conditional quantile forecasts from them are not rejected by the DQ test. This holds for both the GJR GARCH as well as the realized GARCH in margins. With the constant copula model, quantile forecasts are rejected primarily for the lower quantiles, and according to the empirical conditional rates, we can see that it underestimates the risk.

For statistical testing, we employ time-varying Student's t as a benchmark forecaster and test all the other models against it. When looking at the loss functions $\hat{L}_{\alpha,m}$, we can see that the constant copula model is usually rejected against the time-varying Student's t. This is also the case for other two time-varying specifications, and so, the realized GARCH time-varying Student's t copula model appears to have statistically the most accurate quantile forecasts. Interestingly, when looking at the results from semiparametric models, we see fewer rejections, and overall, these models appear to provide few more accurate quantile forecasts. The results remain almost the same when the model is benchmarked to the GJR GARCH specifications. The GJR GARCH overestimates the VaR at all quantiles, while the use of high frequency data help at the right tail of the distribution, where the realized GARCH models outperform the GJR GARCH models. While one would expect high frequency data to improve the forecasts, we note that this may be a feature of the static nature of the forecasts. Even in a static environment, a high frequency measure statistically outperforms the benchmark, and while we are evaluating the VaR forecasts, this result has direct economic implications for the improvement of dynamic hedging.

2.5.2 Time varying diversification benefits

In case the dependence of the assets is strongly changing over time, it needs to translate into the changing of diversification benefits as well. While mean dependence is employed in most of the studies to assess the diversification benefits, independence in tails may translate into higher than anticipated benefits. In case the empirical distribution departs from normality, it is important to also account for this departure when calculating diversification benefits. In the previous section, we have observed that the empirical model we have built captures the quantiles of the return distribution well and is correctly specified. The correct choice of the model for quantiles is also important for identification of diversification benefits, which we will utilize here.

In this respect, we provide an important insight, as literature finds mixed results for oil – stocks relationship depending on whether the increase in oil prices is driven by supply or demand shocks, or whether we are studying relationship to oil importing or exporting country. Fundamentally, shocks in oil prices affect the stock markets through several mechanisms. The literature on the negative association between oil prices and stock market suggest unidirectional causality from oil to stocks. At the micro level, an increase in oil prices will increase the cost of production of the firms using oil as the main production factor, translating to lower earnings, hence stock prices. At the macro level, increase in oil prices may cause pressures on inflation that force central banks to increase interest rates, making bond markets more attractive to investors. On the other hand, increase in oil prices may also cause increase in stock prices through income and wealth effect channels for oil exporting country. In case the increased government revenues are transferred back to the economy, it will result in increase in economic activity and improve stock market performance.

The economic reasons for both negative and positive relationship between stock and oil prices only underline the importance of proper quantification of the diversification benefits, which may be strongly dynamic. Here, we utilize the results form previous sections to correctly quantify the dynamics of diversification benefits, as we found the correlation to strongly vary over time. Unlike VaR, the expected shortfall satisfies the sub-additivity property and is a coherent measure of risk. Motivated by these properties, Christoffersen *et al.* (2012) propose a measure capturing the dynamics in diversification benefits based on expected shortfall. The conditional diversification benefit (CDB) for a given probability level α is defined by

$$CDB_t^\alpha = \frac{\overline{ES}_t^\alpha - ES_t^\alpha}{\overline{ES}_t^\alpha - \underline{ES}_t^\alpha}, \tag{2.29}$$

where ES_t^α is the expected shortfall of the portfolio at hand,

$$ES_t^\alpha \equiv E[Y_t|F_{t-1}, Y_t \leq F_t^{-1}(\alpha)], \text{for } \alpha \in (0,1), \tag{2.30}$$

$\overline{ES}_t^\alpha$ is the upper bound of the portfolio, the expected shortfall being the weighted average of the asset's individual expected shortfalls, and $\underline{ES}_t^\alpha$ the lower bound on the expected shortfall being the inverse cumulative distribution function for the portfolio. In other words, this lower bound corresponds to the case where the portfolio never loses more than its α distribution quantile. The measure is designed to stay within $[0, 1]$ interval and is increasing in the level of diversification benefits. When the conditional diversification benefit (CDB) is equal to zero, there are literarily no benefits from diversification; when it equals one, the benefits from diversification are the highest possible.

Figure 2.3 plots the conditional diversification benefits for the oil and stocks portfolio implied by the two best performing models in the VaR evaluation for $\alpha = 0.05$. The correct dynamic specification of the quantiles from the empirical models is crucial for the CDB, as it mainly captures the potential diversification benefits from tail independence. Hence, to obtain a precise CDB, one needs to first find a model that captures the dynamics in the quantiles correctly.

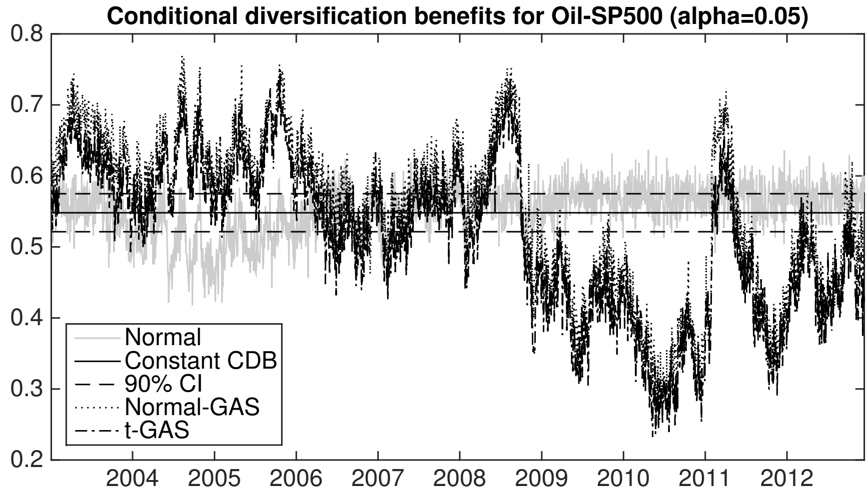

Figure 2.3: Conditional diversification benefits, $CDB_t^{0.05}$ utilizing constant normal, time-varying normal, and Student's t copulas. The dashed line is the constant level of diversification benefits, and in grey, its 90% confidence band.

Similarly to the VaR case, as there is no closed form to our empirical model, we need to rely on the simulations for computing the CDB. Encouraged by the previous results, we compute the CDB for the best performing models with time-varying normal and student's t copulas. As a benchmark, we include the model with a constant copula. Moreover, we report 90% bootstrapped confidence bands computed around a constant level of diversification benefits. Assuming the returns data are independently distributed over time with the same unconditional correlation as the oil and stocks pair, the bootstrap confidence level can be conveniently computed with simulations. We use 10,000 simulations and report the mean value together with the distribution of the constant conditional benefits.

From the Figure 2.3, we can see how greatly the diversification benefits vary over time. Corresponding to the correlations, there are also several identifiable periods where the benefits from diversification were significantly different. Thus, we conduct the same endogeneity test to find whether there is a structural break in the CDS, and the result is that the test identified exactly the same dates as using the correlation. In addition, January 31, 2011 was identified as another structural break.

While in the first period the benefits from diversification were relatively high, in the second period between 2006 and 2008, they decreased corresponding to increasing correlation. In the several years after the 2008 crisis, benefits from diversification between oil and stocks were decreasing rapidly, while we can see some rebound in the last few years.

2.6 Conclusions

This work revisits the oil and stocks dependence with the aim of studying the opportunities of these two assets in portfolio management. We propose utilizing the high frequency data in the copula models by choosing to model the marginal dependence with the realized GARCH of Hansen *et al.* (2012). Based on the recently proposed generalized autoregressive score copula functions (Creal *et al.* 2013), we build a new empirical model for oil and stocks, the realized GARCH time-varying GAS copula.

The modeling strategy is able to capture the time-varying conditional distribution of the oil stocks pair accurately, including the dynamics in the correlation and tails. This also translates into accurate quantile forecasts from the model, which are central to risk management, as they represent value-at-risk. Utilizing the ten years of the data covering several different periods, we study the time-varying correlations, and we find two main endogenous breaks in the dependence structure. Most important, we translate the results into the conditional diversification benefits measure recently proposed by Christoffersen *et al.* (2012). The main result is that the possible benefits from using oil as a diversification tool for stocks have been decreasing rapidly over time, while in the last year of the sample, it displayed some rebound. These results have important implications for the risk industry and portfolio management as commodities have recently become an attractive opportunity for risk diversification in portfolios. According to our results, the benefits may not be as high as in the first half of the sample.

In conclusion, during the period under research, oil and stocks could be used in a well-diversified portfolio less often than common perception would imply. We find substantial evidence of dynamics in tail dependence, which translates into dynamically decreasing diversification benefits from employing oil as a hedging tool for stocks. The empirical results have important implications for portfolio management, which should be explored in the future. It may be useful to think about the improvement in dynamic hedging strategies, which will account for our empirical findings. An interesting venue of research is the inclusion of a threshold or multiple component dependences in the models.

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2.A Copula functions

2.A.1 Normal copula

The Normal copula does not have a simple closed form. For the bivariate case and $|\rho| < 1$, we can approximate it by the double integral:

$$C_\rho^N(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} drds \quad (2.31)$$
$$-1 < \rho < 1$$

where Φ^{-1} is the inverse of standard normal distribution $(c.d.f)$. The correlation is modeled by the transformed variable $\rho_t = (1 - e^{-\kappa_t})(1 + e^{-\kappa_t})^{-1}$, which guarantees that ρ_t will remain in (-1,1). For $\rho = 0$, we obtain the independence copula, and for $\rho = 1$, the comonotonicity copula. For $\rho = -1$ the countermonotonicity copula is obtained. We note that Normal copula has no tail dependence for $\rho < 1$.

2.A.2 Student's t copula

The bivariate Student's t copula is defined by

$$C_{\eta,\rho}^t(u,v) = \int_{-\infty}^{t_\eta^{-1}(u)} \int_{-\infty}^{t_\eta^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho rs + s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} drds \quad (2.32)$$

where $\rho \in (-1,1)$, and $0 < \eta$. The t_η^{-1} is the inverse of t distribution with η degrees of freedom. The correlation parameter of the t copula undergoes the same transformation as in the case of the Normal to guarantee $\rho_t \in (-1,1)$. For the time varying t copula, we allow only the correlation to vary through time, the degrees of freedom η remain constant.

In contrast to the Normal copula, provided that $\rho > -1$, the t copula has symmetric tail dependence given by

$$\lambda^L = \lambda^U = 2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right) \quad (2.33)$$

We utilize the time-varying dynamics of the correlation ρ_t for the time-varying tail dependence λ_t.

2.A.3 Clayton copula

The bivariate Clayton copula is defined as

$$C_\theta^{Cl}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{\frac{-1}{\theta}}, \qquad 0 < \theta < \infty \quad (2.34)$$

In the limit as $\theta \to 0$, we approach the independence copula, and as $\theta \to \infty$, we approach the two-dimensional comonotonicity copula.

2.A.4 (Rotated) Gumbel copula

The Gumbel copula is defined by

$$C_{\delta}^{Gu}(u,v) = \exp\{-((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta}\}, \quad 1 \leq \delta < \infty \tag{2.35}$$

The Gumbel copula parameter is required to be greater than one, and the transformation $\delta_t = 1 + \exp(\kappa_t)$ guarantees this. For $\delta = 1$. The Gumbel copula reduces to the fundamental independence copula:

$$\begin{aligned} C_{\delta}^{Gu}(u,v) &= exp\{-((-\log u)^{1} + (-\log v)^{1})^{1/1}\} \\ &= exp\{\log u + \log v\} \\ &= exp\{\log(uv)\} = uv \end{aligned}$$

The rotated Gumbel copula has the same functional form as the Gumbel copula and is obtained by replacing u and v by *1-u* and *1-v*, respectively.

2.A.5 Symmetrized Joe-Clayton Copula

The SJC copula is obtained from the linear combination of the Joe-Clayton copula (C^{JC}).

$$C^{SJC}(u,v|\tau^U,\tau^L) = 0.5 \cdot (C^{JC}(u,v|\tau^U,\tau^L) + C^{JC}(1-u,1-v|\tau^L,\tau^U) + u + v - 1)$$

where

$$C^{JC}(u,v|\tau^U,\tau^L) = 1 - (1 - \{[1-(1-u)^{\psi}]^{-\gamma} + [1-(1-v)^{\psi}]^{-\gamma} - 1\}^{-1/\gamma})^{1/\psi} \tag{2.36}$$

$$\begin{aligned} \psi &= 1/\log_2(2-\tau^U) \\ \gamma &= -1/\log_2(\tau^L) \\ \tau^U &\in (0,1), \quad \tau^L \in (0,1) \end{aligned}$$

This copula has two parameters, τ^U and τ^L, representing the upper and lower tail dependence, respectively. For more details on this copula, see Patton (2006b).

2.B Figures and Tables

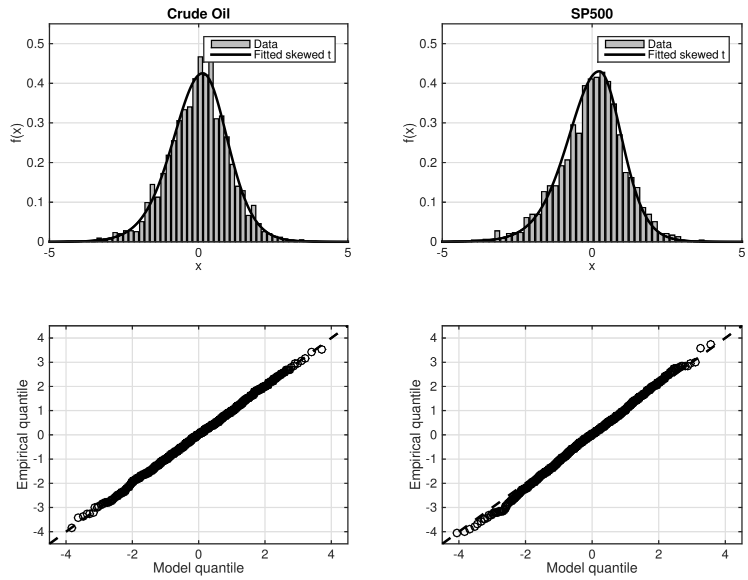

Figure 2.4: First row shows fitted skew-t density and the histogram of standardized residuals for Crude Oil and the S&P 500. In the second row, QQ plot is shown.

	Returns		Realized Volatility	
	Crude Oil	S&P 500	Crude Oil	S&P 500
Mean	0.000	0.000	0.016	0.007
Std dev	0.017	0.010	0.007	0.006
Skewness	-0.083	-0.352	2.395	3.556
Ex. Kurtosis	3.924	10.940	8.049	19.944
Minimum	-0.108	-0.082	0.005	0.001
Maximum	0.123	0.073	0.064	0.076

Table 2.5: Descriptive Statistics for daily oil and stock (S&P 500) returns and realized volatilities ($\sqrt{RV_t}$) over the sample period extending from January 3, 2003 until December 11, 2012.

2.C Algorithms

The forecasting algorithm follows the work from Patton (2013). We do several adjustments to fit the specifics of our model.

Algorithm 1 Out-of-sample estimates

1: $T \leftarrow length(data)$
2: $R \leftarrow floor(0.75 * T)$ # Arbitrary choosing 75% length of the series for in-sample (IS) data.
3: Estimate conditional mean models (ARMA) for the IS data and get 0 mean residuals
4: Estimate Realized GARCH parameters on standardized residuals from ARMA and from realized volatility using the in sample data
5: Get standardized residuals from Realized GARCH
6: Fit *skewed-t* distribution to previous step data
7: Get standard uniform data by applying estimated *skewed-t* distribution (or empirical) to data in line 5
8: Estimate copula parameters for in-sample using the data in line 7
9: $\hat{\mu}_{OOS} \leftarrow$ Forecast the mean for out-of-sample (OOS) using the IS estimated parameters in line 3
10: residuals OOS $\leftarrow returns - \hat{\mu}_{OOS}$ # Get residuals from OOS data (and standardize)
11: Forecast the volatility one-step-ahead using Realized GARCH parameters estimated in line 4
12: Get standardized residuals for OOS using results in lines 10 and 11
13: Get forecast of OOS uniform data by applying *skewed-t* (or empirical) distribution with parameters obtained in line 6
14: Forecast one-step-ahead GAS copula parameters using data from line 13 and estimated IS copula parameters (for constant copulas the parameters will be the same as IS)
15: Get the OOS copula log-likelihoods by evaluating copula log-likelihood functions using data from line 13 and with parameters the ones obtained from IS

Algorithm 2 Out-of-sample VaR (and ES) via simulations

1: T;q;w; # sample size; quantile of interest and portfolio weight
2: set t=1 # start at time t=1
3: get estimated parameters for copula C(·) at time t # these parameters are estimated in steps 8 and 14 in Algorithm 1
4: U ← copula_rnd(parameters;size=S) # generate S observations from copula model. S=5000, thus U is of size 2 × 5000
5: E[,i] ← $quantile(stdresids[,i], U[,i])$ # get the implied standardized residuals from copula model $i \in \{Oil, SP500\}$ with *stdresids* from step 12 in Algorithm 1. Here copula function induces the dependence between *Oil & SP500*. This step can also be interpreted as bootstraping.
6: Y[,i] ← $\hat{\mu} + vol[t] \cdot E[,i]$ # Form simulated returns at time "t" for each asset. Volatility *vol* comes from step 11 in Algorithm 1
7: pf ← $w \cdot Y[,i] + (1-w) \cdot Y[,j]$ # Form the simulated portfolio returns for time "t"
Now use the empirical distribution of simulated portfolio returns to estimate the VaR and ES measures
8: VaR[t] ← $quantile(pf, q)$
9: index ← $(pf <= quantile(pf, q))$ # get index where VaR is violated
10: ES[t] ← $mean(pf[index])$ # Expected Shortfall
11: Repeat steps 3 - 10 until t=T

Chapter 3

Semiparametric nonlinear quantile regression model for financial returns

Abstract

Financial institutions use Value-at-Risk (VaR) as the standard measure of market risk. Despite its simplicity, measuring and forecasting it accurately is a challenging task. Recently, quantile regression models have been used successfully to capture the conditional quantiles of returns. We explore further non-linearities in the data, and propose to use realized measures in the nonlinear quantile regression framework to explain and forecast conditional quantiles of financial returns. In addition, we apply the proposed model to a pool of the most liquid U.S. assets across different industries. The nonlinear quantile regression models are implied by copula specifications and allow us to capture possible nonlinearities, and asymmetries in conditional quantiles of financial returns. Using high frequency data covering most liquid U.S. stocks in seven sectors, we provide ample evidence of asymmetric conditional dependence and different level of dependence characteristic for each industry. The backtesting results of estimated VaR favour our approach.

Keywords: quantile copula regression, realized-volatility, value-at-risk
JEL: C14, C32, C58, F37, G32

3.1 Introduction

A number of important financial decisions require the specification and estimation of the entire portfolio distribution. This is not an easy task in practice considering that the joint distribution is nonelliptic and fat tailed, as is standard

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with financial returns. The difficulty increases further if we take into account that the distribution of financial returns typically changes over time. Modelling of the distribution has direct implications in calculation of the standard measure of the market risk, the Value-at-Risk (VaR). The Basel Committee on Banking Supervision (1996) at the Bank for International Settlements requires financial institutions to use VaR to measure and report the market risk. This measure is just the value of future portfolio returns at a particular quantile of the portfolio distribution. Thus, a suitable model for time varying conditional quantiles is of key importance. Most of conditional quantile models rely on global distributional assumptions though, conditioning on past returns. In this paper, we propose to model the future quantiles of returns using the realized volatility and nonlinear quantile regression framework.

Quantile regression is lively area of research with several recent advances. Koenker (2004) extends the quantile regression to panel data applications. The author introduces a general approach to estimate quantile regression models for longitudinal data by employing ℓ_1 regularization methods. Xiao (2009) propose a cointegration model with quantile-varying coefficients. In this model the cointegrating coefficients are allowed to be affected by shocks received in each period over the innovation quantile. Chen *et al.* (2009) introduce nonlinear-in-parameters quantile autoregression (QAR) models using parametric copulas. They use a copula-based Markov model which due to the stationarity of the process considered requires the specification of only one margin in addition to the copula. Under mild conditions their model allows for global misspecification of parametric copulas and marginals. In a recent paper Cappiello *et al.* (2014) measure the comovements by using regression quantiles. The authors compute the conditional probability that a random variable is lower than a given quantile, when another random variable is also lower than its corresponding quantile, for any set of prespecified quantiles.

Quantile regression models have important applications in risk management, portfolio optimization, and asset pricing. Koenker & Bassett (1978) introduced the regression quantiles more than three decades ago, but only in the last one the financial literature paid more interest to it. Engle & Manganelli (2004) introduce the conditional autoregressive value at risk, which is also known as the CAViaR model. Instead of modelling the whole distribution, the authors model the quantiles directly. In CAViaR model the quantile of the distribution is regressed on its lagged values and a term which plays the role of the news impact curve for GARCH models. The former ensures a smooth change of quantile, while the latter links the quantile with the observable variables that belong to the information set. Under a semi-parametric quantile regression framework Žikeš & Baruník (2014) utilize nonparametric measures of the various components of *ex post* variation in asset prices to study the properties of conditional quantiles of daily asset returns and realized volatility, and forecast their future values. We exploit the ideas discussed in this paper in a nonlinear semiparametric conditional quantile regression framework to estimate the dependency between returns and realized volatility at quantiles of interest.

Our contribution of this paper is twofold: First, we propose to use realized measures in the nonlinear quantile regression framework to explain and forecast conditional quantiles of financial returns. Second, we apply the proposed model to a pool of the most liquid U.S. assets across different industries. The article is structured as follows. Section 2 introduces the copula quantile regression model. Section 3 describes the data under analysis. Section 4 presents an application to real data. Section 5 evaluates quantile forecast and section 6 concludes the article.

3.2 Modelling framework

Let us consider the logarithmic price process that obeys Itô semimartingale

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s,$$

where μ is a predictable process, σ is cadlag and W is the standard Brownian motion. The process is very general, and as Todorov & Tauchen (2011) show, it allows to accommodate stochastic volatility with possible discontinuous sample paths. To capture the conditional quantiles of returns $r_{t+1} = X_{t+1} - X_t$, Žikeš & Baruník (2014) propose a simple linear semiparametric model based on quantile regression. They assume that the α-quantile of future returns conditional on information set Ω_t, can be written as a linear function of its past quadratic variation,[1]

$$q_\alpha(r_{t+1}|\Omega_t) = \beta_0(\alpha) + \beta_\vartheta(\alpha)'\vartheta_t \tag{3.1}$$

where ϑ_t is a measure of quadratic variation $QV_t = \int_0^t \sigma_s^2 \mathrm{d}s$, commonly realized volatility, and $\beta_0(\alpha), \beta_\vartheta(\alpha)$ are vectors of coefficients to be estimated.

To see the connection that exists between the linear quantile model in Equation 3.1 and the logarithmic price process assumed to underly the data, Žikeš & Baruník (2014) argue that the conditional quantile $q_\alpha(r_{t+1}|IV_t)$ can be obtained from the

[1]Note that Žikeš & Baruník (2014) allow more general specification including various components of volatility, and weakly exogenous variables to drive the conditional quantiles in the model.

conditional distribution of r_{t+1} given IV_t in case quadratic variation of the process is equal to integrated variance:

$$f_{r_{t+1}|IV_t}(w_r|IV_t) = \int_0^\infty f_{r_{t+1}|IV_{t+1},IV_t}(w_r|w_{IV}, IV_t) f_{IV_{t+1}|IV_t}(w_{IV}|IV_t)\mathrm{d}w_{IV}$$

where $f_{y|X}(w_y|w_X)$ is the conditional distribution of y given X evaluated at w_y and w_X. For simplicity take the one-factor volatility model where σ_t follows an Ornstein-Uhlenbeck process and $\mu_t = 0$. Meddahi (2003) shows that the integrated volatility IV_t follows an ARMA(1,1) process with non-gaussian innovations. It follows that $f_{IV_{t+1}|IV_t}$ has a non-gaussian density, while $f_{r_{t+1}|IV_{t+1},IV_t}$ is the normal density with zero mean and variance IV_t if there is no leverage effect, or a non-gaussian density otherwise. The implied conditional quantiles of the densities above, $q_\alpha(r_{t+1}|\Omega_t), \quad \alpha \in (0,1)$, can be approximated by linear functions of the current and past values of IV_t and other volatility measures.

The model described by Equation 3.1 is a linear quantile regression proposed by Koenker & Bassett (1978). They show that the parameters can be estimated a solution to the following problem:

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^k} \left(\sum_{t\in\mathcal{T}_\alpha} \alpha \left| r_{t+1} - \beta_0(\alpha) - \beta_\vartheta(\alpha)'\vartheta_t \right| + \sum_{t\in\mathcal{T}_{1-\alpha}} (1-\alpha) \left| r_{t+1} - \beta_0(\alpha) - \beta_\vartheta(\alpha)'\vartheta_t \right| \right) \tag{3.2}$$

with $\mathcal{T}_\alpha = \{t : r_{t+1} \geq \beta_0(\alpha) - \beta_\vartheta(\alpha)'\vartheta_t\}$ and $\mathcal{T}_{1-\alpha}$ its complement.

In the special case where $\alpha = 0.5$, the above quantile regression delivers the least absolute deviation (LAD) model. The LAD model is more robust than ordinary least squares (OLS) estimators whenever the errors have a fat-tailed distribution. The problem defined in Equation 3.2 does not have a closed form-solution, however Portnoy & Koenker (1997) provide computationally fast algorithm which is also implemented in the `quantreg` package for R.

3.2.1 Copula quantile regressions

Conditional quantile functions allow for nonlinear parametric models. Bouyé & Salmon (2009) introduced a general approach to nonlinear quantile regression based on copula models. Using the properties of conditional probability distribution the link between copula functions and conditional quantile functions becomes obvious. Consider a random sample $(x_1, \ldots, x_T)$ and $(y_1, \ldots, y_T)$ from X and Y respectively. The probability distribution of y conditional on x is defined as

$$\alpha(y|x;\delta) = Pr\{Y \leq y, X = x\}$$

$$
\begin{aligned}
&= \boldsymbol{E}(\mathbb{1}_{Y \leq y}|X = x) \\
&= \lim_{\epsilon \to 0} Pr\{Y \leq y | x \leq X \leq x + \epsilon\} \\
&= \lim_{\epsilon \to 0} \frac{\boldsymbol{F}(x+\epsilon, y; \delta) - \boldsymbol{F}(x, y; \delta)}{F_X(x+\epsilon) - F_X(x)} \\
&= \lim_{\epsilon \to 0} \frac{\boldsymbol{C}[F_X(x+\epsilon), F_Y(y); \delta] - \boldsymbol{C}[F_x(x), F_y(y); \delta]}{F_X(x+\epsilon) - F_X(x)}
\end{aligned}
$$

Denoting by $C_1(\cdot, \cdot; \delta)$ the partial derivative of copula function with respect to the first argument, the probability distribution of y conditional on x can also be writen as

$$
\alpha(y|x; \delta) = \frac{\partial C(u, v; \delta)}{\partial u} = C_1[F_X(x), F_Y(y); \delta] \tag{3.3}
$$

where $u = F_X(x)$ and $v = F_Y(y)$. Refer to the Appendix 3.A for the proof. In case Equation 3.3 is invertible with respect to v the relationship between X and quantile of Y can be expressed as

$$
Q_Y(\alpha|x) = \boldsymbol{q}(x, \alpha; \delta) = F_Y^{[-1]}(D(F_X(x), \alpha; \delta)) \tag{3.4}
$$

where D is the partial inverse of C_1 in the second argument and $F_Y^{[-1]}$ the pseudo-inverse of F_Y. It may happen that relationship in Equation 3.3 is not invertible, and if this is the case numerical methods should be used[2]. We can generate observations on Y given X by evaluating Equation 3.4 and replacing α by independent uniformly distributed draws.

The copula quantile regression is a special case of the nonlinear quantile regression, Before we introduce specific copula functions for estimation, we introduce estimation of parameters in general nonlinear regression case with general quantile curve function. Turning to the problem of explaining conditional quantiles of future returns r_{t+1} using its past volatility ϑ_t utilizing nonlinear quantile regression, Equation 3.2 needs to be altered. The parameters of such regression can be estimated as a solution to the following problem:

[2] Gumbel copula is a typical example where numerical methods are required. Its α quantile function is given by the following expression:

$$
\alpha = u^{-1}(-\log(u))^{\theta-1} e^{-\left((-\log(u))^\theta + (-\log(v))^\theta\right)^{1/\theta}} \left((-\log(u))^\theta + (-\log(v))^\theta\right)^{\frac{1}{\theta}-1}
$$

As one can see this expression is not invertible. The implementation of the numerical methods is time consuming though, because finding the root of the non-invertible quantile function above needs extra time in addition to solving the non-linear quantile regression problem.

$$\min_{\delta}\left(\sum_{t\in\mathcal{T}_p}\alpha|r_{t+1}-\boldsymbol{q}(\vartheta_t,\alpha;\boldsymbol{\delta})|+\sum_{t\in\mathcal{T}_{1-\alpha}}(1-\alpha)|r_{t+1}-\boldsymbol{q}(\vartheta_t,\alpha;\boldsymbol{\delta})|\right) \tag{3.5}$$

with $\mathcal{T}_\alpha = \{t : r_{t+1} \geq \boldsymbol{q}(\vartheta_t,\alpha;\boldsymbol{\delta})\}$ and $\mathcal{T}_{1-\alpha}$ its complement. Koenker & Park (1996) developed an interior point algorithm to compute the quantile regression estimates for problems with nonlinear response functions. Their approach to solve the nonlinear problem is by solving a succession of linearized ℓ_1 problems, *i.e.* splitting the nonlinear problem into a set of linear ones.

In our model we regress future returns r_{t+1} on its lagged realized volatility $\vartheta_t = \sqrt{RV_t}$ computed as square root of a sum of squared 5-minute intraday returns

$$RV_t = \sum_{i=1}^{M} r_i^2, \tag{3.6}$$

where i is the 5-minute intraday time interval.

When working with quantile regressions we may face the problem of quantile crossing. The cause of quantile crossing may be due to estimation error, misspecification or both. Some recent papers provide a solution to this problem. Dette & Volgushev (2008) propose a non-parametric estimate of conditional quantiles that avoids quantile crossing. The method uses an initial estimate of the conditional distribution function in the first step and solves the problem of inversion and monotonization with respect to $\alpha \in (0,1)$ simultaneously. Chernozhukov *et al.* (2009; 2010) propose a closely related, but different method to address the problem of quantile crossing[3]. Their method consists in sorting the original estimated non-monotone curve into a monotone rearranged curve. This is a two step procedure: First, a preliminary (parametric) estimate of the conditional quantile curve is isotonized and inverted. Next, the final non-crossing estimates are constructed by an inversion of the curves that are obtained in the first step [4].

3.2.2 Copula quantile functions

Let us now introduce two specific copula quantile functions which we use in this paper, first Normal and then t copula. The bivariate Normal copula function

[3] For a comparison of these approaches refer to Dette & Volgushev (2008).

[4] This method is incorporated in the `quantreg` package in `R`. This package is also used by us in this paper.

can be written as

$$C_\rho^N(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} drds$$

where $\Phi(\cdot)$ is the standard Normal distribution and ρ the linear correlation. The partial derivative with respect to $u = F_X(x)$ is

$$\alpha = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \tag{3.7}$$

The quantile curve implied by copula function is obtained by solving Equation 3.7 for v:

$$v = \Phi\left(\rho\Phi^{-1}(F_X(x)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right) \tag{3.8}$$

The relationship between x and the quantile of y is then

$$Q_{Y|X}(\alpha|x) = F_Y^{[-1]}\left(\Phi\left(\rho\Phi^{-1}(F_X(x)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right) \tag{3.9}$$

where $F_Y^{[-1]}$ is the pseudo-inverse of F_Y. The distributions of F_X and F_Y can be specified either parametrically or non-parametrically. If we assume that F_Y is known only up to a location and scale parameter the quantile curve will have this form

$$Q_{Y|X}(\alpha|x) = \mu + \sigma F_Y^{[-1]}\left(\Phi\left(\rho\Phi^{-1}(F_X(x)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right) \tag{3.10}$$

When the margins are estimated non-parametrically we get a semiparametric copula (quantile) model. The properties of this estimator are established by Chen & Fan (2006). The authors also show that the semiparametric conditional quantile estimators are automatically monotonic across quantiles, a useful property for conditional value-at-risk models. In this work for margins of the returns and for realized volatility we employ the non-parametric empirical distribution F_j introduced by Genest *et al.* (1995), which consists of modeling the marginal distributions by the (rescaled) empirical distribution.

$$\hat{F}_j(z) = \frac{1}{T+1}\sum_{t=1}^{T} \mathbf{1}\{\hat{z}_{j,t} \leq z\}, \quad \hat{z}_{j,t} \in \{r_t, \sqrt{RV_t}\} \tag{3.11}$$

where $\mathbf{1}$ is the indicator function.

The bivariate t copula is expressed as

$$C^t_{\eta,\rho}(u,v) = \int_{-\infty}^{t_\eta^{-1}(u)} \int_{-\infty}^{t_\eta^{-1}(v)} \frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}} \left(1+\frac{r^2-2\rho rs+s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} drds$$

where $\Gamma(\cdot)$ is the Gamma distribution, ρ the linear correlation and η the degrees of freedom parameter. As can bee seen in the expression below, the partial derivative with respect to u is a bit more complicated

$$\alpha = t_{\eta+1}\left(\frac{t_\eta^{-1}(v)-\rho t_\eta^{-1}(u)}{\sqrt{\frac{\left(\eta+[t_\eta^{-1}(u)]^2\right)(1-\rho^2)}{\eta+1}}}\right) \tag{3.12}$$

Following the same steps as previously, the quantile curve implied by t copula is obtained by solving Equation 3.12 for v:

$$v = t_\eta\left[t_{\eta+1}^{-1}(\alpha)\sqrt{(\eta+1)^{-1}\left(1-\rho^2\right)\left(\eta+[t_\eta^{-1}(F_X(x))]^2\right)}+\rho t_\eta^{-1}(F_X(x))\right] \tag{3.13}$$

The relationship between x and the quantile of y is then

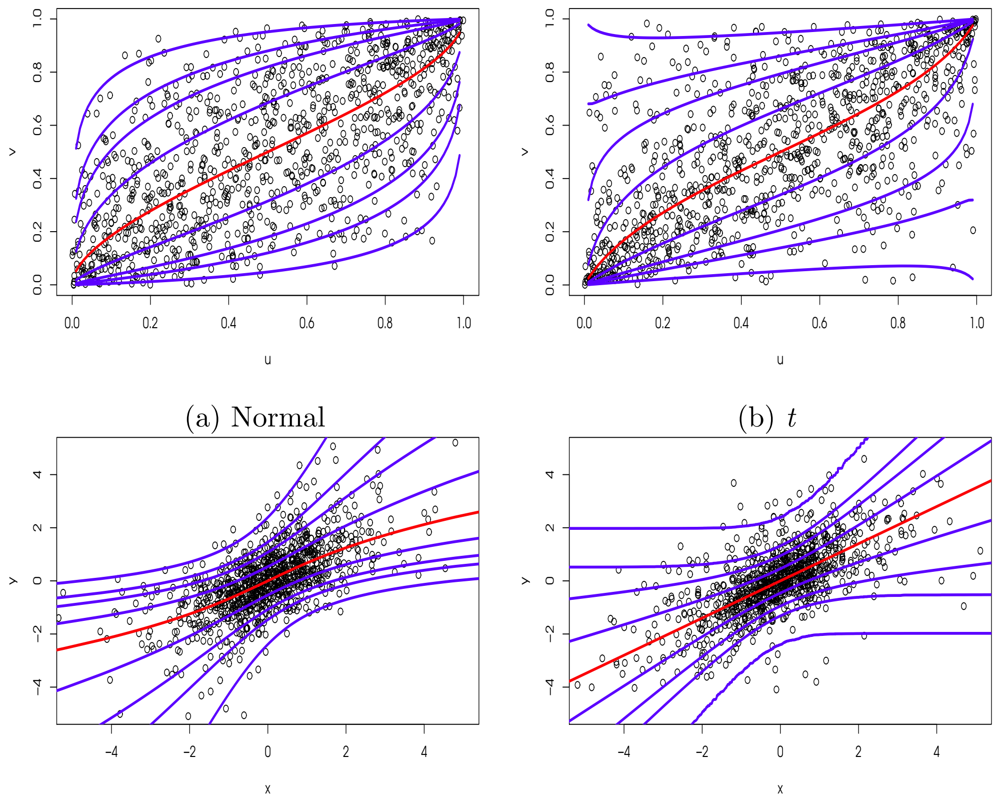
$$Q_{Y|X}(\alpha|x) = F_Y^{[-1]}\left(t_\eta\left[t_{\eta+1}^{-1}(\alpha)\sqrt{(\eta+1)^{-1}\left(1-\rho^2\right)\left(\eta+[t_\eta^{-1}(F_X(x))]^2\right)}+\rho t_\eta^{-1}(F_X(x))\right]\right) \tag{3.14}$$

Again, if we assume that F_Y is known only up to a location and scale parameter the quantile curve will have this form

$$Q_{Y|X}(\alpha|x) = \mu+\sigma F_Y^{[-1]}\left(t_\eta\left[t_{\eta+1}^{-1}(\alpha)\sqrt{(\eta+1)^{-1}\left(1-\rho^2\right)\left(\eta+[t_\eta^{-1}(F_X(x))]^2\right)} +\rho t_\eta^{-1}(F_X(x))\right]\right) \tag{3.15}$$

The theoretical quantile curves of models in Equations 3.8-3.9 and 3.13-3.14 are plotted in Figure 3.1. From this Figure we can see that for the same correlation parameter and same margins different copulas capture different type of dependence (c and d). We know that Normal copula does not have tail dependence for $\rho < 1$, while t copula has tail dependence which is symmetric and is estimated as $\tau^L = \tau^U = 2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$. The tail behaviour of these copulas is obvious if we focus in Figure 3.1 a and b.

Figure 3.1: Theoretical quantile curves of Normal and t copula both with correlation $\rho = 0.7$ and for the latter 3 degrees of freedom. The marginal distributions in c) and d) are $t_3(\cdot)$ and quantiles are $\alpha \in \{.01, .05, .25, .5, .75, .9, .95, .99\}$ in all cases.



(a) Normal (b) t

(c) Normal (d) t

3.3 The Data description

Using the proposed methodology, we study the conditional quantiles of the 21 most liquid U.S. stocks from the seven main market sectors defined in accordance with the Global Industry Classification Standard (GICS).[5] We use three stocks with the highest market capitalization in a sector as representative of the analyzed

[5]Morgan Stanley Capital International (MSCI) and Standard & Poor's developed the GICS. This is a common global classification standard used by global financial community.

Sector	Stocks
Financials	Bank of America Corporation (BAC), Citigroup (C), Wells Fargo & Company (WFC)
Information Technology	Apple (AAPL), Intel Corporation (INTC), Microsoft Corporation (MSFT)
Energy	Chevron Corporation (CVX), Schlumberger Limited (SLB), Exxon Mobil Corporation (XOM)
Consumer Discretionary	Amazon.com (AMZN), Walt Disney Company (DIS), McDonald's Corp. (MCD)
Consumer Staples	Coca-Cola Company (KO), Procter & Gamble Co. (PG), Wal-Mart Stores (WMT)
Telecommunication Services	Comcast Corporation (CMCSA), AT&T (T), Verizon Communications (VZ)
Health Care	Johnson & Johnson (JNJ), Merck & Co. (MRK), Pfizer (PFE)

Table 3.1: Sectors and representative stocks.

sector. The selected stocks account for approximately half of the total capitalization of the sector. The sectors and representative stocks are listed in Table 3.1. The data spans from August 2004 to December 2011. The period under study is very informative because it covers the recent U.S. recession of Dec. 2007 - June 2009 and three years before and after the crisis. The data were obtained from the Price-Data.com.[6]

For the computation of realized measures, we restrict the analysis to 5-minute returns during the 9:30 a.m. to 4:00 p.m. business hours of the New York Stock Exchange (NYSE). The data are time-synchronized by the same time-stamps. To rule out potential estimation bias which could come from low activity we eliminate transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2. Consequently, our data contains 1835 trading days.

In Table 3.2, we present descriptive statistics of the returns and realized volatility for the data that constitute our sample. All daily returns series have excess kurtosis and, as usual, the stocks from Financial sector on average have higher volatility than stocks from the other sectors.

[6] http://www.price-data.com/

3.4 Empirical Results

3.4.1 Full sample results

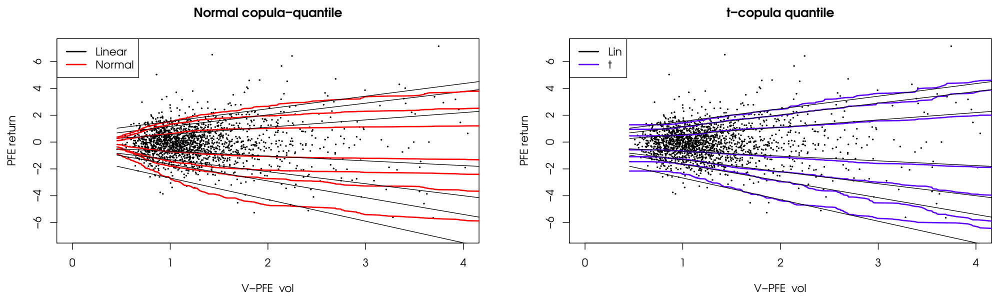
We quantile-regress the returns at time *t+1* conditional on the realized volatility at time t using the full sample of data. We solve the copula quantile regression problem as in Equation 3.5, where the quantile curve function $\mathbf{q}(\cdot)$ follows either Equation 3.10 in the case of Normal copula or Equation 3.15 in the case of t copula. For comparison we also estimate the linear quantile regression as in Equation 3.1. Given that the parameters of linear model are not directly comparable to the nonlinear ones we do not report them here, but they are available on request[7].

The parameters of copula quantile models for the full sample are summarized in Tables 3.3-4.10 in Appendix 3.B. Due to the amount of output results it is difficult to read in this form, thus we further synthesize these results using the boxplots. In Figures 4.1-4.2 in Appendix 3.C, we plot the parameters ρ, μ, σ for both models and the degrees of freedom η for t copula. Let us start with the analysis of correlation parameter ρ. The first thing we notice is that Normal copula on average estimates lower correlation than the t counterpart for all quantiles. In addition we see that the t copula estimates a greater scale of asymmetry when comparing lower ($\alpha = 5\%$) and upper ($\alpha = 95\%$) quantiles. Another important finding is that the distribution of returns is heavy tailed at all quantiles under analysis. This can be seen in Figure 4.2 where the mean of the degrees of freedom is around 5 for all quantiles. The degrees of freedom parameter allows the t copula to capture higher dependence, especially in the tails of distribution. Finally, the location-scale parameters (μ and σ) in general are different than 0 and 1 respectively. Thus, inclusion of these parameters in the copula quantile curves (Equation 3.10 and 3.15) is a good choice.

We comment in more details the results for one time series to save space, but they are similar for all the other 20. In Figure 3.2 we plot the fitted quantile curves for Pfizer using Normal and t copula, and for comparison include the fitted curves from linear quantile regression model. Positive (negative) slope curves correspond to positive (negative) correlation. We notice that there are nonlinear dependencies, especially for the lower quantiles, and lower quantiles seem to be driven by past volatility with larger extend showing asymmetry. The findings from this illustratory stock are uniform across all stocks considered, and support our nonlinear model for conditional quantiles fo returns.

[7] Reporting the parameters for the linear model requires adding 7 more tables to current results, and this would be too much. Besides, this would bring little value. To clarify the comparison of these models, we do it with respect to the VaR performance.

Figure 3.2: Semiparametric quantile regression of Pfizer returns r_{t+1} on its realized volatility $\sqrt{RV_t}$. For both models, linear and nonlinear, the regression is estimated at quantiles $\alpha \in \{.01, .05, .1, .25, .75, .9, .95\}$.



3.4.2 Out-of-sample results

Most of the times we are interested in utilizing the models to conduct predictions and not just to fit the data. For this reason we split the data to have 1335 observations for the in-sample set and leave 500 for the out-of-sample (OOS). This corresponds to dates from January 2010 to December 2012. We estimate the models as previously using the in-sample data and then use the rolling window data by shifting one-step-ahead [8]. This way we obtain the one-step-ahead forecast of the quantile of returns (or the VaR) for a window of 500, or a period of two years. Model parameters for the OOS are represented in Figures 3.5 and 3.6. When looking the parameters of the Normal copula we notice that the estimates have high variance and many outliers. The t counterpart on the other hand produces much stable correlation. This is because the Normal copula is not as flexible as t and thus cannot cope with big changes in dependence. In addition for most assets there is a slight correlation asymmetry when comparing lower and upper quantiles.

An interesting result is large heterogeneity of parameter estimates across industries. Financial industry tend to have largest negative correlation with increase in volatility driving the future lower quantiles the most when compared to other industries. Future quantiles of returns in Health and Cyclicals industries are much less sensitive to increases in volatility. On the opposite, upper quantiles of returns seem to be driven by volatility mostly in Consumer Cyclicals with highest correlation.

[8] For t copula we estimate the degrees of freedom only for the in-sample data and then assume it remains constant throughout the OSS. We could re-estimate it for every rolling window, but the model estimation is already time-hungry and it would increase further more if we do so.

Following the same approach as above we forecast the quantile returns five-step-ahead. As the estimated model parameters are not qualitatively different we do not report them here. However, we will use the obtained forecasts from one and five steps-ahead for comparing model accuracy in the following section. We plot the $VaR_{\alpha=5\%}$ forecasts for one and five steps-ahead in Figures 3.7-3.10 in appendix 3.C. We notice that all models (the linear, Normal and t copula) give similar patterns and capture the conditional quantiles well. In order to distinguish which one performs better we perform statistical testing in the next section.

3.5 Evaluation of quantile forecasts

We evaluate the *absolute* out-of-sample performance of the various conditional quantile models using test originally proposed by Engle & Manganelli (2004), who use the n-th order autoregression

$$I_t = \omega + \sum_{k=1}^{n} \beta_{1k} I_{t-k} + \sum_{k=1}^{n} \beta_{2k} q^{\alpha}_{t-k+1} + u_t, \tag{3.16}$$

where I_{t+1} is 1 if $y_{t+1} < q^{\alpha}_t$, and zero otherwise. While the hit sequence I_t is a binary sequence, u_t is assumed to follow a logistic distribution, and we can estimate it as a simple logit model and test whether $Pr(I_t = 1) = q^{\alpha}_t$. To obtain the p-values, we rely on simulations as suggested by Berkowitz *et al.* (2011), and we refer to this test as a DQ test in the results.

The main motivation of the DQ test is to determine whether the conditional quantiles are correctly dynamically specified; hence, it evaluates the absolute performance of the various models. This approach to evaluating absolute performance of quantile forecasts is only suitable for one-step-ahead forecasts and to the best of our knowledge, there is currently no alternative, reliable test for correct dynamic specification of multi-step conditional quantiles.

To assess the *relative* performance of the models, we evaluate the accuracy of the VaR forecasts statistically by defining the expected loss of the VaR forecast made by a forecaster m as

$$L_{\alpha,m} = E\left[\left(\alpha - 1\left\{y_{t,t+1} < q^{\alpha,m}_{t,t+1}\right\}\right)\left(y_{t,t+1} - q^{\alpha,m}_{t,t+1}\right)\right], \tag{3.17}$$

which was proposed by Giacomini & Komunjer (2005). The tick loss function penalizes quantile violations more heavily, and the penalization increases with the magnitude of the violation. As argued by Giacomini & Komunjer (2005), the tick loss is a natural loss function when evaluating conditional quantile forecasts. To compare the forecast accuracy of the two models, we test the null hypothesis that the expected losses for the models are equal, $H_0 : d = L_{\alpha,1} - L_{\alpha,2} = 0$, against a general alternative. The differences can be tested using Diebold & Mariano (2002) test statistics with Newey-West variance (in case of multi-step-ahead forecasts). Under the null of equal predictive accuracy the test statistics is distributed $N(0, 1)$.

The forecasting models performance for one and for five steps-ahead is summarized in Tables 3.7 - 3.20. We report the unconditional coverage $\hat{C}_\alpha$, the tick loss function $\hat{L}$, DQ test statistics and the simulated p-values. The results are quite mixed. For the one-step-ahead forecast the DQ test rejects the nonlinear quantile regression (NQR) models about 12.3% and 20.4% of times for Normal and t model respectively. While for LRQ the rejection rate is about 13% of times. Next we test the relative model performance, where the NQR models are compared to linear quantile regression (LQR) model. Sometimes the NQR models perform better (with t copula being the better model) sometimes it is the LQR which performs better. We should note that there are situations where the NQR models perform worse than LQR, and at the same time all these models are rejected by DQ test *e.g.* in Table 3.7 for AT&T and quantiles $\alpha = \{.1; .9\}$. In such situations we cannot say which model is the best as none of them passes the DQ.

As we cannot use the DQ test for the five-step-ahead forecast, the models are compared only based on the relative performance. The performance of the NQR models improves significantly, especially for the t copula. The *NQR-t* outperforms the LQR model in 13 cases or about 9% of the times, but there are many times where the tick loss function of NQR models is lower than the LQR counterpart, although this difference is not statistically significant. In conclusion, the models are well specified, and the NQR models seem to outperform LQR when five-step-ahead forecasts are considered.

3.6 Conclusion

This paper proposes to use the nonlinear quantile regression with realized measures of volatility to forecast conditional quantiles of financial assets returns. To make the results robust we apply this methodology on most liquid U.S. stocks in seven sectors. We argue that using the realized volatility under a copula quantile framework is useful, especially in the cases where the quantile dependence is nonlinear. The proposed models capture and forecast the dynamics of quantiles well.

Possible directions for further development would be to study the interdependence between asset returns or using copulas functions which allow for higher dependence in the tails.

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3.A Proofs

Probability distribution of y conditional on x

$$\alpha(y|x) = \frac{\partial C(u,v)}{\partial u}$$

Proof. From

$$P(Y \leq y, X = x) = \frac{\partial F_{X,Y}(x,y)}{\partial x}$$

it follows that

$$\begin{aligned}
P(Y \leq y, X = x) &= \lim_{\epsilon \to 0} Pr\{Y \leq y | x \leq X \leq x + \epsilon\} \\
&= \lim_{\epsilon \to 0} \frac{F(x+\epsilon, y) - F(x,y)}{F_X(x+\epsilon) - F_X(x)} \\
&\approx \frac{(\partial F(x,y)/\partial x) \cdot \epsilon}{f_X(x) \cdot \epsilon} \\
&= \frac{1}{f_X(x)} \frac{\partial C(F_X(x), F_Y(y))}{\partial x} \\
&= \underbrace{\frac{1}{f_X(x)}}_{\bigstar} \frac{\partial C(u,v)}{\partial u} \underbrace{\frac{\partial F_X(x)}{\partial x}}_{\bigstar} \\
&= \partial C(u,v)/\partial u
\end{aligned}$$

where $u = F_X(x)$, $v = F_Y(y)$ and $\bigstar$ terms cancel out. □

Following the same path it is easy to show that

$$\alpha(x|y) = \frac{\partial C(u,v)}{\partial v}$$

3.B Tables

Table 3.2: Descriptive Statistics for daily returns and realized volatility over the sample period extending from August 2004 to December 2011.

Returns

	Information Technology			Consumer Discretionary			Consumer Staples			Telecommunication Services		
	AAPL	INTC	MSFT	AMZN	DIS	MCD	KO	PG	WMT	CMCSA	T	VZ
Mean	−0.0003	−0.0001	−0.0001	0.0015	0.0009	0.0004	0.0001	0.0006	−0.0001	0.0002	−0.0001	−0.0004
Std dev	0.0201	0.0164	0.0140	0.0224	0.0156	0.0121	0.0106	0.0099	0.0108	0.0187	0.0132	0.0127
Skewness	−0.3097	0.0641	0.1483	0.3004	0.4682	0.2967	0.0474	−0.0580	0.4404	0.6274	0.5877	0.5984
Kurtosis	3.2914	3.3402	5.8121	4.4135	6.9769	6.0683	8.1274	6.6594	6.5979	18.2322	9.5964	8.3887
Minimum	−0.1223	−0.0907	−0.0755	−0.1313	−0.0909	−0.0799	−0.0717	−0.0660	−0.0653	−0.1416	−0.0629	−0.0760
Maximum	0.1123	0.0880	0.1102	0.1388	0.1185	0.1035	0.0795	0.0776	0.0762	0.2325	0.1242	0.1118

	Financials			Energy			Health Care		
	BAC	C	WFC	CVX	SLB	XOM	JNJ	MRK	PFE
Mean	−0.0023	−0.0042	−0.0002	0.0001	−0.0002	0.0005	0.0001	0.0000	−0.0006
Std dev	0.0327	0.0341	0.0272	0.0154	0.0215	0.0147	0.0092	0.0152	0.0133
Skewness	−0.4071	−1.7889	0.2458	0.0847	−0.4012	−0.0108	0.0305	−0.1710	0.1302
Kurtosis	13.6287	20.8588	13.0526	11.9675	5.4278	10.3940	9.6599	6.8905	3.2493
Minimum	−0.2509	−0.3468	−0.2081	−0.1296	−0.1552	−0.1261	−0.0803	−0.1092	−0.0696
Maximum	0.2014	0.1992	0.1933	0.1460	0.1253	0.1189	0.0728	0.0919	0.0714

Realized Volatility

	Information Technology			Consumer Discretionary			Consumer Staples			Telecommunication Services		
	AAPL	INTC	MSFT	AMZN	DIS	MCD	KO	PG	WMT	CMCSA	T	VZ
Mean	0.0004	0.0003	0.0002	0.0006	0.0003	0.0002	0.0001	0.0001	0.0002	0.0004	0.0002	0.0002
Std dev	0.0008	0.0005	0.0004	0.0009	0.0005	0.0004	0.0003	0.0004	0.0004	0.0007	0.0005	0.0005
Skewness	11.9764	10.7700	7.6919	7.6707	9.8775	25.3799	10.4895	26.0745	20.7189	12.7904	12.0541	15.4769
Kurtosis	209.4541	191.6730	90.6071	80.7537	151.3225	868.6006	175.7496	895.9773	625.6571	243.4969	242.6791	382.1494
Minimum	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	0.0192	0.0121	0.0070	0.0144	0.0117	0.0157	0.0063	0.0143	0.0126	0.0169	0.0142	0.0148

	Financials			Energy			Health Care		
	BAC	C	WFC	CVX	SLB	XOM	JNJ	MRK	PFE
Mean	0.0009	0.0011	0.0007	0.0003	0.0005	0.0002	0.0001	0.0003	0.0002
Std dev	0.0026	0.0040	0.0017	0.0007	0.0009	0.0007	0.0003	0.0007	0.0004
Skewness	7.9607	10.8488	5.7960	17.1272	8.3253	18.3961	18.5604	13.5083	8.8700
Kurtosis	95.3976	166.7985	45.1057	435.1365	111.8150	490.6232	486.2773	263.8237	125.7655
Minimum	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	0.0489	0.0866	0.0231	0.0207	0.0178	0.0205	0.0090	0.0165	0.0079

Table 3.3: Normal copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Information Technology			
AAPL							
ρ	−0.4192	−0.2492	−0.0484	−0.1355	0.1028	0.0997	0.2561
s.e.	0.0240	0.0168	0.0051	0.0135	0.0191	0.0130	0.0235
μ	0.8851	2.2089	11.5827	0.7282	−0.5176	−3.8602	−0.8675
s.e.	0.4495	0.5859	1.8553	0.3142	0.4138	1.1487	0.4923
σ	1.1128	1.6933	5.8489	1.7924	1.5613	2.7612	1.2622
s.e.	0.1185	0.2150	0.8206	0.3250	0.3969	0.4805	0.1905
INTC							
ρ	−0.2972	−0.2040	−0.1226	−0.0390	0.0734	0.1812	0.2744
s.e.	0.0269	0.0320	0.0164	0.0071	0.0112	0.0168	0.0215
μ	0.6850	1.8858	3.3092	2.5938	−1.9156	−1.5600	−0.9748
s.e.	0.6805	0.5415	0.8749	1.0543	0.6293	0.5261	0.4661
σ	1.0734	1.7545	2.8480	3.8788	3.1750	1.9038	1.3807
s.e.	0.1440	0.1807	0.4885	1.1854	0.7218	0.3305	0.1965
MSFT							
ρ	−0.7595	−0.2579	−0.0569	−0.1342	0.1545	0.2011	0.2017
s.e.	0.0251	0.0155	0.0044	0.0178	0.0341	0.0265	0.0265
μ	−1.2209	0.9211	7.5192	0.6845	−0.1010	−0.9822	−1.3536
s.e.	0.2120	0.3471	1.0359	0.3403	0.2332	0.3519	0.3923
σ	0.8562	1.4089	5.9617	2.1483	1.2656	1.8031	1.6200
s.e.	0.0729	0.1985	0.5751	0.5587	0.3559	0.2781	0.2159
				Consumer Discretionary			
AMZN							
ρ	−0.2694	−0.2585	−0.1037	−0.0334	0.0977	0.2232	0.1648
s.e.	0.0543	0.0527	0.0067	0.0083	0.0234	0.0258	0.0190
μ	−0.1056	0.7437	3.4476	3.0367	−0.6070	−0.6526	−2.2167
s.e.	1.1084	0.5344	0.9889	1.5362	0.7776	0.6692	1.2647
σ	0.9605	1.2113	2.6280	3.6883	1.4843	1.2764	1.6028
s.e.	0.1964	0.2211	0.4508	1.2825	0.6022	0.2754	0.3499
DIS							
ρ	−0.2787	−0.2327	−0.1119	−0.1056	0.1018	0.2100	0.1810
s.e.	0.0221	0.0208	0.0115	0.0210	0.0157	0.0196	0.0134
μ	1.3592	0.9793	2.6505	0.4693	−0.8733	−1.5648	−3.3255
s.e.	0.5004	0.4952	0.5794	0.3018	0.4327	0.5366	0.6732
σ	1.1837	1.4224	2.7667	1.7112	2.1379	2.0538	2.4848
s.e.	0.1453	0.2863	0.3863	0.4453	0.5433	0.3158	0.3106
MCD							
ρ	−0.3595	−0.2018	−0.0698	−0.1186	0.1036	0.0970	0.1036
s.e.	0.0362	0.0224	0.0065	0.0205	0.0145	0.0148	0.0133
μ	0.2461	0.8295	4.2030	0.4059	−0.5565	−3.6565	−5.6282
s.e.	0.5343	0.3514	0.6950	0.2597	0.2395	1.3061	1.3101
σ	0.9794	1.3645	4.1672	1.7746	1.9091	3.7642	4.1155
s.e.	0.1810	0.1674	0.5712	0.4554	0.3657	0.9429	0.7309

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Consumer Staples			
KO							
ρ	−0.2729	−0.1337	−0.0997	−0.0709	0.1546	0.2041	0.2325
s.e.	0.0263	0.0111	0.0086	0.0129	0.0242	0.0166	0.0253
μ	1.1698	2.3614	2.6328	1.0094	−0.0770	−0.6899	−0.5618
s.e.	0.4784	0.6034	0.4814	0.3983	0.1322	0.2844	0.2973
σ	1.3082	2.5148	3.5732	3.1343	1.2265	1.7312	1.3144
s.e.	0.2010	0.3928	0.4886	0.8448	0.2496	0.2906	0.1847
PG							
ρ	−0.1309	−0.2700	−0.0926	−0.1029	0.0414	0.1090	0.2346
s.e.	0.0131	0.0242	0.0139	0.0278	0.0068	0.0123	0.0187
μ	3.2351	0.2981	1.9504	0.2629	−1.6479	−1.7456	−0.6579
s.e.	0.6128	0.2460	0.5154	0.2671	0.6723	0.4308	0.2657
σ	2.0416	1.1824	3.0235	1.6642	4.1110	2.6666	1.4377
s.e.	0.2700	0.1776	0.5346	0.6290	1.3042	0.4108	0.2012
WMT							
ρ	−0.1964	−0.3678	−0.0256	−0.0937	0.1020	0.0813	0.1070
s.e.	0.0191	0.0350	0.0026	0.0201	0.0260	0.0140	0.0115
μ	2.5943	−0.2105	13.0420	0.5692	−0.2201	−2.2288	−2.3101
s.e.	0.5107	0.1969	2.2365	0.3187	0.3139	0.6571	0.7508
σ	1.9904	0.9250	12.1326	2.0629	1.4500	2.9281	2.3572
s.e.	0.2427	0.1302	2.1929	0.6035	0.5652	0.5675	0.5103
				Telecommunication Services			
CMCSA							
ρ	−0.3105	−0.1921	−0.1080	−0.1729	0.1438	0.2427	0.3109
s.e.	0.0228	0.0227	0.0079	0.0242	0.0230	0.0234	0.0182
μ	1.8578	1.7172	4.2584	0.3093	−0.5497	−0.8384	−1.1375
s.e.	0.9078	0.6805	0.5788	0.2480	0.2938	0.4384	0.3813
σ	1.2349	1.5301	3.2686	1.5494	1.6883	1.4516	1.4670
s.e.	0.2754	0.3008	0.2872	0.2861	0.3692	0.2654	0.1708
T							
ρ	−0.2591	−0.2733	−0.1050	−0.1036	0.1632	0.3246	0.2158
s.e.	0.0381	0.0203	0.0146	0.0133	0.0213	0.0291	0.0524
μ	0.1639	0.6642	2.3433	0.9848	−0.0898	−0.1594	−1.0177
s.e.	0.6202	0.3566	0.6612	0.2993	0.1661	0.1529	0.5982
σ	0.8710	1.3151	2.6534	2.5144	1.2236	1.2322	1.4858
s.e.	0.1415	0.2170	0.4990	0.4126	0.2963	0.1416	0.3050
VZ							
ρ	−0.2937	−0.2033	−0.0992	−0.0893	0.1711	0.1881	0.2437
s.e.	0.0223	0.0188	0.0087	0.0156	0.0345	0.0186	0.0203
μ	0.8489	1.4392	3.7811	0.9756	0.0193	−0.9259	−1.2942
s.e.	0.4171	0.4019	0.7558	0.3085	0.1840	0.3415	0.2769
σ	1.1397	1.8094	3.9593	2.4386	1.0820	1.8110	1.8410
s.e.	0.1080	0.2667	0.5583	0.4612	0.3257	0.2923	0.2034

Table 3.4: Normal copula parameters estimated on full sample.

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.25$	$\alpha=0.75$	$\alpha=0.9$	$\alpha=0.95$
	Financials						
BAC							
ρ	−0.4868	−0.3444	−0.2425	−0.0999	0.2628	0.3388	0.3423
s.e.	0.0061	0.0167	0.0168	0.0066	0.0220	0.0264	0.0102
μ	0.7617	1.8663	3.4042	3.3520	−0.0858	−0.6277	−0.9349
s.e.	0.2102	0.3313	0.3469	0.4831	0.0917	0.1825	0.1883
σ	0.8906	1.2574	2.3036	4.3851	1.6004	1.2589	1.1363
s.e.	0.0781	0.1085	0.1311	0.5194	0.1751	0.1294	0.1012
C							
ρ	−0.4895	−0.3460	−0.1966	−0.1601	0.0515	0.2331	0.3046
s.e.	0.0150	0.0133	0.0070	0.0099	0.0068	0.0100	0.0096
μ	0.7102	2.0002	4.2054	1.8556	−2.7925	−1.3930	−1.7823
s.e.	0.3575	0.3121	0.4023	0.1915	0.7074	0.2757	0.3216
σ	0.7683	1.1873	2.3887	2.7794	6.0522	1.8544	1.4905
s.e.	0.1329	0.1067	0.1672	0.1802	1.2717	0.1723	0.1698
WFC							
ρ	−0.4886	−0.3161	−0.1932	−0.1053	0.3026	0.3466	0.3636
s.e.	0.0076	0.0135	0.0138	0.0072	0.0221	0.0167	0.0163
μ	0.5597	1.1831	1.8645	2.5053	−0.0660	−0.8193	−0.8014
s.e.	0.2600	0.2492	0.3404	0.4548	0.0918	0.1982	0.2055
σ	0.9135	1.2044	1.6162	4.3726	1.4490	1.3816	1.1267
s.e.	0.1053	0.0934	0.1970	0.5856	0.1742	0.1264	0.0959
	Energy						
CVX							
ρ	−0.2513	−0.2777	−0.0544	−0.0345	0.1655	0.2104	0.2622
s.e.	0.0318	0.0406	0.0040	0.0058	0.0385	0.0277	0.0212
μ	1.5123	0.4442	7.4870	4.7164	−0.0400	−0.2253	−0.6279
s.e.	0.8756	0.5760	1.3551	1.2268	0.2670	0.4194	0.3868
σ	1.2666	1.1357	5.4868	6.8640	1.0973	1.1482	1.2345
s.e.	0.2379	0.2923	0.7863	1.6082	0.3037	0.3063	0.1778
SLB							
ρ	−0.2616	−0.1612	−0.0995	−0.0824	0.0675	0.1288	0.1807
s.e.	0.0248	0.0136	0.0104	0.0141	0.0121	0.0144	0.0273
μ	2.0458	5.0010	3.7794	1.8692	−1.5403	−2.7518	−2.2746
s.e.	1.5638	1.3703	1.0504	0.7962	0.7854	0.6999	1.0910
σ	1.2970	2.5418	2.4854	2.6743	2.3412	2.2109	1.6793
s.e.	0.3498	0.4570	0.4061	0.6078	0.6677	0.3148	0.3689
XOM							
ρ	−0.4074	−0.3307	−0.1732	−0.0593	0.1317	0.2553	0.2483
s.e.	0.0266	0.0268	0.0205	0.0071	0.0310	0.0214	0.0414
μ	0.0503	0.1780	1.5877	1.7277	−0.2466	−0.7734	−0.9190
s.e.	0.5704	0.2599	0.4562	0.4970	0.2849	0.3118	0.5356
σ	0.8699	1.1063	2.0306	3.6182	1.3728	1.4876	1.3950
s.e.	0.1463	0.1316	0.2906	0.6956	0.3556	0.2384	0.3065

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.25$	$\alpha=0.75$	$\alpha=0.9$	$\alpha=0.95$
	Health Care						
JNJ							
ρ	−0.2860	−0.2930	−0.0874	−0.0438	0.1197	0.1086	0.2421
s.e.	0.0232	0.0385	0.0104	0.0071	0.0216	0.0120	0.0283
μ	0.4794	0.2771	2.3369	1.5557	−0.2489	−2.1492	−0.5880
s.e.	0.4230	0.2547	0.4474	0.3996	0.1676	0.5298	0.1589
σ	1.0453	1.0976	3.5361	4.7527	1.6230	3.4297	1.3920
s.e.	0.2017	0.2035	0.5095	0.9805	0.4035	0.5798	0.1550
MRK							
ρ	−0.2636	−0.2881	−0.0900	−0.0711	0.1101	0.2039	0.2624
s.e.	0.0220	0.0245	0.0081	0.0129	0.0180	0.0214	0.0217
μ	0.0870	0.7326	4.8705	1.1868	−0.8037	−0.8053	−0.5106
s.e.	1.0122	0.2921	0.8442	0.3846	0.4215	0.3838	0.3778
σ	0.8995	1.3037	4.2971	2.8185	2.1461	1.5697	1.1964
s.e.	0.2103	0.1443	0.5836	0.5372	0.5515	0.2418	0.1930
PFE							
ρ	−0.2838	−0.2420	−0.0917	−0.0587	0.1399	0.1520	0.2799
s.e.	0.0364	0.0254	0.0102	0.0123	0.0321	0.0244	0.0280
μ	1.2264	0.9872	2.5471	1.7485	−0.0682	−1.4004	−0.4679
s.e.	0.6583	0.4155	0.6411	0.7179	0.1918	0.4820	0.4512
σ	1.3195	1.4851	2.6484	3.3538	1.2435	2.0611	1.2267
s.e.	0.1936	0.2239	0.3301	1.0037	0.3260	0.3315	0.2471

Table 3.5: t copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Information Technology			
AAPL							
ρ	−0.5804	−0.3131	−0.3387	−0.1358	0.0894	0.2282	0.3526
s.e.	0.0172	0.0359	0.0417	0.0219	0.0180	0.0467	0.0371
η	6.6774	5.7800	11.5568	11.4652	10.5836	8.6033	9.0145
s.e.	1.2411	1.2335	5.7976	4.1581	4.8837	2.7456	4.7557
μ	2.3386	1.2786	0.0570	0.9282	−1.1876	−0.3212	−0.2816
s.e.	1.1840	1.0215	0.3297	0.4317	0.7253	0.6661	0.5941
σ	1.7339	1.3426	1.1145	2.0146	2.1927	1.1751	1.1813
s.e.	0.3444	0.3403	0.1533	0.5146	0.7091	0.2806	0.2589
INTC							
ρ	−0.4708	−0.4129	−0.2182	−0.1655	0.1485	0.3291	0.1646
s.e.	0.0582	0.0452	0.0307	0.0356	0.0135	0.0321	0.0183
η	5.8243	5.3423	6.1218	7.0434	5.4422	5.7386	11.6837
s.e.	1.3717	1.1874	0.9210	3.0085	1.0542	0.8998	2.2810
μ	1.2479	0.1937	1.3395	0.1098	−0.5832	−0.2731	−4.1549
s.e.	1.0378	0.4102	0.6445	0.3641	0.3415	0.2657	1.6261
σ	1.3257	1.1822	1.7664	1.2118	1.7127	1.1788	2.7262
s.e.	0.2613	0.1796	0.3321	0.4540	0.4324	0.1759	0.7665
MSFT							
ρ	−0.5591	−0.2428	−0.1837	−0.1527	0.1831	0.2154	0.2652
s.e.	0.0393	0.0299	0.0201	0.0199	0.0517	0.0271	0.0271
η	4.6579	10.8673	7.5900	5.2910	4.8818	6.1469	9.1475
s.e.	0.7372	2.2845	1.3481	1.0330	1.9430	0.8540	1.8320
μ	0.5581	2.0153	2.3028	0.7239	−0.0950	−1.1138	−1.4483
s.e.	0.3710	0.5710	0.5226	0.3685	0.3125	0.5330	0.4295
σ	1.2301	1.9450	2.5617	2.2151	1.2905	1.7911	1.7274
s.e.	0.1225	0.3112	0.3576	0.5530	0.4712	0.3865	0.2446
				Consumer Discretionary			
AMZN							
ρ	−0.4796	−0.3080	−0.2163	−0.3043	0.2008	0.1127	0.3241
s.e.	0.1503	0.0395	0.0217	0.0982	0.0388	0.0200	0.0750
η	6.0556	4.8652	7.0609	2.5911	3.4113	7.1842	3.9460
s.e.	2.5071	0.7991	1.1240	1.5346	0.8171	0.8941	0.9188
μ	−0.1944	0.0848	1.2207	−0.6695	0.5020	−3.9749	−0.3052
s.e.	1.5367	0.5275	0.7367	0.2264	0.3315	1.0481	0.7845
σ	1.0414	1.0937	1.6166	0.4778	0.6559	2.5226	1.1492
s.e.	0.3399	0.1686	0.3973	0.2210	0.2614	0.4708	0.2334
DIS							
ρ	−0.4943	−0.4226	−0.3291	−0.2258	0.1584	0.2590	0.3281
s.e.	0.0362	0.0382	0.0347	0.0349	0.0228	0.0172	0.0288
η	4.5172	6.0331	6.0108	4.1368	4.5574	5.4376	6.1454
s.e.	0.7226	1.4577	1.0880	1.4413	0.9130	0.4361	0.8521
μ	0.1952	0.2142	0.4637	−0.0867	−0.6636	−1.2645	−1.0162
s.e.	0.4222	0.2931	0.2720	0.1921	0.3311	0.3219	0.3247
σ	1.0391	1.1892	1.4317	0.9955	1.9371	1.8339	1.4776
s.e.	0.1420	0.1768	0.1994	0.3233	0.4173	0.2402	0.1598
MCD							
ρ	−0.5496	−0.3002	−0.4164	−0.1966	0.2696	0.2889	0.2647
s.e.	0.0344	0.0172	0.0324	0.0295	0.0383	0.0500	0.0505
η	4.3626	5.0649	3.3268	4.3773	4.8413	8.6828	6.7880
s.e.	0.8542	0.4058	0.8640	0.9066	1.7976	3.4896	2.9584
μ	0.6235	1.3054	−0.0039	0.0651	0.2324	−0.5320	−0.5066
s.e.	0.4750	0.5151	0.1735	0.1711	0.1214	0.4064	0.5084
σ	1.2913	1.7860	1.1123	1.2389	0.7805	1.4883	1.2736
s.e.	0.2084	0.2957	0.1511	0.3382	0.1763	0.2884	0.3210

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Consumer Staples			
KO							
ρ	−0.5795	−0.2252	−0.2285	−0.1944	0.2150	0.3011	0.2563
s.e.	0.0261	0.0365	0.0244	0.0280	0.0540	0.0418	0.0241
η	4.0095	9.3059	3.5121	3.5392	3.8313	3.7569	9.5709
s.e.	0.2998	2.2861	0.4314	0.6385	1.9446	0.7785	1.8536
μ	0.1690	1.3979	0.4014	0.1592	0.0813	−0.0384	−0.9893
s.e.	0.2766	0.6215	0.1889	0.1775	0.1505	0.2046	0.4660
σ	1.1369	1.9089	1.3148	1.4665	0.9251	1.0065	1.6789
s.e.	0.1426	0.4320	0.1729	0.4053	0.3142	0.2000	0.2893
PG							
ρ	−0.5594	−0.1980	−0.1352	−0.1749	0.1926	0.2684	0.2748
s.e.	0.0405	0.0278	0.0303	0.0357	0.0366	0.0264	0.0268
η	3.8021	4.5034	3.6834	4.3791	4.5274	6.2114	6.4472
s.e.	0.4669	0.4228	0.4347	1.2453	1.2086	0.9910	0.7811
μ	0.4097	0.7931	0.5073	0.0660	−0.0836	−0.3288	−0.8643
s.e.	0.5241	0.3566	0.2749	0.1787	0.2133	0.1896	0.3559
σ	1.2550	1.4937	1.4611	1.2667	1.2388	1.3840	1.5831
s.e.	0.2858	0.3143	0.2409	0.5102	0.4470	0.1934	0.2445
WMT							
ρ	−0.3454	−0.2789	−0.1799	−0.2292	0.1576	0.2125	0.4096
s.e.	0.0586	0.0308	0.0210	0.0384	0.0217	0.0534	0.0547
η	10.4060	4.6786	9.6347	3.9731	4.0116	4.5453	5.0504
s.e.	3.6974	0.7503	1.4443	0.9489	0.5015	1.2663	1.0577
μ	0.4189	1.1465	0.9995	−0.0918	−0.1027	−0.2948	−0.1141
s.e.	0.2554	0.3757	0.5202	0.1444	0.1902	0.2834	0.4215
σ	1.0870	1.7373	1.9231	0.9666	1.2621	1.2755	1.1677
s.e.	0.0611	0.2410	0.4119	0.2657	0.4126	0.2721	0.3079
				Telecommunication Services			
CMCSA							
ρ	−0.5461	−0.2066	−0.2780	−0.1614	0.1750	0.2956	0.3668
s.e.	0.0258	0.0389	0.0244	0.0284	0.0123	0.0572	0.0389
η	4.6493	4.9735	6.8151	6.8463	4.0055	4.9505	7.8370
s.e.	0.7523	0.9807	0.8554	2.1226	0.4684	1.2178	1.6153
μ	1.9627	1.9203	1.1801	0.6054	−0.4012	−0.9208	−0.7611
s.e.	0.8121	0.8389	0.4872	0.2883	0.2827	0.3810	0.6020
σ	1.5697	1.6502	1.7229	1.8914	1.5462	1.5165	1.3317
s.e.	0.2215	0.3790	0.2825	0.3758	0.3496	0.1988	0.2281
T							
ρ	−0.4508	−0.2946	−0.3363	−0.3411	0.1949	0.2754	0.3073
s.e.	0.0563	0.0334	0.0289	0.0420	0.0378	0.0248	0.0629
η	5.4241	4.3334	4.7620	2.9551	4.1061	6.3098	4.1813
s.e.	1.3170	0.6218	0.7192	1.2187	1.2236	0.8341	0.7706
μ	−0.0997	1.0201	0.4209	−0.2474	−0.1983	−0.9139	−0.3624
s.e.	0.7727	0.4704	0.2323	0.1102	0.2798	0.3684	0.4249
σ	0.9547	1.4794	1.3353	0.7925	1.4362	1.7879	1.2070
s.e.	0.2602	0.2623	0.1907	0.1797	0.5132	0.3250	0.2674
VZ							
ρ	−0.5132	−0.1755	−0.3972	−0.2619	0.1579	0.2386	0.2478
s.e.	0.0289	0.0165	0.0233	0.0392	0.0497	0.0394	0.0280
η	8.1971	9.7979	4.4163	3.3925	4.0282	4.9915	7.1829
s.e.	1.9451	1.3541	0.6047	0.9019	1.2565	0.8618	1.4420
μ	0.0980	2.6030	0.2302	−0.1307	−0.1393	−0.8441	−1.5241
s.e.	0.3473	0.4715	0.2004	0.1407	0.2544	0.4956	0.6687
σ	1.0327	2.3608	1.2026	0.8955	1.3615	1.6986	1.8836
s.e.	0.1225	0.2603	0.1679	0.2260	0.4347	0.3919	0.3696

Table 3.6: t copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Financials			
BAC							
ρ	−0.7107	−0.3966	−0.1460	−0.2007	0.3494	0.3834	0.4048
s.e.	0.0151	0.0154	0.0105	0.0107	0.0299	0.0211	0.0271
η	4.2281	4.7458	14.5765	5.3023	2.7614	4.1084	3.8952
s.e.	0.6041	0.2889	1.6010	0.7546	0.6045	0.2826	0.3100
μ	2.3280	2.8346	8.8153	1.5878	0.1623	−0.9097	−1.4861
s.e.	0.4867	0.4255	2.0859	0.2757	0.1125	0.2596	0.6016
σ	1.3622	1.5638	4.3663	2.8095	1.2901	1.4070	1.3214
s.e.	0.1239	0.1515	0.7548	0.3056	0.2387	0.1522	0.2539
C							
ρ	−0.7500	−0.3764	−0.2375	−0.1930	0.0703	0.2410	0.2423
s.e.	0.0132	0.0104	0.0139	0.0176	0.0075	0.0086	0.0109
η	4.6460	4.7496	6.6016	5.6321	21.3892	8.6820	6.2973
s.e.	0.9407	0.2560	0.6653	1.1260	5.9556	0.6977	0.3735
μ	0.7931	2.3899	5.0919	2.0987	−2.6378	−2.8239	−4.2840
s.e.	0.5119	0.4821	0.5169	0.3488	0.7469	0.5065	0.5148
σ	1.1185	1.3041	2.6481	3.1173	5.8228	2.6069	2.1167
s.e.	0.1510	0.1058	0.1668	0.3166	1.3700	0.3476	0.2170
WFC							
ρ	−0.7237	−0.3890	−0.3852	−0.1523	0.2246	0.4592	0.3117
s.e.	0.0151	0.0216	0.0193	0.0119	0.0183	0.0247	0.0110
η	3.2454	5.7105	4.8624	7.8185	4.7135	4.9862	10.0667
s.e.	0.1915	0.4660	0.4402	1.3338	0.8602	0.7106	0.9117
μ	1.7729	2.7371	1.3298	1.7176	−0.5965	−0.3585	−2.3631
s.e.	0.9491	0.3452	0.2674	0.4000	0.2180	0.2232	0.3094
σ	1.4603	1.6747	1.7288	3.3937	2.1775	1.3728	1.6724
s.e.	0.2459	0.1550	0.1615	0.5409	0.3405	0.1429	0.0966
				Energy			
CVX							
ρ	−0.3964	−0.2060	−0.2179	−0.1323	0.2086	0.2535	0.2564
s.e.	0.0519	0.0178	0.0308	0.0222	0.0314	0.0234	0.0597
η	7.0492	9.8606	6.9669	7.6642	3.2236	5.8128	4.2452
s.e.	1.5099	2.0528	1.3979	3.6803	0.9491	0.8896	1.1798
μ	0.4968	4.1416	1.7506	0.5272	0.1618	−1.2527	−0.5512
s.e.	0.3356	1.0621	0.5391	0.4039	0.1959	0.4347	0.6086
σ	1.0952	2.8731	2.1675	1.7304	0.8830	1.9090	1.2421
s.e.	0.1230	0.4580	0.3423	0.5086	0.2822	0.2540	0.2714
SLB							
ρ	−0.4600	−0.4125	−0.2225	−0.1954	0.0963	0.2455	0.3779
s.e.	0.0188	0.0502	0.0309	0.0199	0.0270	0.0296	0.0390
η	5.8404	5.2385	4.6543	5.8327	8.2368	9.0462	9.6765
s.e.	0.6178	1.0816	0.8355	1.4554	2.8531	2.7833	2.9036
μ	0.8860	0.8302	0.4864	0.1246	−0.9879	−0.4725	0.1159
s.e.	1.6930	0.9226	0.5877	0.2804	0.6466	0.3744	0.5186
σ	1.1953	1.2879	1.1697	1.1765	1.9037	1.2478	1.0169
s.e.	0.3461	0.2941	0.2659	0.2258	0.5613	0.1783	0.1689
XOM							
ρ	−0.5552	−0.2590	−0.1863	−0.2068	0.0862	0.2331	0.2317
s.e.	0.0269	0.0347	0.0151	0.0424	0.0232	0.0312	0.0266
η	8.6360	9.7856	10.6215	6.0903	9.2878	6.9456	7.5758
s.e.	3.4628	2.0814	1.6838	3.1400	3.8598	1.4240	1.1270
μ	−0.0601	1.6134	1.5151	−0.0466	−1.5014	−1.1543	−1.9522
s.e.	0.3874	0.8850	0.4506	0.1876	1.0059	0.5337	0.6825
σ	1.0334	1.7218	2.0155	1.0950	2.9148	1.8483	1.9312
s.e.	0.1153	0.4102	0.2781	0.2610	1.2834	0.3588	0.3664

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 0.95$
				Health Care			
JNJ							
ρ	−0.4839	−0.3264	−0.1481	−0.1264	0.1735	0.1491	0.2299
s.e.	0.0901	0.0278	0.0247	0.0247	0.0330	0.0257	0.0172
η	4.0045	6.9429	10.3607	7.7440	4.0989	13.2224	9.4232
s.e.	1.0910	1.3030	2.9106	2.0818	1.1231	2.5597	1.0914
μ	0.1891	0.7596	1.6713	0.4911	−0.1006	−1.6187	−0.8853
s.e.	0.5786	0.3104	0.4448	0.1853	0.1714	0.5727	0.3380
σ	1.0661	1.6554	2.7863	2.3226	1.3188	2.8153	1.6551
s.e.	0.3647	0.2628	0.4739	0.4910	0.4154	0.6893	0.2477
MRK							
ρ	−0.6880	−0.4346	−0.1939	−0.1368	0.3123	0.2658	0.1832
s.e.	0.0692	0.0330	0.0435	0.0186	0.0402	0.0209	0.0181
η	8.8962	5.2899	3.7655	7.9431	2.9654	8.7251	7.4106
s.e.	5.6144	0.7939	0.6427	2.0555	0.6262	1.4986	0.7360
μ	−1.0626	0.2106	0.5983	0.4698	0.3617	−0.3748	−1.8161
s.e.	0.5869	0.2687	0.3871	0.2872	0.1190	0.3136	0.8776
σ	0.8391	1.2278	1.3236	1.8353	0.6789	1.3011	1.7450
s.e.	0.1601	0.1483	0.2538	0.4567	0.1614	0.2031	0.3955
PFE							
ρ	−0.4126	−0.2895	−0.3261	−0.2546	0.1413	0.1873	0.1728
s.e.	0.0966	0.0343	0.0291	0.0654	0.0248	0.0233	0.0235
η	8.1190	7.1473	5.2902	3.6636	5.0498	9.2755	12.3260
s.e.	3.3749	2.0699	0.9652	1.6838	1.2475	1.6768	2.3873
μ	0.6948	1.5051	0.1298	−0.2361	−0.2094	−1.2176	−1.6054
s.e.	0.5383	0.5127	0.3543	0.1594	0.2702	0.4662	0.6939
σ	1.1907	1.7854	1.1855	0.7657	1.4389	1.9065	1.7702
s.e.	0.1529	0.2685	0.2412	0.2627	0.3956	0.3454	0.4376

Table 3.7: **Telecommunication Services**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

CMCSA	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.040	0.076	0.210	0.752	0.908	0.972
$\hat{L}$	**0.041**	0.158	0.263	0.465	0.453	**0.257**	0.153
DQ	2.165	9.015	5.525	16.927	4.456	4.508	10.126
p-val	0.826	0.108	0.355	0.005	0.486	0.479	0.072
NQR–t							
$\hat{C}_\alpha$	0.010	0.050	0.082	0.222	0.746	0.910	0.954
$\hat{L}$	0.040	0.156	0.258	0.460	0.453	0.252	<u>0.151</u>
DQ	0.521	5.954	13.258	9.908	3.836	5.271	7.200
p-val	0.991	0.311	0.021	0.078	0.573	0.384	0.206
LQR							
$\hat{C}_\alpha$	0.006	0.038	0.084	0.224	0.744	0.908	0.958
$\hat{L}$	0.039	0.156	0.259	0.461	0.452	0.251	0.152
DQ	0.625	4.904	11.263	9.165	4.509	5.249	6.382
p-val	0.987	0.428	0.046	0.103	0.479	0.386	0.271
T	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.042	0.084	0.212	0.774	0.924	0.960
$\hat{L}$	0.030	**0.095**	**0.156**	0.260	0.252	**0.146**	0.087
DQ	2.165	9.606	11.758	10.288	5.524	16.864	12.762
p-val	0.826	0.087	0.038	0.067	0.355	0.005	0.026
NQR–t							
$\hat{C}_\alpha$	0.004	0.034	0.076	0.214	0.788	0.936	0.966
$\hat{L}$	**0.030**	0.093	0.150	0.257	**0.254**	**0.144**	<u>0.085</u>
DQ	2.165	8.537	13.035	13.799	12.260	16.804	11.533
p-val	0.826	0.129	0.023	0.017	0.031	0.005	0.042
LQR							
$\hat{C}_\alpha$	0.004	0.030	0.070	0.216	0.780	0.928	0.976
$\hat{L}$	0.029	0.092	0.149	0.256	0.252	0.141	0.087
DQ	2.165	10.395	12.353	10.214	9.084	14.522	20.357
p-val	0.826	0.065	0.030	0.069	0.106	0.013	0.001
VZ	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.006	0.042	0.072	0.212	0.754	0.914	0.958
$\hat{L}$	0.031	**0.102**	**0.165**	0.279	0.270	**0.155**	**0.100**
DQ	0.677	5.986	10.494	13.396	3.761	9.754	10.353
p-val	0.984	0.308	0.062	0.020	0.584	0.083	0.066
NQR–t							
$\hat{C}_\alpha$	0.006	0.044	0.070	0.220	0.726	0.906	0.962
$\hat{L}$	0.028	0.098	0.160	0.276	0.272	0.152	0.093
DQ	0.625	4.592	9.062	15.274	3.859	5.512	6.853
p-val	0.987	0.468	0.107	0.009	0.570	0.357	0.232
LQR							
$\hat{C}_\alpha$	0.002	0.040	0.070	0.212	0.724	0.910	0.964
$\hat{L}$	0.029	0.099	0.160	0.275	0.271	0.151	0.093
DQ	4.700	6.736	10.288	17.995	4.711	6.614	12.255
p-val	0.454	0.241	0.067	0.003	0.452	0.251	0.031

Table 3.8: **Consumer Discretionary**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

AMZN	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.012	0.058	0.110	0.246	0.774	0.904	0.952
$\hat{L}$	**0.061**	0.193	0.319	0.564	0.577	**0.347**	0.222
DQ	0.966	1.659	4.616	3.281	6.409	10.748	7.888
p-val	0.965	0.894	0.465	0.657	0.268	0.057	0.162
NQR–t							
$\hat{C}_\alpha$	0.014	0.056	0.110	0.252	0.770	0.914	0.960
$\hat{L}$	0.057	0.193	0.315	0.564	0.574	0.343	0.219
DQ	1.794	2.370	1.090	3.630	10.132	4.090	6.057
p-val	0.877	0.796	0.955	0.604	0.072	0.537	0.301
LQR							
$\hat{C}_\alpha$	0.010	0.060	0.102	0.256	0.762	0.910	0.960
$\hat{L}$	0.056	0.193	0.316	0.564	0.574	0.342	0.218
DQ	0.521	2.948	2.591	1.829	8.228	9.470	7.270
p-val	0.991	0.708	0.763	0.872	0.144	0.092	0.201

DIS	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.054	0.094	0.242	0.750	0.908	0.972
$\hat{L}$	0.034	0.128	0.219	0.386	0.379	**0.214**	**0.128**
DQ	2.165	2.470	3.303	8.491	2.848	9.416	10.126
p-val	0.826	0.781	0.653	0.131	0.723	0.094	0.072
NQR–t							
$\hat{C}_\alpha$	0.008	0.066	0.098	0.262	0.742	0.912	0.968
$\hat{L}$	0.033	**0.134**	0.220	**0.394**	0.380	<u>0.203</u>	<u>0.121</u>
DQ	0.528	9.579	2.763	7.904	5.275	9.345	8.231
p-val	0.991	0.088	0.737	0.162	0.383	0.096	0.144
LQR							
$\hat{C}_\alpha$	0.008	0.056	0.100	0.258	0.742	0.918	0.970
$\hat{L}$	0.032	0.131	0.221	0.391	0.378	0.206	0.124
DQ	0.528	6.652	3.406	8.984	2.772	7.556	9.004
p-val	0.991	0.248	0.638	0.110	0.735	0.182	0.109

MCD	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.010	0.044	0.090	0.230	0.784	0.920	0.956
$\hat{L}$	0.026	0.088	**0.149**	0.250	<u>0.245</u>	0.141	0.088
DQ	0.457	3.343	4.209	7.422	15.651	10.812	4.461
p-val	0.994	0.647	0.520	0.191	0.008	0.055	0.485
NQR–t							
$\hat{C}_\alpha$	0.004	0.028	0.074	0.212	0.798	0.926	0.962
$\hat{L}$	0.025	**0.092**	**0.148**	0.250	0.249	**0.144**	**0.090**
DQ	2.165	9.724	9.260	7.631	16.153	16.112	4.894
p-val	0.826	0.083	0.099	0.178	0.006	0.007	0.429
LQR							
$\hat{C}_\alpha$	0.006	0.032	0.084	0.208	0.798	0.926	0.960
$\hat{L}$	0.025	0.088	0.143	0.248	0.248	0.141	0.087
DQ	0.677	7.688	4.005	7.783	22.959	13.607	4.436
p-val	0.984	0.174	0.549	0.169	0.000	0.018	0.489

Table 3.9: **Consumer Staples**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

KO	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.016	0.054	0.080	0.226	0.730	0.886	0.934
$\hat{L}$	0.027	0.096	0.152	0.251	0.242	**0.146**	**0.090**
DQ	5.273	3.793	4.392	5.079	3.978	7.229	10.945
p-val	0.383	0.580	0.494	0.406	0.553	0.204	0.052
NQR–t							
$\hat{C}_\alpha$	0.014	0.046	0.082	0.234	0.732	0.900	0.954
$\hat{L}$	0.024	0.093	0.147	0.252	0.240	0.143	0.088
DQ	1.794	7.628	7.248	4.474	2.975	3.694	7.425
p-val	0.877	0.178	0.203	0.483	0.704	0.594	0.191
LQR							
$\hat{C}_\alpha$	0.016	0.048	0.084	0.238	0.730	0.890	0.956
$\hat{L}$	0.025	0.094	0.149	0.250	0.241	0.141	0.086
DQ	5.273	5.003	7.528	3.786	4.481	2.999	8.419
p-val	0.383	0.416	0.184	0.581	0.482	0.700	0.135
PG	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.010	0.054	0.108	0.260	0.740	0.892	0.944
$\hat{L}$	**0.021**	0.078	**0.127**	0.219	0.218	**0.129**	**0.081**
DQ	4.783	5.366	3.762	4.179	2.178	1.493	7.563
p-val	0.443	0.373	0.584	0.524	0.824	0.914	0.182
NQR–t							
$\hat{C}_\alpha$	0.004	0.040	0.098	0.258	0.750	0.906	0.966
$\hat{L}$	0.021	**0.078**	0.124	0.218	0.218	0.123	**0.078**
DQ	2.165	6.561	3.944	1.120	5.113	8.247	9.596
p-val	0.826	0.255	0.557	0.952	0.402	0.143	0.088
LQR							
$\hat{C}_\alpha$	0.006	0.048	0.096	0.266	0.740	0.904	0.956
$\hat{L}$	0.020	0.076	0.124	0.217	0.218	0.124	0.075
DQ	0.625	6.069	6.592	3.022	2.929	2.770	5.494
p-val	0.987	0.300	0.253	0.697	0.711	0.735	0.359
WMT	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.008	0.042	0.066	0.198	0.740	0.892	0.948
$\hat{L}$	0.023	0.088	**0.146**	0.238	**0.237**	**0.142**	**0.085**
DQ	0.528	6.672	10.904	9.091	4.147	13.113	12.194
p-val	0.991	0.246	0.053	0.106	0.528	0.022	0.032
NQR–t							
$\hat{C}_\alpha$	0.012	0.040	0.060	0.190	0.748	0.908	0.958
$\hat{L}$	0.024	**0.090**	**0.144**	0.240	0.234	0.136	0.083
DQ	0.966	13.278	22.010	13.123	3.343	13.708	11.415
p-val	0.965	0.021	0.001	0.022	0.647	0.018	0.044
LQR							
$\hat{C}_\alpha$	0.014	0.040	0.062	0.194	0.736	0.916	0.960
$\hat{L}$	0.023	0.088	0.142	0.239	0.233	0.135	0.083
DQ	1.794	10.615	24.678	10.871	5.367	17.762	11.334
p-val	0.877	0.060	0.000	0.054	0.373	0.003	0.045

Table 3.10: **Energy**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

CVX	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.008	0.048	0.086	0.218	0.742	0.920	0.942
$\hat{L}$	0.032	0.120	0.200	0.354	0.328	0.182	**0.114**
DQ	0.528	3.097	5.103	7.204	6.358	4.948	5.070
p-val	0.991	0.685	0.403	0.206	0.273	0.422	0.407
NQR–t							
$\hat{C}_\alpha$	0.006	0.030	0.086	0.214	0.762	0.936	0.960
$\hat{L}$	0.033	0.122	0.198	0.357	0.327	**0.187**	**0.113**
DQ	0.625	8.748	3.705	8.766	9.360	13.662	6.086
p-val	0.987	0.120	0.593	0.119	0.096	0.018	0.298
LQR							
$\hat{C}_\alpha$	0.006	0.034	0.080	0.210	0.752	0.930	0.966
$\hat{L}$	0.031	0.120	0.197	0.357	0.328	0.181	0.109
DQ	0.625	7.137	4.526	6.363	8.908	11.580	6.641
p-val	0.987	0.211	0.476	0.272	0.113	0.041	0.249
SLB	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.048	0.104	0.246	0.778	0.910	0.960
$\hat{L}$	0.055	0.189	0.317	0.555	0.537	0.297	0.173
DQ	2.165	3.090	1.714	6.479	7.796	2.743	7.098
p-val	0.826	0.686	0.887	0.262	0.168	0.740	0.213
NQR–t							
$\hat{C}_\alpha$	0.006	0.050	0.104	0.246	0.774	0.918	0.968
$\hat{L}$	0.052	**0.191**	0.312	0.552	0.537	0.297	**0.175**
DQ	0.625	2.958	5.302	9.207	9.767	7.251	10.555
p-val	0.987	0.707	0.380	0.101	0.082	0.203	0.061
LQR							
$\hat{C}_\alpha$	0.004	0.042	0.104	0.252	0.774	0.912	0.968
$\hat{L}$	0.051	0.187	0.315	0.552	0.536	0.296	0.174
DQ	2.165	2.551	5.436	9.337	9.156	7.249	10.555
p-val	0.826	0.769	0.365	0.096	0.103	0.203	0.061
XOM	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.006	0.042	0.080	0.218	0.754	0.896	0.918
$\hat{L}$	0.030	0.112	0.185	0.316	0.304	0.171	**0.109**
DQ	0.625	6.284	4.644	8.302	4.540	2.795	17.536
p-val	0.987	0.280	0.461	0.140	0.475	0.732	0.004
NQR–t							
$\hat{C}_\alpha$	0.004	0.034	0.066	0.202	0.772	0.926	0.970
$\hat{L}$	**0.033**	**0.119**	0.185	**0.321**	**0.310**	0.170	**0.107**
DQ	2.165	6.967	11.610	17.769	2.698	6.640	8.944
p-val	0.826	0.223	0.041	0.003	0.746	0.249	0.111
LQR							
$\hat{C}_\alpha$	0.004	0.036	0.056	0.204	0.752	0.924	0.960
$\hat{L}$	0.030	0.116	0.184	0.318	0.305	0.168	0.103
DQ	2.165	6.486	17.044	16.395	3.983	10.348	6.086
p-val	0.826	0.262	0.004	0.006	0.552	0.066	0.298

Table 3.11: **Financials**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

BAC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.034	0.092	0.224	0.798	0.916	0.964
$\hat{L}$	0.079	**0.277**	0.410	0.693	0.682	0.386	**0.239**
DQ	4.700	7.862	9.525	18.985	10.441	4.778	7.500
p-val	0.454	0.164	0.090	0.002	0.064	0.444	0.186
NQR–t							
$\hat{C}_\alpha$	0.012	0.074	0.130	0.264	0.762	0.884	0.928
$\hat{L}$	0.072	**0.258**	0.412	0.689	0.669	0.381	0.230
DQ	0.966	8.162	11.953	14.831	6.278	6.485	6.002
p-val	0.965	0.148	0.035	0.011	0.280	0.262	0.306
LQR							
$\hat{C}_\alpha$	0.012	0.054	0.116	0.302	0.776	0.908	0.940
$\hat{L}$	0.074	0.247	0.408	0.695	0.672	0.384	0.230
DQ	4.729	4.113	7.873	18.698	9.527	2.779	9.004
p-val	0.450	0.533	0.163	0.002	0.090	0.734	0.109
C	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.038	0.080	0.188	0.760	0.918	0.968
$\hat{L}$	**0.074**	**0.240**	0.385	0.661	0.646	0.367	0.228
DQ	2.165	3.457	7.214	18.904	3.659	7.713	9.973
p-val	0.826	0.630	0.205	0.002	0.599	0.173	0.076
NQR–t							
$\hat{C}_\alpha$	0.020	0.068	0.138	0.210	0.754	0.878	0.930
$\hat{L}$	0.059	0.229	0.383	0.650	0.642	**0.374**	0.222
DQ	7.278	5.646	11.310	8.423	5.226	11.960	11.945
p-val	0.201	0.342	0.046	0.134	0.389	0.035	0.036
LQR							
$\hat{C}_\alpha$	0.012	0.050	0.092	0.216	0.738	0.902	0.958
$\hat{L}$	0.062	0.227	0.379	0.652	0.645	0.366	0.223
DQ	0.966	3.008	4.651	9.657	4.822	11.235	7.019
p-val	0.965	0.699	0.460	0.086	0.438	0.047	0.219
WFC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.020	0.068	0.216	0.784	0.934	0.966
$\hat{L}$	**0.060**	**0.191**	**0.317**	0.553	0.532	0.316	0.187
DQ	4.700	15.095	9.827	9.355	9.891	20.233	9.349
p-val	0.454	0.010	0.080	0.096	0.078	0.001	0.096
NQR–t							
$\hat{C}_\alpha$	0.006	0.036	0.126	0.260	0.746	0.916	0.950
$\hat{L}$	**0.053**	0.180	0.307	0.549	0.531	0.308	<u>0.180</u>
DQ	0.625	5.946	5.528	2.432	3.187	11.603	6.409
p-val	0.987	0.311	0.355	0.787	0.671	0.041	0.268
LQR							
$\hat{C}_\alpha$	0.004	0.040	0.104	0.252	0.744	0.906	0.954
$\hat{L}$	0.047	0.180	0.305	0.547	0.532	0.311	0.184
DQ	2.165	4.253	4.162	2.628	4.604	10.960	5.352
p-val	0.826	0.514	0.526	0.757	0.466	0.052	0.375

Table 3.12: **Health Care**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

JNJ	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.008	0.050	0.086	0.242	0.750	0.916	0.940
$\hat{L}$	0.023	0.078	0.124	0.223	0.224	0.137	0.085
DQ	0.528	8.739	9.696	11.787	4.834	1.777	4.591
p-val	0.991	0.120	0.084	0.038	0.436	0.879	0.468
NQR–t							
$\hat{C}_\alpha$	0.010	0.046	0.098	0.238	0.738	0.906	0.956
$\hat{L}$	0.023	0.077	0.125	0.222	0.227	0.134	0.085
DQ	0.521	5.373	7.201	8.620	4.583	1.614	4.235
p-val	0.991	0.372	0.206	0.125	0.469	0.900	0.516
LQR							
$\hat{C}_\alpha$	0.008	0.046	0.092	0.246	0.740	0.908	0.954
$\hat{L}$	0.022	0.077	0.124	0.223	0.227	0.134	0.084
DQ	0.528	5.373	7.770	8.230	4.641	2.110	4.677
p-val	0.991	0.372	0.169	0.144	0.461	0.834	0.457
MRK	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.010	0.064	0.122	0.256	0.762	0.916	0.958
$\hat{L}$	**0.036**	0.120	**0.198**	0.324	0.316	0.185	0.114
DQ	0.521	11.754	6.019	4.120	6.632	4.045	6.776
p-val	0.991	0.038	0.304	0.532	0.249	0.543	0.238
NQR–t							
$\hat{C}_\alpha$	0.002	0.046	0.096	0.250	0.784	0.930	0.978
$\hat{L}$	**0.034**	0.114	0.191	0.322	0.316	0.178	0.107
DQ	4.700	8.810	3.909	2.898	9.327	12.937	13.355
p-val	0.454	0.117	0.563	0.716	0.097	0.024	0.020
LQR							
$\hat{C}_\alpha$	0.004	0.044	0.100	0.240	0.782	0.938	0.976
$\hat{L}$	0.031	0.115	0.191	0.321	0.316	0.180	0.110
DQ	2.165	8.823	5.160	4.349	8.123	15.792	11.610
p-val	0.826	0.116	0.397	0.500	0.150	0.007	0.041
PFE	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.010	0.034	0.086	0.224	0.762	0.918	0.962
$\hat{L}$	**0.035**	0.118	0.199	0.353	0.339	0.185	0.110
DQ	0.521	8.045	4.133	9.155	9.560	9.973	11.253
p-val	0.991	0.154	0.530	0.103	0.089	0.076	0.047
NQR–t							
$\hat{C}_\alpha$	0.006	0.038	0.088	0.220	0.752	0.900	0.952
$\hat{L}$	0.033	0.118	0.195	**0.353**	**0.342**	0.184	0.109
DQ	0.625	8.253	6.215	7.514	7.169	2.850	5.355
p-val	0.987	0.143	0.286	0.185	0.208	0.723	0.374
LQR							
$\hat{C}_\alpha$	0.008	0.038	0.092	0.222	0.748	0.906	0.964
$\hat{L}$	0.033	0.118	0.196	0.352	0.339	0.184	0.109
DQ	0.528	6.321	6.515	7.438	9.738	3.798	8.053
p-val	0.991	0.276	0.259	0.190	0.083	0.579	0.153

Table 3.13: **Information Technology**: OOS 1-step-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated p-val are reported. $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

AAPL	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.008	0.044	0.090	0.246	0.768	0.918	0.954
$\hat{L}$	0.037	0.142	0.235	0.407	0.421	0.250	0.151
DQ	0.528	7.313	12.154	4.184	9.674	4.354	6.992
p-val	0.991	0.198	0.033	0.523	0.085	0.500	0.221
NQR–t							
$\hat{C}_\alpha$	0.004	0.052	0.088	0.242	0.798	0.920	0.966
$\hat{L}$	0.037	0.144	<u>0.231</u>	0.406	**0.427**	0.252	0.153
DQ	2.165	11.821	8.301	5.820	8.896	7.450	6.115
p-val	0.826	0.037	0.140	0.324	0.113	0.189	0.295
LQR							
$\hat{C}_\alpha$	0.004	0.028	0.072	0.232	0.782	0.924	0.956
$\hat{L}$	0.037	0.142	0.236	0.409	0.419	0.252	0.151
DQ	2.165	10.803	12.643	7.312	6.676	6.676	5.293
p-val	0.826	0.055	0.027	0.198	0.246	0.246	0.381
INTC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.024	0.084	0.210	0.736	0.880	0.950
$\hat{L}$	**0.040**	0.136	0.228	0.400	0.396	0.217	0.130
DQ	2.165	11.939	5.714	6.955	1.610	8.670	2.643
p-val	0.826	0.036	0.335	0.224	0.900	0.123	0.755
NQR–t							
$\hat{C}_\alpha$	0.008	0.034	0.080	0.208	0.730	0.890	0.946
$\hat{L}$	0.038	0.137	**0.230**	**0.404**	**0.401**	0.216	**0.132**
DQ	0.528	13.381	9.361	10.628	1.797	3.163	4.850
p-val	0.991	0.020	0.095	0.059	0.876	0.675	0.434
LQR							
$\hat{C}_\alpha$	0.006	0.032	0.078	0.214	0.732	0.904	0.952
$\hat{L}$	0.037	0.136	0.228	0.401	0.397	0.214	0.129
DQ	0.625	9.235	9.069	6.843	1.051	6.436	1.743
p-val	0.987	0.100	0.106	0.233	0.958	0.266	0.883
MSFT	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.006	0.050	0.084	0.258	0.780	0.912	0.962
$\hat{L}$	0.036	**0.134**	0.216	0.369	0.357	0.212	0.130
DQ	0.625	7.825	7.001	2.623	6.287	2.177	6.331
p-val	0.987	0.166	0.221	0.758	0.279	0.824	0.275
NQR–t							
$\hat{C}_\alpha$	0.014	0.048	0.100	0.278	0.770	0.912	0.960
$\hat{L}$	0.036	0.131	0.212	0.365	0.356	0.209	0.129
DQ	4.545	2.224	5.380	5.847	12.139	5.960	6.163
p-val	0.474	0.817	0.371	0.321	0.033	0.310	0.291
LQR							
$\hat{C}_\alpha$	0.014	0.050	0.094	0.268	0.770	0.914	0.962
$\hat{L}$	0.038	0.130	0.211	0.364	0.357	0.210	0.129
DQ	4.545	3.315	5.653	6.091	10.553	2.825	7.831
p-val	0.474	0.651	0.341	0.298	0.061	0.727	0.166

Table 3.14: **Telecommunication Services**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

CMCSA	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.006	0.044	0.088	0.218	0.702	0.880	0.942
$\hat{L}$	0.093	0.350	0.594	1.063	1.066	0.566	**0.329**
DM	0.619	0.154	1.218	1.093	-0.360	1.151	2.423
NQR–t							
$\hat{C}_\alpha$	0.006	0.060	0.106	0.242	0.702	0.872	0.932
$\hat{L}$	0.088	0.347	0.582	1.051	1.067	0.553	0.314
DM	-0.479	-0.306	0.939	0.112	-0.374	-0.546	0.258
LQR							
$\hat{C}_\alpha$	0.016	0.056	0.104	0.232	0.688	0.872	0.938
$\hat{L}$	0.090	0.348	0.579	1.050	1.069	0.556	0.313
T	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.022	0.084	0.230	0.804	0.902	0.934
$\hat{L}$	0.049	0.173	0.298	0.536	0.555	**0.341**	**0.216**
DM	0.979	-0.022	1.094	0.515	1.102	1.653	1.988
NQR–t							
$\hat{C}_\alpha$	0.002	0.022	0.068	0.220	0.808	0.930	0.960
$\hat{L}$	0.048	0.173	0.284	0.528	0.550	0.333	0.205
DM	0.954	-0.333	-1.305	-0.701	-0.002	0.885	0.743
LQR							
$\hat{C}_\alpha$	0.002	0.014	0.060	0.206	0.798	0.928	0.966
$\hat{L}$	0.047	0.173	0.290	0.532	0.550	0.330	0.203
VZ	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.010	0.028	0.054	0.186	0.752	0.912	0.948
$\hat{L}$	0.054	0.194	0.330	0.580	0.601	0.356	0.219
DM	0.967	0.539	1.017	-0.513	-0.535	1.620	1.549
NQR–t							
$\hat{C}_\alpha$	0.004	0.022	0.058	0.196	0.736	0.914	0.954
$\hat{L}$	<u>0.047</u>	<u>0.186</u>	0.320	0.577	0.609	0.339	0.199
DM	-1.913	-1.776	-0.007	-1.011	0.631	-0.463	-0.939
LQR							
$\hat{C}_\alpha$	0.002	0.020	0.056	0.186	0.726	0.920	0.958
$\hat{L}$	0.049	0.190	0.320	0.586	0.605	0.341	0.203

Table 3.15: **Consumer Discretionary**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

AMZN	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.018	0.078	0.134	0.284	0.760	0.878	0.930
$\hat{L}$	0.098	0.402	0.692	1.297	1.423	0.777	0.452
DM	0.138	-0.169	-0.511	-0.173	-0.345	-0.205	0.632
NQR–t							
$\hat{C}_\alpha$	0.016	0.072	0.126	0.284	0.760	0.880	0.928
$\hat{L}$	0.098	0.403	0.703	1.296	1.443	0.786	0.463
DM	0.247	0.009	0.807	-0.315	0.745	0.366	0.815
LQR							
$\hat{C}_\alpha$	0.018	0.066	0.124	0.286	0.754	0.876	0.942
$\hat{L}$	0.097	0.403	0.698	1.299	1.426	0.779	0.442
DIS	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.012	0.058	0.114	0.266	0.764	0.922	0.952
$\hat{L}$	0.076	0.306	0.519	0.934	0.886	0.487	0.282
DM	-1.203	-1.179	-1.428	-1.149	-0.086	1.293	1.013
NQR–t							
$\hat{C}_\alpha$	0.028	0.064	0.114	0.268	0.756	0.904	0.954
$\hat{L}$	0.084	0.313	0.527	<u>0.935</u>	0.890	0.479	0.275
DM	-0.661	-0.635	-1.579	-1.696	0.663	0.890	-1.097
LQR							
$\hat{C}_\alpha$	0.022	0.060	0.118	0.272	0.752	0.904	0.954
$\hat{L}$	0.086	0.317	0.538	0.945	0.887	0.477	0.277
MCD	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.028	0.080	0.130	0.264	0.802	0.930	0.962
$\hat{L}$	**0.061**	0.211	**0.362**	0.588	0.499	0.289	0.175
DM	1.919	1.529	1.849	1.588	-1.293	-0.553	-1.150
NQR–t							
$\hat{C}_\alpha$	0.002	0.048	0.106	0.252	0.812	0.946	0.976
$\hat{L}$	0.056	0.200	0.338	0.574	**0.526**	**0.298**	**0.183**
DM	1.063	1.386	1.316	1.245	2.505	1.857	2.224
LQR							
$\hat{C}_\alpha$	0.014	0.052	0.100	0.212	0.800	0.948	0.976
$\hat{L}$	0.051	0.196	0.331	0.559	0.503	0.292	0.179

Table 3.16: **Consumer Staples**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

KO	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.014	0.056	0.112	0.248	0.686	0.868	0.942
$\hat{L}$	**0.055**	0.194	0.330	0.568	0.576	0.308	0.181
DM	1.957	0.221	0.708	-0.953	0.669	-0.005	1.014
NQR–t							
$\hat{C}_\alpha$	0.012	0.056	0.102	0.252	0.686	0.864	0.948
$\hat{L}$	**0.057**	0.190	0.325	0.576	0.575	0.308	0.176
DM	2.269	-1.205	0.593	0.470	0.588	0.048	0.538
LQR							
$\hat{C}_\alpha$	0.016	0.056	0.100	0.256	0.706	0.896	0.956
$\hat{L}$	0.052	0.193	0.323	0.574	0.571	0.308	0.174
PG	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.026	0.090	0.130	0.288	0.742	0.880	0.934
$\hat{L}$	0.050	**0.177**	0.285	0.495	0.477	0.267	0.162
DM	1.514	2.199	1.600	1.160	0.355	0.985	1.414
NQR–t							
$\hat{C}_\alpha$	0.014	0.064	0.112	0.248	0.754	0.906	0.948
$\hat{L}$	**0.046**	**0.166**	0.274	0.495	0.476	0.259	0.156
DM	2.533	4.099	1.441	1.511	0.799	0.861	1.568
LQR							
$\hat{C}_\alpha$	0.016	0.058	0.118	0.262	0.760	0.910	0.958
$\hat{L}$	0.042	0.161	0.271	0.488	0.474	0.257	0.153
WMT	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.022	0.052	0.080	0.192	0.680	0.870	0.934
$\hat{L}$	0.047	0.170	0.283	0.518	**0.551**	0.287	0.164
DM	0.559	-0.068	-0.529	-1.537	2.601	1.580	1.445
NQR–t							
$\hat{C}_\alpha$	0.014	0.042	0.082	0.186	0.728	0.870	0.950
$\hat{L}$	0.046	0.171	0.289	<u>0.518</u>	**0.533**	0.282	0.157
DM	0.796	0.186	0.442	-1.916	1.809	1.388	0.664
LQR							
$\hat{C}_\alpha$	0.008	0.036	0.070	0.170	0.710	0.888	0.968
$\hat{L}$	0.045	0.170	0.288	0.528	0.525	0.275	0.156

Table 3.17: **Energy**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

CVX	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.012	0.064	0.102	0.238	0.736	0.894	0.940
$\hat{L}$	0.069	0.275	0.460	<u>0.797</u>	0.734	0.396	0.233
DM	-0.686	-0.730	-1.314	-2.525	-0.780	0.218	0.788
NQR–t							
$\hat{C}_\alpha$	0.014	0.062	0.102	0.224	0.774	0.922	0.952
$\hat{L}$	**0.084**	0.288	0.466	<u>0.810</u>	0.748	**0.425**	**0.242**
DM	2.826	1.248	-1.139	-2.035	0.454	1.815	1.923
LQR							
$\hat{C}_\alpha$	0.012	0.056	0.104	0.210	0.726	0.902	0.960
$\hat{L}$	0.070	0.281	0.476	0.830	0.739	0.394	0.225
SLB	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.022	0.068	0.128	0.288	0.780	0.920	0.972
$\hat{L}$	0.101	0.398	0.676	1.233	1.157	0.640	0.375
DM	-1.100	-0.849	-0.850	0.170	0.861	0.716	1.577
NQR–t							
$\hat{C}_\alpha$	0.018	0.074	0.116	0.274	0.780	0.924	0.948
$\hat{L}$	0.106	0.404	0.676	1.229	1.156	0.640	0.382
DM	-0.262	-0.646	-1.145	-0.034	0.387	0.772	0.775
LQR							
$\hat{C}_\alpha$	0.018	0.066	0.120	0.282	0.778	0.928	0.972
$\hat{L}$	0.107	0.411	0.692	1.229	1.155	0.636	0.371
XOM	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.014	0.046	0.090	0.238	0.724	0.890	0.932
$\hat{L}$	<u>0.065</u>	0.250	<u>0.410</u>	0.708	0.688	0.387	**0.239**
DM	-3.030	-1.576	-2.020	-1.340	-0.602	1.520	2.044
NQR–t							
$\hat{C}_\alpha$	0.016	0.048	0.082	0.220	0.736	0.910	0.962
$\hat{L}$	**0.072**	<u>0.253</u>	<u>0.414</u>	<u>0.705</u>	0.700	**0.392**	**0.233**
DM	1.796	-2.325	-2.014	-2.052	1.079	1.913	1.680
LQR							
$\hat{C}_\alpha$	0.014	0.044	0.078	0.200	0.724	0.906	0.960
$\hat{L}$	0.070	0.264	0.426	0.725	0.692	0.376	0.225

Table 3.18: **Financials**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

BAC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.034	0.076	0.240	0.824	0.936	0.984
$\hat{L}$	0.177	**0.494**	**0.788**	1.396	1.461	0.819	**0.498**
DM	1.132	3.624	2.220	0.792	1.279	0.771	1.736
NQR–t							
$\hat{C}_\alpha$	0.004	0.066	0.122	0.266	0.812	0.918	0.968
$\hat{L}$	0.147	0.465	0.777	1.366	1.440	0.803	0.484
DM	-0.059	1.486	1.163	0.232	0.160	-0.416	1.178
LQR							
$\hat{C}_\alpha$	0.006	0.052	0.104	0.296	0.796	0.922	0.972
$\hat{L}$	0.148	0.450	0.751	1.359	1.439	0.808	0.478
C	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.024	0.042	0.148	0.748	0.908	0.952
$\hat{L}$	**0.204**	**0.482**	0.797	**1.434**	1.348	0.782	0.451
DM	6.938	1.884	1.168	2.321	1.001	0.471	-0.317
NQR–t							
$\hat{C}_\alpha$	0.020	0.058	0.078	0.202	0.720	0.900	0.902
$\hat{L}$	0.131	0.455	0.757	1.350	1.347	0.778	0.486
DM	-0.298	-0.744	-1.371	0.150	0.571	0.439	1.347
LQR							
$\hat{C}_\alpha$	0.008	0.024	0.070	0.202	0.748	0.888	0.938
$\hat{L}$	0.134	0.468	0.779	1.348	1.342	0.776	0.454
WFC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.004	0.040	0.092	0.204	0.778	0.930	0.970
$\hat{L}$	**0.118**	**0.395**	0.667	1.154	1.077	0.596	**0.369**
DM	8.562	1.652	1.383	1.001	-0.520	0.539	2.272
NQR–t							
$\hat{C}_\alpha$	0.010	0.062	0.088	0.232	0.746	0.912	0.958
$\hat{L}$	0.094	0.393	0.650	1.147	<u>1.074</u>	<u>0.584</u>	0.351
DM	-1.124	0.853	0.496	1.311	-1.798	-1.691	-0.895
LQR							
$\hat{C}_\alpha$	0.004	0.052	0.088	0.248	0.736	0.900	0.962
$\hat{L}$	0.098	0.381	0.646	1.138	1.085	0.592	0.358

Table 3.19: **Health Care**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

JNJ	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.042	0.072	0.214	0.774	0.916	0.954
$\hat{L}$	**0.042**	0.144	0.255	0.465	0.484	0.288	0.175
DM	2.099	-0.295	-0.051	-1.145	0.440	0.636	-0.390
NQR–t							
$\hat{C}_\alpha$	0.004	0.038	0.082	0.232	0.766	0.918	0.952
$\hat{L}$	0.040	0.144	0.253	<u>0.465</u>	0.485	0.286	0.176
DM	0.368	-1.050	-0.925	-2.047	0.694	0.385	-0.365
LQR							
$\hat{C}_\alpha$	0.000	0.036	0.082	0.232	0.764	0.912	0.966
$\hat{L}$	0.039	0.146	0.255	0.474	0.482	0.284	0.178
MRK	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.020	0.082	0.138	0.268	0.802	0.924	0.958
$\hat{L}$	0.065	0.259	0.427	0.741	<u>0.751</u>	0.417	0.244
DM	0.857	0.979	0.684	0.360	-1.704	-0.612	-0.505
NQR–t							
$\hat{C}_\alpha$	0.012	0.066	0.102	0.240	0.808	0.928	0.972
$\hat{L}$	**0.068**	0.244	0.418	0.732	<u>0.754</u>	0.415	0.252
DM	2.240	0.190	0.746	-0.213	-1.905	-1.123	0.474
LQR							
$\hat{C}_\alpha$	0.010	0.064	0.102	0.238	0.818	0.942	0.970
$\hat{L}$	0.061	0.243	0.414	0.733	0.772	0.425	0.250
PFE	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.026	0.042	0.162	0.770	0.936	0.980
$\hat{L}$	0.067	0.264	0.443	0.717	0.687	0.372	**0.231**
DM	1.322	0.426	0.454	-0.136	-0.213	-0.423	2.287
NQR–t							
$\hat{C}_\alpha$	0.002	0.026	0.044	0.164	0.752	0.908	0.966
$\hat{L}$	0.065	0.260	0.439	0.717	0.687	0.375	0.219
DM	-0.479	-0.775	-0.322	-0.214	-0.075	-0.024	-1.022
LQR							
$\hat{C}_\alpha$	0.004	0.026	0.046	0.168	0.764	0.916	0.974
$\hat{L}$	0.065	0.263	0.440	0.718	0.687	0.375	0.221

Table 3.20: **Information Technology**: 5-days-ahead VaR evaluation. Empirical quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, $\hat{L}$ is moreover tested with Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to LQR (Linear Quantile Regression) while models with significantly less accurate forecasts at 95% level are reported in bold, significantly more accurate as underlined. NQR is Nonlinear Quantile Regression.

AAPL	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.018	0.088	0.146	0.344	0.812	0.908	0.944
$\hat{L}$	0.073	0.302	0.525	0.926	0.894	0.527	0.309
DM	0.743	0.954	1.029	0.747	0.084	0.096	0.279
NQR–t							
$\hat{C}_\alpha$	0.014	0.084	0.160	0.332	0.816	0.912	0.954
$\hat{L}$	0.067	0.276	0.474	0.902	0.895	0.528	0.307
DM	-0.609	-0.336	-0.590	0.067	0.234	0.198	0.166
LQR							
$\hat{C}_\alpha$	0.000	0.050	0.114	0.302	0.830	0.928	0.960
$\hat{L}$	0.069	0.280	0.482	0.901	0.894	0.526	0.306
INTC	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.002	0.020	0.064	0.194	0.694	0.906	0.952
$\hat{L}$	**0.081**	**0.277**	0.473	0.886	0.905	0.504	0.300
DM	3.748	1.701	-0.279	-0.645	-0.329	-0.501	-1.357
NQR–t							
$\hat{C}_\alpha$	0.002	0.022	0.080	0.190	0.708	0.902	0.950
$\hat{L}$	0.077	0.269	0.476	<u>0.882</u>	0.912	0.511	**0.317**
DM	1.522	-0.002	0.377	-2.048	0.784	0.269	1.800
LQR							
$\hat{C}_\alpha$	0.002	0.020	0.072	0.198	0.706	0.896	0.954
$\hat{L}$	0.075	0.269	0.475	0.894	0.909	0.510	0.307
MSFT	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{C}_\alpha$	0.018	0.048	0.092	0.230	0.794	0.912	0.946
$\hat{L}$	0.084	0.296	0.482	0.834	0.864	0.485	0.282
DM	-0.540	0.100	0.739	1.204	1.363	0.616	-0.524
NQR–t							
$\hat{C}_\alpha$	0.020	0.056	0.102	0.252	0.788	0.916	0.944
$\hat{L}$	0.092	0.294	0.474	0.823	0.862	0.483	0.283
DM	1.291	-0.272	0.035	0.023	1.441	0.312	-0.285
LQR							
$\hat{C}_\alpha$	0.018	0.054	0.100	0.268	0.786	0.910	0.948
$\hat{L}$	0.089	0.295	0.474	0.822	0.859	0.481	0.284

3.C Figures

Figure 3.3: Estimated parameters from Normal and t copula using full sample data. For each quantile level we are summarizing the results for 21 assets.

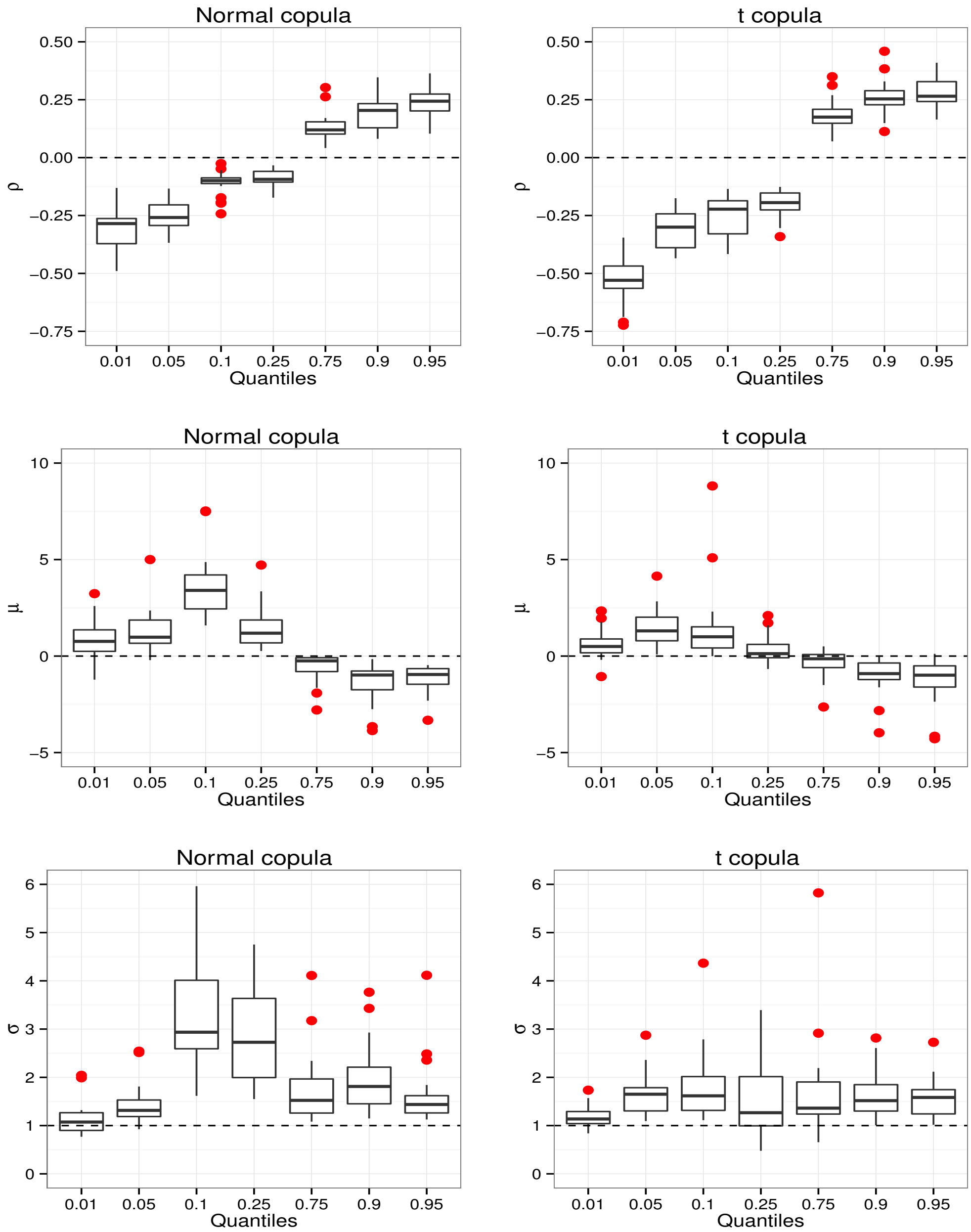


Figure 3.4: Degrees of freedom for t copula using the full sample data. For each quantile level we are summarizing the results for 21 assets.

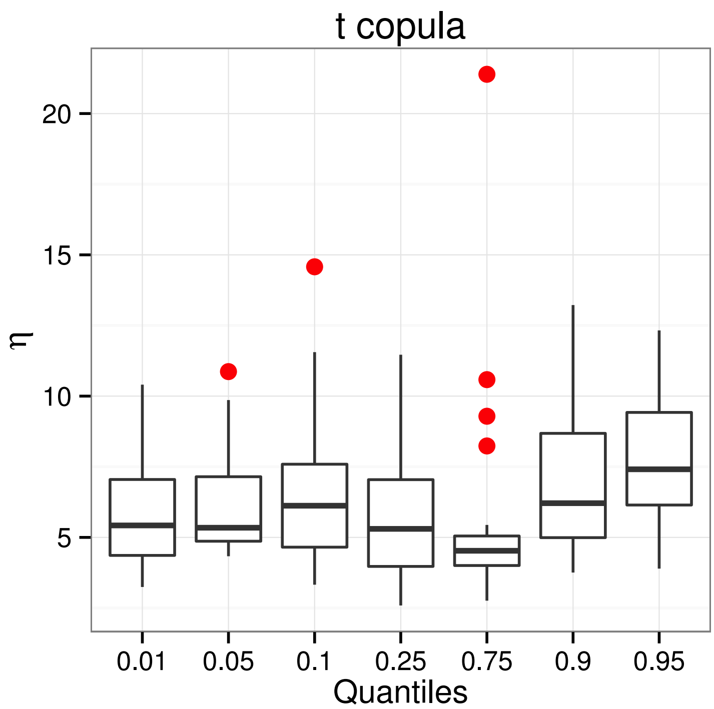


Figure 3.5: Out-of-sample correlation from Normal copula. The in-sample period includes 1335 observations and the out-of-sample 500. We use the one-step-ahead rolling window for a length for approximately two years.

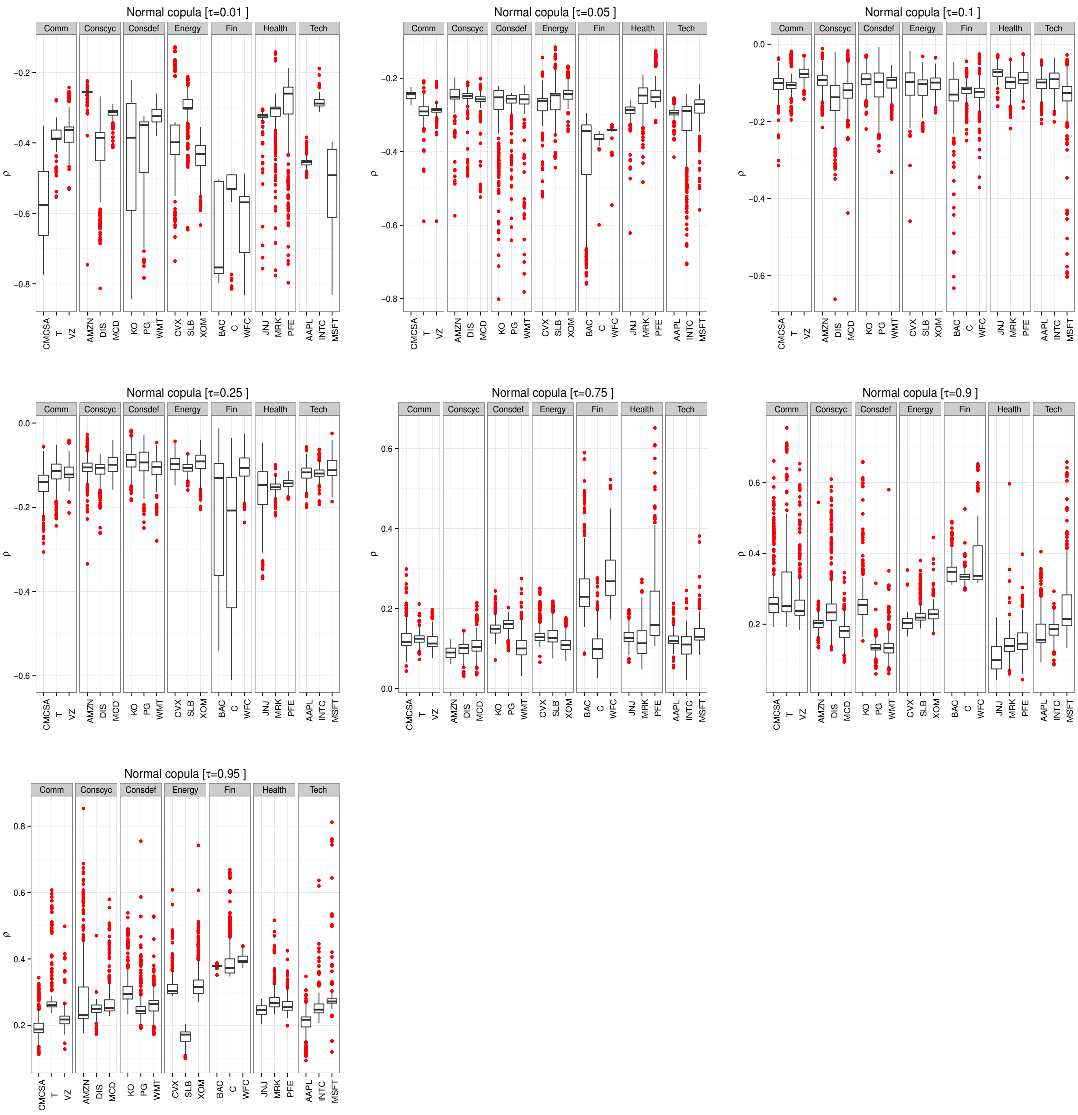


Figure 3.6: Out-of-sample correlation from t copula. The in-sample period includes 1335 observations and the out-of-sample 500. We use the one-step-ahead rolling window for a length for approximately two years and assume that the degrees-of-freedom are constant throughout the out-of-sample window.

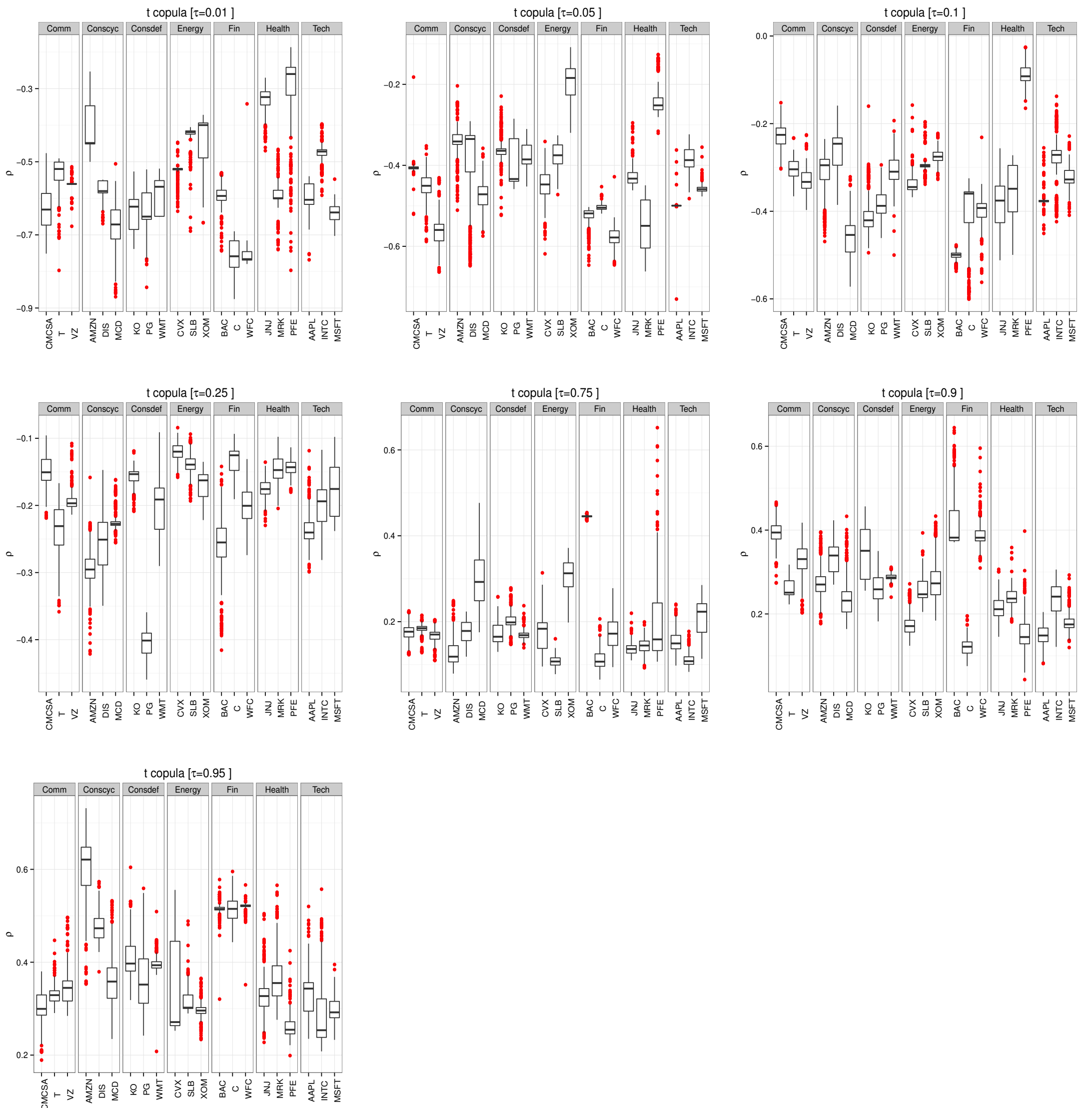


Figure 3.7: One-step-ahead forecast $VaR_{\alpha=5\%}$ from Linear quantile regression and Normal and t copula regression. By row: technology, consumer cyclical, consumer defense and communication.

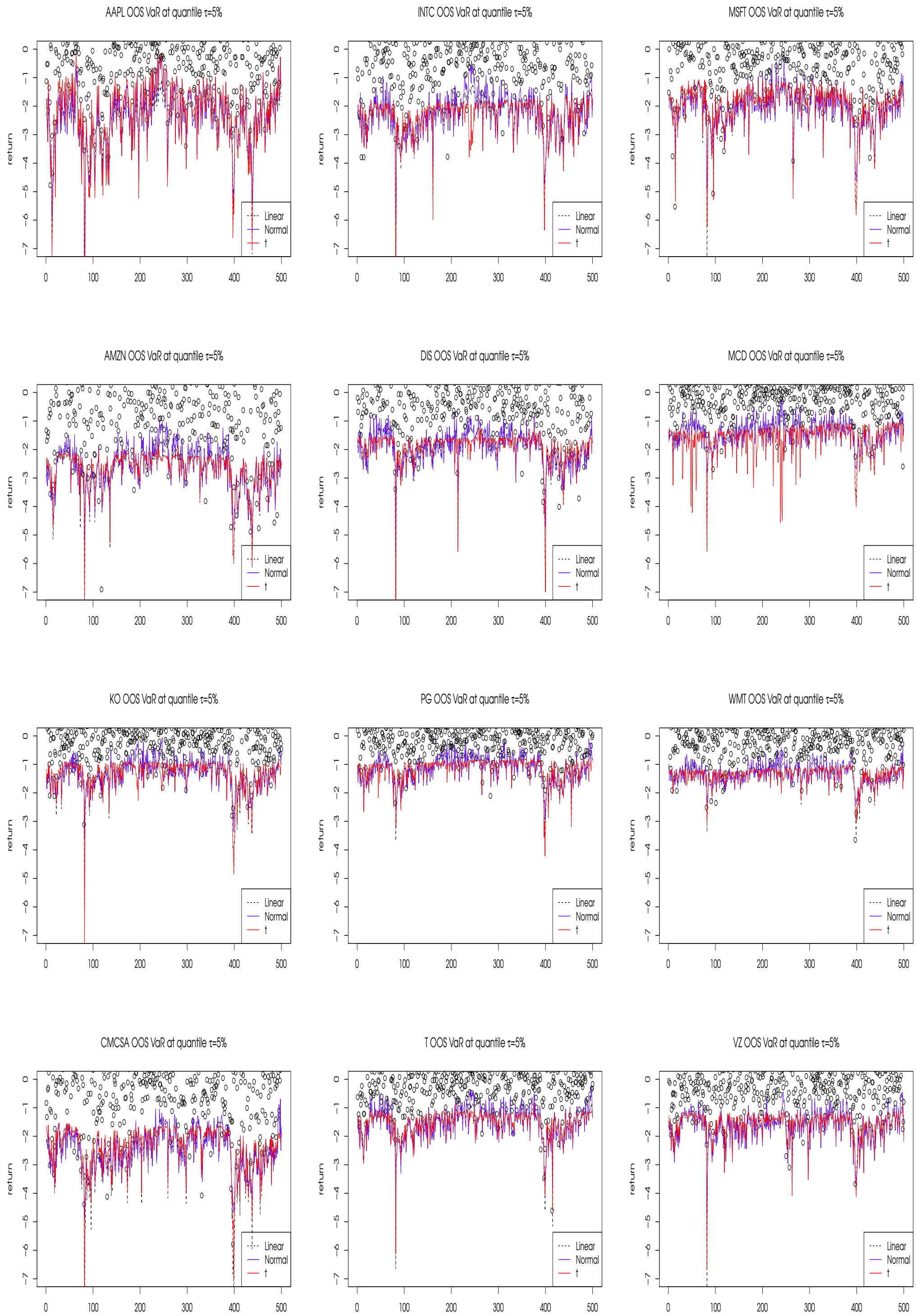


Figure 3.8: One-step-ahead $VaR_{\alpha=5\%}$ from Linear quantile regression and Normal and t copula regression. By row: financial, energy and health.

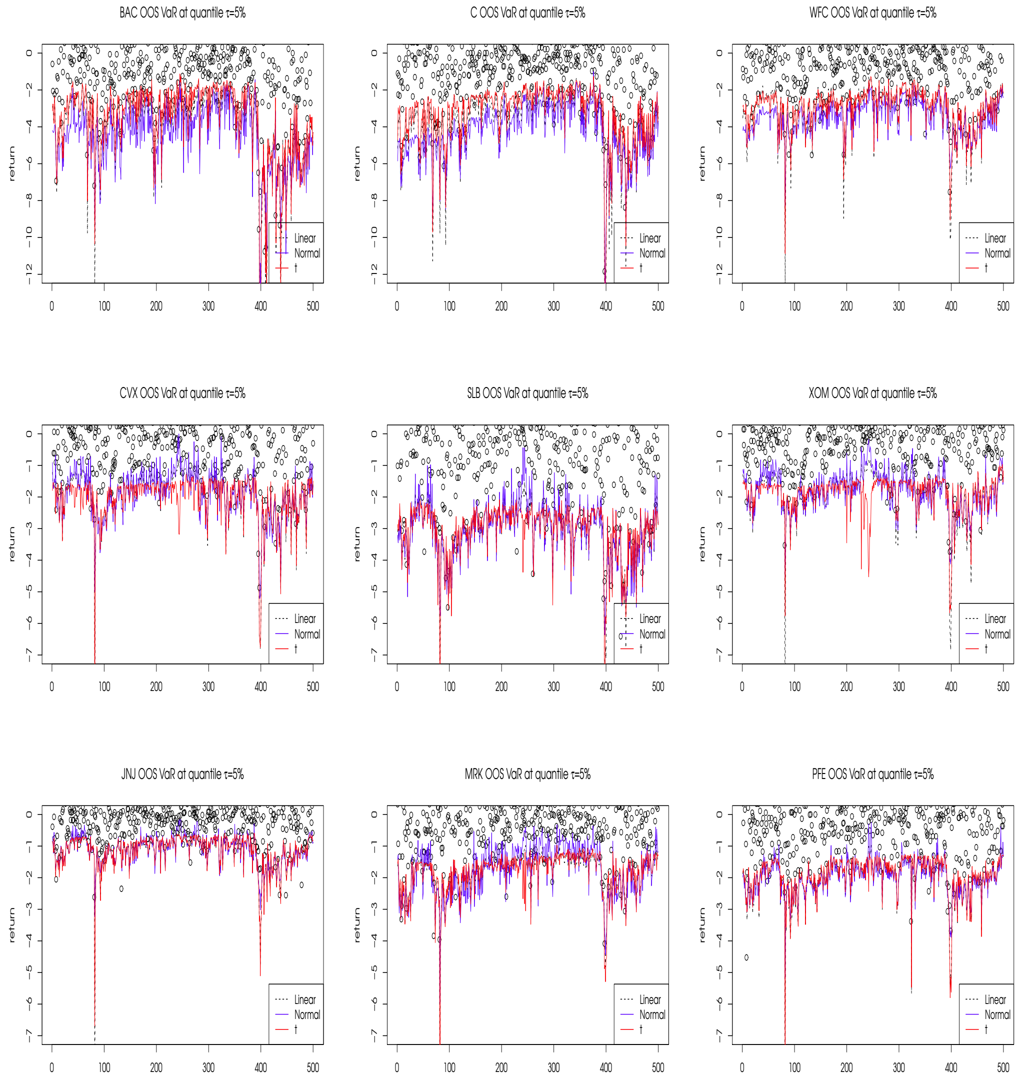


Figure 3.9: Five-step-ahead forecast $VaR_{\alpha=5\%}$ from Linear quantile regression and Normal and t copula regression. By row: technology, consumer cyclical, consumer defense and communication.

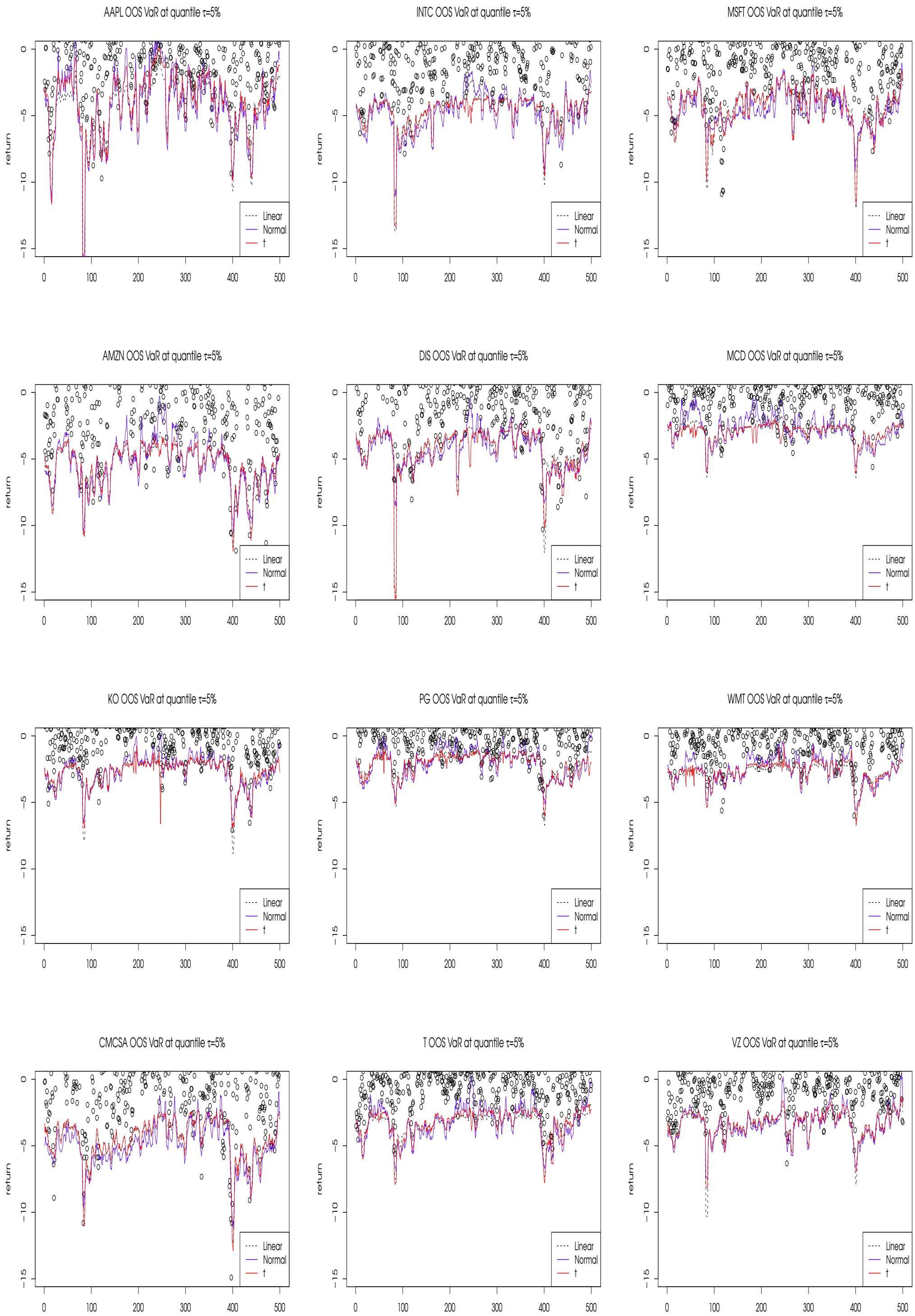
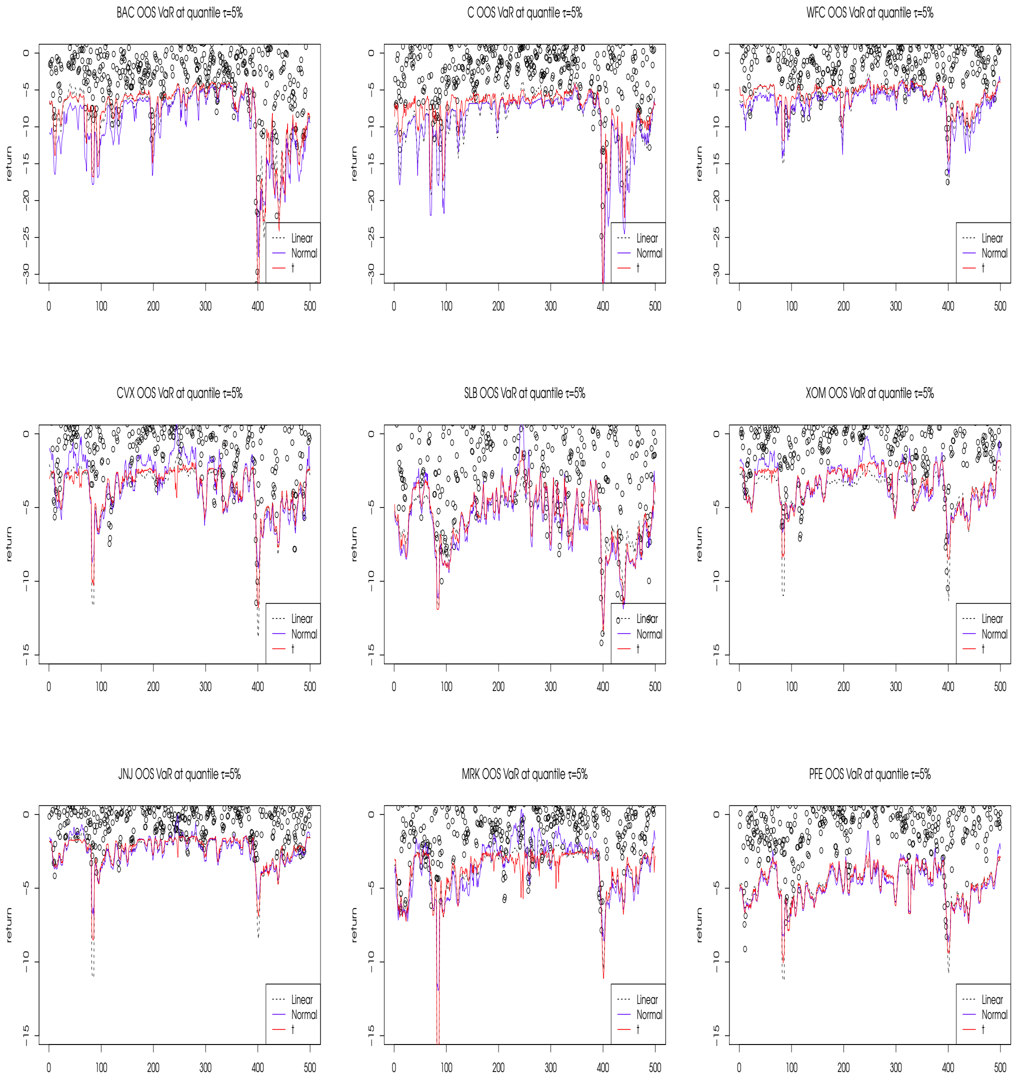


Figure 3.10: Five-step-ahead $VaR_{\alpha=5\%}$ from Linear quantile regression and Normal and t copula regression. By row: financial, energy and health.



Chapter 4

A copula quantile approach to conditional Value-at-Risk estimation

Abstract

We propose to use copula quantile regression models and realized volatility to estimate Value-at-Risk of an institution conditional on some other institution being under financial distress. The model proposed uses copulas from elliptical family, Normal and t copula, and the realized volatility measure which is calculated using 5 minutes returns. Contrary to the literature which studies the systemic risk of financial institutions, we estimate the risk contribution of an institution to some other institution. We apply the model on 21 most liquid U.S. stocks from seven main market sectors. For comparison we use a benchmark model for VaR based on rescaled realized volatility and also use linear quantile regression model. We find that in most cases the average value of $\Delta CoVaR$ from copula models is between the benchmark and the linear model. We also find that stocks from Financial sector have the highest risk spillovers among each other, followed by the Information Technology sector. Consumer Staples industry stocks and Health Care have the lowest risk spillovers.

Keywords: value-at-risk, quantile copula regression, realized-volatility
JEL: C14, C32, C58

4.1 Introduction

Value-at-Risk (VaR) is the standard risk measure used by financial industry. It is also used by regulators to determine the required capital which companies have to set aside to tackle market risk. VaR considers only the market risk of an institution viewed in isolation though. Thus, to avoid this shortcoming, alternative measures which reflect the systemic risk have developed in the recent years.

Adrian & Brunnermeier (2011) introduce the conditional Value-at-Risk (CoVaR). They define $CoVaR^{j|i}$ as the VaR of institution j conditional on institution i being at its VaR. If we denote the two random variables at time t by $R_{j,t}$ and $R_{i,t}$

This paper was co-authored with Jozef Baruník and is in preparation for submission.

the conditioning event for CoVaR is $R_{i,t} = VaR_t^i$. The authors estimate systemic risk by replacing the financial system for the institution *j*. They use daily market equity data and quarterly data for the balance sheet. The CoVaR is estimated by linear quantile regression. After CoVaR introduction a number of systemic risk research papers emerged. Most of them focus in studying financial institutions, where banks and insurance companies prevail. López-Espinosa *et al.* (2012) use the CoVaR approach to identify the main factors behind systemic risk in a set of large international banks. They find that short-term wholesale funding is a key determinant in triggering systemic risk episodes. In a recent paper Karimalis & Nomikos (2014) use copula functions to model CoVaR. They measure the systemic risk contribution for a portfolio of large European banks. The authors use copula functions from Archimedean family, which in most cases offer closed-form expressions for CoVaR estimation[1]. By construction, their CoVaR estimation is constant, but they solve this shortcoming by employing time-varying copula models based on work of Patton (2006).

In literature there is a second definition of CoVaR. Girardi & Ergün (2013) propose the same risk metric, but the conditioning event is that institution i is *at most* at its VaR level. This definition considers more severe distress events for institution i that are further in the tail of losses distribution *i.e* $R_{i,t} \leq VaR_t^i$. Mainik & Schaanning (2014) study the consistency of of CoVaR measure under both definitions with respect to stochastic dependence. The authors show that under stress event $R_{i,t} = VaR_t^i$ CoVaR is not dependence consistent. In particular if (R_j, R_i) is bivariate normal, then CoVaR is not an increasing function of the correlation parameter. Similar issues arise in the bivariate t model and in the model with t margins and a Gumbel copula. On the other hand, they show that the CoVaR estimated under the stress event $R_{i,t} \leq VaR_t^i$ (in all cases) is an increasing function of the dependence parameter.

Alternative systemic risk estimation approaches to CoVaR exist in literature. Acharya *et al.* (2012) use equity returns of financial institutions to calculate systemic expected shortfall (SES). The bank's contribution to this risk is measured by marginal expected shortfall (MES). Brownlees & Engle (2015) introduce the SRISK index to measure the systemic risk contribution of a financial firm. The SRISK index is a function of the firm's size, its degree of leverage and its expected equity loss conditional on a market downturn (LRMES). For LRMES predictions authors propose to use DCC-GARCH model of Engle (2002) or dynamic copula

[1]Gumbel copula is an example where a closed-form solution does not exist. In such cases numerical methods are required. For more details on this issue see the Footnote 2 in Chapter 3 of this thesis.

models as in Patton (2006). For a comprehensive survey on systemic risk measures please refer to Bisias *et al.* (2012). The authors discuss 31 quantitative measures of systemic risk in the economics and finance literature. In addition, they provide an extensive appendix where they bring the definitions of these measures and show how to implement them.

In this work we analyze the risk contribution of institution i on institution j where both institutions belong to the same industry. This allows the study of spillover effects within the industry. For VaR estimation we use the nonlinear quantile copula regression models which we introduced in Chapter 3 of this dissertation. In contrast to the model introduced in 3.2 where we use *own* lagged realized volatility as state variable, here we estimate the VaR using *inter* lagged realized volatility as state variable[2]. We use the same data as in Chapter 3, which consider 21 assets from 7 different industries. Our analysis identifies the risk transmission differences that exists between companies and industries. We compare our results with a benchmark model for VaR based on rescaled realized volatility and also compare with linear quantile regression model. In all cases the CoVaR is estimated using linear quantile regression. Baruník *et al.* (2015) study the same data from a different perspective. They investigate how "good" and "bad" volatility spills over stocks. The authors document asymmetric connectedness of markets at the disaggregate sectoral level, which is in contrast to the symmetric volatility transmission mechanism at the aggregate level.

In this study we propose to use a *two-step* procedure for CoVaR estimation. First step consists in utilizing semiparametric copula-quantile (CQ) regression models to obtain the VaR. We quantile-regress the returns of institution j on lagged realized volatility of institution i. Using this nonlinear framework we obtain the $VaR^{j|i}$. The second step consists in CoVaR estimation, which is done by using the linear quantile regression as in Adrian & Brunnermeier (2011). We consider a total of 21 most liquid U.S. stocks which belong to 7 industries *i.e.* three stock representatives for each industry. The data spans from August 2004 to December 2011.

The organization of the rest of the paper is as follows. Section 4.2 introduces the methodology. Section 4.3 describes the data used in the empirical part. Section 4.4 reports the results and Section 4.5 concludes.

[2]Refer to Footnote 1 in Chapter 1 for explanations on *own* and *inter* lagged realized volatility.

4.2 Methodology

4.2.1 Model for VaR

Let the random variable $r_{i,t}$ represent the return of firm i at time t ($i = 1, \ldots, N, t = 1, \ldots, T$). The Value-at-Risk of the random variable $r_{i,t}$ at confidence level $\alpha \in (0, 1)$ is defined as the α quantile of the return distribution

$$VaR^i_{\alpha,t} = F^{-1}_{i,t}(\alpha) \quad \text{or equivalently} \quad Pr(r_{i,t} \leq VaR^i_{\alpha,t}) = \alpha, \tag{4.1}$$

where $F^{-1}_{i,t}$ is the generalized inverse distribution function of the return distribution $F_{i,t}$, *i.e.* $F^{-1}_{i,t} := inf\{r_{i,t} \in \mathbb{R} : F_{i,t}(r_{i,t}) \geq \alpha\}$. Recall that for VaR we are not following the usual sign convention and report all downside risks in negative numbers.

For VaR estimation we employ the methodology introduced in Chapter 3.2. We assume that the logarithmic price process obeys Itô semimartingale

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s \tag{4.2}$$

where μ is a predictable process, σ is cadlag and W is the standard Brownian motion. The process in Equation 4.2 is very general, and as Todorov & Tauchen (2011) show, it allows to accommodate stochastic volatility with possible discontinuous sample paths.

Žikeš & Baruník (2014) propose a linear semiparametric model for the quantiles of future returns and volatility. They assume that the α-quantile of future returns $r_{t+1} = X_{t+1} - X_t$ conditional on information set Ω_t, can be written as a linear function of the various components of the current and past quadratic variation and weakly exogenous variables,

$$q_\alpha(r_{t+1}|\Omega_t) = \beta_0(\alpha) + \boldsymbol{\beta}_v(\alpha)'\boldsymbol{v}_t + \boldsymbol{\beta}_z(\alpha)'\boldsymbol{z}_t \tag{4.3}$$

where $\boldsymbol{v}_t$ is a measure of quadratic variation $QV_t = \int_0^t \sigma_s^2 \mathrm{d}s = IV_t$, where IV_t is the integrated variance, $\boldsymbol{z}_t$ is a vector of weakly exogenous variables and $\beta_0(\alpha), \boldsymbol{\beta}_v(\alpha), \boldsymbol{\beta}_z(\alpha)$ are vectors of coefficients to be estimated.

To see the connection that exists between the linear quantile model in Equation 4.3 and the logarithmic price process in Equation 4.2, Žikeš & Baruník (2014) consider a simplified case where the Ω_t contains only IV_t, *i.e.* $\boldsymbol{v}_t = IV_t$ and $\boldsymbol{z}_t = 0$. The $q_\alpha(r_{t+1}|IV_t)$ can be obtained from the conditional distribution of r_{t+1} given IV_t

$$f_{r_{t+1}|IV_t}(w_r|IV_t) = \int_0^\infty f_{r_{t+1}|IV_{t+1},IV_t}(w_r|w_{IV}, IV_t) f_{IV_{t+1}|IV_t}(w_{IV}|IV_t)\mathrm{d}w_{IV}$$

where $f_{y|X}(w_y|w_X)$ is the conditional distribution of y given X evaluated at w_y and w_X. For simplicity take the one-factor volatility model where σ_t in Equation 4.2 follows an Ornstein-Uhlenbeck process and $\mu_t = 0$. Meddahi (2003) shows that the integrated volatility IV_t follows an ARMA(1,1) process with non-gaussian innovations. It follows that $f_{IV_{t+1}|IV_t}$ has a non-gaussian density, while $f_{r_{t+1}|IV_{t+1},IV_t}$ is the normal density with zero mean and variance IV_t if there is no leverage effect, or a non-gaussian density otherwise. The implied conditional quantiles of the densities above, $q_\alpha(r_{t+1}|\Omega_t), \quad \alpha \in (0,1)$, can be approximated by linear functions of the current and past values of IV_t and other volatility measures.

Let us formally introduce the linear quantile regression (LQR) proposed by Koenker & Bassett (1978). Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(\mathbf{x}_1, \ldots, \mathbf{x}_T)'$ a random k sample on X. Then, the linear quantile regression definition follows.

Definition 4.1. The α quantile regression is any solution to the following problem:

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^k} \left(\sum_{t\in\mathcal{T}_\alpha} \alpha|y_t - \boldsymbol{x}_t'\boldsymbol{\beta}| + \sum_{t\in\mathcal{T}_{1-\alpha}} (1-\alpha)|y_t - \boldsymbol{x}_t'\boldsymbol{\beta}| \right) \tag{4.4}$$

with $\mathcal{T}_\alpha = \{t : y_t \geq \boldsymbol{x}_t'\boldsymbol{\beta}\}$ and $\mathcal{T}_{1-\alpha}$ its complement.

In the special case where $\alpha = 0.5$, the above quantile regression delivers the least absolute deviation (LAD) model. The LAD model is more robust than ordinary least squares (OLS) estimators whenever the errors have a fat-tailed distribution. The problem defined in Equation 4.4 does not have a closed form-solution, however Portnoy & Koenker (1997) provide computationally fast algorithm which is also implemented in the `quantreg` package for R.

Conditional quantile functions allow for nonlinear parametric models. Bouyé & Salmon (2009) introduced a general approach to nonlinear quantile regression based on copula models. Using the properties of conditional probability distribution the link between copula functions and conditional quantile functions becomes obvious. The conditional probability distribution $Pr(Y \leq y|X = x)$ can be expressed in terms of a copula function as

$$Pr(Y \leq y|X = x) = \frac{\partial C(u, v; \delta)}{\partial u} = C_1[F_X(x), F_Y(y); \delta] = \alpha \tag{4.5}$$

where $C(u, v; \delta)$ is the bivariate copula function with parameters δ[3]. In case Equation 4.5 is invertible with respect to v the relationship between X and quantile of Y can be expressed as

$$Q_{Y|X}(\alpha|x) = \boldsymbol{q}(x, \alpha; \delta) = F_Y^{[-1]}(D(F_X(x), \alpha; \delta)) \tag{4.6}$$

where D is the partial inverse of C_1 in the second argument and $F_Y^{[-1]}$ the pseudo-inverse of F_Y. If relationship in Equation 4.5 is not invertible, the numerical methods are required.

The distributions of F_X and F_Y can be specified either parametrically or non-parametrically. If we assume that F_Y is known only up to a location (μ) and scale (σ) parameter the quantile curve will have this form

$$Q_{Y|X}(\alpha|x) = \boldsymbol{q}(x, \alpha; \mu, \sigma, \delta) = \mu + \sigma F_Y^{[-1]}(D(F_X(x), \alpha; \delta)) \tag{4.7}$$

Thus, the copula quantile curve will have two more parameters which characterize the distribution of Y.

In this work we use Normal and t copula, both belonging to the Elliptical family. The quantile curve of Normal copula has the following form

$$Q_{Y|X}(\alpha|x) = \mu + \sigma F_Y^{[-1]}\left(\Phi\left(\rho\Phi^{-1}(F_X(x)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right) \tag{4.8}$$

while the quantile of t copula has the following

$$Q_{Y|X}(\alpha|x) = \mu + \sigma F_Y^{[-1]}\left(t_\eta\left[t_{\eta+1}^{-1}(\alpha)\sqrt{(\eta+1)^{-1}(1-\rho^2)\left(\eta + [t_\eta^{-1}(F_X(x))]^2\right)} + \rho t_\eta^{-1}(F_X(x))\right]\right) \tag{4.9}$$

For the derivation of these quantile functions please see Section 3.2.2 and Appendix A.2.

The copula quantile regression is a special case of the nonlinear quantile regression. Again, let $(y_1, \ldots, y_T)$ be a random sample on Y and $(x_1, \ldots, x_T)$ a random sample on X.

[3] Refer to Section 3.2.1 for the derivation.

Definition 4.2. The α copula quantile regression $\boldsymbol{q}(\mathbf{x}_t, \alpha; \boldsymbol{\delta})$ is a solution to the following problem:

$$\min_{\delta} \left(\sum_{t \in \mathcal{T}_p} \alpha |y_t - \boldsymbol{q}(\mathbf{x}_t, \alpha; \boldsymbol{\delta})| + \sum_{t \in \mathcal{T}_{1-\alpha}} (1-\alpha) |y_t - \boldsymbol{q}(\mathbf{x}_t, \alpha; \boldsymbol{\delta})| \right) \tag{4.10}$$

with $\mathcal{T}_\alpha = \{t : y_t \geq \boldsymbol{q}(\mathbf{x}_t, \alpha; \boldsymbol{\delta})\}$ and $\mathcal{T}_{1-\alpha}$ its complement.

In our model we regress returns at time *t+1* (r_{t+1}) on lagged Realized volatility at time t ($\sqrt{RV_t}$). The realized variance is computed as a sum of squared 5-minute intraday returns

$$RV_t = \sum_{i=1}^{M} r_i^2, \tag{4.11}$$

where i is the 5-minute intraday time interval. Thus, in the minimization problem of Equation 4.10 we replace y_t with r_{t+1} and x_t with $\sqrt{RV_t}$.

4.2.2 Model for CoVaR

In this work we consider the original definition of CoVaR as in Adrian & Brunnermeier (2011), *i.e.* the conditioning event for institution j is that institution i is exactly at its VaR level of distress.

Definition 4.3 (CoVaR). We denote by $CoVaR^{j|i}(\alpha, \beta, t)$ the VaR of institution j conditional on institution i being under distress ($r_{i,t} = VaR^i_{\alpha,t}$). That is, $CoVaR^{j|i}(\alpha, \beta, t)$ is implicitly defined by the

$$Pr(r_{j,t} \leq CoVaR^{j|i}_{\alpha,\beta,t} | r_{i,t} = VaR^i_{\alpha,t}) = \beta \tag{4.12}$$

We denote institution i's risk contribution to j by

$$\Delta CoVaR^{j|i}(\alpha, \beta, t) = CoVaR^{j|i}(\alpha, \beta, t) - CoVaR^{j|i}(0.5, \beta, t) \tag{4.13}$$

The estimation of CoVaR measure can be computed in various ways. Adrian & Brunnermeier (2011) use quantile regressions, but they also show that other alternatives like GARCH based models are possible. They primarily study the case where $j = system$, *i.e.*, when the return of the portfolio of all financial institutions is at its VaR level. On the other hand, Girardi & Ergün (2013) introduce a new conditioning event to describe the financial distress. They propose for distressed institution being at most at its VaR as opposed to being exactly at its VaR. The authors use a three-step procedure to estimate CoVaR. They employ univariate and multivariate GARCH models and take into consideration the skewness and excess kurtosis using the skewed-t distribution of Hansen (1994) for innovations. Karimalis & Nomikos (2014) use copulas from Archimedean family for CoVaR estimation. The authors employ a bivariate parametric model and use Inference Functions for Margins method to maximize the log-likelihood of the model. Having obtained the copula parameters they get the innovations implied by copula functions and the CoVaR is obtained rescaling Equation 3.4 by conditional mean and standard deviation similar to Equation 3.10[4]. The time-varying VaR is obtained from GJR-GARCH model of Glosten *et al.* (1993). Our approach is different from Karimalis & Nomikos (2014) in several aspects. We use semi-parametric copula quantile regression model for VaR estimation where the *inter* realized volatility drives the quantile returns forecast. Next difference consists in the definition of CoVaR. Karimalis & Nomikos (2014) use both definitions, the original from Adrian & Brunnermeier (2011) and the one from Girardi & Ergün (2013), whereas we use just the former. Finally, we study the VaR of institution j conditional on institution i being at its VaR level, which allows the study of spillover effects within specific industry.

4.2.3 Benchmark model for CoVaR

We compare the results from CoVaR model in subsection 4.2.2 with two alternative models. The first model, which we consider as the benchmark model estimates the VaR by rescaling the realized volatility. The second model estimates the VaR by the use of linear quantile regression (LQR). Both models, the benchmark and the LQR, use the same method as the copula quantile regression for CoVaR estimation. Thus, these models deffer in CoVaR estimation only on how the VaR is estimated. The alternative models have the following form

$$VaR^i_{t+1} = \phi^{-1}(\alpha) \cdot \sqrt{RV^i_t}, \qquad \text{(benchmark model)} \qquad (4.14)$$

$$VaR^i_{t+1} = q_\alpha(r_{t+1}|\Omega) = \beta_0(\alpha) + \beta_1(\alpha)\sqrt{RV^i_t} \qquad \text{(linear model)} \qquad (4.15)$$

where $\phi^{-1}(\cdot)$ is the quantile of standard Normal distribution and the linear model in Equation 4.15 is the linear quantile regression model.

4.3 Data

We use the same data as in Chapter 3.3. Let us refresh the information about this dataset. The data contain 21 most liquid U.S. stocks from the seven main

[4]The authors get the time varying innovations, hence time varying $CoVaR_t$, by estimating a time varying copula following the model introduced by Patton (2006).

market sectors defined in accordance with the Global Industry Classification Standard (GICS). We use three stocks with the highest market capitalization in a sector as representative of the analyzed sector. The selected stocks account for approximately half of the total capitalization of the sector. The sectors and representative stocks are listed in Table 3.1. The data spans from August 2004 to December 2011 and the source is Price-Data.com.[5]

For the computation of realized measures, we restrict the analysis to 5-minute returns during the 9:30 a.m. to 4:00 p.m. business hours of the New York Stock Exchange (NYSE). The data are time-synchronized by the same time-stamps. To rule out potential estimation bias which could come from low activity we eliminate transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2. Consequently, our data contains 1835 trading days. A descriptive statistics for the data can be found in Table 3.2.

4.4 Empirical Results

4.4.1 VaR

Based on methodology introduced in Section 4.2, the first step in CoVaR estimation is VaR estimation. We model the VaR using the copula quantile regression models introduced in Section 4.2.1, and being more specific we use the quantile curves from Equations 4.8 and 4.9. To save space, we report copula-quantile results in Appendix 4.A. We run the quantile copula regression for a set of quantiles $\alpha \in \{0.01, 0.05, 0.1, 0.5, 0.9, 0.95, 0.99\}$. For CoVaR estimation we make use of the results only for quantiles $\alpha = 0.05$ and $\alpha = 0.5$ though. This because it is a standard to use quantile $\alpha = 0.05$ in risk reporting, while the use of quantile $\alpha = 0.5$ is due to the definition of $\Delta CoVaR$. In Table 4.2 in Appendix 4.A we provide the estimated parameters for Normal copula for Information Technology stocks *i.e.* AAPL, INTC and MSFT. For estimating the VaR of each stock within the industry $VaR^i_{\alpha,t}$ is calculated by (copula) quantile regressing the returns of asset i on lagged realized volatility of asset k at quantile α *e.g.* for Information Technology $i, k \in \{AAPL, INTC, MSFT\}$ and $i \neq k$. Thus, for every asset we have two possible regressions (6 for each industry) *e.g.* $R^{AAPL}_{t+1} \sim \sqrt{RV}^{INTC}_t$ and $R^{AAPL}_{t+1} \sim \sqrt{RV}^{MSFT}_t$. Because we cannot arbitrarily omit any of the regressions we consider both of them in the analysis. In Table 4.2 almost all parameters are

[5] http://www.price-data.com/

statistically significant[6]. We notice that for all pairs there is asymmetry in correlations when comparing results of quantiles α with $1-\alpha$. The correlation for the left tail quantiles is higher (in absolute value) than the right tail quantiles. In addition, when we compare the quantile regression of the pairs when assets switch place (or from regressor to regressand), *e.g.* $R^i_{t+1} \sim \sqrt{RV}^k_t$ with $R^k_{t+1} \sim \sqrt{RV}^i_t$, we notice that other than for pair INTC-MSFT for the other pairs the dependence level at specified quantiles is much different.

In the Appendix 4.A.1 we include the Normal copula parameters results for the rest of industries. The correlation asymmetry holds for Consumer Discretionary and Staples stocks, with higher magnitude for pairs $MCD \rightleftarrows AMZN$, $KO \to PG$ and $KO \to WMT$[7]. For the last two pairs it is interesting to see a change in the direction of asymmetry, where the correlation of the right tail quantiles is higher than the respective left tail. For Telecommunications stocks the correlation asymmetry is almost lacking. For Financial stocks we observe a higher level of correlation at all quantiles under study when compared with other industries. In addition, the correlation asymmetry has higher magnitude and it is evident for quantiles $\alpha = 0.01$ and $\alpha = 0.99$. The correlation asymmetry persists also for Health Care and Energy stocks.

In comparison to Normal copula, the t copula allows for tail dependence through the degrees of freedom parameter η. In Appendix 4.A.2 we report estimated parameters from t copula quantile model. Commenting on Information Technology stocks in Table 4.9 we notice that t copula gives correlation estimates which are higher (in absolute value) than Normal copula[8]. In addition the degrees of freedom parameter η is significant at almost all pairs and quantiles[9]. This confirms the stylized fact that asset returns' distribution are heavy tailed. Finally, t copula gives consistent estimates as the Normal one with respect to correlation asymmetry. For Financial stocks in particular we observe the highest correlation levels and degrees of freedom in the rage 3-5. For the rest of industries we observe similar correlation asymmetries as in Information Technology stocks. A synthesis for copula parameter results reported in tables of Section 4.A, for both Normal and t copula, is presented in figures of Section 4.B.1. In these figures, through boxplots,

[6]Insignificant parameters in general correspond to median regression or quantile $\alpha = 0.5$. For this quantile the dependence is very weak and the optimization problem as in Equation 4.10 has converging difficulties.

[7]The arrows here show regression direction *e.g.* $A \to B$ means regression of returns of A on (lagged) realized volatility of B.

[8]Recall that t copula is able to capture higher dependence than Normal copula due to the degrees of freedom parameter.

[9]Again, we experience non-significant parameter for quantile $\alpha = 0.5$.

we see more clearly the asymmetry of correlation and the distribution of parameter estimates at each quantile.

4.4.2 CoVaR

We estimate CoVaR and $\Delta CoVaR$ as in Section 4.2.2. Recall that we use 4 models for CoVaR estimation, two based on copulas, one based on linear quantile regression (LQR) and the last one based on rescaled realized volatility. All these models allow for time-varying VaR and CoVaR. We plot the dynamics of VaR and CoVaR in Figures 4.17 - 4.23 of Appendix 4.B.4. For both risk measures we use the same confidence level *i.e.* $\alpha = \beta = 5\%$. From these plots it is evident that there is a strong relationship between $VaR^i_{\alpha,t}$ and $\Delta CoVaR_{\alpha,\beta,t}$. The $\Delta CoVaR$ achieves its highest (absolute) values during the 2008 and 2009, when crisis was at its peak.

In order to compare the differences among models and to identify the assets with the highest risk contribution we use the averages of $\Delta CoVaR$. In Table 4.1 we rank the average $\Delta CoVaR$ estimated from four different models[10]. The Normal copula model ranks Wells Fargo & Company (WFC) risk spillovers to Citigroup (C) in the top, with an average risk contribution of 4.75%. The next highest contributor is Apple (AAPL), whose risk spillover contributes to Microsoft (MSFT) and Intel Corporation (INTC) by an average of 4.6% and 4.54% respectively. The next top ranks are occupied by Financial industry. We note that WFC highly affects C and Bank of America Corporation (BAC), but not vice-versa. BAC and C risk spillovers towards WFC are close to the mean of the rank with contribution of around 2.7%. Similar risk spillover structure is between AAPL and MSFT and INTC. The latter risk contribution ranks close to the mean of of ranked pairs. For Apple it makes sense to highly contribute to INTC risk because the products of former use the processors of the latter as input. Thus, returns of Apple serve as a proxy for returns of Intel. At first look it might seem nonsense to see that Apple risk spillovers towards Microsoft are high, but not the vice versa. The economic rationale behind this result stands in the fact that Apple sells premium products, the demand of which is nonelastic, while Microsoft market is focused to products which are more affordable to general public, thus more prone to negative shocks of the economy[11]. Energy and Consumer Discretionary stocks have higher

[10]The synthesized results of Table 4.1 are decomposed in Tables 4.16 - 4.19 in Appendix 4.C. In those tables we bring summarized values for VaR and CoVaR for each industry.

[11]Recall that customers of Apple products stay in line for the newest models and pay the premium to get their products. On the other hands PC users use the same Operating System for years and update occasionally. In fact most of PC users postpone or do not even buy updates during their product lifetime.

Table 4.1: This table reports average $\Delta CoVaR^{j|i}_{\alpha,\beta,t}$ based on four different models. For each model we sort the stock pairs according to highest risk transmission. There are 42 pairs in total, *i.e.* 6 pairs for each industry. Each colour represents assets from the same industry. The second asset in the pair is the conditioning variable *e.g.* MSFT←AAPL means the VaR of MSFT conditional on AAPL being at its VaR. All risk measures, *i.e.* VaR and CoVaR, are computed at $\alpha = \beta = 5\%$ and expressed in percentage.

	Normal copula		*t* copula		Benchmark model		Linear model	
	Pair	$\Delta CoVaR$	Pair	$\Delta CoVaR$	Pair	$\Delta CoVaR$	Pair	$\Delta CoVaR$
1	C←WFC	-4.7475	C←WFC	-4.9775	C←WFC	-4.8964	MSFT←AAPL	-5.4773
2	MSFT←AAPL	-4.5977	BAC←WFC	-4.8247	BAC←WFC	-4.5013	INTC←AAPL	-4.7151
3	INTC←AAPL	-4.5369	MSFT←AAPL	-4.3013	C←BAC	-4.2574	C←WFC	-4.7138
4	BAC←WFC	-4.1345	SLB←XOM	-4.2088	BAC←C	-4.1724	BAC←WFC	-4.6508
5	C←BAC	-3.9245	BAC←C	-4.0803	WFC←C	-3.0602	C←BAC	-3.9910
6	BAC←C	-3.6531	INTC←AAPL	-4.0251	WFC←BAC	-2.9651	BAC←C	-3.8761
7	AMZN←MCD	-3.4096	AMZN←MCD	-3.9389	SLB←CVX	-2.3647	AMZN←MCD	-3.7659
8	SLB←XOM	-3.3051	C←BAC	-3.8706	CMCSA←VZ	-2.2746	SLB←XOM	-3.5797
9	SLB←CVX	-3.0459	CVX←XOM	-3.6367	AMZN←MCD	-2.2549	CVX←SLB	-3.2955
10	DIS←MCD	-3.0408	DIS←MCD	-3.3433	AMZN←DIS	-2.2152	CMCSA←T	-3.0624
11	CVX←SLB	-2.9900	CMCSA←T	-3.2802	CVX←XOM	-2.2103	CMCSA←VZ	-3.0028
12	DIS←AMZN	-2.9765	CVX←SLB	-2.9787	SLB←XOM	-2.2057	SLB←CVX	-2.9719
13	MRK←PFE	-2.9422	SLB←CVX	-2.8687	MSFT←INTC	-2.2051	DIS←MCD	-2.9003
14	CMCSA←VZ	-2.8455	CMCSA←VZ	-2.8571	MRK←PFE	-2.0110	AAPL←INTC	-2.8111
15	MRK←JNJ	-2.8367	AMZN←DIS	-2.8383	CVX←SLB	-1.9015	CVX←XOM	-2.7831
16	WFC←C	-2.7777	WFC←C	-2.7841	CMCSA←T	-1.8663	AMZN←DIS	-2.7789
17	WFC←BAC	-2.7140	MRK←PFE	-2.6620	MRK←JNJ	-1.8384	WFC←C	-2.7753
18	CMCSA←T	-2.6000	AAPL←INTC	-2.5454	INTC←AAPL	-1.8009	WFC←BAC	-2.7745
19	XOM←SLB	-2.5427	WFC←BAC	-2.5376	DIS←AMZN	-1.7628	MSFT←INTC	-2.7574
20	MSFT←INTC	-2.5312	MSFT←INTC	-2.4290	XOM←CVX	-1.7431	XOM←SLB	-2.7087
21	AMZN←DIS	-2.5275	XOM←SLB	-2.4151	DIS←MCD	-1.7339	MRK←PFE	-2.7043
22	AAPL←INTC	-2.5206	DIS←AMZN	-2.3823	MSFT←AAPL	-1.5885	AAPL←MSFT	-2.4092
23	AAPL←MSFT	-2.3860	AAPL←MSFT	-2.1841	XOM←SLB	-1.5733	DIS←AMZN	-2.3836
24	INTC←MSFT	-2.3517	MRK←JNJ	-2.1300	T←VZ	-1.5471	MRK←JNJ	-2.3298
25	T←CMCSA	-2.3413	KO←WMT	-2.0976	KO←PG	-1.4974	INTC←MSFT	-2.1847
26	CVX←XOM	-2.3038	INTC←MSFT	-2.0951	INTC←MSFT	-1.4303	T←CMCSA	-2.0873
27	VZ←CMCSA	-2.2275	XOM←CVX	-1.8736	T←CMCSA	-1.3972	KO←WMT	-2.0648
28	MCD←AMZN	-2.2124	T←VZ	-1.8615	PFE←JNJ	-1.3666	T←VZ	-2.0522
29	T←VZ	-2.1034	PG←WMT	-1.8421	KO←WMT	-1.3607	XOM←CVX	-1.9989
30	PFE←JNJ	-2.0794	VZ←T	-1.8229	AAPL←INTC	-1.3604	MCD←AMZN	-1.8687
31	XOM←CVX	-2.0598	PFE←MRK	-1.7989	VZ←CMCSA	-1.3452	PFE←JNJ	-1.8218
32	KO←WMT	-1.8952	VZ←CMCSA	-1.7943	VZ←T	-1.3250	PFE←MRK	-1.8163
33	VZ←T	-1.7802	KO←PG	-1.7732	PFE←MRK	-1.2978	VZ←CMCSA	-1.8059
34	KO←PG	-1.7434	T←CMCSA	-1.7636	WMT←PG	-1.2549	VZ←T	-1.7551
35	PFE←MRK	-1.6876	MCD←AMZN	-1.7073	PG←WMT	-1.2458	PG←WMT	-1.7303
36	PG←WMT	-1.6452	PFE←JNJ	-1.6750	JNJ←PFE	-1.1468	KO←PG	-1.6932
37	MCD←DIS	-1.5111	MCD←DIS	-1.6251	PG←KO	-1.1424	MCD←DIS	-1.6878
38	JNJ←PFE	-1.4745	JNJ←PFE	-1.6007	MCD←AMZN	-1.0807	JNJ←PFE	-1.5575
39	WMT←PG	-1.4654	WMT←PG	-1.3861	AAPL←MSFT	-1.0759	PG←KO	-1.3929
40	WMT←KO	-1.3702	JNJ←MRK	-1.3845	MCD←DIS	-1.0291	JNJ←MRK	-1.3608
41	PG←KO	-1.3666	PG←KO	-1.2563	WMT←KO	-0.9773	WMT←KO	-1.3484
42	JNJ←MRK	-1.1422	WMT←KO	-1.2225	JNJ←MRK	-0.9578	WMT←PG	-1.3303

Legend

Finanials	IT	Energy	Cons. Discretionary	Cons. Staples	Telecommunication	Health Care

than average risk spillover effects among each-other, while Consumer Staples and Health Care stocks have the lowest risk contribution. The latter industries rank is understandable as they produce and sell essential products that people are unable or unwilling to cut out of their budgets regardless of their financial situation.

Going back to our results, the specific ranking of the risk contributors differs significantly among the models used for $\Delta CoVaR$ estimation but is more consistent if we compare ranking of industries. Looking at the table we notice that highest average risk contributors are among Financial sector. In fact, most of top five or all top contributors belong to assets from this sector. For example, for the benchmark model all the Financials make the first six positions. Normal copula, t copula and the benchmark model identify the spillover of Wells Fargo & Company on Citigroup as the highest risk contributor, with very similar values for $\Delta CoVaR$ estimates. While for Financial sector it makes sense to see their assets to have higher risk contribution among each-other, it is interesting to see that Information Technology assets are ranked high. The linear quantile regression (LQR) model ranks the Apple risk spillovers in the top. Based on this model, if Apple is under distress, Microsoft (at a higher extent) and Intel, are both highly affected from its risk contribution. Another interesting result is that the average $\Delta CoVaR$ of the benchmark model have the lowest values when compared with the other models and the linear quantile regression (LQR) model has the highest. Copula quantile regression models have average $\Delta CoVaR$ values between the benchmark and the LQR model.

4.5 Conclusions

In this paper we estimate the VaR of an asset conditional on some other asset being under distress. Following a slightly different approach than current literature, where in the focus is systemic risk, we estimate the risk contribution that an asset has on some other individual asset. This approach allows the study of spillover effects within the industry. We estimate the conditional VaR on a set of 21 most liquid US assets from 7 industries and propose to use nonlinear quantile regression models based on copula theory to estimate CoVaR. The individual risk contribution estimated from these models gives estimates in between the benchmark model which is based on realized volatility and the Linear quantile regression model. We find that assets from Financial industry have the highest risk contribution among each other. Our proposed copula quantile model also identifies Apple to be a high risk contributor to Microsoft and Intel. Finally, assets from Consumer Staples industry and Health Care have the lowest risk contribution.

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4.A Tables

4.A.1 Normal copula parameters

Table 4.2: *Information Technology*: Normal copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
AAPL-INTC							
ρ	−0.3018	−0.2069	−0.1137	−0.3883	0.1422	0.1398	0.2022
s.e.	0.0494	0.0472	0.0306	4.3477	0.0194	0.0267	0.0637
μ	0.0523	0.2828	1.4089	0.0084	−1.4877	−2.2774	−1.2809
s.e.	1.6521	1.0467	1.0084	0.0485	0.9706	1.3031	2.6015
σ	1.1016	1.0963	1.5894	−0.0069	1.6961	1.7022	1.2547
s.e.	0.3184	0.3495	0.4475	0.1144	0.4409	0.4100	0.5310
AAPL-MSFT							
ρ	−0.2022	−0.1068	−0.0474	0.0236	0.1669	0.1277	0.1234
s.e.	0.0353	0.0307	0.0120	1.1751	0.0363	0.0351	0.0598
μ	0.4628	1.9205	4.6724	0.0053	0.0505	−1.0010	−0.8861
s.e.	3.3038	2.5420	2.6543	0.0495	0.6445	1.5744	3.3582
σ	1.0827	1.5968	2.9587	−0.0452	0.9934	1.3247	1.1991
s.e.	0.6484	0.8389	1.0620	2.3836	0.2856	0.5488	0.8062
INTC-AAPL							
ρ	−0.4040	−0.2370	−0.0543	−0.1273	0.1217	0.2999	0.2178
s.e.	0.0267	0.0324	0.0061	0.1373	0.0230	0.0270	0.0237
μ	0.1950	1.4547	9.6440	−0.0103	−2.5042	−0.4639	−2.5896
s.e.	0.3877	0.5864	1.6804	0.0442	0.8382	0.3799	1.3994
σ	1.0149	1.5443	6.2682	0.2381	2.4488	1.2307	1.4997
s.e.	0.1102	0.2080	1.0111	0.2736	0.5570	0.1955	0.3009

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
INTC-MSFT							
ρ	−0.3977	−0.2833	−0.1458	−0.1834	0.1562	0.1830	0.2532
s.e.	0.0246	0.0365	0.0280	0.1858	0.0202	0.0250	0.0339
μ	0.1761	0.1204	1.7209	0.0095	−1.4457	−2.0130	−0.7903
s.e.	0.4792	0.4330	0.9656	0.0457	0.7571	0.8937	0.9453
σ	1.0232	1.1126	1.9881	−0.1337	1.8426	1.7947	1.1236
s.e.	0.1149	0.1917	0.5532	0.2448	0.5567	0.3178	0.2232
MSFT-AAPL							
ρ	−0.5678	−0.2250	−0.1006	−0.0360	0.1214	0.1919	0.2688
s.e.	0.0428	0.0119	0.0192	0.0089	0.0395	0.0323	0.0532
μ	−1.0770	1.2677	2.4011	0.0035	−0.4063	−0.4283	−0.5400
s.e.	0.3526	0.3471	0.7563	0.0311	0.7662	0.6367	0.9505
σ	0.7677	1.5972	2.6279	1.3990	1.3161	1.2427	1.1486
s.e.	0.0932	0.2289	0.4940	1.1010	0.5392	0.3328	0.3222
MSFT-INTC							
ρ	−0.4074	−0.2950	−0.0601	−0.1203	0.1015	0.1690	0.2305
s.e.	0.0380	0.0177	0.0047	0.1742	0.0159	0.0195	0.0247
μ	0.3323	0.5996	6.1525	−0.0192	−2.1877	−1.5339	−1.3391
s.e.	0.3957	0.2460	0.8571	0.0320	0.7413	0.7539	0.7042
σ	1.0348	1.2943	5.1246	0.1919	2.5346	1.7097	1.2661
s.e.	0.1221	0.1396	0.5698	0.4252	0.5660	0.3836	0.2122

Table 4.3: *Consumer Discretionary*: Normal copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
AMZN-DIS							
ρ	−0.2302	−0.1949	−0.0809	0.0216	0.0957	0.1900	0.1670
s.e.	0.0226	0.0274	0.0124	0.0348	0.0124	0.0455	0.0482
μ	2.5660	0.8830	3.5374	0.0277	−4.2823	−1.3921	−1.3033
s.e.	0.9939	1.0964	1.3705	0.1173	1.4822	1.0701	1.9179
σ	1.4936	1.2796	2.6173	0.7514	2.6617	1.3887	1.1423
s.e.	0.1996	0.3745	0.6826	2.0731	0.5497	0.3007	0.3165
AMZN-MCD							
ρ	−0.3032	−0.2557	−0.1272	0.0149	0.1070	0.1951	0.1535
s.e.	0.0586	0.0428	0.0177	0.1855	0.0130	0.0261	0.0353
μ	0.2724	0.1958	2.0872	0.0548	−3.3348	−1.7327	−4.9249
s.e.	1.0942	0.8299	0.8500	0.1508	1.5219	0.8931	1.8886
σ	1.0392	1.0862	1.9976	0.2205	2.3446	1.4941	1.6137
s.e.	0.2000	0.3131	0.4030	2.7038	0.6562	0.2732	0.3587
DIS-AMZN							
ρ	−0.2386	−0.1733	−0.1068	−0.0144	0.1255	0.1721	0.2037
s.e.	0.0243	0.0199	0.0135	0.2234	0.0151	0.0173	0.0309
μ	1.4499	2.6172	2.2345	0.0733	−3.2836	−2.7928	−2.2100
s.e.	0.9410	0.8265	0.6382	0.1620	0.7525	0.9634	1.5971
σ	1.2686	2.2662	2.4924	−0.1709	3.0548	2.1981	1.4351
s.e.	0.2648	0.4256	0.4621	2.7328	0.5336	0.3905	0.3614

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
DIS-MCD							
ρ	−0.1960	−0.1436	−0.1252	−0.0290	0.1804	0.2028	0.2168
s.e.	0.0321	0.0158	0.0145	0.1704	0.0181	0.0176	0.0150
μ	2.2141	3.3323	1.6334	0.0568	−1.9383	−2.0906	−2.6558
s.e.	1.1852	0.9849	0.4045	0.0898	0.5781	0.5151	0.8883
σ	1.4559	2.6769	2.0469	0.1288	2.2199	1.8631	1.4605
s.e.	0.3003	0.5402	0.2841	1.2799	0.3591	0.2610	0.2169
MCD-AMZN							
ρ	−0.2481	−0.2226	−0.0806	0.0275	0.1369	0.1668	0.1439
s.e.	0.0312	0.0264	0.0108	0.0984	0.0246	0.0309	0.0466
μ	0.8400	0.2885	2.4849	0.0438	−0.8793	−0.8842	−0.8492
s.e.	1.2131	0.3516	0.8017	0.0734	0.6469	0.7791	1.2501
σ	1.2681	1.1515	2.8681	0.2125	1.6566	1.4779	1.2003
s.e.	0.4602	0.1977	0.5513	1.4219	0.5203	0.4893	0.3970
MCD-DIS							
ρ	−0.2697	−0.2006	−0.1466	0.1433	0.1454	0.2075	0.2056
s.e.	0.0325	0.0342	0.0169	0.1675	0.0475	0.0341	0.0441
μ	0.5597	0.4046	1.2078	0.0531	−0.6855	−0.7603	−0.3343
s.e.	0.7957	0.6003	0.5006	0.0284	0.8496	0.7444	0.7459
σ	1.1725	1.2065	1.9196	−0.1723	1.4782	1.4537	1.0361
s.e.	0.2829	0.2888	0.3830	0.3029	0.6668	0.4140	0.2986

Table 4.4: *Consumer Staples*: Normal copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
KO-PG							
ρ	−0.2171	−0.1741	−0.0976	−0.0209	0.2055	0.2037	0.2916
s.e.	0.0131	0.0131	0.0092	0.2549	0.0177	0.0313	0.0284
μ	1.8808	1.1478	2.2749	0.0217	−0.7804	−0.9719	−1.1547
s.e.	0.4458	0.4259	0.6103	0.0486	0.3115	0.4989	0.4680
σ	1.5545	1.6722	3.2332	−0.0790	1.7973	1.5843	1.3645
s.e.	0.1904	0.2845	0.6039	1.7422	0.3054	0.2793	0.1634
KO-WMT							
ρ	−0.2668	−0.2117	−0.1091	−0.2074	0.2099	0.2084	0.3695
s.e.	0.0327	0.0198	0.0119	1.1005	0.0220	0.0258	0.0513
μ	0.7997	0.8319	1.9967	0.0221	−0.3787	−0.7991	0.0579
s.e.	0.4426	0.3496	0.4251	0.0235	0.2819	0.4397	0.4091
σ	1.2215	1.4714	2.9722	−0.0204	1.4033	1.4676	0.9543
s.e.	0.1581	0.2574	0.4587	0.2269	0.3124	0.2932	0.1625
PG-KO							
ρ	−0.1840	−0.2018	−0.0939	−0.0045	0.1158	0.1455	0.2088
s.e.	0.0531	0.0311	0.0089	0.5389	0.0089	0.0202	0.0190
μ	1.6277	1.2082	2.1377	0.0728	−1.7513	−1.7293	−1.5831
s.e.	0.7927	0.4234	0.5415	0.5080	0.4438	0.5745	0.8119
σ	1.4605	1.8747	3.2391	−0.0630	2.6936	2.1101	1.4860
s.e.	0.3208	0.3049	0.4922	6.9210	0.4063	0.3129	0.3151

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
PG-WMT							
ρ	−0.2713	−0.2361	−0.1026	−0.1406	0.1333	0.2297	0.2234
s.e.	0.0274	0.0422	0.0119	0.1874	0.0136	0.0248	0.0197
μ	0.4777	0.2723	1.3943	0.0812	−1.2239	−0.4264	−0.9934
s.e.	0.2756	0.3760	0.4950	0.0350	0.3851	0.4576	0.6929
σ	1.1354	1.1655	2.4589	−0.1552	2.1708	1.3099	1.2837
s.e.	0.0900	0.2609	0.4564	0.3719	0.3533	0.2969	0.2618
WMT-KO							
ρ	−0.2164	−0.2588	−0.0995	−0.1177	0.0765	0.1444	0.1974
s.e.	0.0267	0.0280	0.0088	0.0908	0.0130	0.0212	0.0186
μ	2.1614	0.2751	1.9574	−0.0356	−2.5910	−1.6846	−2.6383
s.e.	0.8160	0.3023	0.4557	0.0236	1.1290	0.4544	1.6875
σ	1.8430	1.1544	2.7145	−0.2965	3.2276	2.0039	1.8978
s.e.	0.3222	0.1983	0.3865	0.2872	1.0533	0.3196	0.6491
WMT-PG							
ρ	−0.2275	−0.2061	−0.0996	−0.1636	0.0690	0.1274	0.1638
s.e.	0.0279	0.0211	0.0096	0.6551	0.0112	0.0151	0.0222
μ	2.6355	0.8259	2.1598	−0.0305	−2.7801	−1.4899	−3.2000
s.e.	0.7870	0.3464	0.4461	0.0249	0.8986	0.5270	1.5893
σ	1.9825	1.4751	2.8670	−0.0483	3.3785	1.8732	2.0791
s.e.	0.3252	0.1833	0.3588	0.3013	0.7366	0.3228	0.6154

Table 4.5: *Telecommunication Services*: Normal copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
CMCSA-T							
ρ	−0.2892	−0.2403	−0.1811	−0.1588	0.1589	0.2317	0.1810
s.e.	0.0302	0.0311	0.0214	0.1649	0.0155	0.0199	0.0210
μ	0.5777	0.7066	1.4168	−0.0048	−1.3133	−1.5753	−3.3138
s.e.	0.9266	0.5938	0.5676	0.0456	0.4946	0.7113	1.1396
σ	1.0198	1.2787	1.7753	0.1268	1.7121	1.6023	1.5512
s.e.	0.2455	0.2389	0.3098	0.2722	0.2855	0.2757	0.2062
CMCSA-VZ							
ρ	−0.2296	−0.2557	−0.1106	−0.1911	0.1582	0.2350	0.2128
s.e.	0.0259	0.0223	0.0107	0.4559	0.0168	0.0234	0.0220
μ	2.7956	0.5172	2.8640	0.0176	−1.6233	−1.9124	−2.7712
s.e.	1.5422	0.3361	0.6753	0.0451	0.4561	0.6093	0.8434
σ	1.5375	1.1802	2.5564	−0.0553	1.8196	1.6691	1.4722
s.e.	0.3644	0.1442	0.3775	0.2453	0.2303	0.2528	0.1840
T-CMCSA							
ρ	−0.3245	−0.2226	−0.1798	−0.0854	0.1004	0.1911	0.3005
s.e.	0.0234	0.0172	0.0152	0.0405	0.0180	0.0355	0.0268
μ	0.1521	1.1921	1.3137	−0.0182	−2.2852	−1.3076	−0.4049
s.e.	0.4456	0.5193	0.3227	0.0264	1.1036	0.5861	0.7470
σ	1.0175	1.5970	1.9760	0.6401	2.7649	1.7282	1.0242
s.e.	0.1359	0.3093	0.2421	0.4922	0.8464	0.3901	0.2664

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
T-VZ							
ρ	−0.2597	−0.2358	−0.1696	−0.1362	0.2413	0.2397	0.2596
s.e.	0.0520	0.0228	0.0184	0.0853	0.0235	0.0175	0.0227
μ	0.0841	0.8735	1.1975	−0.0174	−1.1558	−1.1774	−1.3821
s.e.	0.6194	0.3597	0.3790	0.0349	0.3144	0.3663	0.4878
σ	0.9248	1.4350	1.9270	0.2939	1.9074	1.5937	1.2655
s.e.	0.1603	0.1994	0.3020	0.3134	0.2512	0.2277	0.2148
VZ-CMCSA							
ρ	−0.2431	−0.2829	−0.1605	−0.0888	0.1163	0.2813	0.2492
s.e.	0.0241	0.0405	0.0211	0.0379	0.0147	0.0300	0.0197
μ	0.7632	0.2062	1.4174	−0.0301	−2.2303	−0.6762	−0.9731
s.e.	0.4456	0.2812	0.5804	0.0271	0.9422	0.4130	0.7381
σ	1.1390	1.1318	2.1258	0.5225	2.8362	1.4659	1.1823
s.e.	0.1441	0.1576	0.4613	0.4782	0.7355	0.2774	0.2551
VZ-T							
ρ	−0.2836	−0.2554	−0.1201	−0.1358	0.0885	0.2393	0.2046
s.e.	0.0306	0.0276	0.0131	0.0546	0.0096	0.0204	0.0231
μ	0.3330	0.5732	3.5643	−0.0308	−3.4365	−0.9265	−1.0626
s.e.	0.5599	0.4101	0.6214	0.0267	1.1059	0.3784	0.6018
σ	0.9879	1.3543	3.7717	0.5138	3.8306	1.4975	1.2012
s.e.	0.1716	0.2371	0.5308	0.2843	0.8328	0.2658	0.1945

Table 4.6: *Financials*: Normal copula parameters estimated on full sample.

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
BAC-C							
ρ	−0.4898	−0.3415	−0.1616	−0.0883	0.2666	0.3127	0.3691
s.e.	0.0124	0.0191	0.0129	0.0101	0.0102	0.0141	0.0116
μ	1.0091	1.7262	4.6336	−0.0288	−1.7085	−1.0404	−1.7173
s.e.	0.1973	0.2766	0.6074	0.0377	0.2034	0.2359	0.3359
σ	0.9299	1.1412	2.5833	1.8730	2.0586	1.1999	1.0157
s.e.	0.0571	0.1047	0.2587	0.6600	0.1311	0.1301	0.1126
BAC-WFC							
ρ	−0.4989	−0.2988	−0.1940	−0.1072	0.1407	0.2089	0.2649
s.e.	0.0101	0.0148	0.0087	0.0175	0.0072	0.0058	0.0063
μ	0.8276	1.9702	3.6565	−0.0651	−3.9266	−3.0406	−3.4353
s.e.	0.2326	0.4117	0.3563	0.0353	0.5615	0.3611	0.4496
σ	0.9141	1.2591	2.3514	1.4520	3.0609	1.8321	1.1318
s.e.	0.0836	0.1333	0.1752	0.5119	0.3109	0.1382	0.1243
C-BAC							
ρ	−0.4266	−0.3505	−0.2007	−0.1223	0.1958	0.3280	0.3906
s.e.	0.0204	0.0219	0.0080	0.0079	0.0160	0.0146	0.0137
μ	1.3907	1.4705	4.1262	0.0641	−2.2801	−1.1147	−1.1513
s.e.	0.4505	0.4752	0.3932	0.0649	0.4407	0.2407	0.4689
σ	0.8599	1.0958	2.4250	1.9100	2.1572	1.2732	0.8983
s.e.	0.1407	0.1363	0.1595	0.4799	0.2735	0.1012	0.1148

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
C-WFC							
ρ	−0.4256	−0.3271	−0.1540	−0.1251	0.1996	0.3010	0.3346
s.e.	0.0106	0.0236	0.0076	0.0080	0.0120	0.0176	0.0049
μ	1.5864	1.8386	4.5412	0.0763	−2.1263	−1.2492	−2.1101
s.e.	0.4165	0.3479	0.5302	0.0574	0.3857	0.3077	0.3492
σ	0.9038	1.1811	2.3055	2.0393	2.1587	1.3121	1.0749
s.e.	0.1330	0.1080	0.1940	0.4469	0.2151	0.1315	0.1080
WFC-BAC							
ρ	−0.4869	−0.2991	−0.1400	−0.0796	0.1395	0.2260	0.3103
s.e.	0.0124	0.0172	0.0104	0.0410	0.0086	0.0105	0.0039
μ	0.4065	1.3970	3.7640	−0.0294	−3.6765	−2.5140	−2.8862
s.e.	0.3367	0.2932	0.4361	0.0380	0.3966	0.3619	0.3555
σ	0.9104	1.2435	2.5098	−0.4230	2.5214	1.5155	1.1671
s.e.	0.0984	0.1085	0.2450	0.5113	0.1811	0.1293	0.0879
WFC-C							
ρ	−0.4898	−0.3133	−0.1705	−0.0818	0.2016	0.3186	0.3808
s.e.	0.0102	0.0156	0.0141	0.1616	0.0098	0.0127	0.0071
μ	0.5631	1.2163	2.9849	−0.0331	−2.4272	−1.4480	−1.9590
s.e.	0.2103	0.2604	0.4477	0.0430	0.3104	0.2107	0.3371
σ	0.9318	1.2135	2.1692	0.1204	2.0544	1.2950	1.0705
s.e.	0.0947	0.1188	0.2373	0.5547	0.1947	0.0978	0.0979

Table 4.7: *Energy*: Normal copula parameters estimated on full sample.

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
CVX-SLB							
ρ	−0.2938	−0.2768	−0.0833	−0.0060	0.1475	0.1859	0.2498
s.e.	0.0401	0.0314	0.0103	0.0217	0.0280	0.0235	0.0207
μ	0.7555	0.7907	3.2223	0.2580	−0.5963	−1.1129	−2.6660
s.e.	0.9831	0.6673	0.8053	0.8672	0.7263	0.6131	1.0366
σ	1.1706	1.3534	2.9332	−2.3242	1.3042	1.4925	1.7428
s.e.	0.2512	0.3520	0.4506	12.3864	0.4046	0.3614	0.3793
CVX-XOM							
ρ	−0.2287	−0.1744	−0.0980	−0.0802	0.0951	0.1654	0.1733
s.e.	0.0370	0.0173	0.0088	0.6943	0.0141	0.0198	0.0331
μ	1.8817	3.0908	3.4911	0.0691	−1.8511	−2.4462	−3.2165
s.e.	1.0772	1.0102	0.6507	0.0825	1.0602	0.7429	1.6556
σ	1.3538	2.2894	3.1008	0.0415	2.1149	2.0813	1.7127
s.e.	0.3497	0.5143	0.3814	0.5968	0.6988	0.3813	0.5230
SLB-CVX							
ρ	−0.2700	−0.2026	−0.1032	−0.0176	0.1952	0.1958	0.2064
s.e.	0.0379	0.0195	0.0129	0.0124	0.0317	0.0341	0.0380
μ	1.4628	2.0515	3.2993	−0.0492	−0.3671	−1.1816	−2.0961
s.e.	1.5369	1.0646	0.9291	0.1036	0.6366	0.8214	3.4373
σ	1.2141	1.6561	2.3128	1.9378	1.2071	1.3529	1.4049
s.e.	0.3291	0.3575	0.4512	1.7277	0.2715	0.2464	0.6784

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
SLB-XOM							
ρ	−0.2915	−0.2252	−0.1871	−0.0152	0.1009	0.1775	0.2083
s.e.	0.0302	0.0264	0.0290	0.0077	0.0113	0.0237	0.0529
μ	1.7308	1.3181	1.2728	−0.0880	−2.9267	−1.5797	−2.9742
s.e.	1.2409	1.0366	0.6418	0.1179	1.1845	0.7914	3.2800
σ	1.2618	1.4015	1.5104	2.9786	2.2709	1.4726	1.5588
s.e.	0.2826	0.3924	0.2616	2.0523	0.5994	0.2800	0.6293
XOM-CVX							
ρ	−0.3456	−0.2246	−0.0868	−0.0556	0.0480	0.1342	0.1719
s.e.	0.0297	0.0144	0.0084	0.0515	0.0066	0.0222	0.0253
μ	−0.1445	1.2888	3.8250	0.0373	−8.0802	−3.0127	−3.6629
s.e.	0.7383	0.4093	1.0220	0.0928	1.6566	1.3440	1.0393
σ	0.8133	1.5784	3.4312	0.5571	6.3133	2.3681	1.8558
s.e.	0.2444	0.2259	0.6358	0.7794	1.1772	0.5828	0.3066
XOM-SLB							
ρ	−0.3600	−0.3004	−0.1526	−0.0690	0.2665	0.2059	0.2424
s.e.	0.0583	0.0447	0.0176	0.5277	0.0376	0.0186	0.0326
μ	0.2347	0.2989	1.0143	0.0920	−0.4703	−1.5642	−1.6220
s.e.	1.1705	0.5472	0.3239	0.0733	0.3246	0.5211	0.9633
σ	0.9630	1.1750	1.6264	0.0528	1.3227	1.7139	1.3586
s.e.	0.2906	0.2341	0.2541	0.6711	0.2413	0.2745	0.2872

Table 4.8: *Health Care*: Normal copula parameters estimated on full sample.

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
JNJ-MRK							
ρ	−0.3576	−0.2501	−0.1599	0.0056	0.1572	0.2867	0.3712
s.e.	0.0397	0.0301	0.0196	0.0016	0.0263	0.0490	0.0527
μ	−0.0744	0.4198	0.4040	0.0018	−0.6003	−0.0780	0.6082
s.e.	0.3362	0.2320	0.2492	0.0208	0.3837	0.2418	0.4918
σ	0.8512	1.3027	1.4421	12.6582	1.6400	1.0434	0.8377
s.e.	0.1436	0.1983	0.2572	37.1725	0.4421	0.2128	0.1793
JNJ-PFE							
ρ	−0.2567	−0.2144	−0.0876	0.0209	0.0971	0.2173	0.2344
s.e.	0.0466	0.0335	0.0102	0.0148	0.0101	0.0235	0.0397
μ	0.2458	0.4224	1.7367	0.0038	−2.1685	−0.8518	−0.3788
s.e.	0.4200	0.5055	0.5118	0.0199	0.4697	0.4198	1.1707
σ	1.0364	1.3332	2.8181	1.0115	3.3308	1.6495	1.1140
s.e.	0.2064	0.4018	0.5976	1.8863	0.5556	0.3021	0.5082
MRK-JNJ							
ρ	−0.3811	−0.3084	−0.1156	−0.1711	0.1522	0.1860	0.2656
s.e.	0.0977	0.0348	0.0193	0.7588	0.0234	0.0295	0.0331
μ	−0.5559	0.3588	2.0893	0.0154	−1.1245	−1.2435	−0.7549
s.e.	0.8834	0.3348	0.7380	0.0317	0.5905	0.6839	0.8950
σ	0.8155	1.1945	2.4225	−0.0512	1.7689	1.5079	1.1047
s.e.	0.2163	0.1373	0.5197	0.2959	0.3764	0.3327	0.2642

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
MRK-PFE							
ρ	−0.3011	−0.2487	−0.1040	−0.0301	0.1141	0.1992	0.1821
s.e.	0.0647	0.0310	0.0128	0.0530	0.0126	0.0233	0.0213
μ	−0.6110	0.5085	2.7058	0.0201	−2.1410	−1.6425	−3.4688
s.e.	1.1298	0.3873	0.9486	0.0363	0.7592	0.6205	2.0479
σ	0.7644	1.2643	2.8082	−0.6637	2.4067	1.7247	1.8495
s.e.	0.2908	0.1872	0.6346	1.4241	0.5150	0.2895	0.5810
PFE-JNJ							
ρ	−0.2479	−0.2363	−0.1484	−0.0519	0.1253	0.1629	0.2092
s.e.	0.0397	0.0256	0.0227	0.0152	0.0160	0.0122	0.0527
μ	1.6450	0.9596	1.0225	−0.0320	−1.9691	−1.9270	−0.2407
s.e.	0.8058	0.5334	0.6003	0.0542	0.5845	0.5690	0.8891
σ	1.3972	1.4867	1.6755	0.6288	2.4635	1.9119	0.9347
s.e.	0.2570	0.2868	0.3593	0.6601	0.4266	0.2947	0.3698
PFE-MRK							
ρ	−0.2654	−0.2456	−0.1317	−0.0468	0.1389	0.2474	0.2514
s.e.	0.0515	0.0265	0.0142	0.0085	0.0221	0.0264	0.0500
μ	0.4508	0.7209	1.3551	0.0000	−1.5884	−0.6011	−0.2263
s.e.	1.3110	0.5312	0.5801	0.0563	0.7220	0.4585	0.6525
σ	1.1184	1.3660	1.8824	1.1536	2.2012	1.2444	1.0367
s.e.	0.3845	0.2390	0.3284	0.7537	0.4551	0.2170	0.1852

4.A.2 t-copula parameters

Table 4.9: *Information Technology*: t-copula parameters estimated on full sample.

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
AAPL-INTC							
ρ	−0.3941	−0.1278	−0.1742	−0.0849	0.2013	0.2715	0.3409
s.e.	0.0836	0.0425	0.0504	0.4565	0.0451	0.0817	0.1581
η	7.7205	18.7120	6.6327	10.8098	10.0339	5.2677	6.4130
s.e.	2.5719	10.0473	1.9229	741.0530	3.6140	1.7872	4.5568
μ	0.1604	2.4298	0.4361	0.0066	−0.6131	−0.1714	−0.0866
s.e.	1.7539	2.1865	0.9788	0.0500	0.8293	0.9056	1.9587
σ	1.1061	1.7579	1.1819	0.0795	1.3005	1.0515	1.0568
s.e.	0.3436	0.6387	0.4094	0.4276	0.3687	0.2854	0.4770
AAPL-MSFT							
ρ	−0.3994	−0.3205	−0.2098	−0.0015	0.1787	0.2135	0.3188
s.e.	0.1156	0.0677	0.0756	0.0232	0.0807	0.0669	0.0910
η	4.0346	3.9905	4.8708	8.8790	5.6198	6.8023	4.8121
s.e.	1.1412	0.6994	1.5119	2324.5796	2.7556	2.2181	1.6639
μ	0.1243	0.0900	0.0910	0.0502	−0.2471	−0.2009	−0.5728
s.e.	1.2409	0.8959	0.6473	0.7129	0.8913	1.2162	2.2842
σ	1.1211	1.1070	1.0319	−1.9584	1.0818	1.0483	1.1764
s.e.	0.2338	0.2844	0.2844	28.9189	0.3693	0.3951	0.5321
INTC-AAPL							
ρ	−0.3951	−0.4724	−0.2151	−0.1177	0.1188	0.2672	0.3246
s.e.	0.0318	0.0402	0.0199	0.1669	0.0248	0.0363	0.0318
η	6.2286	7.0362	13.7402	3.4729	19.4200	7.8651	8.0456
s.e.	0.9944	2.4298	2.6852	6.1236	6.5501	1.8975	1.1501
μ	1.0377	−0.2923	1.3657	−0.0069	−3.0118	−0.8330	−2.3959
s.e.	0.9418	0.3211	0.7008	0.0450	1.0456	0.7983	1.0857
σ	1.2449	0.9783	1.8151	0.1753	2.7701	1.4363	1.5183
s.e.	0.2521	0.1826	0.3902	0.2453	0.5751	0.3554	0.2996

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
INTC-MSFT							
ρ	−0.3749	−0.2742	−0.1794	0.0922	0.2803	0.2603	0.3838
s.e.	0.0281	0.0385	0.0285	0.1374	0.0412	0.0348	0.0597
η	4.7682	5.9725	5.7730	7.0816	4.7484	6.9183	5.5225
s.e.	0.7175	1.0953	0.7667	37.3714	1.3206	1.4877	1.5045
μ	2.5420	1.0869	1.2730	0.0018	−0.1316	−1.1998	−0.9547
s.e.	1.0321	0.8048	0.6678	0.0408	0.4911	0.7252	1.2102
σ	1.6531	1.4661	1.6636	0.3375	1.0649	1.5287	1.2087
s.e.	0.2888	0.3525	0.3967	0.4725	0.2903	0.3012	0.3754
MSFT-AAPL							
ρ	−0.4422	−0.3270	−0.2317	−0.1977	0.1412	0.2509	0.3647
s.e.	0.0363	0.0279	0.0577	0.1886	0.0435	0.0470	0.0396
η	4.2981	5.4196	5.0348	4.0492	5.6595	5.5625	6.4122
s.e.	0.7114	1.3039	1.4926	8.5748	1.4990	1.1722	1.5157
μ	0.5769	0.4790	0.1035	−0.0140	−0.8041	−1.0802	−0.7860
s.e.	0.6154	0.5054	0.4139	0.0309	0.6709	0.7514	0.9555
σ	1.2009	1.2802	1.1295	0.2350	1.5706	1.6188	1.2696
s.e.	0.1623	0.2831	0.3145	0.2230	0.4425	0.3629	0.2333
MSFT-INTC							
ρ	−0.4186	−0.2622	−0.2166	0.5454	0.2698	0.2343	0.3923
s.e.	0.0437	0.0255	0.0317	1.1010	0.0592	0.0373	0.0540
η	8.3864	7.8105	5.9034	22.4841	4.4070	5.7390	5.1074
s.e.	2.8781	1.0434	1.2785	1418.3915	1.1478	1.1895	1.1646
μ	0.7897	1.9430	0.9747	−0.0164	−0.4346	−1.4513	−0.6577
s.e.	0.6517	0.5983	0.3600	0.0313	0.3728	0.5992	0.5686
σ	1.1837	1.9198	1.6243	−0.0396	1.3486	1.7233	1.1633
s.e.	0.1914	0.2729	0.2614	0.0919	0.2458	0.3125	0.1540

Table 4.10: *Consumer Discretionary*: t-copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
AMZN-DIS							
ρ	−0.6081	−0.4461	−0.2356	0.0271	0.1853	0.3060	0.3202
s.e.	0.0729	0.0414	0.0544	0.0999	0.0354	0.0475	0.1858
η	6.0654	4.3560	4.1945	3.1796	3.1462	6.7294	4.0232
s.e.	2.5887	0.7737	1.0237	11.3754	0.4363	1.4954	2.2739
μ	−0.9324	−0.0070	0.4888	0.0514	−0.2986	−0.8315	0.1127
s.e.	0.6229	0.4893	0.7086	0.0905	0.5934	0.8832	1.7865
σ	0.9917	1.1595	1.2420	0.3217	1.1418	1.2788	0.9688
s.e.	0.1478	0.1535	0.3365	1.2347	0.2329	0.2740	0.3104
AMZN-MCD							
ρ	−0.4307	−0.3335	−0.3473	0.0006	0.1544	0.2122	0.3027
s.e.	0.0932	0.0769	0.0417	0.0066	0.0155	0.0323	0.1093
η	5.4688	3.9616	4.7686	0.7197	6.8489	5.5982	3.9959
s.e.	1.7103	1.0277	1.2324	1.5498	1.0359	0.7934	0.8906
μ	0.6107	0.5715	−0.1869	0.0361	−2.5902	−1.1507	−0.2565
s.e.	0.8408	0.8697	0.4827	0.1265	1.0311	1.1184	1.3163
σ	1.1825	1.2754	0.9949	0.8563	2.0131	1.3139	1.0486
s.e.	0.1584	0.2567	0.2604	1.6188	0.4159	0.2758	0.2449
DIS-AMZN							
ρ	−0.3626	−0.3294	−0.2626	0.0247	0.1573	0.1815	0.2434
s.e.	0.0958	0.0564	0.0325	0.0543	0.0340	0.0300	0.0618
η	5.4802	4.7122	5.5212	1.2603	3.6530	9.5712	5.0761
s.e.	1.7786	0.9065	0.8291	2.3912	0.4663	1.7575	0.9853
μ	1.4857	0.2049	0.4150	0.0515	−0.7077	−2.2417	−0.5642
s.e.	0.9085	0.5944	0.4055	0.0518	0.4220	0.7353	1.4501
σ	1.3415	1.1205	1.3619	0.1994	1.3861	1.9731	1.0391
s.e.	0.2751	0.2775	0.2800	0.5214	0.2210	0.3068	0.2685

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
DIS-MCD							
ρ	−0.4628	−0.3367	−0.1661	0.0419	0.2361	0.2070	0.2594
s.e.	0.0332	0.0410	0.0437	0.0923	0.0256	0.0175	0.0328
η	4.5391	5.6541	5.1201	1.0368	6.8039	9.9698	9.2381
s.e.	0.8077	1.3732	1.4339	1.4335	1.2080	2.1395	1.9284
μ	0.0863	0.6088	0.4800	0.0597	−1.3150	−2.1336	−3.5283
s.e.	0.4289	0.4937	0.6603	0.0400	0.5170	0.5459	1.2610
σ	1.0622	1.3829	1.2936	0.1719	1.9018	1.9581	1.5692
s.e.	0.1150	0.2295	0.4530	0.2615	0.3678	0.2443	0.2741
MCD-AMZN							
ρ	−0.3668	−0.2971	−0.2493	−0.0141	0.2010	0.2706	0.3564
s.e.	0.0700	0.0666	0.0480	0.0927	0.0414	0.0687	0.0374
η	6.2910	3.6398	4.2812	7.9979	6.5799	2.8223	3.7746
s.e.	2.5887	0.7998	0.9674	308.1843	1.6432	0.5263	0.5637
μ	0.5109	0.1900	0.0833	0.0803	−0.7733	0.0222	−0.3429
s.e.	0.9637	0.3790	0.2367	0.2024	0.5055	0.4277	1.1510
σ	1.2311	1.1261	1.0215	−0.5018	1.6170	1.0203	1.1289
s.e.	0.3469	0.2222	0.1913	2.9682	0.4150	0.1551	0.4783
MCD-DIS							
ρ	−0.3920	−0.2441	−0.2308	−0.1726	0.2338	0.2402	0.3253
s.e.	0.0376	0.0640	0.0399	0.3097	0.0645	0.0744	0.0335
η	4.8791	3.7219	6.2660	6.2089	6.8874	4.1767	5.2737
s.e.	0.6364	0.7675	1.2175	61.9561	2.2394	1.0323	0.8319
μ	1.6598	0.1100	0.4556	0.0404	−0.0445	−0.2521	−1.2646
s.e.	0.8176	0.4522	0.3234	0.0332	0.5884	0.5501	1.4966
σ	1.6176	1.1075	1.4248	0.1130	1.0475	1.1672	1.4446
s.e.	0.3292	0.2631	0.2808	0.2594	0.4599	0.3353	0.5989

Table 4.11: *Consumer Staples*: t-copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
KO-PG							
ρ	−0.5467	−0.2943	−0.2705	0.0834	0.2949	0.2587	0.4221
s.e.	0.0442	0.0332	0.0308	0.3204	0.0256	0.0334	0.0290
η	4.8455	5.1834	5.5410	3.8298	5.8506	4.7753	5.6583
s.e.	1.6019	0.6657	1.2008	38.0262	0.7162	0.5178	0.6172
μ	−0.0709	0.8910	0.5298	0.0120	−0.3583	−1.1917	−0.4970
s.e.	0.4254	0.3003	0.2737	0.0206	0.2432	0.4666	0.4375
σ	1.0247	1.5652	1.6094	0.0978	1.4212	1.8070	1.1230
s.e.	0.2024	0.1925	0.3083	0.5096	0.2691	0.3105	0.1605
KO-WMT							
ρ	−0.4098	−0.2951	−0.3062	−0.5980	0.2469	0.2977	0.3009
s.e.	0.0573	0.0243	0.0272	1.9021	0.0434	0.0575	0.0662
η	3.3061	5.7516	4.0527	9.3581	4.2463	4.0025	7.3824
s.e.	0.6025	0.8859	0.3788	455.9928	0.8404	0.6545	2.5548
μ	0.0138	0.7287	0.2913	0.0161	−0.3723	−0.2522	−0.8921
s.e.	0.4620	0.2236	0.1786	0.0215	0.2978	0.3546	0.5546
σ	1.0085	1.5096	1.3872	−0.0155	1.4468	1.1537	1.2606
s.e.	0.2226	0.1622	0.1783	0.0758	0.2876	0.1968	0.1929
PG-KO							
ρ	−0.3988	−0.3439	−0.2420	−0.0869	0.2224	0.3122	0.4638
s.e.	0.0258	0.0406	0.0280	4.2427	0.0416	0.0359	0.0472
η	7.1027	5.3880	6.0663	3.4218	5.8594	6.2189	5.4041
s.e.	0.7485	0.8725	1.2302	649.8348	1.3853	1.0649	0.8397
μ	0.7763	0.3024	0.4868	0.0666	−0.4465	−0.5096	−0.8004
s.e.	0.3165	0.2951	0.2056	0.0402	0.3331	0.3809	0.8170
σ	1.2105	1.1954	1.6073	−0.0069	1.4544	1.3703	1.4948
s.e.	0.1312	0.2173	0.2270	0.4394	0.2897	0.2598	0.3721

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
PG-WMT							
ρ	−0.4007	−0.3416	−0.2708	−0.0322	0.1607	0.2095	0.3062
s.e.	0.0764	0.0432	0.0290	0.0795	0.0086	0.0319	0.0236
η	7.5025	3.8608	4.4161	6.1874	7.6194	6.2295	8.3555
s.e.	2.7191	0.5318	0.8562	91.8275	1.0360	0.9353	1.3728
μ	0.0870	0.2064	0.3537	0.1223	−1.0382	−0.5904	−1.4729
s.e.	0.3718	0.2286	0.2218	0.0829	0.3474	0.4406	1.5490
σ	1.0032	1.1667	1.5094	−0.6716	2.0199	1.3120	1.5228
s.e.	0.1204	0.1785	0.2295	1.0579	0.3365	0.2895	0.6150
WMT-KO							
ρ	−0.4429	−0.3197	−0.2521	0.0839	0.2271	0.3384	0.4959
s.e.	0.0837	0.0395	0.0310	0.0843	0.0384	0.0739	0.0281
η	6.4868	4.9253	5.9577	4.4153	6.1286	4.9677	5.4838
s.e.	2.1599	0.5149	0.9044	16.7027	1.8411	1.4524	1.1593
μ	0.3728	0.2095	0.2592	−0.0166	−0.3995	−0.1833	−0.5919
s.e.	0.3550	0.3509	0.2067	0.0293	0.3730	0.3769	0.6813
σ	1.1444	1.1439	1.2908	0.3074	1.3462	1.1480	1.3422
s.e.	0.1470	0.2909	0.1692	0.4572	0.3456	0.2450	0.3421
WMT-PG							
ρ	−0.4881	−0.2836	−0.1647	−0.0206	0.2725	0.2729	0.4351
s.e.	0.0479	0.0509	0.0322	0.0192	0.0498	0.0815	0.0285
η	5.7325	4.7119	6.7667	4.3601	7.6452	4.3695	6.9597
s.e.	1.0964	0.7841	1.5980	18.5389	2.8727	1.3665	1.8585
μ	0.2747	0.7513	0.8762	0.0009	0.2044	−0.3595	−0.2297
s.e.	0.2271	0.6067	0.3499	0.0470	0.2854	0.4908	0.4841
σ	1.1085	1.4156	1.6835	1.1713	0.8710	1.2359	1.1603
s.e.	0.0716	0.3568	0.2888	1.2447	0.2724	0.3042	0.1579

Table 4.12: *Telecommunication Services*: t-copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
CMCSA-T							
ρ	−0.5189	−0.3450	−0.3665	−0.2941	0.3417	0.3483	0.3570
s.e.	0.0287	0.0497	0.0421	0.4479	0.0343	0.0432	0.0331
η	5.2748	3.8115	3.7396	100.0000	4.5331	4.3670	4.0317
s.e.	0.5698	0.6213	0.7971	0.0000	0.6769	0.7898	0.6285
μ	1.1086	0.5198	0.1480	0.0101	0.1061	−0.8146	−1.5816
s.e.	1.0340	0.6070	0.3064	0.0473	0.2125	0.6913	2.2965
σ	1.3253	1.2137	1.2086	0.0640	1.0111	1.3101	1.3670
s.e.	0.2549	0.2968	0.1558	0.1400	0.1066	0.3056	0.6403
CMCSA-VZ							
ρ	−0.5148	−0.3935	−0.3651	−0.0679	0.3487	0.2526	0.3553
s.e.	0.0310	0.0367	0.0371	0.3127	0.0514	0.0368	0.0351
η	4.7581	3.9434	3.7016	7.5269	5.7447	5.1168	4.1458
s.e.	0.6058	0.5197	0.6101	235.5652	1.8243	0.6341	0.4976
μ	1.2101	1.1310	0.1278	0.0133	0.0869	−1.3643	−1.1852
s.e.	0.7665	0.5691	0.4161	0.0449	0.3036	0.8823	2.7437
σ	1.3399	1.4952	1.1494	−0.1925	1.0559	1.4908	1.2580
s.e.	0.1848	0.2550	0.2522	0.6772	0.1697	0.3404	0.8577
T-CMCSA							
ρ	−0.6395	−0.4168	−0.3037	−0.1955	0.2287	0.2463	0.3841
s.e.	0.0361	0.0555	0.0337	0.2175	0.0357	0.0828	0.0474
η	5.3816	5.5822	4.6792	5.7862	5.0016	4.3711	3.7748
s.e.	1.2604	1.3707	0.9217	27.4960	1.0009	1.2833	0.6374
μ	−0.4801	0.0118	0.4444	−0.0190	−0.5325	−0.2954	−0.9330
s.e.	0.4074	0.1941	0.2967	0.0307	0.4429	0.6103	0.9720
σ	1.0345	1.0398	1.4059	0.1979	1.5205	1.2183	1.2776
s.e.	0.2120	0.0965	0.2400	0.2022	0.3322	0.3453	0.3639

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
T-VZ							
ρ	−0.4882	−0.2908	−0.1838	−0.2171	0.2216	0.2692	0.4385
s.e.	0.0895	0.0356	0.0197	0.1896	0.0224	0.0407	0.0494
η	4.1183	4.3306	8.0529	13.9711	5.7602	4.5236	4.8737
s.e.	1.0027	0.6515	1.0155	221.9730	0.8701	0.7835	0.6438
μ	0.0353	0.5053	1.2872	−0.0224	−1.1807	−0.7779	−0.8148
s.e.	0.3944	0.4286	0.4803	0.0387	0.4695	0.5358	0.6259
σ	0.9979	1.2570	2.0014	0.1878	1.9427	1.4228	1.1437
s.e.	0.1775	0.2252	0.3652	0.2187	0.3864	0.3210	0.2001
VZ-CMCSA							
ρ	−0.3317	−0.3412	−0.1869	−0.0866	0.1851	0.3255	0.3614
s.e.	0.0960	0.0486	0.0315	0.0381	0.0679	0.0345	0.0506
η	6.9976	6.4544	4.3069	100.0000	3.4135	3.6738	4.4027
s.e.	2.8600	1.1615	0.6621	0.0000	0.7166	0.4385	0.9766
μ	1.3286	0.4771	0.7976	−0.0201	−0.1567	−0.2503	−1.2240
s.e.	1.3540	0.3610	0.4241	0.0296	0.4410	0.3912	1.3381
σ	1.2847	1.3343	1.5282	0.5662	1.1226	1.1472	1.2957
s.e.	0.3151	0.2103	0.3561	0.4820	0.3387	0.2162	0.4762
VZ-T							
ρ	−0.4280	−0.3455	−0.2264	−0.1856	0.2131	0.3091	0.3560
s.e.	0.0835	0.0455	0.0419	0.1065	0.0232	0.0428	0.0301
η	3.1025	6.5807	8.1545	7.7244	5.1088	4.1033	4.5785
s.e.	1.0238	1.1980	2.6379	40.8954	0.6532	0.6441	0.7729
μ	0.1595	0.5346	1.0816	−0.0396	−0.9928	−0.3548	−0.8045
s.e.	0.5280	0.3294	0.4513	0.0264	0.5237	0.4585	0.9285
σ	1.0577	1.3471	1.9117	0.3351	1.8653	1.1972	1.1547
s.e.	0.1322	0.1676	0.3521	0.2244	0.4540	0.2863	0.3702

Table 4.13: *Financials*: t-copula parameters estimated on full sample.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
BAC-C							
ρ	−0.6092	−0.5082	−0.2404	−0.0658	0.2731	0.2816	0.5044
s.e.	0.0323	0.0152	0.0193	0.0059	0.0198	0.0091	0.0083
η	5.1322	3.6773	4.6995	100.0000	4.3899	6.4381	6.3814
s.e.	0.7578	0.1893	0.4527	0.0000	0.3938	0.3350	0.3175
μ	3.3529	1.6557	2.8622	0.0462	−1.4308	−2.8550	−3.3500
s.e.	0.5976	0.2539	0.6261	0.0464	0.4267	0.6632	0.3807
σ	1.1818	1.3384	1.9432	2.7294	1.6103	1.7085	1.2675
s.e.	0.1094	0.1029	0.2935	0.7993	0.2323	0.2581	0.0777
BAC-WFC							
ρ	−0.6708	−0.3809	−0.2234	−0.1155	0.3605	0.3435	0.6407
s.e.	0.0102	0.0173	0.0194	0.0235	0.0126	0.0146	0.0166
η	4.3438	5.2202	5.5846	100.0000	6.3733	4.4799	3.5123
s.e.	0.3522	0.4056	0.5427	0.0000	0.4251	0.2577	0.1640
μ	3.1615	3.1109	3.7092	−0.0663	−0.8119	−2.3349	−2.4277
s.e.	0.5037	0.3053	0.9813	0.0356	0.2443	0.4496	0.8990
σ	1.4565	1.6396	2.2969	1.4580	1.5338	1.5622	1.4119
s.e.	0.0987	0.1023	0.4109	0.5042	0.1351	0.1677	0.1940
C-BAC							
ρ	−0.6199	−0.4226	−0.2305	−0.1804	0.2878	0.2726	0.4479
s.e.	0.0249	0.0155	0.0149	0.0140	0.0168	0.0146	0.0220
η	3.9095	3.9890	7.1142	100.0000	5.6909	6.2101	5.6386
s.e.	0.3700	0.2360	0.7663	0.0000	0.6024	0.3690	0.5140
μ	4.5702	1.7429	5.3978	−0.0329	−2.1465	−2.7861	−4.7330
s.e.	1.6834	0.5141	0.8973	0.0444	0.4468	0.6607	1.4371
σ	1.4473	1.2678	2.7176	1.5247	2.1437	1.7349	1.4651
s.e.	0.3027	0.1293	0.3299	0.3203	0.2811	0.2191	0.1767

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$
C-WFC							
ρ	−0.6418	−0.5182	−0.4063	−0.2886	0.2700	0.3058	0.3916
s.e.	0.0177	0.0167	0.0198	0.0395	0.0423	0.0173	0.0209
η	4.3168	3.5652	4.9179	57.7579	4.0342	7.7337	5.6259
s.e.	0.2902	0.1901	0.6168	617.0949	0.5129	0.6344	0.3345
μ	3.5512	2.0403	1.6664	−0.1816	−0.6095	−2.6485	−5.0457
s.e.	0.7651	0.4972	0.2933	0.0335	0.7290	0.4877	1.4072
σ	1.3725	1.4136	1.6767	0.8019	1.2674	1.7982	1.4392
s.e.	0.1501	0.1637	0.1555	0.2374	0.4254	0.1541	0.2162
WFC-BAC							
ρ	−0.6050	−0.6047	−0.2385	−0.2281	0.1811	0.4574	0.5286
s.e.	0.0346	0.0189	0.0186	0.1716	0.0161	0.0187	0.0110
η	4.0990	3.5547	4.1239	10.1991	5.7504	6.6267	4.5168
s.e.	0.4877	0.3350	0.4403	69.2091	0.5747	0.9925	0.2945
μ	2.2312	0.4315	2.0602	−0.0124	−2.8667	−0.9389	−2.8773
s.e.	1.3657	0.2881	0.5857	0.0487	0.6729	0.2977	0.8688
σ	1.3069	1.3457	1.8226	−0.1576	2.1599	1.3266	1.2951
s.e.	0.2761	0.1244	0.2837	0.1790	0.3194	0.1251	0.1812
WFC-C							
ρ	−0.6360	−0.4064	−0.3634	−0.1451	0.3315	0.3985	0.5587
s.e.	0.0223	0.0270	0.0176	1.3098	0.0272	0.0121	0.0159
η	3.1887	4.2440	5.1003	3.1906	4.0573	4.8506	4.9526
s.e.	0.2731	0.3718	0.4808	71.0041	0.4211	0.3018	0.3670
μ	2.8629	1.5460	1.3170	−0.0318	−0.9843	−1.3037	−3.3277
s.e.	1.0285	0.4589	0.3234	0.0458	0.3219	0.4402	0.6433
σ	1.4887	1.3615	1.7621	0.0202	1.4442	1.3003	1.4333
s.e.	0.2082	0.1861	0.1870	0.2206	0.2004	0.1729	0.1302

Table 4.14: *Energy*: t-copula parameters estimated on full sample.

CVX-SLB

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4285	−0.2952	−0.1917	0.0125	0.1370	0.2189	0.3470
s.e.	0.0514	0.0315	0.0407	0.0283	0.0119	0.0343	0.0601
η	5.9577	5.9865	5.7372	1.1099	7.1709	5.8081	7.7720
s.e.	1.4900	1.0373	1.6203	1.0572	1.0867	0.9945	1.9260
μ	0.1885	1.9198	0.6516	0.0266	−2.8547	−1.0773	−0.7736
s.e.	0.3205	0.6238	0.7631	0.0518	0.6763	0.5540	0.8808
σ	1.0542	1.9111	1.4014	0.7657	2.8422	1.4370	1.1080
s.e.	0.0814	0.2968	0.4462	0.5341	0.4426	0.2698	0.2357

SLB-XOM

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4324	−0.3460	−0.2190	−0.1702	0.2284	0.1523	0.2311
s.e.	0.0269	0.0426	0.0308	0.1076	0.0378	0.0711	0.1420
η	5.8729	6.8816	8.3675	4.8353	9.7633	5.2302	6.5226
s.e.	1.3733	1.5239	1.8104	9.4595	3.3286	1.2786	3.2939
μ	0.6996	1.0827	0.8374	−0.0125	−0.4060	−0.4376	−1.2432
s.e.	1.9964	0.8425	0.6144	0.0579	0.5153	1.3303	3.5210
σ	1.1336	1.3608	1.3483	0.2669	1.2257	1.1270	1.2380
s.e.	0.4456	0.2587	0.2729	0.1550	0.2172	0.3708	0.7401

CVX-XOM

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.3697	−0.2848	−0.2604	−0.0292	0.1498	0.2655	0.3064
s.e.	0.0389	0.0332	0.0319	0.0614	0.0098	0.0655	0.0371
η	9.4907	6.7285	5.3883	100.0000	7.4652	4.4893	4.8485
s.e.	2.2810	1.0185	0.7988	0.0000	0.6146	1.0291	0.5881
μ	1.1165	2.1093	1.0977	0.0288	−3.3156	−0.3117	−0.5190
s.e.	0.3934	0.6507	0.4563	0.1454	0.8036	0.5296	1.1587
σ	1.2333	2.0415	1.6990	0.4451	3.1823	1.1191	1.0327
s.e.	0.1212	0.3517	0.2934	1.4660	0.4812	0.2611	0.4099

XOM-CVX

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4393	−0.3131	−0.2374	−0.2076	0.1778	0.2603	0.2720
s.e.	0.0303	0.0985	0.0379	0.2078	0.0163	0.0311	0.0438
η	5.9477	5.6742	6.5839	100.0000	7.0889	4.8911	5.4711
s.e.	0.9646	2.8890	1.1588	0.0000	1.0272	1.0623	1.3549
μ	0.2207	0.1771	0.6095	0.0690	−1.8986	−0.5312	−0.6969
s.e.	0.6407	0.5904	0.4480	0.0447	0.5417	0.4103	0.4994
σ	1.0861	1.1164	1.4313	0.1694	2.3233	1.1896	1.1052
s.e.	0.1945	0.2952	0.3091	0.2090	0.3645	0.2075	0.1663

SLB-CVX

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4379	−0.3074	−0.1992	−0.1674	0.2540	0.2319	0.3709
s.e.	0.0487	0.0253	0.0433	0.2221	0.0284	0.0500	0.1180
η	5.2786	6.8860	6.0897	3.4238	9.4498	8.9075	4.2650
s.e.	1.0946	1.3895	1.6504	11.2905	2.1898	2.8503	1.2413
μ	0.8955	1.3873	0.9004	0.0190	−0.0270	−0.7997	−1.9741
s.e.	1.8064	1.0022	0.7610	0.0543	0.4314	1.1113	2.1185
σ	1.1652	1.4433	1.3617	0.1494	1.0719	1.2439	1.4758
s.e.	0.3585	0.3185	0.3162	0.2031	0.1863	0.3647	0.4936

XOM-SLB

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.3592	−0.3159	−0.2019	−0.1197	0.1356	0.2799	0.3591
s.e.	0.0907	0.0588	0.0509	0.2637	0.0119	0.0479	0.0609
η	4.1754	5.4335	6.6450	100.0000	8.1115	4.4443	5.1398
s.e.	1.1479	1.1733	1.9993	0.0000	1.0189	1.0236	1.0787
μ	0.1946	0.2514	0.7802	0.0832	−2.8297	−0.5532	−0.3555
s.e.	0.5978	0.4579	0.7005	0.0528	0.7357	0.3847	0.5925
σ	1.0865	1.1207	1.5256	0.0838	2.9620	1.2740	1.0619
s.e.	0.1467	0.2168	0.4660	0.3426	0.5872	0.1959	0.1397

Table 4.15: *Health Care*: t-copula parameters estimated on full sample.

JNJ-MRK

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4057	−0.3003	−0.1618	−0.0489	0.3259	0.1873	0.3784
s.e.	0.1075	0.0442	0.0413	1.5574	0.0356	0.0860	0.0775
η	3.9995	4.6836	3.3134	16.9292	4.8582	4.8475	5.6663
s.e.	1.0939	0.8319	0.5100	0.0000	1.2305	1.5532	1.3380
μ	0.1020	0.4159	0.3466	−0.0008	−0.0069	−0.0948	0.0041
s.e.	0.6619	0.3256	0.2862	0.0197	0.1717	0.4346	1.0217
σ	1.0619	1.3275	1.3460	0.0119	1.0841	1.0687	1.0941
s.e.	0.3053	0.2706	0.3392	0.7807	0.2129	0.3532	0.4711

MRK-PFE

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4211	−0.3749	−0.3508	−0.0372	0.2761	0.1706	0.3857
s.e.	0.1220	0.0698	0.0641	0.0656	0.0338	0.0439	0.0755
η	3.5197	3.3929	4.3190	2.1781	8.8416	5.9820	7.2204
s.e.	1.1319	0.6269	1.0819	4.6262	1.8663	1.1855	2.7809
μ	0.1245	−0.0296	−0.1739	0.0235	−0.1913	−0.7640	0.1699
s.e.	0.9344	0.4189	0.3540	0.0328	0.3681	0.6826	0.7310
σ	1.0638	1.0732	0.8903	−0.4182	1.1713	1.3043	0.9765
s.e.	0.2304	0.1910	0.2543	0.6839	0.2615	0.4005	0.2134

JNJ-PFE

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4493	−0.4449	−0.4059	0.0205	0.1689	0.1908	0.2014
s.e.	0.0500	0.0250	0.0648	0.0266	0.0500	0.0429	0.1369
η	3.4241	3.4109	2.6166	1.8831	3.4316	9.0889	8.0286
s.e.	0.6649	0.6312	0.5041	4.3461	0.6941	2.3148	5.6685
μ	0.6772	0.1513	−0.0889	0.0024	−0.2407	−1.0044	−0.6014
s.e.	0.5694	0.2654	0.1532	0.0214	0.2246	0.5483	1.6099
σ	1.2893	1.2534	1.0249	0.6274	1.2284	1.7279	1.1782
s.e.	0.3074	0.2280	0.1771	1.2439	0.2652	0.3877	0.5999

PFE-JNJ

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.5341	−0.3602	−0.2162	−0.0732	0.3312	0.2084	0.3782
s.e.	0.0394	0.0410	0.0369	0.0776	0.0408	0.0370	0.1556
η	5.5814	6.2800	7.9359	6.2729	5.8079	6.5144	3.4365
s.e.	0.9350	1.5549	1.7619	71.0738	2.1954	1.2281	0.8568
μ	0.4099	0.4866	0.4425	−0.0526	−0.0772	−0.8518	−0.3970
s.e.	0.4425	0.5627	0.4485	0.0439	0.2210	0.5838	0.8320
σ	1.1833	1.3266	1.3240	0.2991	1.1505	1.3825	1.1101
s.e.	0.1021	0.2905	0.3493	0.4647	0.1763	0.3380	0.2363

MRK-JNJ

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.4177	−0.2558	−0.1850	−0.0441	0.2148	0.2698	0.2683
s.e.	0.2320	0.0250	0.0228	0.6151	0.0209	0.0417	0.1301
η	3.4736	6.6219	11.0402	8.9364	8.8457	14.6844	5.1001
s.e.	1.0474	0.9580	3.9363	1200.9245	1.4737	5.6752	1.0186
μ	0.0542	0.7437	1.0676	0.0170	−0.9916	−0.6490	−0.2090
s.e.	0.9687	0.4614	0.5629	0.0334	0.4928	0.5577	1.5157
σ	1.1024	1.3774	1.7919	0.0688	1.7149	1.2937	1.0126
s.e.	0.2307	0.1941	0.3873	0.9953	0.2975	0.2777	0.5353

PFE-MRK

	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.95$	$\alpha=0.99$
ρ	−0.5044	−0.4744	−0.2368	0.0631	0.3632	0.2826	0.4055
s.e.	0.0697	0.0442	0.0333	0.0331	0.0623	0.0361	0.0411
η	4.3457	4.2681	5.8239	72.3943	5.5033	5.1581	6.5804
s.e.	1.1015	1.0641	1.1170	3423.2660	1.6058	0.7319	1.2744
μ	0.0058	−0.0886	0.9792	−0.1445	0.2304	−0.5626	−1.3408
s.e.	0.6699	0.3095	0.4899	0.0707	0.2022	0.3567	1.2798
σ	1.1452	1.0651	1.6984	−1.0642	0.9592	1.2713	1.4090
s.e.	0.2056	0.1795	0.3676	0.7443	0.1642	0.2193	0.3707

4.B Figures

4.B.1 Copula parameters' summary

Figure 4.1: Estimated parameters from Normal and t copula using full sample data. For each quantile level we are summarizing the results for 21 assets. Each boxplot includes 42 observations. *Note:* Recall from Section 4.4.1 that for each stock within the same industry there are two possible regressions.

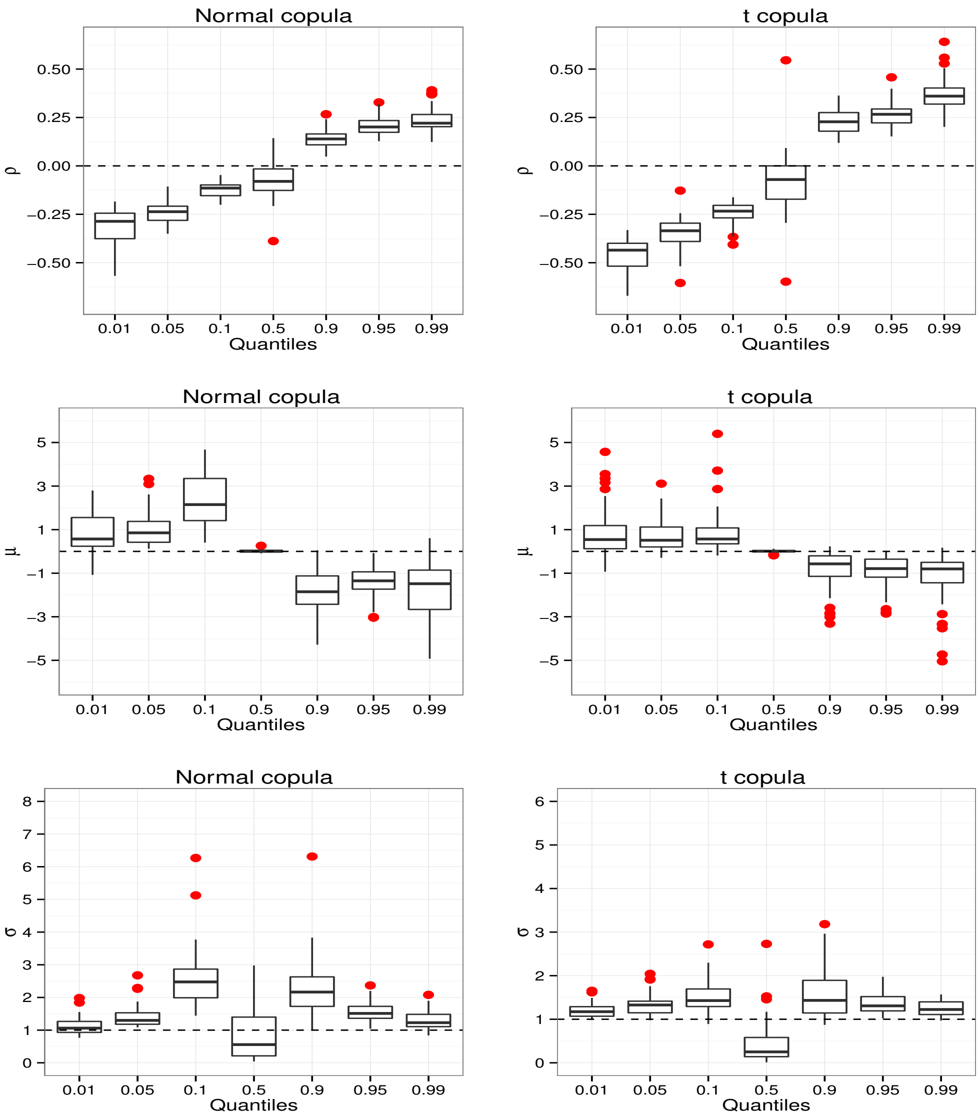
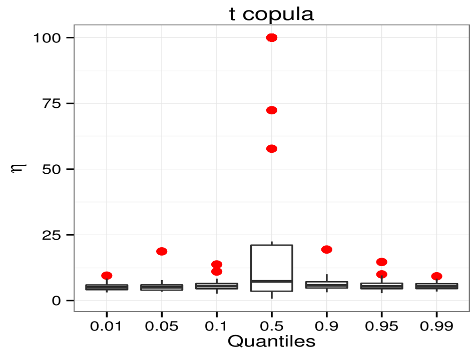


Figure 4.2: Degrees of freedom for t copula using the full sample data. For each quantile level we are summarizing the results for 21 assets. Each boxplot includes 42 observations. *Note:* Recall from Section 4.4.1 that for each stock within the same industry there are two possible regressions.



4.B.2 Normal copula fit

Figure 4.3: *Information Technology*: Normal copula

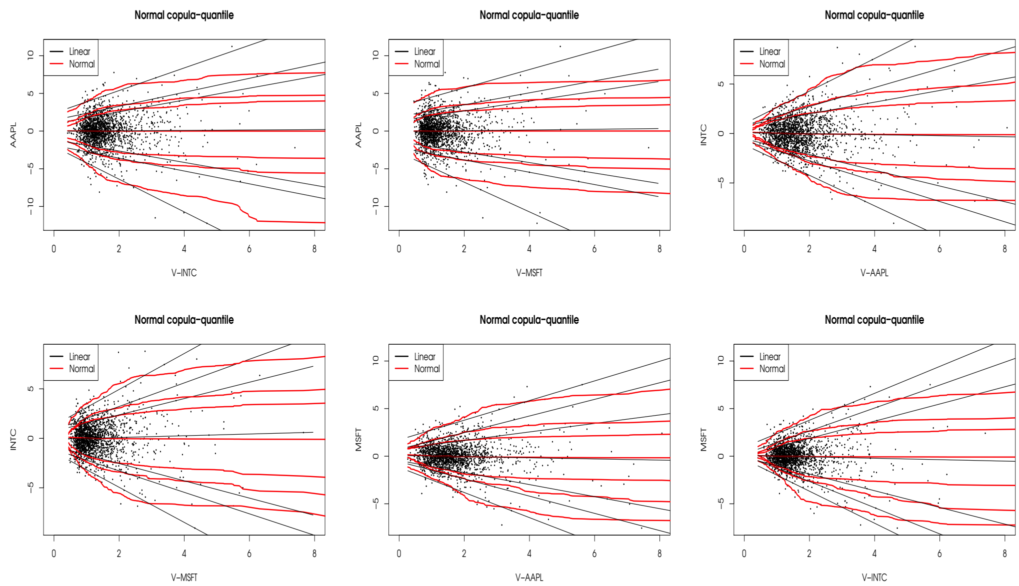


Figure 4.4: *Consumer Discretionary*: Normal copula

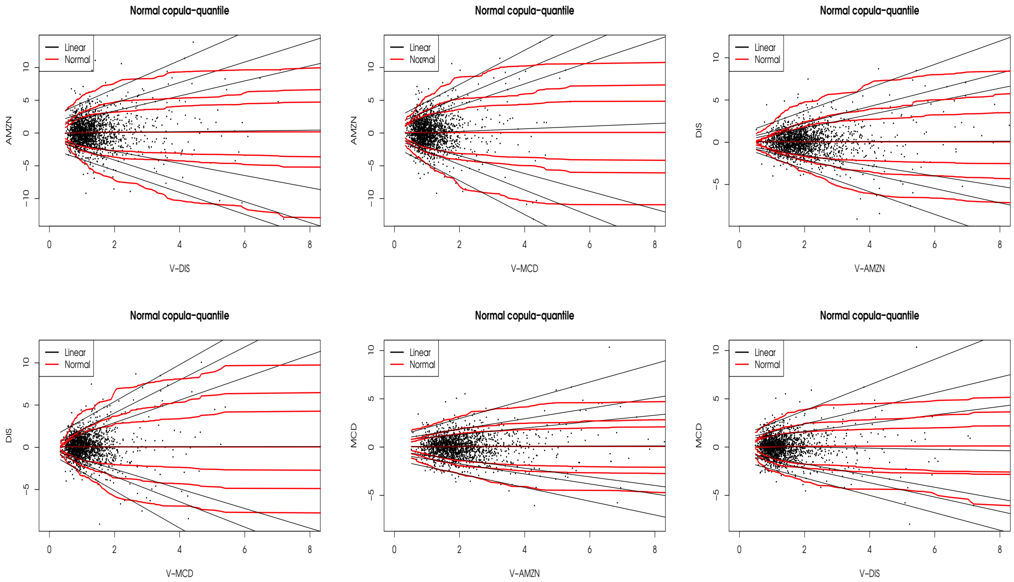


Figure 4.5: *Consumer Staples*: Normal copula

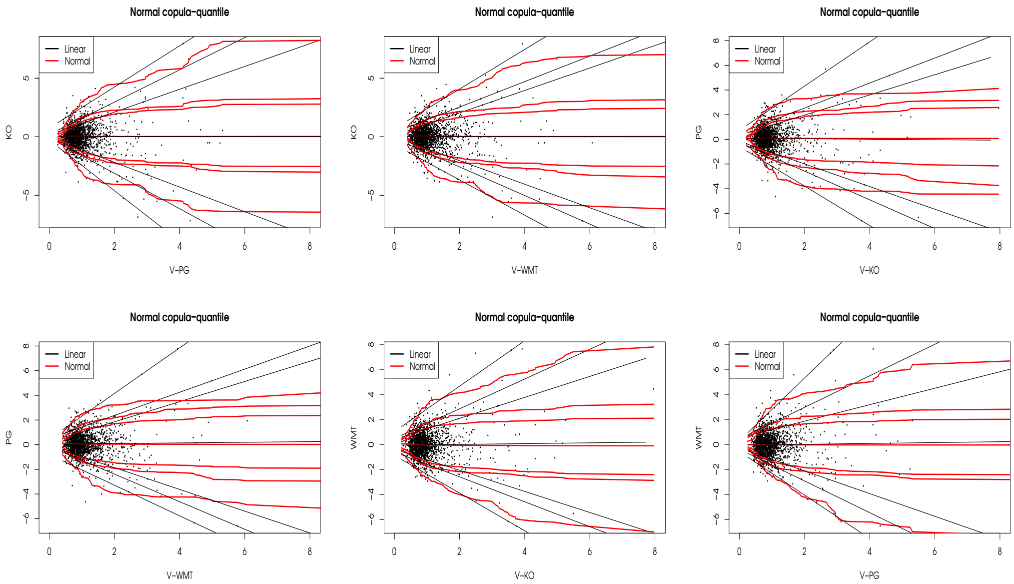


Figure 4.6: *Telecommunication Services*: Normal copula

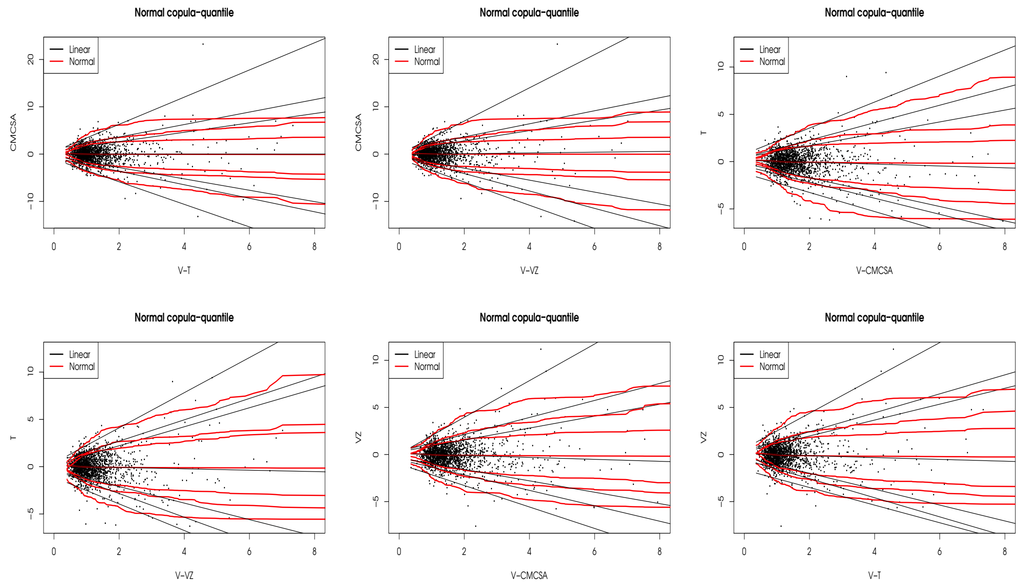


Figure 4.7: *Financials*: Normal copula

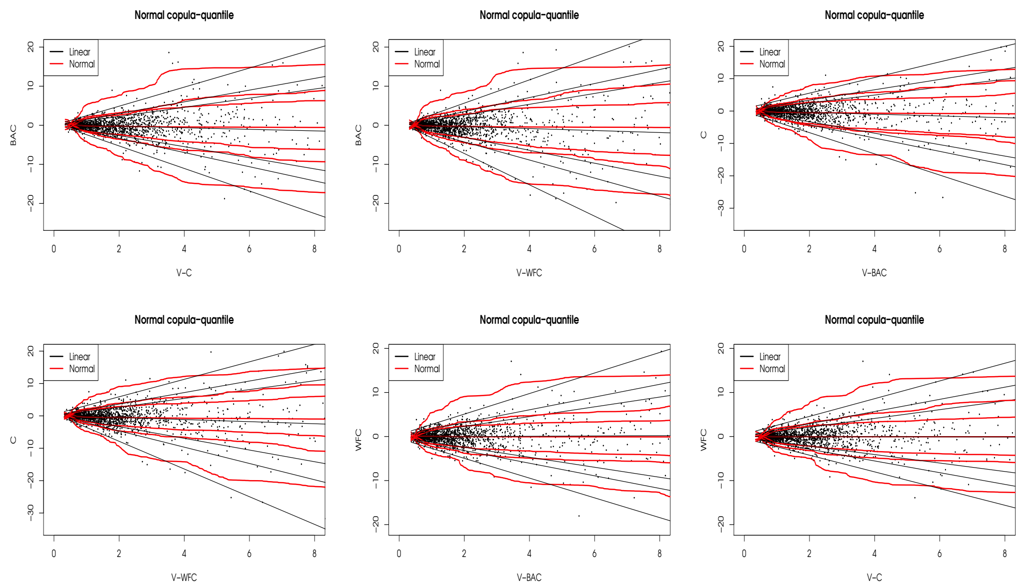


Figure 4.8: *Energy*: Normal copula

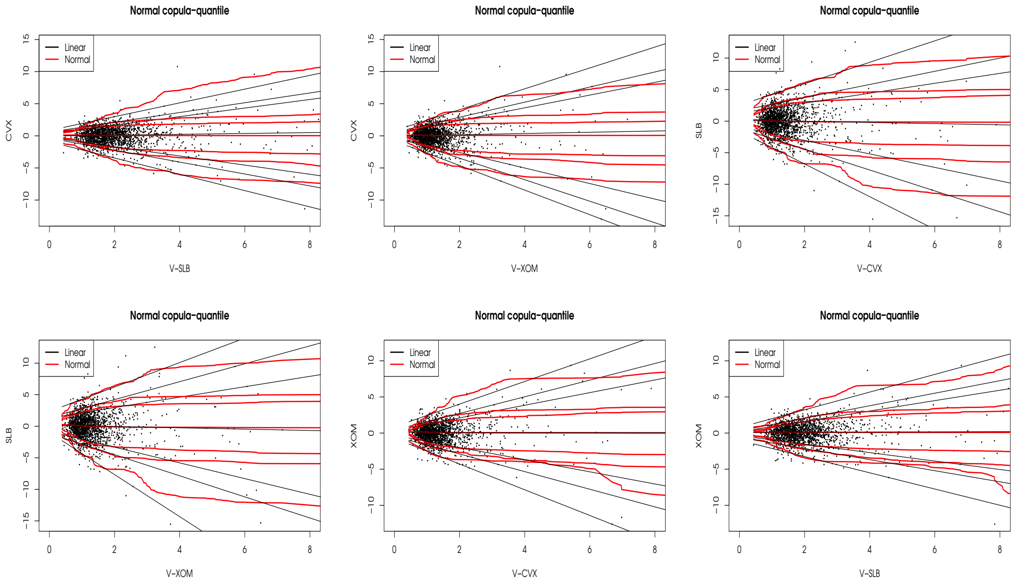
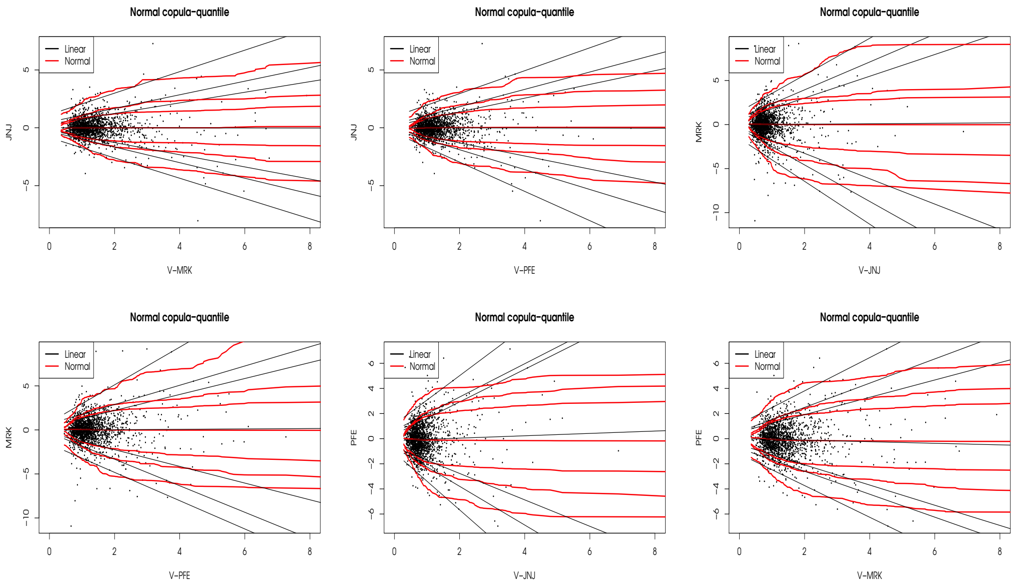


Figure 4.9: *Health Care*: Normal copula



4.B.3 t-copula fit

Figure 4.10: *Information Technology*: t copula

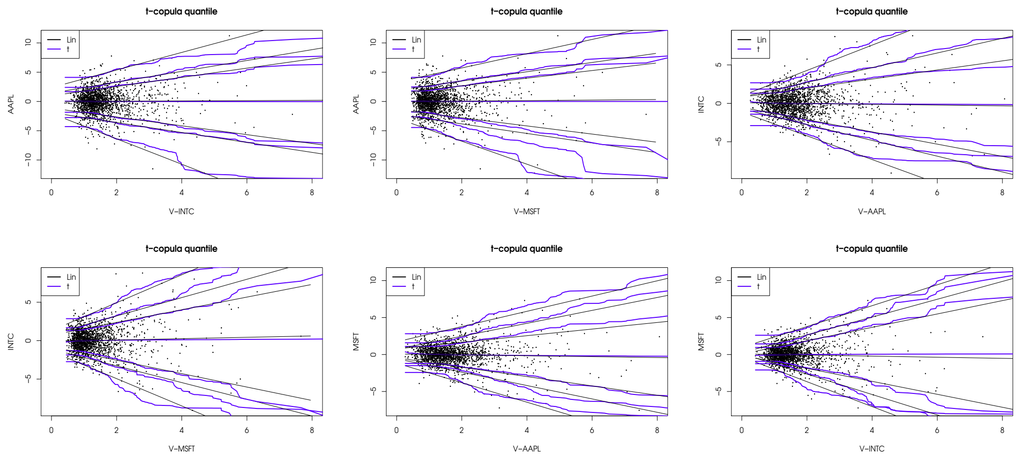


Figure 4.11: *Consumer Discretionary*: t copula

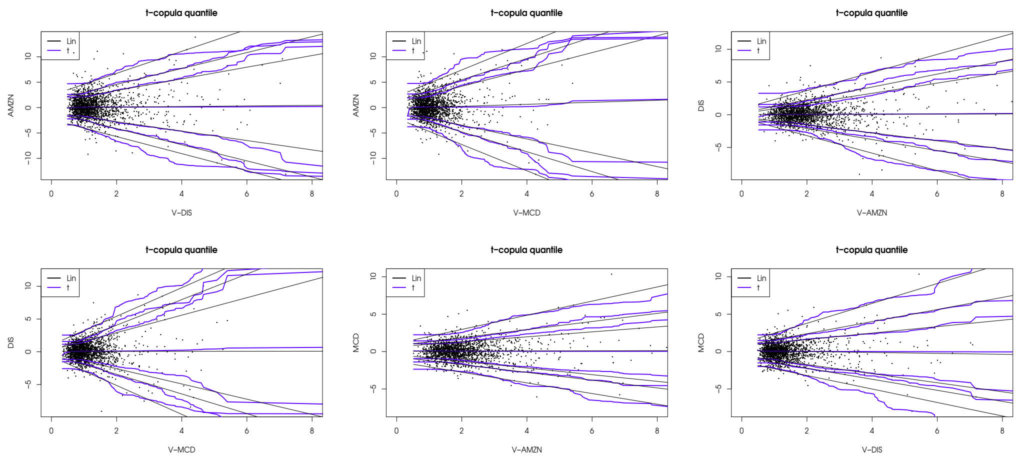


Figure 4.12: *Consumer Staples*: *t* copula

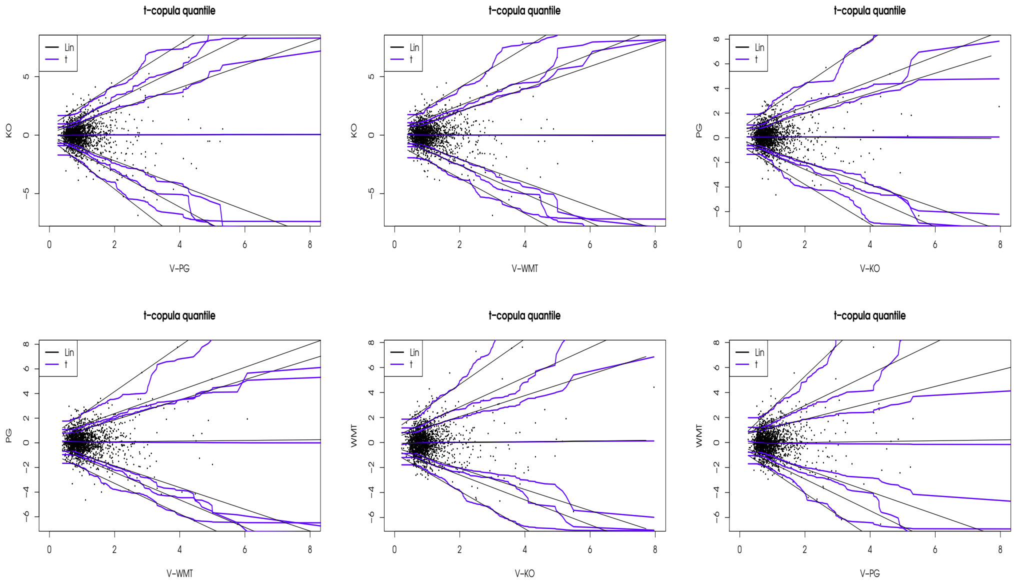


Figure 4.13: *Telecommunication Services*: *t* copula

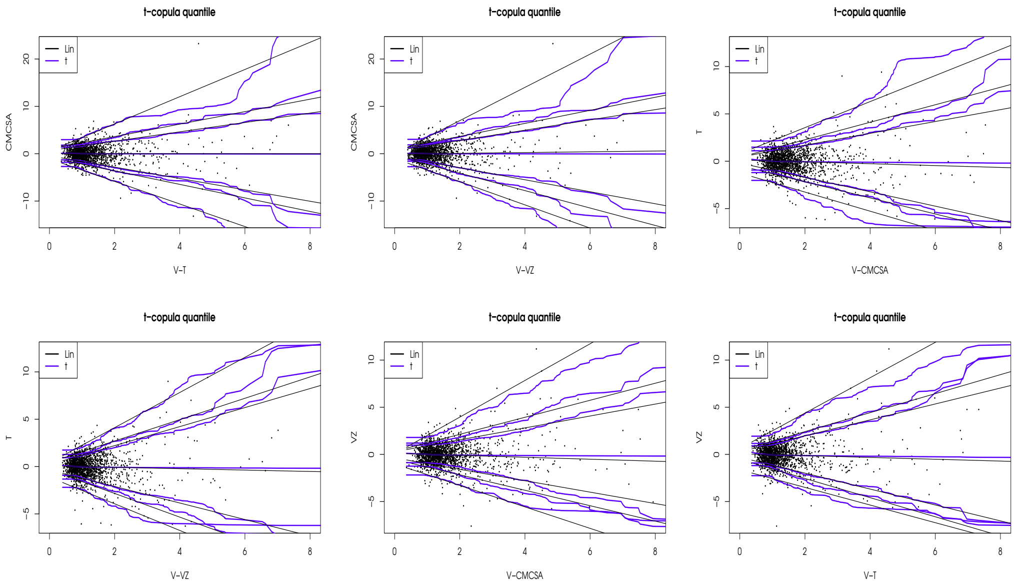


Figure 4.14: *Financials*: t copula

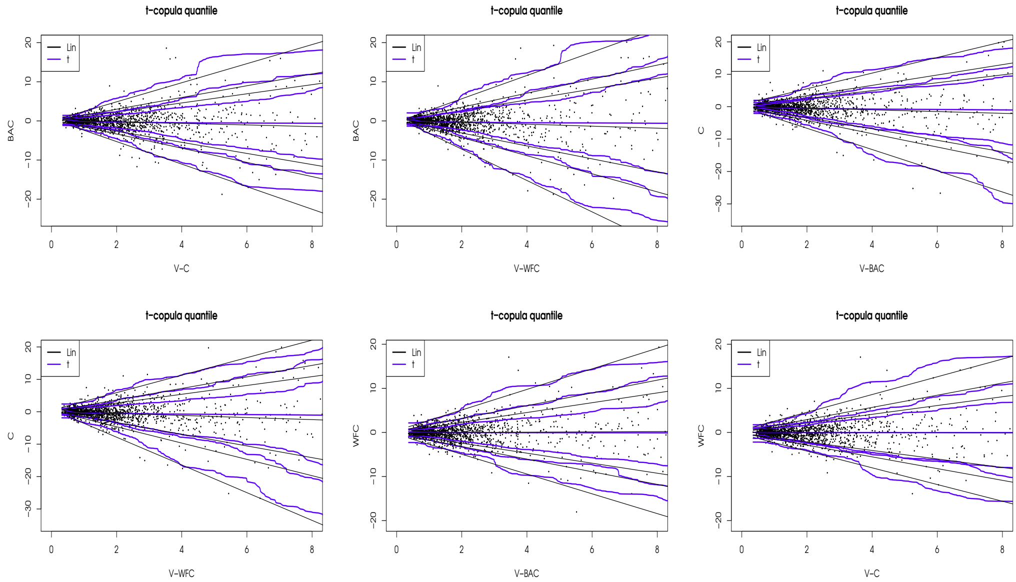


Figure 4.15: *Energy*: t copula

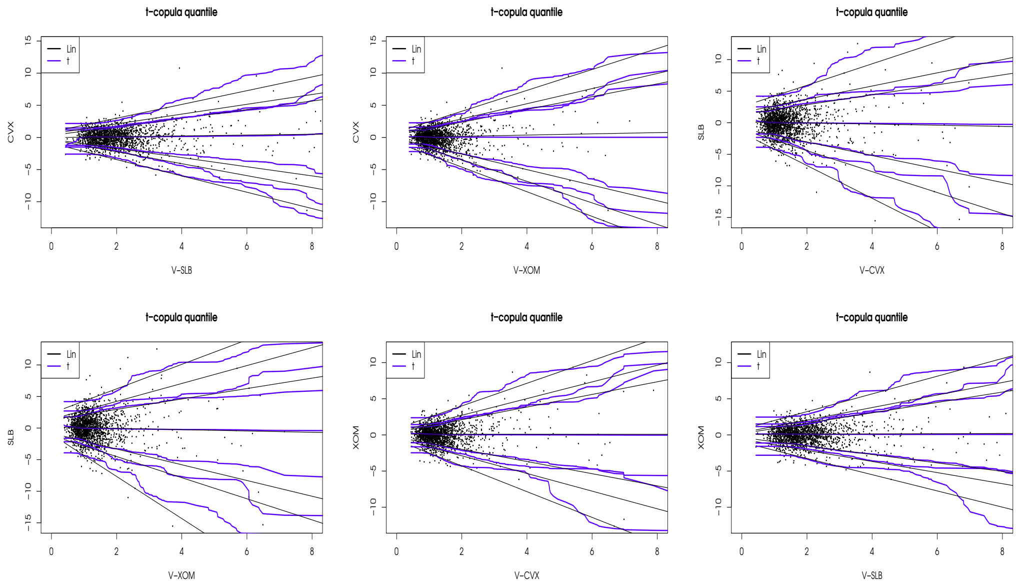
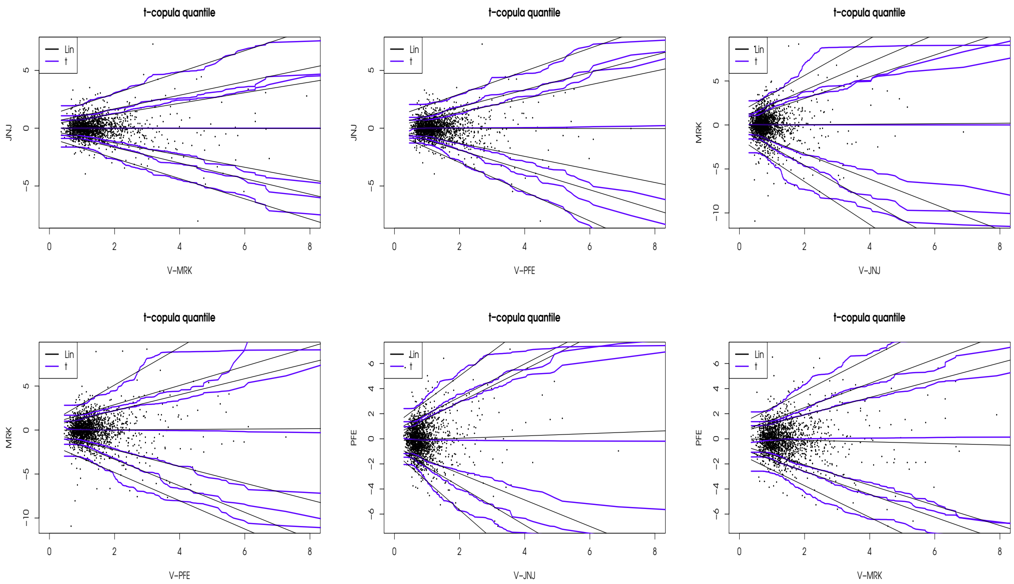


Figure 4.16: *Health Care*: t copula



4.B.4 CoVaR dynamics

Figure 4.17: *Information Technology*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR^{j|i}_{5\%}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR^{j|i}_{5\%}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{AAPl, INTC, MSFT\}$ and $i \neq j$, $j \neq k$.

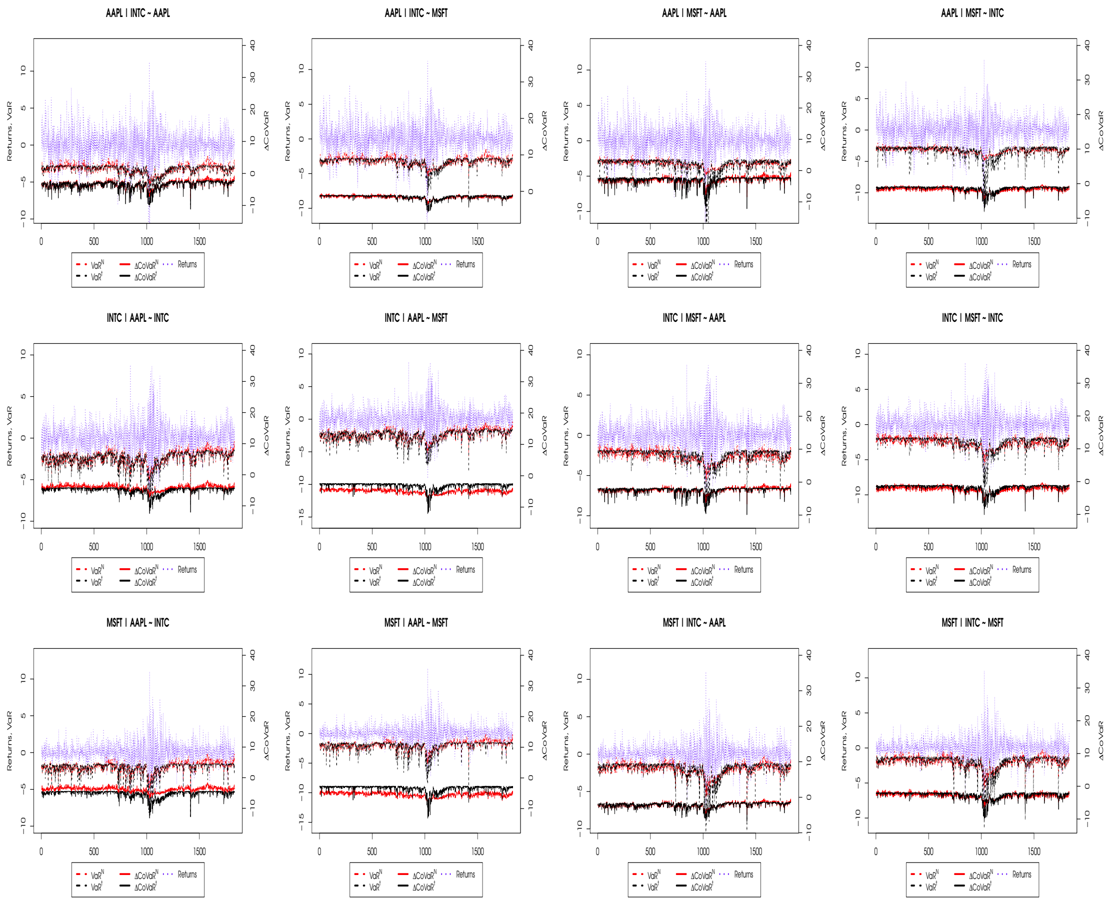


Figure 4.18: *Consumer Discretionary*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR^{j|i}_{5\%}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR^{j|i}_{5\%}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{AMZN, DIS, MCD\}$ and $i \neq j$, $j \neq k$.

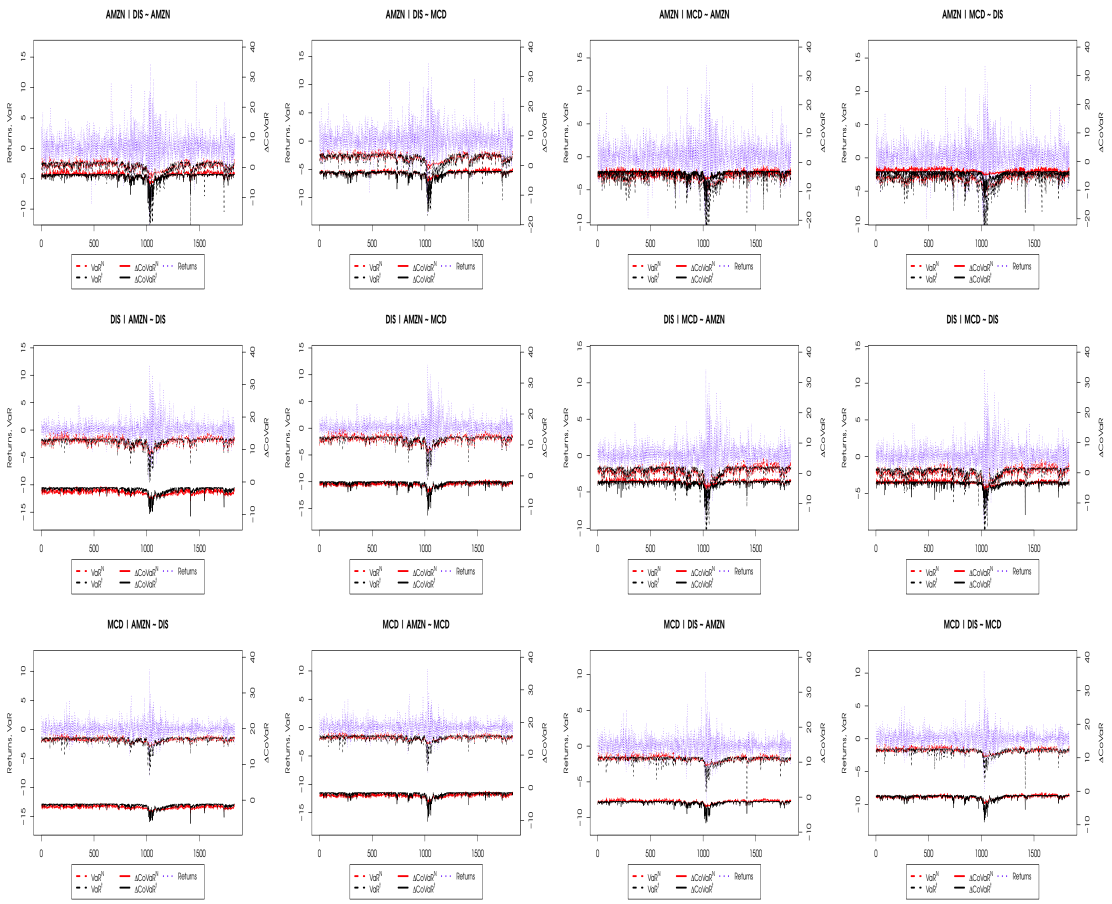


Figure 4.19: *Consumer Staples*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR^{j|i}_{5\%}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR^{j|i}_{5\%}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{KO, PG, WMT\}$ and $i \neq j$, $j \neq k$.

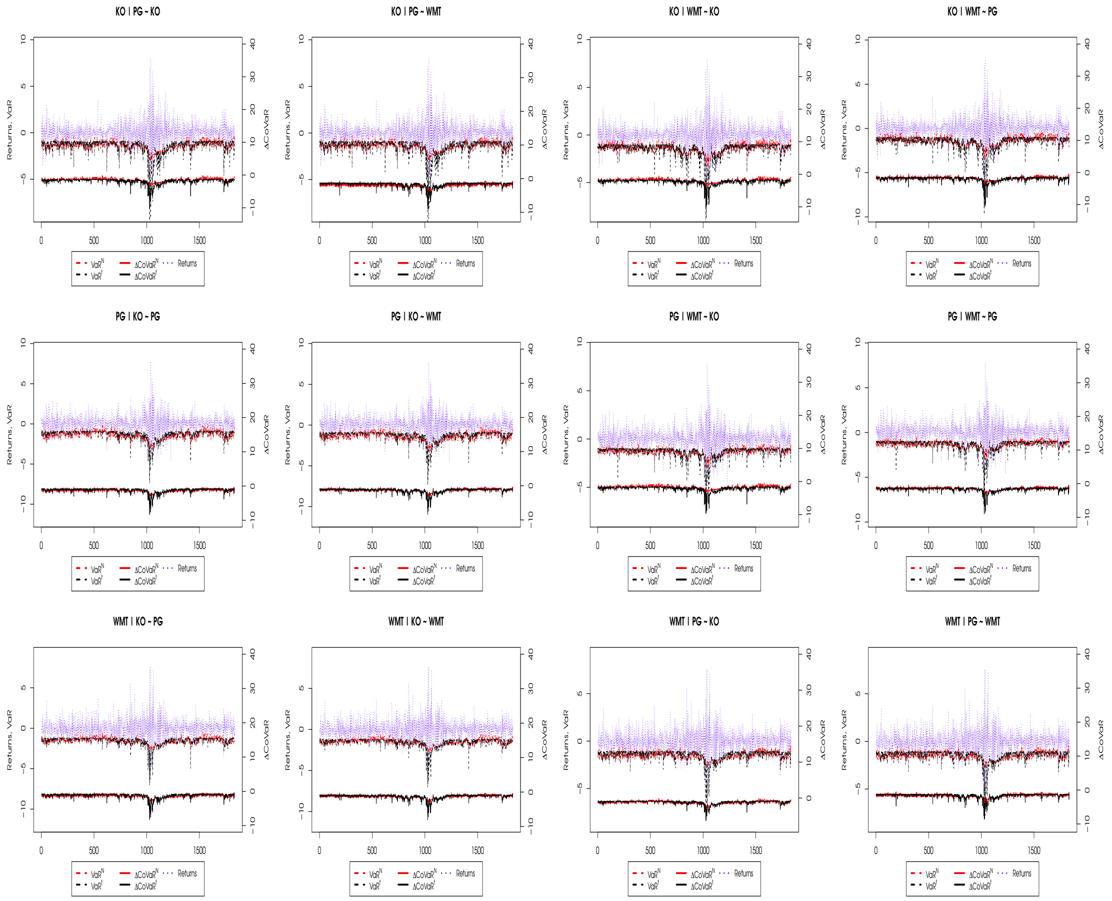


Figure 4.20: *Telecommunication Services*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR^{j|i}_{5\%}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR^{j|i}_{5\%}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{CMCSA, T, VZ\}$ and $i \neq j$, $j \neq k$.

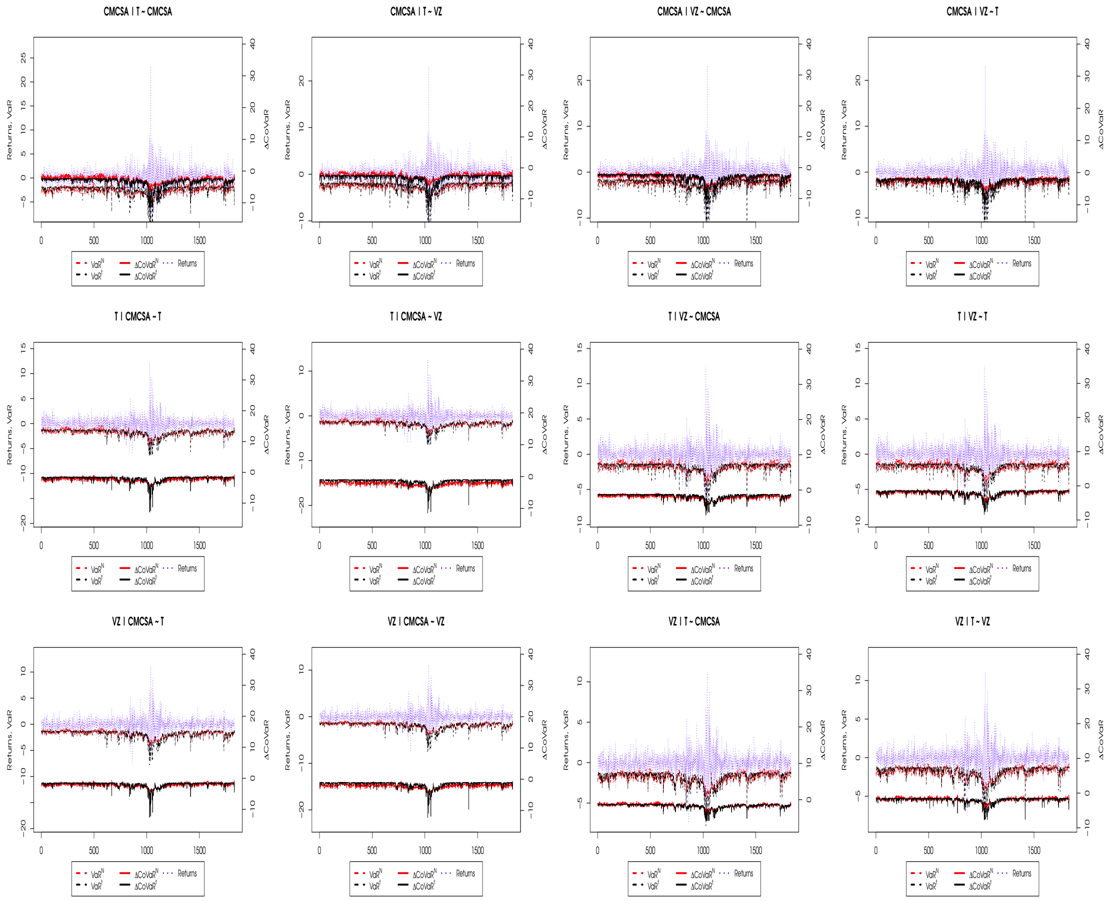


Figure 4.21: *Financials*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR_{5\%}^{j|i}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR_{5\%}^{j|i}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{BAC, C, WFC\}$ and $i \neq j$, $j \neq k$.

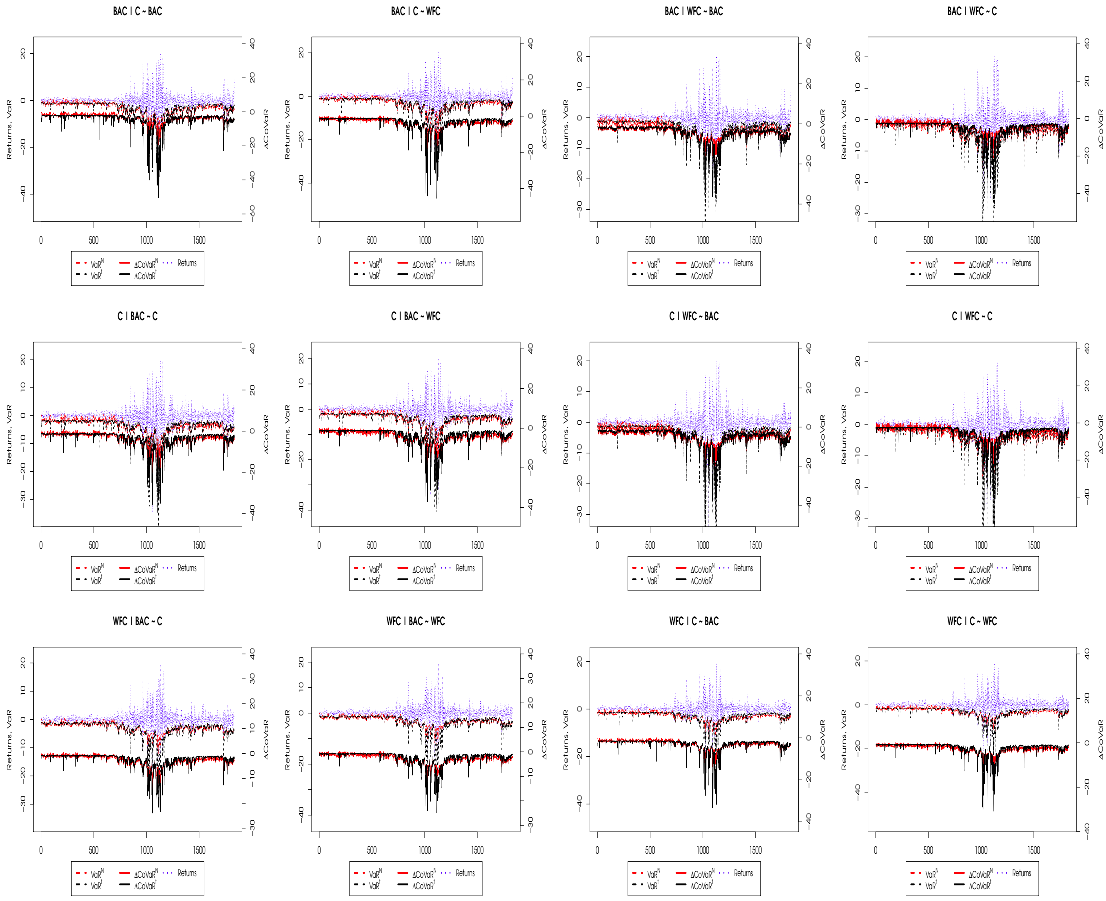


Figure 4.22: *Energy*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR^{j|i}_{5\%}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR^{j|i}_{5\%}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{CVX, SLB, XOM\}$ and $i \neq j$, $j \neq k$.

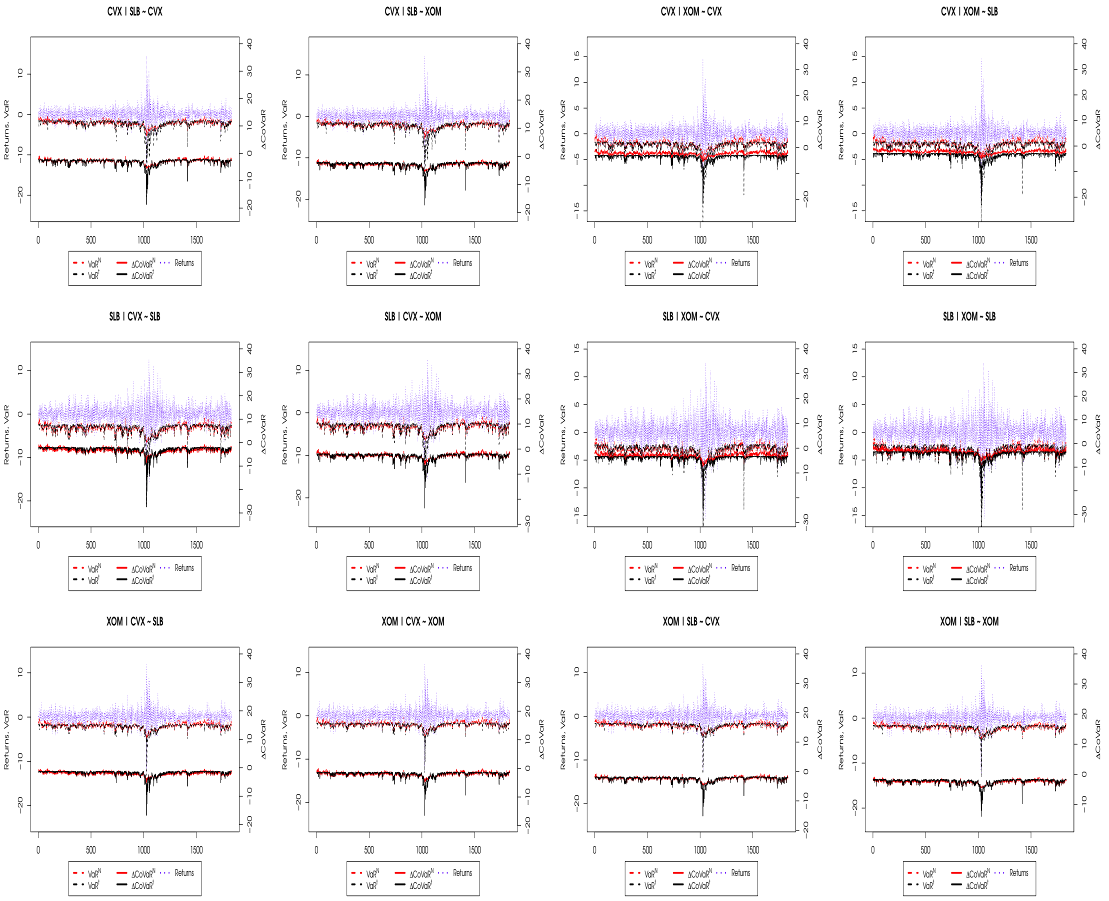
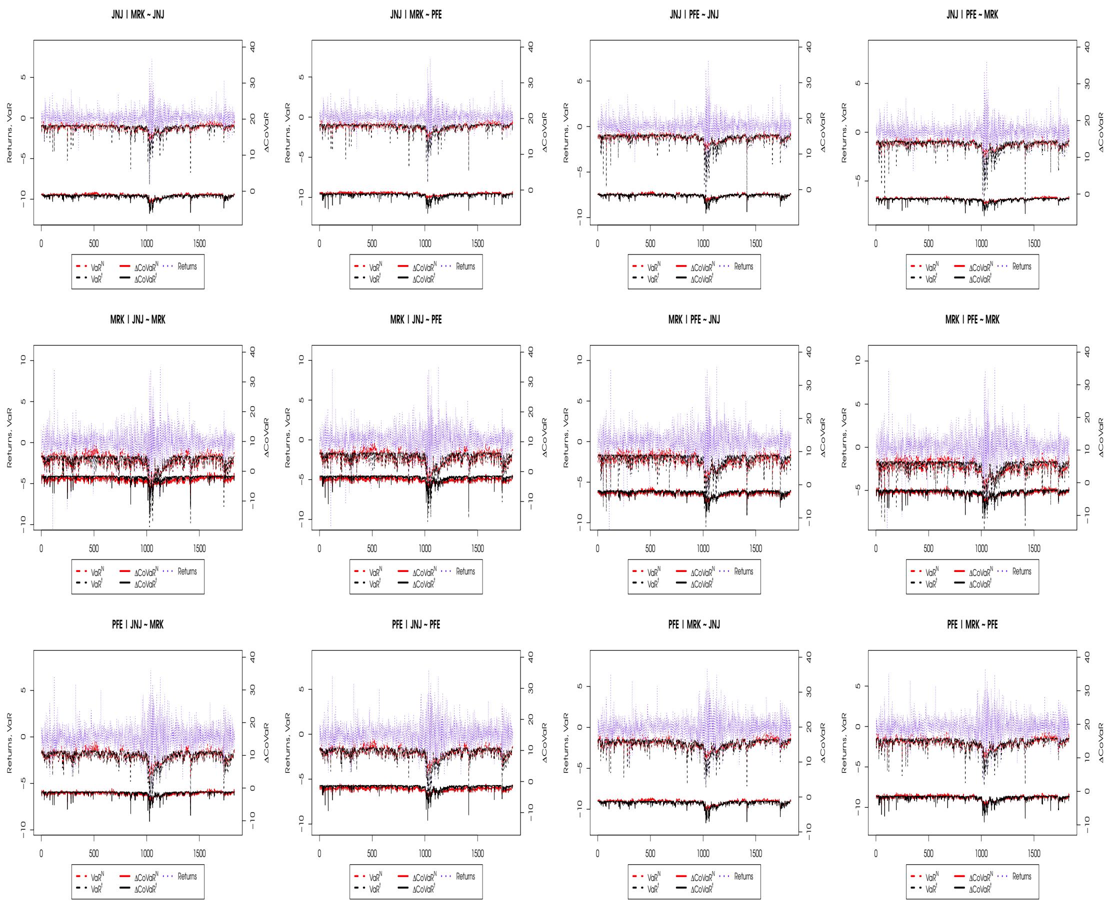


Figure 4.23: *Health Care*: The figure shows returns (in blue), 5% VaR^j from Normal and t copula quantile regression (dashed lines red and black respectively), $\Delta CoVaR_{5\%}^{j|i}$ from Normal and t copula quantile regression (red and black lines respectively). The $\Delta CoVaR_{5\%}^{j|i}$ gives us the VaR contribution of institution i on institution j. In the figure titles we have the label of the form $(j|i \sim k)$, which means that we are estimating the $VaR^j|VaR^i$ *i.e.* $CoVaR$, and the VaR^i is calculated by (copula) quantile regressing the returns of i on realized volatility of k, where $i, j, k \in \{JNJ, MRK, PFE\}$ and $i \neq j$, $j \neq k$.



4.C CoVaR models comparison

Table 4.16: Summary statistics for estimated risk measures using nonlinear quantile regression models. In the asset pairs the first one represents asset j and the second asset i *e.g.* for pair AAPL-INTC we have *j-i*. The $VaR^{i,j}$ is calculated by (copula) quantile regressing the returns of asset *i,j* on realized volatility of asset k *e.g.* for Information Technology $k \in \{AAPL, INTC, MSFT\}$ and $i, j \neq k$. Thus we have two VaR measures for the same asset i or j. The summary statistics are based on both these measures *e.g.* the summary reported in the table for $VaR^{C_N,j}_{5\%,t}$ for AAPL includes the results of $VaR^{C_N,AAPL\sim INTC}_{5\%,t} + VaR^{C_N,AAPL\sim MSFT}_{5\%,t}$. This is why we have the same numbers for VaR in the table. We have similar situation for the $\Delta CoVaR^{j|i}$, where the VaR^i is calculated by (copula) quantile regressing the returns of asset i on realized realized volatility of asset k. Continuing the example on Information Technology $k \in \{AAPL, INTC, MSFT\}$ and $i \neq k$. All numbers are in percentage, C_N, C_t represent Normal and t copula respectively, and B and L represent the benchmark and linear quantile regression.

	Information Technology			
	AAPL-INTC		AAPL-MSFT	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0268	2.0087	−0.0268	2.0087
$VaR^{C_N,j}_{5\%,t}$	−3.2590	0.5349	−3.2590	0.5349
$VaR^{C_t,j}_{5\%,t}$	−3.3123	0.8776	−3.3123	0.8776
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.5206	1.2166	−2.3860	1.0565
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.5454	1.4296	−2.1841	1.3118
	INTC-AAPL		INTC-MSFT	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0090	1.6404	−0.0090	1.6404
$VaR^{C_N,j}_{5\%,t}$	−2.5129	0.7506	−2.5129	0.7506
$VaR^{C_t,j}_{5\%,t}$	−2.5004	0.9344	−2.5004	0.9344
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−4.5369	0.8510	−2.3517	0.8241
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−4.0251	1.3419	−2.0951	1.0617
	MSFT-AAPL		MSFT-INTC	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0096	1.4026	−0.0096	1.4026
$VaR^{C_N,j}_{5\%,t}$	−2.0256	0.7379	−2.0256	0.7379
$VaR^{C_t,j}_{5\%,t}$	−2.0494	1.0410	−2.0494	1.0410
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−4.5977	1.0894	−2.5312	0.8255
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−4.3013	1.4039	−2.4290	0.9601
	Consumer Cyclical			
	AMZN-DIS		AMZN-MCD	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.1497	2.2343	0.1497	2.2343
$VaR^{C_N,j}_{5\%,t}$	−2.9838	0.6678	−2.9838	0.6678
$VaR^{C_t,j}_{5\%,t}$	−3.1187	1.3009	−3.1187	1.3009
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.5275	0.8256	−3.4096	0.7270
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.8383	1.2253	−3.9389	1.1572
	DIS-AMZN		DIS-MCD	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.0857	1.5579	0.0857	1.5579
$VaR^{C_N,j}_{5\%,t}$	−2.1698	0.7302	−2.1698	0.7302
$VaR^{C_t,j}_{5\%,t}$	−2.1103	0.9202	−2.1103	0.9202
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.9765	0.7125	−3.0408	0.6474
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.3823	0.9874	−3.3433	0.9796
	MCD-AMZN		MCD-DIS	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.0394	1.2084	0.0394	1.2084
$VaR^{C_N,j}_{5\%,t}$	−1.7903	0.3727	−1.7903	0.3727
$VaR^{C_t,j}_{5\%,t}$	−1.8024	0.5401	−1.8024	0.5401
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.2124	0.5415	−1.5111	0.5413
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.7073	0.7522	−1.6251	0.7106

	Information Technology			
	AAPL-INTC		AAPL-MSFT	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0268	2.0087	−0.0268	2.0087
$VaR^{B,j}_{5\%,t}$	−2.8858	1.6697	−2.8858	1.6697
$VaR^{L,j\|i}_{5\%,t}$	−3.2672	0.6426	−3.2672	0.6426
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.3604	0.6795	−1.0759	0.6034
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.8111	1.4292	−2.4092	1.5867
	INTC-AAPL		INTC-MSFT	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0090	1.6404	−0.0090	1.6404
$VaR^{B,j}_{5\%,t}$	−2.6321	1.3148	−2.6321	1.3148
$VaR^{L,j\|i}_{5\%,t}$	−2.5312	0.9299	−2.5312	0.9299
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.8009	1.0420	−1.4303	0.8022
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−4.7151	1.0241	−2.1847	1.0061
	MSFT-AAPL		MSFT-INTC	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	−0.0096	1.4026	−0.0096	1.4026
$VaR^{B,j}_{5\%,t}$	−2.1677	1.2158	−2.1677	1.2158
$VaR^{L,j\|i}_{5\%,t}$	−2.1315	1.0147	−2.1315	1.0147
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.5885	0.9191	−2.2051	1.1015
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−5.4773	1.1230	−2.7574	1.1384
	Consumer Cyclical			
	AMZN-DIS		AMZN-MCD	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.1497	2.2343	0.1497	2.2343
$VaR^{B,j}_{5\%,t}$	−3.4230	1.7934	−3.4230	1.7934
$VaR^{L,j\|i}_{5\%,t}$	−3.1486	1.3026	−3.1486	1.3026
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−2.2152	1.3460	−2.2549	1.2581
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.7789	1.2000	−3.7659	1.1905
	DIS-AMZN		DIS-MCD	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.0857	1.5579	0.0857	1.5579
$VaR^{B,j}_{5\%,t}$	−2.2875	1.3899	−2.2875	1.3899
$VaR^{L,j\|i}_{5\%,t}$	−2.1631	0.9456	−2.1631	0.9456
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.7628	0.9235	−1.7339	0.9674
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.3836	1.0342	−2.9003	0.9169
	MCD-AMZN		MCD-DIS	
	Mean	Std.Dev.	Mean	Std.Dev.
X_t^j	0.0394	1.2084	0.0394	1.2084
$VaR^{B,j}_{5\%,t}$	−1.8991	1.0596	−1.8991	1.0596
$VaR^{L,j\|i}_{5\%,t}$	−1.7794	0.5965	−1.7794	0.5965
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.0807	0.5662	−1.0291	0.6253
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.8687	0.9035	−1.6878	0.8113

Table 4.17: Summary statistics for estimated risk measures using nonlinear quantile regression models. In the asset pairs the first one represents asset j and the second asset i *e.g.* for pair KO-PG we have j-i. The $VaR^{i,j}$ is calculated by (copula) quantile regressing the returns of asset i,j on realized volatility of asset k *e.g.* for Consumer Defensive $k \in \{KO, PG, WMT\}$ and $i, j \neq k$. Thus we have two VaR measures for the same asset i or j. The summary statistics are based on both these measures *e.g.* the summary reported in the table for $VaR^{C_N,j}_{5\%,t}$ for KO includes the results of $VaR^{C_N,KO\sim PG}_{5\%,t} + VaR^{C_N,KO\sim WMT}_{5\%,t}$. This is why we have the same numbers for VaR in the table. We have similar situation for the $\Delta CoVaR^{j|i}$, where the VaR^i is calculated by (copula) quantile regressing the returns of asset i on realized realized volatility of asset k. Continuing the example on Information Technology $k \in \{KO, PG, WMT\}$ and $i \neq k$. All numbers are in percentage, C_N, C_t represent Normal and t copula respectively, and B and L represent the benchmark and linear quantile regression.

	Consumer Defensive			
	KO-PG		KO-WMT	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0103	1.0565	0.0103	1.0565
$VaR^{C_N,j}_{5\%,t}$	−1.4157	0.4428	−1.4157	0.4428
$VaR^{C_t,j}_{5\%,t}$	−1.4739	0.8064	−1.4739	0.8064
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−1.7434	0.5692	−1.8952	0.4934
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.7732	0.7969	−2.0976	0.7997
	PG-KO		PG-WMT	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0624	0.9851	0.0624	0.9851
$VaR^{C_N,j}_{5\%,t}$	−1.3646	0.4347	−1.3646	0.4347
$VaR^{C_t,j}_{5\%,t}$	−1.3099	0.6112	−1.3099	0.6112
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−1.3666	0.4233	−1.6452	0.4347
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.2563	0.6859	−1.8421	0.7290
	WMT-KO		WMT-PG	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0121	1.0774	−0.0121	1.0774
$VaR^{C_N,j}_{5\%,t}$	−1.5506	0.3780	−1.5506	0.3780
$VaR^{C_t,j}_{5\%,t}$	−1.5039	0.5552	−1.5039	0.5552
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−1.3702	0.4262	−1.4654	0.4924
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.2225	0.6668	−1.3861	0.6719
	Telecommunication Services			
	CMCSA-T		CMCSA-VZ	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0201	1.8655	0.0201	1.8655
$VaR^{C_N,j}_{5\%,t}$	−2.6018	0.7634	−2.6018	0.7634
$VaR^{C_t,j}_{5\%,t}$	−2.5225	1.3769	−2.5225	1.3769
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.6000	0.8379	−2.8455	0.9481
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−3.2802	1.3957	−2.8571	1.3127
	T-CMCSA		T-VZ	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0142	1.3164	−0.0142	1.3164
$VaR^{C_N,j}_{5\%,t}$	−1.8462	0.6278	−1.8462	0.6278
$VaR^{C_t,j}_{5\%,t}$	−1.8215	0.7974	−1.8215	0.7974
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.3413	0.6800	−2.1034	0.6579
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.7636	0.9697	−1.8615	0.8572
	VZ-CMCSA		VZ-T	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0418	1.2736	−0.0418	1.2736
$VaR^{C_N,j}_{5\%,t}$	−1.8833	0.6232	−1.8833	0.6232
$VaR^{C_t,j}_{5\%,t}$	−1.8559	0.8615	−1.8559	0.8615
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.2275	0.6469	−1.7802	0.5772
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.7943	0.9717	−1.8229	0.7903

	Consumer Defensive			
	KO-PG		KO-WMT	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0103	1.0565	0.0103	1.0565
$VaR^{B,j}_{5\%,t}$	−1.6138	0.9566	−1.6138	0.9566
$VaR^{L,j\|i}_{5\%,t}$	−1.5528	0.8388	−1.5528	0.8388
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.4974	0.9034	−1.3607	0.7656
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.6932	0.7996	−2.0648	0.8985
	PG-KO		PG-WMT	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0624	0.9851	0.0624	0.9851
$VaR^{B,j}_{5\%,t}$	−1.6308	0.9839	−1.6308	0.9839
$VaR^{L,j\|i}_{5\%,t}$	−1.3984	0.6891	−1.3984	0.6891
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.1424	0.6772	−1.2458	0.7010
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.3929	0.7379	−1.7303	0.7512
	WMT-KO		WMT-PG	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0121	1.0774	−0.0121	1.0774
$VaR^{B,j}_{5\%,t}$	−1.8358	1.0329	−1.8358	1.0329
$VaR^{L,j\|i}_{5\%,t}$	−1.6164	0.6748	−1.6164	0.6748
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−0.9773	0.5793	−1.2549	0.7571
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.3484	0.7155	−1.3303	0.6350
	Telecommunication Services			
	CMCSA-T		CMCSA-VZ	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0201	1.8655	0.0201	1.8655
$VaR^{B,j}_{5\%,t}$	−2.7010	1.5787	−2.7010	1.5787
$VaR^{L,j\|i}_{5\%,t}$	−2.6707	1.3179	−2.6707	1.3179
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.8663	1.1984	−2.2746	1.4267
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−3.0624	1.3784	−3.0028	1.2985
	T-CMCSA		T-VZ	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0142	1.3164	−0.0142	1.3164
$VaR^{B,j}_{5\%,t}$	−2.1481	1.3793	−2.1481	1.3793
$VaR^{L,j\|i}_{5\%,t}$	−1.8739	0.8968	−1.8739	0.8968
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.3972	0.8166	−1.5471	0.9704
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.0873	1.0586	−2.0522	0.8593
	VZ-CMCSA		VZ-T	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0418	1.2736	−0.0418	1.2736
$VaR^{B,j}_{5\%,t}$	−2.0340	1.2758	−2.0340	1.2758
$VaR^{L,j\|i}_{5\%,t}$	−1.8629	0.8191	−1.8629	0.8191
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.3452	0.7862	−1.3250	0.8508
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.8059	0.9037	−1.7551	0.7806

Table 4.18: Summary statistics for estimated risk measures using nonlinear quantile regression models. In the asset pairs the first one represents asset j and the second asset i *e.g.* for pair BAC-C we have j-i. The $VaR^{i,j}$ is calculated by (copula) quantile regressing the returns of asset i,j on realized volatility of asset k *e.g.* for Financials $k \in \{BAC, C, WFC\}$ and $i, j \neq k$. Thus we have two VaR measures for the same asset i or j. The summary statistics are based on both these measures *e.g.* the summary reported in the table for $VaR^{C_N,j}_{5\%,t}$ for BAC includes the results of $VaR^{C_N,BAC\sim C}_{5\%,t} + VaR^{C_N,BAC\sim WFC}_{5\%,t}$. This is why we have the same numbers for VaR in the table. We have similar situation for the $\Delta CoVaR^{j|i}$, where the VaR^i is calculated by (copula) quantile regressing the returns of asset i on realized realized volatility of asset k. Continuing the example on Information Technology $k \in \{BAC, C, WFC\}$ and $i \neq k$. All numbers are in percentage, C_N, C_t represent Normal and t copula respectively, and B and L represent the benchmark and linear quantile regression.

Financials	BAC-C Mean	BAC-C Std.Dev.	BAC-WFC Mean	BAC-WFC Std.Dev.
X_t^j	−0.2287	3.2662	−0.2287	3.2662
$VaR^{C_N,j}_{5\%,t}$	−3.5153	2.5213	−3.5153	2.5213
$VaR^{C_t,j}_{5\%,t}$	−3.6270	4.1774	−3.6270	4.1774
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−3.6531	2.4349	−4.1345	2.2968
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−4.0803	4.8295	−4.8247	4.4158
	C-BAC Mean	Std.Dev.	**C-WFC** Mean	Std.Dev.
X_t^j	−0.4165	3.4154	−0.4165	3.4154
$VaR^{C_N,j}_{5\%,t}$	−3.7100	2.4754	−3.7100	2.4754
$VaR^{C_t,j}_{5\%,t}$	−3.8720	4.4056	−3.8720	4.4056
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−3.9245	2.7817	−4.7475	2.6371
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−3.8706	4.5434	−4.9775	4.5553
	WFC-BAC Mean	Std.Dev.	**WFC-C** Mean	Std.Dev.
X_t^j	−0.0189	2.7202	−0.0189	2.7202
$VaR^{C_N,j}_{5\%,t}$	−3.0109	1.6453	−3.0109	1.6453
$VaR^{C_t,j}_{5\%,t}$	−3.1319	2.8500	−3.1319	2.8500
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.7140	1.9244	−2.7777	1.8528
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.5376	2.9787	−2.7841	3.2953
Energy	**CVX-SLB** Mean	Std.Dev.	**CVX-XOM** Mean	Std.Dev.
X_t^j	0.0114	1.5384	0.0114	1.5384
$VaR^{C_N,j}_{5\%,t}$	−2.1110	0.6879	−2.1110	0.6879
$VaR^{C_t,j}_{5\%,t}$	−2.3312	1.2602	−2.3312	1.2602
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.9900	0.7326	−2.3038	0.6720
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.9787	1.1728	−3.6367	1.1700
	SLB-CVX Mean	Std.Dev.	**SLB-XOM** Mean	Std.Dev.
X_t^j	−0.0180	2.1495	−0.0180	2.1495
$VaR^{C_N,j}_{5\%,t}$	−3.3431	0.9044	−3.3431	0.9044
$VaR^{C_t,j}_{5\%,t}$	−3.2637	1.3489	−3.2637	1.3489
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−3.0459	1.0325	−3.3051	0.9748
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.8687	1.5376	−4.2088	1.3675
	XOM-CVX Mean	Std.Dev.	**XOM-SLB** Mean	Std.Dev.
X_t^j	0.0511	1.4661	0.0511	1.4661
$VaR^{C_N,j}_{5\%,t}$	−2.0919	0.6549	−2.0919	0.6549
$VaR^{C_t,j}_{5\%,t}$	−2.0847	0.7165	−2.0847	0.7165
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.0598	0.6520	−2.5427	0.6227
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.8736	0.9974	−2.4151	0.9508

Financials	BAC-C Mean	BAC-C Std.Dev.	BAC-WFC Mean	BAC-WFC Std.Dev.
X_t^j	−0.2287	3.2662	−0.2287	3.2662
$VaR^{B,j}_{5\%,t}$	−3.4063	3.5345	−3.4063	3.5345
$VaR^{L,j\|i}_{5\%,t}$	−4.0146	4.3168	−4.0146	4.3168
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−4.1724	4.4315	−4.5013	4.1964
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−3.8761	3.8197	−4.6508	4.5421
	C-BAC Mean	Std.Dev.	**C-WFC** Mean	Std.Dev.
X_t^j	−0.4165	3.4154	−0.4165	3.4154
$VaR^{B,j}_{5\%,t}$	−3.7836	4.0186	−3.7836	4.0186
$VaR^{L,j\|i}_{5\%,t}$	−4.4367	4.4916	−4.4367	4.4916
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−4.2574	4.4175	−4.8964	4.5648
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−3.9910	4.1336	−4.7138	4.6007
	WFC-BAC Mean	Std.Dev.	**WFC-C** Mean	Std.Dev.
X_t^j	−0.0189	2.7202	−0.0189	2.7202
$VaR^{B,j}_{5\%,t}$	−3.2013	2.9845	−3.2013	2.9845
$VaR^{L,j\|i}_{5\%,t}$	−3.2915	3.1642	−3.2915	3.1642
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−2.9651	3.0767	−3.0602	3.2502
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.7745	2.8730	−2.7753	2.7317
Energy	**CVX-SLB** Mean	Std.Dev.	**CVX-XOM** Mean	Std.Dev.
X_t^j	0.0114	1.5384	0.0114	1.5384
$VaR^{B,j}_{5\%,t}$	−2.3468	1.4215	−2.3468	1.4215
$VaR^{L,j\|i}_{5\%,t}$	−2.3060	1.2054	−2.3060	1.2054
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.9015	0.9947	−2.2103	1.3838
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−3.2955	1.2660	−2.7831	1.1622
	SLB-CVX Mean	Std.Dev.	**SLB-XOM** Mean	Std.Dev.
X_t^j	−0.0180	2.1495	−0.0180	2.1495
$VaR^{B,j}_{5\%,t}$	−3.3654	1.7604	−3.3654	1.7604
$VaR^{L,j\|i}_{5\%,t}$	−3.4224	1.4077	−3.4224	1.4077
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−2.3647	1.4324	−2.2057	1.3810
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.9719	1.6038	−3.5797	1.5344
	XOM-CVX Mean	Std.Dev.	**XOM-SLB** Mean	Std.Dev.
X_t^j	0.0511	1.4661	0.0511	1.4661
$VaR^{B,j}_{5\%,t}$	−2.1698	1.3585	−2.1698	1.3585
$VaR^{L,j\|i}_{5\%,t}$	−2.1814	0.9465	−2.1814	0.9465
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.7431	1.0558	−1.5733	0.8230
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.9989	1.0605	−2.7087	1.0434

Table 4.19: Summary statistics for estimated risk measures using nonlinear quantile regression models. In the asset pairs the first one represents asset j and the second asset i *e.g.* for pair JNJ-MRK we have *j-i*. The $VaR^{i,j}$ is calculated by (copula) quantile regressing the returns of asset i,j on realized volatility of asset k *e.g.* for Health Care $k \in \{JNJ, MRK, PFE\}$ and $i, j \neq k$. Thus we have two VaR measures for the same asset i or j. The summary statistics are based on both these measures *e.g.* the summary reported in the table for $VaR^{C_N,j}_{5\%,t}$ for JNJ includes the results of $VaR^{C_N,JNJ\sim MRK}_{5\%,t} + VaR^{C_N,JNJ\sim PFE}_{5\%,t}$. This is why we have the same numbers for VaR in the table. We have similar situation for the $\Delta CoVaR^{j|i}$, where the VaR^i is calculated by (copula) quantile regressing the returns of asset i on realized realized volatility of asset k. Continuing the example on Information Technology $k \in \{JNJ, MRK, PFE\}$ and $i \neq k$. All numbers are in percentage, C_N, C_t represent Normal and t copula respectively, and B and L represent the benchmark and linear quantile regression.

	Health Care			
	JNJ-MRK		JNJ-PFE	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0107	0.9193	0.0107	0.9193
$VaR^{C_N,j}_{5\%,t}$	−1.2303	0.3581	−1.2303	0.3581
$VaR^{C_t,j}_{5\%,t}$	−1.2622	0.6414	−1.2622	0.6414
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−1.1422	0.4328	−1.4745	0.3838
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.3845	0.6153	−1.6007	0.5810
	MRK-JNJ		MRK-PFE	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0012	1.5244	0.0012	1.5244
$VaR^{C_N,j}_{5\%,t}$	−2.2939	0.8205	−2.2939	0.8205
$VaR^{C_t,j}_{5\%,t}$	−2.2700	1.0097	−2.2700	1.0097
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.8367	0.8490	−2.9422	0.7697
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−2.1300	1.0864	−2.6620	0.9867
	PFE-JNJ		PFE-MRK	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0573	1.3253	−0.0573	1.3253
$VaR^{C_N,j}_{5\%,t}$	−2.0483	0.5362	−2.0483	0.5362
$VaR^{C_t,j}_{5\%,t}$	−2.0848	0.7625	−2.0848	0.7625
$\Delta CoVaR^{C_N,j\|i}_{5\%,t}$	−2.0794	0.7084	−1.6876	0.6402
$\Delta CoVaR^{C_t,j\|i}_{5\%,t}$	−1.6750	0.8756	−1.7989	0.8125

	Health Care			
	JNJ-MRK		JNJ-PFE	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0107	0.9193	0.0107	0.9193
$VaR^{B,j}_{5\%,t}$	−1.4584	0.8995	−1.4584	0.8995
$VaR^{L,j\|i}_{5\%,t}$	−1.2983	0.5980	−1.2983	0.5980
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−0.9578	0.5948	−1.1468	0.6003
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.3608	0.6559	−1.5575	0.6555
	MRK-JNJ		MRK-PFE	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	0.0012	1.5244	0.0012	1.5244
$VaR^{B,j}_{5\%,t}$	−2.3359	1.4507	−2.3359	1.4507
$VaR^{L,j\|i}_{5\%,t}$	−2.3021	1.0942	−2.3021	1.0942
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.8384	1.1339	−2.0110	1.0526
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−2.3298	1.0683	−2.7043	1.1150
	PFE-JNJ		PFE-MRK	
	Mean	Std.Dev.	Mean	Std.Dev.
X^j_t	−0.0573	1.3253	−0.0573	1.3253
$VaR^{B,j}_{5\%,t}$	−2.1959	1.1494	−2.1959	1.1494
$VaR^{L,j\|i}_{5\%,t}$	−2.0713	0.8246	−2.0713	0.8246
$\Delta CoVaR^{B,j\|i}_{5\%,t}$	−1.3666	0.8429	−1.2978	0.8060
$\Delta CoVaR^{L,j\|i}_{5\%,t}$	−1.8218	0.8408	−1.8163	0.8773

Chapter 5
Conclusion

In my dissertation I study the nonlinear dependence in financial time series from different perspectives. First, I investigate the dependence between oil and stocks with the aim to explore opportunities in portfolio management. Using high frequency data and copula models I capture the time-varying conditional distribution of the oil stocks pair accurately, including the dynamics in the correlation and tails. This also converts into accurate quantile forecasts from the model, which are central to risk management, as they represent value-at-risk. These results then are translated into the conditional diversification benefits measure which is proposed recently by Christoffersen *et al.* (2012). I find that possible benefits from using oil as a diversification tool for stocks have been decreasing rapidly over time, while in the last year of the sample under study, it displayed some rebound. These results have important implications for the risk industry and portfolio management as commodities have recently become an attractive opportunity for risk diversification in portfolios.

Next I focus in modelling and forecasting the conditional quantiles of financial assets returns. I explore further non-linearities in the data, and propose to use realized measures in the nonlinear quantile regression framework to explain and forecast conditional quantiles of financial returns. The nonlinear quantile regression models are implied by copula specifications and allow me to capture possible nonlinearities and asymmetries in conditional quantiles of financial returns. Modelling of the conditional quantiles has direct implications in calculation of the standard measure of the market risk, the Value-at-Risk (VaR). I apply this methodology to estimate and forecast the VaR of 21 most liquid U.S. stocks which com from seven sectors. I find that using the realized volatility under a copula quantile framework is useful, especially in the cases where the quantile dependence is nonlinear.

Finally I consider the VaR of an asset conditional on some other asset being under distress or the conditional Value-at-Risk (CoVaR). I follow a slightly different approach than current literature, where in the focus is systemic risk. I estimate the risk contribution that an asset has on some other individual asset, which allows the study of risk spillovers among assets. I estimate the conditional VaR on the same data as previously and propose to use nonlinear quantile regression models based on copula theory to estimate CoVaR. The individual risk contribution estimated from these models gives estimates in between the benchmark model which is based on realized volatility and the Linear quantile regression model. I find that assets from Financial industry have the highest risk contribution among each other. The model I propose identifies Apple to be a high risk contributor to Microsoft and Intel. While for assets from Consumer Staples industry and Health Care I find that they have the lowest risk spillovers.

Response to Reviewers

for the *Dissertation Defense*

December 21, 2015

on manuscript
Essays in Financial Econometrics
by Krenar Avdulaj

I thank the reviewers for insightful comments on the pre-defense version of my dissertation. Since the reviewers suggest that the dissertation can be submitted without major changes, I have made minor adjustments in the text.

Response to Comments from Prof. Ing. Evžen Kočenda PhD.

I thank Prof. Kočenda for his kind assessment of my research. Prof. Kočenda has several suggestions for thesis improvement, mainly for the third essay. I have taken all these comments in consideration while preparing the final version of the dissertation.

Response to Comments from Prof. Tiziana Di Matteo

I am grateful to Prof. Di Matteo for her kind words on my dissertation. Prof. Di Matteo has few minor suggestions with respect to organization and coherent presentation of the dissertation. I considered all her suggestions which resulted in further improvement of my thesis. Thank you!

Response to Comments from Doc. RNDr. Jiří Witzany Ph.D.

I thank Doc. RNDr. Witzany for his kind assessment of my research. Below I answer his questions/comments (*"Q"* refers to question or comment and *"A"* is my answer).

- *Q: The first paper focuses on oil-stock dependence and the diversification benefits. My understanding of the results of the realized GARCH with time varying copula is that the diversification benefits are lower than commonly* *believed (2.6 Conclusions and Figure 2.3). But conclusions at the end of Section 2.4 say the opposite: "Our results have serious implications for investors as they suggest that diversification possibilities may be even larger than commonly perceived from the mere dynamics of the correlations." I would like to ask the author to clarify the inconsistent interpretations of the results.*

A: In fact this is some misunderstanding caused by the text. Up to the end of Section 2.4 we are considering the cumulative results which do not take into consideration conditional diversification benefit (CDB).

- *Q: According to Section 2.5.1, it appears that the investigated diversified portfolio of stocks and oil is equally weighted and the weights do not change over time. However, the changing volatilities and correlations (copula parameters) allow re-balancing of the portfolio optimizing the diversification benefit, for example, measured by the diversification index proposed in the paper. The changing volatilities and dependence structure may just cause the equally weighted portfolio being less optimal, not necessarily implying a lower diversification benefit on an optimally diversified portfolio. I would like the author to comment this objection.*

  A: From the literature we know that diversification on an equally weighted portfolio in the case of two assets cannot be (easily) beaten using a dynamic asset allocation strategy. The transaction costs often overcome the benefits. In addition, Christoffersen *et al.* (2012) compare the CDB using equally weighted and optimally weighted portfolios. They find that the difference is nonzero, but not very large. They claim that relatively modest differences between optimal and equal-weighted diversification benefits suggest that the $1/N$ style portfolios recently advocated in a normal setting may work relatively well in our nonnormal context as well.

- *Q: A formal remark concerns the quantile definition (2.26) implicitly assuming that the cdf is continuous increasing which does not have to be necessarily the case (e.g. in case of an empirical cdf).*

  A: Yes, this is the underlying assumption. I made it clear in the final version of the dissertation.

- *Q: The second paper uses high frequency data and the nonlinear quantile regression framework to study conditional quantiles of returns on a pool of the most liquid US assets across different industries. A formal remark is that sometimes there are notions or shortcuts that are firstly used and only later defined in the text. For example, IVt, integrated variance, is firstly used in*

Section 3.2, but more precisely defined in 4.2. Similarly, the shortcut LQR is firstly used in Section 3.5 but more specifically defined in Section 4.2 (it would be useful to mention it already in Section 3.2). Figure 3.2 shows dependence of a set of quantiles of a stock returns on its realized volatility. I have not found (in the text preceding the figure) any specification of the probabilities for which the quantiles are calculated.

A: This is true. I introduced the acronyms when they first appear in the text and also add the missing information in the text.

- *Q: Finally, the last paper focuses on Conditional Value at Risk estimated using the nonlinear quantile copula regression technique and using the same dataset as the second paper. Already in the introduction, the concept of VaR is used in the nonstandard convention where the values are negative (equal to the respective quantile) while the standard convention is to report VaR as a positive number. This is explained later, in Section 4.2. I recommend to explain this change of convention already in the introduction in order to avoid confusion. I am not sure that the methodology section explains the notion of "inter lagged realized volatility" as opposed to "own lagged realized volatility" used already in section 4.1?*

  The "benchmark" model is based on VaR estimated by rescaling the realized volatility, but still using the same linear quantile regression for CoVaR estimation (Section 4.2.3). It is surprising why the author does not use as a basic benchmark a simpler and easier to implement model, e.g. based on constant correlations and multivariate normality, or DCC GARCH, etc.?

  A: This is correct, I introduced the Value-at-Risk (VaR) convention earlier in the text and also explain *"own"* and *"inter"* volatility.

  Regarding the choice of the benchmark model I chose realized volatility because of three main reasons:

  1. The parametric models tend to overestimate the risk.
  2. Given that the realized volatility is the best in the market, why to use another metrics?
  3. We already are using realized volatility in our model so it is easily implemented.

- *Q: Besides the minor comments above there is a more general practical question I would like to ask. It is obvious that the complex realized GARCH dynamical copula and quantile regression modeling framework is technically very* *demanding in terms of presentation and implementation. On the other hand, it brings a better precision of the VaR estimations, conditional dependence measures, portfolio diversification, etc. Does the author think that, from the practical point of view (of banks, financial institutions, and investors), the benefits out-weight the "costs"?*

A: I think that all modelling framework I have in my dissertation can be easily implemented. The most demanding from the computational point of view is the provision of inference via bootstrapping and simulations. However, this is done only once to show that the model estimates are significant. Besides, using C code for the bottlenecks of estimation significantly reduces the (computational) costs.

References

Christoffersen, P., V. Errunza, K. Jacobs, & H. Langlois (2012): "Is the potential for international diversification disappearing? a dynamic copula approach." *Review of Financial Studies* **25(12)**: pp. 3711–3751.

Appendix A

Copula properties

In this appendix I provide some properties of bivariate copula functions which I have used in my dissertation.

A.1 Tail Dependence

If the limit

$$\lim_{\epsilon\to 0} Pr[U \leq \epsilon | V \leq \epsilon] = \lim_{\epsilon\to 0} Pr[V \leq \epsilon | U \leq \epsilon] = \lim_{\epsilon\to 0} \frac{C(\epsilon,\epsilon)}{\epsilon} = \tau^L \tag{A.1}$$

exists, then the copula C exhibits lower tail dependence if $\tau^L \in (0, 1]$ and no lower tail dependence if $\tau^L = 0$. The lower quantile-quantile dependence measure is defined as

$$\tau^L(\epsilon) = \frac{C(\epsilon,\epsilon)}{\epsilon} \tag{A.2}$$

Similarly, if the limit

$$\begin{aligned}\lim_{\delta\to 1} Pr[U > \delta | V > \delta] &= \lim_{\delta\to 1} Pr[V > \delta | U > \delta] \\ &= \lim_{\delta\to 1} \frac{1 - 2\delta + C(\delta,\delta)}{1-\delta} = \tau^U \end{aligned}\tag{A.3}$$

exists, then the copula C exhibits upper tail dependence if $\tau^U \in (0, 1]$ and no upper tail dependence if $\tau^U = 0$. The upper quantile-quantile dependence measure is defined as

$$\tau^U(\delta) = \frac{1 - 2\delta + C(\delta,\delta)}{1-\delta} \tag{A.4}$$

A.2 Elliptical copulas

A.2.1 Normal copula

$$C^N_\rho(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} drds$$

where

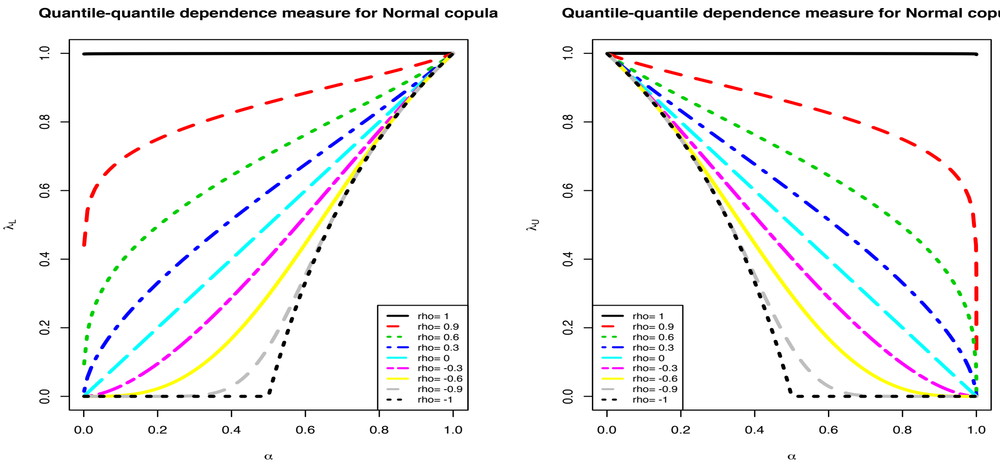
$$\rho \in (-1, 1)$$

and upper and lower tail dependence

$$\tau^L = \tau^U = 2 \lim_{r \to -\infty} \Phi \left(r \frac{\sqrt{1-\rho}}{1+\rho} \right) = 0$$

For $\rho = 0$ we obtain the independence copula, while for $\rho = 1$ the comonotonicity one. For $\rho = -1$ the countermonotonicity copula is obtained. We note that Normal copula has no tail dependence for $\rho < 1$.

In Figure A.1 we plot the (*upper* and *lower*) quantile-quantile dependence for Normal copula. The plots are generated using the Equation A.2 for *a)* and Equation A.4 for *b)*. From these plots it is clear that Normal copula has 0 tail dependence for all $\rho < 1$.



(a) Lower tail

(b) Upper tail

Figure A.1: Quantile-quantile dependence for Normal copula.

Partial derivative(s)

Assign

$$g(r, s) = \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{ -\frac{r^2 - 2\rho r s + s^2}{2(1-\rho^2)} \right\}$$

and

$$b_1 = \Phi^{-1}(u), \quad b_2 = \Phi^{-1}(v)$$

Then,

$$\begin{aligned}
C_1^N(u,v) =& \frac{\partial C^N(u,v)}{\partial u} \\
=& \frac{\partial}{\partial u} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} g(r,s)\, dr\, ds \\
=& \frac{\partial b_1}{\partial u} \frac{\partial}{\partial b_1} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} g(r,s)\, dr\, ds \\
=& \frac{1}{\phi(b_1)} \frac{\partial}{\partial b_1} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} g(r,s)\, dr\, ds \\
=& \frac{1}{\phi(b_1)} \int_{-\infty}^{b_2} \left[\frac{\partial}{\partial b_1} \int_{-\infty}^{b_1} g(r,s)\, ds \right] dr \\
=& \frac{1}{\phi(b_1)} \int_{-\infty}^{b_2} g(b_1,s)\, ds \\
=& \frac{1}{\phi(b_1)} \int_{-\infty}^{b_2} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{ -\frac{b_1^2 - 2\rho b_1 s + s^2}{2(1-\rho^2)} \right\} ds \\
=& \frac{1}{\phi(b_1)} \int_{-\infty}^{b_2} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{ -\frac{(s-\rho b_1)^2 + (b_1^2 - \rho^2 b_1^2)}{2(1-\rho^2)} \right\} ds \\
=& \frac{1}{\phi(b_1)} \int_{-\infty}^{b_2} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{ -\frac{b_1^2(1-\rho^2)}{2(1-\rho^2)} \right\} \exp\left\{ -\frac{(s-\rho b_1)^2}{2(1-\rho^2)} \right\} ds \\
=& \frac{1}{\phi(b_1)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b_1^2}{2} \right) \int_{-\infty}^{b_2} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{ \frac{-(s-\rho b_1)^2}{2(1-\rho^2)} \right\} ds \\
=& \frac{1}{\phi(b_1)} \phi(b_1) \Phi\left(\frac{b_2 - \rho b_1}{\sqrt{1-\rho^2}} \right) \\
=& \Phi\left(\frac{b_2 - \rho b_1}{\sqrt{1-\rho^2}} \right) \\
=& \Phi\left(\frac{\Phi^{-1}(v) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}} \right)
\end{aligned}$$

The derivative w.r.t. v can be derived in similar way. However, due to symmetricity we can get $C_2(u,v) = \dfrac{\partial C(u,v)}{\partial v}$ by just replacing u for v and *vice-versa* in the result above.

A.2.2 Student's t

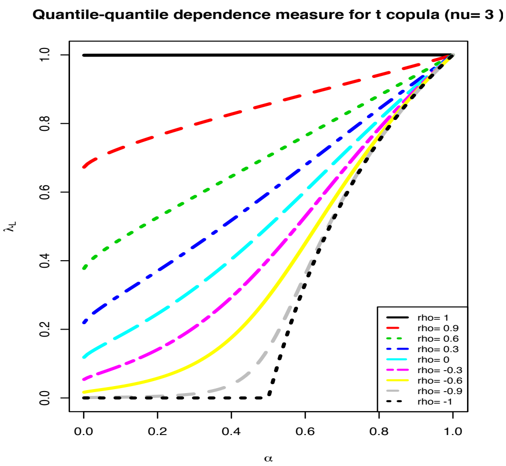
$$C^t_{\eta,\rho}(u,v) = \int_{-\infty}^{t_\eta^{-1}(u)} \int_{-\infty}^{t_\eta^{-1}(v)} \frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho rs + s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} drds$$

where

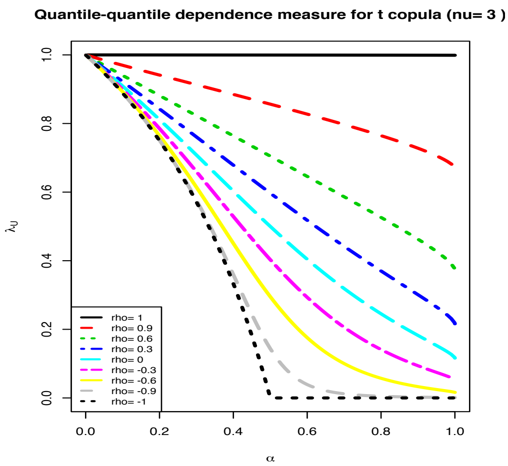
$$\rho \in (-1,1), \quad 0 < \eta$$

In contrast to Normal copula, provided that $\rho > -1$, the t copula has symmetric tail dependence given by:

$$\tau^L = \tau^U = 2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$$



(a) Lower tail



(b) Upper tail

Figure A.2: Quantile-quantile dependence for t copula.

In Figure A.2 we plot the (*upper* and *lower*) quantile-quantile dependence for t copula. The plots are generated using quantile-quantile dependence measure as in Equation A.2 for *a)* and as in Equation A.4 for *b)*. From these plots we see that t copula has tail dependence for all $\rho > -1$, and this dependence is symmetric.

Partial derivative(s)

Assign

$$g(r,s) = \frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{r^2-2\rho rs+s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}}$$
$$f_\eta(s) = \frac{\Gamma(\frac{\eta+1}{2})}{\Gamma(\frac{\eta}{2})\sqrt{\pi\eta}}\left(1+\frac{s^2}{\eta}\right)^{-\frac{\eta+1}{2}}$$

and

$$b_1 = t_\eta^{-1}(u), \quad b_2 = t_\eta^{-1}(v)$$

Then,

$$\begin{aligned}
C_1^t(u,v) =& \frac{\partial C^t(u,v)}{\partial u}\\
=& \frac{\partial}{\partial u}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2} g(r,s)\,dr\,ds\\
=& \frac{\partial b_1}{\partial u}\frac{\partial}{\partial b_1}\int_{-\infty}^{b_1}\int_{-\infty}^{b_2} g(r,s)\,dr\,ds\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\left[\frac{\partial}{\partial b_1}\int_{-\infty}^{b_1} g(r,s)\,ds\right]\,dr\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2} g(b_1,s)ds\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{b_1^2-2\rho b_1 s+s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} ds\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{(s-\rho b_1)^2+(b_1^2-\rho^2 b_1^2)}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} ds\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{(s-\rho b_1)^2}{\eta(1-\rho^2)}+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+2}{2}} ds\\
=& \frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{(s-\rho b_1)^2(\eta+b_1^2)}{\eta(1-\rho^2)(\eta+b_1^2)}+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+2}{2}} ds
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{\eta(s-\rho b_1)^2}{\eta(1-\rho^2)(\eta+b_1^2)}+\frac{b_1^2(s-\rho b_1)^2}{\eta(1-\rho^2)(\eta+b_1^2)}+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+2}{2}}ds\\
=&\frac{1}{f_\eta(b_1)}\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{(s-\rho b_1)^2}{(1-\rho^2)(\eta+b_1^2)}\right)^{-\frac{\eta+2}{2}}\left(1+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+2}{2}}ds\\
=&\frac{1}{f_\eta(b_1)}\frac{\Gamma(\frac{\eta+1}{2})\sqrt{\pi(1-\rho^2)(\eta+b_1^2)}}{\Gamma(\frac{\eta}{2})\pi\eta\sqrt{1-\rho^2}}\left(1+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+1}{2}}\left(1+\frac{b_1^2}{\eta}\right)^{-\frac{1}{2}}\\
&\cdot\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta+1}{2})\sqrt{\pi(1-\rho^2)(\eta+b_1^2)}}\left(1+\frac{(s-\rho b_1)^2}{(1-\rho^2)(\eta+b_1^2)}\right)^{-\frac{\eta+2}{2}}ds\\
=&\frac{1}{f_\eta(b_1)}\underbrace{\frac{\Gamma(\frac{\eta+1}{2})\sqrt{\eta+b_1^2}}{\Gamma(\frac{\eta}{2})\eta\sqrt{\pi}}\sqrt{\frac{\eta}{\eta+b_1^2}}\left(1+\frac{b_1^2}{\eta}\right)^{-\frac{\eta+1}{2}}}_{f_\eta(b_1)}\\
&\cdot\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta+1}{2})\sqrt{\pi(1-\rho^2)(\eta+b_1^2)}}\left(1+\frac{(s-\rho b_1)^2}{(1-\rho^2)(\eta+b_1^2)}\right)^{-\frac{\eta+2}{2}}ds\\
=&\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\eta+2}{2})}{\Gamma(\frac{\eta+1}{2})\sqrt{\pi(1-\rho^2)(\eta+b_1^2)}}\left(1+\frac{(s-\rho b_1)^2}{(1-\rho^2)(\eta+b_1^2)}\right)^{-\frac{\eta+2}{2}}ds
\end{aligned}
$$

Replace,

$$
\begin{aligned}
\nu =&\eta+1\\
\mu =&\rho b_1\\
\sigma^2 =&\frac{\eta+b_1^2}{\eta+1}(1-\rho^2)
\end{aligned}
$$

Then

$$
C_1^t(u,v)=\int_{-\infty}^{b_2}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\frac{(\eta+1)(1-\rho^2)(\eta+b_1^2)}{\eta+1}}}\left(1+\frac{(s-\rho b_1)^2}{\frac{(\eta+1)(1-\rho^2)(\eta+b_1^2)}{\eta+1}}\right)^{-\frac{\nu+1}{2}}ds
$$

$$
\begin{aligned}
&= \int_{-\infty}^{b_2} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma} \left(1 + \frac{(s-\rho b_1)^2}{\frac{(\eta+1)(1-\rho^2)(\eta+b_1^2)}{\eta+1}}\right)^{-\frac{\nu+1}{2}} ds \\
&= \int_{-\infty}^{b_2} \frac{\Gamma(\frac{\nu+1}{2})}{\sigma\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} \left(1 + \frac{1}{\nu}\left(\frac{s-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}} ds \\
&= \int_{-\infty}^{b_2} \frac{1}{\sigma} f_\nu \left(\frac{s-\mu}{\sigma}\right) ds \\
&= \int_{-\infty}^{(b_2-\mu)/\sigma} f_\nu(y)dy \\
&= t_\nu \left(\frac{b_2-\mu}{\sigma}\right)
\end{aligned}
$$

Setting back the values for b_1, b_2, σ, μ and ν we recover the expression

$$
\begin{aligned}
C_1^t(u,v) &= t_\nu \left(\frac{b_2-\mu}{\sigma}\right) \\
&= t_{\eta+1}\left(\frac{t_\eta^{-1}(v) - \rho t_\eta^{-1}(u)}{\sqrt{\frac{\left(\eta+[t_\eta^{-1}(u)]^2\right)(1-\rho^2)}{\eta+1}}}\right)
\end{aligned}
$$

The derivative w.r.t. v can be derived in similar way. However, due to symmetricity we can get $C_2(u,v) = \dfrac{\partial C(u,v)}{\partial v}$ by just replacing u for v and *vice-versa* in the result above.

A.3 Archimedian copulas

A.3.1 Clayton copula

$$
C_\delta^{Cl}(u,v) = \left(u^{-\delta} + v^{-\delta} - 1\right)^{\frac{-1}{\delta}}
$$

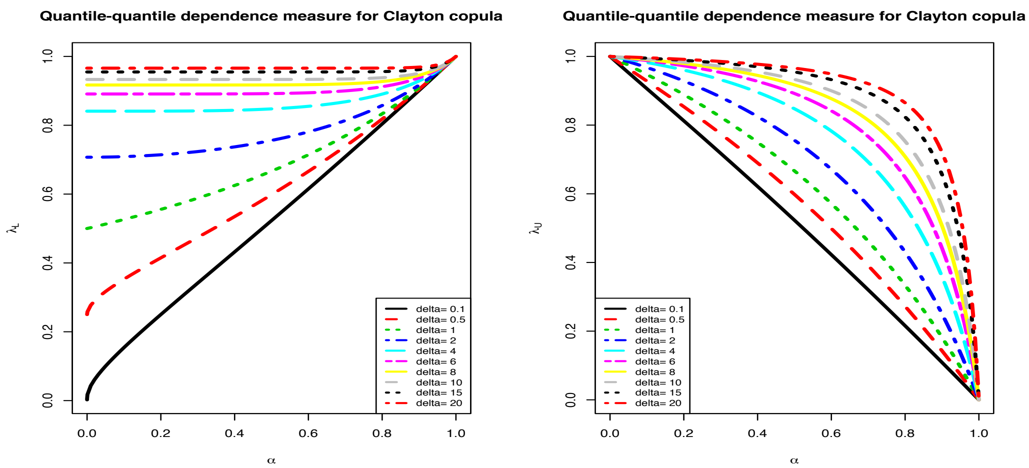
where

$$
0 < \delta < \infty
$$

and lower and upper tail dependence

$$\tau^L = 2^{-\frac{1}{\delta}}, \quad \tau^U = 0$$

For $\delta \to 0$ implies independence, while $\delta \to \infty$ perfect dependence. This can also be seen in Figure A.3 *a)*, where for parameter $\delta = 0.1$ the tail dependence is almost 0 and for parameter $\delta = 20$ the tail dependence is close to 1. Figure A.3 *b)* just shows that, indeed, Clayton copula does not have upper tail dependence. Both plots are produced using quantile-quantile definitions in Section A.1. Note that Clayton copula allows for negative dependence for $\delta \in (-1, 0)$, however this form of dependence is not used in empirical work.



(a) Lower tail (b) Upper tail

Figure A.3: Quantile-quantile dependence for Clayton copula.

Partial derivative(s)

$$\begin{aligned} C_1^{Cl}(u,v) =& \frac{\partial C^{Cl}(u,v)}{\partial u} \\ =& \frac{\partial}{\partial u}\left(u^{-\delta} + v^{-\delta} - 1\right)^{\frac{-1}{\delta}} \\ =& u^{-\delta-1}\left(u^{-\delta} + v^{-\delta} - 1\right)^{\frac{-1}{\delta}-1} \end{aligned}$$

A.3.2 Gumbel copula

$$C_\delta^{Gu}(u,v) = \exp\{-([-\log u]^\delta + [-\log v]^\delta)^{1/\delta}\}$$

where

$$1 \leq \delta < \infty$$

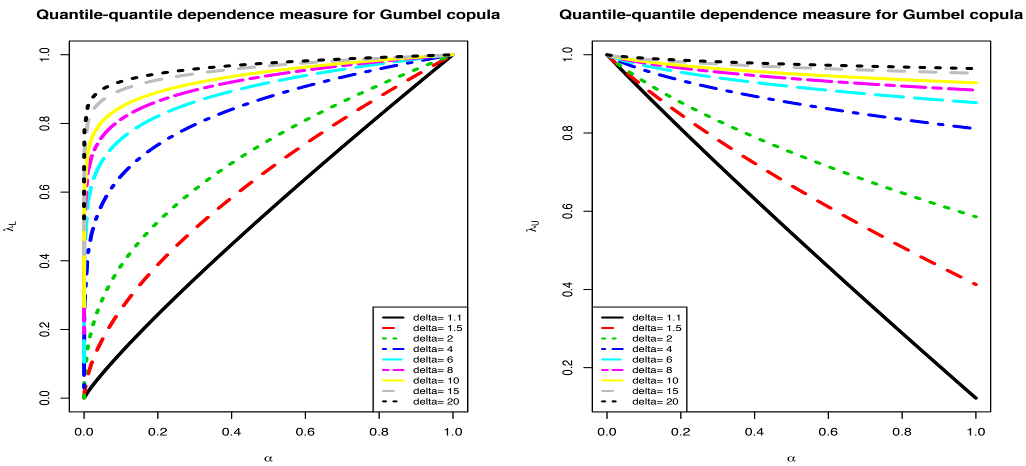
and lower and upper tail dependence

$$\tau^L = 0, \quad \tau^U = 2 - 2^{1/\delta}$$

For $\delta = 1$ Gumbel copula reduces to the fundamental independence copula:

$$\begin{aligned} C_\delta^{Gu}(u, v) &= exp\{-((-\log u)^1 + (-\log v)^1)^{1/1}\} \\ &= exp\{\log u + \log v\} \\ &= exp\{\log(uv)\} = uv \end{aligned}$$

In Figure A.4 we plot the quantile-quantile dependence for Gumbel copula. From *b)* we see that for parameter $\delta = 1.1$ we get almost 0 dependence. Figure *a)* shows that Gumbel copula does not have lower tail dependence.



(a) Lower tail (b) Upper tail

Figure A.4: Quantile-quantile dependence for Gumbel copula.

Partial derivative(s)

$$C_1^{Gu}(u, v) = \frac{\partial C^{Gu}(u, v)}{\partial u}$$

$$=\frac{\partial}{\partial u}\exp\{-([-\log u]^{\delta}+[-\log v]^{\delta})^{1/\delta}\}$$

$$=C^{Gu}(u,v)\frac{1}{u}(-\log u)^{\delta-1}([-\log u]^{\delta}+[-\log v]^{\delta})^{\frac{1}{\delta}-1}$$

A.3.3 Rotated Gumbel copula

$$C_{\delta}^{RGu}(1-u,1-v)=\exp\{-([-\log(1-u)]^{\delta}+[-\log(1-v)]^{\delta})^{1/\delta}\}+u+v-1$$

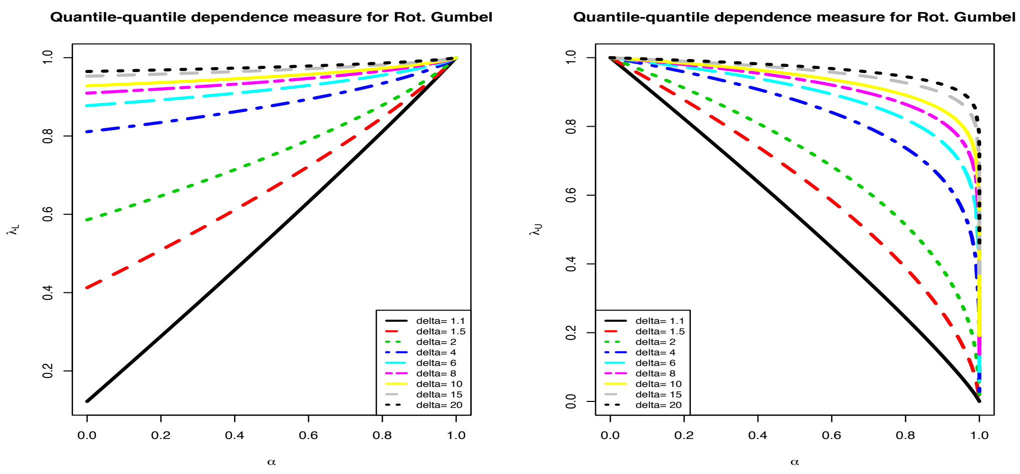
where

$$1\leq\delta<\infty$$

and lower and upper tail dependence

$$\tau^{L}=2-2^{1/\delta},\quad \tau^{U}=0$$

In Figure A.5 we plot the quantile-quantile dependence for Rotated Gumbel copula. As the name suggests this copula is just the rotated Gumbel. Thus, when compared to Gumbel its tails are switched. From Figure A.5 *a)* we see that for parameter $\delta = 1.1$ we get almost 0 dependence, which increases with δ. Whereas from Figure *b)* we see that Rotated Gumbel copula does not have upper tail dependence.



(a) Lower tail (b) Upper tail

Figure A.5: Quantile-quantile dependence for Rotated Gumbel copula.

Partial derivative(s)

$$
\begin{aligned}
C_1^{RGu}(1-u,1-v) =& \frac{\partial C^{Gu}(1-u,1-v)}{\partial(1-u)} \\
=& \frac{\partial u}{\partial(1-u)} \frac{\partial C^{RGu}(1-u,1-v)}{\partial u} \\
=& \frac{1}{\frac{\partial(1-u)}{\partial u}} \frac{\partial C^{RGu}(1-u,1-v)}{\partial u} \\
=& -1[\exp\{-([-\log(1-u)]^\delta + [-\log(1-v)]^\delta)^{\frac{1}{\delta}}\} \\
& \cdot (-1)([-\log(1-u)]^\delta + [-\log(1-v)]^\delta)^{\frac{1}{\delta}-1}[-\log(1-u)]^{\delta-1}\frac{-1}{1-u} + 1] \\
=& -\exp\{-([-\log(1-u)]^\delta + [-\log(1-v)]^\delta)^{\frac{1}{\delta}}\} \\
& \cdot ([-\log(1-u)]^\delta + [-\log(1-v)]^\delta)^{\frac{1}{\delta}-1}[-\log(1-u)]^{\delta-1}\frac{1}{1-u} - 1
\end{aligned}
$$