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BACHELOR THESIS

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The Helly numbers of systems of sets with bounded algebraic and topological complexity

Department of Applied Mathematics

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Study programme: Computer Science
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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, May 19, 2016 Jakub Sosnovec

Název práce: Hellyho čísla systémů množin s omezenou algebraickou a topologickou složitostí

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Abstrakt: Maehara dokázal, že je-li $\mathcal{F}$ systém alespoň $d+3$ sfér v $\mathbb{R}^d$ takový, že každých $d+1$ sfér z $\mathcal{F}$ má neprázdný průnik, pak celý systém $\mathcal{F}$ má neprázdný průnik. V této práci rozšiřujeme jeho výsledek Hellyho typu ve dvou směrech.

Nejprve ukážeme platnost analogické věty pro systémy pseudosfér, tedy systémy množin splňující, že průnik každého neprázdného podsystému je homeomorfní sféře nějaké dimenze nebo je prázdný.

Dále využijeme toho, že sféru v $\mathbb{R}^d$ lze vyjádřit jako nulovou množinu reálného polynomu. Je-li $\mathcal{P}$ množina polynomů, pak Hellyho číslo systému nulových množin polynomů z $\mathcal{P}$ je omezeno dimenzí vektorového prostoru generovaného $\mathcal{P}$. Pro systémy sfér ovšem Maeharův výsledek dává silnější odhad. Ukážeme některé obecné postačující podmínky pro lepší odhad Hellyho čísel v tomto kontextu.

Klíčová slova: Hellyho číslo, nulové množiny polynomů, systémy pseudosfér, kombinatorická geometrie

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Abstract: Maehara has shown that a family $\mathcal{F}$ of at least $d+3$ spheres in $\mathbb{R}^d$ has a nonempty intersection if every $d+1$ spheres from $\mathcal{F}$ have a nonempty intersection. We extend this Helly-type result in two directions.

On the one hand, we show an analogous theorem holds for families of pseudospheres, i.e., systems of sets such that the intersection of any nonempty subsystem is homeomorphic to a sphere of some dimension or is empty.

On the other hand, a sphere in $\mathbb{R}^d$ can be expressed as the zero set of a real polynomial. For a set of polynomials $\mathcal{P}$, the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is bounded by the dimension of the vector space generated by $\mathcal{P}$. For spheres, however, Maehara's result gives a stronger bound. We show some general sufficient assumptions that allow better bounds on the Helly numbers in this context.

Keywords: Helly number, zero sets of polynomials, systems of pseudospheres, combinatorial geometry

I would like to thank my supervisor Martin Tancer for his guidance and all the valuable advice. I would also like to express gratitude to my parents for supporting me during my studies. Last but not least, I would like to thank my friends for their patient listening.

Contents

Introduction

Helly's theorem [1] is a fundamental result in discrete and combinatorial geometry describing the way convex sets intersect. It states that if a finite family $\mathcal{F}$ of at least $d+1$ convex sets in $\mathbb{R}^d$ satisfies that every $d+1$ sets from $\mathcal{F}$ have a point in common, then all sets in $\mathcal{F}$ have a point in common. In this thesis, we present similiar theorems about some families of not necessarily convex sets.

The Helly number

In the literature, there are various ways how to define the Helly number. We have chosen the following definition (used in [2]), because it allows our results to be expressed in the most natural way, as we will explain later.

Definition 1. *Let $\mathcal{F}$ be a nonempty finite family of arbitrary sets. If $\mathcal{F}$ has an empty intersection, then the Helly number of $\mathcal{F}$ is defined as the size of the largest subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ has an empty intersection and all its proper subfamilies have nonempty intersections; if $\mathcal{F}$ has a nonempty intersection, then its Helly number is* 1*.*

Since this is a rather unintuitive definition, we will now clarify the motivation behind it by reformulating the original Helly's theorem. We claim that the Helly's theorem, as stated above, is equivalent to the assertion that the Helly number of any finite family of convex sets in $\mathbb{R}^d$ is at most $d+1$:

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that every $d+1$ sets from $\mathcal{F}$ have a point in common and suppose that $\mathcal{F}$ has an empty intersection. Assuming the Helly number of $\mathcal{F}$ is at most $d+1$, we obtain that there exists a subset $\mathcal{G} \subseteq \mathcal{F}$ of size at most $d+1$ with an empty intersection. This yields a contradiction with the assumption on $\mathcal{F}$.

Reversely, let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. The case that $\mathcal{F}$ has a nonempty intersection is trivial, suppose the opposite. Let $\mathcal{G} \subseteq \mathcal{F}$ be the largest subfamily such that $\mathcal{G}$ has an empty intersection and all its proper subfamilies have nonempty intersections. By applying the Helly's theorem on $\mathcal{G}$, we obtain that the size of $\mathcal{G}$ is at most $d+1$, as desired.

In fact, it is easy to observe that using an analogous proof, we can derive the following general statement. For a finite family of sets $\mathcal{F}$, these two assertions are equivalent:

(a) The Helly number of $\mathcal{F}$ is at most h.

(b) If every subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size at most h has a nonempty intersection, then $\mathcal{F}$ has a nonempty intersection.

The results of this thesis bound the Helly numbers of certain families of sets as in (a). The above observation thus provides a reformulation of our results and a way to understand them with no need to use the explicit definition of the Helly number.

Known results

Helly's theorem has spawned various versions and modifications for possibly non-convex sets, such as topological Helly theorem [3] or Helly-type theorem for unions of convex sets [4]. The foundation of our work are the following (reformulated) results of Maehara [5].

Theorem 2. *The Helly number of a family of spheres in $\mathbb{R}^d$ is at most $d + 2$.*

Theorem 3. *The Helly number of a family of at least $d + 3$ different spheres in $\mathbb{R}^d$ is at most $d + 1$.*

Theorem 3 is tight in the following way. There exists a family of $d+2$ spheres such that every $d + 1$ of them intersect but the whole family doesn't intersect; however, the theorem claims that such a family is always maximal in the sense that it is impossible to add another sphere to it while keeping the property that every $d + 1$ of them intersect (see Fig. 1). This motivated us to seek more cases of families where assuming greater size may reduce the Helly number.

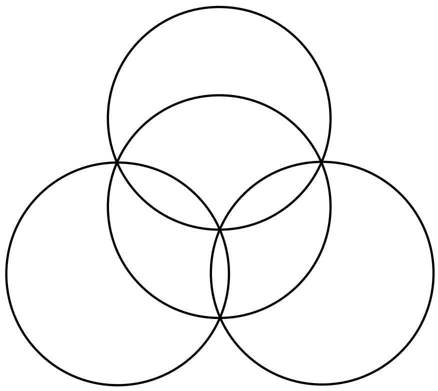

Figure 1: A configuration of four circles such that every three of them intersect but all four do not. It is not possible to add a different circle while preserving the property that every three circles intersect.

The setting of spheres in $\mathbb{R}^d$ can be generalized as follows. A *family of pseudo-spheres* is defined as a family $\mathcal{F}$ of subsets of $\mathbb{R}^d$ such that for any nonempty sub-family $\mathcal{G} \subseteq \mathcal{F}$, the intersection $\bigcap \mathcal{G}$ is homeomorphic to a k-dimensional sphere for some $k \in \{0, \ldots, d - 1\}$, to a single point, or is empty. It follows from Goaoc et al. [2] that the Helly number of a family of pseudospheres is bounded.

Since spheres in the Euclidian space can be expressed as zero sets of quadratic polynomials, it is a natural generalization to further study Helly-type theorems for zero sets of polynomials of bounded degree. Motzkin [6] first showed that the Helly number of such families is bounded by the dimension of the vector space of polynomials. We mainly use an analogous version of a result of Deza and Frankl [7]:

Let $\mathbb{R}[x_1, \ldots, x_d]$ denote the vector space of real polynomials in d variables and $\mathbb{R}_k[x_1, \ldots, x_d]$ the subspace containing polynomials of degree at most k. Moreover, $D_{d,k}$ denotes the dimension of $\mathbb{R}_k[x_1, \ldots, x_d]$.

Theorem 9. *The Helly number of a family of zero sets of polynomials from $\mathbb{R}_k[x_1, \ldots, x_d]$ is at most $D_{d,k}$.*

Their proof uses linear independence of polynomials as vectors in the space of polynomials. We discuss the technique later and use it extensively.

Our contributions

A set $A \subset \mathbb{R}^d$ is said to be an *affine sphere* if it is an intersection of a $(d-1)$-dimensional sphere in $\mathbb{R}^d$ and an affine subspace of $\mathbb{R}^d$ and it contains at least two points. Note that a sphere in $\mathbb{R}^d$ is also an affine sphere in $\mathbb{R}^d$ and a family of affine spheres is also a family of pseudosphere (defined above).

In Chapter 1, we first discuss Helly numbers of families of affine spheres as an intermediate generalization. Then, we extend Maehara's results to families of pseudospheres, showing analogous bounds on the respective Helly numbers.

Theorem 7. *The Helly number of a family of pseudospheres in $\mathbb{R}^d$ with at least $d+3$ different elements is at most $d+1$.*

We proceed with Chapter 2, where we study families of zero sets of polynomials and bound their Helly numbers. First, we observe that the bound from Deza and Frankl's result can be reduced if we choose an appropriate subspace of the space of all polynomials of bounded degree.

Theorem 10. *Let $\mathcal{P} \subseteq \mathbb{R}[x_1, \dots, x_d]$ be a set of polynomials and $D_{\mathcal{P}}$ the dimension of the linear hull of $\mathcal{P}$. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D_{\mathcal{P}}$.*

The polynomials whose zero set is a sphere in $\mathbb{R}^d$ are of the type $p(x_1, \dots, x_d) = \sum_{i=1}^{d}(x_i - \beta_i)^2 + \gamma$ for some reals $\beta_1, \dots, \beta_d, \gamma$. Moreover, the linear hull containing all these polynomials is a proper subspace of $\mathbb{R}_2[x_1, \dots, x_d]$ and its dimension is $d+2$ (a possible basis is $\{1, x_1, \dots, x_d, \sum_{i=1}^{d} x_i^2\}$). We conclude that Theorem 2 is a special case of Theorem 10. The natural goal is to show a generalized version of Theorem 3.

Let us observe the following property of spheres. If $\mathcal{S}$ is a family of d spheres such that no sphere from $\mathcal{S}$ can be removed without changing the intersection of the whole family ($\mathcal{S}$ is independent, in some sense), then the intersection of $\mathcal{S}$ is at most two points. This motivates a similar assumption on the space of polynomials.

Definition 12. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \dots, x_d]$ be a D-dimensional space of polynomials. We say that $\mathbb{P}$ has the k-points-property if the following holds for all $\ell \in \{0, 1, \dots, k\}$. If $\mathcal{Q} \subseteq \mathbb{P}$ is a linearly independent set of $D - \ell$ polynomials, then $Z(\mathcal{Q})$ is at most ℓ points.*

Using the definition, we can say that the space of polynomials that define spheres has the 2-intersection-property. We further discuss the k-points-property in Section 2.3. A sufficient condition for a space of polynomials to satisfy the property is also shown and used to derive that $\mathbb{R}_k[x_1, \dots, x_d]$ has the k-points-property for every $k \in \mathbb{N}$.

We observe that the assumption in Theorem 3, that the spheres are different, naturally translates to linear independence of polynomials. This allows us to present the desired generalization of Theorem 3.

Theorem 15. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the 2-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of at least $D+1$ polynomials and suppose the polynomials from $\mathcal{P}$ are pairwise linearly independent. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D-1$.*

We further seek a more general bound on the Helly number under stronger assumption. In Construction 2, we show that it is necessary to strengthen the assumptions on linear independence, as well as the size of the set. The most general result of the thesis is the following.

Theorem 16. *For all positive integers k, D, there exists a constant $N = N(k, D)$ such that the following holds. Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the k-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of at least N polynomials and suppose that every k polynomials from $\mathcal{P}$ are linearly independent. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D-k+1$.*

The bound on the necessary size $N(k, D)$ in the above theorem is very poor. We present a version of Theorem 16, which additionally assumes that $\mathcal{P}$ is in general position (i.e., an even stronger assumption on linear independence) and obtains a polynomial upper bound on the size of $\mathcal{P}$ sufficient to bound the Helly number by $D-k+1$.

Finally, we study the situation that no assumptions on linear independence are allowed. In that case, it follows from Construction 2 that the Helly number can be arbitrarily big. Nevertheless, we can still show a result about the structure of intersections by bounding the piercing number of the family of zero sets.

1. Affine spheres and pseudospheres

In this chapter, we first introduce spheres and affine spheres in $\mathbb{R}^d$ and present some fundamental properties. Then we state and give a proof of a version of a theorem of Maehara [5]. Following that, we define a family of pseudospheres and further generalize the previous results.

1.1 Preliminaries

A set S is a *sphere* in $\mathbb{R}^d$ if there exists a point $z \in \mathbb{R}^d$ and a positive parameter r such that S consists of all points of $\mathbb{R}^d$ with Euclidian distance r from z. Thus a sphere in $\mathbb{R}^1$ is any set of two different points, a sphere in $\mathbb{R}^2$ is a circle and a sphere in $\mathbb{R}^3$ is the surface of a ball.

Furthermore, a set A is an *affine sphere* in $\mathbb{R}^d$ if A is an intersection of a sphere and a flat (i.e., an affine subspace) and it contains at least two points. Clearly, any sphere is also an affine sphere. For instance, if S is a sphere in $\mathbb{R}^d$ and h is a hyperplane intersecting S, then $S \cap h$ is either a single point, or an affine sphere in $\mathbb{R}^d$ (see Fig. 1.1).

For a set $X \subseteq \mathbb{R}^d$, let $\mathrm{aff}(X)$ denote the affine hull of X and $\dim(\mathrm{aff}(X))$ the dimension of the flat $\mathrm{aff}(X)$. Then, the *dimension* of an affine sphere A is defined as $\dim(\mathrm{aff}(A)) - 1$ and denoted $\dim(A)$. This definition respects the convention of a d-dimensional sphere as a sphere in $\mathbb{R}^{d+1}$. For example, the dimension of an affine sphere formed by two different points in $\mathbb{R}^d$ is 0, regardless of the value of d.

A fundamental tool when working with affine spheres is the following observation. Let S and R be affine spheres in $\mathbb{R}^d$ such that their intersection is nonempty and not a single point. Then $R \cap S$ is also an affine sphere in $\mathbb{R}^d$ and its dimension is at most the minimum of $\dim(S)$ and $\dim(R)$, with equality attained only in the case that one of the spheres is contained in the other.

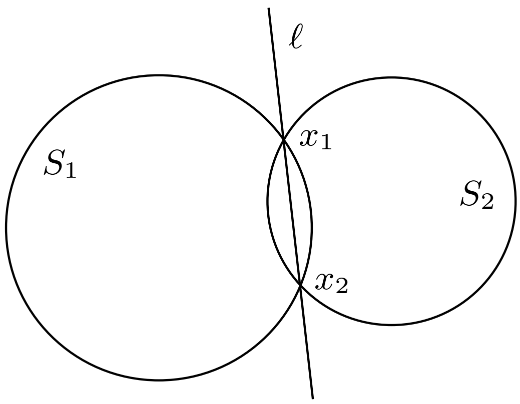

Figure 1.1: S_1 and S_2 are 1-dimensional spheres in $\mathbb{R}^2$ and their intersection $\{x_1, x_2\}$ is a 0-dimensional affine sphere in $\mathbb{R}^2$, as it is an intersection of a line ℓ and the sphere S_1.

1.2 Families of affine spheres

Let us now recall the two theorems of Maehara [5]:

Theorem 2. *The Helly number of a family of spheres in $\mathbb{R}^d$ is at most $d+2$.*

Theorem 3. *The Helly number of a family of at least $d+3$ different spheres in $\mathbb{R}^d$ is at most $d+1$.*

The proof of Theorem 2 by Maehara proceeds by induction on d and uses stereographic projection. We show the following generalization with a different, arguably simpler, proof.

Theorem 4. *The Helly number of a family of affine spheres in $\mathbb{R}^d$ is at most $d+2$.*

Proof. Let $\mathcal{F}$ be a family of affine spheres in $\mathbb{R}^d$ and suppose that every $d+2$ affine spheres from $\mathcal{F}$ have a nonempty intersection. Our goal is to show that the intersection of the whole family $\mathcal{F}$ is nonempty. We give a proof by contradiction, therefore we assume that $\bigcap \mathcal{F} = \emptyset$.

Let $\mathcal{G} = \{G_1, \dots, G_k\} \subseteq \mathcal{F}$ be an inclusion-minimal subfamily such that $\bigcap \mathcal{G} = \bigcap \mathcal{F} = \emptyset$. We denote $C_\ell = \bigcap_{i=1}^{\ell} G_i$ and claim that

$$C_1 \supsetneq C_2 \supsetneq \cdots \supsetneq C_k. \tag{1.1}$$

If we have $C_{j-1} = C_j$ for some $j \in \{2, \dots, k\}$, then

$$\bigcap(\mathcal{G} \setminus \{G_j\}) = \bigcap \mathcal{G} = \emptyset,$$

yielding a contradiction with the choice of $\mathcal{G}$. Hence the equation (1.1) holds.

Consequently, $C_1, \dots, C_{k-1}$ are nonempty, while $C_k = \emptyset$. Moreover, C_{k-1} may be a single point, but the sets $C_1, \dots, C_{k-2}$ are surely affine spheres, as they are intersections of affine spheres and contain at least two points. Combined with (1.1), we derive that $\dim(C_1), \dots, \dim(C_{k-2})$ is a decreasing sequence of integers. It follows from the fact $\dim(C_{k-2}) \geq 0$ that

$$d-1 \geq \dim(C_1) \geq \dim(C_2) + 1 \geq \cdots \geq \dim(C_{k-2}) + k - 3 \geq k-3,$$

therefore $k \leq d+2$. We obtain that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ of size at most $d+2$ with an empty intersection, which is a contradiction with the assumption. □

A version of Theorem 3 generalized to affine spheres holds as well. We do not explicitly state it here, as it will be a direct corollary of a more general Theorem 7.

The key properties of affine spheres used in the proof of Theorem 4 are the following.

(a) If an affine sphere A_1 is a proper subset of another affine sphere A_2, then the dimension of A_1 is strictly less than the dimension of A_2.

(b) The intersection of any number of affine spheres is an affine sphere, a single point, or is empty.

The combinatorial nature of these properties motivated us to seek an appropriate generalization of affine spheres, such that analogous Helly-type theorems would hold.

1.3 Families of pseudospheres

A family $\mathcal{F}$ of subsets of $\mathbb{R}^d$ is a *family of pseudospheres* if for any nonempty subfamily $\mathcal{G} \subseteq \mathcal{F}$, the intersection $\bigcap \mathcal{G}$ is homeomorphic to a k-dimensional sphere for some $k \in \{0, \dots, d-1\}$, to a single point, or is empty.

Moreover, we define the *rank* of an intersection $I = \bigcap \mathcal{G}$ of a nonempty subfamily $\mathcal{G}$ of a family of pseudospheres $\mathcal{F}$ in the following way. If I is homeomorphic to a k-dimensional sphere, then its rank is defined as k; the rank of a single point is defined as -1 and the rank of an empty set is defined as -2. Let $\mathrm{rk}(I)$ denote the rank of I. It is intuitively clear that S^k is not homeomorphic to S^ℓ for $k \neq \ell$ and thus $\mathrm{rk}(I)$ is well-defined; see Lemma 5 for further discussion.

Clearly, a family of affine spheres $\mathcal{A}$ is also a family of pseudospheres. For any nonempty $\mathcal{A}' \subseteq \mathcal{A}$, if $I = \bigcap \mathcal{A}'$ is an affine sphere (i.e., contains at least two points), then the dimension of I is equal to the rank of I. Furthermore, the combinatorial properties of spheres and affine spheres are preserved:

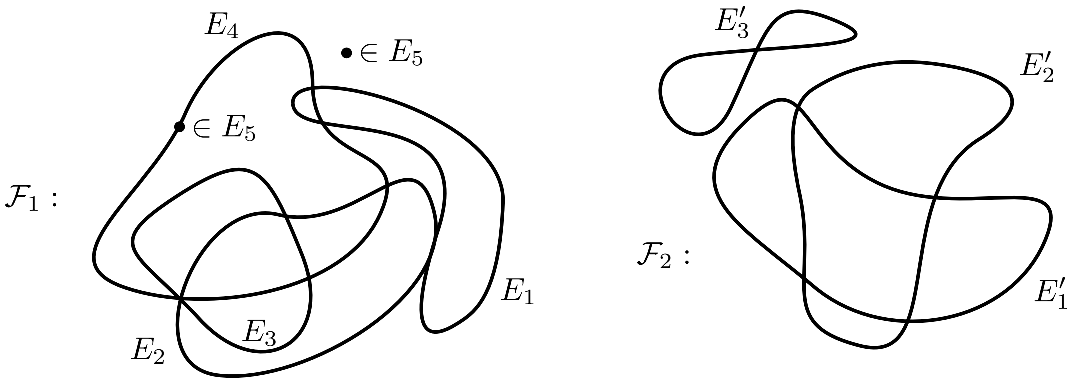

Figure 1.2: The family $\mathcal{F}_1$ is a family of pseudospheres, whereas the family $\mathcal{F}_2$ is not a family of pseudospheres, since E'_3 and the intersection of E'_1 and E'_2 are not homeomorphic to a sphere of any dimension.

Lemma 5. *Let $\mathcal{F}$ be a family of pseudospheres, $\mathcal{G}_1$ and $\mathcal{G}_2$ subfamilies of $\mathcal{F}$ such that $\mathcal{G}_1 \supseteq \mathcal{G}_2$ and denote $I_1 = \bigcap \mathcal{G}_1$ and $I_2 = \bigcap \mathcal{G}_2$. Then the following assertions are satisfied:*

(a) The rank of I_1 is well-defined.

(b) The rank of I_1 is at most the rank of I_2.

(c) The rank of I_1 is equal to the rank of I_2 if and only if $I_1 = I_2$.

Proof. Let S^d denote the unit sphere in $\mathbb{R}^{d+1}$ centered in the origin. Abusing the notation, S^k also denotes the projection of S^d into the first $k+1$ coordinates in $\mathbb{R}^{d+1}$ for $k \leq d$, i.e., a k-dimensional affine sphere in $\mathbb{R}^{d+1}$. Thus we can write $S^k \subseteq S^\ell$ for $k \leq \ell$.

Obviously, any k-dimensional affine sphere in $\mathbb{R}^d$ for $0 \leq k \leq d-1$ is homeomorphic to S^k. Recall that the homeomorphism relation is an equivalence on subsets of $\mathbb{R}^d$. Also, a restriction of a homeomorphism is a homeomorphism; we will use this as follows. If A is a subset of B and B is homeomorphic to C, then A is homeomorphic to a subset of C.

It is common knowledge that S^k is not homeomorphic to S^ℓ for all $k \neq \ell$. We will, however, need a strictly stronger assertion.

$$\textit{If } A \subseteq S^d \textit{ is homeomorphic to } S^d\textit{, then } A = S^d. \qquad (\star)$$

A detailed proof and examination of $(\star)$ using homology groups can be found in Hatcher's book [8], page 170.

Let us now show how to infer Lemma 5 from $(\star)$:

(a) Our goal is to show that $\mathrm{rk}(I_1)$ is a unique integer. First, the case $\mathrm{rk}(I_1) < 1$ is trivial, as then I_1 is a finite set and thus can be homeomorphic only to sets of equal size. Suppose that I_1 is homeomorphic to both S^k and S^ℓ for some $k, \ell \in \mathbb{N}$. This translates to a homeomorphism between S^k and S^ℓ by transitivity. Hence $k = \ell$, as required.

(b) Let I_1 be homeomorphic to S^k and I_2 homeomorphic to S^ℓ; we aim to show that $k \leq \ell$. Since $I_1 \subseteq I_2$, by transitivity we obtain a homeomorphism between S^k and a subset $A \subseteq S^\ell$. If $k \geq \ell$, then $A \subseteq S^\ell \subseteq S^k$ and we apply $(\star)$ to get $A = S^k$. Therefore $\ell = k$ and (b) follows.

(c) Suppose that both I_1 and I_2 are homeomorphic to S^k. Since $I_1 \subseteq I_2$, we have that I_1 and a subset A of S^k are homeomorphic through the homeomorphism of I_2. We obtain $A = S^k$ by $(\star)$, hence $I_1 = I_2$ holds. The other implication follows trivially from (a).

□

With Lemma 5 in hand, it is straightforward to translate the proof of Theorem 4 into the setting of families of pseudospheres and obtain that an analogous version holds.

Theorem 6. *The Helly number of a family of pseudospheres in $\mathbb{R}^d$ is at most $d + 2$.*

Let us now observe that the number $d + 2$ in the statement of Theorem 6 cannot be reduced without further assumptions.

Construction 1. We construct a family $\mathcal{S}$ of $d + 2$ spheres in $\mathbb{R}^d$ such that the intersection of $\mathcal{S}$ is empty and the intersection of any proper subfamily of $\mathcal{S}$ is nonempty.

We say that a finite set of points $Y \subset \mathbb{R}^d$ is in *general position* if no $m + 2$ points from Y lie in an m-dimensional flat for any $m \in \{0, 1, \ldots, d - 1\}$. Recall that any $d+1$ points in general position uniquely determine a $(d-1)$-dimensional sphere that intersects them.

Let $X = \{x_1, x_2, \ldots, x_{d+2}\} \subset \mathbb{R}^d$ be a set of $d + 2$ points in general position that do not lie on a single sphere. We define S_i to be the $(d-1)$-dimensional sphere determined by $X \setminus \{x_i\}$ for $i \in \{1, \ldots, d + 2\}$. Finally, set $\mathcal{S} = \{S_1, \ldots, S_{d+2}\}$; see Fig. 1.3 for an illustration.

It is clear that $x_i \in \bigcap(\mathcal{S} \setminus \{S_i\})$, hence the intersection of any proper subfamily of $\mathcal{S}$ is nonempty. We claim that the intersection of $\mathcal{S}$ is empty. To show this, define $C_\ell = \bigcap_{i=1}^{\ell} S_i$ and observe that $x_j \in C_{j-1}$ and $x_j \notin C_j$, deriving that $C_{j-1} \supsetneq C_j$ for all $j \in \{2, \ldots, d + 2\}$. By Lemma 5, we obtain that $\mathrm{rk}(C_1), \mathrm{rk}(C_2), \ldots, \mathrm{rk}(C_{d+2})$ is a decreasing sequence of $d + 2$ integers. Since $\mathrm{rk}(C_1) = d - 1$ and $\mathrm{rk}(C_{d+2}) \geq -2$, we conclude that $\mathrm{rk}(C_{d+2}) = -2$ and $\bigcap \mathcal{S} = C_{d+2} = \emptyset$.

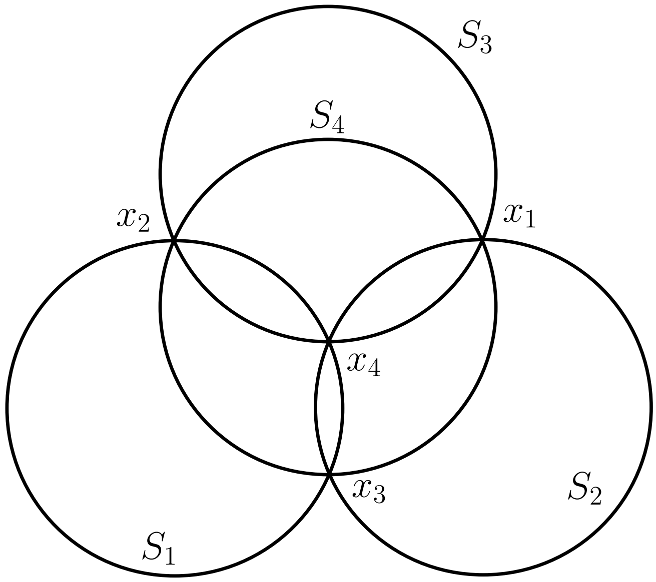

Figure 1.3: The family $\mathcal{S}$ in Construction 1 for $d = 2$.

We have constructed a family of pseudospheres satisfying that every $(d+1)$-element subfamily has a nonempty intersection, while the whole family has an empty intersection. As is the case with spheres, we will show that an additional assumption on the size of the family will avoid all such families. Thus an analogous version of Theorem 3 of Maehara holds for families of pseudospheres. We are now ready to state the main result of this chapter.

Theorem 7. *The Helly number of a family of pseudospheres in $\mathbb{R}^d$ with at least $d+3$ different elements is at most $d+1$.*

Before proceeding with the proof, we need to obtain some properties of the "bad" families, such as the one in Construction 1.

Lemma 8. *Let $\mathcal{G}$ be a family of pseudospheres in $\mathbb{R}^d$ of size $d+2$ such that the intersection of $\mathcal{G}$ is empty and the intersection of any proper subfamily of $\mathcal{G}$ is nonempty. Then the rank of the intersection of a k-element subfamily of $\mathcal{G}$ is exactly $d-k$ for any $1 \leq k \leq d+2$.*

Proof. Let $\mathcal{H} \subseteq \mathcal{G}$ be a subfamily of size k and denote $I = \bigcap \mathcal{H}$. We proceed by induction on $\alpha = d - k + 2$ (where d is fixed). If $\alpha = 0$, then $I = \bigcap \mathcal{G}$, which is empty by the assumption. Thus the rank of I is -2, as required.

Now, let $\alpha \geq 1$ and suppose that the proposition holds for all subfamilies of greater size. Choose $E \in \mathcal{F} \setminus \mathcal{H}$ arbitrarily and define $\mathcal{H}^+ = \mathcal{H} \cup \{E\}$ and $I^+ = \bigcap \mathcal{H}^+$. The rank of I^+ is $d-k-1$ by the induction hypothesis. If $I = I \cap E = I^+$, then the intersection of $\mathcal{G} \setminus \{E\}$ is empty, a contradiction with the assumption. Hence I is a proper superset of I^+ and $\mathrm{rk}(I) > \mathrm{rk}(I^+) = d-k-1$ by Lemma 5.

Suppose that $\mathrm{rk}(I) \geq d-k+1$; order the elements $\mathcal{H} = \{E_1, \dots, E_k\}$ arbitrarily and denote $C_\ell = \bigcap_{i=1}^{\ell} E_i$. The sequence $\mathrm{rk}(C_1), \mathrm{rk}(C_2), \dots, \mathrm{rk}(C_k)$ is a non-increasing sequence of k integers by Lemma 5. Combining the fact that $\mathrm{rk}(C_1) \leq d-1$ and the assumption $\mathrm{rk}(C_k) = \mathrm{rk}(I) \geq d-k+1$ gives us the existence of an index $j \in \{1, \dots, k-1\}$ such that $\mathrm{rk}(C_j) = \mathrm{rk}(C_{j+1})$. We obtain $C_j = C_{j+1}$ by Lemma 5, hence the intersection of $\mathcal{G} \setminus \{E_{j+1}\}$ is empty, a contradiction. □

Proof of Theorem 7. Let $\mathcal{F}$ be the family of pseudospheres. Our goal is to show that any $(d+2)$-element subfamily of $\mathcal{F}$ has a nonempty intersection. Then we can apply Theorem 6 and obtain that the intersection of $\mathcal{F}$ is nonempty.

Let $\mathcal{G}$ be an arbitrary subfamily of $\mathcal{F}$ of size $d+2$ and suppose that the intersection of $\mathcal{G}$ is empty. Observe that $\mathcal{G}$ satisfies the assumption of Lemma 8. Denote $\mathcal{G} = \{E_1, \dots, E_{d+2}\}$ and choose points $X = \{x_1, \dots, x_{d+2}\}$ so that

$$x_i \in \bigcap(\mathcal{G} \setminus \{E_i\}).$$

Clearly, $x_i \neq x_j$ for any $i \neq j$, as otherwise $x_i \in \bigcap \mathcal{G}$. We denote $\mathcal{G}_{i,j} = \mathcal{G} \setminus \{E_i, E_j\}$ for $i, j \in \{1, \dots, d+2\}$, $i \neq j$. By Lemma 8, the rank of $\bigcap \mathcal{G}_{i,j}$ is 0, therefore $\bigcap \mathcal{G}_{i,j}$ is a set of two points. Since $x_i, x_j \in \bigcap \mathcal{G}_{i,j}$, we obtain

$$\bigcap \mathcal{G}_{i,j} = \{x_i, x_j\} \text{ for all } i \neq j.$$

We use the assumption on the size of $\mathcal{F}$ to choose some $T \in \mathcal{F} \setminus \mathcal{G}$. Also, we assume that any $d+1$ elements from $\mathcal{F}$ have a nonempty intersection, deriving that T has to intersect $\bigcap \mathcal{G}_{i,j}$. Hence, for every $i \neq j$, at least one of the points x_i and x_j is contained in T. We obtain that at most one point from X is not contained in T. With no loss of generality, we assume $X \setminus \{x_1\} \subseteq T$.

Since E_1 and T are different, the rank of $E_1 \cap T$ is at most $d-2$ by Lemma 5. Let $\mathcal{H} \subseteq \mathcal{G} \setminus \{E_1\}$ be an inclusion-minimal subfamily such that

$$T \cap E_1 \cap \left(\bigcap \mathcal{H}\right) = \emptyset.$$

The size of $\mathcal{H}$ is at least d by the assumption. Without loss of generality, we assume $\mathcal{H} = \{E_2, E_3, \dots, E_k\}$. We claim that $k < d+2$; denote

$$C_\ell = T \cap \left(\bigcap_{i=1}^{\ell} E_i\right) \quad \text{for } \ell \geq 1.$$

We have $C_i \supsetneq C_{i+1}$ from the minimality of $\mathcal{H}$, therefore the sequence $\mathrm{rk}(C_1), \mathrm{rk}(C_2), \dots, \mathrm{rk}(C_k)$ is decreasing. Since $\mathrm{rk}(C_1) \leq d-2$ and $\mathrm{rk}(C_k) = -2$, we obtain that $k \leq d+1$, as desired.

We have shown that $E_{d+2} \notin \mathcal{H}$. Hence

$$T \cap E_1 \cap E_2 \cap \dots \cap E_{d+1} \subseteq T \cap E_1 \cap \left(\bigcap \mathcal{H}\right) = \emptyset.$$

But $x_{d+2} \in T$ and $x_{d+2} \in E_i$ for all $i \in \{1, \dots, d+1\}$, we have obtained a contradiction. □

Remark. It should be noted that the first half of the above proof of Theorem 7 proceeds in a way similiar to Maehara's [5] proof of Theorem 3. In particular, the definition of the points X and the conclusion that the intersection of the examined $(d+1)$-element subfamily is exactly $\{x_i, x_j\}$ are key components of our proof and both are adopted from Maehara's method. However, the conclusion of Maehara's proof is specific to the setting of spheres. The second half of our proof, from the definition of the subfamily $\mathcal{H}$, is thus new.

2. Zero sets of polynomials

Recall that a sphere in $\mathbb{R}^d$ can be expressed as the zero set of a quadratic polynomial. In this chapter, we seek to generalize Maehara's results in the setting of zero sets of polynomials, which are also called hypersurfaces or varieties.

First, we introduce necessary notation and establish some fundamentals regarding real polynomials of multiple variables. Then we state a known result by Deza and Frankl [7], including a proof of a variation of the theorem. Following that, we discuss necessary properties of spaces of polynomials and we show some propositions further reducing the Helly number under some rather strong assumptions. We also present an example showing these assumptions are necessary. Finally, we discuss the piercing number, a notion related to Helly numbers.

2.1 Preliminaries

Let $\mathbb{R}[x_1, \ldots, x_d]$ denote the vector space of real polynomials in d variables. Moreover, $\mathbb{R}_k[x_1, \ldots, x_d]$ is the subspace of $\mathbb{R}[x_1, \ldots, x_d]$ containing all polynomials of degree at most k. For a set $\mathcal{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$, *the zero set* of $\mathcal{P}$ is defined as

$$Z(\mathcal{P}) = \{x \in \mathbb{R}^d \mid p(x) = 0 \text{ for all } p \in \mathcal{P}\}.$$

If p is a polynomial, we write $Z(p)$ for $Z(\{p\})$.

Note that $Z(\mathcal{P}) \supseteq Z(\mathcal{Q})$ for any sets $\mathcal{P} \subseteq \mathcal{Q}$. Furthermore, $p \cdot q$ is a polynomial with zero set $Z(p) \cup Z(q)$ for any polynomials $p, q \in \mathbb{R}[x_1, \ldots, x_d]$. We say that a set $X \subseteq \mathbb{R}^d$ is a *hypersurface* if there exists a polynomial p such that $Z(p) = X$.

For example, the zero set of the polynomial $p(x, y) = x^2 + y^2 - 1$ is the unit circle. In general, spheres in $\mathbb{R}^d$ are hypersurfaces. A hyperplane can be defined as the zero set of a linear polynomial. However, the boundary of a box in $\mathbb{R}^d$ is not a hypersurface, hence, families of pseudospheres do not, in general, consist of hypersurfaces.

Let $D_{d,k}$ denote the dimension of the space $\mathbb{R}_k[x_1, \ldots, x_d]$. Since the set of monomials provides a possible basis, we can obtain $D_{d,k} = \binom{d+k}{d}$ by a simple combinatorial argument. For a set of polynomials $\mathcal{P}$, let $\operatorname{span} \mathcal{P}$ denote the linear hull of $\mathcal{P}$.

Following the usual conventions, a set of polynomials $\mathcal{Q} = \{q_1, \ldots, q_\ell\}$ is *linearly independent* if $\sum_{i=1}^{\ell} \alpha_i q_i = 0$ (where 0 denotes the zero polynomial) implies $\alpha_1 = \cdots = \alpha_\ell = 0$ for any $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$.

We will use the following observation extensively. Let $\mathcal{P}$ be a set of polynomials and suppose that $q = \alpha_1 p_1 + \cdots + \alpha_\ell p_\ell$ is a linear combination of polynomials $p_1, \ldots, p_\ell \in \mathcal{P}$. Then we have $Z(\mathcal{P}) \subseteq Z(q)$, since $q(x) = \alpha_1 p_1(x) + \cdots + \alpha_\ell p_\ell(x) = 0$ for any $x \in Z(\mathcal{P})$. Stated in the counterpositive, $Z(\mathcal{P}) \cap Z(r) \subsetneq Z(\mathcal{P})$ implies that r is linearly independent of $\mathcal{P}$.

2.2 Basic bound on the Helly number

The cornerstone theorem of this chapter is the following result of Deza and Frankl [7].

Theorem 9. *The Helly number of a family of zero sets of polynomials from $\mathbb{R}_k[x_1, \ldots, x_d]$ is at most $D_{d,k}$.*

We know from Maehara's Theorem 2 that the Helly number of a family of spheres in $\mathbb{R}^d$ is at most $d+2$. Moreover, spheres are zero sets of polynomials of degree two. But the dimension of $\mathbb{R}_2[x_1, \ldots, x_d]$ is $D_{d,2} = \binom{d+2}{2} > d+2$, so the bound from Theorem 9 is much weaker. In fact, a slight modification of the proof of Theorem 9 by Deza and Frankl [7] gives the following, stronger theorem.

Theorem 10. *Let $\mathcal{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a set of polynomials and $D_{\mathcal{P}}$ the dimension of the linear hull of $\mathcal{P}$. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D_{\mathcal{P}}$.*

We will later show that in the special case of $\mathcal{P}$ being a set of polynomials that define spheres, Theorem 10 attains the bound from Theorem 2.

We include a proof here, as it is simple and instructive. For this purpose, let us first show a sufficient condition for linear independence of a set of polynomials. We will use this later as well.

Lemma 11. *Let $\mathcal{Q} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a set of polynomials and suppose that $Z(\mathcal{Q}) \subsetneq Z(\mathcal{Q} \setminus \{q\})$ for all $q \in \mathcal{Q}$. Then $\mathcal{Q}$ is linearly independent.*

Proof. Let us denote $\mathcal{Q} = \{q_1, \ldots, q_k\}$. By the assumption, there exist points $x_1, \ldots, x_k \in \mathbb{R}^d$ such that $x_i \notin Z(q_i)$ and $x_i \in Z(q_j)$ for all indices $i \neq j$. We can reformulate this as $q_i(x_i) \neq 0$ and $q_i(x_j) = 0$ for all $i \neq j$.

Suppose we have $\sum_{i=1}^k \alpha_i q_i = 0$ for some reals $\alpha_1, \ldots, \alpha_k$ which are not all zero. Without loss of generality, we assume $\alpha_1 \neq 0$. Evaluating the linear combination at x_1, we obtain

$$\sum_{i=1}^{k} \alpha_i q_i(x_1) = \alpha_1 \underbrace{q_1(x_1)}_{\neq 0} + \sum_{i=2}^{k} \alpha_i \underbrace{q_i(x_1)}_{=0} \neq 0,$$

which is a contradiction. Therefore the set $\mathcal{Q}$ is linearly independent. □

Proof of Theorem 10. Let $\mathcal{P}$ be a given set of polynomials, $D_{\mathcal{P}}$ the dimension of $\operatorname{span} \mathcal{P}$ and $\mathcal{Z} = \{Z(p) \mid p \in \mathcal{P}\}$ the family of zero sets of polynomials from $\mathcal{P}$. Suppose that every $D_{\mathcal{P}}$ sets from $\mathcal{Z}$ have a point in common. In other words, $Z(\mathcal{R})$ is nonempty for every subset $\mathcal{R} \subseteq \mathcal{P}$ of size at most $D_{\mathcal{P}}$. Our goal is to show that the intersection of $\mathcal{Z}$ is nonempty. Since $\bigcap \mathcal{Z} = Z(\mathcal{P})$, we only need to show $Z(\mathcal{P}) \neq \emptyset$.

Let $\mathcal{Q} \subseteq \mathcal{P}$ be an inclusion-minimal subset satisfying $Z(\mathcal{Q}) = Z(\mathcal{P})$. By the minimality of $\mathcal{Q}$ and Lemma 11, $\mathcal{Q}$ linearly independent. It is well known that the size of a linearly independent set is bounded by the dimension of the vector space. Therefore, the size of $\mathcal{Q}$ is at most $D_{\mathcal{P}}$. Finally, we use the assumption to obtain $\emptyset \neq Z(\mathcal{Q}) = Z(\mathcal{P})$, as desired. □

Recall from the introduction that the linear hull of the set of polynomials that define spheres is $\mathbb{P}_S = \operatorname{span}\{1, x_1, \ldots, x_d, \sum_{i=1}^d x_i^2\}$. It is a proper subspace of $\mathbb{R}_2[x_1, \ldots, x_d]$ of dimension $d+2$. Theorem 2 is thus a special case of Theorem 10 for $\mathcal{P} \subseteq \mathbb{P}_S$ and $D_{\mathcal{P}} \leq d+2$.

2.3 Assumptions on spaces of polynomials

Viewing Theorem 10 as a more general version of Theorem 2, the goal of this section is to show a generalized version of Theorem 3, i.e., reduce the bound on the Helly number to $D_{\mathcal{P}} - 1$ under certain assumptions.

First, note that a simple assumption on the size of $\mathcal{P}$ is not sufficient, as $Z(p) = Z(\alpha p)$ for any real $\alpha \neq 0$, therefore different polynomials can have identical zero sets. To avoid this issue, we will additionally require that the polynomials in $\mathcal{P}$ are pairwise linearly independent, as will be discussed later. In general, this assumption is still not sufficient; consider a set of polynomials $\mathcal{P} \subseteq \mathbb{R}_1[x_1, x_2]$ which defines a family L of lines in $\mathbb{R}^2$ in general position. Then the bound $D_{2,1} = 3$ from Theorem 10 cannot be improved, as any two lines from L intersect, but the whole family may not. Hence, we will also need an assumption on the space of polynomials.

A crucial property of families of spheres (and pseudospheres) used in the previous chapter is the following. If $\mathcal{F}$ is a family of d spheres such that

$$\bigcap \mathcal{F} \subsetneq \bigcap (\mathcal{F} \setminus \{E\}) \text{ for all } E \in \mathcal{F}, \tag{$\otimes$}$$

then the intersection of $\mathcal{F}$ is at most two points. For a set of polynomials $\mathcal{P}$, if the family of zero sets of polynomials from $\mathcal{P}$ satisfies ($\otimes$), then $\mathcal{P}$ is linearly independent by Lemma 11. If we assume that $Z(\mathcal{P})$ is also at most few points, we may be able to use the approach of Theorem 3 to derive the desired generalization.

This correspondence motivates the following definition.

Definition 12. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials. We say that $\mathbb{P}$ has the k-points-property if the following holds for all $\ell \in \{0, 1, \ldots, k\}$. If $\mathcal{Q} \subseteq \mathbb{P}$ is a linearly independent set of $D - \ell$ polynomials, then $Z(\mathcal{Q})$ is at most ℓ points.*

Note that the property is monotone, meaning that $\mathbb{P}$ having the k-points-property implies that $\mathbb{P}$ has the m-points-property for all $m \leq k$.

As discussed above, the space of polynomials that define spheres has the 2-points-property. Furthermore, we will show that the space of all polynomials of bounded degree satisfies the property for general k.

Lemma 13. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a space of polynomials. If $\mathbb{R}_k[x_1, \ldots, x_d]$ is a subspace of $\mathbb{P}$, then $\mathbb{P}$ has the k-points-property.*

We postpone the proof, as it will be an easy corollary of Lemma 14, which gives a sufficient condition for a space to have the k-points-property.

Let us now clarify the role of the space $\mathbb{P}$ in our further results. Through the rest of this chapter, we view $\mathbb{P}$ as a given space with the k-points-property for appropriate k and we bound the Helly number of zero sets of polynomials from a subset of $\mathbb{P}$ with a function of the dimension of $\mathbb{P}$. Given only a set of polynomials $\mathcal{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$, the best bounds are therefore attained when $\mathbb{P}$ is chosen as a space that has the k-points-property and contains $\mathcal{P}$ and whose dimension is the least.

However, it is not clear how to determine this space $\mathbb{P}$ from the given set $\mathcal{P}$ or what its dimension is. We can always work with $\mathbb{R}_k[x_1, \ldots, x_d]$, which has the k-points-property by Lemma 13, but the dimension $D_{d,k}$ grows fast and the resulting bounds on the Helly numbers may not be as strong – as we have seen on the example of spheres.

We were not able to obtain a complete solution to this problem in this thesis. There is, however, a natural sufficient condition for a space to satisfy the k-points-property.

Lemma 14. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a space of polynomials. Suppose that for every $\ell \in \{0, 1, \ldots, k\}$ and every set of $\ell + 1$ points $X \subseteq \mathbb{R}^d$, there exists a polynomial $p \in \mathbb{P}$ such that $Z(p)$ contains exactly ℓ points from X. Then $\mathbb{P}$ has the k-points-property.*

Proof. Let D denote the dimension of $\mathbb{P}$. We proceed by induction on k. For $k = 0$, the assumption says that no point is contained in all polynomials from $\mathbb{P}$, hence $Z(\mathbb{P})$ is empty. If $\mathcal{Q} \subseteq \mathbb{P}$ is a set of D linearly independent polynomials, then $\mathcal{Q}$ is a basis of $\mathbb{P}$. This means that $Z(\mathcal{Q}) \subseteq Z(p)$ for all $p \in \mathbb{P}$, we obtain that $Z(\mathcal{Q}) = Z(\mathbb{P}) = \emptyset$. Therefore $\mathbb{P}$ has the 0-points-property.

Suppose that $k \geq 1$. By the induction hypothesis, $\mathbb{P}$ has the $(k-1)$-points-property. Let $\mathcal{Q} \subseteq \mathbb{P}$ be a set of $D - k$ linearly independent polynomials. We only need to verify that $Z(\mathcal{Q})$ is at most k points. Suppose that $Z(\mathcal{Q})$ contains at least $k + 1$ different points. Using the assumption, there exists a polynomial $p \in \mathbb{P}$ whose zero set intersects exactly k of them. Since $Z(p)$ avoids at least one of the points in $Z(\mathcal{Q})$, p is linearly independent of $\mathcal{Q}$. We conclude that $\mathcal{Q} \cup \{p\}$ is a linearly independent set of $D - k + 1$ polynomials whose zero set contains k different points. This contradicts the $(k-1)$-points-property of $\mathbb{P}$. □

Proof of Lemma 13. We only need to show the space $\mathbb{R}_k[x_1, \ldots, x_d]$ satisfies the assumptions of Lemma 14.

Let X be a set of $\ell + 1$ points for some $\ell \leq k$. We denote $X = \{x_1, \ldots, x_{\ell+1}\}$ and choose some hyperplanes $h_1, \ldots h_\ell$ such that h_i contains x_i and avoids $x_{\ell+1}$ for all $i \in \{1, \ldots, \ell\}$. Let q_i be the linear polynomial defining h_i as its zero set. Finally, set $p = \prod_{i=1}^{\ell} q_i$ and observe that

$$x_{\ell+1} \notin Z(p) = \bigcup_{i=1}^{\ell} h_i \supseteq \{x_1, \ldots, x_\ell\}.$$

It is also clear that $p \in \mathbb{R}_k[x_1, \ldots, x_d]$, as it is a product of $\ell \leq k$ linear polynomials. □

Using the observation that spaces of spheres have the 2-points-property, let us now present the desired generalization of Theorem 3.

Theorem 15. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the 2-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of at least $D + 1$ polynomials and suppose the polynomials from $\mathcal{P}$ are pairwise linearly independent. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D - 1$.*

We argue that the assumption on linear independence of pairs of polynomials from $\mathcal{P}$ naturally corresponds to the assumption that no two spheres are the same. The zero set of the zero polynomial is the whole space (so we need not consider it) and if two polynomials are multiples of each other, then their zero sets are equal. The proof of Theorem 15 is a combination of the proofs of Theorems 10 and 7.

Proof of Theorem 15. We assume that the zero sets of every $D-1$ polynomials from $\mathcal{P}$ intersect. Let $\mathcal{Q} \subseteq \mathcal{P}$ be a subset of size D. If we show that $Z(\mathcal{Q})$ is nonempty for any such choice of $\mathcal{Q}$, then $Z(\mathcal{P})$ is nonempty by Theorem 10.

Suppose for a contradiction that $Z(\mathcal{Q}) = \emptyset$. We denote $\mathcal{Q} = \{q_1, \ldots, q_D\}$ and choose $r \in \mathcal{P} \setminus \mathcal{Q}$ arbitrarily. As the zero set of any proper subset of $\mathcal{Q}$ is nonempty by the assumption, we can apply Lemma 11 to obtain that $\mathcal{Q}$ is linearly independent. Therefore, $\mathcal{Q}$ forms a basis of $\mathbb{P}$ and r can thus be expressed as a linear combination $r = \sum_{i=1}^{D} \alpha_i q_i$ for some reals $\alpha_1, \ldots, \alpha_D$.

Let $x_1, \ldots, x_D \in \mathbb{R}^d$ be points such that $x_i \in Z(\mathcal{Q} \setminus \{q_i\})$, which is nonempty by the assumption. The points $x_1, \ldots, x_D$ are all different, as otherwise $Z(\mathcal{Q})$ would be nonempty. Since $\mathbb{P}$ satisfies the 2-points-property, we have $Z(\mathcal{Q} \setminus \{q_i\}) = \{x_i\}$ and moreover, $Z(\mathcal{Q} \setminus \{q_i, q_j\}) = \{x_i, x_j\}$ for any $i \neq j$.

By the assumption, $Z(r)$ intersects the zero set of every choice of $D-2$ polynomials from $\mathcal{P}$. In particular, $Z(r)$ has to contain at least one of the points x_i, x_j for every $i \neq j$. This implies that at most one of the points $x_1, \ldots, x_D$ is not contained in $Z(r)$. With no loss of generality, we assume $x_i \in Z(r)$ for every $i \geq 2$.

Observe that $x_i \in Z(r)$ if and only if $\alpha_i = 0$. Therefore $\alpha_i = 0$ for all $i \geq 2$. We obtain that $r = \alpha_1 q_1$ which is a contradiction with the assumption that every two polynomials from $\mathcal{P}$ are linearly independent. $\square$

2.4 Stronger linear independence

The key features of Theorems 3, 7 and 15 are the assumptions on the sizes of the sets. As a continuation of Theorem 15, one can ask for better bounds under stronger assumptions. Specifically, if we assume that $\mathbb{P}$ has the 3-points-property and the size of $\mathcal{P}$ is sufficient, can we bound the Helly number by $D-2$? We will later present an example showing this is not the case.

All is not lost, however. If we additionally strengthen the assumption on linear independence of polynomials in $\mathcal{P}$ and we require the set $\mathcal{P}$ to be sufficiently huge, we can reduce the bound further.

Theorem 16. *For all positive integers k, D, there exists a constant $N = N(k, D)$ such that the following holds. Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the k-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of at least N polynomials and suppose that every k polynomials from $\mathcal{P}$ are linearly independent. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D-k+1$.*

Observe that Theorem 15 is a special case of the above with $k = 2$ and $N(2, D) = D + 1$. The assumption in Theorem 15 that pairs of polynomials are linearly independent was rather natural and followed nicely from the situation of simple spheres. Theorem 16, however, assumes that every k polynomials are linearly independent, which is a direct generalization, but there seems to be no straightforward geometrical insight explaining this assumption. As mentioned, we will see it is necessary in Construction 2.

Proof of Theorem 16. Let us assume that (a) every k polynomials from $\mathcal{P}$ are linearly independent, (b) the zero sets of every $D-k+1$ polynomials from $\mathcal{P}$

intersect and (c) $Z(\mathcal{P})$ is empty. We will bound the size of $\mathcal{P}$ by $N(k, D) - 1$ and thus prove the theorem.

Let $\mathcal{Q} \subseteq \mathcal{P}$ be a basis of the linear hull of $\mathcal{P}$. This choice immediately implies that $Z(\mathcal{Q}) = Z(\mathcal{P}) = \emptyset$. We set $m = |\mathcal{Q}| - D + k$, then m is at least 2, as otherwise $|\mathcal{Q}| \leq D - k + 1$ and $Z(\mathcal{Q}) \neq \emptyset$ by the assumption, which is a contradiction. Also, $m \leq k$, simply because $|\mathcal{Q}| \leq D$. We now define a set of points

$$X = \bigcup_{\substack{\mathcal{R} \subseteq \mathcal{Q} \\ |\mathcal{R}| = D - k}} Z(\mathcal{R}),$$

where the sets $Z(\mathcal{R})$ are always at most k points by the k-points-property of $\mathbb{P}$. Therefore $|X| \leq k\binom{|\mathcal{Q}|}{D-k} \leq k\binom{D}{D-k}$.

Let $\overline{\mathcal{Q}}$ denote $\mathcal{P} \setminus \mathcal{Q}$, we will bound the size of $\overline{\mathcal{Q}}$ and thus the size of $\mathcal{P}$. We claim that for every different $r_1, \ldots, r_m \in \overline{\mathcal{Q}}$ there exist indices $i \neq j$ such that $Z(r_i) \cap X \neq Z(r_j) \cap X$. Suppose for a contradiction that $r_1, \ldots, r_m \in \overline{\mathcal{Q}}$ satisfy

$$Z(r_1) \cap X = Z(r_2) \cap X = \cdots = Z(r_m) \cap X.$$

Since $m \leq k$, we assumed that $r_1, \ldots, r_m$ are linearly independent. Hence there exists $\mathcal{R} \subset \mathcal{Q}$ such that $\mathcal{R} \cup \{r_1, \ldots, r_m\}$ is again a basis of $\operatorname{span} \mathcal{P}$. Observe that the size of $\mathcal{R}$ is exactly $D - k$, therefore $Z(\mathcal{R}) \neq \emptyset$ and moreover, $Z(\mathcal{R} \cup \{r_i\}) \neq \emptyset$ for every $i \in \{1, \ldots, m\}$. But from the choice of $r_1, \ldots, r_m$ and the fact that $Z(\mathcal{R}) \subseteq X$, we derive

$$\emptyset \neq Z(\mathcal{R} \cup \{r_1\}) = Z(\mathcal{R}) \cap Z(r_1) \cap \cdots \cap Z(r_m) = Z(\mathcal{R} \cup \{r_1, \ldots, r_m\}).$$

We have found a basis of $\mathcal{P}$ whose zero set is nonempty, this is a contradiction.

We obtain that the same intersection pattern $Z(r) \cap X$ can be attained by at most $m - 1 \leq k - 1$ polynomials from $\overline{\mathcal{Q}}$. Therefore $|\overline{\mathcal{Q}}| \leq (k-1)2^{|X|}$, which is again finite. Finally, we bound the size of $\mathcal{P} = \mathcal{Q} \cup \overline{\mathcal{Q}}$ and set $N(k, D)$.

$$|\mathcal{P}| \leq D + (k-1)2^{|X|} \leq D + (k-1)2^{k\binom{D}{D-k}} = N(k, D) - 1$$

$\square$

The bound on $N(k, D)$ in Theorem 16 is indeed very poor. Our method allows us to greatly reduce the bound if we assume that the set $\mathcal{P}$ is in general position, i.e., every $D_{\mathcal{P}}$ polynomials from $\mathcal{P}$ are linearly independent, where $D_{\mathcal{P}}$ is the dimension of the linear hull of $\mathcal{P}$.

Theorem 17. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the k-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of at least $D + k^2 - 2k + 1$ polynomials and suppose that $\mathcal{P}$ is in general position. Then the Helly number of the family of zero sets of polynomials from $\mathcal{P}$ is at most $D - k + 1$.*

Proof. We assume that the zero sets of every $D - k + 1$ polynomials from $\mathcal{P}$ intersect and that $\mathcal{P}$ is in general position. Moreover, suppose that $Z(\mathcal{P})$ is empty. Our goal is to show that the size of $\mathcal{P}$ is at most $D + k^2 - 2k$.

Let $\mathcal{R}$ be an arbitrary subset of $\mathcal{P}$ of size $D - k$. Note that $\mathcal{R}$ is linearly independent by the general position of $\mathcal{P}$. We set $X = Z(\mathcal{R})$, which is nonempty by the assumption. We denote $\overline{\mathcal{R}} = \mathcal{P} \setminus \mathcal{R}$ and $X = \{x_1, \ldots, x_\ell\}$; note that $\ell \leq k$ by the k-points property of $\mathbb{P}$. Finally, let $\mathcal{Q}_i$ for $i \in \{1, \ldots, \ell\}$ be the subset of $\overline{\mathcal{R}}$ containing the polynomials whose zero set contains x_i.

We claim that the size of $\mathcal{Q}_i$ is at most $k-1$ for every $i \in \{1, \ldots, \ell\}$. Suppose for a contradiction that $|\mathcal{Q}_j| \geq k$, then the size of $\mathcal{R} \cup \mathcal{Q}_j$ is at least D. Therefore $\mathcal{R} \cup \mathcal{Q}_j$ contains a basis of the linear hull of $\mathcal{P}$ by the assumption on general position. But $x_j \in Z(\mathcal{R} \cup \mathcal{Q}_j)$, which contradicts $Z(\mathcal{P}) = \emptyset$.

We obtain that each point x_i is contained in the zero set of at most $k-1$ polynomials from $\overline{\mathcal{R}}$. Conversely, each polynomial $q \in \overline{\mathcal{R}}$ has to contain at least one of $x_1, \ldots, x_\ell$, as $\mathcal{R} \cup \{q\}$ is a set of $D - k + 1$ polynomials from $\mathcal{P}$ and its zero set is nonempty by the assumption. This yields

$$|\mathcal{P}| = |\mathcal{R}| + |\overline{\mathcal{R}}| \leq D - k + (k-1)k = D + k^2 - 2k.$$

□

Let us now present a construction showing that the assumption on linear independence of polynomials from $\mathcal{P}$ in Theorem 16 is indeed necessary.

Construction 2. We construct a set of polynomials $\mathcal{P} \subset \mathbb{R}_3[x_1, x_2]$ that satisfies the following:

(a) $Z(\mathcal{P}) = \emptyset$

(b) Every 2 polynomials in $\mathcal{P}$ are linearly independent.

(c) The zero set of every $\mathcal{Q} \subset \mathcal{P}$ of size at most $D_{2,3} - 2 = 8$ is nonempty.

(d) $\mathcal{P}$ is infinite.

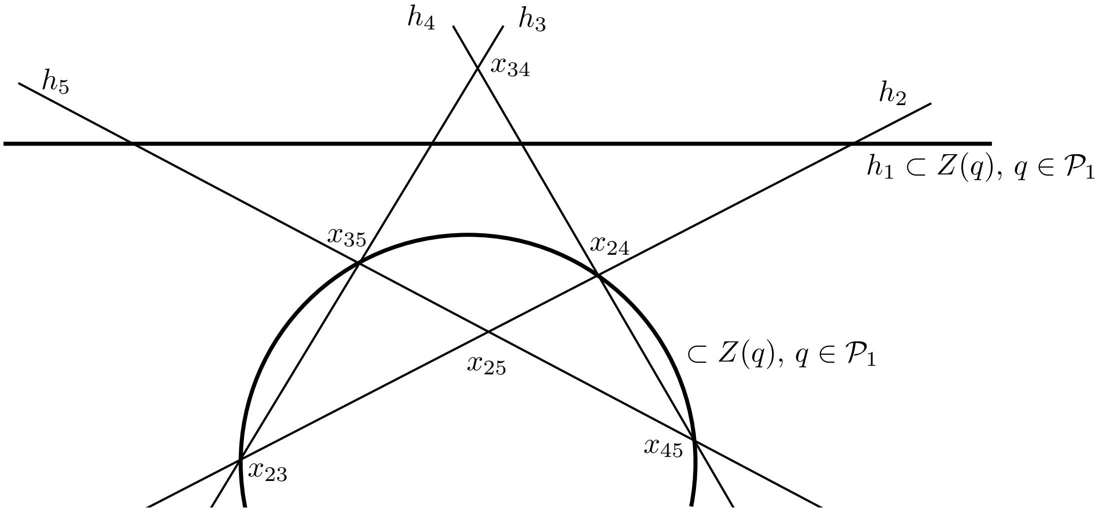

Figure 2.1: An illustration of Construction 2. Zero sets of polynomials from $\mathcal{P}_0$ are unions of three of the lines $h_1, \ldots, h_5$. The zero set of a polynomial from $\mathcal{P}_1$ is drawn bold and consists of the line h_1 and a curve intersecting the points $x_{23}, x_{35}, x_{24}, x_{45}$.

This constitutes a counterexample to a version of Theorem 16 with the assumption that every k polynomials from $\mathcal{P}$ are linearly independent relaxed to

$k-1$. In particular $k=3$ and $d=2$ and $\mathbb{P}=\mathbb{R}_3[x_1,x_2]$ here, but the construction can be generalized.

Let h_1,h_2,h_3,h_4,h_5 be lines in general position, meaning that every two intersect in a point and no three intersect. Choose $p_1,\dots,p_5 \in \mathbb{R}_1[x_1,x_2]$ so that p_i defines h_i, i.e., $Z(p_i)=h_i$. Let $x_{ij} \in \mathbb{R}^2$ denote the intersection point of h_i and h_j. We now define

$$\mathcal{P}_0 = \{p_{ijk} = p_i \cdot p_j \cdot p_k \mid i,j,k \in \{1,\dots,5\},\ i<j<k\}.$$

Clearly, $|\mathcal{P}_0| = \binom{5}{3} = 10$ and $Z(\mathcal{P}_0)=\emptyset$, as the zero set of each polynomial from $\mathcal{P}_0$ intersects all of the points x_{ij} except for one. For instance, the point x_{12} is not contained in $Z(p_{345})$. Furthermore, removing any polynomial from $\mathcal{P}_0$, say p_{123}, will result in a nonempty zero set: $x_{45} \in Z(p_{ijk})$ for every $\{i,j,k\} \neq \{1,2,3\}$. Therefore the zero set of every subset $\mathcal{Q} \subset \mathcal{P}_0$ of size at most $D_{2,3}-1 = 9$ is nonempty. Also, note that the polynomials from $\mathcal{P}_0$ are pairwise linearly independent, since they have different zero sets.

It remains to add infinitely many polynomials and preserve the property that the zero set of any eight polynomials is nonempty. We define

$$\mathcal{P}_1 = \{\alpha p_{125} + (\alpha+1)p_{134} \mid 0 \neq \alpha \in \mathbb{R}^+\},$$

and set $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ (see Fig. 2.1). Observe that for all $q \in \mathcal{P}_1$ we have $x_{ij} \in Z(q)$ for all i,j except for x_{34} and x_{25}. Our goal is to show that $Z(\mathcal{Q})$ is nonempty for every $\mathcal{Q} \subset \mathcal{P}$ of size 8. Note that

$$Z(\mathcal{P}_0 \setminus \{p_{125}, p_{134}\}) = \{x_{34}, x_{25}\},$$

which are the only points the zero sets of polynomials from $\mathcal{P}_1$ avoid. A point x_{ij} different from x_{34} and x_{25} will therefore appear in the zero set of any seven polynomials from $\mathcal{P}_0$. This point is also contained in $Z(q)$ for all $q \in \mathcal{P}_1$. Hence the zero set of $\mathcal{Q}$ is nonempty for every $\mathcal{Q} \subset \mathcal{P}$ of size at most 8. It is clear that no two polynomials in $\mathcal{P}$ are linearly dependent. Therefore $\mathcal{P}$ indeed satisfies all the assumptions, but $Z(\mathcal{P})=\emptyset$ and it is infinite.

2.5 The piercing number

As we have seen above, the assumption on linear independence of polynomials is necessary for bounds on the Helly number better than D. Indeed, there exist collections of polynomials satisfying the property that the zero sets of every $D-k+1$ polynomials intersect and yet the family of all zero sets has an empty intersection. However, we can still show some meaningful observations about the structure of these collections. In particular, there exists a set of few points such that the zero set of every polynomial contains at least one of them.

The *piercing number* $P(\mathcal{P})$ of a set of polynomials $\mathcal{P} \subseteq \mathbb{R}[x_1,\dots,x_d]$ is defined as the minimum size of a set of points $Y \subset \mathbb{R}^d$ such that $Z(p)$ contains at least one point from Y for every $p \in \mathcal{P}$.

The theorems above conclude that $Z(\mathcal{P}) \neq \emptyset$ for appropriate sets $\mathcal{P}$, which means the piercing number $P(\mathcal{P})$ is exactly 1. We now omit the assumption on linear independence from Theorem 16, at the cost of weaker conclusion.

Theorem 18. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the k-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of polynomials and suppose that $Z(\mathcal{Q})$ is nonempty for every $\mathcal{Q} \subseteq \mathcal{P}$ of size at most $D-k+1$. Then the piercing number of $\mathcal{P}$ is at most k.*

Proof. Let $\mathcal{P}$ be given and the assumptions satisfied. Let $\mathcal{Q} \subseteq \mathcal{P}$ be a basis of the linear hull of $\mathcal{P}$. If $|\mathcal{Q}| \leq D-k+1$, we obtain $Z(\mathcal{P}) = Z(\mathcal{Q}) \neq \emptyset$ directly from the assumption and therefore $P(\mathcal{P}) = 1$.

Otherwise, we define $\mathcal{R} \subseteq \mathcal{Q}$ to be a subset of size $D-k$, then $|Z(\mathcal{R})| \leq k$ by the k-points-property of $\mathbb{P}$. Also, for any $p \in \mathcal{P}$ we have $Z(\mathcal{R}) \cap Z(p) \neq \emptyset$ by the assumption. Therefore the zero set of any polynomial in $\mathcal{P}$ contains at least one point from $Z(\mathcal{R})$. We obtain

$$P(\mathcal{P}) \leq |Z(\mathcal{R})| \leq k.$$

□

We can now use the bound on the piercing number to derive that there exists a large subset with nonempty zero set. This result is in the spirit of the fractional Helly theorem.

Corollary 19. *Let $\mathbb{P} \subseteq \mathbb{R}[x_1, \ldots, x_d]$ be a D-dimensional space of polynomials with the k-points-property. Let $\mathcal{P} \subseteq \mathbb{P}$ be a set of polynomials and suppose that $Z(\mathcal{Q})$ is nonempty for every $\mathcal{Q} \subseteq \mathcal{P}$ of size at most $D-k+1$. Then there exists a subset $\mathcal{R} \subseteq \mathcal{P}$ of size at least $\frac{1}{k}|\mathcal{P}|$ such that the zero set of $\mathcal{R}$ is nonempty.*

Open problems and further directions

In Section 2.3, we introduced the k-points-property, a monotone property of spaces of polynomials motivated by an observation about families of spheres. It is an open problem to characterize spaces of polynomials with the k-points-property. We also ask how to determine the least dimension of a space with the k-points-property containing a given set of polynomials. It seems that techniques from algebraic geometry will be needed for satisfactory answers to these problems.

In Theorem 16, we provided a general bound on the Helly number of families of hypersurfaces that are sufficiently big and satisfy strong assumptions on linear independence. As discussed before, the bound on the least size $N(k, D)$ is nowhere close to tight. A careful analysis is likely to improve the bound considerably. We know that $N(2, D) = D + 1$ and conjecture the following.

Conjecture 1. *The number $N(k, D)$ grows linearly in $D + k$.*

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