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MASTER THESIS

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Classes of modules arising in contemporary algebraic geometry

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague on 30th July 2015 Alexander Slávik

Název práce: Třídy modulů inspirované moderní algebraickou geometrií

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Abstrakt: V kategorii modulů nad noetherovskými nebo Dedekindovými obory zkoumáme vlastnosti velmi plochých a kontraadjustovaných modulů, tj. modulů z odpovídajících tříd v kotorzním páru $(\mathcal{VF}, \mathcal{CA})$ generovaném množinou všech modulů tvaru $R[s^{-1}]$. Dále definujeme lokálně velmi ploché moduly a vyšetřujeme jejich vztah k velmi plochým modulům jako analogii vztahu projektivních a plochých Mittag-Lefflerových modulů. Pro noetherovské obory ukazujeme, že třída všech velmi plochých modulů je pokrývající, právě když třída všech lokálně velmi plochých modulů je předpokrývající, právě když spektrum příslušného okruhu je konečné. Pro obory mohutnosti menší než 2^ω je toto dále ekvivalentní výroku, že třída všech kontraadjustovaných modulů je pokrývající.

Klíčová slova: velmi plochý modul, kontraadjustovaný modul, pokrytí, předpokrytí, obal

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Abstract: In the setting of Noetherian or Dedekind domains, we investigate the properties of very flat and contraadjusted modules. These are the modules from the respective classes in the cotorsion pair $(\mathcal{VF}, \mathcal{CA})$ generated by the set of all modules of the form $R[s^{-1}]$. Furthermore, we introduce the concept of locally very flat modules and pursue the analogy of their relation to very flat modules and the relation between projective and flat Mittag-Leffler modules. It is shown that for Noetherian domains, the class of all very flat modules is covering, if and only if the class of all locally very flat modules is precovering, if and only if the spectrum of the ring is finite; for domains of cardinality less than 2^ω, this is further equivalent to the class of all contraadjusted modules being enveloping.

Keywodrs: very flat module, contraadjusted module, cover, precover, envelope

Contents

A note about conventions

All rings are associative with a unit. Further, except for Section 1, all rings are assumed to be commutative. The ring shall be usually called R; thus, whenever we speak about modules without specifying the ring, we mean R-modules. If R is a domain, then Q denotes its field of fractions.

Notation used in the text

${}^{\perp}\mathcal{S}$	the class of all modules M such that $\operatorname{Ext}^1_R(M,S)=0$ for all $S\in\mathcal{S}$
$\mathcal{S}^{\perp}$	the class of all modules M such that $\operatorname{Ext}^1_R(S,M)=0$ for all $S\in\mathcal{S}$
$R[s^{-1}]$	the localization of R in the multiplicative set $\{1,s,s^2,\dots\}$
$R_{(p)}$	the localization of R in the prime ideal p
$E(M)$	the injective envelope (hull) of M
$\widehat{M}_s$	the completion of M in the principal ideal sR (i.e. $\varprojlim_{i<\omega} M/s^iM$)
$\operatorname{Ass}_R M$	the set of associated primes of M
$\operatorname{Spec} R$	the spectrum of R (i.e. the set of all prime ideals of R)
$\operatorname{mSpec} R$	the set of all maximal ideals of R
$\operatorname{card} X$	the cardinality of X

Introduction

Quasi-coherent sheaves on a scheme X can be studied as certain representations of the quiver whose vertices correspond to affine open subschemes of X, [3]. The key role here is played by various classes of flat modules, notably the projective and Mittag-Leffler ones. These correspond to (infinite dimensional) vector bundles and Drinfeld vector bundles on X, respectively, [4].

Further classes of flat modules were introduced in [4] in order to build new monoidal model category structures on complexes of quasi-coherent sheaves, and hence provide for new ways of computing cohomology. The idea is to follow the approach of Hovey [10] which relies on existence of approximations (precovers). However, while projective and flat modules always provide for approximations [2], flat Mittag-Leffler modules over non-perfect rings do not. The latter fact is not a curious exception: it is just a 0-dimensional instance of a more general phenomenon arising in (infinite dimensional) tilting theory for all n-tilting modules that are not Σ-pure-split [1].

In [14], Positselski introduced the dual notion of a contraherent cosheaf on X, again as a certain representation of the quiver of affine open subschemes of X. The modules relevant here are the *contraadjusted* and *very flat* ones. These are the classes the nature of which we investigate, mostly in the setting of domains. Our primary interest are the approximation properties of the classes.

In Section 2, we show that the class of all very flat modules is covering quite rarely; for Noetherian domains, this is equivalent to the finiteness of the spectrum of the ring.

Section 3 deals with a more pathological class: We define the *locally very flat modules*, an analogy to flat Mittag-Leffler modules. We show that for Dedekind domains, these classes exhibit very similar behavior. Further, over a Noetherian domain, the class of all locally very flat modules is not precovering unless (again) the spectrum is finite.

Finally, Section 4 focuses on contraadjusted modules. We show the connection of completeness and contraadjustedness for torsion-free modules. The existence of contraadjusted envelopes is somewhat harder to grasp; the (negative) result we have here is only for Noetherian domains of cardinality less than 2^ω. Finally, we provide a characterization of torsion contraadjusted modules over Dedekind domains.

1. Basic notions and key facts

This section covers most of the (standard) definitions and propositions used in the following sections. For further information we refer to [9].

Definition 1.1. Let $\mathcal{A}$ be a class of modules. We put

$$\mathcal{A}^{\perp} = \{B \in \text{Mod-}R \mid \operatorname{Ext}^1_R(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$$

and

$${}^{\perp}\mathcal{A} = \{B \in \text{Mod-}R \mid \operatorname{Ext}^1_R(B, A) = 0 \text{ for all } A \in \mathcal{A}\}.$$

Definition 1.2. A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

Definition 1.3. Let $\mathcal{A}$ be a class of modules and M a module. A module $A \in \mathcal{A}$ together with a homomorphism $f\colon A \to M$ is called *$\mathcal{A}$-precover of M* if for every $A' \in \mathcal{A}$ and $f'\colon A' \to M$ there is $g\colon A' \to A$ such that $f' = fg$. An $\mathcal{A}$-precover is *special* if it is surjective and its kernel is an element of $\mathcal{A}^{\perp}$. Finally, an *$\mathcal{A}$-cover* is an $\mathcal{A}$-precover with the following additional property: Whenever we have $f = fg$ for some $g\colon A \to A$, then g is an automorphism of A.

Dually, we define the *$\mathcal{A}$-preenvelope*, *special $\mathcal{A}$-preenvelope*, and *$\mathcal{A}$-envelope*.

A class $\mathcal{A}$ is called *precovering* if every module has an $\mathcal{A}$-precover; in the same fashion we define *special precovering class*, *covering class*, etc.

Lemma 1.4 (Salce Lemma, [9, 5.20])**.** *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the class $\mathcal{A}$ is special precovering if and only if $\mathcal{B}$ is special preenveloping. In such a case, the cotorsion pair is called* complete.

Theorem 1.5 ([9, 6.11])**.** *Let $\mathcal{S}$ be any set of modules. Then $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a complete cotorsion pair (which is called the cotorion pair* generated by *$\mathcal{S}$).*

Definition 1.6. Let M be a module and $\mathcal{A}$ a class of modules. We say that M is *$\mathcal{A}$-filtered* or a *transfinite extension* of modules from $\mathcal{A}$ if there is an ordinal-indexed increasing chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that

(i) $M_0 = 0$, $M_\sigma = M$,

(ii) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for α limit (i.e. the chain is *continuous*), and

(iii) $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$ for each $\alpha < \sigma$.

A class $\mathcal{A}$ is called *deconstructible* if it coincides with the class of all $\mathcal{S}$-filtered modules for some set of modules $\mathcal{S}$.

Lemma 1.7 (Eklof Lemma, [9, 6.2])**.** *Let M, N be modules. If M is ${}^{\perp}N$-filtered, then $M \in {}^{\perp}N$. In particular, the class $\mathcal{P}_i$ of all modules of projective dimension at most i is closed under transfinite extensions for every $i < \omega$ (this is often called* Auslander Lemma*).*

Theorem 1.8 ([9, 6.14]). *If $\mathcal{S}$ is a set of modules, then the class ${}^{\perp}(\mathcal{S}^{\perp})$ consists of direct summands of extensions of free modules by $\mathcal{S}$-filtered modules.*

Finally, we recall the recent result due to Šaroch:

Theorem 1.9 ([1, 3.2]). *Let $\mathcal{F}$ be a class of countably presented modules and B a module which is a countable direct limit of modules from $\mathcal{F}$. Let $\mathcal{L}_{\mathcal{F}}$ be the class of all locally $\mathcal{F}$-free modules, i.e. modules possessing a system $\mathcal{E}$ of submodules from $\mathcal{F}$ closed under unions of countable chains and such that every countable subset of the module is contained in an element of $\mathcal{E}$. If B is not a direct summand in a module from $\mathcal{L}_{\mathcal{F}}$, then B has no $\mathcal{L}_{\mathcal{F}}$-precover.*

The module B from the preceding theorem is usually referred to as a *Bass module*. The theorem, among others, implies the fact that unless the ring is (right) perfect, the class of all flat Mittag-Leffler modules is not precovering, for this is precisely obtained if one takes $\mathcal{C}$ to be the class of all countably presented projective modules.

2. Very flat modules

This section introduces very flat modules. Many facts concerning this class may be found in [14], we recall only those needed for our results. The main result of the section is Theorem 2.9 dealing with the covering property of very flat modules over domains.

Definition 2.1. Let R be a ring and $s \in R$. We shall denote by $R[s^{-1}]$ the localization $S^{-1}R$ in the multiplicative set $S = \{1, s, s^2, \dots\}$. We also put

$$\mathcal{L} = \{R[s^{-1}] \mid s \in R\}.$$

Note that $R[s^{-1}]$ is a ring on one hand, and an R-module on the other hand. In most cases, we shall view it as an R-module; should it be viewed as a ring, it will be explicitly stressed or clear from the context.

To start with, note that for each $s \in R$, $R[s^{-1}]$ has the following two-term free resolution,

$$0 \longrightarrow R^{(\omega)} \xrightarrow{f_s} R^{(\omega)} \longrightarrow R[s^{-1}] \longrightarrow 0, \tag{1}$$

f being defined on the free generators as $f_s(1_i) = 1_i - 1_{i+1}s$, therefore $R[s^{-1}]$ is flat (the exact sequence is pure) and of projective dimension at most 1.

Definition 2.2. A module M is *very flat*, provided that $M \in \mathcal{VF}$, where $(\mathcal{VF}, \mathcal{CA})$ denotes the (complete) cotorsion pair generated by the set $\mathcal{L}$. The modules in the class $\mathcal{CA}$ are called *contraadjusted*.

Lemma 2.3. *The class $\mathcal{VF}$ consists of direct summands of $\mathcal{L}$-filtered modules; these are all flat and of projective dimension at most* 1.

Proof. This is a direct consequence of Theorem 1.8 and Lemma 1.7. □

Lemma 2.4. *The class $\mathcal{VF}$ is closed under tensor products.*

Proof. First notice that for any $r, s \in R$, we have $R[r^{-1}] \otimes_R R[s^{-1}] \cong R[(rs)^{-1}]$, so the set $\mathcal{L}$ is closed under tensor products.

Now let V be $\mathcal{L}$-filtered module with $\mathcal{L}$-filtration $(V_\alpha \mid \alpha \leq \sigma)$ and $s \in R$; let us proceed by induction. Knowing that $V_\alpha \otimes_R R[s^{-1}]$ is $\mathcal{L}$-filtered and assuming that $V_{\alpha+1}/V_\alpha \cong R[r^{-1}]$ for some $r \in R$, tensoring the pure exact sequence

$$0 \longrightarrow V_\alpha \longrightarrow V_{\alpha+1} \longrightarrow R[r^{-1}] \longrightarrow 0$$

by $R[s^{-1}]$ shows that $V_{\alpha+1} \otimes_R R[s^{-1}]$ is an extension of an $\mathcal{L}$-filtered module by $R[(rs)^{-1}]$, hence $\mathcal{L}$-filtered. For the limit steps, it suffices to utilize the fact that the tensor product commutes with direct limits.

If V, W are two $\mathcal{L}$-filtered modules, we may proceed in the same way, replacing $R[s^{-1}]$ with W. When passing from $V_\alpha \otimes_R W$ to $V_{\alpha+1} \otimes_R W$, one uses the observation from the preceding paragraph.

Finally, for two general very flat modules V, W, by Lemma 2.3, we have V', W' such that $V \oplus V'$ and $W \oplus W'$ are $\mathcal{L}$-filtered. Therefore,

$$(V \oplus V') \otimes_R (W \oplus W') \cong (V \otimes_R W) \oplus (V' \otimes_R W) \oplus (V \otimes_R W') \oplus (V' \otimes_R W')$$

is $\mathcal{L}$-filtered, and we conclude that $V \otimes_R W$ is a direct sumand of an $\mathcal{L}$-filtered module, hence very flat. □

From now on, we shall focus on the case when R is a domain. In such a case, each $R[s^{-1}]$ is a subring/submodule of the fraction field Q containing R and consisting of (equivalence classes of) the fractions whose denominators are products of powers of s. In particular, if r divides s, then $R[r^{-1}] \subseteq R[s^{-1}]$.

We begin with an observation concerning the relation of very flat modules to the modules from $\mathcal{L}$. For future purposes, we formulate it in a more general form, dealing with submodules of very flat modules.

Lemma 2.5. *Let M be a torsion-free module of rank κ, which is a submodule of a very flat module. Then there is a filtration $(M_\alpha \mid \alpha \leq \sigma)$, where σ is an ordinal of cardinality κ and $M_{\alpha+1}/M_\alpha$ is a non-zero submodule of $R[s_\alpha^{-1}]$ for some non-zero $s_\alpha \in R$.*

If κ is finite, there is $0 \neq s \in R$ such that $M \otimes_R R[s^{-1}]$ is a torsion-free $R[s^{-1}]$-module of rank κ; this is further finitely generated, provided that R is Noetherian.

Proof. As M is a submodule of a very flat module, it is a submodule of an $\mathcal{L}$-filtered one by Lemma 2.3. Hence let $M \subseteq V$, where V is equipped with an $\mathcal{L}$-filtration $(V_\alpha \mid \alpha \leq \sigma)$. Put $M_\alpha = M \cap V_\alpha$; then $M_{\alpha+1}/M_\alpha$ embeds into $V_{\alpha+1}/V_\alpha \cong R[s_\alpha^{-1}]$. The chain $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ is readily seen to be continuous. We may also force all the factors to be non-zero by omitting those terms of $\mathcal{M}$ equal to some previous term. As $M_{\alpha+1}/M_\alpha$ is a torsion-free rank 1 module, in order to ranks to match, the chain has to have length κ.

If the rank is finite, say n, put $s = \prod_{i<n} s_i$. Then $R[s_i^{-1}] \otimes_R R[s^{-1}] \cong R[s^{-1}]$, so $0 \neq M_{i+1}/M_i \otimes_R R[s^{-1}]$ is isomorphic to an ideal in $R[s^{-1}]$ for each $i < n$. Thus $M \otimes_R R[s^{-1}]$ is a torsion-free $R[s^{-1}]$-module of rank n. If further R is Noetherian, then so is $R[s^{-1}]$, hence all the ideals are finitely generated and the same applies to $M \otimes_R R[s^{-1}]$. □

The following type of domains will be of particular interest, playing the role of the "trivial case" for very flat modules:

Definition 2.6. A domain R is said to be a *G-domain* (cf. [11]), if $Q = R[s^{-1}]$ for some $s \in R$.

It is easy to see that a domain with a finite spectrum is a G-domain: It suffices to take an element s belonging to all non-zero primes (e.g. pick a non-zero element from each prime and multiply them) and we have $Q = R[s^{-1}]$, since rR intersects the set $\{1, s, s^2 \dots\}$ for each non-zero $r \in R$. For Noetherian domains, this is in fact a characterization:

Lemma 2.7 ([11, Theorems 144 & 146])**.** *Let R be a Noetherian domain. The following are equivalent:*

(i) R is a G-domain.

(ii) The spectrum of R is finite.

(iii) The spectrum of R is finite and all the non-zero prime ideals are maximal.

Let us put G-domains into our context:

Lemma 2.8. *A domain R is a G-domain if and only if Q is very flat.*

Proof. Since Q has rank 1, the chain constructed in Lemma 2.5 consists only of two elements, 0 and Q, and Q is a submodule in $R[s^{-1}]$ for some $0 \neq s \in R$, whence $Q = R[s^{-1}]$. □

As a right class of a cotorsion pair generated by set, the class $\mathcal{VF}$ is always special precovering. However, the case when it is covering happens only for G-domains.

Theorem 2.9. *Let R be a domain. The following are equivalent:*

(i) Q has a $\mathcal{VF}$-cover,

(ii) Q is very flat.

Proof. Assume that $f\colon V \to Q$ is the $\mathcal{VF}$-cover of Q. Pick $s \in R$ and consider the map

$$f_s = f \otimes_R R[s^{-1}]\colon V \otimes_R R[s^{-1}] \longrightarrow Q \otimes_R R[s^{-1}] \cong Q.$$

By Lemma 2.4, $V \otimes_R R[s^{-1}]$ is very flat; hence, by the precovering property of V, there is $g\colon V \otimes_R R[s^{-1}] \to V$ such that $f_s = fg$.

On the other hand, tensoring the short exact sequence

$$0 \longrightarrow R \longrightarrow R[s^{-1}] \longrightarrow R[s^{-1}]/R \longrightarrow 0$$

by the flat module V, one obtains the inclusion morphism $i\colon V \to V \otimes_R R[s^{-1}]$ (exploiting $V \otimes_R R \cong V$). The cokernel of this morphism is $V \otimes_R (R[s^{-1}]/R)$, which is a torsion module (every element is annihilated by a power of s). Now clearly $f = f_s i = fgi$ and by the covering property of V, we infer that gi is an automorphism of V. This means that i is a split inclusion; however, it has to be an isomorphism, since the torsion module $V \otimes_R (R[s^{-1}]/R)$ cannot be (isomorphic to) a submodule of the torsion-free module $V \otimes_R R[s^{-1}]$.

We now see that $V \cong V \otimes_R R[s^{-1}]$ for each $s \in R$. As the tensor product commutes with direct limits, we may pass to the directed union $Q = \bigcup_{s \in R} R[s^{-1}]$ (the directed system ordered by divisibility with the morphisms being inclusions) and obtain $V \cong V \otimes_R Q$. This implies that the very flat module V is the direct sum of copies of Q, forcing Q to be very flat.

The reverse implications is obvious. □

For Noetherian domains, we see that the class $\mathcal{VF}$ is yet another one which supports the affirmative answer to the open question whether covering classes are always closed under direct limits:

Corollary 2.10. *The following are equivalent for a Noetherian domain R:*

(i) *R is a G-domain.*

(ii) *Each flat module is very flat.*

(iii) *The class $\mathcal{VF}$ is covering.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 2.7, the Krull dimension of R is at most 1. Therefore, R is almost perfect in the sense of [9, 7.55]. By [9, 7.56], this implies that the class of all flat modules is generated by the single module Q (i.e. all flat modules are strongly flat). Since we assume that Q is very flat, we conclude that all flat modules are very flat.

(ii) $\Rightarrow$ (iii): The class of all flat modules is always covering, cf. [9, 8.1].

(iii) $\Rightarrow$ (i): Follows from Theorem 2.9 and Lemma 2.8. □

Finally, we turn our attention to the case of Dedekind domains. The direct consequence of heredity is that the class $\mathcal{VF}$ is closed under submodules. Lemma 2.5 then shows that the rank 1 very flat modules are precisely the submodules of the modules from $\mathcal{L}$. If we denote this set of submodules $\mathcal{S}$, then Lemma 2.5 again implies that the very flat modules are precisely the $\mathcal{S}$-filtered ones.

For finite rank modules we can do even better:

Lemma 2.11. *Let R be a Dedekind domain, M a torsion free module of finite rank and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a pure-exact sequence. Then M is very flat if and only if both M', M'' are.

Proof. The if part follows from the class $\mathcal{VF}$ being closed under extensions.

For the only-if part, note that by heredity, M' is very flat. Further, by Lemma 2.5, there is non-zero $s \in R$ such that $M \otimes_R R[s^{-1}]$ is a finitely generated torsion-free $R[s^{-1}]$-module; the same clearly applies to $M'' \otimes_R R[s^{-1}]$. The domain $R[s^{-1}]$ being Dedekind, too, the finitely generated torsion-free modules are projective, so we infer that $M'' \otimes_R R[s^{-1}]$ is a projective $R[s^{-1}]$-module, and as such a submodule of a free $R[s^{-1}]$-module, say $R[s^{-1}]^{(t)}$. We have

$$M'' \subseteq M'' \otimes_R R[s^{-1}] \subseteq R[s^{-1}]^{(t)}$$

as R-modules. Since clearly $R[s^{-1}]^{(t)} \in \mathcal{VF}$, heredity implies $M'' \in \mathcal{VF}$. □

3. Locally very flat modules

In this section we deal with locally very flat modules, which are obtained by replacing "projective" with "very flat" in the definition of $\aleph_1$-projective modules, which are usually referred to as flat Mittag-Leffler modules, cf. [9, 3.19]. By the results from [1], the class of all flat Mittag-Leffler modules is precovering if and only if the ring is (right-) perfect, which is the result we are going to mimic.

Definition 3.1. A module M is said to be *locally very flat* provided there exists a set $\mathcal{T}$ consisting of countably presented very flat submodules of M such that each countable subset of M is contained in an element of $\mathcal{T}$, and $\mathcal{T}$ is closed under unions of countable chains. The set $\mathcal{T}$ is said to *witness* the local very flatness of M.

The class of all locally very flat modules is denoted by $\mathcal{LV}$. Note that a countably generated module is locally very flat, iff it is very flat.

The class $\mathcal{VF}$ is a particular instance of so-called *locally $\mathcal{F}$-free modules* in the sense of [15]. As such, it is closed under transfinite extensions.

To efficiently show that certain modules are not very flat, we employ the machinery of associated primes. For a module M, we shall denote the set of associated primes $\operatorname{Ass}_R M$.

Let R be a Noetherian domain. Then

$$E(Q/R) = \bigoplus_{p \in P} E(R/p)^{(\alpha_p)} \tag{2}$$

where $P \subseteq \operatorname{Spec} R$ and $\alpha_p = \mu_1(p, R)$ is the first Bass invariant of R at p (see [5, §9.2]). For each i, let P_i denote the set of all prime ideals of height i. Since R is a domain, we have $P_1 \subseteq P$ by [5, 9.2.13].

Let $s \in R$ and $O(s) = \{p \in P_1 \mid s \in p\}$. Then each $p \in O(s)$ is a minimal prime over sR, so the set $O(s)$ is finite. Moreover, for each $p \in P_1$, we have $(R/p) \otimes_R R[s^{-1}] = 0$, iff $p \in O(s)$. Indeed, $s \in p$ implies

$$(r + p) \otimes t/s^k = (rs + p) \otimes t/s^{k+1} = 0,$$

while if $s \notin p$, then $p \otimes_R R[s^{-1}]$ is a prime ideal in $R[s^{-1}]$, and

$$(R/p) \otimes_R R[s^{-1}] \cong R[s^{-1}]/(p \otimes_R R[s^{-1}]) \neq 0.$$

Lemma 3.2. *Let R be a Noetherian domain. Let M be a finite rank submodule of a very flat module, and F be its free submodule of the same rank. Then the set $P_1 \cap \operatorname{Ass}_R M/F$ is finite.*

Proof. By Lemma 2.5, there is $0 \neq s \in R$ such that $M \otimes_R R[s^{-1}]$ is a finitely generated $R[s^{-1}]$-module, whence $A = \operatorname{Ass}_{R[s^{-1}]}((M/F) \otimes_R R[s^{-1}])$ is finite.

Let $p \in P_1 \cap \operatorname{Ass}_R M/F$, that is, $R/p \subseteq M/F$. If $p \notin O(s)$, then

$$R[s^{-1}]/(p \otimes_R R[s^{-1}]) \subseteq (M/F) \otimes_R R[s^{-1}]$$

so $p \otimes_R R[s^{-1}] \in A$. It follows that

$$\operatorname{card}(P_1 \cap \operatorname{Ass}_R M/F) \leq \operatorname{card} A + \operatorname{card} O(s)$$

is finite. □

Example 3.3. The abelian group $\mathbb{Z}^\kappa$ ($\kappa \geq \omega$), so-called Baer-Specker group, is well-known not to be free, but it is flat Mittag-Leffler ([9, 3.35]), hence locally very flat. To see that it is not very flat, we use the refined version of Quillen's small object argument from [6, Theorem 2] to obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C \longrightarrow V \longrightarrow 0$$

with V very flat and C contraadjusted, both with cardinality at most 2^ω. As C is an extension of very flat groups, it is very flat; as such, it cannot be cotorsion, for this would imply having the localization $\mathbb{Z}_{(p)}$ as a subgroup for some prime p, contradicting Lemma 3.2. Now [8, 1.2(4)] implies that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\omega, C) \neq 0$. Hence no infinite direct product of copies of $\mathbb{Z}$ is very flat.

We now proceed to construct so-called Bass module:

Lemma 3.4. *Let R be a Noetherian domain such that* $\operatorname{Spec} R$ *is infinite. Then there is an overring B of R such that B is a countable directed union of modules from $\mathcal{L}$, but B cannot be a submodule of a very flat module. Further, B can be chosen so that $U \cap \operatorname{Ass}_R B/R = \emptyset$ for a given finite set $U \subseteq P_1$.*

Proof. First note that the set P_1 is infinite, too, because of Lemma 2.7. Further, for $p, q \in P_1$, $p \subseteq q$ or $p \supseteq q$ if and only if $p = q$. Therefore, by prime avoidance [11, Theorem 81], for finite $O \subseteq P_1$, $q \in O$ if and only if $q \subseteq \bigcup_{p \in O} p$.

Let $U \subseteq P_1$ be finite. We define sequences $(r_i \mid i < \omega)$ and $(s_i \mid i < \omega)$ of elements of R as follows: $r_0 = 1$, and if $r_0, \dots, r_i$ are defined, we let $O_i = U \cup \bigcup_{j \leq i} O(r_j)$. Then O_i is a finite subset of P_1. Let $s_i = \prod_{k \leq i} r_k$. Then $s_i \in \bigcap_{p \in O_i} p$ and $s_i \notin q$ for each $q \in P_1 \setminus O_i$. In particular, $s_i^{-1} + R$ has zero q-th component in the decomposition of $E(Q/R)$ from (2) for each $q \in P_1 \setminus O_i$. It follows that $P_1 \cap \operatorname{Ass}_R(s_i^{-1}R/R) \subseteq O_i$. Since P_1 is infinite, by prime avoidance, we can find $q_i \in P_1$ such that $q_i \notin O_i$. We take $r_{i+1} \in q_i \setminus \bigcup_{p \in O_i} p$.

Since $r_{i+1} \notin p$, the p-th component of $r_{i+1}^{-1} + R \in E(Q/R)$ is zero, for each $p \in O_i$. Since $r_{i+1}^{-1}R/R \cong R/r_{i+1}R$ maps surjectively onto R/q_i, there is a non-zero homomorphism $\varphi\colon E(r_{i+1}^{-1}R/R) \to E(R/q_i)$. By [5, 3.3.8], this implies that the q_i-th component of $r_{i+1}^{-1} + R \in E(Q/R)$ is non-zero, and $q_i \in P_1 \cap \operatorname{Ass}_R(r_{i+1}^{-1}R/R)$.

By construction,

$$R \subseteq R[s_0^{-1}] \subseteq R[s_1^{-1}] \subseteq \ldots \subseteq R[s_i^{-1}] \subseteq R[s_{i+1}^{-1}] \subseteq \ldots$$

is an increasing chain of very flat submodules of Q. Moreover, for each $i < \omega$, $R[s_i^{-1}] \neq R[s_{i+1}^{-1}]$, because $q_i \in \mathrm{Ass}_R(s_{i+1}^{-1}R/R) \setminus \mathrm{Ass}_R(s_i^{-1}R/R)$.

Let B denote the union of the chain above. Then B has rank one and $P_1 \cap \mathrm{Ass}_R B/R = \bigcup_{i<\omega}(P_1 \cap \mathrm{Ass}_R(s_i^{-1}R/R)) \supseteq \{q_i \mid i < \omega\}$ is infinite. From Lemma 3.2, we conclude that B is not a submodule of a very flat module.

For the final claim, note that for each $q \in U$ and $i < \omega$, $r_i \notin q$, and so $s_i \notin q$ as well. Thus $U \cap \mathrm{Ass}_R(s_i^{-1}R/R) = \emptyset$, and consequently, $U \cap \mathrm{Ass}_R B/R = \emptyset$. □

Remark 3.5. The construction above may be viewed geometrically: In the affine scheme $\mathrm{Spec}\, R$, we construct a descending chain of principal open subsets, the intersection of which is not open, cf. [14, Lemma 1.2.4]

Theorem 3.6. *For a Noetherian domain R, the class $\mathcal{LV}$ is precovering if and only if the spectrum of R is finite.*

Proof. By Corollary 2.10, the finiteness of $\mathrm{Spec}\, R$ implies that the classes of very flat and flat modules coincide, hence the intermediate class $\mathcal{LV}$ equals both of them and therefore is even covering. On the other hand, we may use the module B constructed in Lemma 3.4 together with Theorem 1.9 to show that B has no $\mathcal{LV}$-precover. □

Finally, we examine the nature of locally very flat modules for Dedekind domains. The heredity implies that the class $\mathcal{LV}$ is closed under submodules: As $\mathcal{VF}$ is closed under submodules, the system witnessing local very flatness of a submodule is simply obtained by intersecting the modules in the witnessing system of the larger module. Moreover, $\mathrm{Spec}\, R$ is finite, iff R is a discrete valuation ring, cf. [12, p.86].

In this case, the analogy between flat Mittag-Leffler modules and the locally very flat ones goes much further: for example, Definition 3.1 can equivalently be formulated using pure submodules in M, and we also have an analogy of Pontryagin's criterion for abelian groups (cf. [9, 3.14]):

Theorem 3.7. *Let R be a Dedekind domain and M be a module. Then the following conditions are equivalent:*

- *(i) For each finite subset F of M, there exists a countably generated pure submodule N of M such that N is very flat and contains F.*
- *(ii) For each countable subset C of M, there exists a countably generated pure submodule N of M such that N is very flat and contains C.*
- *(iii) Each finite rank submodule of M is very flat.*
- *(iv) Each countably generated submodule of M is very flat.*
- *(v) M is locally very flat.*

Proof. (i) ⇒ (ii): Let $C = \{c_i \mid i < \omega\}$. By induction, we define a pure chain $\mathcal{M} = (M_i \mid i < \omega)$ of very flat submodules of M of finite rank such that

$\{c_j \mid j < i\} \subseteq M_i$ for each $i < \omega$ as follows: $M_0 = 0$, and if M_i is defined, then there is a finitely generated free submodule $G \trianglelefteq M_i + c_i R$. By (i), there is also a countably generated pure submodule D of M such that D is very flat and contains G. If D is of finite rank, put $M_{i+1} = D$; otherwise we may assume $D \trianglelefteq Q^{(\omega)}$ and let $M_{i+1} = D \cap Q^{(k)}$, where $k < \omega$ is chosen so that $G \subseteq D \cap Q^{(k)}$. Now M_{i+1} is a finite rank pure and very flat submodule of D, and hence also $M_i + c_i R \subseteq M_{i+1}$. By Lemma 2.11, M_{i+1}/M_i is very flat of finite rank, hence countably generated. Moreover, $\mathcal{M}$ is a $\mathcal{VF}$-filtration of $N = \bigcup_{i<\omega} M_i$. We conclude that N is a countably generated very flat and pure submodule of M containing the set C.

(ii) $\Rightarrow$ (iii): Let G be a finite rank submodule of M. Then $F \trianglelefteq G$ for a finitely generated free module F. By (ii), there is a countably generated very flat pure submodule N of M containing F. Then also $G \subseteq N$, whence G is very flat.

(iii) $\Rightarrow$ (iv): Let C be a countably generated submodule of M of countable rank. W.l.o.g., $R^{(\omega)} \trianglelefteq C \trianglelefteq Q^{(\omega)}$. For each $n < \omega$, let $C_n = C \cap Q^{(n)}$. By assumption, for each $n < \omega$, C_n is a very flat pure submodule of C, whence C_{n+1}/C_n is very flat by Lemma 2.11, and so is C.

(iv) $\Rightarrow$ (v): If (iv) holds, then the set $\mathcal{T}$ of *all* countably generated submodules of M witnesses the local very flatness of M.

(v) $\Rightarrow$ (i): First, (v) clearly implies (iv), since each countably generated submodule of M is contained in a (very flat) module from $\mathcal{T}$.

In order to prove that (iv) $\Rightarrow$ (i), we let F be a finite subset of M and G be a pure submodule of M of finite rank, say n, such that $F \subseteq G$. Then $R^{(n)} \trianglelefteq G \trianglelefteq Q^{(n)}$. It suffices to prove that G is countably generated.

If this is not the case, we let $G_i = G \cap Q^{(i)}$ for each $i \leq n$, and let $k < n$ be the largest index such that G_k is countably generated (and hence very flat). Then $H = G_{k+1}/G_k$ is a torsion-free module of rank one, so w.l.o.g. $R \subseteq H \subseteq Q$, but H is not countably generated. Hence $\operatorname{Ass}_R H/R$ is uncountable.

Let $\{p_i \mid i < \omega\}$ be a set of distinct elements of $\operatorname{Ass}_R H/R$. We can choose $g_0 \in G_{k+1}$ such that $g_0 + G_k = 1 \in R$, and for each $i < \omega$, $g_{i+1} \in G_{k+1}$ such that

$$(\langle g_{i+1} + G_k \rangle + \langle g_0 + G_k \rangle)/\langle g_0 + G_k \rangle = R/p_i \subseteq Q/R.$$

Let G' be the submodule of G_{k+1} generated by $G_k \cup \{g_i \mid i < \omega\}$. Since G' is countably generated, it is very flat, and so is its rank one pure-epimorphic image $H' = G'/G_k = \langle g_i + G_k \mid i < \omega \rangle$ (see Lemma 2.11). By the definition of H', $R \subseteq H' \subseteq H$, and $p_i \in \operatorname{Ass}_R H'/R$ for each $i < \omega$. So $\operatorname{Ass}_R H'/R$ is infinite, in contradiction with Lemma 3.2. $\square$

4. Contraadjusted modules

Recall that a module C is contraadjusted if $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for each $s \in R$. This can be easily rephrased using the short exact sequence (1):

Lemma 4.1. *A module M is contraadjusted, if and only if for each $s \in R$ and for each sequence $(m_i \mid i < \omega)$ of elements of M, the countable system of linear equations with unknowns x_i*

$$x_i - sx_{i+1} = m_i \quad (i < \omega) \tag{3}$$

has a solution in M.

Proof. Applying the contravariant functor $\mathrm{Hom}_R(-, M)$ to (1), one sees that the condition $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ is equivalent to the map $\mathrm{Hom}_R(f_r, M)$ being surjective. The latter condition easily translates into the solvability of the countable system (3). □

Proposition 4.2. *If $f\colon R \to S$ is a ring homomorphism and C is a contraadjusted S-module, then it is also contraadjusted as an R-module.*

Proof. This follows immediately from Lemma 4.1, since the solvability of each system of the form (3) still holds. □

Proposition 4.3. *If R is a semiprime Goldie ring (e.g. a domain), then every divisible module (i.e. $sM = M$ for each non zero-divisor) is contraadjusted.*

Proof. By [9, 9.1], for semiprime Goldie rings, $\mathrm{Ext}^1_R(P, D) = 0$ whenever P has projective dimension ≤ 1 and D is divisible, so the claim follows from Lemma 2.3. □

If M is a module and $0 \neq s \in R$, we let $\widehat{M_s}$ be the completion of M in the ideal sR, i.e. the module

$$\varprojlim_{i<\omega} M/s^iM$$

(the maps between the modules being $m + s^iM \mapsto m + s^{i+1}M$). We further denote by c_s the canonical morphism $M \to \widehat{M_s}$. The following lemma shows that for torsion-free modules over domains, the property of being contraadjusted can be translated to some form of completeness:

Lemma 4.4. *Let R be a domain and M a torsion-free module and $0 \neq s \in R$. Then $\mathrm{Ext}^1_R(R[s^{-1}, M]) = 0$ if and only if the canonical homomorphism c_s is surjective.*

Proof. Assume that c_s is surjective and let $m_0, m_1, \ldots$ be a sequence of elements of M; we shall check the solvability of the system (3). In $\widehat{M_s}$ viewed as a submodule of the product $\prod_{i<\omega} M/s^iM$, consider the element

$$\Big(\sum_{k<i} m_k s^k + s^iM \;\Big|\; i < \omega\Big);$$

let x_0 be any of its preimages in c_s. Now the elements $x_1, x_2, \ldots$ can be simply constructed by a recurrence: By the definition of x_0, we have $x_0 - m_0 \in sM$, so there is $x_1 \in M$ such that $x_0 - sx_1 = m_0$. Given x_1, we observe that

$$s(x_1 - m_1) = x_0 - m_0 - sm_1 \in s^2 M;$$

since M is torsion-free, we infer that $x_1 - m_1 \in sM$ and proceed as before to find $x_2, x_3, \ldots$.

On the other hand, assume the solvability of (3) and pick an element $(t_i + s^i M \mid i < \omega)$ in $\widehat{M}_s$. Put $m_0 = t_0$ and $m_i = s^{-i}(t_i - t_{i-1})$ for $i > 0$; the division by s is possible because of the definition of inverse limit and the torsion-freeness of M. Let $x_0, x_1, \ldots$ be the solution of the system (3) with the given right-hand side $m_0, m_1, \ldots$. It is now easy to check $x_0 - t_0 \in sM$, $x_0 - t_1 \in s^2 M$, etc. Hence x_0 is the sought preimage of the element of the completion. □

The kernel of the homomorphism c_s above is the intersection $\bigcap_{i<\omega} s^i M$, which is for torsion-free M an $R[s^{-1}]$-module. Thus, roughly said, there are two reasons for contraadjustedness of torsion-free modules: divisibility and completeness.

We will now be interested in how much "redundancy" is in the set $\mathcal{L}$. For example, if R is a Dedekind domain and $r, s \in R$ are such that r divides s, then clearly

$$\operatorname{Ext}^1_R(R[s^{-1}], M) = 0 \quad \Longrightarrow \quad \operatorname{Ext}^1_R(R[r^{-1}], M) = 0$$

for any module M, so for testing contraadjustedness it suffices to take "large" s. The generalization of this fact is the following:

Proposition 4.5. *Let R be a domain and $r, s \in R$ such that $rR + sR = R$. Then*

$$\operatorname{Ext}^1_R(R[(rs)^{-1}], M) = 0 \quad \Longrightarrow \quad \operatorname{Ext}^1_R(R[r^{-1}], M) = 0$$

for any module M.

Proof. We start with a classical observation: If $a, b \in R$ are such that $ar + bs = 1$, then for any $k < \omega$ there are also $c, d \in R$ such that $cr^k + ds^k = 1$; these are obtained by expanding the left-hand side of the equality $(ar + bs)^{2k} = 1$.

Next, applying the contravariant functor $\operatorname{Hom}_R(-, M)$ to the short exact sequence

$$0 \longrightarrow R[r^{-1}] \longrightarrow R[(rs)^{-1}] \longrightarrow R[(rs)^{-1}]/R[r^{-1}] \longrightarrow 0$$

shows that it suffices to prove that the projective dimension of the module $R[(rs)^{-1}]/R[r^{-1}]$ is at most 1, too. This would follow if we prove the isomorphism

$$R[s^{-1}]/R \cong R[(rs)^{-1}]/R[r^{-1}].$$

For this it is enough to show that $R[r^{-1}] + R[s^{-1}] = R[(rs)^{-1}]$ (as R-submodules of the module $R[(rs)^{-1}]$) and $R[r^{-1}] \cap R[s^{-1}] = R$, the rest is the standard isomorphism

$$R[s^{-1}]/(R[r^{-1}] \cap R[s^{-1}]) \cong (R[r^{-1}] + R[s^{-1}])/R[r^{-1}].$$

The equality $R[r^{-1}] + R[s^{-1}] = R[(rs)^{-1}]$ is easy: Observe that $1/(rs)^k = a/s^k + b/r^k$, where $a, b \in R$ are chosen so that $ar^k + bs^k = 1$.

Finally, the fact $R[r^{-1}] \cap R[s^{-1}] = R$ can be viewed this way: Assume that $x/r^k = y/s^k$ for some $x, y \in R$, $k < \omega$. Again, take $a, b \in R$ such that $ar^k + bs^k = 1$. Now

$$x = x(ar^k + bs^k) = r^k(xa + by),$$

thus $x/r^k \in R$ as desired. □

Example 4.6. Let k be a field and $R = k[w, z]$, the ring of polynomials in two variables. Put $M = \widehat{R}_{wz}$. As R is Noetherian, M is wzR-complete, so by Lemma 4.4, $\operatorname{Ext}^1_R(R[(wz)^{-1}], M) = 0$. However, it is easy to show that the system (3) with $s = w$ and $m_i = 1$ for all $i < \omega$ has no solution in M, thus $\operatorname{Ext}^1_R(R[w^{-1}], M) \neq 0$.

Our next goal will be to examine the existence of $\mathcal{CA}$-envelopes.

Lemma 4.7. *Let M be an R-module, which is an $R[s^{-1}]$-module for some non-zero $s \in R$. Then there is a $\mathcal{CA}$-preenvelope of M (in the category of R-modules), which is an $R[s^{-1}]$-module.*

Proof. It suffices to construct a special $\mathcal{CA}$-preenvelope

$$0 \longrightarrow M \longrightarrow C \longrightarrow V \longrightarrow 0$$

in the category of $R[s^{-1}]$-modules. By Proposition 4.2, we see that the C is a contraadjusted R-module. Likewise, V is a very flat R-module, because $R[s^{-1}][(r/s^k)^{-1}] \cong R[(rs)^{-1}]$ as R-modules for each $0 \neq r \in R$, □

Proposition 4.8. *Assume that R is a Noetherian domain with infinite spectrum. Let B be the module constructed in Lemma 3.4. If the $\mathcal{CA}$-envelope of B exists, it is equal to B.*

Proof. The module B is just the union of the rings of the form $R[s^{-1}]$, hence it is a module over each of them. By Lemma 4.7, it has a $\mathcal{CA}$-preenvelope which is a module over $R[s^{-1}]$ for each such ring in the union. If the $\mathcal{CA}$-envelope exists, it is a direct sumand in each such preenvelope, hence a B-module (note that to verify that a torsion-free module M is an $R[s^{-1}]$-module, it suffices to check $sM = M$, which is preserved under pure submodules).

Assume that C is the $\mathcal{CA}$-envelope of B. By Wakamatsu lemma [9, 5.13], C/B is very flat; however, as a factor of B-modules, it is a B-module. So unless $C/B = 0$, B is a submodule of a very flat module. However, this is not possible by Lemma 3.2. □

Corollary 4.9. *Let R be a Noetherian domain of cardinality less than 2^ω with infinite spectrum. Then the class of all contraadjusted modules is not enveloping.*

Proof. The module B from Lemma 3.4 has the same cardinality as the ring R; we may also assume that $B \neq Q$ using the final claim from Lemma 3.4. This implies that there is a non-zero $s \in R$ such that $\widehat{B}_s$ is non-zero. However, the latter module has always cardinality at least 2^ω, thus the canonical map c_s cannot be onto. By Lemma 4.4, B is not contraadjusted, so Lemma 4.8 shows that B has no $\mathcal{CA}$-envelope. □

As in the previous sections, we now turn our attention to Dedekind domains R. Our main goal is the characterization of torsion contraadjusted modules. Most of the facts concerning the used apparatus are covered in [13].

Recall that in the setting of Dedekind domains, we have the isomorphism

$$Q/R \cong \bigoplus_{p \in \operatorname{mSpec} R} E(R/p).$$

Also, a module M is cotorsion if and only if $\operatorname{Ext}^1_R(Q, M) = 0$. (cf. [13, 7.1]). Every reduced module M has a cotorsion envelope

$$\operatorname{Ext}^1_R(Q/R, M) \cong \prod_{p \in \operatorname{mSpec} R} \operatorname{Ext}^1_R(E(R/p), M)$$

(cf. [7, VIII.6]) and the inclusion is given via the short exact sequence

$$0 \longrightarrow M \cong \operatorname{Hom}_R(R, M) \longrightarrow \operatorname{Ext}^1_R(Q/R, M) \longrightarrow \operatorname{Ext}^1_R(Q, M) \longrightarrow 0,$$

which is obtained by applying $\operatorname{Hom}_R(-, M)$ to the sequence

$$0 \longrightarrow R \longrightarrow Q \longrightarrow Q/R \longrightarrow 0.$$

It is also worth noting that each module $\operatorname{Ext}^1_R(E(R/p), M)$ is cotorsion and a module over the ring $\widehat{R}_{(p)}$, i.e. the completion of the localization of R in p.

In the following, we shall always view M as a submodule of its cotorsion envelope.

Definition 4.10. Let P be a set of prime ideals of the Dedekind domain R. By $R[P^{-1}]$ we shall denote the (unique) subring of Q such that its factor by R (as R-modules) is isomorphic to $\bigoplus_{p \in P} E(R/p)$. For $P = \{p\}$ we abbreviate $R[\{p\}^{-1}]$ to $R[p^{-1}]$.

Note that if the set $P \subseteq \operatorname{mSpec} R$ is finite and product of its elements is a principal ideal sR, then $R[P^{-1}] \cong R[s^{-1}]$ (using Definition 2.1 for the latter module).

Lemma 4.11. *Let P be a finite subset of* $\operatorname{mSpec} R$. *Then P can be extended to finite $P' \subseteq$* $\operatorname{mSpec} R$ *such that the product of primes in P' is a principal ideal.*

Proof. Pick any non-zero t in the intersection of P; the principal ideal tR factors into the product of prime ideals, among which have to be the primes from P. We let P' to be the set of the prime ideals in the product. □

Lemma 4.12. *Let M be a reduced module over a Dedekind domain R and P a subset of* $\operatorname{mSpec} R$*. The following statements are equivalent:*

(i) $\operatorname{Ext}^1_R(R[P^{-1}], M) = 0$,

(ii) the projection $\operatorname{Ext}^1_R(Q/R, M) \to \prod_{p \in P} \operatorname{Ext}^1_R(E(R/p), M)$*, restricted to M, is surjective.*

Proof. Let us apply the functor $\operatorname{Hom}_R(-, M)$ to the short exact sequence

$$0 \longrightarrow R \longrightarrow R[P^{-1}] \longrightarrow \bigoplus_{p \in P} E(R/p) \longrightarrow 0.$$

As M is reduced, we have

$$\operatorname{Hom}_R\Big(\bigoplus_{p \in P} E(R/p), M\Big) = 0,$$

so the result is

$$0 \longrightarrow \operatorname{Hom}_R(R[P^{-1}], M) \longrightarrow \operatorname{Hom}_R(R, M) \longrightarrow \\ \longrightarrow \prod_{p \in P} \operatorname{Ext}^1_R(E(R/p), M) \longrightarrow \operatorname{Ext}^1_R(R[P^{-1}], M) \longrightarrow 0.$$

As $\operatorname{Hom}_R(R[P^{-1}], M)$ corresponds to the elements of M divisible by all $p \in P$, the corresponding submodule of the cotorsion envelope is

$$\operatorname{Hom}_R(R, M) \cap \prod_{p \notin P} \operatorname{Ext}^1_R(E(R/p), M).$$

(We exploit the purity of M in its cotorsion envelope.) Consequently, the map $\operatorname{Hom}_R(R, M) \to \prod_{p \in P} \operatorname{Ext}^1_R(E(R/p), M)$ is the restriction in question. We infer that this map is onto if and only if $\operatorname{Ext}^1_R(R[P^{-1}], M) = 0$. □

Using the preceding lemmas, we may characterize the contraadjusted modules via the relation to their cotorsion envelope:

Corollary 4.13. *Let M be a reduced module over a Dedekind domain R. Then M is contraadjusted, if and only if for every finite set $P \subseteq \operatorname{mSpec} R$, the projection $\operatorname{Ext}^1_R(Q/R, M) \to \prod_{p \in P} \operatorname{Ext}^1_R(E(R/p), M)$ restricted to M is onto.*

Proof. If P is a finite subset of $\operatorname{mSpec} R$, then we may extend it to a set of primes with principal product. Therefore, $R[P^{-1}] \subseteq R[s^{-1}]$ for some $s \in R$ and $\operatorname{Ext}^1_R(R[s^{-1}], M) = 0$ implies $\operatorname{Ext}^1_R(R[P^{-1}], M) = 0$ (heredity). On the other hand, given non-zero $s \in R$, we have $R[P^{-1}] = R[s^{-1}]$, where P is the (finite) set of primes in the decomposition of sR. The rest is just Lemma 4.12. □

Note that this property may be stated in a "topological" way: If each component $\operatorname{Ext}^1_R(E(R/p), M)$ is equipped with the discrete topology and their product $\operatorname{Ext}^1_R(Q/R, M)$ with the product topology, then the module M is contraadjusted if and only if it is dense in $\operatorname{Ext}^1_R(Q/R, M)$.

Example 4.14. If
$$M \cong \prod_{p \in \operatorname{mSpec} R} \operatorname{Ext}^1_R(E(R/p), M)$$
is a cotorsion module, then
$$\bigoplus_{p \in \operatorname{mSpec} R} \operatorname{Ext}^1_R(E(R/p), M)$$
is a contraadujsted module; it is cotorsion if and only if R is a discrete valuation ring.

The previous example is in fact the sought defining property of (reduced) torsion contraadjusted modules:

Theorem 4.15. *Let M be a reduced torsion contraadjusted module over a Dedekind domain R. Then*
$$M \cong \bigoplus_{p \in \operatorname{mSpec} R} \operatorname{Ext}^1_R(E(R/p), M),$$
each module in the sum being p-primary and bounded.

Proof. The module M being torsion, it decomposes into a direct sum of p-primary components, so we may w.l.o.g. assume that M is p-primary. In such a case, $\operatorname{Hom}_R(R[p^{-1}], M) = 0$, which yields $M \cong \operatorname{Ext}^1_R(E(R/p), M)$ in the light of Lemma 4.12. Since this module is cotorsion (it is a direct summand of $\operatorname{Ext}^1_R(Q/R, M)$), together with the fact that M is reduced we conclude that M is bounded, cf. [13, 7.8]. □

Note that in the proof above, we did not need the "full power" of M being contraadjusted, only the fact that $\operatorname{Ext}^1_R(R[p^{-1}], M) = 0$ for every $p \in \operatorname{mSpec} R$ was used. This is, in general, not sufficient: For example, if $R = \mathbb{Z}$, then the class ${}^\perp(\{R[p^{-1}] \mid p \in \operatorname{mSpec} \mathbb{Z}\}^\perp)$ does not contain the (very flat) groups of the form $\mathbb{Z}[s^{-1}]$, where s has more than one prime divisor.

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