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DIPLOMOVÁ PRÁCE

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Greenovy funkce proudů v anomálním sektoru kvantové chromodynamiky

Ústav částicové a jaderné fyziky

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Green functions of currents in the odd-intrinsic parity sector of QCD

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To my parents

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague July 31, 2015 Tomáš Kadavý

Název práce: Greenovy funkce proudů v anomálním sektoru kvantové chromodynamiky

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Abstrakt: Greenovy funkce proudů umožňují studium resonancí v nízkoenergetické limitě QCD. Jejich použitím můžeme konstruovat např. amplitudy či rozpadové šířky procesů. Porovnáním teoretických předpovědí s experimentálními měřeními můžeme stanovit hodnoty parametrů teorie a získat tak ucelenější obraz o chování QCD procesů. V námi zkoumaném sektoru liché parity existuje celkem pětice netriviální tří-bodových Greenových funkcí, neboli korelátorů. Korelátory VVP a VAS již byly studovány v [1], zatímco výpočet AAP byl proveden v [2]. V této práci se tedy zabýváme dosud nestudovanými Greenovými funkcemi VVA a AAA přičemž předkládáme kompletní výpočet přispívajících Feynmanových diagramů v antisymetrickém tensorovém formalismu, který splňuje vysokoenergetické chování v rámci OPE, jak bylo rovněž ukázáno v [2]. Získané výsledky poté aplikujeme na studium fenomenologie. Dále, v práci se rovněž zabýváme studiem dvou čtyř-bodových Greenových funkcí, $VVPP$ a $VVVV$, přičemž uvádíme pouze popis našich výpočtů, které, vzhledem k jejich komplikované tensorové struktuře, nemohou být v této práci obsaženy ve své kompletní formě. Na závěr pak uvádíme popis kódu 'Mercury', který jsme napsali za účelem zjednodušení složitých výpočtů. Jeho funkčnost demonstrujeme na výpočtech mnoha typů vertexů pro různé Lagrangiány.

Klíčová slova: Kvantová chromodynamika, Greenovy funkce proudů, chirální poruchová teorie, resonanční chirální teorie.

Title: Green functions of currents in the odd-intrinsic parity sector of QCD

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Abstract: Green functions of currents allow us to study resonances in the low-energy limit of QCD. Using them, we can construct amplitudes or decay widths of processes. By a comparison of the theoretical predictions with experimental measurements we can determine the values of parameters of the theory and obtain a more comprehensive understanding about the behaviour of QCD processes. In our investigated topic, an odd-parity sector of QCD, exist five nontrivial three-point Green functions, also referred to as correlators. Correlators VVP and VAS have already been studied in [1], whilst a calculation of AAP was provided in [2]. In this thesis we therefore deal with the Green functions VVA and AAA that have not yet been studied before. We present a complete calculation of the contributing Feynman diagrams in the antisymmetric tensor formalism that satisfies high-energy behaviour within the OPE framework, as was shown also in [2]. The obtained results are submitted to phenomenological studies. Further, we also present an introduction to our calculations of the four-point Green functions $VVPP$ and $VVVV$. The calculations were carried out both in the antisymmetric tensor and vector formalism but due to the complicated tensor structure of the results, the calculations can not be shown here in their complete form. At the end we also give a description of the 'Mercury' code that we wrote to be able to simplify our calculations as much as possible. We demonstrate the functionality of the code on calculations of many types of vertices with various contributing Lagrangians.

Keywords: Quantum chromodynamics, Green functions of currents, chiral perturbation theory, resonance chiral theory.

Contents

Introduction

"Per aspera ad astra."

– Latin proverb

The Standard model is a very succesful theory that describes interactions of elementary particles. Over many years it has faultlessly passed many experiments. For instance, let us remember the discovery of the Higgs boson about three years ago. This finding ended the search for particles contained in the model.

Although the Standard model seems to work very accurately, it is still obvious that it is not a final theory but rather a low-energy approximation of a more general theory because we still encounter problems that are not solved within the Standard model.

One of the parts of the Standard model is Quantum chromodynamics that describes strong interactions between quarks and hadrons. However, at low energies, the quarks and gluons are confined into hadrons - mesons and baryons. Therefore, it is obvious that we need a different approach to describe the interactions at low energy. One of the possible techniques includes so called Chiral perturbation theory and, for rather higher energies, Resonance chiral theory. These methods are the cornerstones of our work that rests in a low-energy region of QCD and we will devote detailed attention to them in the following chapters.

Motivation

A general motivation of this thesis is to contribute to the systematic study of odd-intrinsic parity sector of QCD by calculations of contributions of resonances into three-point and four-point Green functions. Using this approach, it is possible to obtain parameters that are suitable for being verified by experiments.

This paper is supposed to be an extension of an article [1] and a bachelor thesis [2], where VVP, VAS and AAP correlators were studied. Following this journey, we present here calculations of new Green functions that have not yet been studied extensively in the literature before.

Outline

This thesis consists of the following. In the first chapter 1 we present a basic introduction into the quantum chromodynamics and cover various topics, from a construction of the general QCD Lagrangian to generating functional to symmetry breaking.

In the second chapter 2, we pay our attention to the low-energy region of QCD and its description using chiral perturbation theory and resonance chiral theory. We also deal with a description of the resonance fields using different formalisms and study an anomalous WZW Lagrangian in detail.

The third chapter 3 deals with a general description of the Green functions of currents. We also present a discussion regarding the topology of the correlators and revise the Green functions that have already been studied in the past.

The fourth and fifth chapters 4-5 contain original calculations of the three-point Green functions VVA and AAA in the antisymmetric tensor formalism. We present the full procedure, from a determination of all the contributing Lagrangians to the calculation of individual vertices and diagrams. In the case of VVA correlator, we also determine several constraints for some of the coupling constants and calculate the decay of the $f_1(1285)$ meson.

The sixth and seventh chapters 6-7 represent a short introduction to our original calculations of the four-point Green functions $VVPP$ and $VVVV$. Despite their complicated structure, we studied both of these correlators in the vector and antisymmetric tensor formalism. Since the results are very difficult to present in a comprehensible form due to the very extensive tensor structure, we just mention their basic properties and present some of the phenomenological applications that we will pay our attention to in future studies.

This thesis also includes a set of appendices, where we pursue an effort to give a deeper description of the topics that could not be mentioned in detail in the main chapters. Appendix A deals with mathematical aspects of the theory we used, whilst Appendix B contains a general expansion of the chiral operators in its individual terms. In Appendix C we present a formalism used to describe the spin one particles. Appendix D presents a survey of the Feynman rules for Lagrangians we used extensively in this thesis and also for the ones that could be useful in future studies. We deal with various Lagrangians, throughout the vertices that come both from χPT and RχT. The results in this appendix were obtained by our algorithm, many of them have not been studied before. Finally, in Appendix E, we give a complete description of the original algorithm 'Mercury' in FeynCalc, that we wrote and used for our calculations.

For clarity in the description of this topic, the reader is recommended to browse this thesis in the following order, to get a complete understanding of the studied issues:

1. chapters 1, 2, 3.
2. appendices A, B, C, D.
3. chapters 4, 5, 6, 7.
4. appendix E.

Convention

Throughout this thesis, we use the "West coast convention" for the metric tensor,

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+,-,-,-)\,,$$

and for the Levi-Civita tensor we take

$$\varepsilon_{0123} = 1\,.$$

1. Quantum chromodynamics

Quantum chromodynamics (QCD) is a non-abelian gauge theory of the strong interactions, based on the color SU(3) as the underlying gauge group, with the fundamental degrees of freedom, gluons and quarks which are asymptotically free at high energies. However, at low energies the quarks and gluons are coupled into hadrons. Therefore, since quarks have not been observed as asymptotically free states, the meaning of quark masses and their numerical values are tightly connected with the method by which they are extracted from hadronic properties.

QCD is a very important part of the Standard model and a lot of experimental evidences benefit for its excellent validity. Let us start this paper with a basic introduction to the mathematical background of QCD so we can build our calculations on these principles afterwards.

1.1 QCD Lagrangian

We present the quark colour triplet as the basic building block [3], [4], [5]

$$q_f = \begin{pmatrix} q_f^r \\ q_f^g \\ q_f^b \end{pmatrix} , \tag{1.1}$$

where f stands for the flavour of the quark. The triplet transforms as

$$q_f \to U(x) q_f \, , \tag{1.2}$$

where $U(x)$ is an element of SU(3) group that we can write in the form

$$U(x) = \exp\left(-i \sum_{a=1}^{8} \theta_a(x) \frac{\lambda_a}{2} \right) , \tag{1.3}$$

where θ_a is one of the eight parameters $\theta = (\theta_1, \ldots \theta_8)$ that describes the gauge transformation above and λ^a represents Gell-Mann matrices.

The SU(3) invariant Lagrangian can be written in the form [3], [4], [5]

$$\mathcal{L}_q = \sum_f \overline{q}_f (i\gamma^\mu \nabla_\mu - m_f) q_f \, , \tag{1.4}$$

where ∇_μ is the covariant derivative and

$$\nabla_\mu g_f = \partial_\mu q_f - iq \mathcal{A}_\mu(x) q_f \, , \tag{1.5}$$

where

$$\mathcal{A}_\mu(x) = \sum_{a=1}^{8} \frac{\lambda^a}{2} \mathcal{A}_\mu^a(x) \tag{1.6}$$

is the octet of SU(3) gauge fields. The octet transforms as

$$\mathcal{A}_\mu(x) \to U(x) \mathcal{A}_\mu(x) U^\dagger(x) - \frac{i}{g} \partial_\mu U(x) U^\dagger(x) \, . \tag{1.7}$$

As an invariant object made out of gluon fields we present nonabelian stress tensor [3], [4]

$$\mathcal{G}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \tag{1.8}$$

that transforms as

$$\mathcal{G}_{\mu\nu} \to U(x)\mathcal{G}_{\mu\nu}U^{\dagger}(x)\,. \tag{1.9}$$

The only nontrivial scalar, considering dimensions equal to or less than 4, made out of (1.4) and (1.8) is the contraction of the two stress tensors. Thus we have the complete QCD Lagrangian [3], [4]

$$\mathcal{L}_{QCD} = \sum_f \overline{q}_f(i\gamma^\mu\nabla_\mu - m_f)q_f - \frac{1}{4}\sum_{a=1}^{8}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{\mu\nu,a}\,. \tag{1.10}$$

There are six quarks in the flavour sector. With respect to their masses we can divide them into two parts consisting of quarks with masses less or greater than 1 GeV, which is so called hadron scale Λ_H. Schematically, we have

$$m_u, m_d, m_s \ll 1\,\mathrm{GeV} < m_c, m_b, m_t\,. \tag{1.11}$$

In the low-energy region only the first three quarks are necessary to be taken into account. The approximation, based on the massless quarks, is called chiral limit. In this case (1.10) has the form [3], [4]

$$\mathcal{L}^0_{QCD} = \sum_{f=u,d,s} \overline{q}_f i\gamma^\mu\nabla_\mu g_f - \frac{1}{4}\sum_{a=1}^{8}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{\mu\nu,a}\,. \tag{1.12}$$

Massless QCD Lagrangian (1.12) is invariant under SU(3) and even possesses U(3) symmetry.

In order to exhibit the global symmetries of (1.12), we consider the chirality matrix γ_5, also known as the fifth Dirac matrix, and projection operators

$$P_L = \frac{1}{2}(1+\gamma_5)\,, \qquad P_R = \frac{1}{2}(1-\gamma_5)\,, \tag{1.13}$$

These operators satisfy their expected properties, such as idempotent

$$P_L^2 = P_L\,, \qquad P_R^2 = P_R\,, \tag{1.14}$$

orthogonality

$$P_L P_R = 0\,, \qquad P_R P_L = 0 \tag{1.15}$$

and completeness

$$P_L + P_R = 1\,. \tag{1.16}$$

The properties (1.14)-(1.16) guarantee that P_L and P_R project from the quark field q to its chiral components q_L and q_R,

$$q_L = P_L q, \qquad q_R = P_R q\,, \tag{1.17}$$

where

$$q = q_L + q_R\,. \tag{1.18}$$

With the respect to the properties of Dirac matrices (see Appendix A), we can rewrite the massless QCD Lagrangian (1.12)

$$\mathcal{L}^0_{QCD} = \sum_{f=u,d,s}(\overline{q}_{R,f}i\gamma^\mu\nabla_\mu q_{R,f} + \overline{q}_{L,f}i\gamma^\mu\nabla_\mu q_{L,f}) - \frac{1}{4}\sum_{a=1}^{8}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{\mu\nu,a}\,. \tag{1.19}$$

Now we can see that (1.12) is also invariant under the independent transformations of chiral components q_L and q_R,

$$q_L \to U_L q_L \,, \qquad q_R \to U_R q_R \,, \tag{1.20}$$

where U_L and U_R are unitary 3×3 matrices. We say that (1.19) has a classical global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry.

The result of Noether's theorem is that there are 18 conserved currents associated with the mentioned transformations of left-handed and right-handed quarks. The SU(3) currents ($a = 1 \ldots 8$) are defined as

$$L^{\mu,a} = \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L \,, \tag{1.21}$$

$$R^{\mu,a} = \bar{q}_R \gamma^\mu \frac{\lambda^a}{2} q_R \,, \tag{1.22}$$

with

$$\partial_\mu L^{\mu,a} = 0 \,, \tag{1.23}$$

$$\partial_\mu R^{\mu,a} = 0 \,, \tag{1.24}$$

and the U(1) singlet currents ($a = 0$) are

$$V^\mu = \bar{q}\gamma^\mu q \,, \tag{1.25}$$

$$A^\mu = \bar{q}\gamma^\mu \gamma_5 q \,. \tag{1.26}$$

The singlet currents are conserved on the classical level, however, after the quantization, the axial current is not conserved anymore. Also, the Lagrangian $\mathcal{L}^0_{QCD}$ is invariant under the local group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ on the quantum level.

Instead of (1.21) and (1.22) it is more useful to take into account their linear combinations

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q \,, \tag{1.27}$$

$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q \,, \tag{1.28}$$

with the parity transformations

$$V^{\mu,a}(\mathbf{x},t) \to V^a_\mu(-\mathbf{x},t) \,, \tag{1.29}$$

$$A^{\mu,a}(\mathbf{x},t) \to -A^a_\mu(-\mathbf{x},t) \,. \tag{1.30}$$

In addition to the vector (1.27) and axial-vector (1.28) currents, it is also appropriate to define the SU(3) densities, specifically the scalar density

$$S^a = \bar{q}\frac{\lambda^a}{2} q \,, \tag{1.31}$$

and the pseudoscalar density

$$P^a = i\bar{q}\gamma_5 \frac{\lambda^a}{2} q \,. \tag{1.32}$$

The U(1) singlets are defined as

$$S = \bar{q}q \,, \tag{1.33}$$

$$P = i\bar{q}\gamma_5 q \,. \tag{1.34}$$

The densities transform under parity transformations [3]

$$S^a(\mathbf{x},t) \to S^a(-\mathbf{x},t)\,, \tag{1.35}$$
$$P^a(\mathbf{x},t) \to -P^a(-\mathbf{x},t)\,. \tag{1.36}$$

We also define the external fields

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda^a}{2} v_a^\mu\,, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda^a}{2} a_a^\mu\,, \quad s = \sum_{a=1}^{8} \frac{\lambda^a}{2} s_a\,, \quad p = \sum_{a=1}^{8} \frac{\lambda^a}{2} p_a\,, \tag{1.37}$$

represented in the flavour sector by Hermitian 3×3 matrices. By coupling vector and axial-vector currents to external fields we can get

$$\mathcal{L} = \mathcal{L}^0_{QCD} + \mathcal{L}_{\text{ext}} \tag{1.38}$$
$$= \mathcal{L}^0_{QCD} + \overline{q}\gamma_\mu\left(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu\right)q - \overline{q}(s - i\gamma_5 p)q\,. \tag{1.39}$$

Here we added the single vector current $v^\mu_{(s)}$ together with other external fields, however we omitted the axial-vector single current $a^\mu_{(s)}$ which has an anomaly. It is easy to notice that we can recover the ordinary three flavor QCD Lagrangian by setting

$$v^\mu = v^\mu_{(s)} = p = 0 \tag{1.40}$$

and

$$s = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{1.41}$$

in (1.39). The diagonal matrix (1.41) is usually called the mass matrix, denoted as $\mathcal{M}$, i.e. $s = \mathcal{M}$ is required in this case to recover the ordinary QCD Lagrangian.

If we define vector and axial-vector currents in the form

$$v^\mu = \frac{1}{2}(r^\mu + l^\mu)\,, \qquad a^\mu = \frac{1}{2}(r^\mu - l^\mu)\,, \tag{1.42}$$

Lagrangian (1.39) now has the form

$$\begin{aligned}\mathcal{L} =& \mathcal{L}^0_{QCD} + \overline{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_L + \overline{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_R \\ &- \overline{q}_R(s+ip)q_L - \overline{q}_L(s-ip)q_R\,.\end{aligned} \tag{1.43}$$

1.2 Green functions of currents

The amplitudes of physical processes can be calculated using LSZ reduction formula from the Green functions, the vacuum expectation values of the time ordered products of the quantum fields:

$$\begin{aligned}&\langle 0|\mathrm{T}\big[\widetilde{\mathcal{O}}_1(p_1)\ldots\widetilde{\mathcal{O}}_n(0)\big]|0\rangle = \\ &= \int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_{n-1}\, e^{i(p_1x_1+\ldots+p_{n-1}x_{n-1})}\langle 0|\mathrm{T}\big[\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(0)\big]|0\rangle\,.\end{aligned} \tag{1.44}$$

In our case, the operators $\mathcal{O}_i(x)$ stand for any of the currents (1.27), (1.28) or densities (1.31), (1.32). Since we have a set of four possible operators which we choose from, considering a general n-point Green function, we have a total number of

$$C'_n(4) = \binom{n+3}{n} \tag{1.45}$$

different Green functions for the given n because the choosing of the operators is basically a combination with repetitions.

Easily, one can find that we have 20 possible three-point, 35 four-point Green functions etc. Of course, a lot of them are trivially zero. For example, there are only five nontrivial three-point Green functions: VVP, VAS, AAP, VVA and AAA. Moreover, individual Green functions are connected with other ones due to the chiral Ward identities.

1.3 Chiral Ward identities

The Green functions are connected through the ward identities that reflect the symmetry properties of a given theory on the quantum level. The knowledge of the identities allows us to determine the structure of the Green functions and other important properties.

The divergences of chiral Green functions (1.44) correspond to the linear combinations of other Green functions. These relations are called chiral Ward identities, explicitly

$$\begin{aligned}\partial^x_\mu\langle 0|\mathrm{T}\big[J^\mu(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big]|0\rangle &= \langle 0|\mathrm{T}\big[\big(\partial^x_\mu J^\mu(x)\big)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big]|0\rangle \\ &+\sum_{i=1}^n \delta(x^0-x^i)\langle 0|\mathrm{T}\big[\mathcal{O}_1(x_1)\ldots[J_0(x),\mathcal{O}_i(x_i)]\ldots\mathcal{O}_n(x_n)\big]|0\rangle\,, \end{aligned} \tag{1.46}$$

where $J^\mu(x)$ stands for any of the Noether currents. Notice that here we let the divergence act on the right side of (1.44). Integration out over coordinates, which eliminates the divergence, we obtain chiral Ward identites for the Green functions in the impulse representation, i.e. in the form of the left side of (1.44).

To evaluate chiral Ward identities it is necessary to know the equal-time commutation relations among currents V, A and densities S, P [3], [4].

$$[V^a_0(t,\boldsymbol{x}), V^b_\mu(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}V^c_\mu(t,\boldsymbol{x})\,, \tag{1.47}$$
$$[V^a_0(t,\boldsymbol{x}), V_\mu(t,\boldsymbol{y})] = 0\,, \tag{1.48}$$
$$[V^a_0(t,\boldsymbol{x}), A^b_\mu(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}A^c_\mu(t,\boldsymbol{x})\,, \tag{1.49}$$
$$[V^a_0(t,\boldsymbol{x}), S^b(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}S^c(t,\boldsymbol{x})\,, \tag{1.50}$$
$$[V^a_0(t,\boldsymbol{x}), S^0(t,\boldsymbol{y})] = 0\,, \tag{1.51}$$
$$[V^a_0(t,\boldsymbol{x}), P^b(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}P^c(t,\boldsymbol{x})\,, \tag{1.52}$$
$$[V^a_0(t,\boldsymbol{x}), P^0(t,\boldsymbol{y})] = 0\,, \tag{1.53}$$
$$[A^a_0(t,\boldsymbol{x}), V^b_\mu(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}A^c_\mu(t,\boldsymbol{x})\,, \tag{1.54}$$
$$[A^a_0(t,\boldsymbol{x}), V_\mu(t,\boldsymbol{y})] = 0\,, \tag{1.55}$$
$$[A^a_0(t,\boldsymbol{x}), A^b_\mu(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})f^{abc}V^c_\mu(t,\boldsymbol{x})\,, \tag{1.56}$$
$$[A^a_0(t,\boldsymbol{x}), S^b(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\left[d^{abc}P^c(t,\boldsymbol{x}) + \sqrt{\frac{2}{3}}\delta^{ab}P^0(t,\boldsymbol{x})\right], \tag{1.57}$$
$$[A^a_0(t,\boldsymbol{x}), S^0(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\sqrt{\frac{2}{3}}P^a(t,\boldsymbol{x})\,, \tag{1.58}$$
$$[A^a_0(t,\boldsymbol{x}), P^b(t,\boldsymbol{y})] = -i\delta^3(\boldsymbol{x}-\boldsymbol{y})\left[d^{abc}S^c(t,\boldsymbol{x}) + \sqrt{\frac{2}{3}}\delta^{ab}S^0(t,\boldsymbol{x})\right], \tag{1.59}$$
$$[A^a_0(t,\boldsymbol{x}), P^0(t,\boldsymbol{y})] = -i\delta^3(\boldsymbol{x}-\boldsymbol{y})\sqrt{\frac{2}{3}}S^a(t,\boldsymbol{x})\,. \tag{1.60}$$

1.4 Divergences of currents

The divergences of the currents (1.25), (1.26) and (1.27), (1.28) are possible to express depending on the mass matrix, usually denoted as

$$\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)\,. \tag{1.61}$$

The explicit forms of the divergences are as follows [3], [4]:

$$\partial_\mu V^{\mu,a} = i\bar{q}\left[\mathcal{M}, \frac{\lambda^a}{2}\right]q\,, \tag{1.62}$$

$$\partial_\mu A^{\mu,a} = i\bar{q}\gamma_5\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\}q\,, \tag{1.63}$$

$$\partial_\mu V^\mu = 0\,, \tag{1.64}$$

$$\partial_\mu A^\mu = 2i\bar{q}\mathcal{M}\gamma_5 q + \frac{3g^2}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{G}^{\mu\nu,a}\mathcal{G}^{\rho\sigma,a}\,. \tag{1.65}$$

The last term is a well known anomaly due to the violation of the axial-vector current conservation.

For the purpose of an evaluation of the chiral Ward identities, it will be convenient to express the divergences of the currents (1.62) and (1.63) in a more suitable way. Let us start with the mass matrix. It is very easy to express it as a linear combination of the Gell-Mann matrices, such as

$$\mathcal{M} = \frac{m_u + m_d + m_s}{\sqrt{6}}\lambda^0 + \frac{m_u - m_d}{2}\lambda^3 + \frac{\frac{1}{2}(m_u + m_d) - m_s}{\sqrt{3}}\lambda^8\,. \tag{1.66}$$

Using easy matrix manipulations, explained in detail in Appendix A, one can arrive at

$$\left[\mathcal{M}, \frac{\lambda^a}{2}\right] = \left[i(m_u - m_d)f^{3ab} + \frac{i}{\sqrt{3}}(m_u + m_d - 2m_s)f^{8ab}\right]\frac{\lambda^b}{2}\,, \tag{1.67}$$

$$\begin{aligned}\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\} = &\left[\frac{\sqrt{2}}{3}(m_u + m_d - 2m_s)\delta^{8a} + \sqrt{\frac{2}{3}}(m_u - m_d)\delta^{3a}\right]\frac{\lambda^0}{2} \\ &+ \left[\frac{1}{\sqrt{3}}(m_u + m_d - 2m_s)d^{8ab} + (m_u - m_d)d^{3ab}\right]\frac{\lambda^b}{2} \\ &+ \frac{2}{3}(m_u + m_d + m_s)\frac{\lambda^a}{2}\,.\end{aligned} \tag{1.68}$$

It might be helpful to notice the possible non-vanishing structure constants in the previous expressions. Easily, one can find that the totally symmetric tensor d^{8ab} in (1.68) has the non-zero contributions only for $a = b$. Considering this fact, we can rewrite (1.68) in the form

$$\begin{aligned}\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\} = &\left[\frac{\sqrt{2}}{3}(m_u + m_d - 2m_s)\delta^{8a} + \sqrt{\frac{2}{3}}(m_u - m_d)\delta^{3a}\right]\frac{\lambda^0}{2} \\ &+ \left[\frac{2}{3}(m_u + m_d + m_s) + \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s)d^{8aa}\right]\frac{\lambda^a}{2} \\ &+ (m_u - m_d)d^{3ab}\frac{\lambda^b}{2}\,.\end{aligned} \tag{1.69}$$

Knowing the expressions (1.67) and (1.69) above, we can finally find the relations for divergences (1.62) and (1.63) in the form

$$\partial_\mu V^{\mu,a} = \alpha^{ab}S^b\,, \tag{1.70}$$

$$\partial_\mu A^{\mu,a} = \beta_1^a P^0 + \beta_2 P^a + \beta_3^{ab}P^b\,, \tag{1.71}$$

where we denoted

$$\alpha^{ab} = -(m_u - m_d) f^{3ab} - \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s) f^{8ab}\,, \tag{1.72}$$

$$\beta_1^a = \frac{\sqrt{2}}{3}(m_u + m_d - 2m_s)\delta^{8a} + \sqrt{\frac{2}{3}}(m_u - m_d)\delta^{3a}\,, \tag{1.73}$$

$$\beta_2 = \frac{2}{3}(m_u + m_d + m_s) + \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s) d^{8aa}\,, \tag{1.74}$$

$$\beta_3^{ab} = (m_u - m_d) d^{3ab}\,. \tag{1.75}$$

1.5 Generating functional and the anomaly

The Green functions associated with the vector, axial-vector, scalar and pseudoscalar currents are generated by the functional [6]

$$\exp\big(iZ[v,a,s,p,\theta]\big) = \langle 0_{\text{out}}|0_{\text{in}}\rangle_{v,a,s,p,\theta}\,, \tag{1.76}$$

where $\langle 0_{\text{out}}|0_{\text{in}}\rangle$ is the vacuum-to-vacuum transition amplitude in the presence of external fields, determined by the Lagrangian (1.39) with the included operator in the form

$$-\frac{1}{16\pi^2}\theta(x)\langle G_{\mu\nu}\widehat{G}^{\mu\nu}\rangle\,, \tag{1.77}$$

where

$$G_{\mu\nu} = i[\nabla_\mu, \nabla_\nu] \tag{1.78}$$

is the gluon field strength (1.8) with the covariant derivative (1.5) and $\theta(x)$ is the vacuum angle.

The meaning behind the vacuum-to-vacuum transition amplitude is the following. Let us consider that, in the remote past, the system was in the ground state and consider the evolution in the presence of the external fields. Then, the vacuum-to-vacuum transition amplitude represents the probability amplitude for the system to wind up in the ground state when $x^0 \to \infty$ [6].

It is convenient to collect the currents into the universal definition of the generating functional [6],

$$\begin{aligned}\exp\big(iZ[v,a,s,p,\theta]\big) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n\, f^{i_1}_{\mu_1}(x_1)\ldots f^{i_n}_{\mu_n}(x_n)\times \\ \times\big\langle 0\big|\mathrm{T}\big[J^{\mu_1}_{i_1}(x_1)\ldots J^{\mu_n}_{i_n}(x_n)\big]\big|0\big\rangle\,,\end{aligned} \tag{1.79}$$

where $f^i_\mu(x)$ is a set of external fields. In our case of the Lagrangian (1.39), we can write the generating functional in the form

$$\exp\big(iZ[v,a,s,p]\big) = \big\langle 0\big|\mathrm{T}\bigg[\exp\bigg(i\int \mathrm{d}^4x\,\mathcal{L}_{\text{ext}}(x)\bigg)\bigg]\big|0\big\rangle\,. \tag{1.80}$$

By expanding the generating functional around $v_\mu = a_\mu = s = p = 0$, $\theta(x) = \theta_0$ one can obtain the Green functions of QCD with massless u, d, s quarks. However, the Green functions of the real world, i.e. with the mass quarks, are obtained by expanding the functional around $v_\mu = a_\mu = p = 0$, $s = \mathcal{M}$ and $\theta(x) = \theta_0$. In other words, the n-point Green functions are obtained by variation with respect to corresponding external sources, for example (we consciously do not consider $\theta(x)$ anymore)

$$\big\langle 0\big|\mathrm{T}\big[A^a_\mu(x)A^b_\nu(0)\big]\big|0\big\rangle = (-i)^2\frac{\delta^2}{\delta a^\mu_a(x)\delta a^\nu_b(0)}\exp\big(iZ[v,a,s,p]\big)\bigg|_{\substack{v=a=p=0\\ s=\mathcal{M}}}\,. \tag{1.81}$$

Formally, the vacuum-to-vacuum amplitude is invariant with the respect to local $\mathrm{U}(3)\times\mathrm{U}(3)$ transformations

$$q(x)\to V_R(x)\frac{1+\gamma_5}{2}q(x)+V_L(x)\frac{1-\gamma_5}{2}q(x)\,, \tag{1.82}$$

which generate a gauge transformation of the external fields

$$v_\mu+a_\mu\to v'_\mu+a'_\mu=V_R(x)(v_\mu+a_\mu)V_R^\dagger(x)+iV_R(x)\partial_\mu V_R^\dagger(x)\,, \tag{1.83}$$
$$v_\mu-a_\mu\to v'_\mu-a'_\mu=V_L(x)(v_\mu-a_\mu)V_L^\dagger(x)+iV_L(x)\partial_\mu V_L^\dagger(x)\,. \tag{1.84}$$

and

$$s+ip\to s'+ip'=V_R(x)(s+ip)V_L^\dagger(x)\,, \tag{1.85}$$
$$s-ip\to s'-ip'=V_L(x)(s-ip)V_R^\dagger(x)\,. \tag{1.86}$$

For an infinitesimal chiral transformation

$$V_R(x)=1+i\alpha(x)+i\beta(x)+\dots\,, \tag{1.87}$$
$$V_L(x)=1+i\alpha(x)-i\beta(x)+\dots\,, \tag{1.88}$$

where

$$\alpha(x)=\alpha^a(x)\frac{\lambda^a}{2}\,,\qquad \beta(x)=\beta^a(x)\frac{\lambda^a}{2}\,, \tag{1.89}$$

the change in the external fields is given by

$$\delta v_\mu=\partial_\mu\alpha+i[\alpha,v_\mu]+i[\beta,a_\mu]\,, \tag{1.90}$$
$$\delta a_\mu=\partial_\mu\beta+i[\alpha,a_\mu]+i[\beta,v_\mu]\,, \tag{1.91}$$
$$\delta s=i[\alpha,s]-\{\beta,p\}\,, \tag{1.92}$$
$$\delta p=i[\alpha,p]+\{\beta,s\}\,. \tag{1.93}$$

However, the anomalies of the fermion determinant break chiral invariance [12], i.e. the generating functional is not invariant under the transformations (1.83)-(1.86). The change in Z can be schematically given as

$$Z[v',a',s',p']=Z[v,a,s,p]+\delta Z[v,a,s,p,V_LV_R^\dagger] \tag{1.94}$$

where [12]

$$\delta Z=-\frac{N_C}{16\pi^2}\int \mathrm{d}^4x\,\langle\beta(x)\Omega(x)\rangle \tag{1.95}$$

with

$$\Omega(x)=\varepsilon^{\alpha\beta\mu\nu}\bigg(v_{\alpha\beta}v_{\mu\nu}+\frac{4}{3}\nabla_\alpha a_\beta\nabla_\mu a_\nu+\frac{2i}{3}\{v_{\alpha\beta},a_\mu a_\nu\}+\frac{8i}{3}a_\mu v_{\alpha\beta}a_\nu+\frac{4}{3}a_\alpha a_\beta a_\mu a_\nu\bigg)\,, \tag{1.96}$$

where we have defined

$$v_{\alpha\beta}=\partial_\alpha v_\beta-\partial_\beta v_\alpha-i[v_\alpha,v_\beta]\,, \tag{1.97}$$
$$\nabla_\alpha a_\beta=\partial_\alpha a_\beta-i[v_\alpha,a_\beta]\,. \tag{1.98}$$

An assumption to obtain (1.95) explicitly is that one simultaneously transforms the external field $\theta(x)$ as

$$\delta\theta(x)=-2\langle\beta(x)\rangle\,. \tag{1.99}$$

Let us get back to (1.96) and rewrite the expression as

$$\Omega(x)=\varepsilon^{\alpha\beta\mu\nu}\Omega_{\alpha\beta\mu\nu}\,. \tag{1.100}$$

1.6 Symmetry breaking

1.6.1 Explicit symmetry breaking

So far we have not considered the quarks to have masses whatsoever. Taking them into account, the symmetry is explicitly broken due to the presence of the mass term in the Lagrangian (1.43), specifically by the existence of the part

$$\mathcal{L}_M = -\overline{q}_R s\, q_L - \overline{q}_L s\, q_R \tag{1.101}$$
$$= -\overline{q}\mathcal{M}q\,, \tag{1.102}$$

where we denoted mass matrix in its usual form (1.61) which is contained in the scalar field s, cf. (1.41).

Depending on the masses of the quarks in the mass matrix, we can introduce the following special cases [7], [8]:

1. $m_u = m_d = m_s = 0$: the octet vector and axial-vector currents are conserved, the symmetry group is $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$.

2. $m_u = m_d = m_s \neq 0$: vector current is conserved and the Lagrangian is invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$.

3. $m_u = m_d = 0$: the symmetry is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{SV} \times \mathrm{U}(1)_V$, where $\mathrm{U}(1)_{SV}$ represents the conservation of the strangeness.

4. $m_u = m_d \neq 0$: the chiral limit of the lightest quarks with the symmetry $\mathrm{SU}(2)_V \times \mathrm{U}(1)_{SV} \times \mathrm{U}(1)_V$.

For the general values of m_u, m_d, m_s we have no flavor symmetry, except for $\mathrm{U}(1)_V$, which is always present and represents the conservation of the baryon number.

1.6.2 Spontaneous symmetry breaking

The symmetry group of QCD with the massless quarks, $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$, is spontaneously broken to $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ due to the presence of an order parameter in QCD. According to the Goldstone theorem, to each generator, which does not anihilate the vacuum state, there corresponds one massless Goldstone boson. Therefore, an octet of these particles appears in the spectrum of QCD.

The generators of $\mathrm{SU}(3)_V$ symmetry are [3]

$$Q_V^a(t) = \int \mathrm{d}^3x\, V_0^a(x,t) = \int q^\dagger(x,t)\frac{\lambda^a}{2}q(x,t) \tag{1.103}$$

and satisfy the equal-time commutation relations with the $\mathrm{SU}(3)_V$ octet of scalar densities

$$[Q_V^a(t), S^b(y)] = i f^{abc} S^c(y)\,. \tag{1.104}$$

This relation can be used to express the scalar density in the form

$$S^a(y) = -\frac{i}{3} f^{abc}[Q_V^b(t), S^c(y)]\,. \tag{1.105}$$

Since the vacuum is invariant under $\mathrm{SU}(3)_V$, we can write the vacuum expectation value of the scalar density to be zero, i.e.

$$\langle 0|S^a(y)|0\rangle = 0\,. \tag{1.106}$$

Let us consider $a = 3$ and $a = 8$. Then, we get

$$\langle \overline{u}u \rangle - \langle \overline{d}d \rangle = 0\,, \tag{1.107}$$

$$\langle \overline{u}u \rangle + \langle \overline{d}d \rangle - 2\langle \overline{s}s \rangle = 0\,, \tag{1.108}$$

which leads to

$$\langle \overline{u}u \rangle = \langle \overline{d}d \rangle = \langle \overline{s}s \rangle\,. \tag{1.109}$$

Assuming the non-vanishing singlet scalar density, we simply obtain [3]

$$\langle 0|S|0 \rangle = \langle \overline{q}q \rangle = 3\langle \overline{u}u \rangle \neq 0\,. \tag{1.110}$$

For the equal-time commutation relation

$$i[Q_A^a(t), P^a(y)] = d^{aac} S^c(x,t) + \frac{2}{3} S(x,t)\,, \tag{1.111}$$

the vacuum expectation value is easily

$$i\langle 0|[Q_A^a(t), P^a(y)]|0\rangle = \frac{2}{3}\langle \overline{q}q \rangle \neq 0\,. \tag{1.112}$$

In conclusion, we have found that the order parameter in QCD is $\langle \overline{q}q \rangle$.

2. Low-energy region of QCD

An approach, based on the strong interaction in terms of dynamical quarks and gluons, fails in the low-energy region of hadronic spectrum, i.e. for energies less than 2 GeV, where QCD becomes non-perturbative. An alternative way out would be to replace QCD at low energies with a new theory that would take other relevant degrees of freedom into account, i.e. mesons and baryons. However, the situation here is not that simple since we do not know such a theory in the full low-energy spectrum.

Nevertheless, in the region of energies typically less than M_ρ, with M_ρ standing for the mass of the $\rho(770)$ meson, we have an effective field theory of QCD, called Chiral perturbation theory.

2.1 Chiral perturbation theory

Spontaneous breaking of the chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry down to $\mathrm{SU}(3)_V$ in QCD leads to the presence of Goldstone bosons [3]. Identifying them with the octet of pseudoscalar mesons (π, K, η), as the lightest hadronic observable states, we can construct Chiral perturbation theory (χPT). Then, the effective Lagrangian is expressed in terms of the mentioned hadronic degrees of freedom [16]. The construction of such a Lagrangian is now our task.

2.1.1 Chiral operators and the χPT Lagrangian

A general formalism of how to build effective Lagrangians with spontaneous symmetry breaking was proposed in [9], where a suitable way of parametrization of the Goldstone boson was provided. We can use this formalism to construct an effective Lagrangian to describe the interaction among Goldstone bosons, the lightest pseudoscalar multiplet. Considering the fact that there is a mass gap separating the pseudoscalar octet from the rest of the hadronic spectrum, one can construct an effective field theory containing only these modes.

Let us recall the spontaneous chiral symmetry breaking

$$G \equiv \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \longrightarrow H \equiv \mathrm{SU}(3)_V \, . \tag{2.1}$$

Denoting $\phi^a (a = 1 \ldots 8)$ the coordinates describing the Goldstone fields in the coset space G/H, a coset representative $u_{R,L}(\phi)$ is chosen. The transformation of coordinates, carrying the Goldstone modes, under a chiral transformation $g = (g_\mathrm{L}, g_\mathrm{R}) \in G$ is given by

$$u_L(\phi) \xrightarrow{G} g_L u_L(\phi) h^\dagger(g, \phi) \, , \tag{2.2}$$

$$u_R(\phi) \xrightarrow{G} g_R u_R(\phi) h^\dagger(g, \phi) \, , \tag{2.3}$$

where $h(g, \phi) \in H$ is the compensator field. We can take the choice of a coset representative such that

$$u_R(\phi) = u_L^\dagger(\phi) \equiv u(\phi) \, . \tag{2.4}$$

The explicit form of parametrization can be written in the form of the basic building block of χPT, i.e. [10], [11], [12]

$$u(\phi) = \exp\left(\frac{i}{\sqrt{2}F}\phi\right) , \tag{2.5}$$

where the low-energy parameter F is the pion decay constant and

$$\phi = \sum_{a=1}^{8} \frac{\lambda^a}{\sqrt{2}}\phi^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \tag{2.6}$$

is the matrix of the octet of pseudoscalar meson fields, i.e. Goldstone bosons, under chiral SU(3) transformations.

The octet of pseudoscalar fields can be easily extended to include the η' meson, the singlet counterpart of the Goldstone boson octet. Although the η' is not a Goldstone boson due to the axial U(1) symmetry which is explicitly broken through the chiral anomaly of the strong interactions, it combines with the Goldstone bosons to a nonet at the level of the effective theory. The extension of the η' to the nonet is possible due to the ninth Gell-Mann matrix (see Appendix A).

The matrix (2.6) also says that physical fields representing the octet of pseudoscalar mesons are linear combinations of real fields ϕ^a. It is also clear that physical fields on the diagonal positions, π^0 and η_8, are singlets and for this reason they are representable only by a single field, specifically:

$$\pi^0 = \phi_3\,, \qquad \eta_8 = \phi_8\,. \tag{2.7}$$

For remaining components of dublets we can find the following field representation:

$$\pi^+ = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,, \qquad \pi^- = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \tag{2.8}$$

$$K^+ = \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5)\,, \qquad K^- = \frac{1}{\sqrt{2}}(\phi_4 + i\phi_5)\,, \tag{2.9}$$

$$K^0 = \frac{1}{\sqrt{2}}(\phi_6 - i\phi_7)\,, \qquad \overline{K}^0 = \frac{1}{\sqrt{2}}(\phi_6 + i\phi_7)\,. \tag{2.10}$$

Obviously, every couple in (2.8)-(2.10) makes up a mutually complex conjugated pair, which we have to take into account to find $\phi^\dagger$. It means that during every complex transposition of ϕ, for example, physical field π^+ in ϕ changes to π^- and takes the same place as π^- had in the original matrix ϕ. This is just a simple consequence of complex transposition, which is just a composition of matrix transposition and complex conjugation. The same principle applies on every physical field in ϕ. This satisfies the fact that $\phi \equiv \phi^\dagger$, which is also just a simple consequence of hermiticity of Gell-Mann matrices λ^a and reality of fields ϕ^a. Knowing how to work under the complex transposition, we will be able to express a few terms of an expansion of the building block and covariant tensors. We will return to this point later on.

Mesonic chiral Lagrangians can be constructed by taking traces of products of chiral operators X that either transform as [9], [10], [11], [12]

$$X \xrightarrow{G} h(g,\phi)Xh^\dagger(g,\phi) \tag{2.11}$$

or remain invariant under chiral transformations. The simplest such operators are

$$u_\mu = u_\mu^\dagger = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\right] \tag{2.12}$$

and

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \tag{2.13}$$

where

$$\chi = 2B_0(s + ip) \tag{2.14}$$

and B_0 is a constant not restricted by chiral symmetry and related with the quark condensate.

The χPT Lagrangian can be written in the form [10], [11], [12]

$$\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}^{(2)}_{\chi\mathrm{PT}} + \mathcal{L}^{(4)}_{\chi\mathrm{PT}} + \mathcal{L}^{(4)}_{\mathrm{WZW}} + \dots\,, \tag{2.15}$$

where

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle\,, \tag{2.16}$$

$$\begin{aligned}\mathcal{L}^{(4)}_{\chi\mathrm{PT}} = {} & L_1\langle u_\mu u^\mu\rangle^2 + L_2\langle u_\mu u^\nu\rangle\langle u^\mu u_\nu\rangle + L_3\langle u_\mu u^\mu u_\nu u^\nu\rangle + L_4\langle u_\mu u^\mu\rangle\langle\chi_+\rangle \\ & + L_5\langle u_\mu u^\mu \chi_+\rangle + L_6\langle\chi_+\rangle^2 + L_7\langle\chi_-\rangle^2 + \frac{L_8}{2}\langle\chi_-^2 + \chi_+^2\rangle \\ & - iL_9\langle f_+^{\mu\nu} u_\mu u_\nu\rangle + \frac{L_{10}}{4}\langle f_{+\mu\nu} f_+^{\mu\nu} - f_{-\mu\nu} f_-^{\mu\nu}\rangle \\ & + iL_{11}\langle\chi_-(\nabla_\mu u^\mu + \frac{i}{2}\chi_-)\rangle - L_{12}\langle(\nabla_\mu u^\mu + \frac{i}{2}\chi_-)^2\rangle \\ & + \frac{1}{2}H_1\langle f_{+\mu\nu} f_+^{\mu\nu} + f_{-\mu\nu} f_-^{\mu\nu}\rangle + \frac{1}{4}H_2\langle\chi_+^2 - \chi_-^2\rangle\,.\end{aligned} \tag{2.17}$$

The term $\mathcal{L}^{(4)}_{\mathrm{WZW}}$ is the Wess-Zumino-Witten Lagrangian and we will pay special attention to it in the next chapter.

We see that the lowest order χPT Lagrangian (2.16) in chiral limit contains only two unknown constants: F and B_0 which is contained in the operator (2.13). However, the situation in the next-to-leading order $\mathcal{O}(p^4)$ is a bit complicated and there we already have 10 constants. The reader may notice that (2.17) contains 14 constants but the constants L_{11}, L_{12} vanish when the equations of motion are used and the constants H_1, H_2 are needed only for the renormalization.

In the Lagrangian (2.17) we were required to introduce other additional operators in order to have the most general Lagrangian which has to be invariant under parity, charge conjugation and the local chiral transformations. Up to and including $\mathcal{O}(p^6)$, the operators

$$f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu} u \tag{2.18}$$

and

$$h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu \tag{2.19}$$

are suitable choices to satisfy the requested conditions. In (2.18),

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i\,[\ell^\mu, \ell^\nu]\,, \tag{2.20}$$

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i\,[r^\mu, r^\nu]\,, \tag{2.21}$$

are the left and right non-Abelian field-strength tensors and the covariant derivative in (2.19) is defined by

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]\,, \tag{2.22}$$

where the chiral connection is

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - i\ell_\mu)u^\dagger\right]\,. \tag{2.23}$$

An important note regarding the basic properties of the building blocks and the chiral power counting has to be made. Simply,

$$u \sim \mathcal{O}(1)\,, \tag{2.24}$$

$$\partial_\mu, \ell_\mu, r_\mu \sim \mathcal{O}(p)\,, \tag{2.25}$$

$$s, p \sim \mathcal{O}(p^2)\,, \tag{2.26}$$

whilst the constructed operators above have other additional properties, that are shown in the table 2.1 below.

operator	P	C	h.c.	power counting
u_μ	$-u^\mu$	u_μ^T	u_μ	$\mathcal{O}(p)$
$h_{\mu\nu}$	$-h^{\mu\nu}$	$h_{\mu\nu}^T$	$h_{\mu\nu}$	$\mathcal{O}(p^2)$
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$	$\mathcal{O}(p^2)$
$f_{\mu\nu\pm}$	$\pm f_\pm^{\mu\nu}$	$\mp f_{\mu\nu\pm}^T$	$f_{\mu\nu\pm}$	$\mathcal{O}(p^2)$

Table 2.1: The P, C and hermiticity properties of the chiral operators.

2.1.2 Expansion of the chiral operators

Calculations of chiral Lagrangians require one to expand the basic building block and the chiral operators in several terms sufficient for the given case. For example, in the case of three-point Green functions, it is sufficient to take into account only terms linear in the fields, i.e. the basic building block of χPT reads

$$u = \mathbb{1} + \frac{i}{\sqrt{2}F}\phi + \dots\,, \tag{2.27}$$

which gives us the following forms for the chiral operators:

$$u_\mu \simeq -\frac{\sqrt{2}}{F}\partial_\mu\phi + 2a_\mu + \dots\,, \tag{2.28}$$
$$h_{\mu\nu} \simeq -\frac{2\sqrt{2}}{F}\partial_\mu\partial_\nu\phi + 2(\partial_\mu a_\nu + \partial_\nu a_\mu) + \dots\,, \tag{2.29}$$
$$\chi_+ \simeq 4B_0 s + \dots\,, \tag{2.30}$$
$$\chi_- \simeq 4iB_0 p + \dots\,, \tag{2.31}$$
$$f_+^{\mu\nu} \simeq 2(\partial^\mu v^\nu - \partial^\nu v^\mu) + \dots\,, \tag{2.32}$$
$$f_-^{\mu\nu} \simeq -2(\partial^\mu a^\nu - \partial^\nu a^\mu) + \dots \tag{2.33}$$

The chiral connection is

$$\Gamma_\mu \simeq -iv_\mu + \dots \tag{2.34}$$

so we have the covariant derivative in the form

$$\nabla_\mu X \simeq \partial_\mu X - i[v_\mu, X] + \dots \tag{2.35}$$

Expansions of the chiral operators above are listed in Appendix B, where we considered all terms that consist of the external sources that are coupled to three pseudoscalar fields at most.

2.2 Wess-Zumino-Witten Lagrangian

The leading order of the pure Goldstone-boson part of the odd-intrinsic parity sector starts at $\mathcal{O}(p^4)$, as we have already seen in (2.15), and the parameters are set entirely by the chiral anomaly. The Wess-Zumino-Witten Lagrangian contributes to the anomalous term and is generated by the need of coupling the Goldstone bosons to the external gauge invariant sources. The full action at next-to-leading order is written as [13]

$$\begin{aligned} S(u,\ell,r)_{\mathrm{WZW}} = &- i\frac{N_C}{48\pi^2}\int \mathrm{d}^4x\, \varepsilon_{\mu\nu\alpha\beta}\langle W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbb{1},\ell,r)^{\mu\nu\alpha\beta}\rangle \\ &- i\frac{N_C}{240\pi^2}\int \mathrm{d}\sigma^{ijklm}\,\langle \Sigma_i^L\Sigma_j^L\Sigma_k^L\Sigma_l^L\Sigma_m^L\rangle \end{aligned} \tag{2.36}$$

where

$$\begin{aligned}W(U,\ell,r)_{\mu\nu\alpha\beta} = &- i\Sigma^L_\mu \ell_\nu U^\dagger r_\alpha U \ell_\beta + \Sigma^L_\mu U^\dagger \partial_\nu r_\alpha U \ell_\beta - i\Sigma^L_\mu \ell_\nu \ell_\alpha \ell_\beta \\ &+ iU\partial_\mu \ell_\nu \ell_\alpha U^\dagger r_\beta + i\partial_\mu r_\nu U \ell_\alpha U^\dagger r_\beta + \frac{1}{2}\Sigma^L_\mu \ell_\nu \Sigma^L_\alpha \ell_\beta \\ &+ U\ell_\mu \ell_\nu \ell_\alpha U^\dagger r_\beta - \Sigma^L_\mu \Sigma^L_\nu U^\dagger r_\alpha U \ell_\beta - i\Sigma^L_\mu \Sigma^L_\nu \Sigma^L_\alpha \ell_\beta \\ &+ \frac{1}{4} U\ell_\mu U^\dagger r_\nu U \ell_\alpha U^\dagger r_\beta + \Sigma^L_\mu \ell_\nu \partial_\alpha \ell_\beta + \Sigma^L_\mu \partial_\nu \ell_\alpha \ell_\beta \\ &- (L \leftrightarrow R)\end{aligned} \tag{2.37}$$

with the parametrization

$$U = u^2 \tag{2.38}$$

and

$$\Sigma^L_\mu = U^\dagger \partial_\mu U\,, \qquad \Sigma^R_\mu = U\partial_\mu U^\dagger\,. \tag{2.39}$$

The symbol $(L \leftrightarrow R)$ in (2.37) stands for the interchanges $U \leftrightarrow U^\dagger$, $\ell_\mu \leftrightarrow r_\mu$ and $\Sigma^L_\mu \leftrightarrow \Sigma^R_\mu$. The tensor $W(\mathbb{1},\ell,r)^{\mu\nu\alpha\beta}$ has the same meaning as $W(U,\ell,r)^{\mu\nu\alpha\beta}$ but in this case the building block (2.5) does not contain any pseudoscalar mesons, i.e. $\phi = \mathbb{0}$ and for this reason $U = u^2 = \mathbb{1}$. In this special case we then have $\Sigma^L_\mu = \Sigma^R_\mu = \mathbb{0}$.

Knowing the full action it is possible to extract the Wess-Zumino-Witten Lagrangian. Moving on from the parametrization (2.38) to (2.5) we can get the Lagrangian in the form [13]

$$\begin{aligned}\mathcal{L}^{(4)}_{WZW} = &- i\frac{N_C}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\langle W(u,\ell,r)_{\mu\nu\alpha\beta} - W(\mathbb{1},\ell,r)_{\mu\nu\alpha\beta}\rangle \\ &+ \frac{N_C}{48\pi^2 F}\varepsilon^{\mu\nu\alpha\beta}\int_0^1 \mathrm{d}\xi\, \langle \sigma^\xi_\mu \sigma^\xi_\nu \sigma^\xi_\alpha \sigma^\xi_\beta \phi\rangle\,,\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}W_{\mu\nu\alpha\beta}(u,\ell,r) = \quad & L_\mu L_\nu L_\alpha R_\beta + \frac{1}{4} L_\mu R_\nu L_\alpha R_\beta + iL_{\mu\nu} L_\alpha R_\beta + iR_{\mu\nu} L_\alpha R_\beta \\ &- i\sigma_\mu L_\nu R_\alpha L_\beta + \sigma_\mu R_{\nu\alpha} L_\beta - \sigma_\mu \sigma_\nu R_\alpha L_\beta + \sigma_\mu L_\nu L_{\alpha\beta} \\ &+ \sigma_\mu L_{\nu\alpha} L_\beta - i\sigma_\mu L_\nu L_\alpha L_\beta + \frac{1}{2}\sigma_\mu L_\nu \sigma_\alpha L_\beta - i\sigma_\mu \sigma_\nu \sigma_\alpha L_\beta \\ &- (L \leftrightarrow R)\,,\end{aligned} \tag{2.41}$$

with $L \leftrightarrow R$ standing also for $\sigma \leftrightarrow \sigma^\dagger$ interchange. In (2.41) we have denoted

$$L_\mu = u\,\ell_\mu u^\dagger, \qquad L_{\mu\nu} = u(\partial_\mu \ell_\nu)u^\dagger\,, \tag{2.42}$$

$$R_\mu = u^\dagger r_\mu u, \qquad R_{\mu\nu} = u^\dagger(\partial_\mu r_\nu)u \tag{2.43}$$

and

$$\sigma_\mu = \{u^\dagger, \partial_\mu u\}\,, \qquad \sigma^\dagger_\mu = \{u, \partial_\mu u^\dagger\}\,. \tag{2.44}$$

The power ξ in (2.40) denotes a change of the building block u to

$$u^\xi = \exp\left(\frac{i\xi}{\sqrt{2}F}\phi\right). \tag{2.45}$$

It is important to notice that both definitions (2.37) and (2.41) are equivalent due to the cyclic property of the matrix trace operation.

We can neglect the term under the integral sign in (2.40) as a higher order contribution and consider only the upper line of the equation. In this case, considering that for $W(\mathbb{1},\ell,r)_{\mu\nu\alpha\beta}$ we have $\sigma_\mu = \mathbb{0}$, we have, in total, 10 contributing terms in the means of vector and axial-vector external sources and pseudoscalar fields. However, there are only two types of contributions if we consider only the three-point vertices. In this case, the result can be written in the form

$$\begin{aligned}\mathcal{L}^{(4)}_{\mathrm{WZW}} = & \frac{N_C}{12\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)a_\nu(\partial_\alpha a_\beta)\rangle\varepsilon^{\mu\nu\alpha\beta} \\ & + \frac{N_C}{12\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)v_\nu(\partial_\alpha v_\beta)\rangle\varepsilon^{\mu\nu\alpha\beta} + \frac{N_C}{6\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\rangle\varepsilon^{\mu\nu\alpha\beta}\,.\end{aligned} \tag{2.46}$$

The detailed calculation can be found in Appendix D.

2.3 Resonance chiral theory

Taking the large-N_C limit we can construct the effective theory of QCD for an intermediate energy region that also satifies all symetries of the underlying theory [8], [17]. This effective theory is called Resonance chiral theory (RχT) and is relevant for energies $M_\rho \leq E \leq 2\,\mathrm{GeV}$. For bigger energies R$\chi$T loses its convergence and can not be properly used because of the higher masses that become significant in hadron dynamics.

Since we are in an intermediate region there is no suitable expansion parameter. At lower energies we expand in small momenta while at high energies we can use the $1/N_C$ expansion. In the resonance sector we are required to combine both expansion parameters in order to get phenomenologically relevant results.

RχT increases the number of degrees of freedom of Chiral perturbation theory by including massive U(3) multiplets of vector $V(1^{--})$, axial-vector $A(1^{++})$, scalar $S(0^{++})$ and pseudoscalar $P(0^{-+})$ resonances, denoted generically as a nonet field R. We assume that this field can be decomposed into singlet R^0 and octet R^a such as [17], [18]

$$R = \frac{1}{\sqrt{3}}R^0 + \sum_{a=1}^{8}\frac{\lambda^a}{\sqrt{2}}R^a\,, \tag{2.47}$$

where $R = V, A, S, P$. For instance, let us show the nonets that are considered in this thesis:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix}, \tag{2.48}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 & K^{*0} \\ K^{*-} & \overline{K}^{*0} & -\frac{2}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 \end{pmatrix}, \tag{2.49}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}}a_1^0 + \frac{1}{\sqrt{6}}f_1^8 + \frac{1}{\sqrt{3}}f_1^1 & a_1^+ & K_{1A}^+ \\ a_1^- & -\frac{1}{\sqrt{2}}a_1^0 + \frac{1}{\sqrt{6}}f_1^8 + \frac{1}{\sqrt{3}}f_1^1 & K_{1A}^0 \\ K_{1A}^- & \overline{K}_{1A}^0 & -\frac{2}{\sqrt{6}}f_1^8 + \frac{1}{\sqrt{3}}f_1^1 \end{pmatrix}. \tag{2.50}$$

2.3.1 RχT Lagrangian

A procedure of constructing the RχT lagrangian consists of the following steps [17], [18]:

1. In order to recover at low energies the results of χPT, to consider chiral symmetry seems to be a reasonable choice. On account of large-N_C, the mesons are contained in the U(3) multiplets shown above and the operators with only one trace over flavour space are considered.

2. To have a meaningful effective description, it is necessary to properly match the incipient theory with the underlying theory. In this case, the underlying theory is QCD with asymptotic behaviour that sets in at $E \sim 2\,\mathrm{GeV}$. Therefore, the RχT should recover the short-distance behaviour of QCD. This demand simplifies our model because it excludes all interactions with large numbers of derivatives that tend to violate asymptotic behaviour of form factors. The matching also allows us to determine relations between coupling constants in the model Lagrangian (see (2.68)). Without further discussion, let us introduce such relations [17]:

$$F_V = 2G_V = \sqrt{2}F_A = \sqrt{2}F\,, \tag{2.51}$$

$$c_m = c_d = \sqrt{2}d_m = \frac{F}{2}\,, \tag{2.52}$$

$$M_A = \sqrt{2}M_V\,, \tag{2.53}$$

$$M_P \simeq \sqrt{2}M_S\,. \tag{2.54}$$

   It is obvious now that all parameters are determined by the pion decay constant F and the characteristic masses of the vector and scalar multiplets, M_V and M_S.

3. Some approximations are needed to construct the effective Lagrangian. As the number of meson states is infinite at large-N_C, the most common approximation is the cut in the number of resonances, leaving us to consider only the lightest states. Also, the contributions of the higher resonances are suppressed by their masses.

4. It is known that $\mathcal{L}^{(4)}_{\chi\mathrm{PT}}$ is largely saturated by the resonance exchanges generated by the linear terms in the resonance field. Hence, the explicit introduction of the terms constructed with no resonances and chiral operators of $\mathcal{O}(p^4)$ would amount to include an overlap between both contributions. Thus our theory stands for a complete resonance saturation of the χPT Lagrangian.

To summarize the procedure above, to construct the RχT Lagrangian one takes into account so called Single resonance approximation where just the lightest resonances are considered. In this approach, the Goldstone bosons are coupled to massive U(3) multiplets. Then, the construction follows the path of using the operators that transform similarly as in (2.11) but in our case we look for tensors that obey the transformation

$$X \xrightarrow{G'} h(g,\phi)Xh^\dagger(g,\phi)\,, \tag{2.55}$$

where

$$G' \equiv \mathrm{U}(3)_L \times \mathrm{U}(3)_R\,. \tag{2.56}$$

However, the situation here is a little bit tricky. Since one of the lightest resonances are one-spin particles, we are allowed to use different formalisms to describe them. Although the solution of any calculation should be independent of used formalism, since we work with perturbation theories, we provide calculations only up to a given order which makes different formalisms nonequivalent.

To obtain a more complete picture, in the construction of the RχT Lagrangians we will distinguish within the most used formalisms which are vector and antisymmetric tensor formalism.

2.3.2 Vector field formalism

The first formalism to describe one-spin resonances is the vector (Proca) formalism. The resonances $\widehat{R}_\mu = \widehat{V}_\mu, \widehat{A}_\mu$ carry one Lorentz index and are defined in the usual way, together with the field made out of the covariant derivative acting on the resonance [14], [15], [17]:

$$\widehat{R}_\mu = \sum_{a=1}^{8} \frac{\lambda^a}{\sqrt{2}} \widehat{R}^a_\mu \,, \tag{2.57}$$

$$\widehat{R}_{\mu\nu} = \nabla_\mu \widehat{R}_\nu - \nabla_\nu \widehat{R}_\mu \,. \tag{2.58}$$

The chiral order of the resonances in the vector formalism is $R_\mu \sim \mathcal{O}(p^3)$ and $R_{\mu\nu} \sim \mathcal{O}(p^2)$. Then, the resonance Lagrangian up to $\mathcal{O}(p^8)$ in the vector formalism is given by [14], [15], [17]

$$\mathcal{L}^{(8)}_{\mathrm{res}} = \mathcal{L}^{(6)}_V + \mathcal{L}^{(6)}_A + \mathcal{L}^{(8)}_{VV} + \mathcal{L}^{(8)}_{AA} + \mathcal{L}^{(8)}_{VA} \tag{2.59}$$

where the individual terms are the following (we omit mass and kinetic terms)

$$\begin{aligned}
\mathcal{L}^{(6)}_V =& - \frac{f_V}{2\sqrt{2}} \langle \widehat{V}_{\mu\nu} f^{\mu\nu}_+ \rangle - \frac{i g_V}{2\sqrt{2}} \langle \widehat{V}_{\mu\nu} [u^\mu, u^\nu] + i\alpha_V \langle \widehat{V}_\mu [u_\nu, f^{\mu\nu}_-] \rangle \rangle && (2.60)\\
&+ \beta_V \langle \widehat{V}_\mu [u^\mu, \chi_-] \rangle + i\theta_V \langle \widehat{V}^\mu u^\nu u^\alpha u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} + h_V \langle \widehat{V}^\mu \{u^\nu, f^{\alpha\beta}_+\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \\
\mathcal{L}^{(6)}_A =& - \frac{f_A}{2\sqrt{2}} \langle \widehat{A}_{\mu\nu} f^{\mu\nu}_- \rangle + i\alpha_A \langle \widehat{A}_\mu [u_\nu, f^{\mu\nu}_+] \rangle + \gamma^{(1)}_A \langle \widehat{A}_\mu u_\nu u^\mu u^\nu \rangle && (2.61)\\
&+ \gamma^{(2)}_A \langle \widehat{A}_\mu \{u^\mu, u^\nu u_\nu\} \rangle + \gamma^{(3)}_A \langle \widehat{A}_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \gamma^{(4)}_A \langle \widehat{A}_\mu u^\mu \rangle \langle u^\nu u_\nu \rangle \\
&+ h_A \langle \widehat{A}^\mu \{u^\nu, f^{\alpha\beta}_-\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \\
\mathcal{L}^{(8)}_{VV} =& \ \frac{\delta^{(1)}_V}{2} \langle \widehat{V}_\mu \widehat{V}^\mu u_\nu u^\nu \rangle + \frac{\delta^{(2)}_V}{2} \langle \widehat{V}_\mu u_\nu \widehat{V}^\mu u^\nu \rangle + \frac{\delta^{(3)}_V}{2} \langle \widehat{V}_\mu \widehat{V}_\nu u^\mu u^\nu \rangle && (2.62)\\
&+ \frac{\delta^{(4)}_V}{2} \langle \widehat{V}_\mu \widehat{V}_\nu u^\nu u^\mu \rangle + \frac{\delta^{(5)}_V}{2} \langle \widehat{V}_\mu u^\mu \widehat{V}_\nu u^\nu + \widehat{V}_\mu u_\nu \widehat{V}^\nu u^\mu \rangle + \frac{\kappa_V}{2} \langle \widehat{V}_\mu \widehat{V}^\mu \chi_+ \rangle \\
&+ \frac{i\phi_V}{2} \langle \widehat{V}_\mu [\widehat{V}_\nu, f^{\mu\nu}_+] \rangle + \frac{\sigma_V}{2} \langle \widehat{V}^\mu \{u^\nu, \widehat{V}^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \\
\mathcal{L}^{(8)}_{AA} =& \ \frac{\delta^{(1)}_A}{2} \langle \widehat{A}_\mu \widehat{A}^\mu u_\nu u^\nu \rangle + \frac{\delta^{(2)}_A}{2} \langle \widehat{A}_\mu u_\nu \widehat{A}^\mu u^\nu \rangle + \frac{\delta^{(3)}_A}{2} \langle \widehat{A}_\mu \widehat{A}_\nu u^\mu u^\nu \rangle && (2.63)\\
&+ \frac{\delta^{(4)}_A}{2} \langle \widehat{A}_\mu \widehat{A}_\nu u^\nu u^\mu \rangle + \frac{\delta^{(5)}_A}{2} \langle \widehat{A}_\mu u^\mu \widehat{A}_\nu u^\nu + \widehat{A}_\mu u_\nu \widehat{A}^\nu u^\mu \rangle + \frac{\kappa_A}{2} \langle \widehat{A}_\mu \widehat{A}^\mu \chi_+ \rangle \\
&+ \frac{i\phi_A}{2} \langle \widehat{A}_\mu [\widehat{A}_\nu, f^{\mu\nu}_+] \rangle + \frac{\sigma_A}{2} \langle \widehat{A}^\mu \{u^\nu, \widehat{A}^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \\
\mathcal{L}^{(8)}_{VA} =& \ iA^{(1)} \langle \widehat{V}_\mu [\widehat{A}_\nu, f^{\mu\nu}_-] \rangle + iA^{(2)} \langle \widehat{V}_\mu [u_\nu, \widehat{A}^{\mu\nu}] \rangle + iA^{(3)} \langle \widehat{A}_\mu [u_\nu, \widehat{V}^{\mu\nu}] \rangle && (2.64)\\
&+ B \langle \widehat{V}_\mu [\widehat{A}^\mu, \chi_-] \rangle + H \langle \widehat{V}^\mu \{\widehat{A}^\nu, f^{\alpha\beta}_+\} \rangle \varepsilon_{\mu\nu\alpha\beta} + iZ^{(1)} \langle u^\mu u^\nu \{\widehat{A}^\alpha, \widehat{V}^\beta\} \rangle \varepsilon_{\mu\nu\alpha\beta} \\
&+ iZ^{(2)} \langle u^\mu \widehat{A}^\nu u^\alpha \widehat{V}^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,.
\end{aligned}$$

2.3.3 Antisymmetric tensor field formalism

The second formalism is the antisymmetric tensor field formalism. Using the large-N_C approach, there is no limit to the number of resonances that can be included in the effective Lagrangians. Hence, we can construct the RχT Lagrangian as an expansion in the number of resonance fields, i.e. [1], [17]

$$\mathcal{L}_{\mathrm{R}\chi\mathrm{T}} = \mathcal{L}_{GB} + \mathcal{L}_{R_1 R_1,\mathrm{kin}} + \sum_{R_1} \mathcal{L}_{R_1} + \sum_{R_1,R_2} \mathcal{L}_{R_1,R_2} + \sum_{R_1,R_2,R_3} \mathcal{L}_{R_1,R_2,R_3} + \dots \,, \tag{2.65}$$

where the dots denote terms with four or more resonances, indices R_i run over all resonances fields $R_i = V, A, S, P$ and

$$\mathcal{L}_{R_1 R_1,\text{kin}} = -\frac{1}{2}\langle \nabla^\mu R_{\mu\nu} \nabla_\alpha R^{\alpha\nu} \rangle + \frac{1}{4} M_R^2 \langle R_{\mu\nu} R^{\mu\nu} \rangle + \frac{1}{2}\langle \nabla^\alpha R' \nabla_\alpha R' \rangle - \frac{1}{2} M_{R'}^2 \langle R' R' \rangle \tag{2.66}$$

is the kinetic term, where we denoted $R_{\mu\nu} = V_{\mu\nu}, A_{\mu\nu}$ and $R' = S, P$ to avoid confusion with the previous notation R_i.

In (2.65), the term $\mathcal{L}_{GB}$ contains only Goldstone bosons and external sources and includes terms with the same structure as χPT Lagrangian (2.16) but differs in the value of the couplings as $\mathcal{L}_{GB}$ belongs to the theory where resonances are active degrees of freedom.

operator	P	C	h.c.	power counting
$V_{\mu\nu}$	$V^{\mu\nu}$	$-V_{\mu\nu}^T$	$V_{\mu\nu}$	$\mathcal{O}(p^2)$
$A_{\mu\nu}$	$-A^{\mu\nu}$	$A_{\mu\nu}^T$	$A_{\mu\nu}$	$\mathcal{O}(p^2)$
S	S	S^T	S	$\mathcal{O}(p)$
P	$-P$	P^T	P	$\mathcal{O}(p)$

Table 2.2: The P, C and hermiticity properties of the resonances in the antisymmetric tensor formalism.

Also, there is another expansion of the RχT Lagrangian, based on the ordering according to the contribution to the chiral coupling constants. Here, the resonance fields are effectively of the order $\mathcal{O}(p^2)$ and the chiral building blocks are counted as usual. Now we can write

$$\mathcal{L}_{\text{R}\chi\text{T}} = \mathcal{L}_{GB}^{(2)} + \mathcal{L}_{GB}^{(4)} + \mathcal{L}_{RR,\text{kin}}^{(4)} + \mathcal{L}_{RR,\text{kin}}^{(6)} + \mathcal{L}_{R}^{(4)} + \mathcal{L}_{GB}^{(6)} + \mathcal{L}_{R}^{(6)} + \mathcal{L}_{RR'}^{(6)} + \mathcal{L}_{RR'R''}^{(6)} + \ldots \tag{2.67}$$

The couplings of the massive U(3) multiplets with the pseudoscalar fields and external sources in the leading order of $1/N_C$ are given by the linear interaction resonance Lagrangian from (2.67), more specifically

$$\mathcal{L}_R^{(4)} = \mathcal{L}_V^{(4)} + \mathcal{L}_A^{(4)} + \mathcal{L}_S^{(4)} + \mathcal{L}_P^{(4)}\,, \tag{2.68}$$

with the following contributing terms [17]:

$$\mathcal{L}_V^{(4)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i G_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu] \rangle\,, \tag{2.69}$$

$$\mathcal{L}_A^{(4)} = \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu} \rangle\,, \tag{2.70}$$

$$\mathcal{L}_S^{(4)} = c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi_+ \rangle\,, \tag{2.71}$$

$$\mathcal{L}_P^{(4)} = i d_m \langle P \chi_- \rangle + \frac{i d_{m0}}{N_F}\langle P \rangle \langle \chi_- \rangle\,. \tag{2.72}$$

It is important to mention that the second term in (2.72) represents the η' exchanges that we do not consider in this paper. A short remark on this is made in appendix E.

In the antisymmetric tensor formalism with Lagrangians up to $\mathcal{O}(p^6)$, we will use the independent operator basis, relevant in the odd-intrinsic parity sector, which was formulated in [1]. Every operator has the form

$$\mathcal{O}_i^X = \varepsilon^{\mu\nu\alpha\beta} \widehat{\mathcal{O}}^X_{i\,\mu\nu\alpha\beta}, \tag{2.73}$$

with all the contributing operators shown in the tables 2.3-2.5 below.

R	i	Operator $\mathcal{O}^R_{i\,\mu\nu\alpha\beta}$	Vertex structure
V	1	$i\langle V^{\mu\nu}(h^{\alpha\sigma}u_\sigma u^\beta - u^\beta u_\sigma h^{\alpha\sigma})\rangle$	$Va^3, Va^2\phi, Va\phi^2, V\phi^3$
	2	$i\langle V^{\mu\nu}(u_\sigma h^{\alpha\sigma}u^\beta - u^\beta h^{\alpha\sigma}u_\sigma)\rangle$	$Va^3, Va^2\phi, Va\phi^2, V\phi^3$
	3	$i\langle V^{\mu\nu}(u_\sigma u^\beta h^{\alpha\sigma} - h^{\alpha\sigma}u^\beta u_\sigma)\rangle$	$Va^3, Va^2\phi, Va\phi^2, V\phi^3$
	4	$i\langle[V^{\mu\nu}, \nabla^\alpha\chi_+]u^\beta\rangle$	$Vsa, Vs\phi$
	5	$i\langle V^{\mu\nu}[f_-^{\alpha\beta}, u_\sigma u^\sigma]\rangle$	$Va^3, Va^2\phi, Va\phi^2$
	6	$i\langle V^{\mu\nu}(f_-^{\alpha\sigma}u^\beta u_\sigma - u_\sigma u^\beta f_-^{\alpha\sigma})\rangle$	$Va^3, Va^2\phi, Va\phi^2$
	7	$i\langle V^{\mu\nu}(u_\sigma f_-^{\alpha\sigma}u^\beta - u^\beta f_-^{\alpha\sigma}u_\sigma)\rangle$	$Va^3, Va^2\phi, Va\phi^2$
	8	$i\langle V^{\mu\nu}(f_-^{\alpha\sigma}u_\sigma u^\beta - u^\beta u_\sigma f_-^{\alpha\sigma})\rangle$	$Va^3, Va^2\phi, Va\phi^2$
	9	$\langle V^{\mu\nu}\{\chi_-, u^\alpha u^\beta\}\rangle$	$Va^2p, Vap\phi, Vp\phi^2$
	10	$\langle V^{\mu\nu}u^\alpha\chi_- u^\beta\rangle$	$Va^2p, Vap\phi, Vp\phi^2$
	11	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, f_-^{\beta\sigma}\}\rangle g_{\rho\sigma}$	Vva
	12	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, h^{\beta\sigma}\}\rangle g_{\rho\sigma}$	$Vva, Vv\phi$
	13	$i\langle V^{\mu\nu}f_+^{\alpha\beta}\rangle\langle\chi_-\rangle$	Vvp
	14	$i\langle V^{\mu\nu}\{f_+^{\alpha\beta}, \chi_-\}\rangle$	Vvp
	15	$i\langle V^{\mu\nu}[f_-^{\alpha\beta}, \chi_+]\rangle$	Vas
	16	$\langle V^{\mu\nu}\{\nabla^\alpha f_+^{\beta\sigma}, u_\sigma\}\rangle$	$Vva, Vv\phi$
	17	$\langle V^{\mu\nu}\{\nabla_\sigma f_+^{\alpha\sigma}, u^\beta\}\rangle$	$Vva, Vv\phi$
	18	$\langle V^{\mu\nu}u^\alpha u^\beta\rangle\langle\chi_-\rangle$	$Va^2p, Vap\phi, Vp\phi^2$
A	1	$\langle A^{\mu\nu}[u^\alpha u^\beta, u_\sigma u^\sigma]\rangle$	$Aa^4, Aa^3\phi, Aa^2\phi^2, Aa\phi^3, A\phi^4$
	2	$\langle A^{\mu\nu}[u^\alpha u^\sigma u^\beta, u^\sigma]\rangle$	$Aa^4, Aa^3\phi, Aa^2\phi^2, Aa\phi^3, A\phi^4$
	3	$\langle A^{\mu\nu}\{\nabla^\alpha h^{\beta\sigma}, u_\sigma\}\rangle$	$Aa^2, Aa\phi, A\phi^2$
	4	$i\langle A^{\mu\nu}[f_+^{\alpha\beta}, u^\sigma u_\sigma]\rangle$	$Ava^2, Ava\phi, Av\phi^2$
	5	$i\langle A^{\mu\nu}(f_+^{\alpha\sigma}u_\sigma u^\beta - u^\beta u_\sigma f_+^{\alpha\sigma})\rangle$	$Ava^2, Ava\phi, Av\phi^2$
	6	$i\langle A^{\mu\nu}(f_+^{\alpha\sigma}u^\beta u_\sigma - u_\sigma u^\beta f_+^{\alpha\sigma})\rangle$	$Ava^2, Ava\phi, Av\phi^2$
	7	$i\langle A^{\mu\nu}(u_\sigma f_+^{\alpha\sigma}u^\beta - u^\beta f_+^{\alpha\sigma}u_\sigma)\rangle$	$Ava^2, Ava\phi, Av\phi^2$
	8	$\langle A^{\mu\nu}\{f_-^{\alpha\sigma}, h^{\beta\sigma}\}\rangle$	$Aa^2, Aa\phi$
	9	$i\langle A^{\mu\nu}f_-^{\alpha\beta}\rangle\langle\chi_-\rangle$	Aap
	10	$i\langle A^{\mu\nu}u^\alpha\rangle\langle\nabla^\beta\chi_-\rangle$	$Aap, A\phi p$
	11	$i\langle A^{\mu\nu}\{f_-^{\alpha\beta}, \chi_-\}\rangle$	Aap
	12	$i\langle A^{\mu\nu}\{\nabla^\alpha\chi_-, u^\beta\}\rangle$	$Aap, A\phi p$
	13	$\langle A^{\mu\nu}[\chi_+, u^\alpha u^\beta]\rangle$	$Asa^2, Asa\phi, As\phi^2$
	14	$i\langle A^{\mu\nu}[f_+^{\alpha\beta}, \chi_+]\rangle$	Avs
	15	$\langle A^{\mu\nu}\{\nabla^\alpha f_-^{\beta\sigma}, u_\sigma\}\rangle$	$Aa^2, Aa\phi$
	16	$\langle A^{\mu\nu}\{\nabla_\sigma f_-^{\alpha\sigma}, u^\beta\}\rangle$	$Aa^2, Aa\phi$
P	1	$\langle P\{f_-^{\mu\nu}, f_-^{\alpha\beta}\}\rangle$	Pa^2
	2	$i\langle Pu^\alpha f_+^{\mu\nu}u^\beta\rangle$	$Pva^2, Pva\phi, Pv\phi^2$
	3	$i\langle P\{f_+^{\mu\nu}, u^\alpha u^\beta\}\rangle$	$Pva^2, Pva\phi, Pv\phi^2$
	4	$\langle Pu^\mu u^\nu u^\alpha u^\beta\rangle$	$Pa^4, Pa^3\phi, Pa^2\phi^2, Pa\phi^3, P\phi^4$
	5	$\langle P\{f_+^{\mu\nu}, f_+^{\alpha\beta}\}\rangle$	Pv^2
S	1	$\langle S[f_-^{\alpha\beta}, u^\mu u^\nu]\rangle$	$Sa^3, Sa^2\phi, Sa\phi^2$
	2	$i\langle S[f_+^{\mu\nu}, f_-^{\alpha\beta}]\rangle$	Sav

Table 2.3: Monomials with one resonance field and possible vertex structures for four-point Green functions at most.

RR	i	Operator $\mathcal{O}^{RR}_{i\,\mu\nu\alpha\beta}$	Vertex structure
VV	1	$i\langle V^{\mu\nu}V^{\alpha\beta}\rangle\langle\chi_-\rangle$	VVp
	2	$i\langle\{V^{\mu\nu},V^{\alpha\beta}\}\chi_-\rangle$	VVp
	3	$\langle\{\nabla_\sigma V^{\mu\nu},V^{\alpha\sigma}\}u^\beta\rangle$	$VVa, VV\phi$
	4	$\langle\{\nabla^\beta V^{\mu\nu},V^{\alpha\sigma}\}u_\sigma\rangle$	$VVa, VV\phi$
AA	1	$i\langle A^{\mu\nu}A^{\alpha\beta}\rangle\langle\chi_-\rangle$	AAp
	2	$i\langle\{A^{\mu\nu},A^{\alpha\beta}\}\chi_-\rangle$	AAp
	3	$\langle\{\nabla_\sigma A^{\mu\nu},A^{\alpha\sigma}\}u^\beta\rangle$	$AAa, AA\phi$
	4	$\langle\{\nabla^\beta A^{\mu\nu},A^{\alpha\sigma}\}u_\sigma\rangle$	$AAa, AA\phi$
SA	1	$i\langle[A^{\mu\nu},S]f_+^{\alpha\beta}\rangle$	ASv
	2	$\langle A^{\mu\nu}[S,u^\alpha u^\beta]\rangle$	$ASa^2, ASa\phi, AS\phi^2$
SV	1	$i\langle[V^{\mu\nu},S]f_-^{\alpha\beta}\rangle$	VSa
	2	$i\langle[V^{\mu\nu},\nabla^\alpha S]u^\beta\rangle$	$VSa, VS\phi$
VA	1	$i\langle V^{\mu\nu}[A^{\alpha\beta},u^\sigma u_\sigma]\rangle$	$AVa^2, AVa\phi, AV\phi^2$
	2	$i\langle V^{\mu\nu}(A^{\alpha\sigma}u_\sigma u^\beta - u^\beta u_\sigma A^{\alpha\sigma})\rangle$	$AVa^2, AVa\phi, AV\phi^2$
	3	$i\langle V^{\mu\nu}(A^{\alpha\sigma}u^\beta u_\sigma - u_\sigma u^\beta A^{\alpha\sigma})\rangle$	$AVa^2, AVa\phi, AV\phi^2$
	4	$i\langle V^{\mu\nu}(u_\sigma A^{\alpha\sigma}u^\beta - u^\beta A^{\alpha\sigma}u_\sigma)\rangle$	$AVa^2, AVa\phi, AV\phi^2$
	5	$\langle\{V^{\mu\nu},A^{\alpha\rho}\}f_+^{\beta\sigma}\rangle g_{\rho\sigma}$	AVv
	6	$i\langle[V^{\mu\nu},A^{\alpha\beta}]\chi_+\rangle$	AVs
PA	1	$\langle\{A^{\mu\nu},P\}f_-^{\alpha\beta}\rangle$	APa
	2	$\langle\{A^{\mu\nu},\nabla^\alpha P\}u^\beta\rangle$	$APa, AP\phi$
PV	1	$i\langle\{V^{\mu\nu},P\}u^\alpha u^\beta\rangle$	$VPa^2, VPa\phi, VP\phi^2$
	2	$i\langle V^{\mu\nu}u^\alpha P u^\beta\rangle$	$VPa^2, VPa\phi, VP\phi^2$
	3	$\langle\{V^{\mu\nu},P\}f_+^{\alpha\beta}\rangle$	VPv

Table 2.4: Monomials with two resonance fields and possible vertex structures for four-point Green functions at most.

RRR	i	Operator $\mathcal{O}^{RRR}_{i\,\mu\nu\alpha\beta}$	Vertex structure
VVP	-	$\langle V^{\mu\nu}V^{\alpha\beta}P\rangle$	VVP
VAS	-	$i\langle[V^{\mu\nu},A^{\alpha\beta}]S\rangle$	VAS
AAP	-	$\langle A^{\mu\nu}A^{\alpha\beta}P\rangle$	AAP

Table 2.5: Monomials with three resonance fields and possible vertex structures for four-point Green functions at most.

Thus, the Lagrangian is

$$\mathcal{L}^{(6,\mathrm{odd})}_{R\chi T} = \sum_X \sum_i \kappa_i^X \mathcal{O}_i^X , \tag{2.74}$$

where X stands for resonance fields contributing to the Lagrangian, i.e. the single fields V, A, S, P, the combinations of two fields $VV, AA, SA, SV, VA, PA, PV$ and three fields VVP, VAS, AAP.

In what follows we will strictly use the same notation as in [1] and consider this as a referent basis, especially with indices i standing for a serial number of the Lagrangian.

3. Green functions in the odd-intrinsic parity sector

Before we make some steps ahead, we should establish some ground rules to work with. First of all, let us introduce the difference between external and internal vertices.

- External vertex: a vertex where at least one external source is present.
- Internal vertex: a vertex where no external source is present.
- n-point diagram: a representation of only just n external sources where all the sources are connected, either within the contact terms or by propagators.

Although we do not consider Feynman diagrams with loops, the n-point diagrams can contain more than one external source in the external vertices. And since we require at least one resonance exchange between the external vertices, the n-point diagram can not be made up by only one contact term, that would contain all n external sources, i.e. can not be made up by so called one n-contact term. In other words, the n-point diagrams can be made of diagrams with $(n-1)$-contact terms at most.

To calculate the Feynman diagrams contributing to the n-point Green functions, one must find the appropriate topology of the graphs first. Basically, there are two main possibilities of how to establish the topology.

1. First of all, one can find all topological combinations of how to connect four points in the planar topology, regardless of the existing vertices of the χPT and RχT and then select only the suitable ones.
2. The second approach is to find all existing diagrams based strictly on the knowledge of χPT/RχT vertices, i.e. 'gluing' the individual vertices of the graphs together untill the full set of graphs is complete.

Although it may be seen as the method of trial and error, we would actually prefer the first method. The reason is that in the closed channels (see section 3.2) it is actually possible to have contributions via the resonances of the other types than the ones entering through the external vertices. This feature gives us a rich set of contributing diagrams and it could be very difficult to cover all the options from within the second method above.

3.1 Three-point Green functions

Considering the general definition (1.44), we can define the three-point Green functions as the vacuum expectation values of the time ordered products of the quantum fields:

$$\begin{aligned} &\langle 0|\mathrm{T}\big[\widetilde{\mathcal{O}}_1(p_1)\widetilde{\mathcal{O}}_2(p_2)\widetilde{\mathcal{O}}_3(0)\big]|0\rangle = \\ &= \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2\, e^{i(p_1x_1+p_2x_2)}\,\langle 0|\mathrm{T}\big[\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(0)\big]|0\rangle\,, \end{aligned} \tag{3.1}$$

where the operators $\widetilde{\mathcal{O}}$ stand for any of the currents (1.27), (1.28) or densities (1.31), (1.32). Despite the fact that we have 20 possible three-point Green functions in total, in the odd-intrinsic parity sector of QCD exist only five nontrivial three-point correlators. Three of them, VVP, VAS and AAP have been already studied in the past. In this chapter, we will focus primarily on the study of the two remaining Green functions:

VVA and AAA.

Also, in this chapter we take all the 4-momenta as ingoing to the vertices, i.e. the law of 4-momentum conservation can be written in the form

$$p + q + r = 0\,. \tag{3.2}$$

As it will be brighter later, structures consisting of the scalar product of the 4-momenta associated with the appropriate vertices will develop in our calculations of the Feynman rules. Using (3.2), it is more convenient to use the relevant combinations of the Lorentz invariants instead:

$$p\cdot q = \frac{1}{2}(-p^2 - q^2 + r^2)\,, \tag{3.3}$$

$$p\cdot r = \frac{1}{2}(-p^2 + q^2 - r^2)\,, \tag{3.4}$$

$$q\cdot r = \frac{1}{2}(p^2 - q^2 - r^2)\,, \tag{3.5}$$

that are easily obtained by multiplying (3.2) by all 4-momenta individually and solving the system of equations formed.

In our future calculations, we will deal with contractions of components of 4-momenta with Levi-Civita tensor. For simplicity, we will use the notation (A.72)-(A.73). In our case, considering (3.2), we can explicitly write the shortened notation in the form

$$\varepsilon_{\mu\nu(p)(r)} = \varepsilon_{\mu\nu(p)(-p-q)} = \varepsilon_{\mu\nu(p)(-q)} = -\varepsilon_{\mu\nu(p)(q)}\,, \tag{3.6}$$

$$\varepsilon_{\mu\nu(q)(r)} = \varepsilon_{\mu\nu(q)(-p-q)} = \varepsilon_{\mu\nu(q)(-p)} = -\varepsilon_{\mu\nu(q)(p)} = \varepsilon_{\mu\nu(p)(q)}\,, \tag{3.7}$$

$$\varepsilon_{\mu(p)(q)(r)} = \varepsilon_{\mu(p)(q)(-p-q)} = -\varepsilon_{\mu(p)(q)(p)} - \varepsilon_{\mu(p)(q)(q)} = 0\,. \tag{3.8}$$

3.1.1 Topology of the Feynman diagrams

The topology of the three-point diagrams is very simple. Just for now, let us assume that a full line in figures 3.1-3.2 stands either for the pseudoscalar or resonance fields.

1-contact diagrams This type of Feynman diagrams describes all three individual external sources that do not constitute a multiple contact vertex and, therefore, are coupled together through at least two (three at most) propagators.

Considering all possible combinations, we can draw the full set of Feynman diagrams with the couplings of pseudoscalar fields and resonances, without its physical relevance yet.

Figure 3.1: A general 1-contact topology of the three-point Green functions.

2-contact diagrams Unlike the previous case, this type of Feynman diagrams describes the topology that include two external sources coupled together in one vertex which is connected with the remaining external source through only one propagator.

Figure 3.2: A general 2-contact topology of the three-point Green functions.

Before we advance to the study of the new Green functions, it is appropriate to introduce some familiar results.

3.1.2 Familiar results

As we have already mentioned, the Green functions that have been already studied in the literature contain the following correlators: VVP, VAS and AAP. All three of them can be written in the simple forms

$$\big(\Pi_{VVP}(p,q;r)\big)^{abc}_{\mu\nu} = \Pi_{VVP}(p^2,q^2;r^2)d^{abc}\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta\,. \tag{3.9}$$
$$\big(\Pi_{VAS}(p,q;r)\big)^{abc}_{\mu\nu} = \Pi_{VAS}(p^2,q^2;r^2)f^{abc}\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta\,, \tag{3.10}$$
$$\big(\Pi_{AAP}(p,q;r)\big)^{abc}_{\mu\nu} = \Pi_{AAP}(p^2,q^2;r^2)d^{abc}\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta\,, \tag{3.11}$$

due to the fact that the external sources, that generate the correlators, carry only two different Lorentz indices together, it makes the tensor structure easy to express. The high-energy behaviour within the OPE framework can be written as

$$\Pi_{VVP}\big(((\lambda p)^2,(\lambda q)^2;(\lambda r)^2\big) = \frac{B_0F^2}{2\lambda^4}\frac{p^2+q^2+r^2}{p^2q^2r^2} + \mathcal{O}\bigg(\frac{1}{\lambda^6}\bigg)\,, \tag{3.12}$$
$$\Pi_{VAS}\big(((\lambda p)^2,(\lambda q)^2;(\lambda r)^2\big) = \frac{B_0F^2}{2\lambda^4}\frac{p^2-q^2-r^2}{p^2q^2r^2} + \mathcal{O}\bigg(\frac{1}{\lambda^6}\bigg)\,, \tag{3.13}$$
$$\Pi_{AAP}\big((\lambda p)^2,(\lambda q)^2;(\lambda r)^2\big) = \frac{B_0F^2}{2\lambda^4}\frac{p^2+q^2-r^2}{p^2q^2r^2} + \mathcal{O}\bigg(\frac{1}{\lambda^6}\bigg)\,. \tag{3.14}$$

By comparing this behaviour with the calculated Green functions, one can obtain important relations for the coupling constants, as we will see later.

VVP **Green function** The first correlator is a very important example in the odd-intrinsic parity sector of QCD, with a lot of important phenomenological applications. A full calculation [1] reads:

$$\begin{aligned}
\Pi_{VVP}(p^2,q^2;r^2) &= -\frac{B_0N_C}{8\pi^2r^2} - \frac{64B_0d_m\kappa_5^P}{r^2-M_P^2} + \frac{4B_0F_V^2\kappa_3^{VV}(p^2+q^2)}{(p^2-M_V^2)(q^2-M_V^2)r^2} \\
&+ \frac{4B_0F_V^2(8\kappa_2^{VV}-\kappa_3^{VV})}{(p^2-M_V^2)(q^2-M_V^2)} - \frac{16\sqrt{2}B_0d_mF_V\kappa_3^{PV}}{(p^2-M_V^2)(r^2-M_P^2)} - \frac{16\sqrt{2}B_0d_mF_V\kappa_3^{PV}}{(q^2-M_V^2)(r^2-M_P^2)} \\
&- \frac{2\sqrt{2}B_0F_V}{(p^2-M_V^2)r^2}\big[p^2(\kappa_{16}^V+2\kappa_{12}^V) - q^2(\kappa_{16}^V-2\kappa_{17}^V+2\kappa_{12}^V) - r^2(8\kappa_{14}^V+\kappa_{16}^V+2\kappa_{12}^V)\big] \\
&- \frac{2\sqrt{2}B_0F_V}{(q^2-M_V^2)r^2}\big[q^2(\kappa_{16}^V+2\kappa_{12}^V) - p^2(\kappa_{16}^V-2\kappa_{17}^V+2\kappa_{12}^V) - r^2(8\kappa_{14}^V+\kappa_{16}^V+2\kappa_{12}^V)\big] \\
&- \frac{16B_0d_mF_V^2\kappa^{VVP}}{(p^2-M_V^2)(q^2-M_V^2)(r^2-M_P^2)}\,.
\end{aligned} \tag{3.15}$$

Using the comparison with the necessarily fulfilled high-energy behaviour allows us to extract the following constraints for the coupling constants:

$$\kappa_2^{VV} - \frac{d_m \kappa_3^{PV}}{\sqrt{2}F_V} = \frac{F^2}{32F_V^2} - \frac{N_C M_V^2}{512\pi^2 F_V^2}\,, \tag{3.16}$$

$$8\kappa_2^{VV} - \kappa_3^{VV} = \frac{F^2}{8F_V^2}\,, \tag{3.17}$$

$$\kappa_{16}^V + 2\kappa_{12}^V = -\frac{N_C}{32\sqrt{2}\pi^2 F_V}\,, \tag{3.18}$$

$$\kappa_{17}^V = -\frac{N_C}{64\sqrt{2}\pi^2 F_V}\,, \tag{3.19}$$

$$\kappa_{14}^V = \frac{N_C}{256\sqrt{2}\pi^2 F_V}\,, \tag{3.20}$$

$$\kappa_5^P = 0\,. \tag{3.21}$$

As it is obvious, in this case we have two free parameters: κ_3^{PV} and κ^{VVP}.

Different phenomenological aspects were already discussed in [1], namely the $\mathcal{F}_{\pi^0\gamma\gamma}$ formfactor and its applications to the decays $\rho \to \pi\gamma, \pi(1300) \to \gamma\gamma$ and $\pi(1300) \to \rho\gamma$, as well as the π^0-pole contribution to the muon $g-2$ factor.

VAS **Green function** A full calculation of the VAS, without the tensor structure, simply reads [1]

$$\begin{aligned}\Pi_{VAS}(p^2,q^2;r^2) = {} & \frac{8\sqrt{2}B_0 F_V(\kappa_4^V - 2\kappa_{15}^V)}{p^2 - M_V^2} + \frac{16\sqrt{2}B_0 F_A \kappa_{14}^A}{q^2 - M_A^2} + \frac{32 B_0 c_m \kappa_2^S}{r^2 - M_S^2} \\ & + \frac{16\sqrt{2}B_0 F_A c_m \kappa_1^{SA}}{(q^2 - M_A^2)(r^2 - M_S^2)} - \frac{8\sqrt{2}B_0 F_V c_m (2\kappa_1^{SV} + \kappa_2^{SV})}{(p^2 - M_V^2)(r^2 - M_S^2)} \\ & - \frac{16 B_0 F_A F_V \kappa_6^{VA}}{(p^2 - M_V^2)(q^2 - M_A^2)} + \frac{16 B_0 F_A F_V c_m \kappa^{VAS}}{(q^2 - M_A^2)(p^2 - M_V^2)(r^2 - M_S^2)}\end{aligned} \tag{3.22}$$

and the high energy behaviour dictates the following coupling constants constraints

$$2\kappa_1^{SV} + \kappa_2^{SV} = \frac{F^2}{16\sqrt{2}c_m F_V}\,, \tag{3.23}$$

$$\kappa_4^V - 2\kappa_{15}^V = 0\,, \tag{3.24}$$

$$\kappa_1^{SA} = \frac{F^2}{32\sqrt{2}c_m F_A}\,, \tag{3.25}$$

$$\kappa_6^{VA} = \frac{F^2}{32 F_A F_V}\,, \tag{3.26}$$

$$\kappa_2^S = 0\,, \tag{3.27}$$

$$\kappa_{14}^A = 0\,. \tag{3.28}$$

In this case we have only one free parameter, the coupling constant κ^{VAS}. This implies that we can connect all processes of the type

$$(V: \rho, \omega, K^*, \gamma \ldots) \sim (A: a_1, f_1, K_1, GB, W \ldots) \sim (S: \sigma, \kappa, a_0, f_0, H \ldots) \tag{3.29}$$

via a single parameter. However, these processes are very rare and have not yet been studied experimentally. For more details, see also [1].

AAP **Green function** This Green function was a subject of study in [2]. There we provided the complete calculations of this correlator both in vector and antisymmetric tensor field formalism. The result in the vector formalism is quite easy, explicitly

$$\Pi_{AAP}(p^2,q^2;r^2) = -\frac{B_0 N_C}{24\pi^2 r^2} + \frac{4\sqrt{2}B_0 f_A h_A p^2}{(p^2-M_A^2)r^2} + \frac{4\sqrt{2}B_0 f_A h_A q^2}{(q^2-M_A^2)r^2} - \frac{4B_0 f_A^2 \sigma_A p^2 q^2}{(p^2-M_A^2)(q^2-M_A^2)r^2}\,, \tag{3.30}$$

whilst the antisymmetric tensor field formalism is a bit complicated due to the richer set of contributing Lagrangians, leading to the result

$$\begin{aligned}\Pi_{AAP}(p^2,q^2,r^2) = &-\frac{B_0 N_C}{24\pi^2 r^2} - \frac{64 B_0 d_m \kappa_1^P}{r^2-M_P^2} + \frac{32 B_0 F_A^2 \kappa_2^{AA}}{(p^2-M_A^2)(q^2-M_A^2)} \\ &+ \frac{4B_0 F_A^2 \kappa_3^{AA}}{(p^2-M_A^2)(q^2-M_A^2)r^2}(p^2+q^2-r^2) \\ &- \frac{16 B_0 F_A^2 d_m \kappa^{AAP}}{(p^2-M_A^2)(q^2-M_A^2)(r^2-M_P^2)} \\ &+ \frac{4\sqrt{2}B_0 F_A}{r^2(q^2-M_A^2)}\left[\frac{1}{2}(p^2-q^2+r^2)(\kappa_3^A+2\kappa_8^A+\kappa_{15}^A) - p^2\kappa_{16}^A\right] \\ &- \frac{4\sqrt{2}B_0 F_A}{r^2(p^2-M_A^2)}\left[\frac{1}{2}(p^2-q^2-r^2)(\kappa_3^A+2\kappa_8^A+\kappa_{15}^A) + q^2\kappa_{16}^A\right] \\ &- \frac{8\sqrt{2}B_0 F_A d_m}{(p^2-M_A^2)(r^2-M_P^2)}(2\kappa_1^{AP}+\kappa_2^{AP}) \\ &- \frac{8\sqrt{2}B_0 F_A d_m}{(q^2-M_A^2)(r^2-M_P^2)}(2\kappa_1^{AP}+\kappa_2^{AP}) \\ &+ \frac{8\sqrt{2}B_0 F_A}{p^2-M_A^2}\left(2\kappa_{11}^A+\kappa_{12}^A\right) + \frac{8\sqrt{2}B_0 F_A}{q^2-M_A^2}\left(2\kappa_{11}^A+\kappa_{12}^A\right).\end{aligned} \tag{3.31}$$

In order to satisfy the high-energy behaviour for vector formalism, we obtain the condition

$$-\frac{N_C}{24\pi^2} - 4f_A^2\sigma_A + 8\sqrt{2}f_A h_A = 0\,, \tag{3.32}$$

from which it is obvious that vector formalism is not consistent with OPE at order $1/\lambda^4$. However, the antisymmetric tensor formalism satisfies the OPE well and gives us the following constraints on the coupling constants:

$$2\kappa_1^{AP} + \kappa_2^{AP} - \frac{2\sqrt{2}F_A}{d_m}\kappa_2^{AA} = 0\,, \tag{3.33}$$

$$\kappa_3^A + 2\kappa_8^A + \kappa_{15}^A - 2\kappa_{16}^A = 0\,, \tag{3.34}$$

$$64 d_m \kappa_1^P + 8\sqrt{2}F_A\kappa_{16}^A = -\frac{N_C}{24\pi^2}\,, \tag{3.35}$$

$$2\kappa_{11}^A + \kappa_{12}^A + \frac{1}{2}\kappa_{16}^A = 0\,, \tag{3.36}$$

$$\kappa_3^{AA} - 8\kappa_2^{AA} = \frac{F^2}{8F_A^2}\,, \tag{3.37}$$

$$\kappa^{AAP} = 0\,. \tag{3.38}$$

3.2 Four-point Green functions

The motivation to study the four-point Green functions is given mostly by the interesting phenomenological applications, such as the conjecture that $VVPP$ Green functions in the antisymmetric tensor formalism for π^0 Compton-like scattering violates the Froissart bound while the vector formalism preserves it. Another example can be the hadronic $VVVV$ correlator that contributes to the anomalous magnetic moment.

As usual, let us start with the general definition of the four-point Green functions:

$$\begin{aligned} &\langle 0|\mathrm{T}\big[\widetilde{\mathcal{O}}_1(p_1)\widetilde{\mathcal{O}}_2(p_2)\widetilde{\mathcal{O}}_3(p_3)\widetilde{\mathcal{O}}_4(0)\big]|0\rangle = \\ &\quad = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4x_3\,e^{i(p_1x_1+p_2x_2+p_3x_3)}\,\langle 0|\mathrm{T}\big[\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(0)\big]|0\rangle\,. \end{aligned} \tag{3.39}$$

Equivalently as in the case before, for four-point Green functions we take all the 4-momenta as ingoing to the vertices, i.e. the law of conservation of energy takes the form

$$p+q+r+s=0\,. \tag{3.40}$$

Besides the obvious invariants p^2, q^2, r^2, s^2 we can introduce the Mandelstam variables S, T and U, typical for four-particle processes, in the form

$$S=(p+q)^2=(r+s)^2\,, \tag{3.41}$$
$$T=(p+r)^2=(q+s)^2\,, \tag{3.42}$$
$$U=(p+s)^2=(q+r)^2\,, \tag{3.43}$$

that can easily give us the following set of useful formulas:

$$p\cdot q=\frac{1}{2}(S-p^2-q^2)\,, \tag{3.44}$$
$$r\cdot s=\frac{1}{2}(S-r^2-s^2)\,, \tag{3.45}$$
$$p\cdot r=\frac{1}{2}(T-p^2-r^2)\,, \tag{3.46}$$
$$q\cdot s=\frac{1}{2}(T-q^2-s^2)\,, \tag{3.47}$$
$$p\cdot s=\frac{1}{2}(U-p^2-s^2)\,, \tag{3.48}$$
$$q\cdot r=\frac{1}{2}(U-q^2-r^2)\,. \tag{3.49}$$

3.2.1 Topology of the Feynman diagrams

The topology of the 4-point Feynman diagrams is a bit complicated. Instead of listing all types of diagrams, let us mention that the complete set consists of ten types of diagrams, some of the examples can be found in Chapters 6 and 7. Here we only introduce some basic properties.

1-contact diagrams This type of Feynman diagrams describes all four individual external sources that do not constitute a multiple contact vertex and, therefore, are coupled together through at least three (five at most) propagators.

2-contact diagrams Unlike the previous case, this type of the Feynman diagrams describes the topology that include two external sources coupled together in one vertex which is connected with the two remaining external sources through at least two (three at most) propagators.

3-contact diagrams Finally, a topology that consists of one 3-contact vertex, where three external sources are coupled together. The last one is connected with the contact vertex through one propagator.

3.2.2 Froissart bound

Regarding one of the motivation topics to study four-point Green functions, the Froissart bound is a very general property of the behaviour of total particle scattering cross sections at very high energy. More specifically, the bound states that the total cross section of four-particle scattering does not increase faster than

$$\sigma_{tot} \lesssim \ln^2 S\,. \tag{3.50}$$

Violation of this limit would mean violation of unitarity. In what follows we concisely derive the Froissart bound from the principles of quantum mechanics.

4. VVA Green function

The standard definition of the VVA Green function, i.e. the correlator of two vector and one axial-vector current, is

$$\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = i\int \mathrm{d}^4x\,\mathrm{d}^4y\,e^{i(px+qy)}\,\big\langle 0\big|\mathrm{T}\big[V^a_\mu(x)V^b_\nu(y)A^c_\rho(0)\big]\big|0\big\rangle\,. \tag{4.1}$$

A calculation of (4.1) will be the task we will deal with in this chapter. To provide such a calculation, it is necessary to determine which currents and chiral building blocks we will need.

Looking at the definition (4.1) it is obvious that we will need Lagrangians that consist of vector and axial-vector currents. Given the structure of the independent operator basis up to $\mathcal{O}(p^6)$ one can find that only the linear terms in external sources and pseudoscalar fields of the expansions of the chiral building blocks are needed. Therefore, the covariant derivative can be simply identified with the standard derivative and one can easily find (see chapter B) that the suitable operators are $u^\mu, h^{\mu\nu}$ and $f_\pm^{\mu\nu}$. The building block $f_+^{\mu\nu}$ is the key ingredient for the VVA Green function since it is the only chiral operator that contributes linearly as a vector external source in its expansion. Obviously, since we do not have any scalar or pseudoscalar sources in this case, we do not consider $\chi_\pm$.

Knowing the suitable building blocks, let us start with an introduction of the independent operator basis contributing to this correlator.

4.1 Independent operator basis up to $\mathcal{O}(p^6)$

Before we start to determine the independent operator basis of $\mathcal{O}(p^6)$, let us summarize contributions of the Lagrangians of the lower powers.

Contribution up to $\mathcal{O}(p^2)$

$a\phi$ vertex First of all, we will need the contributions from χPT. Up to $\mathcal{O}(p^2)$ there is only one contribution of the lowest χPT Lagrangian and that is a contribution of the coupling between an axial-vector external source and a pseudoscalar field. The relevant part of the Lagrangian (2.16) is

$$\mathcal{L}^{(2)}_\chi = -\frac{F}{\sqrt{2}}\langle\{\partial_\mu\phi, a^\mu\}\rangle\,. \tag{4.2}$$

Contributions up to $\mathcal{O}(p^4)$

Vertex $vv\phi$ A contribution of the anomalous Wess-Zumino-Witten Lagrangian (2.40) comes by the term

$$\mathcal{L}^{(4)}_{\mathrm{WZW}} = \frac{N_C}{12\sqrt{2}\pi^2 F}\big\langle(\partial_\mu\phi)a_\nu(\partial_\alpha a_\beta)\big\rangle\varepsilon^{\mu\nu\alpha\beta} \tag{4.3}$$

and couples two vector external sources with the pseudoscalar field.

Vertex Vv The contributions of the couplings between vector or axial-vector external sources with the resonances comes from the Lagrangian (2.68). More specifically, a contribution to the vertex consisted of vector external source and vector resonance is given by the Lagrangian (2.69) with the relevant part

$$\mathcal{L}^{(4)}_V = \frac{F_V}{\sqrt{2}}\langle V_{\mu\nu}(\partial^\mu v^\nu - \partial^\nu v^\mu)\rangle\,. \tag{4.4}$$

Vertex $\boldsymbol{Aa}$ As in the previous case, a contribution to the coupling between axial-vector external source and axial-vector resonance is given by the Lagrangian (2.70) with the relevant part

$$\mathcal{L}_A^{(4)} = -\frac{F_A}{\sqrt{2}} \langle A_{\mu\nu} (\partial^\mu a^\nu - \partial^\nu a^\mu) \rangle \,. \tag{4.5}$$

Contributions up to $\mathcal{O}(p^6)$

Now, we can finally determine the independent operator basis of $\mathcal{O}(p^6)$. We need operators that couple vector or axial-vector resonances together with the building blocks $u^\mu, h^{\mu\nu}$ and $f_\pm^{\mu\nu}$. Having considered all possible couplings from the tables 2.3-2.5, the contributing Lagrangians of $\mathcal{O}(p^6)$ can be schematically designed with respect to the resonance exchanges, such as

$$\mathcal{L}^V = \mathcal{L}_{11}^V + \mathcal{L}_{12}^V + \mathcal{L}_{16}^V + \mathcal{L}_{17}^V \,, \tag{4.6}$$

$$\mathcal{L}^{VV} = \mathcal{L}_3^{VV} + \mathcal{L}_4^{VV} \,, \tag{4.7}$$

$$\mathcal{L}^{VA} = \mathcal{L}_5^{VA} \,, \tag{4.8}$$

with the corresponding operators shown in the table below.

i	$\mathcal{O}^V_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{VV,VA}_{i\,\mu\nu\alpha\beta}$
11	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, f_-^{\beta\sigma}\}\rangle g_{\rho\sigma}$	3	$\langle\{\nabla_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}u^\beta\rangle$
12	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, h^{\beta\sigma}\}\rangle g_{\rho\sigma}$	4	$\langle\{\nabla^\beta V^{\mu\nu}, V^{\alpha\sigma}\}u_\sigma\rangle$
16	$\langle V^{\mu\nu}\{\nabla^\alpha f_+^{\beta\sigma}, u_\sigma\}\rangle$	5	$\langle\{V^{\mu\nu}, A^{\alpha\rho}\}f_+^{\beta\sigma}\rangle g_{\rho\sigma}$
17	$\langle V^{\mu\nu}\{\nabla_\sigma f_+^{\alpha\sigma}, u^\beta\}\rangle$		

Table 4.1: Monomials contributing into VVA Green function

Now we modify all contributing operators by rewriting them into the individual terms of vector and axial-vector external sources, pseudoscalar and resonance fields. In doing so, we will realize that we will have a rich set of vertices, given to the possible expansion of the chiral operators. To simplify the writing, we use the notation (D.1).

Vertex $\boldsymbol{Vva}$ The first possibility consists of one vector resonance field coupled to vector and axial-vector external sources. The relevant building blocks are $f_-^{\beta\sigma}, h^{\beta\sigma}, u_\sigma, u^\beta$ as axial-vector sources.

$$\mathcal{L}_{11}^V \simeq -4\kappa_{11}^V \langle V^{\mu\nu}\{\partial^\alpha v^\rho - \partial^\rho v^\sigma, \partial^\beta a^\sigma - \partial^\sigma a^\beta\}\rangle g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \tag{4.9}$$

$$= -2\sqrt{2}\kappa_{11}^V \sum_{(a,b)} d^{abc} V_a^{\mu\nu} (\partial^\alpha v_b^\rho - \partial^\rho v_b^\alpha)(\partial^\beta a_c^\sigma - \partial^\sigma a_c^\beta) g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \,, \tag{4.10}$$

$$\mathcal{L}_{12}^V \simeq 4\kappa_{12}^V \langle V^{\mu\nu}\{\partial^\alpha v^\rho - \partial^\rho v^\alpha, \partial^\beta a^\sigma + \partial^\sigma v^\beta\}\rangle g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \tag{4.11}$$

$$= 2\sqrt{2}\kappa_{12}^V \sum_{(a,b)} d^{abc} V_a^{\mu\nu} (\partial^\alpha v_b^\rho - \partial^\rho v_b^\alpha)(\partial^\beta a_c^\sigma + \partial^\sigma a_c^\beta) g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \,, \tag{4.12}$$

$$\mathcal{L}_{16}^V \simeq 4\kappa_{16}^V \langle V^{\mu\nu}\{\partial^\alpha\partial^\beta v^\sigma - \partial^\alpha\partial^\sigma v^\beta, a_\sigma\}\rangle \varepsilon_{\mu\nu\alpha\beta} \tag{4.13}$$

$$= 2\sqrt{2}\kappa_{16}^V \sum_{(a,b)} d^{abc} V_a^{\mu\nu} (\partial^\alpha\partial^\beta v_b^\sigma - \partial^\alpha\partial^\sigma v_b^\beta) a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta} \,, \tag{4.14}$$

$$\mathcal{L}_{17}^V \simeq 4\kappa_{17}^V \langle V^{\mu\nu}\{\partial_\sigma\partial^\alpha v^\sigma - \partial_\sigma\partial^\sigma v^\alpha, a^\beta\}\rangle \varepsilon_{\mu\nu\alpha\beta} \tag{4.15}$$

$$= 2\sqrt{2}\kappa_{17}^V \sum_{(a,b)} d^{abc} V_a^{\mu\nu} (\partial_\sigma\partial^\alpha v_b^\sigma - \partial_\sigma\partial^\sigma v_b^\alpha) a_c^\beta \varepsilon_{\mu\nu\alpha\beta} \,. \tag{4.16}$$

Vertex $V v \phi$ The second possibility is to consider building blocks $h^{\beta\sigma}, u_\sigma, u^\beta$ as pseudoscalar fields. Since the operator $f_-^{\beta\sigma}$ obviously does not contribute as a pseudoscalar field, the Lagrangian $\mathcal{L}_{11}$ is trivially omitted here.

$$\mathcal{L}_{12}^V \simeq -\frac{4\sqrt{2}\kappa_{12}^V}{F}\langle V^{\mu\nu}\{\partial^\alpha v^\rho - \partial^\rho v^\alpha, \partial^\beta\partial^\sigma\phi\}\rangle g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \tag{4.17}$$

$$= -\frac{4\sqrt{2}\kappa_{12}^V}{F}\sum_{(a,b)} d^{abc}V_a^{\mu\nu}(\partial^\alpha v_b^\rho - \partial^\rho v_b^\alpha)(\partial^\beta\partial^\sigma\phi_c)g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta}\,, \tag{4.18}$$

$$\mathcal{L}_{16}^V \simeq -\frac{2\sqrt{2}\kappa_{16}^V}{F}\langle V^{\mu\nu}\{\partial^\alpha\partial^\beta v^\sigma - \partial^\alpha\partial^\sigma v^\beta, \partial_\sigma\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.19}$$

$$= -\frac{2\sqrt{2}\kappa_{16}^V}{F}\sum_{(a,b)} d^{abc}V_a^{\mu\nu}(\partial^\alpha\partial^\beta v_b^\sigma - \partial^\alpha\partial^\sigma v_b^\beta)(\partial_\sigma\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,, \tag{4.20}$$

$$\mathcal{L}_{17}^V \simeq -\frac{2\sqrt{2}\kappa_{17}^V}{F}\langle V^{\mu\nu}\{\partial_\sigma\partial^\alpha v^\sigma - \partial_\sigma\partial^\sigma v^\alpha, \partial^\beta\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.21}$$

$$= -\frac{2\sqrt{2}\kappa_{17}^V}{F}\sum_{(a,b)} d^{abc}V_a^{\mu\nu}(\partial_\sigma\partial^\alpha v_b^\sigma - \partial_\sigma\partial^\sigma v_b^\alpha)(\partial^\beta\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,. \tag{4.22}$$

Vertex VVa The third possibility is to consider u^β, u_σ as axial-vector sources.

$$\mathcal{L}_3^{VV} \simeq 2\kappa_3^{VV}\langle\{\partial_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}a^\beta\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.23}$$

$$= 2\kappa_3^{VV}\sum_{(a,b)} d^{abc}(\partial_\sigma V_a^{\mu\nu})V_b^{\alpha\sigma}a_c^\beta\varepsilon_{\mu\nu\alpha\beta}\,, \tag{4.24}$$

$$\mathcal{L}_4^{VV} \simeq 2\kappa_4^{VV}\langle\{\partial^\beta V^{\mu\nu}, V^{\alpha\sigma}\}a_\sigma\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.25}$$

$$= 2\kappa_4^{VV}\sum_{(a,b)} d^{abc}(\partial^\beta V_a^{\mu\nu})V_b^{\alpha\sigma}a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta}\,. \tag{4.26}$$

Vertex $VV\phi$ The fourth possibility is to consider u^β, u_σ as pseudoscalar fields.

$$\mathcal{L}_3^{VV} \simeq -\frac{\sqrt{2}\kappa_3^{VV}}{F}\langle\{\partial_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}\partial^\beta\phi\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.27}$$

$$= -\frac{2\kappa_3^{VV}}{F}\sum_{(a,b)} d^{abc}(\partial_\sigma V_a^{\mu\nu})V_b^{\alpha\sigma}(\partial^\beta\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,, \tag{4.28}$$

$$\mathcal{L}_4^{VV} \simeq -\frac{\sqrt{2}\kappa_4^{VV}}{F}\langle\{\partial^\beta V^{\mu\nu}, V^{\alpha\sigma}\}\partial_\sigma\phi\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.29}$$

$$= -\frac{2\kappa_4^{VV}}{F}\sum_{(a,b)} d^{abc}(\partial^\beta V_a^{\mu\nu})V_b^{\alpha\sigma}(\partial_\sigma\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,. \tag{4.30}$$

Vertex VAv The last possibility is quite simple, beacuse the operator $f_+^{\beta\sigma}$ represents linearly only a vector source.

$$\mathcal{L}_5^{VA} \simeq 2\kappa_5^{VA}\langle\{V^{\mu\nu}, A^{\alpha\rho}\}(\partial^\beta v^\sigma - \partial^\sigma v^\beta)\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{4.31}$$

$$= 2\kappa_5^{VA}\sum_{(a,b)} d^{abc}V_a^{\mu\nu}(\partial^\beta v_b^\sigma - \partial^\sigma v_b^\beta)A_c^{\alpha\rho}g_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta}\,. \tag{4.32}$$

These possibilities give us together five Feynman diagrams, which we will calculate now.

4.2 Feynman rules

Here we present all Feynman rules of contributing vertices. First of all, let us start with the propagators needed in this case.

Tensor propagator The kinetic and mass terms form the tensor propagator

$$i(\Delta_R(p))^{ab}_{\alpha\beta\rho\sigma} = -\frac{i\delta^{ab}}{M_R^2(p^2-M_R^2)}\left[g_{\alpha\rho}g_{\beta\sigma}(M_R^2-p^2)+g_{\alpha\rho}p_\beta p_\sigma - g_{\alpha\sigma}p_\beta p_\rho\right] - (\alpha\leftrightarrow\beta)\,, \tag{4.33}$$

where R stands for any resonance field that carry two Lorentz indices in the antisymmetric tensor formalism, of course, i.e. $R = V, A$.

Pseudoscalar propagator The kinetic term comes from $\mathcal{L}_\chi^{(2)}$ whilst the mass term does not exist since pseudoscalars are massless in the chiral limit. Then, the pseudoscalar propagator has the form

$$i(\Delta_P(r))^{ab} = \frac{i}{r^2}\delta^{ab}\,. \tag{4.34}$$

Figure 4.1: Tensor (left) and pseudoscalar (right) propagators.

Now we will present the Feynman rules for the Lagrangians up to $\mathcal{O}(p^6)$. An important note regarding the fixation of the notation is in order. Wherever we will have an external source with the Lorentz index μ, we will assign it the group index a and 4-momentum p, i.e. we will fix the trio (μ, a, p). Similarly, for the external sources with Lorentz indices ν and ρ we will have fixed choices (ν, b, q) and (ρ, c, r).

Vertex WZW This vertex consists of two vector sources and one pseudoscalar. The contributing Lagrangian is the anomaly Wess-Zumino-Witten Lagrangian (4.3). The Feynman rule is due to Bose statistic

$$(V_{WZW})^{abd}_{\mu\nu} = -i\frac{N_C}{8\pi^2 F}d^{abd}\varepsilon_{\mu\nu(p)(q)}\,. \tag{4.35}$$

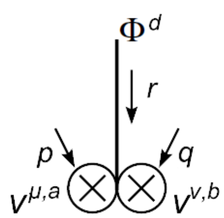


Figure 4.2: Feynman diagram of Wess-Zumino-Witten $vv\phi$ vertex.

Vertex 1 This vertex consists of a pseudoscalar field coupled to an axial-vector external source. The contributing Lagrangian is (4.2) with the Feynman rule for this vertex

$$(V_1)^{ad}_\mu = Fp_\mu\delta^{ad}\,. \tag{4.36}$$

Vertex 2 This vertex consists of axial-vector source and axial-vector resonance. The contributing Lagrangian is (4.5). The Feynman rule is

$$(V_2)^{ad}_{\mu\alpha\beta} = -\frac{F_A}{2}(p_\alpha g_{\mu\beta} - p_\beta g_{\mu\alpha})\delta^{ad}\,. \tag{4.37}$$

Vertex 3 This vertex consists of vector source and vector resonance. The contributing Lagrangian is (4.4). The Feynman rule is

$$(V_3)^{ad}_{\mu\alpha\beta} = \frac{F_V}{2}(p_\alpha g_{\mu\beta} - p_\beta g_{\mu\alpha})\delta^{ad}\,. \tag{4.38}$$

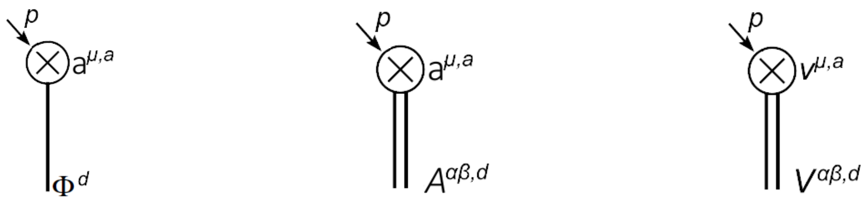


Figure 4.3: Feynman diagrams of vertices 1 (left), 2 (middle) and 3 (right).

Vertex 4 This vertex consists of axial-vector source, vector source and vector resonance. The contributing Lagrangians are (4.10)-(4.16). The second permutation of this vertex is simply obtained by the interchange $(\mu, a, p) \leftrightarrow (\nu, b, q)$. this behaviour is typical for every vertex in the case of VVA Green function. The Feynman rule of this vertex is

$$\begin{aligned}
(V_4^1)^{acd}_{\mu\rho\alpha\beta} =& - 2i\sqrt{2}d^{acd}\Big[(\kappa^V_{11} - \kappa^V_{12})g_{\mu\rho}\varepsilon_{\alpha\beta(p)(q)} + (\kappa^V_{11} - \kappa^V_{12})p_\rho\varepsilon_{\alpha\beta\mu(r)} \qquad (4.39)\\
&+ \kappa_{16}p_\rho\varepsilon_{\alpha\beta\mu(p)} + \frac{1}{2}(p^2 - q^2 + r^2)(\kappa^V_{11} + \kappa^V_{12})\varepsilon_{\alpha\beta\mu\rho} - p^2\kappa^V_{17}\varepsilon_{\alpha\beta\mu\rho}\\
&- \kappa^V_{17}p_\mu\varepsilon_{\alpha\beta\rho(p)} - (\kappa^V_{11} + \kappa^V_{12})r_\mu\varepsilon_{\alpha\beta\rho(p)}\Big]\,,\\
(V_4^2)^{bcd}_{\nu\rho\alpha\beta} =& \quad 2i\sqrt{2}d^{bcd}\Big[(\kappa^V_{11} - \kappa^V_{12})g_{\nu\rho}\varepsilon_{\alpha\beta(p)(q)} - (\kappa^V_{11} - \kappa_{12})q_\rho\varepsilon_{\alpha\beta\nu(r)} \qquad (4.40)\\
&- \kappa_{16}q_\rho\varepsilon_{\alpha\beta\nu(q)} + \frac{1}{2}(p^2 - q^2 - r^2)(\kappa^V_{11} + \kappa^V_{12})\varepsilon_{\alpha\beta\nu\rho} + q^2\kappa_{17}\varepsilon_{\alpha\beta\nu\rho}\\
&+ \kappa^V_{17}q_\nu\varepsilon_{\alpha\beta\rho(q)} + (\kappa^V_{11} + \kappa^V_{12})r_\nu\varepsilon_{\alpha\beta\rho(q)}\Big]\,.
\end{aligned}$$

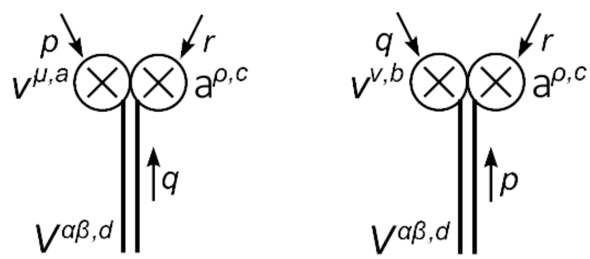


Figure 4.4: Feynman diagrams of vertex 4 (both permutations).

Vertex 5 This vertex consists of vector source, vector resonance and pseudoscalar. The contributing Lagrangians are (4.18)-(4.22). The Feynman rule is

$$
\begin{aligned}
(V_5^1)^{ade}_{\mu\alpha\beta} = & -\frac{2\sqrt{2}d^{ade}}{F}\Big[\frac{1}{2}(p^2-q^2+r^2)\kappa^V_{16}\varepsilon_{\alpha\beta\mu(p)} + 2\kappa^V_{12} r_\mu \varepsilon_{\alpha\beta(p)(q)} \qquad (4.41)\\
& + \kappa^V_{17} p_\mu \varepsilon_{\alpha\beta(p)(q)} - (p^2-q^2+r^2)\kappa^V_{12}\varepsilon_{\alpha\beta\mu(r)} + p^2\kappa^V_{17}\varepsilon_{\alpha\beta\mu(r)}\Big] ,\\
(V_5^2)^{bde}_{\nu\alpha\beta} = & \quad \frac{2\sqrt{2}d^{bde}}{F}\Big[\frac{1}{2}(p^2-q^2-r^2)\kappa^V_{16}\varepsilon_{\alpha\beta\nu(q)} + 2\kappa^V_{12} r_\nu \varepsilon_{\alpha\beta(p)(q)} \qquad (4.42)\\
& + \kappa^V_{17} q_\nu \varepsilon_{\alpha\beta(p)(q)} - (p^2-q^2-r^2)\kappa^V_{12}\varepsilon_{\alpha\beta\nu(r)} - q^2\kappa^V_{17}\varepsilon_{\alpha\beta\nu(r)}\Big] .
\end{aligned}
$$

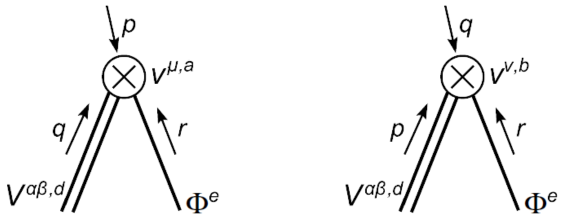


Figure 4.5: Feynman diagrams of vertex 5 (both permutations).

Vertex 6 This vertex consists of axial-vector source and two vector resonances. The contributing Lagrangians are (4.24)-(4.26). The Feynman rule is

$$
\begin{aligned}
(V_6)^{cde}_{\rho\alpha\beta\gamma\delta} = & -\kappa^{VV}_3 d^{cde}(p_\gamma\varepsilon_{\alpha\beta\delta\rho} - p_\delta\varepsilon_{\alpha\beta\gamma\rho} + q_\alpha\varepsilon_{\beta\gamma\delta\rho} - q_\beta\varepsilon_{\alpha\gamma\delta\rho}) \qquad (4.43)\\
& -\kappa^{VV}_4 d^{cde}(g_{\gamma\rho}\varepsilon_{\alpha\beta\delta(p)} - g_{\delta\rho}\varepsilon_{\alpha\beta\gamma(p)} + g_{\alpha\rho}\varepsilon_{\beta\gamma\delta(q)} - g_{\beta\rho}\varepsilon_{\alpha\gamma\delta(q)}) .
\end{aligned}
$$

Vertex 7 This vertex consists of two vector resonances and one pseudoscalar. The contributing Lagrangians are (4.28)-(4.30). The Feynman rule is

$$
\begin{aligned}
(V_7)^{def}_{\alpha\beta\gamma\delta} = & -\frac{i\kappa^{VV}_3 d^{def}}{F}(p_\gamma\varepsilon_{\alpha\beta\delta(r)} - p_\delta\varepsilon_{\alpha\beta\gamma(r)} + q_\alpha\varepsilon_{\beta\gamma\delta(r)} - q_\beta\varepsilon_{\alpha\gamma\delta(r)}) \qquad (4.44)\\
& -\frac{i\kappa^{VV}_4 d^{def}}{F}(r_\gamma\varepsilon_{\alpha\beta\delta(p)} - r_\delta\varepsilon_{\alpha\beta\gamma(p)} + r_\alpha\varepsilon_{\beta\gamma\delta(q)} - r_\beta\varepsilon_{\alpha\gamma\delta(q)}) .
\end{aligned}
$$

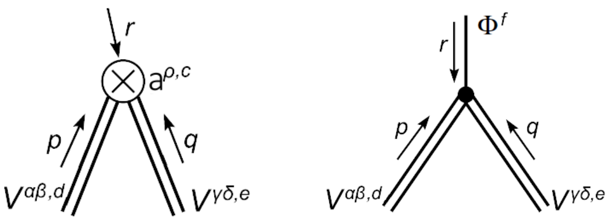


Figure 4.6: Feynman diagrams of vertex 6 (left) and vertex 7 (right).

Vertex 8 This vertex consists of vector source, vector resonance and axial-vector resonance. The contributing Lagrangian is (4.32). The Feynman rule is

$$
(V_8^1)^{ade}_{\mu\alpha\beta\gamma\delta} = -\kappa^{VA}_5 d^{ade}(g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(p)} - g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(p)} - p_\gamma\varepsilon_{\alpha\beta\delta\mu} + p_\delta\varepsilon_{\alpha\beta\gamma\mu}) , \qquad (4.45)
$$

$$
(V_8^2)^{bde}_{\nu\alpha\beta\gamma\delta} = -\kappa^{VA}_5 d^{bde}(g_{\gamma\nu}\varepsilon_{\alpha\beta\delta(q)} - g_{\delta\nu}\varepsilon_{\alpha\beta\gamma(q)} - q_\gamma\varepsilon_{\alpha\beta\delta\nu} + q_\delta\varepsilon_{\alpha\beta\gamma\nu}) . \qquad (4.46)
$$

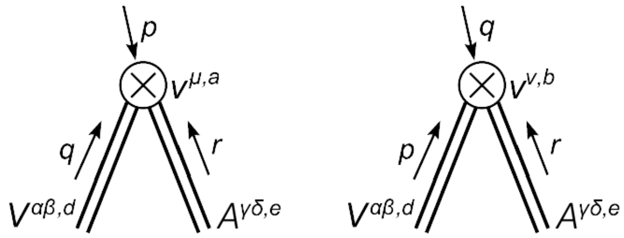


Figure 4.7: Feynman diagrams of vertex 8 (both permutations).

Subdiagram 1 This subdiagram consists of vertex 1 (4.36) and pseudoscalar propagator (4.34). The Feynman rule is

$$(S_1)^{cd}_{\rho} = (V_1)^{ce}_{\rho} i\Delta_P(r)^{de} \tag{4.47}$$

$$= \frac{iF}{r^2} r_\rho \delta^{cd} . \tag{4.48}$$

Subdiagram 2 This subdiagram consists of vertex 2 (4.37) and tensor propagator (4.33). The Feynman rule is

$$(S_2)^{cd}_{\rho\alpha\beta} = (V_2)^{ce}_{\rho\gamma\delta} i\big(\Delta_A(r)\big)^{de}_{\gamma\delta\alpha\beta} \tag{4.49}$$

$$= \frac{iF_A}{r^2 - M_A^2}(r_\alpha g_{\rho\beta} - r_\beta g_{\rho\alpha})\delta^{ac} . \tag{4.50}$$

Subdiagram 3 This subdiagram consists of vertex 3 (4.38) and tensor propagator (4.33). The Feynman rule is

$$(S_3)^{ad}_{\mu\alpha\beta} = (V_3)^{ae}_{\mu\gamma\delta} i(\Delta_V(p))^{de}_{\gamma\delta\alpha\beta} \tag{4.51}$$

$$= -\frac{iF_V}{p^2 - M_V^2}(p_\alpha g_{\mu\beta} - p_\beta g_{\mu\alpha})\delta^{ad} . \tag{4.52}$$

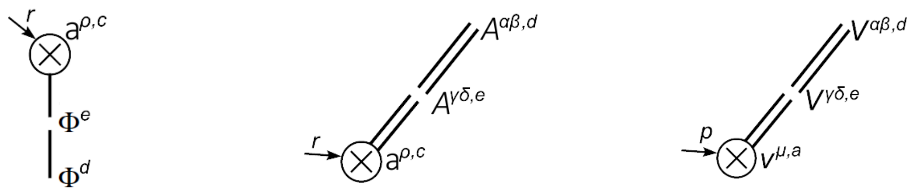


Figure 4.8: Feynman diagrams of subdiagrams 1 (left), 2 (middle) and 3 (right).

4.3 Feynman diagrams

Diagram χ This diagram consists of vertex WZW (4.35) and subdiagram 1 (4.48). The Feynman rule is

$$(\Pi_\chi)^{abc}_{\mu\nu\rho} = (V_{WZW})^{abd}_{\mu\nu}(S_1)^{cd}_{\rho} \tag{4.53}$$

$$= \frac{N_C}{8\pi^2 r^2} d^{abc} \varepsilon_{\mu\nu(p)(q)} r_\rho . \tag{4.54}$$

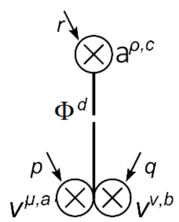


Figure 4.9: Feynman diagram χ.

Diagram 1 This diagram consists of vertex 4 (4.39)-(4.40) and subdiagram 3 (4.52). The Feynman rules of both permutations are as follows

$$(\Pi_1^1)^{abc}_{\mu\nu\rho} = (V_4^1)^{acd}_{\mu\rho\alpha\beta}(S_3)^{bd}_{\nu\alpha\beta} \tag{4.55}$$

$$= -\frac{4\sqrt{2}F_V d^{abc}}{q^2 - M_V^2}\left[\frac{1}{2}(p^2 - q^2 + r^2)(\kappa_{11}^V + \kappa_{12}^V)\varepsilon_{\mu\nu\rho(q)} - p^2\kappa_{17}^V\varepsilon_{\mu\nu\rho(q)}\right. \tag{4.56}$$

$$\left. + \kappa_{17}^V p_\mu \varepsilon_{\nu\rho(p)(q)} + (\kappa_{11}^V + \kappa_{12}^V) r_\mu \varepsilon_{\nu\rho(p)(q)} - (\kappa_{11}^V - \kappa_{12}^V - \kappa_{16}^V) p_\rho \varepsilon_{\mu\nu(p)(q)}\right],$$

$$(\Pi_1^2)^{abc}_{\mu\nu\rho} = (V_4^2)^{bcd}_{\nu\rho\alpha\beta}(S_3)^{ad}_{\mu\alpha\beta} \tag{4.57}$$

$$= -\frac{4\sqrt{2}F_V d^{abc}}{p^2 - M_V^2}\left[\frac{1}{2}(p^2 - q^2 - r^2)(\kappa_{11}^V + \kappa_{12}^V)\varepsilon_{\mu\nu\rho(p)} + q^2\kappa_{17}^V\varepsilon_{\mu\nu\rho(p)}\right. \tag{4.58}$$

$$\left. - \kappa_{17}^V q_\nu \varepsilon_{\mu\rho(p)(q)} - (\kappa_{11}^V + \kappa_{12}^V) r_\nu \varepsilon_{\mu\rho(p)(q)} - (\kappa_{11}^V - \kappa_{12}^V - \kappa_{16}^V) q_\rho \varepsilon_{\mu\nu(p)(q)}\right].$$

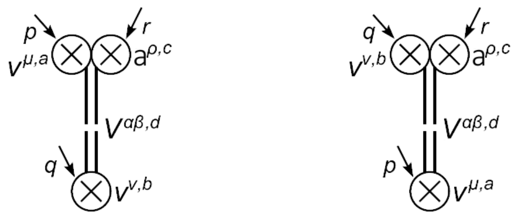


Figure 4.10: Feynman diagram 1 (both permutations).

Diagram 2 This diagram consists of vertex 5 (4.41)-(4.42), subdiagrams 1 (4.48) and 3 (4.52). The Feynman rules of both permutations are as follows

$$(\Pi_2^1)^{abc}_{\mu\nu\rho} = (V_5^1)^{ade}_{\mu\alpha\beta}(S_1)^{ce}_{\rho}(S_3)^{bd}_{\nu\alpha\beta} \tag{4.59}$$

$$= \frac{4\sqrt{2}F_V d^{abc}}{(q^2 - M_V^2)r^2}\left[\frac{1}{2}(-p^2 + q^2 - r^2)(2\kappa_{12}^V + \kappa_{16}^V) + p^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)} r_\rho\,, \tag{4.60}$$

$$(\Pi_2^2)^{abc}_{\mu\nu\rho} = (V_5^2)^{bde}_{\nu\alpha\beta}(S_1)^{ce}_{\rho}(S_3)^{ad}_{\mu\alpha\beta} \tag{4.61}$$

$$= \frac{4\sqrt{2}F_V d^{abc}}{(p^2 - M_V^2)r^2}\left[\frac{1}{2}(p^2 - q^2 - r^2)(2\kappa_{12}^V + \kappa_{16}^V) + q^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)} r^\rho\,. \tag{4.62}$$

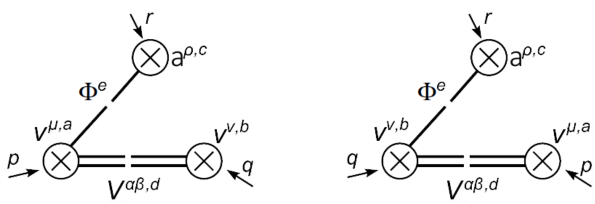


Figure 4.11: Feynman diagram 2 (both permutations).

Diagram 3 This diagram consists of vertex 6 (4.43) and two subdiagrams 3 (4.52). The Feynman rules of both permutations are as follows

$$(\Pi_3)^{abc}_{\mu\nu\rho} = (V_6)^{cde}_{\rho\alpha\beta\gamma\delta}(S_3)^{ad}_{\mu\alpha\beta}(S_3)^{be}_{\nu\gamma\delta} \tag{4.63}$$

$$= \frac{4F_V^2\kappa_3^{VV}d^{abc}}{(p^2-M_V^2)(q^2-M_V^2)}\times \tag{4.64}$$

$$\times\left[\frac{1}{2}(p^2+q^2-r^2)(\varepsilon_{\mu\nu\rho(p)}-\varepsilon_{\mu\nu\rho(q)})-q_\mu\varepsilon_{\nu\rho(p)(q)}+p_\nu\varepsilon_{\mu\rho(p)(q)}\right].$$

Diagram 4 This diagram consists of vertex 7 (4.44), subdiagram 1 (4.48) and two subdiagrams 3 (4.52). The Feynman rules of both permutations are as follows

$$(\Pi_4)^{abc}_{\mu\nu\rho} = (V_7)^{def}_{\alpha\beta\gamma\delta}(S_1)^{cf}_{\rho}(S_3)^{ad}_{\mu\alpha\beta}(S_3)^{be}_{\nu\gamma\delta} \tag{4.65}$$

$$= \frac{4F_V^2\kappa_3^{VV}d^{abc}}{(p^2-M_V^2)(q^2-M_V^2)r^2}(-p^2-q^2+r^2)\varepsilon_{\mu\nu(p)(q)}r_\rho\,. \tag{4.66}$$

Lagrangian (4.30) does not contribute to this diagram after all.

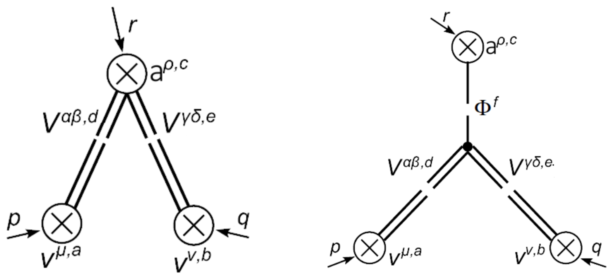


Figure 4.12: Feynman diagrams 3 (left) and 4 (right).

Diagram 5 This diagram consists of vertex 8 (4.45)-(4.46) and subdiagrams 2 (4.50) and 3 (4.52). The Feynman rules of both permutations are as follows

$$(\Pi_5^1)^{abc}_{\mu\nu\rho} = (V_8^1)^{ade}_{\mu\alpha\beta\gamma\delta}(S_2)^{ce}_{\rho\gamma\delta}(S_3)^{bd}_{\nu\alpha\beta} \tag{4.67}$$

$$= \frac{4F_AF_V\kappa_5^{VA}d^{abc}}{(q^2-M_V^2)(r^2-M_A^2)}\left[\frac{1}{2}(p^2-q^2+r^2)\varepsilon_{\mu\nu\rho(q)}-p_\rho\varepsilon_{\mu\nu(p)(q)}+r_\mu\varepsilon_{\nu\rho(p)(q)}\right], \tag{4.68}$$

$$(\Pi_5^2)^{abc}_{\mu\nu\rho} = (V_8^2)^{bde}_{\nu\alpha\beta\gamma\delta}(S_2)^{ce}_{\rho\gamma\delta}(S_3)^{ad}_{\mu\alpha\beta} \tag{4.69}$$

$$= \frac{4F_AF_V\kappa_5^{VA}d^{abc}}{(p^2-M_V^2)(r^2-M_A^2)}\left[\frac{1}{2}(p^2-q^2-r^2)\varepsilon_{\mu\nu\rho(p)}-q_\rho\varepsilon_{\mu\nu(p)(q)}-r_\nu\varepsilon_{\mu\rho(p)(q)}\right]. \tag{4.70}$$

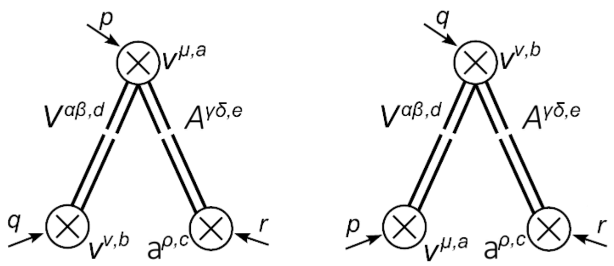


Figure 4.13: Feynman diagram 5 (both permutations).

Taking all contributions together, we have established all Feynman diagrams that contribute to (4.1).

4.4 Ward identities

Generally, the Ward identities reflect the global or gauge symmetries of the theory. In our case, the determination of the identites tells us not only if we did not forget any Feynman diagrams in our calculations, i.e. if the topology of the studied Green function is complete, but also it allows us to verify the conservation of the vector current and nonconservation of the axial current. Hence, the Ward identities have the general forms

$$\{p^\mu, q^\nu, r^\rho\}\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho}\,, \tag{4.71}$$

which we will calculate.

Considering the vector Ward identities first, it is very easy to verify that the results are null for all contributing Feynman diagrams, i.e.

$$\{p^\mu; q^\nu\}(\Pi_\chi)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.72}$$
$$\{p^\mu; q^\nu\}(\Pi_1^1)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.73}$$
$$\{p^\mu; q^\nu\}(\Pi_1^2)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.74}$$
$$\{p^\mu; q^\nu\}(\Pi_2^1)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.75}$$
$$\{p^\mu; q^\nu\}(\Pi_2^2)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.76}$$
$$\{p^\mu; q^\nu\}(\Pi_3)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.77}$$
$$\{p^\mu; q^\nu\}(\Pi_4)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.78}$$
$$\{p^\mu; q^\nu\}(\Pi_5^1)^{abc}_{\mu\nu\rho} = \{0;0\}\,, \tag{4.79}$$
$$\{p^\mu; q^\nu\}(\Pi_5^2)^{abc}_{\mu\nu\rho} = \{0;0\}\,. \tag{4.80}$$

Then we can simply write

$$p^\mu\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = 0\,, \tag{4.81}$$
$$q^\nu\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = 0\,. \tag{4.82}$$

The axial Ward identites are little tricky. Not only that some contributions are nonzero after multiplication by r^ρ, but we can also notice the fact that some diagrams compensate one another in order to reconstruct the anomaly term that causes the nonconservation of the axial current. All individual results are listed below.

$$r^\rho(\Pi_\chi)^{abc}_{\mu\nu\rho} = \frac{N_C d^{abc}}{8\pi^2}\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.83}$$
$$r^\rho(\Pi_1^1)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}d^{abc}F_V}{q^2-M_V^2}\left[\frac{1}{2}(p^2-q^2+r^2)(2\kappa_{12}^V+\kappa_{16}^V) - p^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.84}$$
$$r^\rho(\Pi_1^2)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}d^{abc}F_V}{p^2-M_V^2}\left[\frac{1}{2}(-p^2+q^2+r^2)(2\kappa_{12}^V+\kappa_{16}^V) - q^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.85}$$
$$r^\rho(\Pi_2^1)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}d^{abc}F_V}{q^2-M_V^2}\left[\frac{1}{2}(-p^2+q^2-r^2)(2\kappa_{12}^V+\kappa_{16}^V) + p^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.86}$$
$$r^\rho(\Pi_2^2)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}d^{abc}F_V}{p^2-M_V^2}\left[\frac{1}{2}(p^2-q^2-r^2)(2\kappa_{12}^V+\kappa_{16}^V) + q^2\kappa_{17}^V\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.87}$$

$$r^\rho(\Pi_3)^{abc}_{\mu\nu\rho} = \frac{4F_V^2 d^{abc}\kappa_3^{VV}}{(p^2-M_V^2)(q^2-M_V^2)}(p^2+q^2-r^2)\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.88}$$

$$r^\rho(\Pi_4)^{abc}_{\mu\nu\rho} = \frac{4F_V^2 d^{abc}\kappa_3^{VV}}{(p^2-M_V^2)(q^2-M_V^2)}(-p^2-q^2+r^2)\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.89}$$

$$r^\rho(\Pi_5^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{4.90}$$

$$r^\rho(\Pi_5^2)^{abc}_{\mu\nu\rho} = 0\,. \tag{4.91}$$

Now, it is obvious which diagrams compensate one another. More specifically, we see that

$$r^\rho(\Pi_1^1)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_2^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{4.92}$$

$$r^\rho(\Pi_1^2)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_2^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{4.93}$$

$$r^\rho(\Pi_3)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_4)^{abc}_{\mu\nu\rho} = 0\,. \tag{4.94}$$

Knowing that, we can finally establish that axial Ward identity takes the form

$$r^\rho\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = \frac{N_C}{8\pi^2}d^{abc}\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.95}$$

i.e. the axial Ward identity is determined only by the anomalous term, as expected.

knowing the expected results of the Ward identities, let us rewrite the standard definition (4.1) of the VVA correlator as

$$\big(\Pi_{VVA}(p,q;r)\big)^{abc}_{\mu\nu\rho} \equiv d^{abc}\Pi_{\mu\nu\rho}(p,q;r)\,, \tag{4.96}$$

where we have already separated the tensor structure and the part that comes from the traces over flavor space. The Ward identities restrict the general decomposition of the tensor part in (4.96) into four terms,

$$\Pi_{\mu\nu\rho}(p,q;r) = w_L\varepsilon_{\mu\nu(p)(q)}r_\rho + w_T^{(1)}\Pi^{(1)}_{\mu\nu\rho} + w_T^{(2)}\Pi^{(2)}_{\mu\nu\rho} + w_T^{(3)}\Pi^{(3)}_{\mu\nu\rho}\,, \tag{4.97}$$

where w_L is the longitudinal part, entirely fixed by the anomaly term (4.54), and a trio of formfactors

$$w_T^{(1)} \equiv w_T^{(1)}\big(p^2,q^2,r^2\big) = +w_T^{(1)}\big(p^2,q^2,r^2\big)\,, \tag{4.98}$$

$$w_T^{(2)} \equiv w_T^{(2)}\big(p^2,q^2,r^2\big) = -w_T^{(2)}\big(p^2,q^2,r^2\big)\,, \tag{4.99}$$

$$w_T^{(3)} \equiv w_T^{(3)}\big(p^2,q^2,r^2\big) = -w_T^{(3)}\big(p^2,q^2,r^2\big)\,, \tag{4.100}$$

where we also introduced their properties under the Bose symmetry, and that stand by the transversal tensors [20]

$$\begin{aligned}\Pi^{(1)}_{\mu\nu\rho} \equiv \Pi^{(1)}_{\mu\nu\rho}(p,q;r) = p_\nu\varepsilon_{\mu\rho(p)(q)} - q_\mu\varepsilon_{\nu\rho(p)(q)} - \frac{p^2+q^2-r^2}{r^2}\varepsilon_{\mu\nu(p)(q)}r_\rho \\ - \frac{1}{2}(-p^2-q^2+r^2)(\varepsilon_{\mu\nu\rho(p)} - \varepsilon_{\mu\nu\rho(q)})\,,\end{aligned} \tag{4.101}$$

$$\Pi^{(2)}_{\mu\nu\rho} \equiv \Pi^{(2)}_{\mu\nu\rho}(p,q;r) = \left[(p-q)_\rho + \frac{p^2-q^2}{r^2}r_\rho\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{4.102}$$

$$\Pi^{(3)}_{\mu\nu\rho} \equiv \Pi^{(3)}_{\mu\nu\rho}(p,q;r) = p_\nu\varepsilon_{\mu\rho(p)(q)} + q_\mu\varepsilon_{\nu\rho(p)(q)} + \frac{1}{2}(-p^2-q^2+r^2)\varepsilon_{\mu\nu\rho(r)}\,. \tag{4.103}$$

Extraction of the formfactors

Now, our task will be an extraction of the formfactors $w_T^{(1)}, w_T^{(2)}, w_T^{(3)}$. Although we already know the form of w_L, we will include it in our procedure just to be sure we will be able to recover the correct result.

Let us remind ourselves that in section 4.3 we have provided the final result (4.96). To compare the result with the decomposition (4.97) in order to obtain the formfactors, a simple method comes to mind. The principal is to multiply the expressions (4.96),(4.97) by the components of 4-momenta p, q, r that will not make the results vanish.

Easily, the components p^ν and q^μ are clear choices. However, due to many nonzero terms in (4.83)-(4.91), we will also take the components p^ρ, q^ρ into account. The reason why we do not consider r^ρ at once is that we are looking for four formfactors so we need a system of four equations in order to be able to solve it. If we did not want to recover w_L formfactor, a system of three equations, considering only r^ρ instead of both p^ρ and q^ρ, would be sufficient.

Following the procedure above, using (4.96) we have

$$\begin{aligned} p^\nu\Pi_{\mu\nu\rho}(p,q;r) =& -\frac{2\sqrt{2}F_V}{p^2-M_V^2}\Big[(\kappa_{11}^V+\kappa_{12}^V)(p^2-q^2+r^2)+\kappa_{17}^V(p^2+q^2-r^2)\Big] \\ &+\frac{2\sqrt{2}F_V}{q^2-M_V^2}\Big[(\kappa_{11}^V+\kappa_{12}^V)(p^2-q^2+r^2)-2p^2\kappa_{17}^V\Big] \\ &+\frac{2F_V^2\kappa_3^{VV}(3p^2+q^2-r^2)}{(p^2-M_V^2)(q^2-M_V^2)}-\frac{2F_AF_V\kappa_5^{VA}(p^2-q^2)(p^2-q^2+r^2)}{(p^2-M_V^2)(q^2-M_V^2)(r^2-M_A^2)}, \end{aligned} \tag{4.104}$$

$$\begin{aligned} q^\mu\Pi_{\mu\nu\rho}(p,q;r) =& \frac{2\sqrt{2}F_V}{p^2-M_V^2}\Big[(\kappa_{11}^V+\kappa_{12}^V)(p^2-q^2-r^2)+2q^2\kappa_{17}^V\Big] \\ &-\frac{2\sqrt{2}F_V}{q^2-M_V^2}\Big[(\kappa_{11}^V+\kappa_{12}^V)(p^2-q^2-r^2)-\kappa_{17}^V(p^2+q^2-r^2)\Big] \\ &-\frac{2F_V^2\kappa_3^{VV}(p^2+3q^2-r^2)}{(p^2-M_V^2)(q^2-M_V^2)}+\frac{2F_AF_V\kappa_5^{VA}(p^2-q^2)(p^2-q^2-r^2)}{(p^2-M_V^2)(q^2-M_V^2)(r^2-M_A^2)}, \end{aligned} \tag{4.105}$$

$$\begin{aligned} p^\rho\Pi_{\mu\nu\rho}(p,q;r) =& \frac{1}{16r^2}\Bigg\{\frac{16F_V(p^2-q^2)}{(p^2-M_V^2)(q^2-M_V^2)(r^2-M_A^2)}\times \\ &\times\bigg[2p^2\Big[r^2\big(-F_A\kappa_5^{VA}+\sqrt{2}M_A^2(-\kappa_{11}^V+\kappa_{12}^V+\kappa_{16}^V-\kappa_{17}^V)+F_V\kappa_3^{VV} \\ &+\sqrt{2}M_V^2\kappa_{17}^V-\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V)\big)+\sqrt{2}(r^2)^2(\kappa_{11}^V-\kappa_{12}^V-\kappa_{16}^V+\kappa_{17}^V) \\ &+M_A^2\big(-F_V\kappa_3^{VV}-\sqrt{2}M_V^2\kappa_{17}^V+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V)\big)\Big] \\ &+(q^2-r^2)\Big[r^2\big(2(-F_A\kappa_5^{VA}+F_V\kappa_3^{VV}+\sqrt{2}M_V^2\kappa_{17}^V) \\ &+\sqrt{2}M_A^2(\kappa_{16}^V-2\kappa_{11}^V)+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\big) \\ &-M_A^2\big(2(F_V\kappa_3^{VV}+\sqrt{2}M_V^2\kappa_{17}^V)+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\big) \\ &+\sqrt{2}(r^2)^2(2\kappa_{11}^V-\kappa_{16}^V)\Big]+\sqrt{2}(p^2)^2(r^2-M_A^2)(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\bigg] \\ &-\frac{N_C(p^2-q^2+r^2)}{\pi^2}\Bigg\}, \end{aligned} \tag{4.106}$$

$$
\begin{aligned}
q^\rho\Pi_{\mu\nu\rho}(p,q;r) = &-\frac{1}{16r^2}\Bigg\{\frac{16F_V(p^2-q^2)}{(p^2-M_V^2)(q^2-M_V^2)(r^2-M_A^2)}\times \\
&\times\Bigg[2p^2\Big[r^2\big(-F_A\kappa_5^{VA}+\sqrt{2}M_A^2(-\kappa_{11}^V+\kappa_{12}^V+\kappa_{16}^V-\kappa_{17}^V)+F_V\kappa_3^{VV} \\
&+\sqrt{2}M_V^2\kappa_{17}^V-\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V)\big)+\sqrt{2}(r^2)^2(\kappa_{11}^V-\kappa_{12}^V-\kappa_{16}^V+\kappa_{17}^V) \\
&+M_A^2\big(-F_V\kappa_3^{VV}-\sqrt{2}M_V^2\kappa_{17}^V+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V)\big)\Big] \\
&+(q^2-r^2)\Big[r^2\big(2(-F_A\kappa_5^{VA}+F_V\kappa_3^{VV}+\sqrt{2}M_V^2\kappa_{17}^V) \\
&+\sqrt{2}M_A^2(\kappa_{16}^V-2\kappa_{11}^V)+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\big) \\
&-M_A^2\big(2(F_V\kappa_3^{VV}+\sqrt{2}M_V^2\kappa_{17}^V)+\sqrt{2}q^2(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\big) \\
&+\sqrt{2}(r^2)^2(2\kappa_{11}^V-\kappa_{16}^V)\Big] \\
&+\sqrt{2}(p^2)^2(r^2-M_A^2)(2\kappa_{12}^V+\kappa_{16}^V-2\kappa_{17}^V)\Bigg] \\
&+\frac{N_C(p^2-q^2-r^2)}{\pi^2}\Bigg\}.
\end{aligned} \tag{4.107}
$$

On the other hand, using (4.97) we get a simpler system of equations:

$$
p^\nu\Pi_{\mu\nu\rho}(p,q;r) = \frac{1}{2}p^2(3w_T^{(1)}+w_T^{(3)})+\frac{1}{2}(q^2-r^2)(w_T^{(1)}-w_T^{(3)})\,, \tag{4.108}
$$

$$
q^\mu\Pi_{\mu\nu\rho}(p,q;r) = -\frac{1}{2}p^2(w_T^{(1)}+w_T^{(3)})+\frac{1}{2}q^2(w_T^{(3)}-3w_T^{(1)})+\frac{1}{2}r^2(w_T^{(1)}+w_T^{(3)})\,, \tag{4.109}
$$

$$
\begin{aligned}
p^\rho\Pi_{\mu\nu\rho}(p,q;r) = &-\frac{p^2}{2}(w_L+w_T^{(1)}-2w_T^{(2)}-w_T^{(3)})+\frac{(p^2)^2}{2r^2}(w_T^{(1)}-w_T^{(2)}) \\
&+\frac{p^2q^2}{r^2}w_T^{(2)}-\frac{q^2-r^2}{2r^2}\Big[q^2(w_T^{(1)}+w_T^{(2)})-r^2(w_L+w_T^{(2)}+w_T^{(3)})\Big]\,,
\end{aligned} \tag{4.110}
$$

$$
\begin{aligned}
q^\rho\Pi_{\mu\nu\rho}(p,q;r) = &\frac{p^2}{2}(w_L+w_T^{(1)}-2w_T^{(2)}-w_T^{(3)})+\frac{(p^2)^2}{2r^2}(w_T^{(2)}-w_T^{(1)}) \\
&-\frac{p^2q^2}{r^2}w_T^{(2)}-\frac{q^2}{2}(w_L+w_T^{(1)}+2w_T^{(2)}+w_T^{(3)}) \\
&+\frac{(q^2)^2}{2r^2}(w_T^{(1)}+w_T^{(2)})+\frac{r^2}{2}(-w_L+w_T^{(2)}+w_T^{(3)})\,.
\end{aligned} \tag{4.111}
$$

By comparing appropriate expressions in both systems, we are able to determine the final results for the formfactors:

$$
w_L = \frac{N_C}{8\pi^2r^2}\,, \tag{4.112}
$$

$$
w_T^{(1)} = -\frac{2\sqrt{2}F_V\left[\kappa_{17}^V(p^2+q^2-2M_V^2)-\sqrt{2}F_V\kappa_3^{VV}\right]}{(p^2-M_V^2)(q^2-M_V^2)}\,, \tag{4.113}
$$

$$
w_T^{(2)} = -\frac{2\sqrt{2}F_V(p^2-q^2)(2\kappa_{12}^V+\kappa_{16}^V-\kappa_{17}^V)}{(p^2-M_V^2)(q^2-M_V^2)}\,, \tag{4.114}
$$

$$
w_T^{(3)} = \frac{2\sqrt{2}F_V(p^2-q^2)}{(p^2-M_V^2)(q^2-M_V^2)}\left(2\kappa_{11}^V+2\kappa_{12}^V-\kappa_{17}^V-\frac{\sqrt{2}F_A\kappa_5^{VA}}{r^2-M_A^2}\right). \tag{4.115}
$$

4.5 Coupling constants constraints

To be able to extract some relations for the coupling constants we will construct the formactor in the form [20]

$$w_T(Q^2) = -16\pi^2\left[w_T^{(1)}(-Q^2,0,-Q^2) + w_T^{(3)}(-Q^2,0,-Q^2)\right]. \tag{4.116}$$

Using (4.98),(4.100), we simply obtain

$$w_T^{(1)}(-Q^2,0,-Q^2) = \frac{2\sqrt{2}F_V\left[\kappa_{17}^V(Q^2+2M_V^2)+\sqrt{2}F_V\kappa_3^{VV}\right]}{M_V^2(Q^2+M_V^2)}, \tag{4.117}$$

$$w_T^{(3)}(-Q^2,0,-Q^2) = -\frac{2\sqrt{2}F_VQ^2}{M_V^2(Q^2+M_V^2)}\left(2\kappa_{11}^V+2\kappa_{12}^V-\kappa_{17}^V+\frac{\sqrt{2}F_A\kappa_5^{VA}}{Q^2+M_A^2}\right), \tag{4.118}$$

i.e.

$$\begin{aligned} w_T(Q^2) = &-\frac{2\sqrt{2}F_V}{M_V^2(Q^2+M_V^2)}\left[Q^2\left(2\kappa_{11}^V+2\kappa_{12}^V+\frac{\sqrt{2}F_A\kappa_5^{VA}}{Q^2+M_A^2}\right)-\sqrt{2}F_V\kappa_3^{VV}\right] \\ &+\frac{4\sqrt{2}F_V\kappa_{17}^V}{M_V^2}. \end{aligned} \tag{4.119}$$

In anticipation of what will follow, we will expand (4.119) into a series in the terms of Q^2 up to $\mathcal{O}(1/Q^8)$. But first, let us simplify the previous expression by substituing for κ_{17}^V from one of the coupling constraints for VVP Green function, [1], (3.19):

$$\kappa_{17}^V = -\frac{N_C}{64\sqrt{2}\pi^2F_V}. \tag{4.120}$$

Then, the series reads

$$\begin{aligned} w_T(Q^2) = &-\frac{4F_V}{M_V^2}\left[\frac{N_C}{64\pi^2F_V}+\sqrt{2}(\kappa_{11}^V+\kappa_{12}^V)\right] \\ &+\frac{4F_V}{Q^2}\left[\frac{F_V\kappa_3^{VV}-F_A\kappa_5^{VA}}{M_V^2}+\sqrt{2}(\kappa_{11}^V+\kappa_{12}^V)\right] \\ &-\frac{4F_V}{Q^4}\left[F_V\kappa_3^{VV}+\sqrt{2}M_V^2(\kappa_{11}^V+\kappa_{12}^V)-F_A\kappa_5^{VA}\left(1+\frac{M_A^2}{M_V^2}\right)\right] \\ &+\frac{4F_V}{Q^6}\left[M_V^2(F_V\kappa_3^{VV}-F_A\kappa_5^{VA})-F_AM_A^2\kappa_5^{VA}\left(1+\frac{M_A^2}{M_V^2}\right)+\sqrt{2}M_V^4(\kappa_{11}^V+\kappa_{12}^V)\right] \\ &+\mathcal{O}\left(\frac{1}{Q^8}\right). \end{aligned} \tag{4.121}$$

The result for (4.116) up to $\mathcal{O}(1/Q^8)$ can also be obtained from the OPE framework, in which we have [21], [22]

$$w_T(Q^2) = \frac{N_C}{Q^2}+\frac{128\pi^3\alpha_s\chi\langle\overline{q}q\rangle^2}{9Q^6}+\mathcal{O}\left(\frac{1}{Q^8}\right). \tag{4.122}$$

By a comparison between (4.121) and (4.122) one can easily obtain the following constraints for the coupling constants:

$$\kappa_{11}^V+\kappa_{12}^V = -\frac{N_C}{64\sqrt{2}\pi^2F_V}, \tag{4.123}$$

$$\kappa_3^{VV} = -\frac{N_CM_V^4}{64\pi^2M_A^2F_V^2} \tag{4.124}$$

and

$$\kappa_5^{VA} = \kappa_3^{VV}\frac{F_V}{F_A} \tag{4.125}$$
$$= -\frac{N_C M_V^4}{64\pi^2 M_A^2 F_V F_A}\,. \tag{4.126}$$

Also, looking at (3.17), we can realize that knowing the constraint (4.124) for κ_3^{VV}, we can extract a relation for κ_2^{VV}

$$\kappa_2^{VV} = \frac{1}{64\pi^2}\left(F^2 - \frac{N_C M_V^4}{8\pi^2 M_A^2}\right), \tag{4.127}$$
$$\tag{4.128}$$

and then, from (3.16), also a constraint for κ_3^{PV}:

$$\kappa_3^{PV} = -\frac{F^2}{32\sqrt{2}d_m F_V}\left[1 + \frac{N_C M_V^2}{8\pi^2 F^2}\left(\frac{M_V^2}{M_A^2} - 1\right)\right]. \tag{4.129}$$

By the determination of (4.129) we can also obtain a relation for the deviation δ_{BL} from the form of κ_3^{PV} if we take the Brodsky-Lepage behaviour [1], [23], [24] of the $\mathcal{F}_{\pi^0\gamma\gamma}$ formfactor into account (see (4.139)). Hence, the prediction is

$$\delta_{\mathrm{BL}} = \frac{N_C M_V^2}{8\pi^2 F^2}\left(\frac{M_V^2}{M_A^2} - 1\right). \tag{4.130}$$

Finally, let us also introduce the numerical values of the fully obtained parameters:

$$\kappa_3^{VV} = -0.067\,, \tag{4.131}$$
$$\kappa_5^{VA} = -0.086\,, \tag{4.132}$$
$$\kappa_3^{PV} = 0.017\,. \tag{4.133}$$

and

$$\delta_{\mathrm{BL}} = -1.342\,. \tag{4.134}$$

We do not calculate errors of the obatined parameters since we get the values purely by matching two theoretical approaches. Also, the only parameter that comes with an uncertainty is F_V but it is very small, less than two orders. To obtain the parameters (4.131)-(4.134) we used the following numerical values:

$$M_V = 0.775\,\mathrm{GeV}\,, \tag{4.135}$$
$$F = 92.22\,\mathrm{MeV}\,, \tag{4.136}$$
$$F_V = (146.3 \pm 1.2)\,\mathrm{MeV}\,, \tag{4.137}$$
$$d_m = 26\,\mathrm{MeV}\,. \tag{4.138}$$

Now, a discussion involving the parameter δ_{BL} is in order. Knowing the values (4.133)-(4.134), one can apply them to study a particular example that could be verified either by other theoretical consequences or experiments. In our case, a $\pi^0\gamma\gamma$ formfactor is a suitable tool. The formfactor is originally determined as

$$\mathcal{F}_{\pi^0\gamma\gamma}^{\mathrm{R}\chi\mathrm{T}}(p^2, q^2; 0) = \frac{F}{3(p^2 - M_V^2)(q^2 - M_V^2)}\times \tag{4.139}$$
$$\times\left[(p^2 + q^2)\left(1 + 32\sqrt{2}\frac{d_m F_V}{F^2}\kappa_3^{PV}\right) - \frac{N_C M_V^4}{4\pi^2 F^2}\right].$$

Substituing from the expression (4.129) we can get

$$\mathcal{F}^{\mathrm{R}\chi\mathrm{T}}_{\pi^0\gamma\gamma}(p^2,q^2;0) = -\frac{F}{3(p^2-M_V^2)(q^2-M_V^2)}\left[\delta_{BL}(p^2+q^2)+\frac{N_C M_V^4}{4\pi^2F^2}\right], \tag{4.140}$$

from which we can extract the experimentally measured object

$$\mathcal{F}^{\mathrm{R}\chi\mathrm{T}}_{\pi^0\gamma\gamma}(0,-Q^2;0) = \frac{F}{3(p^2-M_V^2)(q^2-M_V^2)}\left(Q^2\delta_{BL}-\frac{N_C M_V^4}{4\pi^2F^2}\right), \tag{4.141}$$

which is sensitive to the value of δ_{BL}.

The formfactor (4.141) is depicted in Fig. 4.14 for our value (4.134) and for two other values, based on [1]. The figure also contains experimentally obtained values of $\mathcal{F}^{\mathrm{R}\chi\mathrm{T}}_{\pi^0\gamma\gamma}(0,-Q^2;0)$ from experiments BABAR [25], BELLE [26] and CLEO [27].

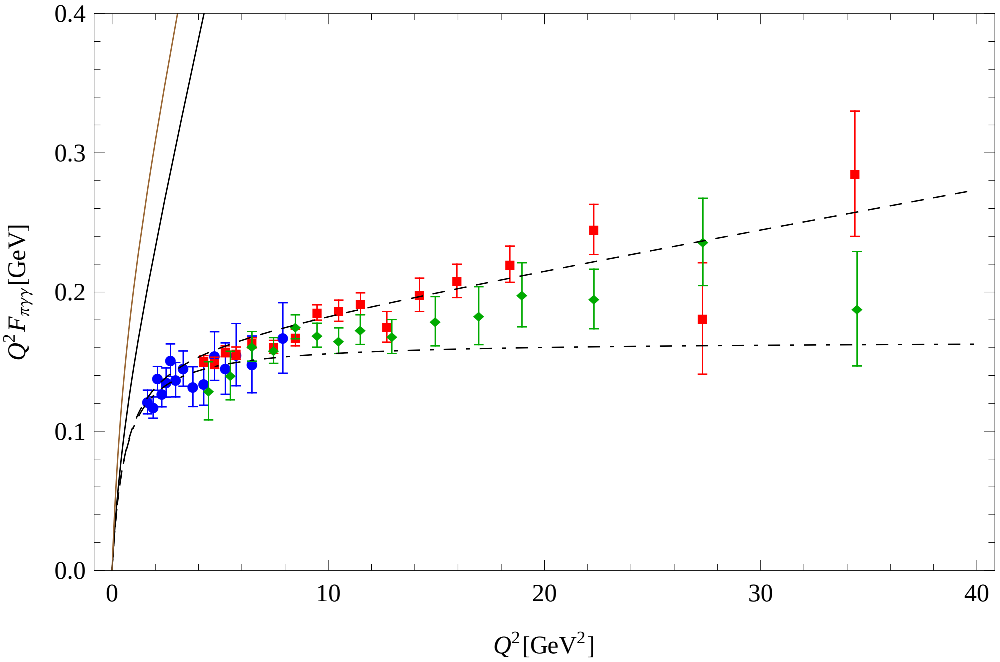


Figure 4.14: A plot of BABAR (red), BELLE (green) and CLEO (blue) data fitted with the formfactor $\mathcal{F}^{\mathrm{R}\chi\mathrm{T}}_{\pi^0\gamma\gamma}(0,-Q^2;0)$ (4.141) using the modified Brodsky-Lepage condition. The full black line represents our fit with $\delta_{\mathrm{BL}} = -1.342$ (4.134) and the full brown line is a fit using the LMD behaviour of the formfactor (4.145). The dashed line stands for $\delta_{\mathrm{BL}} = -0.055$ and the dot-dashed line for $\delta_{\mathrm{BL}} = 0$.

We can clearly see that the formfactor (4.141) does not agree with the experimental data. To get a full notion, one should discuss first if we even have a sufficiently consistent theoretical model to describe such a behaviour. In other words, is it sufficient not to add any other resonance fields and still have an agreement with the experiments? Obviously, not. The reason is that our formfactor (4.141) for the value (4.134) is very close to the behaviour of the formfactor $\mathcal{F}^{\mathrm{LMD}}(p^2,q^2;r^2)$ describing the lowest meson dominance (LMD) [15], [28] which is defined through the formfactor $\mathcal{F}^{\mathrm{VMD}}(p^2,q^2;r^2)$ of the vector meson dominance (VMD) [15] as

$$\mathcal{F}^{\mathrm{VMD}}(p^2,q^2;r^2) = -\frac{N_C}{8\pi^2F}\frac{M_V^4}{(p^2-M_V^2)(q^2-M_V^2)}, \tag{4.142}$$

$$\mathcal{F}^{\mathrm{LMD}}(p^2,q^2;r^2) = \mathcal{F}^{\mathrm{VMD}}(p^2,q^2;r^2)\left[1-\frac{4\pi^2F^2(p^2+q^2)}{N_C M_V^4}\right]. \tag{4.143}$$

In our case, both expression take the forms

$$\mathcal{F}^{\mathrm{VMD}}(0,-Q^2;0) = -\frac{N_C}{8\pi^2 F}\frac{M_V^2}{Q^2+M_V^2}\,, \tag{4.144}$$

$$\mathcal{F}^{\mathrm{LMD}}(0,-Q^2;0) = -\frac{N_C}{8\pi^2 F}\frac{M_V^2}{Q^2+M_V^2}\left(1+\frac{4\pi^2F^2}{N_C M_V^4}Q^2\right). \tag{4.145}$$

To support our explanation, the LMD fomrfactor (4.145) is also depicted in Fig. 4.14.

Finally, let us also mention that by using (4.129), one can determine more specific relations for two low-energy constants:

$$C_7^W = \frac{F^2}{32M_V^2}\left(\frac{1}{2M_V^2}+\frac{1}{M_P^2}\right)+\frac{d_m F_V^2}{2M_P^2M_V^4}\kappa^{VVP}+\frac{N_C}{256\pi^2M_P^2}\left(\frac{M_V^2}{M_A^2}-1\right), \tag{4.146}$$

$$C_{22}^W = \frac{N_C}{128\pi^2}\left(\frac{1}{M_V^2}+\frac{1}{M_A^2}\right). \tag{4.147}$$

4.6 Phenomenology

As a phenomenological example of the VVA Green function, we can study a decay of the axial-vector meson $f_1(1285)$,

$$f_1(1285) \to \rho + \gamma\,, \tag{4.148}$$

with the branching ratio [29]

$$\mathrm{Br} = 0.055 \pm 0.013\,. \tag{4.149}$$

An assignment of the 4-momenta is exactly the same as previously in this chapter, i.e. the axial-vector meson $f_1(1285)$ carries 4-momentum r whilst vector states ρ and γ carry 4-momenta p and q, respectively.

Since the photon in the final state is massless, we are required to take the decay width in the form[1]

$$\Gamma_{f\to\rho\gamma} = \frac{1}{3}\frac{1}{8\pi}\sum_{\text{pol.}}|\mathcal{M}|^2\frac{M_f^2-M_\rho^2}{2M_f^3}\,, \tag{4.150}$$

with the obvious designation of the masses of the particles involved. Also, we have considered the decaying meson to be unpolarized.

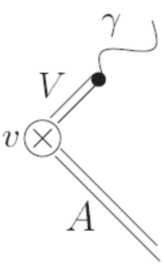


Figure 4.15: A scheme of the decay $f_1(1285) \to \rho\gamma$: axial-vector resonance represents the decaying $f_1(1285)$ meson and the vector external source stands for the ρ meson.

[1]For the sake of simplicity in equations, we will designate $f_1(1285) \equiv f$.

The matrix element

Before we make the step to contructing the matrix element itself, let us explain why we can consider the decay channel that includes the electromagnetic interaction. The reason is simple, considering the external sources (1.37) we have defined in Chapter 1. Since the quarks carry an electric charge, considering the vector-like nature of the electromagnetic current, we can construct the external sources in the form

$$v_\mu = -e\mathcal{A}_\mu Q\,, \tag{4.151}$$

$$a_\mu = 0\,, \tag{4.152}$$

where we have explicitly excluded the contribution of the axial-vector external source to the interaction. In (4.151), we have designated $\mathcal{A}_\mu$ as an electromagnetic field and Q stands for the quark charge matrix

$$Q = \frac{1}{3}\mathrm{diag}(2,-1,-1)\,. \tag{4.153}$$

It is useful to express the quark charge matrix in terms of the Gell-Mann matrices. This simply leads to the expression for (4.151) in the form

$$v_\mu = -e\mathcal{A}_\mu\left(T^3 + \frac{1}{\sqrt{3}}T^8\right). \tag{4.154}$$

In order to have a nonzero contribution to the matrix element, i.e. to be able to reconstruct the experimentally observed decay (4.148), we are now capable of resolving that the matrix element has the form

$$i\mathcal{M} = ie\varepsilon^\mu(p)\left[\frac{1}{M_VF_V}\varepsilon^\nu(q)(q^2-M_V^2)\right]\left[\frac{1}{M_AF_A}\varepsilon^\rho(r)(r^2-M_A^2)\right]\Pi^{338}_{\mu\nu\rho}\,, \tag{4.155}$$

where $\Pi^{338}_{\mu\nu\rho}$ is given by (4.1) for the particular choice of group indices due to the physical nature of the process. The contributing part of $\Pi^{338}_{\mu\nu\rho}$ is determined only by the expression proportional to the coupling constant κ_5^{VA}, i.e.

$$\Pi^{338}_{\mu\nu\rho} = \frac{1}{\sqrt{3}}d^{338}\Pi^{(3)}_{\mu\nu\rho}w_T^{(3)} \tag{4.156}$$

$$= -\frac{4F_AF_V\kappa_5^{VA}(p^2-q^2)}{3(p^2-M_V^2)(q^2-M_V^2)(r^2-M_A^2)}\times \tag{4.157}$$

$$\times\left[p_\nu\varepsilon_{\mu\rho(p)(q)} + q_\mu\varepsilon_{\nu\rho(p)(q)} + \frac{1}{2}(-p^2-q^2+r^2)\varepsilon_{\mu\nu\rho(r)}\right].$$

For completeness, let us mention that the $1/\sqrt{3}$ in (4.156) is a relict from (4.154).

Calculation of the decay width

The last expression is necessary to put back to the matrix element. Since this is a quite difficult, we restrict ourselves only to the result. But before we do that, let us only mention a short note regarding the polarization sums. Indeed, the following expressions are useful. First of all, let us deal with the massless photon,

$$\sum_{\text{pol.}}\varepsilon^\mu(p)\varepsilon^{*\mu'}(p) = -g^{\mu\mu'} - \frac{1}{(\eta\cdot p)^2}p^\mu p^{\mu'} + \frac{1}{\eta\cdot p}(p^\mu\eta^{\mu'} + \eta^\mu p^{\mu'})\,, \tag{4.158}$$

where

$$\eta = (1,0,0,0)\,, \tag{4.159}$$

and with the polarisation sums for the mass particles ρ and $f_1(1285)$,

$$\sum_{\text{pol.}} \varepsilon^{\nu}(q)\varepsilon^{*\nu'}(q) = -g^{\nu\nu'} + \frac{q^{\nu}q^{\nu'}}{M_\rho^2}\,, \tag{4.160}$$

$$\sum_{\text{pol.}} \varepsilon^{\rho}(q)\varepsilon^{*\rho'}(q) = -g^{\rho\rho'} + \frac{r^{\rho}r^{\rho'}}{M_f^2}\,. \tag{4.161}$$

Now, let us remind that the contributing part of (4.1) is transversal. In this case, only the metric tensors in all polarization sums contribute, leaving the other parts to vanish. This fact greatly simplifies the calculation. But still, the procedure is complicated due to the difficult tensor structure and many non-vanishing terms. For simplicity, we will skip the whole calculation.

Then, the decay width of this channel is

$$\Gamma_{f\to\rho\gamma} = \frac{e^2 M_A^2 M_\rho^{10}(M_f^2 - M_\rho^2)}{72\pi M_f^3 M_V^6 (M_A^2 - 2M_\rho^2)^2}(\kappa_5^{VA})^2\,. \tag{4.162}$$

Comparing the result with the branching ratio and the total decay width of the $f_1(1285)$ meson, one can determine the coupling constant κ_5^{VA}:

$$\kappa_5^{VA} = -0.062 \pm 0.030\,, \tag{4.163}$$

where we have already chosen the negative value due to the already known sign from (4.124). We see that both values agree on the given order.

5. AAA Green function

The standard definition of the correlator consisting of three axial-vector currents is

$$\left(\Pi_{AAA}(p,q;r)\right)^{abc}_{\mu\nu\rho} = i\int \mathrm{d}^4x\,\mathrm{d}^4y\, e^{i(px+qy)}\,\langle 0|\mathrm{T}\left[A^a_\mu(x)A^b_\nu(y)A^c_\rho(0)\right]|0\rangle\,. \tag{5.1}$$

The strategy of the calculation of (5.1) is the same as in the previous chapter. The only difference is in the bulding blocks, we will need only $u^\mu, h^{\mu\nu}$ and $f_-^{\mu\nu}$. The assignment of the indices is exactly the same, we still hold the following threesomes: (μ,a,p), (ν,b,q) and (ρ,c,r).

5.1 Independent operator basis up to $\mathcal{O}(p^6)$

Now, we can easily start working on the contributing operators. As in the previous case, we will also distinguish between contributions of the individual orders.

Contribution up to $\mathcal{O}(p^2)$

$a\phi$ vertex A very important contribution to AAA correlator comes from χPT. Up to $\mathcal{O}(p^2)$, we will need the coupling between an axial-vector external source and a pseudoscalar field. The relevant part of the Lagrangian (2.16) is

$$\mathcal{L}^{(2)}_\chi = -\frac{F}{\sqrt{2}}\langle\{\partial_\mu\phi, a^\mu\}\rangle\,. \tag{5.2}$$

Contributions up to $\mathcal{O}(p^4)$

Vertex $aa\phi$ We have also another contribution from χPT, more specifically from the anomalous Wess-Zumino-Witten Lagrangian (2.40). The contributing part has the form

$$\mathcal{L}^{(4)}_{\mathrm{WZW}} = \frac{N_C}{12\sqrt{2}\pi^2 F}\big\langle(\partial_\mu\phi)v_\nu(\partial_\alpha v_\beta)\big\rangle\varepsilon^{\mu\nu\alpha\beta} + \frac{N_C}{6\sqrt{2}\pi^2 F}\big\langle(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\big\rangle\varepsilon^{\mu\nu\alpha\beta} \tag{5.3}$$

and couples two vector external sources with the pseudoscalar field.

Vertex Aa First resonance contribution to the AAA Green function comes as a coupling between the axial-vector external source and the axial-vector resonance. The relevant part of the Lagrangian (2.70) is

$$\mathcal{L}^{(4)}_A = -\frac{F_A}{\sqrt{2}}\langle A_{\mu\nu}(\partial^\mu a^\nu - \partial^\nu a^\mu)\rangle\,. \tag{5.4}$$

Contributions up to $\mathcal{O}(p^6)$

Finally, the contributions from Lagrangians up to $\mathcal{O}(p^6)$. The AAA Green function can consist only one or two axial resonance fields. Therefore, a set of all possible contributions from tables 2.3-2.5 is easily ascertainable. Then, we can schematically write the relevant Lagrangians in the form

$$\mathcal{L}^A = \mathcal{L}^A_3 + \mathcal{L}^A_8 + \mathcal{L}^A_{15} + \mathcal{L}^A_{16}\,, \tag{5.5}$$
$$\mathcal{L}^{AA} = \mathcal{L}^{AA}_3 + \mathcal{L}^{AA}_4\,, \tag{5.6}$$

where all the individual operators are listed in the table below.

i	$\mathcal{O}^{A}_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{AA}_{i\,\mu\nu\alpha\beta}$
3	$\langle A^{\mu\nu}\{\nabla^\alpha h^{\beta\sigma}, u_\sigma\}\rangle$	3	$\langle\{\nabla_\sigma A^{\mu\nu}, A^{\alpha\sigma}\}u^\beta\rangle$
8	$\langle A^{\mu\nu}\{f_-^{\alpha\sigma}, h^{\beta\sigma}\}\rangle$	4	$\langle\{\nabla^\beta A^{\mu\nu}, A^{\alpha\sigma}\}u_\sigma\rangle$
15	$\langle A^{\mu\nu}\{\nabla^\alpha f_-^{\beta\sigma}, u_\sigma\}\rangle$		
16	$\langle A^{\mu\nu}\{\nabla_\sigma f_-^{\alpha\sigma}, u^\beta\}\rangle$		

Table 5.1: Monomials contributing into AAA Green function

Now we can use the table 5.1 and rewrite the Lagrangians into its individual terms. For simplicity, we will distinguish between all possible topological contributions. To simplify the writing, we use the notation (D.2).

Vertex $A\phi\phi$ First of all, let us turn our attention to the vertex consisted of one axial-vector resonance and two pseudoscalar fields. The only possible contribution could arise from the Lagrangian below:

$$\mathcal{L}_3^A \simeq \frac{4\kappa_3^A}{F^2}\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta\partial^\sigma\phi, \partial_\sigma\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.7}$$

$$= \frac{4\sqrt{2}\kappa_3^A}{F^2}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta\partial^\sigma\phi_b)(\partial_\sigma\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.8}$$

Given the structure of derivatives of the pseudoscalar field, coming from the term $\nabla^\alpha h^{\beta\sigma}$, the Feynman rule is trivially zero due to the product of the symmetric combination of 4-momenta, carrying Lorentz indices, and antisymmetric Levi-Civita tensor. Hence, there are not any diagrams consisted of three axial-vector external sources coupled together through one axial-vector resonance and two pseudoscalar fields.

Vertex Aaa The first nontrivial possibility is to consider all chiral building blocks $h^{\mu\nu}$, $f_-^{\mu\nu}$ and u^μ to be axial-vector external sources. Relevant Lagrangians are then

$$\mathcal{L}_3^A \simeq 4\kappa_3^A\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta a^\sigma + \partial^\alpha\partial^\sigma a^\beta, a_\sigma\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.9}$$

$$= 2\sqrt{2}\kappa_3^A\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta a_b^\sigma + \partial^\alpha\partial^\sigma a_b^\beta)a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.10}$$

$$\mathcal{L}_8^A \simeq -4\kappa_8^A\langle A^{\mu\nu}\{\partial^\alpha a^\sigma - \partial^\sigma a^\alpha, \partial^\beta a^\sigma + \partial^\sigma a^\beta\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.11}$$

$$= -2\sqrt{2}\kappa_8^A\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha a_b^\sigma - \partial^\sigma a_b^\alpha)(\partial^\beta a_c^\sigma + \partial^\sigma a_c^\beta)\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.12}$$

$$\mathcal{L}_{15}^A \simeq -4\kappa_{15}^A\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta a^\sigma - \partial^\alpha\partial^\sigma a^\beta, a_\sigma\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.13}$$

$$= -2\sqrt{2}\kappa_{15}^A\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta a_b^\sigma - \partial^\alpha\partial^\sigma a_b^\beta)a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.14}$$

$$\mathcal{L}_{16}^A \simeq -4\kappa_{16}^A\langle A^{\mu\nu}\{\partial_\sigma\partial^\alpha a^\sigma - \partial_\sigma\partial^\sigma a^\alpha, a^\beta\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.15}$$

$$= -2\sqrt{2}\kappa_{16}^A\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial_\sigma\partial^\alpha a_b^\sigma - \partial_\sigma\partial^\sigma a_b^\alpha)a_c^\beta\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.16}$$

Vertex $Aa\phi$ This is a little complicated. The nontrivial contribution of the operator $\mathcal{O}_3^A$ is to consider $h^{\beta\sigma}$ to be an axial-vector sources and u_σ as a pseudoscalar field. In the operators $\mathcal{O}_8^A, \mathcal{O}_{15}^A, \mathcal{O}_{16}^A$ we consider $h^{\beta\sigma}, u_\sigma, u^\beta$ to be pseudoscalar fields.

$$\mathcal{L}_3^A \simeq -\frac{2\sqrt{2}\kappa_3^A}{F}\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta a^\sigma + \partial^\alpha\partial^\sigma a^\beta, \partial_\sigma\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.17}$$

$$= -\frac{2\sqrt{2}\kappa_3^A}{F}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta a_b^\sigma + \partial^\alpha\partial^\sigma a_b^\beta)\partial_\sigma\phi_c\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.18}$$

$$\mathcal{L}_8^A \simeq \frac{4\sqrt{2}\kappa_8^A}{F}\langle A^{\mu\nu}\{\partial^\alpha a^\sigma - \partial^\sigma a^\alpha, \partial^\beta\partial^\sigma\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.19}$$

$$= \frac{4\sqrt{2}\kappa_8^A}{F}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha a_b^\sigma - \partial^\sigma a_b^\alpha)\partial^\beta\partial^\sigma\phi_c\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.20}$$

$$\mathcal{L}_{15}^A \simeq \frac{2\sqrt{2}\kappa_{15}^A}{F}\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta a^\sigma - \partial^\alpha\partial^\sigma a^\beta, \partial_\sigma\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.21}$$

$$= \frac{2\sqrt{2}\kappa_{15}^A}{F}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta a_b^\sigma - \partial^\alpha\partial^\sigma a_b^\beta)\partial_\sigma\phi_c\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.22}$$

$$\mathcal{L}_{16}^A \simeq \frac{2\sqrt{2}\kappa_{16}^A}{F}\langle A^{\mu\nu}\{\partial_\sigma\partial^\alpha a^\sigma - \partial_\sigma\partial^\sigma a^\alpha, \partial^\beta\phi\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.23}$$

$$= \frac{2\sqrt{2}\kappa_{16}^A}{F}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial_\sigma\partial^\alpha a_b^\sigma - \partial_\sigma\partial^\sigma a_b^\alpha)\partial^\beta\phi_c\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.24}$$

The additional possibility of considering $h^{\beta\sigma}$ to be a pseudoscalar field (and, of course, u_σ to be an axial-vector source) gives a zero contribution. It is obvious, given the Lagrangian to be

$$\mathcal{L}_3^A \simeq -\frac{4\sqrt{2}\kappa_3^A}{F}\langle A^{\mu\nu}\{\partial^\alpha\partial^\beta\partial^\sigma\phi, a_\sigma\}\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.25}$$

$$= -\frac{4\sqrt{2}\kappa_3^A}{F}\sum_{(a,b,c)} d^{abc}A_a^{\mu\nu}(\partial^\alpha\partial^\beta\partial^\sigma\phi_b)a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.26}$$

Vertex AAa The third possibility is to consider building blocks u^β, u_σ to be axial-vector sources.

$$\mathcal{L}_3^{AA} \simeq 2\kappa_3^{AA}\langle\{\partial_\sigma A^{\mu\nu}, A^{\alpha\sigma}\}a^\beta\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.27}$$

$$= 2\kappa_3^{AA}\sum_{(a,b,c)} d^{abc}(\partial_\sigma A_a^{\mu\nu})A_b^{\alpha\sigma}a_c^\beta\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.28}$$

$$\mathcal{L}_4^{AA} \simeq 2\kappa_4^{AA}\langle\{\partial^\beta A^{\mu\nu}, A^{\alpha\sigma}\}a_\sigma\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.29}$$

$$= 2\kappa_4^{AA}\sum_{(a,b,c)} d^{abc}(\partial^\beta A_a^{\mu\nu})A_b^{\alpha\sigma}a_{\sigma,c}\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.30}$$

Vertex $AA\phi$ Finally, the last possibility is to consider building blocks u^β, u_σ to be pseudoscalar fields.

$$\mathcal{L}_3^{AA} \simeq -\frac{\sqrt{2}\kappa_3^{AA}}{F}\langle\{\partial_\sigma A^{\mu\nu}, A^{\alpha\sigma}\}\partial^\beta\phi\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.31}$$

$$= -\frac{2\kappa_3^{AA}}{F}\sum_{(a,b,c)} d^{abc}(\partial_\sigma A_a^{\mu\nu})A_b^{\alpha\sigma}(\partial^\beta\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,, \tag{5.32}$$

$$\mathcal{L}_4^{AA} \simeq -\frac{\sqrt{2}\kappa_4^{AA}}{F}\langle\{\partial^\beta A^{\mu\nu}, A^{\alpha\sigma}\}\partial_\sigma\phi\rangle\varepsilon_{\mu\nu\alpha\beta} \tag{5.33}$$

$$= -\frac{2\kappa_4^{AA}}{F}\sum_{(a,b,c)} d^{abc}(\partial^\beta A_a^{\mu\nu})A_b^{\alpha\sigma}(\partial_\sigma\phi_c)\varepsilon_{\mu\nu\alpha\beta}\,. \tag{5.34}$$

5.2 Feynman rules

As in previous case, we also here present all Feynman rules of contributing vertices. To be consistent with the previous chapter, we repeat two types of propagators that we will use here, too.

Tensor propagator The kinetic and mass terms form the tensor propagator that in the antisymmtric tensor formalis takes the form

$$i(\Delta_R(p))^{ab}_{\alpha\beta\rho\sigma} = -\frac{i\delta^{ab}}{M_R^2(p^2-M_R^2)}\left[g_{\alpha\rho}g_{\beta\sigma}(M_R^2-p^2)+g_{\alpha\rho}p_\beta p_\sigma - g_{\alpha\sigma}p_\beta p_\rho\right] - (\alpha\leftrightarrow\beta)\,, \tag{5.35}$$

where in this case we only consider $R = A$.

Pseudoscalar propagator The kinetic term comes from $\mathcal{L}_\chi^{(2)}$ and pseudoscalars are massless in the chiral limit. Then, the pseudoscalar propagator is

$$i(\Delta_P(p))^{ab} = \frac{i}{p^2}\delta^{ab}\,. \tag{5.36}$$

Figure 5.1: Tensor (left) and pseudoscalar (right) propagators.

Vertex WZW This vertex consists of two axial sources and a pseudoscalar. The contributing Lagrangian is the anomaly Wess-Zumino-Witten Lagrangian (5.3). The Feynman rule is due to Bose statistic

$$(V^1_{WZW})^{abd}_{\mu\nu} = -\,i\frac{N_C}{24\pi^2F}\varepsilon_{\mu\nu(p)(q)}d^{abd}\,, \tag{5.37}$$

$$(V^2_{WZW})^{abd}_{\mu\nu} = \;\; i\frac{N_C}{24\pi^2F}\varepsilon_{\mu\rho(p)(q)}d^{acd}\,, \tag{5.38}$$

$$(V^3_{WZW})^{abd}_{\mu\nu} = -\,i\frac{N_C}{24\pi^2F}\varepsilon_{\nu\rho(p)(q)}d^{bcd}\,. \tag{5.39}$$

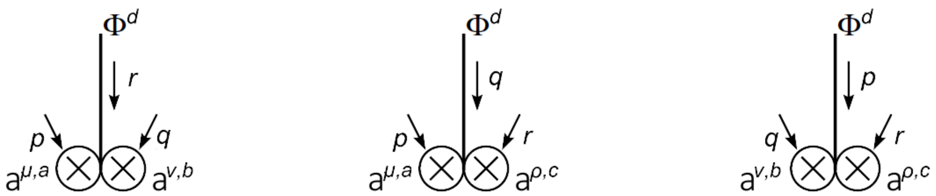


Figure 5.2: Feynman diagram of Wess-Zumino-Witten $aa\phi$ vertex (all permutations).

Vertex 1 This vertex consists of axial source and a pseudoscalar. The contributing Lagrangian is (4.2). The Feynman rule is

$$(V_1)^{ad}_{\mu} = F p_\mu \delta^{ad} \,. \tag{5.40}$$

Vertex 2 This vertex consists of axial source and axial resonance. The contributing Lagrangian is (4.5). The Feynman rule is

$$(V_2)^{ad}_{\mu\alpha\beta} = -\frac{F_A}{2}(p_\alpha g_{\mu\beta} - p_\beta g_{\mu\alpha})\delta^{ad} \,. \tag{5.41}$$

Figure 5.3: Feynman diagrams of vertices 1 (left) and 2 (right).

Vertex 3 This vertex consists of two axial sources and axial resonance. The contributing Lagrangians are (5.10)-(5.16). The Feynman rules of all permutations are as follows

$$\begin{aligned}(V_3^1)^{abd}_{\mu\nu\alpha\beta} =& -2i\sqrt{2}d^{abd}\big[(p^2-q^2)\kappa^A_{16}\varepsilon_{\alpha\beta\mu\nu} - (\kappa^A_3+\kappa^A_{15})(p_\nu\varepsilon_{\alpha\beta\mu(p)} + q_\mu\varepsilon_{\alpha\beta\nu(q)}) \quad (5.42)\\ &+ \kappa^A_{16}(p_\mu\varepsilon_{\alpha\beta\nu(p)} + q_\nu\varepsilon_{\alpha\beta\mu(q)}) + 2\kappa^A_8(q_\mu\varepsilon_{\alpha\beta\nu(p)} + p_\nu\varepsilon_{\alpha\beta\mu(q)})\big] \,,\\ (V_3^2)^{acd}_{\mu\rho\alpha\beta} =& -2i\sqrt{2}d^{acd}\big[(p^2-r^2)\kappa^A_{16}\varepsilon_{\alpha\beta\mu\rho} - (\kappa^A_3+\kappa^A_{15})(p_\rho\varepsilon_{\alpha\beta\mu(p)} + r_\mu\varepsilon_{\alpha\beta\rho(r)}) \quad (5.43)\\ &+ \kappa^A_{16}(p_\mu\varepsilon_{\alpha\beta\rho(p)} + r_\rho\varepsilon_{\alpha\beta\mu(r)}) + 2\kappa^A_8(r_\mu\varepsilon_{\alpha\beta\rho(p)} + p_\rho\varepsilon_{\alpha\beta\mu(r)})\big] \,,\\ (V_3^3)^{bcd}_{\nu\rho\alpha\beta} =& -2i\sqrt{2}d^{bcd}\big[(q^2-r^2)\kappa^A_{16}\varepsilon_{\alpha\beta\nu\rho} - (\kappa^A_3+\kappa^A_{15})(q_\rho\varepsilon_{\alpha\beta\nu(q)} + r_\nu\varepsilon_{\alpha\beta\rho(r)}) \quad (5.44)\\ &+ \kappa^A_{16}(q_\nu\varepsilon_{\alpha\beta\rho(q)} + r_\rho\varepsilon_{\alpha\beta\nu(r)}) + 2\kappa^A_8(r_\nu\varepsilon_{\alpha\beta\rho(q)} + q_\rho\varepsilon_{\alpha\beta\nu(r)})\big] \,.\end{aligned}$$

We can see that every one of the Feynamn rules are interchangeable either under the permutations $(\mu,a,p) \leftrightarrow (\nu,b,q), (\mu,a,p) \leftrightarrow (\rho,c,r)$ or $(\nu,b,q) \leftrightarrow (\rho,c,r)$. This feature is typical for AAA Green functions and we will be able to observe it in the next vertices as well.

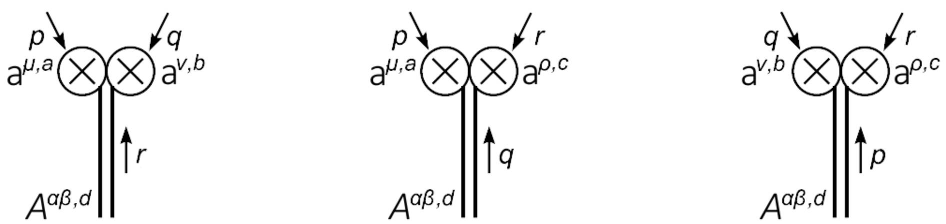


Figure 5.4: Feynman diagrams of vertex 3 (all permutations).

Vertex 4 This vertex consists of axial source, axial resonance and pseudoscalar. The contributing Lagrangians are (5.18)-(5.24). The Feynman rules of all permutations are as follows

$$\begin{aligned}(V_4^1)^{ade}_{\mu\alpha\beta} =& -\frac{2\sqrt{2}d^{ade}}{F}\bigg[\frac{1}{2}(\kappa^A_{15}+\kappa^A_3)(-p^2+q^2-r^2)\varepsilon_{\alpha\beta\mu(p)} - p^2\kappa^A_{16}\varepsilon_{\alpha\beta\mu(r)} \quad (5.45)\\ &- \kappa^A_8(-p^2+q^2-r^2)\varepsilon_{\alpha\beta\mu(r)} - (\kappa^A_{16}p_\mu + 2\kappa^A_8 r_\mu)\varepsilon_{\alpha\beta(p)(q)}\bigg] \,,\end{aligned}$$

$$
\begin{aligned}
(V_4^2)^{ade}_{\mu\alpha\beta} = &- \frac{2\sqrt{2}d^{ade}}{F}\Bigg[\frac{1}{2}(\kappa_3^A + \kappa_{15}^A)(-p^2 - q^2 + r^2)\varepsilon_{\alpha\beta\mu(p)} - p^2\kappa_{16}^A\varepsilon_{\alpha\beta\mu(q)} \\
&- \kappa_8^A(-p^2 - q^2 + r^2)\varepsilon_{\alpha\beta\mu(q)} + (\kappa_{16}^A p_\mu + 2\kappa_8^A q_\mu)\varepsilon_{\alpha\beta(p)(q)}\Bigg] ,
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
(V_4^3)^{bde}_{\nu\alpha\beta} = &- \frac{2\sqrt{2}d^{bde}}{F}\Bigg[\frac{1}{2}(\kappa_3^A + \kappa_{15}^A)(p^2 - q^2 - r^2)\varepsilon_{\alpha\beta\nu(q)} - q^2\kappa_{16}^A\varepsilon_{\alpha\beta\nu(r)} \\
&- \kappa_8^A(p^2 - q^2 - r^2)\varepsilon_{\alpha\beta\nu(r)} + (\kappa_{16}^A q_\nu + 2\kappa_8^A r_\nu)\varepsilon_{\alpha\beta(p)(q)}\Bigg] ,
\end{aligned}
\tag{5.47}
$$

$$
\begin{aligned}
(V_4^4)^{bde}_{\nu\alpha\beta} = &- \frac{2\sqrt{2}d^{bde}}{F}\Bigg[\frac{1}{2}(\kappa_3^A + \kappa_{15}^A)(-p^2 - q^2 + r^2)\varepsilon_{\alpha\beta\nu(q)} - q^2\kappa_{16}^A\varepsilon_{\alpha\beta\nu(p)} \\
&- \kappa_8^A(-p^2 - q^2 + r^2)\varepsilon_{\alpha\beta\nu(p)} - (\kappa_{16} q_\nu + 2\kappa_8^A p_\nu)\varepsilon_{\alpha\beta(p)(q)}\Bigg] ,
\end{aligned}
\tag{5.48}
$$

$$
\begin{aligned}
(V_4^5)^{cde}_{\rho\alpha\beta} = &- \frac{2\sqrt{2}d^{cde}}{F}\Bigg[\frac{1}{2}(\kappa_3^A + \kappa_{15}^A)(p^2 - q^2 - r^2)\varepsilon_{\alpha\beta\rho(r)} - r^2\kappa_{16}^A\varepsilon_{\alpha\beta\rho(q)} \\
&- (\kappa_{16}^A r_\rho + 2\kappa_8^A q_\rho)\varepsilon_{\alpha\beta(p)(q)} - \kappa_8^A(p^2 - q^2 - r^2)\varepsilon_{\alpha\beta\rho(q)}\Bigg] ,
\end{aligned}
\tag{5.49}
$$

$$
\begin{aligned}
(V_4^6)^{cde}_{\rho\alpha\beta} = &- \frac{2\sqrt{2}d^{cde}}{F}\Bigg[\frac{1}{2}(\kappa_3^A + \kappa_{15}^A)(-p^2 + q^2 - r^2)\varepsilon_{\alpha\beta\rho(r)} - r^2\kappa_{16}^A\varepsilon_{\alpha\beta\rho(p)} \\
&- \kappa_8^A(-p^2 + q^2 - r^2)\varepsilon_{\alpha\beta\rho(p)} + (\kappa_{16} r_\rho + 2\kappa_8^A p_\rho)\varepsilon_{\alpha\beta(p)(q)}\Bigg] .
\end{aligned}
\tag{5.50}
$$

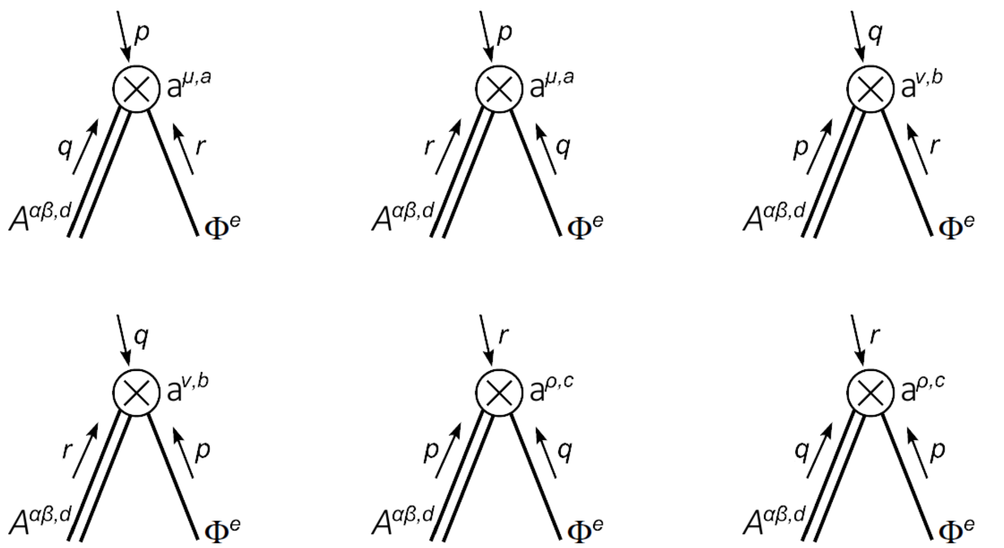


Figure 5.5: Feynman diagrams of vertex 4 (all permutations).

Vertex 5 This vertex consists of axial source and two axial resonances. The contributing Lagrangians are (5.28)-(5.30). The Feynman rules of all permutations are as follows

$$
\begin{aligned}
(V_5^1)^{ade}_{\mu\alpha\beta\gamma\delta} = &- \kappa_3^{VV} d^{ade}(q_\gamma\varepsilon_{\alpha\beta\delta\mu} - q_\delta\varepsilon_{\alpha\beta\gamma\mu} + r_\alpha\varepsilon_{\beta\gamma\delta\mu} - r_\beta\varepsilon_{\alpha\gamma\delta\mu}) \\
&- \kappa_4^{VV} d^{ade}(g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(q)} - g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(q)} + g_{\alpha\mu}\varepsilon_{\beta\gamma\delta(r)} - g_{\beta\mu}\varepsilon_{\alpha\gamma\delta(r)}) ,
\end{aligned}
\tag{5.51}
$$

$$
\begin{aligned}
(V_5^2)^{bde}_{\nu\alpha\beta\gamma\delta} = &- \kappa_3^{VV} d^{bde}(p_\gamma\varepsilon_{\alpha\beta\delta\nu} - p_\delta\varepsilon_{\alpha\beta\gamma\nu} + r_\alpha\varepsilon_{\beta\gamma\delta\nu} - r_\beta\varepsilon_{\alpha\gamma\delta\nu}) \\
&- \kappa_4^{VV} d^{bde}(g_{\gamma\nu}\varepsilon_{\alpha\beta\delta(p)} - g_{\delta\nu}\varepsilon_{\alpha\beta\gamma(p)} + g_{\alpha\nu}\varepsilon_{\beta\gamma\delta(r)} - g_{\beta\nu}\varepsilon_{\alpha\gamma\delta(r)}) ,
\end{aligned}
\tag{5.52}
$$

$$
\begin{aligned}
(V_5^3)^{cde}_{\rho\alpha\beta\gamma\delta} = &- \kappa_3^{VV} d^{cde}(p_\gamma \varepsilon_{\alpha\beta\delta\rho} - p_\delta \varepsilon_{\alpha\beta\gamma\rho} + q_\alpha \varepsilon_{\beta\gamma\delta\rho} - q_\beta \varepsilon_{\alpha\gamma\delta\rho}) \\
&- \kappa_4^{VV} d^{cde}(g_{\gamma\rho}\varepsilon_{\alpha\beta\delta(p)} - g_{\delta\rho}\varepsilon_{\alpha\beta\gamma(p)} + g_{\alpha\rho}\varepsilon_{\beta\gamma\delta(q)} - g_{\beta\rho}\varepsilon_{\alpha\gamma\delta(q)})\,.
\end{aligned}
\tag{5.53}
$$

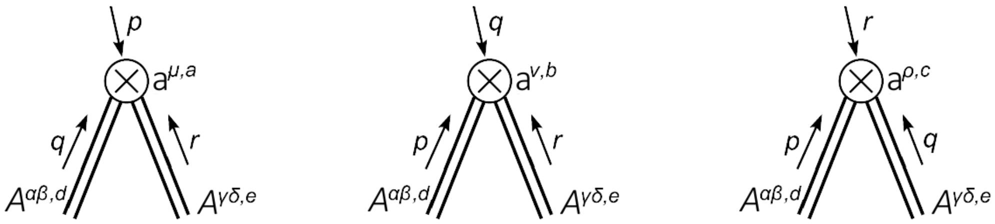


Figure 5.6: Feynman diagrams of vertex 5 (all permutations).

Vertex 6 This vertex consists of two axial resonances and a pseudoscalar. The contributing Lagrangians are (5.32)-(5.34). The Feynman rules of all permutations are as follows

$$
\begin{aligned}
(V_6^1)^{def}_{\alpha\beta\gamma\delta} = &- \frac{i\kappa_3^{VV} d^{def}}{F}(q_\gamma \varepsilon_{\alpha\beta\delta(p)} - q_\delta \varepsilon \alpha\beta\gamma(p) + r_\alpha \varepsilon_{\beta\gamma\delta(p)} - r_\beta \varepsilon_{\alpha\gamma\delta(p)}) \\
&- \frac{i\kappa_4^{VV} d^{def}}{F}(p_\gamma \varepsilon_{\alpha\beta\delta(q)} - p_\delta \varepsilon_{\alpha\beta\gamma(q)} + p_\alpha \varepsilon_{\beta\gamma\delta(r)} - p_\beta \varepsilon_{\alpha\gamma\delta(r)})\,,
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
(V_6^2)^{def}_{\alpha\beta\gamma\delta} = &- \frac{i\kappa_3^{VV} d^{def}}{F}(p_\gamma \varepsilon_{\alpha\beta\delta(q)} - p_\delta \varepsilon_{\alpha\beta\gamma(q)} + r_\alpha \varepsilon_{\beta\gamma\delta(q)} - r_\beta \varepsilon_{\alpha\gamma\delta(q)}) \\
&- \frac{i\kappa_4^{VV} d^{def}}{F}(q_\gamma \varepsilon_{\alpha\beta\delta(p)} - q_\delta \varepsilon_{\alpha\beta\gamma(p)} + q_\alpha \varepsilon_{\beta\gamma\delta(r)} - q_\beta \varepsilon_{\alpha\gamma\delta(r)})\,,
\end{aligned}
\tag{5.55}
$$

$$
\begin{aligned}
(V_6^3)^{def}_{\alpha\beta\gamma\delta} = &- \frac{i\kappa_3^{VV} d^{def}}{F}(p_\gamma \varepsilon_{\alpha\beta\delta(r)} - p_\delta \varepsilon_{\alpha\beta\gamma(r)} + q_\alpha \varepsilon_{\beta\gamma\delta(r)} - q_\beta \varepsilon_{\alpha\gamma\delta(r)}) \\
&- \frac{i\kappa_4^{VV} d^{def}}{F}(r_\gamma \varepsilon_{\alpha\beta\delta(p)} - r_\delta \varepsilon_{\alpha\beta\gamma(p)} + r_\alpha \varepsilon_{\beta\gamma\delta(q)} - r_\beta \varepsilon_{\alpha\gamma\delta(q)})\,.
\end{aligned}
\tag{5.56}
$$

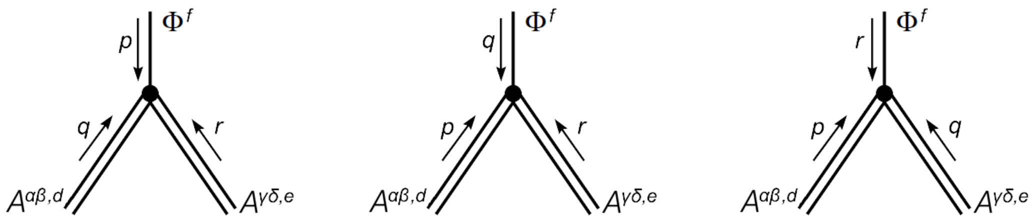


Figure 5.7: Feynman diagrams of vertex 6 (all permutations).

In order to simplify our calculations as much as possible, we now construct subdiagrams consisting of one vertex and one propagator. In the case of AAA correlator we have the following subdiagrams.

Subdiagram 1 This subdiagram consists of vertex 1 (5.40) and pseudoscalar propagator (5.36). The Feynman rule is

$$
(S_1)^{cd}_\rho = (V_1)^{ce}_\rho i\Delta_P(r)^{de} \tag{5.57}
$$

$$
= \frac{iF}{r^2} r_\rho \delta^{cd}\,. \tag{5.58}
$$

Subdiagram 2 This subdiagram consists of vertex 2 (5.41) and tensor propagator (5.35). The Feynman rule is

$$(S_2)^{cd}_{\rho\alpha\beta} = (V_2)^{ce}_{\rho\gamma\delta} i\big(\Delta_A(r)\big)^{de}_{\gamma\delta\alpha\beta} \tag{5.59}$$

$$= \frac{iF_A}{r^2 - M_A^2}(r_\alpha g_{\rho\beta} - r_\beta g_{\rho\alpha})\delta^{ac}\,. \tag{5.60}$$

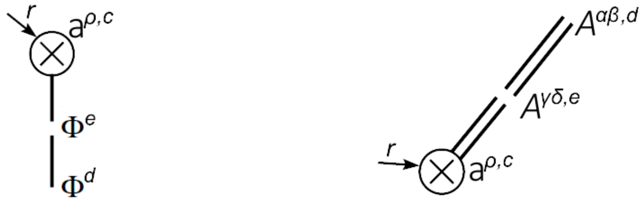


Figure 5.8: Feynman diagrams of subdiagrams 1 (left) and 2 (right).

5.3 Feynman diagrams

Diagram χ This diagram consists of vertex WZW (5.37)-(5.39) and subdiagram 1 (5.58). The Feynman rules of all variants are as follows

$$(\Pi^1_\chi)^{abc}_{\mu\nu\rho} = (V^1_{WZW})^{abd}_{\mu\nu}(S_1)^{cd}_\rho \tag{5.61}$$

$$= \frac{N_C}{24\pi^2 r^2}\varepsilon_{\mu\nu(p)(q)} r_\rho d^{abc}\,, \tag{5.62}$$

$$(\Pi^2_\chi)^{abc}_{\mu\nu\rho} = (V^2_{WZW})^{acd}_{\mu\rho}(S_1)^{bd}_\nu \tag{5.63}$$

$$= -\frac{N_C}{24\pi^2 q^2}\varepsilon_{\mu\rho(p)(q)} q_\nu d^{abc}\,, \tag{5.64}$$

$$(\Pi^3_\chi)^{abc}_{\mu\nu\rho} = (V^3_{WZW})^{bcd}_{\nu\rho}(S_1)^{ad}_\mu \tag{5.65}$$

$$= \frac{N_C}{24\pi^2 p^2}\varepsilon_{\nu\rho(p)(q)} p_\mu d^{abc}\,. \tag{5.66}$$

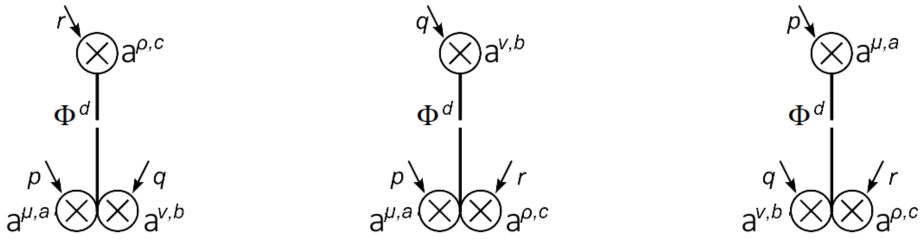


Figure 5.9: Feynman diagram χ (all permutations).

Diagram 1 This diagram consists of vertex 3 (5.42)-(5.44) and subdiagram 2 (5.60). The Feynman rules of all permutations are as follows

$$(\Pi^1_1)^{abc}_{\mu\nu\rho} = (V^1_3)^{abd}_{\mu\nu\alpha\beta}(S_2)^{cd}_{\rho\alpha\beta} \tag{5.67}$$

$$= -\frac{4\sqrt{2}F_A d^{abc}}{r^2 - M_A^2}\Big[(\kappa^A_3 + 2\kappa^A_8 + \kappa^A_{15})(\varepsilon_{\nu\rho(p)(q)}q_\mu - p_\nu\varepsilon_{\mu\rho(p)(q)}) \tag{5.68}$$

$$+ \kappa^A_{16}(p_\mu\varepsilon_{\nu\rho(p)(q)} - q_\nu\varepsilon_{\mu\rho(p)(q)}) + \kappa^A_{16}(p^2 - q^2)\varepsilon_{\mu\nu\rho(r)}\Big]\,,$$

$$(\Pi_1^2)^{abc}_{\mu\nu\rho} =(V_3^2)^{acd}_{\mu\rho\alpha\beta}(S_2)^{bd}_{\nu\alpha\beta} \tag{5.69}$$

$$= -\frac{4\sqrt{2}F_A d^{abc}}{q^2 - M_A^2}\Big[(\kappa_3^A + 2\kappa_8^A + \kappa_{15}^A)(r_\mu \varepsilon_{\nu\rho(p)(q)} + p_\rho \varepsilon_{\mu\nu(p)(q)}) \tag{5.70}$$

$$+ \kappa_{16}^A(p_\mu \varepsilon_{\nu\rho(p)(q)} + r_\rho \varepsilon_{\mu\nu(p)(q)}) + \kappa_{16}(r^2 - p^2)\varepsilon_{\mu\nu\rho(q)}\Big],$$

$$(\Pi_1^3)^{abc}_{\mu\nu\rho} =(V_3^3)^{bcd}_{\nu\rho\alpha\beta}(S_2)^{ad}_{\mu\alpha\beta} \tag{5.71}$$

$$= -\frac{4\sqrt{2}F_A d^{abc}}{p^2 - M_A^2}\Big[(q_\rho \varepsilon_{\mu\nu(p)(q)} - r_\nu \varepsilon_{\mu\rho(p)(q)})(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) \tag{5.72}$$

$$- \kappa_{16}^A(q_\nu \varepsilon_{\mu\rho(p)(q)} - r_\rho \varepsilon_{\mu\nu(p)(q)}) - \kappa_{16}^A(r^2 - q^2)\varepsilon_{\mu\nu\rho(p)}\Big].$$

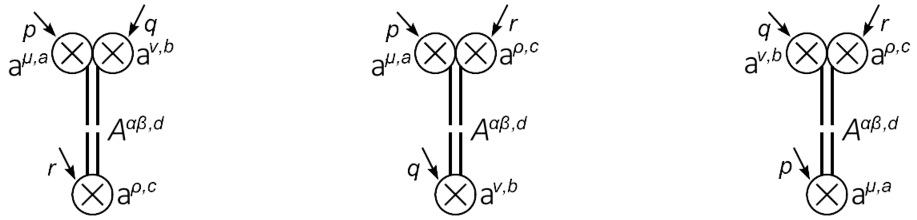


Figure 5.10: Feynman diagram 1 (all permutations).

Diagram 2 This diagram consists of vertex 4 (5.45)-(5.50) and subdiagrams 1 (5.58) and 2 (5.60). The Feynman rules of all permutations are as follows

$$(\Pi_2^1)^{abc}_{\mu\nu\rho} = (V_4^1)^{ade}_{\mu\alpha\beta}(S_1)^{ce}_{\rho}(S_2(q))^{bd}_{\nu\alpha\beta} \tag{5.73}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(q^2 - M_A^2)r^2} r_\rho \varepsilon_{\mu\nu(p)(q)}\left[\frac{-p^2 + q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + p^2\kappa_{16}^A\right], \tag{5.74}$$

$$(\Pi_2^2)^{abc}_{\mu\nu\rho} = (V_4^2)^{ade}_{\mu\alpha\beta}(S_1)^{be}_{\nu}(S_2(r))^{cd}_{\rho\alpha\beta} \tag{5.75}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(r^2 - M_A^2)q^2} q_\nu \varepsilon_{\mu\rho(p)(q)}\left[\frac{p^2 + q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) - p^2\kappa_{16}^A\right], \tag{5.76}$$

$$(\Pi_2^3)^{abc}_{\mu\nu\rho} = (V_4^3)^{bde}_{\nu\alpha\beta}(S_1)^{ce}_{\rho}(S_2(p))^{ad}_{\mu\alpha\beta} \tag{5.77}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(p^2 - M_A^2)r^2} r_\rho \varepsilon_{\mu\nu(p)(q)}\left[\frac{p^2 - q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + q^2\kappa_{16}^A\right], \tag{5.78}$$

$$(\Pi_2^4)^{abc}_{\mu\nu\rho} = (V_4^4)^{bde}_{\nu\alpha\beta}(S_1)^{ae}_{\mu}(S_2(r))^{cd}_{\rho\alpha\beta} \tag{5.79}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(r^2 - M_A^2)p^2} p_\mu \varepsilon_{\nu\rho(p)(q)}\left[\frac{-p^2 - q^2 + r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + q^2\kappa_{16}^A\right], \tag{5.80}$$

$$(\Pi_2^5)^{abc}_{\mu\nu\rho} = (V_4^5)^{cde}_{\rho\alpha\beta}(S_1)^{be}_{\nu}(S_2(p))^{ad}_{\mu\alpha\beta} \tag{5.81}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(p^2 - M_A^2)q^2} q_\nu \varepsilon_{\mu\rho(p)(q)}\left[\frac{-p^2 + q^2 + r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) - r^2\kappa_{16}^A\right], \tag{5.82}$$

$$(\Pi_2^6)^{abc}_{\mu\nu\rho} = (V_4^6)^{cde}_{\rho\alpha\beta}(S_1)^{ae}_{\mu}(S_2(q))^{bd}_{\nu\alpha\beta} \tag{5.83}$$

$$= \frac{4\sqrt{2}F_A d^{abc}}{(q^2 - M_A^2)p^2} p_\mu \varepsilon_{\nu\rho(p)(q)}\left[\frac{-p^2 + q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + r^2\kappa_{16}^A\right]. \tag{5.84}$$

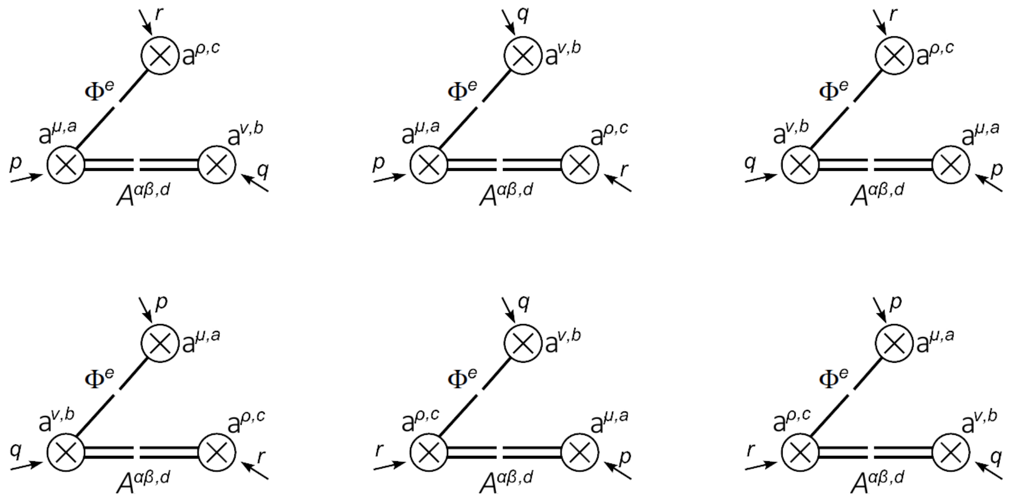


Figure 5.11: Feynman diagram 2 (all permutations).

Diagram 3 This diagram consists of vertex 5 (5.51)-(5.53) and two subdiagrams 2 (5.60). The Feynman rules of all permutations are as follows

$$(\Pi_3^1)^{abc}_{\mu\nu\rho} = (V_5^1)^{ade}_{\mu\alpha\beta\gamma\delta}(S_2)^{bd}_{\nu\alpha\beta}(S_2)^{ce}_{\rho\gamma\delta} \tag{5.85}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(q^2-M_A^2)(M_A^2-r^2)}\times \tag{5.86}$$

$$\times\left[\frac{-p^2+q^2+r^2}{2}(\varepsilon_{\mu\nu\rho(r)}-\varepsilon_{\mu\nu\rho(q)})-r_\nu\varepsilon_{\mu\rho(p)(q)}+q_\rho\varepsilon_{\mu\nu(p)(q)}\right],$$

$$(\Pi_3^2)^{abc}_{\mu\nu\rho} = (V_5^2)^{bde}_{\nu\alpha\beta\gamma\delta}(S_2)^{ad}_{\mu\alpha\beta}(S_2)^{ce}_{\rho\gamma\delta} \tag{5.87}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(r^2-M_A^2)}\times \tag{5.88}$$

$$\times\left[\frac{p^2-q^2+r^2}{2}(\varepsilon_{\mu\nu\rho(r)}-\varepsilon_{\mu\nu\rho(p)})-r_\mu\varepsilon_{\nu\rho(p)(q)}-p_\rho\varepsilon_{\mu\nu(p)(q)}\right],$$

$$(\Pi_3^3)^{abc}_{\mu\nu\rho} = (V_5^3)^{cde}_{\rho\alpha\beta\gamma\delta}(S_2)^{ad}_{\mu\alpha\beta}(S_2)^{be}_{\nu\gamma\delta} \tag{5.89}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(q^2-M_A^2)}\times \tag{5.90}$$

$$\times\left[\frac{p^2+q^2-r^2}{2}(\varepsilon_{\mu\nu\rho(p)}-\varepsilon_{\mu\nu\rho(q)})-q_\mu\varepsilon_{\nu\rho(p)(q)}+p_\nu\varepsilon_{\mu\rho(p)(q)}\right],$$

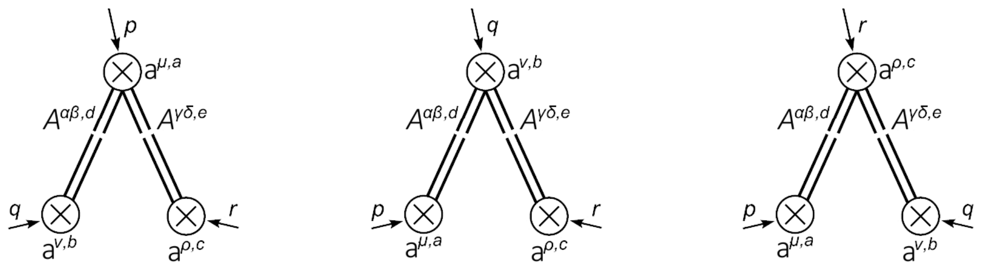


Figure 5.12: Feynman diagram 3 (all permutations).

Diagram 4 This diagram consists of vertex 6 (5.54)-(5.56), subdiagram 1 (5.58) and two subdiagrams 2 (5.58). The Feynman rules of all permutations are as follows

$$(\Pi_4^1)^{abc}_{\mu\nu\rho} = (V_6^1)^{def}_{\alpha\beta\gamma\delta}(S_1)^{af}_{\mu}(S_2)^{bd}_{\nu\alpha\beta}(S_2)^{ce}_{\rho\gamma\delta} \tag{5.91}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{p^2(q^2-M_A^2)(r^2-M_A^2)}(p^2-q^2-r^2)p_\mu\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.92}$$

$$(\Pi_4^2)^{abc}_{\mu\nu\rho} = (V_6^2)^{def}_{\alpha\beta\gamma\delta}(S_1)^{bf}_{\nu}(S_2)^{ad}_{\mu\alpha\beta}(S_2)^{ce}_{\rho\gamma\delta} \tag{5.93}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)q^2(r^2-M_A^2)}(p^2-q^2+r^2)q_\nu\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.94}$$

$$(\Pi_4^3)^{abc}_{\mu\nu\rho} = (V_6^3)^{def}_{\alpha\beta\gamma\delta}(S_1)^{cf}_{\rho}(S_2)^{ad}_{\mu\alpha\beta}(S_2)^{be}_{\nu\gamma\delta} \tag{5.95}$$

$$= \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(q^2-M_A^2)r^2}(-p^2-q^2+r^2)r_\rho\varepsilon_{\mu\nu(p)(q)}\,. \tag{5.96}$$

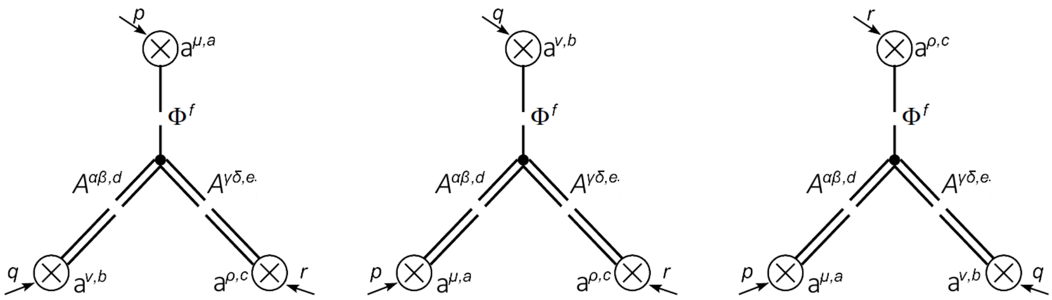


Figure 5.13: Feynman diagram 4 (all permutations).

Taking all contributions together, we have finally calculated all possible Feynman diagrams that contribute to (5.1) in the antisymmetric tensor formalism.

5.4 Ward identities

Having the AAA Green function calculated, we can also study its property in the sense of Ward identities. Since we have an anomalous correlator consisted of three axial-vector currents, we can expect not to have these currents conserved on the quantum level. To verify that, let us proceed in the following way. We can easily find out the results below:

$$p^\mu(\Pi_\chi^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.97}$$

$$p^\mu(\Pi_\chi^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.98}$$

$$p^\mu(\Pi_\chi^3)^{abc}_{\mu\nu\rho} = \frac{N_C}{24\pi^2}\varepsilon_{\nu\rho(p)(q)}d^{abc}\,, \tag{5.99}$$

$$p^\mu(\Pi_1^1)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{r^2-M_A^2}\left[\frac{p^2+q^2-r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)-q^2\kappa_{16}^A\right]\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.100}$$

$$p^\mu(\Pi_1^2)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{q^2-M_A^2}\left[\frac{p^2-q^2+r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)-r^2\kappa_{16}^A\right]\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.101}$$

$$p^\mu(\Pi_1^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.102}$$

$$p^\mu(\Pi_2^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.103}$$

$$p^\mu(\Pi_2^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.104}$$

$$p^\mu(\Pi_2^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.105}$$

$$p^\mu(\Pi_2^4)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{r^2 - M_A^2}\left[\frac{-p^2 - q^2 + r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + q^2\kappa_{16}^A\right]\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.106}$$
$$p^\mu(\Pi_2^5)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.107}$$
$$p^\mu(\Pi_2^6)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{q^2 - M_A^2}\left[\frac{-p^2 + q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + r^2\kappa_{16}^A\right]\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.108}$$
$$p^\mu(\Pi_3^1)^{abc}_{\mu\nu\rho} = -\frac{4F_A^2\kappa_3^{VV}d^{abc}}{(q^2 - M_A^2)(r^2 - M_A^2)}(p^2 - q^2 - r^2)\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.109}$$
$$p^\mu(\Pi_3^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.110}$$
$$p^\mu(\Pi_3^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.111}$$
$$p^\mu(\Pi_4^1)^{abc}_{\mu\nu\rho} = \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(q^2 - M_A^2)(r^2 - M_A^2)}(p^2 - q^2 - r^2)\varepsilon_{\nu\rho(p)(q)}\,, \tag{5.112}$$
$$p^\mu(\Pi_4^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.113}$$
$$p^\mu(\Pi_4^3)^{abc}_{\mu\nu\rho} = 0\,. \tag{5.114}$$

We see that the nonzero expressions cancel each other out,

$$p^\mu(\Pi_1^1)^{abc}_{\mu\nu\rho} + p^\mu(\Pi_2^4)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.115}$$
$$p^\mu(\Pi_1^2)^{abc}_{\mu\nu\rho} + p^\mu(\Pi_2^6)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.116}$$
$$p^\mu(\Pi_3^1)^{abc}_{\mu\nu\rho} + p^\mu(\Pi_4^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.117}$$

except for the term that arises from the anomalous part of the contributions. Similarly, we have

$$q^\nu(\Pi_\chi^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.118}$$
$$q^\nu(\Pi_\chi^2)^{abc}_{\mu\nu\rho} = -\frac{N_C}{24\pi^2}\varepsilon_{\mu\rho(p)(q)}d^{abc}\,, \tag{5.119}$$
$$q^\nu(\Pi_\chi^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.120}$$
$$q^\nu(\Pi_1^1)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{r^2 - M_A^2}\left[\frac{-p^2 - q^2 + r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + p^2\kappa_{16}\right]\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.121}$$
$$q^\nu(\Pi_1^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.122}$$
$$q^\nu(\Pi_1^3)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{p^2 - M_A^2}\left[\frac{p^2 - q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) + r^2\kappa_{16}^A\right]\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.123}$$
$$q^\nu(\Pi_2^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.124}$$
$$q^\nu(\Pi_2^2)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{r^2 - M_A^2}\left[\frac{p^2 + q^2 - r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) - p^2\kappa_{16}^A\right]\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.125}$$
$$q^\nu(\Pi_2^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.126}$$
$$q^\nu(\Pi_2^4)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.127}$$
$$q^\nu(\Pi_2^5)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_A d^{abc}}{p^2 - M_A^2}\left[\frac{-p^2 + q^2 + r^2}{2}(\kappa_{15}^A + \kappa_3^A + 2\kappa_8^A) - r^2\kappa_{16}^A\right]\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.128}$$
$$q^\nu(\Pi_2^6)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.129}$$
$$q^\nu(\Pi_3^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.130}$$
$$q^\nu(\Pi_3^2)^{abc}_{\mu\nu\rho} = \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2 - M_A^2)(r^2 - M_A^2)}(-p^2 + q^2 - r^2)\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.131}$$

$$q^\nu(\Pi_3^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.132}$$
$$q^\nu(\Pi_4^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.133}$$
$$q^\nu(\Pi_4^2)^{abc}_{\mu\nu\rho} = \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(r^2-M_A^2)}(p^2-q^2+r^2)\varepsilon_{\mu\rho(p)(q)}\,, \tag{5.134}$$
$$q^\nu(\Pi_4^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.135}$$

where we can also notice that a lot of non-zero contributions compensate each other,

$$q^\nu(\Pi_1^1)^{abc}_{\mu\nu\rho} + q^\nu(\Pi_2^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.136}$$
$$q^\nu(\Pi_1^3)^{abc}_{\mu\nu\rho} + q^\nu(\Pi_2^5)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.137}$$
$$q^\nu(\Pi_3^2)^{abc}_{\mu\nu\rho} + q^\nu(\Pi_4^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.138}$$

and only the anomalous term will now vanish. Thirdly, the last part of the Ward identites lead to the following results.

$$r^\rho(\Pi_\chi^1)^{abc}_{\mu\nu\rho} = \frac{N_C}{24\pi^2}\varepsilon_{\mu\nu(p)(q)}d^{abc}\,, \tag{5.139}$$
$$r^\rho(\Pi_\chi^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.140}$$
$$r^\rho(\Pi_\chi^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.141}$$
$$r^\rho(\Pi_1^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.142}$$
$$r^\rho(\Pi_1^2)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{q^2-M_A^2}\left[\frac{p^2-q^2+r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)-p^2\kappa_{16}^A\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{5.143}$$
$$r^\rho(\Pi_1^3)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{p^2-M_A^2}\left[\frac{-p^2+q^2+r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)-q^2\kappa_{16}^A\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{5.144}$$
$$r^\rho(\Pi_2^1)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{q^2-M_A^2}\left[\frac{-p^2+q^2-r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)+p^2\kappa_{16}^A\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{5.145}$$
$$r^\rho(\Pi_2^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.146}$$
$$r^\rho(\Pi_2^3)^{abc}_{\mu\nu\rho} = \frac{4\sqrt{2}F_Ad^{abc}}{p^2-M_A^2}\left[\frac{p^2-q^2-r^2}{2}(\kappa_{15}^A+\kappa_3^A+2\kappa_8^A)+q^2\kappa_{16}^A\right]\varepsilon_{\mu\nu(p)(q)}\,, \tag{5.147}$$
$$r^\rho(\Pi_2^4)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.148}$$
$$r^\rho(\Pi_2^5)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.149}$$
$$r^\rho(\Pi_2^6)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.150}$$
$$r^\rho(\Pi_3^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.151}$$
$$r^\rho(\Pi_3^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.152}$$
$$r^\rho(\Pi_3^3)^{abc}_{\mu\nu\rho} = \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(q^2-M_A^2)}(p^2+q^2-r^2)\varepsilon_{\mu\nu(p)(q)}\,, \tag{5.153}$$
$$r^\rho(\Pi_4^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.154}$$
$$r^\rho(\Pi_4^2)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.155}$$
$$r^\rho(\Pi_4^3)^{abc}_{\mu\nu\rho} = \frac{4F_A^2\kappa_3^{VV}d^{abc}}{(p^2-M_A^2)(q^2-M_A^2)}(-p^2-q^2+r^2)\varepsilon_{\mu\nu(p)(q)}\,. \tag{5.156}$$

Only the anomalous term will not be eliminated here also,

$$r^\rho(\Pi_1^2)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_2^1)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.157}$$
$$r^\rho(\Pi_1^3)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_2^3)^{abc}_{\mu\nu\rho} = 0\,, \tag{5.158}$$
$$r^\rho(\Pi_3^3)^{abc}_{\mu\nu\rho} + r^\rho(\Pi_4^3)^{abc}_{\mu\nu\rho} = 0\,. \tag{5.159}$$

Therefore, we have determined that the Ward identites for the AAA Green function take the form

$$p^{\mu}\big(\Pi_{AAA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = \frac{N_C}{24\pi^2}\varepsilon_{\nu\rho(p)(q)}d^{abc}\,, \tag{5.160}$$

$$q^{\nu}\big(\Pi_{AAA}(p,q;r)\big)^{abc}_{\mu\nu\rho} = -\frac{N_C}{24\pi^2}\varepsilon_{\mu\rho(p)(q)}d^{abc}\,, \tag{5.161}$$

$$r^{\rho}\big(\Pi_{AAA}(p,q;r)\big)_{\mu\nu\rho} = \frac{N_C}{24\pi^2}\varepsilon_{\mu\nu(p)(q)}d^{abc}\,, \tag{5.162}$$

that coincide with the non-conservations of the axial-vector currents on the quantum level.

5.5 Phenomenology

A phenomenology study regarding the AAA Green function could not be done due to the lack of experimentally relevant data for our purpose. We will certainly return to this point in future studies and complete the task.

6. $VVPP$ Green function

Before we will define all required ingredients, let us stress properly enough that in this and the next chapter we do not present any full calculations of the four-point Green functions. Although we have obtained the results of the $VVPP$ and $VVVV$ correlators both in the vector and the antisymmetric tensor formalims, the final expressions are very complicated due to the complex tensor structure that it is not appropriate to show them here in their full lenght. For this reason, we present only the set of all contributing Feynman diagrams for Lagrangians of various chiral orders. The study of the results in detailed will be dealt with in our future papers.

The standard definition of the $VVPP$ correlator is

$$\big(\Pi_{VVPP}(p,q,r;s)\big)^{abcd}_{\mu\nu} = \big\langle 0\big|\mathrm{T}\big[\widetilde{V}^a_\mu(p)\widetilde{V}^b_\nu(q)\widetilde{P}^c(r)\widetilde{P}^d(0)\big]\big|0\big\rangle \tag{6.1}$$

$$= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,e^{i(px+qy+rz)}\,\big\langle 0\big|\mathrm{T}\big[V^a_\mu(x)V^b_\nu(y)P^c(z)P^d(0)\big]\big|0\big\rangle\,. \tag{6.2}$$

As we have mentioned several times, the topology of the four-point Green functions in general is complicated. Therefore, we are required to take the higher expansions of the chiral operators into account. Specifically, we need the following operators that couple vector external sources with pseudoscalars,

$$u_\mu = -\frac{\sqrt{2}}{F}\partial_\mu\phi - \frac{i\sqrt{2}}{F}[\phi, v_\mu]\,, \tag{6.3}$$

$$h_{\mu\nu} = -\frac{\sqrt{2}}{F}\partial_\mu\partial_\nu\phi - \frac{i\sqrt{2}}{F}[\partial_\mu\phi, v_\nu] - \frac{i\sqrt{2}}{F}[\phi, \partial_\mu v_\nu] + (\mu\leftrightarrow\nu)\,, \tag{6.4}$$

$$f_-^{\mu\nu} = \frac{i\sqrt{2}}{F}[\phi, \partial^\mu v^\nu - \partial^\nu v^\mu]\,, \tag{6.5}$$

$$f_+^{\mu\nu} = 2(\partial^\mu v^\nu - \partial^\nu v^\mu) - 2i[v^\mu, v^\nu] + \frac{1}{F^2}\phi(\partial^\mu v^\nu - \partial^\nu v^\mu)\phi - \frac{1}{2F^2}\{\phi^2, \partial^\mu v^\nu - \partial^\nu v^\mu\}\,, \tag{6.6}$$

and the building blocks that couple pseudoscalar external sources with the pseudoscalars,

$$\chi_- = 4iB_0 p\,, \tag{6.7}$$

$$\chi_+ = \frac{2\sqrt{2}B_0}{F}\{\phi, p\}\,. \tag{6.8}$$

In this case, we are also required to take the chiral connection and the covariant derivative in the forms

$$\Gamma_\mu = -iv_\mu\,, \tag{6.9}$$

$$\nabla_\mu X = \partial_\mu X - i[v_\mu, X]\,. \tag{6.10}$$

To give a detailed description of the construction of the contributing Feynmand diagrams, it is useful to present again the propagators that are needed. Except for the tensor and pseudoscalar propagators, we will also use pseudoscalar and scalar resonance propagators.

Pseudoscalar resonance propagator The kinetic and mass terms form the pseudoscalar resonance propagator.

$$i(\Delta_P(p, M_P))^{ab} = \frac{i}{p^2 - M_P^2}\delta^{ab}\,. \tag{6.11}$$

Scalar resonance propagator Similarly as in the previous case, the scalar resonance propagator is formed by the kinetic and mass terms.

$$i(\Delta_S(p, M_S))^{ab} = \frac{i}{p^2 - M_S^2}\delta^{ab}\,. \tag{6.12}$$

Figure 6.1: Pseudoscalar (left) and scalar (right) resonance propagators.

Both previous propagators represent resonances that do not carry any Lorentz indices, therefore they do not dependent on the used formalism. However, since we calculate the $VVPP$ and $VVVV$ Green functions also in the vector formalism, we need to introduce the appropriate propagator for the vector resonances in that case, similarly as we have done in the case of the antisymmetric tensor formalism.

Vector propagator Vector propagator is formed by the kinetic and mass terms [7]:

$$i(\Delta_R(p, M_R))^{ab}_{\mu\nu} = -\frac{i}{p^2 - M_R^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_R^2}\right)\delta^{ab}\,, \tag{6.13}$$

where R stands for the resonance. In our case, we consider only the vector resonances, i.e. $R = V$.

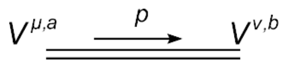


Figure 6.2: Vector propagator.

Similarly as in the previous chapters, it is very useful to define subdiagrams to keep our calculations as simple as possible. As an analogical case to subdiagram 3 (4.52) in the antisymmetric tensor formalism, we can introduce the subdiagram in the vector formalism.

Subdiagram 3' This subdiagram consists of vertex (D.77) and vector propagator (6.13). The Feynman rule is

$$(S_3')^{ab}_{\mu\alpha} = -\frac{f_V}{p^2 - M_V^2}(p^2 g_{\mu\alpha} - p_\mu p_\alpha)\delta^{ab}\,. \tag{6.14}$$

Other subdiagrams consist of pseudoscalar external sources coupled either to pseudoscalar field or pseudoscalar resonance.

Subdiagram 4 This subdiagram consists of vertex (D.32) and pseudoscalar propagator (4.34). The Feynman rule is

$$(S_4)^{ab} = -\frac{FB_0}{p^2}\delta^{ab}\,. \tag{6.15}$$

Subdiagram 5 This subdiagram consists of vertex (D.133) and pseudoscalar resonance propagator (6.11). The Feynman rule is

$$(S_5)^{ab} = \frac{2\sqrt{2}B_0 d_m}{p^2 - M_P^2}\delta^{ab}\,. \tag{6.16}$$

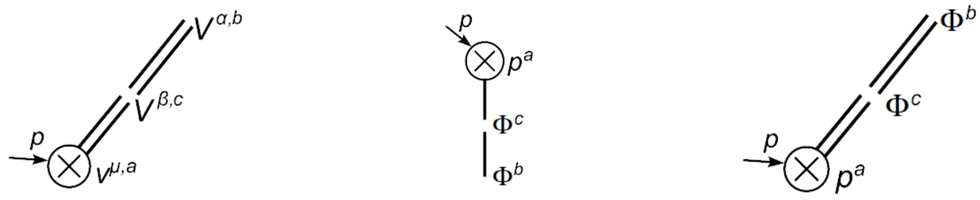


Figure 6.3: Subdiagram 3' (left), 4 (middle) and 5 (right).

Before we advance to construct the Feynman diagrams, let us mention that we do not consider any multiple permutations of the particular diagrams shown as examples. In other words, the Feynman diagrams presented in this and next chapters are alway one of the complete specific set of the Feynman diagrams that differ from one another typically by interchanges $v_\mu^a \leftrightarrow v_\nu^b$ and $p^c \leftrightarrow p^d$ in the case of $VVPP$ Green function and $v_\mu^a \leftrightarrow v_\nu^b \leftrightarrow v_\rho^c \leftrightarrow v_\sigma^d$ in the case of $VVVV$ Green function. The simultaneous interchanges of corresponding 4-momenta is implicitly assumed. More specifically, one topology of the Feynman diagrams then constitutes a complete set of typically six or twelve particular diagrams.

Also, for simplicity, we do not show the Feynman diagrams with explicitly highlighted subdiagrams and propagators included in the Feynman diagrams. Not only that it would lead to more complicated pictures but it is unnecessary for our case here. Instead, we only indicate what resonance or pseudoscalar contributes in that channel. In future studies, where we will work with detailed study of these correlators, we will pay more attention to technicalities as we did in Chapters 4 and 5.

6.1 Non-resonance contribution up to $\mathcal{O}(p^2)$

Considering the lowest possible contribution into $VVPP$ Green function, we have to start with the chiral Lagrangian up to $\mathcal{O}(p^2)$, i.e. with the Lagrangian (2.16):

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle\,. \tag{6.17}$$

The lowest contribution consists only of two different topologies of the Feynman diagrams, one with two possible permutations.

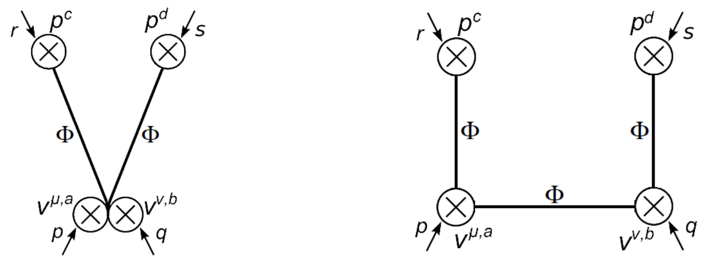


Figure 6.4: Feynman diagrams 1 and 2 in the χPT up to $\mathcal{O}(p^2)$.

6.2 Vector formalism up to $\mathcal{O}(p^6)$

Here we consider all possible diagrams given by the resonance Lagrangian in the vector formalism up to $\mathcal{O}(p^6)$. The relevant part of the Lagrangians (2.60)-(2.61) can be written in the form

$$\begin{aligned}\mathcal{L} = &-\frac{f_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu} f_+^{\mu\nu}\rangle - \frac{f_A}{2\sqrt{2}}\langle \widehat{A}_{\mu\nu} f_-^{\mu\nu}\rangle - \frac{ig_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu}[u^\mu, u^\nu]\rangle \\ &+ i\alpha_V\langle \widehat{V}_\mu[u_\nu, f_-^{\mu\nu}]\rangle + \beta_V\langle \widehat{V}_\mu[u^\mu, \chi_-]\rangle + h_V\langle \widehat{V}^\mu\{u^\nu, f_+^{\alpha\beta}\}\rangle\varepsilon_{\mu\nu\alpha\beta}\,.\end{aligned} \tag{6.18}$$

Note that we can also consider a term that consists of an axial-vector resonance. This may seem odd but remember we use a higher expansion (6.5) of the $f_-^{\mu\nu}$ operator in order to avoid a presence of the axial-vector external source. Then, the axial-vector resonance is fully coupled to vector sources and pseudoscalars.

Then, the contributing Feynman diagrams are the following.

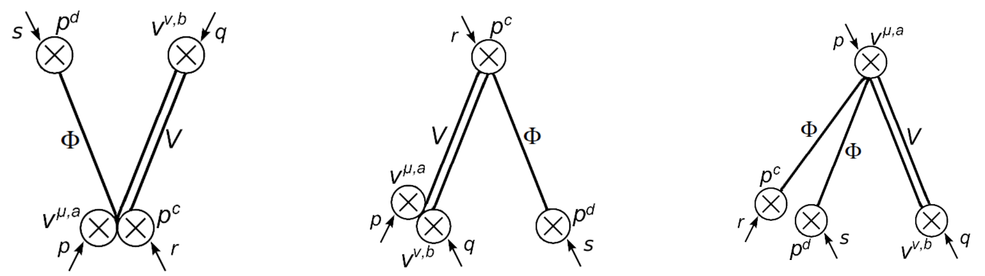


Figure 6.5: Feynman diagrams 1 (left), 2 (middle) and 3 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

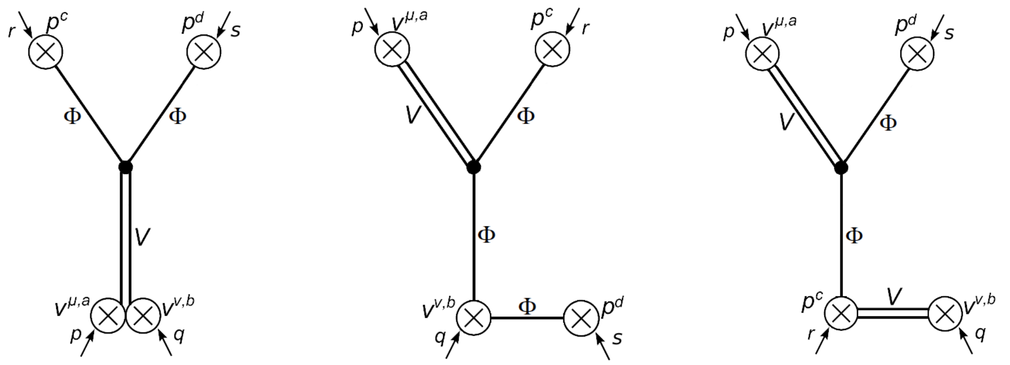


Figure 6.6: Feynman diagrams 4 (left), 5 (middle) and 6 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

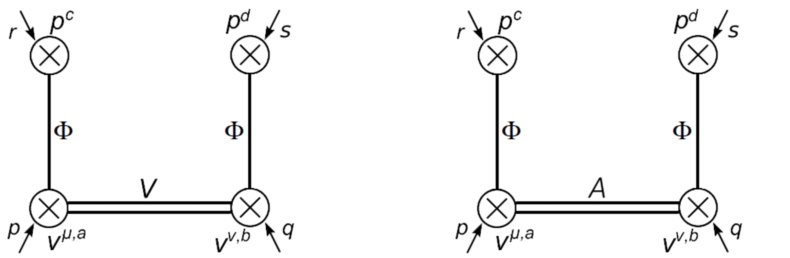


Figure 6.7: Feynman diagrams 7 (left) and 8 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

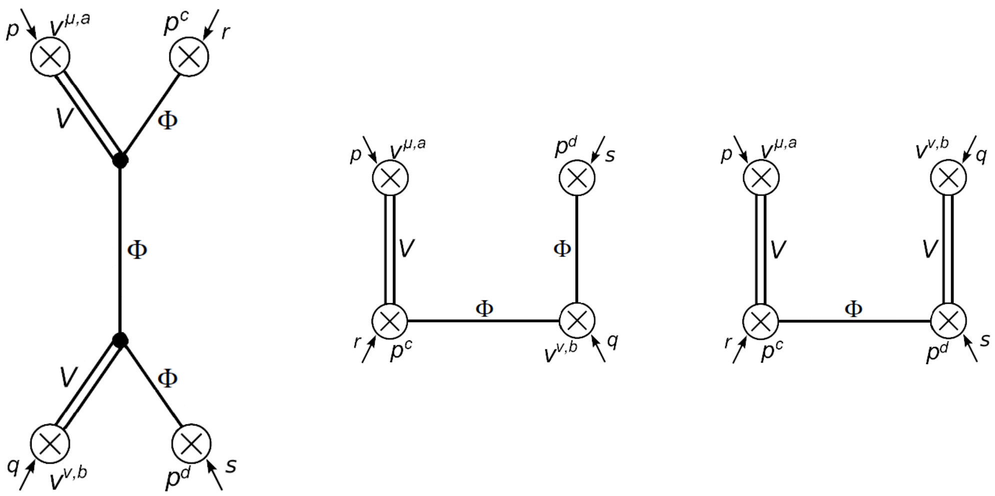


Figure 6.8: Feynman diagrams 9 (left), 10 (middle) and 11 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

6.3 Antisymmetric tensor formalism up to $\mathcal{O}(p^4)$

The lowest resonance Lagrangian (2.68) in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$ has the following relevant part:

$$\mathcal{L}^{(4)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu}\rangle \,. \tag{6.19}$$

The contributing Feynman diagrams are the following.

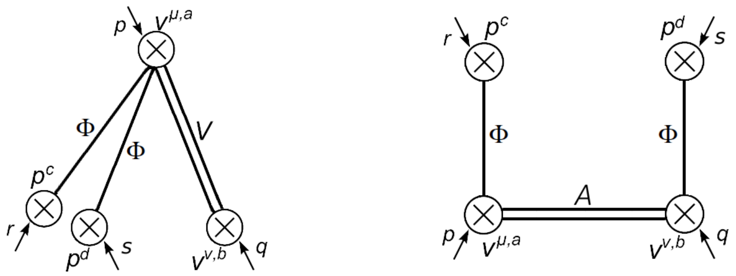


Figure 6.9: Feynman diagrams 1 (left) and 2 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.

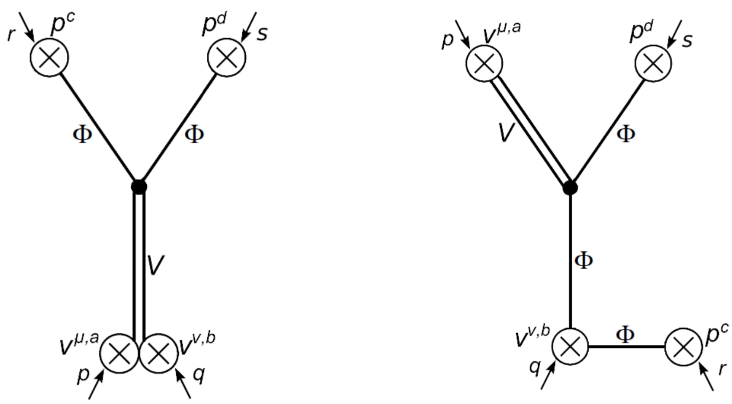


Figure 6.10: Feynman diagrams 3 (left) and 4 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.

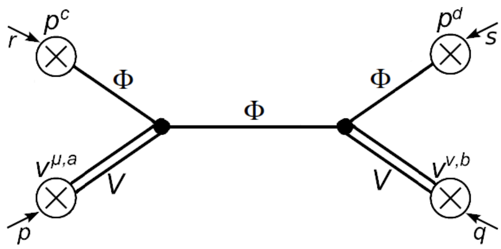


Figure 6.11: Feynman diagram 5 in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.

6.4 Antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

Finally, the resonance Lagrangian in the tensor formalism up to $\mathcal{O}(p^6)$ has the relevant part consisted of two Lagrangians, one of the order p^4 and second of the order p^6, i.e.

$$\mathcal{L}^{(4)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + i d_m \langle P\chi_-\rangle \,, \tag{6.20}$$

$$\mathcal{L}^{(6)} = \mathcal{L}_{12}^V + \mathcal{L}_{14}^V + \mathcal{L}_{16}^V + \mathcal{L}_{17}^V + \mathcal{L}_5^P + \mathcal{L}_2^{VV} + \mathcal{L}_3^{VV} + \mathcal{L}_2^{SV} + \mathcal{L}_3^{PV} + \mathcal{L}^{VVP} \,, \tag{6.21}$$

where the contributing operators of the order p^6 are shown in the table below.

i	$\mathcal{O}^{V,P}_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{VV,SV,PV}_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{VVP}_{i\,\mu\nu\alpha\beta}$
5	$\langle P\{f_+^{\mu\nu}, f_+^{\alpha\beta}\}\rangle$	2	$i\langle\{V^{\mu\nu}, V^{\alpha\beta}\}\chi_-\rangle$	-	$\langle V^{\mu\nu}V^{\alpha\beta}P\rangle$
12	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, h^{\beta\sigma}\}\rangle g_{\rho\sigma}$	2	$i\langle[V^{\mu\nu}, \nabla^\alpha S]u^\beta\rangle$		
14	$i\langle V^{\mu\nu}\{f_+^{\alpha\beta}, \chi_-\}\rangle$	3	$\langle\{V^{\mu\nu}, P\}f_+^{\alpha\beta}\rangle$		
16	$\langle V^{\mu\nu}\{\nabla^\alpha f_+^{\beta\sigma}, u_\sigma\}\rangle$	3	$\langle\{\nabla_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}u^\beta\rangle$		
17	$\langle V^{\mu\nu}\{\nabla_\sigma f_+^{\alpha\sigma}, u^\beta\}\rangle$				

Table 6.1: Monomials contributing into $VVPP$ Green function

The contributing Feynman diagrams are the following.

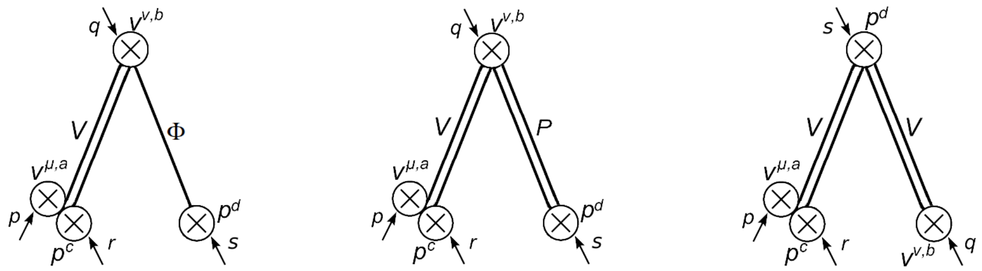


Figure 6.12: Feynman diagrams 1 (left), 2 (middle) and 3 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

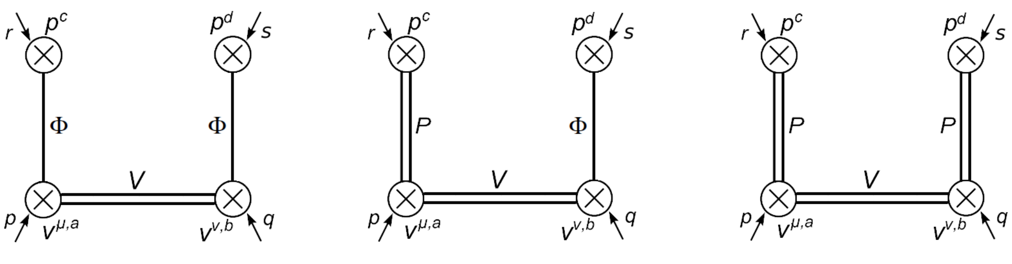


Figure 6.13: Feynman diagrams 4 (left), 5 (middle) and 6 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

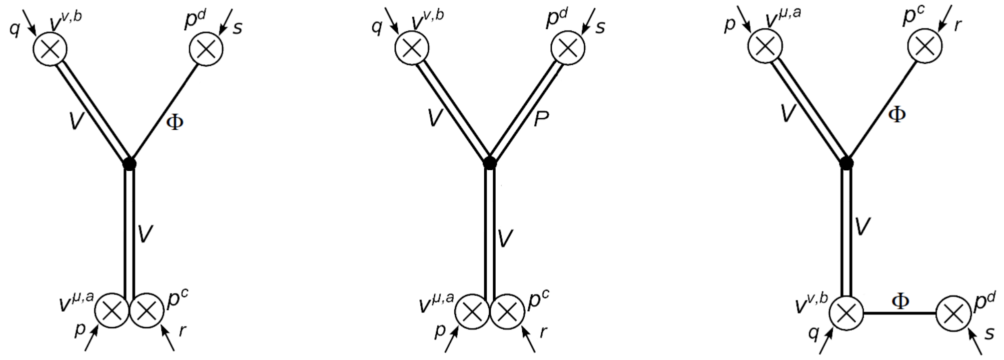


Figure 6.14: Feynman diagrams 7 (left), 8 (middle) and 9 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

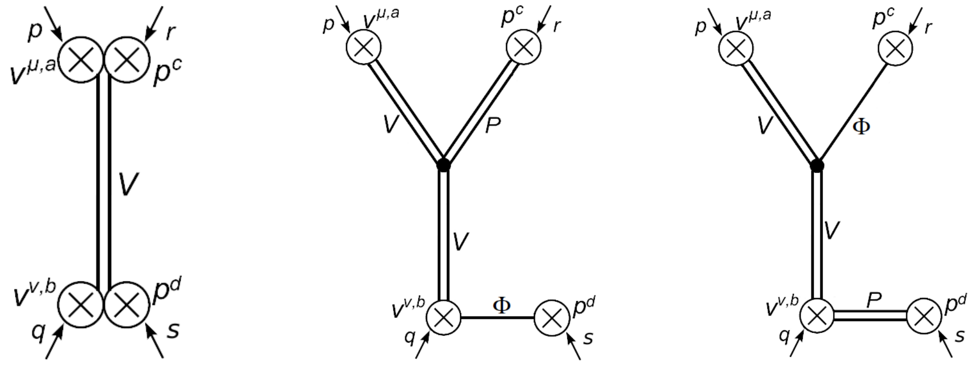


Figure 6.15: Feynman diagrams 10 (left), 11 (middle) and 12 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

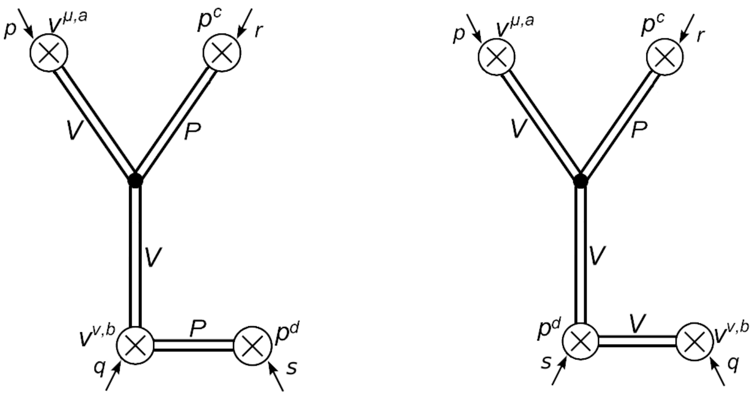


Figure 6.16: Feynman diagrams 13 (left) and 14 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

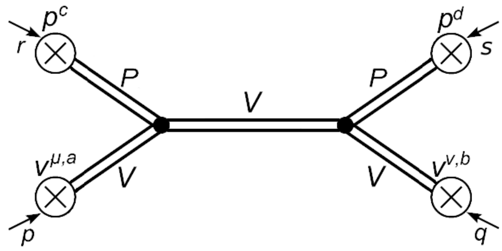
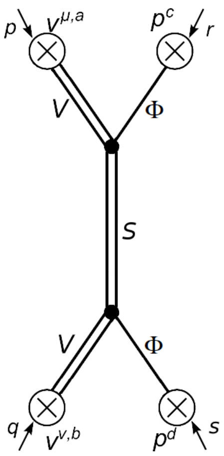


Figure 6.17: Feynman diagram 15 in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.



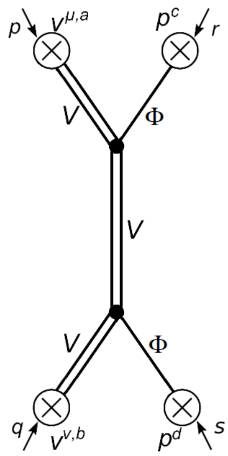


Figure 6.18: Feynman diagrams 16 (left) and 17 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

A summary of this chapter is stated in the table below.

contribution	$VVPP$
χPT $\mathcal{O}(p^2)$	2
RχT $\mathcal{O}(p^4)$ (tensor form.)	5
RχT $\mathcal{O}(p^6)$ (tensor form.)	17
RχT $\mathcal{O}(p^6)$ (vector. form.)	11

Table 6.2: A number of contributing Feynman diagrams into $VVPP$ Green functions.

6.5 Phenomenology

Just for a motivation of the study of the $VVPP$ Green functions, we can introduce the Compton-like scattering of Goldstone bosons within the RχT.

Compton-like scattering of the Goldstone bosons

The on-shell matrix element of the Compton-like process in the chiral limit can be written in the form [8]

$$\big(A(p,q,r;s)\big)^{abcd}_{\mu\nu} = \big\langle \phi^c(r)\big|\mathrm{T}\big[\widetilde{V}^a_\mu(p)\widetilde{V}^b_\nu(0)\big]\big|\phi^d(s)\big\rangle \tag{6.22}$$

$$= -\lim_{r^2,s^2\to 0}\frac{r^2s^2}{B_0^2F^2}\big\langle 0\big|\mathrm{T}\big[\widetilde{V}^a_\mu(p)\widetilde{V}^b_\nu(q)\widetilde{P}^c(r)\widetilde{P}^d(0)\big]\big|0\big\rangle\,, \tag{6.23}$$

i.e.

$$\big(A(p,q,r;s)\big)^{abcd}_{\mu\nu} = -\lim_{r^2,s^2\to 0}\frac{r^2s^2}{B_0^2F^2}\big(\Pi_{VVPP}(p,q,r;s)\big)^{abcd}_{\mu\nu}\,. \tag{6.24}$$

The pseudoscalar density satisfies

$$\langle 0|P^a(0)|\phi^b(s)\rangle = B_0F\delta^{ab} \tag{6.25}$$

and $|\phi^a(p)\rangle$ stands for the Goldstone boson state. Then, the amplitude of the Compton-like scattering of the Goldstone bosons can be written in the form [8]

$$i\mathcal{M}^{abcd}_{\kappa\lambda}(p,q,r,;s) = \lim_{r^2,s^2\to 0}\varepsilon^{*\mu}(p,\kappa)\varepsilon^{\nu}(q,\lambda)\big(A(p,q,r;s)\big)^{abcd}_{\mu\nu}\,. \tag{6.26}$$

Then, by calculating (6.1), one can study this important phenomenological example. However, the total number of all contributing Feynman diagrams to (6.1) is reduced due to the physical nature of the process - we are interested in the diagrams that consist of the external legs of the pseudoscalar bosons. The number of all relevant diagrams is not that high and detailed analysis can be found in [8].

7. $VVVV$ Green function

The standard definition of the $VVVV$ correlator is

$$\big(\Pi_{VVVV}(p,q,r;s)\big)^{abcd}_{\mu\nu\rho\sigma} = \big\langle 0\big|\mathrm{T}\big[\widetilde{V}^a_\mu(p)\widetilde{V}^b_\nu(q)\widetilde{V}^c_\rho(r)\widetilde{V}^d_\sigma(0)\big]\big|0\big\rangle \tag{7.1}$$

$$= \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,e^{i(px+qy+rz)}\,\big\langle 0\big|\mathrm{T}\big[V^a_\mu(x)V^b_\nu(y)V^c_\rho(z)V^d_\sigma(0)\big]\big|0\big\rangle\,. \tag{7.2}$$

Similarly as in the previous chapter, the following chiral operators contribute to this correlator:

$$u_\mu = -\frac{\sqrt{2}}{F}\partial_\mu\phi - \frac{i\sqrt{2}}{F}[\phi, v_\mu]\,, \tag{7.3}$$

$$h_{\mu\nu} = -\frac{\sqrt{2}}{F}\partial_\mu\partial_\nu\phi - \frac{i\sqrt{2}}{F}[\partial_\mu\phi, v_\nu] - \frac{i\sqrt{2}}{F}[\phi, \partial_\mu v_\nu] + (\mu\leftrightarrow\nu)\,, \tag{7.4}$$

$$f_+^{\mu\nu} = 2(\partial^\mu v^\nu - \partial^\nu v^\mu) - 2i[v^\mu, v^\nu]\,, \tag{7.5}$$

whilst the building blocks, that couple scalar and pseudoscalar external sources with the pseudoscalars, do not,

$$\chi_\pm = 0\,. \tag{7.6}$$

Also in this case, it is necessary to take the chiral connection and the covariant derivative as

$$\Gamma_\mu = -iv_\mu\,, \tag{7.7}$$

$$\nabla_\mu X = \partial_\mu X - i[v_\mu, X]\,. \tag{7.8}$$

All required propagators and subdiagrams have been defined in the previous chapter. Now we can finally introduce all contributing Feynman diagrams.

7.1 Non-resonance contribution up to $\mathcal{O}(p^4)$

The lowest non-resonance contribution into the $VVVV$ Green function comes from the anomalous Wess-Zumino-Witten vertex (D.73). The only contributing diagram is shown below.

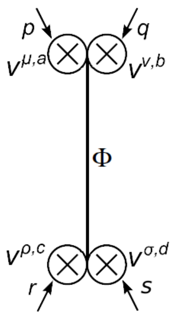


Figure 7.1: Diagram 1 in the χPT up to $\mathcal{O}(p^4)$.

7.2 Vector formalism up to $\mathcal{O}(p^6)$

The relevant part of the resonance Lagrangian up to $\mathcal{O}(p^6)$ in the vector formalism is

$$\mathcal{L}_V = -\frac{f_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu} f_+^{\mu\nu}\rangle + h_V \langle \widehat{V}^{\mu}\{u^{\nu}, f_+^{\alpha\beta}\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,. \tag{7.9}$$

The contributing Feynman diagrams are the following.

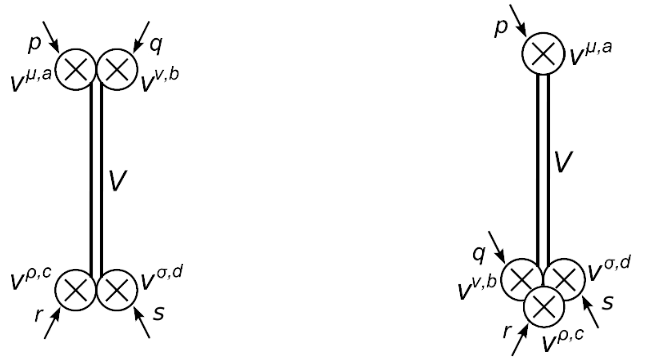


Figure 7.2: Diagrams 1 (left) and 2 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

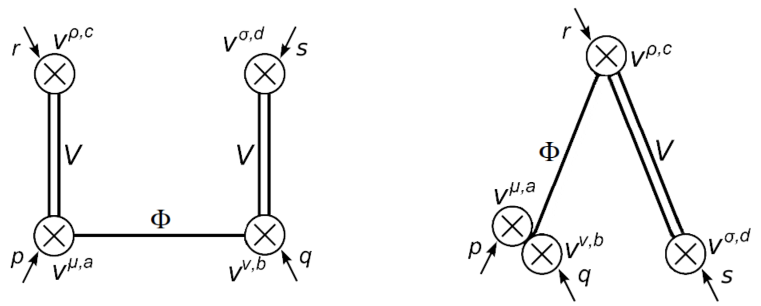


Figure 7.3: Diagrams 3 (left) and 4 (right) in the vector formalism up to $\mathcal{O}(p^6)$.

7.3 Antisymmetric tensor formalism up to $\mathcal{O}(p^4)$

The only relevant part of the Lagrangian (2.68) in the antisymmetric tensor formalism is

$$\mathcal{L}^{(4)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle\,. \tag{7.10}$$

Taking the Lagrangian above into account, we are able to construct only one Feynman diagram with the same type of contact vertices, connected through vector resonance propagator. The contributing Feynman diagram is the following.

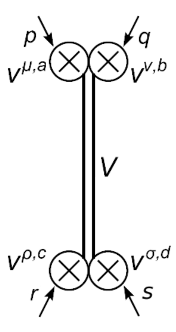


Figure 7.4: Diagram 1 in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.

7.4 Antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

The resonance Lagrangian in the tensor formalism up to $\mathcal{O}(p^6)$ has the relevant part consisted of two Lagrangians, one of the order p^4 and second of the order p^6. More specifically,

$$\mathcal{L}^{(4)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle \,, \tag{7.11}$$

$$\mathcal{L}^{(6)} = \mathcal{L}_{12}^{V} + \mathcal{L}_{16}^{V} + \mathcal{L}_{17}^{V} + \mathcal{L}_{5}^{P} + \mathcal{L}_{3}^{VV} + \mathcal{L}_{4}^{VV} + \mathcal{L}_{5}^{VA} + \mathcal{L}_{3}^{PV} + \mathcal{L}^{VVP} \,, \tag{7.12}$$

where the contributing operators of the order p^6 are shown in the table below.

i	$\mathcal{O}^{P,V}_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{PV,VV,VA}_{i\,\mu\nu\alpha\beta}$	i	$\mathcal{O}^{VVP}_{i\,\mu\nu\alpha\beta}$
5	$\langle P\{f_+^{\mu\nu}, f_+^{\alpha\beta}\}\rangle$	3	$\langle\{V^{\mu\nu}, P\}f_+^{\alpha\beta}\rangle$	-	$\langle V^{\mu\nu}V^{\alpha\beta}P\rangle$
12	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, h^{\beta\sigma}\}\rangle g_{\rho\sigma}$	3	$\langle\{\nabla_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}u^\beta\rangle$		
16	$\langle V^{\mu\nu}\{\nabla^\alpha f_+^{\beta\sigma}, u_\sigma\}\rangle$	4	$\langle\{\nabla^\beta V^{\mu\nu}, V^{\alpha\sigma}\}u_\sigma\rangle$		
17	$\langle V^{\mu\nu}\{\nabla_\sigma f_+^{\alpha\sigma}, u^\beta\}\rangle$	5	$\langle\{V^{\mu\nu}, A^{\alpha\rho}\}f_+^{\beta\sigma}\rangle g_{\rho\sigma}$		

Table 7.1: Monomials contributing into $VVVV$ Green function

The Lagrangian above allow us to construct the following set of Feynman diagrams.

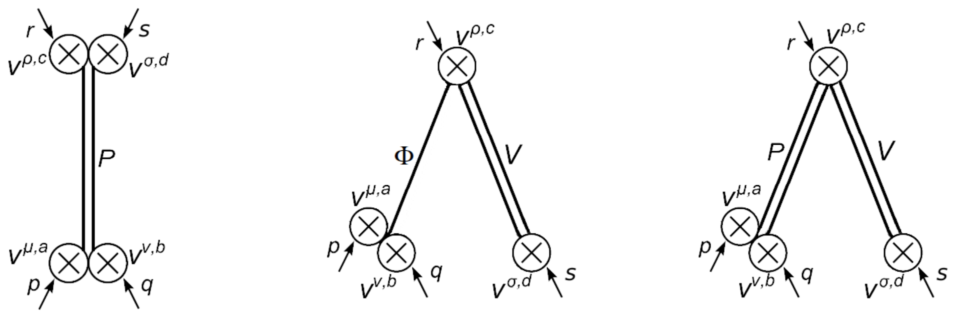


Figure 7.5: Diagram 1 (left), 2 (middle) and 3 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

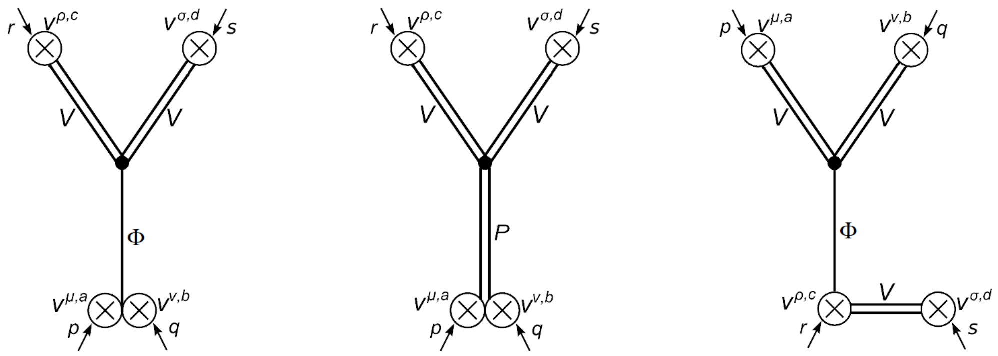


Figure 7.6: Diagram 4 (left), 5 (middle) and 6 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

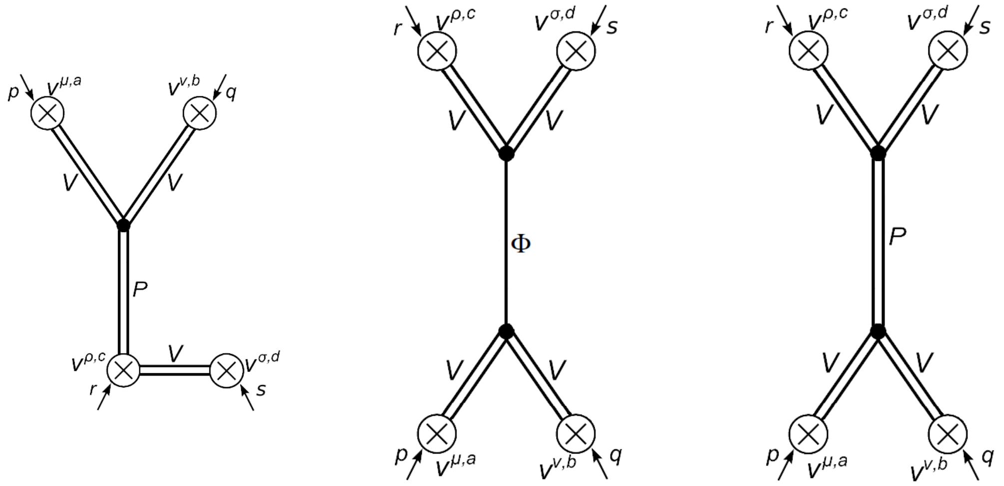


Figure 7.7: Diagram 7 (left), 8 (middle) and 9 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

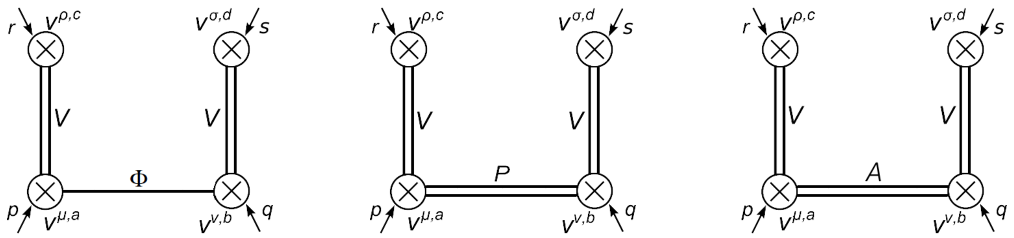


Figure 7.8: Diagram 10 (left), 11 (middle) and 12 (right) in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

To summarize the construction of all the contributing Feynman diagrams, let us mention right away the number of diagrams that are need to get a complete contribution for different chiral orders.

contribution	$VVVV$
χPT $\mathcal{O}(p^4)$	1
RχT $\mathcal{O}(p^4)$ (tensor form.)	1
RχT $\mathcal{O}(p^6)$ (tensor form.)	12
RχT $\mathcal{O}(p^6)$ (vector. form.)	4

Table 7.2: A number of contributing Feynman diagrams into $VVVV$ Green functions.

7.5 Phenomenology

Although unduly complicated, the $VVVV$ correlator is a interesting object to study. For instance, it represents hadronic contribution into the light-by-light scattering [30], [31], [32], [33], [34]. This calculation, regarding our Lagrangian up to $\mathcal{O}(p^6)$ in the odd-intrinsic parity sector, has not yet been provided (at least to our knowledge). We plan to return to this point later on. Here we only present a basic note on the anomalous magnetic moment itself. In what follows we inherit the same notation as in [34].

Anomalous magnetic moment

The muon anomalous magnetic moment,

$$a = \frac{g-2}{2}\,, \tag{7.13}$$

belongs to the most precisely measured physical quantities in particle physics. Recently, it was was measured with a remarkable accuracy by the E821 experiment at Brookhaven National Laboratory with the current rescaled result [35]

$$a = (116\,592\,089 \pm 63) \times 10^{-11}\,. \tag{7.14}$$

However, there has been found a significant and still persisting discrepancy between the theoretical prediction and the measured data. One of the greatest source of the theoretical uncertainty stems from the so called hadronic light-by-light (HLbL) contribution. This process is depicted in Fig. 7.9 below.

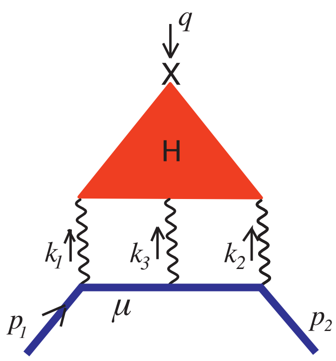


Figure 7.9: Hadronic light–by–light scattering contribution into anomalous magnetic moment (taken from [34]).

The muon anomaly can then be extracted as follows [34]

$$a^{\mathrm{HLbL}} = -i\frac{e^6}{48m_\mu}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_1^2k_2^2k_3^2}\left[\frac{\partial}{\partial q^\mu}\Pi_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)\right]_{q=0}\times$$
$$\times\,\mathrm{Tr}\left\{(\not p + m_\mu)[\gamma^\mu,\gamma^\lambda](\not p + m_\mu)\gamma^\nu\frac{1}{\not p + \not k_2 - m_\mu}\gamma^\rho\frac{1}{\not p - \not k_1 - m_\mu}\gamma^\sigma\right\}. \tag{7.15}$$

Since this is not an easy calculation (the number of the independent contributing tensor structures is 138 up to $\mathcal{O}(p^6)$), we will finish our discussion here. Detailed study can be found for example in [33], [34]. However, we plan to return to this point in future studies with our calculations of the tensor $\Pi_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)$, i.e. (7.1) in our notation.

Conclusion

To conclude this thesis, let us remember the topic we dealt with and summarize the main results we have obtained in our work.

In the first two chapters we have studied the basics of quantum chromodynamics and introduced the description of the QCD at low energies, the chiral perturbation theory and resonance chiral theory.

In the third chapter we have dealt with the general properties of the Green functions of currents and presented some familiar results that have been already obtained.

The fourth chapter contains our original calculation of the VVA Green function. Extracting the formfactors from its result, we were succesful in obtaining some constraints for the coupling constants. We have also studied its phenomenology on the decay $f_1(1285) \to \rho\gamma$ from which we have recovered similar value for one of the coupling constants.

The fifth chapter contains our original calculation of AAA correlator. Since the experimental data are not readily available for our study, we did not deal with the phenomenological predictions.

After finishing studies of the VVA and AAA Green functions, we have devoted our time to study the four-point Green functions $VVPP$ and $VVVV$ in Chapters 6 and 7. As we have mentioned, the calculations were carried out both in the antisymmetric tensor and vector field formalism, up to various chiral orders of the Lagrangians. To accomplish this difficult task we had to extensively use our algorithm Mercury (see Appendix E). Although we were successful and calculated all possible diagrams in the antisymmetric tensor and vector formalisms with Lagrangians up to $\mathcal{O}(p^6)$, we had to make a decision regarding the publication of the results. Since these correlators, especially $VVVV$, have a very difficult tensor structure, the results are inconveniently long for the purpose of this thesis. In addition, we had to deal with dozens of different types of contributing Feynman diagrams, not to mention the permutations of the indices that would take the total number of diagrams even higher. For this reason, we have only mentioned basic properties of these correlators and introduced all contributing Feynman diagrams. Also, we have briefly mentioned the phenomenological studies of these correlator that we will devote our time to in the future.

To illustrate the difficulty of the previous task, let us mention that all calculations of the four-point Green functions took several hours just to simplify the results slightly and extract individual formfactors[2].

To make the description of the topic clear, we have also included several appendices. Some of them, D and E to be more specific, contain original results that have been obtained in order to make our calculations as understandable as possible. For this reason, we have wrote an algorithm in FeynCalc that allows us to carry out difficult calculations in a very simple way. The source code and the files are also attached to the thesis.

Future studies

As we have mentioned previously, not only will we want to carry on in the systematic study of the odd-intrinsic parity sector itself, we will also pay our attention especially to phenomenological studies of the VVA and AAA Green functions in the connection with some new experimantal data that would allow us to widen our knowledge of this topic.

[2]Used computers: Intel Core i5 CPU 2.80 GHz 64 bit, 16 GB RAM and Intel Core i7 CPU 2.20 GHz 64 bit, 8 GB RAM.

Also, we will try to use our obtained results from $VVPP$ and $VVVV$ Green functions on some phenomenologically releavant applications, such as Compton-like scattering in case of $VVPP$ correlator, or the hadronic light-by-light scattering in the case of $VVVV$ Green function.

Final remark

In conclusion, let us mention that the results of this thesis were presented in progress during the work by a poster at MesonNet International Workshop 2014 in Frascati, Italy [36], [37] and by talks at two international conferences, XI$^{\text{th}}$ Quark confinement and the hadron spectrum 2014 in Saint-Petersburg, Russia [38] and 18$^{\text{th}}$ High-energy physics international conference in QCD 2015 in Montpellier, France [39]. Also, during the first year of the masters studies, the ongoing results were presented at 5$^{\text{th}}$ Czech-Slovak student scientific conference in Prague.

A. Mathematical appendix

A.1 Gell-Mann matrices

The Gell-Mann matrices are one of the possible representations of the infinitesimal generators of the special unitary group SU(3) with dimension eight. Generally, this representation gives us a set of eight linearly independent hermitian generators T^a, where $a = 1 \dots 8$. The element of the SU(3) group can be written in the form

$$U(x) = \exp\left(-i \sum_{a=1}^{8} \theta^a(x) T^a \right) . \tag{A.1}$$

These generators satisfy the following (anti)commutation relations [40]:

$$[T^a, T^b] = i f^{abc} T^c \,, \tag{A.2}$$

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab} + d^{abc} T^c \,, \tag{A.3}$$

where d^{abc} is totally symmetric and f^{abc} is antisymmetric tensor. The relations (A.2)-(A.3) can be inverted into the definitions of these tensors:

$$f^{abc} = -2i \big\langle [T^a, T^b] T^c \big\rangle \,, \tag{A.4}$$

$$d^{abc} = 2 \big\langle \{T^a, T^b\} T^c \big\rangle \,. \tag{A.5}$$

Non-zero elements of the tensors above are shown in the tables A.1 and A.2 below. The particular choice of the generators T^a is related to the hermitian Gell-Mann matrices λ^a due the form

$$T^a = \frac{\lambda^a}{2} \,, \tag{A.6}$$

where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{A.7}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{A.8}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{A.9}$$

abc	118	146	157	228	247	256	338	344
d_{abc}	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$
abc	355	366	377	448	558	668	778	888
d_{abc}	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

Table A.1: Totally symmetric non-vanishing structure constants of SU(3).

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Table A.2: Totally antisymmetric non-vanishing structure constants of SU(3).

It is useful to add one additional matrix T^0, despite the fact that it is not a part of SU(3) but corresponds to U(1). For this matrix we choose

$$T^0 = \frac{1}{\sqrt{6}}\mathbb{1}\,. \tag{A.10}$$

It is also useful to define the symbol h^{abc} as follows

$$h^{abc} = d^{abc} + i f^{abc} \tag{A.11}$$

with the identities for index permutations:

$$h^{abc} = h^{bca} = h^{cab}\,, \qquad h^{bac} = h^{acb} = h^{cba}\,, \tag{A.12}$$

$$h^{abc} + h^{bac} = 2d^{abc}\,, \qquad h^{abc} - h^{bac} = 2i f^{abc}\,. \tag{A.13}$$

Jacobi identites

The Jacobi identities for the coefficients (A.4)-(A.5) are as follow:

$$f^{abk} f^{kcl} + f^{bck} f^{kal} + f^{cak} f^{kbl} = 0\,, \tag{A.14}$$

$$d^{abk} f^{kcl} + d^{bck} f^{kal} + d^{cak} f^{kbl} = 0\,. \tag{A.15}$$

Another useful identities are

$$f^{abk} f^{kcl} = \frac{2}{3}(\delta^{ac}\delta^{bl} - \delta^{al}\delta^{bc}) + d^{ack}d^{blk} - d^{alk}d^{bck}\,, \tag{A.16}$$

$$d^{abk} d^{kcl} = \frac{1}{3}(\delta^{ac}\delta^{bl} + \delta^{al}\delta^{bc} - \delta^{ab}\delta^{cl} + f^{ack}f^{blk} + f^{alk}f^{bck})\,. \tag{A.17}$$

Products of T^a matrices

In the following it will be useful to introduce a symbol for a product of n matrices T^a defined as:

$$T^{a_1\ldots a_n} = T^{a_1}\ldots T^{a_n} \tag{A.18}$$

with a usual choice of $a_1 = a, a_2 = b, a_3 = c$ etc. For our purposes the following formulas are useful:

$$T^{ab} = \frac{1}{6}\delta^{ab} + \frac{1}{2}h^{abk}T^k\,, \tag{A.19}$$

$$T^{abc} = \frac{1}{6}\delta^{ab}T^c + \frac{1}{12}h^{abc} + \frac{1}{4}h^{abk}h^{kcl}T^l\,, \tag{A.20}$$

$$T^{abcd} = \frac{1}{36}\delta^{ab}\delta^{cd} + \frac{1}{24}h^{abk}h^{kcd} + \frac{1}{12}(h^{abc}T^d + \delta^{ab}h^{cdk}T^k)\,, \tag{A.21}$$

$$+\frac{1}{8}h^{abk}h^{kcl}h^{ldm}T^m\,.$$

For the same reason as above we also show here the formulas for a commutator and an anticommutator of four T^a matrices at most. For simplier notation, let us define a symbol for a commutator of one and other $n-1$ matrices T^a for $n > 1$:

$$T_-^{a_1\ldots a_n} = [T^{a_1}, T^{a_2\ldots a_n}]\,, \tag{A.22}$$

and equivalently for an anticommutator:

$$T_+^{a_1\dots a_n} = \{T^{a_1}, T^{a_2\dots a_n}\}\,. \tag{A.23}$$

The following formulas will be useful.

$$T_-^{ab} = i f^{abk} T^k\,, \tag{A.24}$$

$$T_+^{ab} = \frac{1}{3}\delta^{ab} + d^{abk} T^k\,, \tag{A.25}$$

$$T_-^{abc} = \frac{1}{6}(\delta^{ab}T^c - \delta^{bc}T^a) + \frac{1}{4}(h^{abk}h^{kcl} - h^{bck}h^{kal})T^l\,, \tag{A.26}$$

$$T_+^{abc} = \frac{1}{6}(\delta^{ab}T^c + \delta^{bc}T^a + h^{abc}) + \frac{1}{4}(h^{abk}h^{kcl} + h^{bck}h^{kal})T^l\,, \tag{A.27}$$

$$\begin{aligned} T_-^{abcd} = \;& \frac{1}{12}(h^{abc}T^d - h^{bcd}T^a) + \frac{1}{12}(\delta^{ab}h^{cdk} - \delta^{bc}h^{dak})T^k \\ &+ \frac{1}{36}(\delta^{ab}\delta^{cd} - \delta^{bc}\delta^{da}) + \frac{1}{24}(h^{abk}h^{kcd} - h^{bck}h^{kda}) \\ &+ \frac{1}{8}(h^{abk}h^{kcl}h^{ldm} - h^{bck}h^{kdl}h^{lam})T^m\,, \end{aligned} \tag{A.28}$$

$$\begin{aligned} T_+^{abcd} = \;& \frac{1}{12}(h^{abc}T^d + h^{bcd}T^a) + \frac{1}{12}(\delta^{ab}h^{cdk} + \delta^{bc}h^{dak})T^k \\ &+ \frac{1}{36}(\delta^{ab}\delta^{cd} + \delta^{bc}\delta^{da}) + \frac{1}{24}(h^{abk}h^{kcd} + h^{bck}h^{kda}) \\ &+ \frac{1}{8}(h^{abk}h^{kcl}h^{ldm} + h^{bck}h^{kdl}h^{lam})T^m\,. \end{aligned} \tag{A.29}$$

Traces of T^a matrices

Knowing the basic properties above we can derive traces of products of the T^a matrices. Trivial calculations based on (A.2) and (A.3) lead to the following relations:

$$\langle T^a \rangle = 0\,, \tag{A.30}$$

$$\langle T^{ab} \rangle = \frac{1}{2}\delta^{ab}\,, \tag{A.31}$$

$$\langle T^{abc} \rangle = \frac{1}{4}h^{abc}\,, \tag{A.32}$$

$$\langle T^{abcd} \rangle = \frac{1}{8}h^{abk}h^{kcd} + \frac{1}{12}\delta^{ab}\delta^{cd}\,, \tag{A.33}$$

that allow us to calculate the traces occurring in the (anti)commutators:

$$\langle T_-^{ab} \rangle = 0\,, \tag{A.34}$$

$$\langle T_+^{ab} \rangle = \delta^{ab}\,, \tag{A.35}$$

$$\langle T_-^{abc} \rangle = 0\,, \tag{A.36}$$

$$\langle T_+^{abc} \rangle = \frac{1}{2}h^{abc}\,, \tag{A.37}$$

$$\langle T_-^{abcd} \rangle = \frac{1}{12}(\delta^{ab}\delta^{cd} - \delta^{bc}\delta^{da}) + \frac{1}{8}(h^{abk}h^{kcd} - h^{bck}h^{kda})\,, \tag{A.38}$$

$$\langle T_+^{abcd} \rangle = \frac{1}{12}(\delta^{ab}\delta^{cd} + \delta^{bc}\delta^{da}) + \frac{1}{8}(h^{abk}h^{kcd} + h^{bck}h^{kda})\,. \tag{A.39}$$

A.2 Dirac matrices

Dirac gamma matrices, $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$, are a set of conventional matrices with specific anticommutation relations. These matrices ensure that they generate a matrix representation of the Clifford algebra $\mathcal{C}\ell_{1,3}(\mathcal{R})$ and they are defined as [40]

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.40}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A.41}$$

It is useful to define the product of the four gamma matrices as follows:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A.42}$$

Traces of Dirac matrices

The matrices shown above satisfy a lot of important properties, but we will introduce here only the identities of traces of products of Dirac matrices. In this case we present following identities [40]:

$$\langle \gamma^\mu \rangle = 0\,, \tag{A.43}$$

$$\langle \gamma_5 \rangle = 0\,, \tag{A.44}$$

$$\langle \gamma^\mu \gamma^\nu \rangle = 4g^{\mu\nu}\,, \tag{A.45}$$

$$\langle \gamma^\mu \gamma^\nu \gamma_5 \rangle = 0\,, \tag{A.46}$$

$$\langle \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \rangle = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,, \tag{A.47}$$

$$\langle \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \rangle = -4i\varepsilon^{\mu\nu\rho\sigma}\,, \tag{A.48}$$

$$\begin{aligned}\langle \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \gamma^\kappa \gamma_5 \rangle = 4i(&g^{\mu\nu}\varepsilon^{\rho\sigma\lambda\kappa} - g^{\mu\rho}\varepsilon^{\nu\sigma\lambda\kappa} + g^{\nu\rho}\varepsilon^{\mu\sigma\lambda\kappa} \\ &- g^{\sigma\kappa}\varepsilon^{\mu\nu\rho\lambda} + g^{\sigma\lambda}\varepsilon^{\mu\nu\rho\kappa} + g^{\lambda\kappa}\varepsilon^{\mu\nu\rho\sigma})\,.\end{aligned} \tag{A.49}$$

Feynman slash notation

Let us introduce and an arbitrary four-vector a_μ. Then we will denote a product of Dirac matrix γ^μ and a_μ in the Feynman slash notation as follows:

$$\not{a} \equiv \gamma^\mu a_\mu\,. \tag{A.50}$$

Using the anticommutators of the gamma matrices, one can show that for any four-vectors a_μ and b_μ

$$\not{a}\not{a} = a^2 \tag{A.51}$$

and

$$\not{a}\not{b} + \not{b}\not{a} = 2(a \cdot b)\,. \tag{A.52}$$

Further identities can be read off directly from the gamma matrix identities by replacing the metric tensor with inner products. For example:

$$\langle \not{a}\not{b} \rangle = 4(a \cdot b)\,, \tag{A.53}$$

$$\langle \not{a}\not{b}\not{c}\not{d} \rangle = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]\,, \tag{A.54}$$

$$\langle \not{a}\not{b}\not{c}\not{d}\gamma_5 \rangle = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma\,. \tag{A.55}$$

A.3 Matrix algebra

A tricky task to deal with is derivatives of a matrix with its components that are dependent on a parameter. For this case, let us assume a general square matrix $X(t)$, dependent on a variable t, and a parameter $\alpha \in \mathbb{R}$. The first derivative of the exponential matrix can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{X(t)} = \int_0^1 e^{\alpha X(t)} \frac{\mathrm{d}X(t)}{\mathrm{d}t} e^{(1-\alpha)X(t)}\, \mathrm{d}\alpha\,. \tag{A.56}$$

The key ingredient that we will need greatly is the derivative $\partial_\mu u$, where u is defined by (2.5). Analogically to (A.56), we can designate

$$X(t) \to \beta\phi\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \to \partial_\mu\,, \tag{A.57}$$

where ϕ is the matrix of pseudoscalar fields (2.6) and

$$\beta \equiv \frac{i}{\sqrt{2}F}\,. \tag{A.58}$$

It will be also appropriate to use the Taylor series to expand the exponential matrices under the integral sign:

$$e^{\alpha\beta\phi} \simeq \mathbb{1} + \alpha\beta\phi + \frac{1}{2}\alpha^2\beta^2\phi^2 + \mathcal{O}(\phi^3)\,, \tag{A.59}$$

$$e^{(1-\alpha)\beta\phi} \simeq \mathbb{1} + (1-\alpha)\beta\phi + \frac{1}{2}(1-\alpha)^2\beta^2\phi^2\mathcal{O}(\phi^3)\,, \tag{A.60}$$

where we expanded the series only up to $\mathcal{O}(\phi^3)$ since we want to have $\partial_\mu u$ (and other chiral building blocks) up to $\mathcal{O}(\phi^4)$. We have therefore

$$\partial_\mu u = \int_0^1 e^{\alpha\beta\phi}\beta(\partial_\mu\phi)e^{(1-\alpha)\beta\phi}\,\mathrm{d}\alpha \tag{A.61}$$

$$\simeq \int_0^1 \Big[\beta(\partial_\mu\phi) + (1-\alpha)\beta^2(\partial_\mu\phi)\phi + \alpha\beta^2\phi(\partial_\mu\phi) + \frac{1}{2}\alpha^2\beta^3\phi^2(\partial_\mu\phi) \tag{A.62}$$

$$+ \frac{1}{2}(1-\alpha)^2\beta^3(\partial_\mu\phi)\phi^2 + \alpha(1-\alpha)\beta^3\phi(\partial_\mu\phi)\phi\Big]\,\mathrm{d}\alpha + \mathcal{O}(\phi^4)$$

$$= \beta(\partial_\mu\phi) + \frac{1}{2}\beta^2\{\partial_\mu\phi,\phi\} + \frac{1}{6}\beta^3\{\partial_\mu\phi,\phi^2\} + \frac{1}{6}\beta^3\phi(\partial_\mu\phi)\phi + \mathcal{O}(\phi^4)\,. \tag{A.63}$$

Substituting back for (A.58) we get

$$\partial_\mu u \simeq \frac{i}{\sqrt{2}F}\partial_\mu\phi - \frac{1}{4F^2}\{\partial_\mu\phi,\phi\} - \frac{i}{12\sqrt{2}F^3}\{\partial_\mu\phi,\phi^2\} \tag{A.64}$$

$$- \frac{i}{12\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi + \mathcal{O}(\phi^4)\,,$$

which is a correct way of treating matrices. The same result can be obtained faster by expanding (2.5) in the Taylor series and then derive term by term.

Since ϕ and $\partial_\mu\phi$ generally do not commute,

$$[\partial_\mu\phi, \phi] \neq 0\,, \tag{A.65}$$

a wrong way of the calculation of $\partial_\mu u$ above would be the following treatment:

$$\begin{aligned}
\partial_\mu u &= \partial_\mu \left[\exp\left(\frac{i}{\sqrt{2}F}\phi \right) \right] && \text{(A.66)}\\
&\neq \exp\left(\frac{i}{\sqrt{2}F}\phi \right) \frac{i}{\sqrt{2}F}\partial_\mu\phi = \frac{i}{\sqrt{2}F} u\,\partial_\mu\phi && \text{(A.67)}\\
&\simeq \frac{i}{\sqrt{2}F}\partial_\mu\phi - \frac{1}{2F^2}\phi(\partial_\mu\phi) - \frac{i}{4\sqrt{2}F^3}\phi^2(\partial_\mu\phi) + \mathcal{O}(\phi^4)\,. && \text{(A.68)}
\end{aligned}$$

Once again let us repeat that the result (A.64) is correct whilst (A.68) is not.

A.4 Levi-Civita tensor

Levi-Civita tensor (or ε-symbol) in four dimensions, assuming

$$\varepsilon_{0123} = 1\,, \tag{A.69}$$

has the following important properties:

$$\begin{aligned}
\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\gamma\delta} &= - \begin{vmatrix} g_\mu^\rho & g_\nu^\rho & g_\alpha^\rho & g_\beta^\rho \\ g_\mu^\sigma & g_\nu^\sigma & g_\alpha^\sigma & g_\beta^\sigma \\ g_\mu^\gamma & g_\nu^\gamma & g_\alpha^\gamma & g_\beta^\gamma \\ g_\mu^\delta & g_\nu^\delta & g_\alpha^\delta & g_\beta^\delta \end{vmatrix}\,, && \text{(A.70)}\\
\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\alpha\beta} &= -2(g_\mu^\rho g_\nu^\sigma - g_\mu^\sigma g_\nu^\rho)\,. && \text{(A.71)}
\end{aligned}$$

The examples of a contractions of components of 4-momenta with Levi-Civita tensor can be written in the simplified forms such as

$$\begin{aligned}
\varepsilon_{\mu\nu\alpha(p)} &\equiv \varepsilon_{\mu\nu\alpha\beta}p^\beta\,, && \text{(A.72)}\\
\varepsilon_{\mu\nu(p)(q)} &\equiv \varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta && \text{(A.73)}
\end{aligned}$$

etc.

Schouten identity

A very important example of the usage of such a tensor is the Schouten identity. For any 4-vector p_μ one has

$$p_\mu\varepsilon_{\nu\rho\sigma\tau} + p_\nu\varepsilon_{\rho\sigma\tau\mu} + p_\rho\varepsilon_{\sigma\tau\mu\nu} + p_\sigma\varepsilon_{\tau\mu\nu\rho} + p_\tau\varepsilon_{\mu\nu\rho\sigma} = 0\,. \tag{A.74}$$

B. General expansion of the building blocks

Now we will perform the procedure to find out the structure of the chiral tensors. Let us start with the basic building block (2.5). Using the Taylor series (see also section A.3), we can expand the matrix exponential in the infinite series of the powers of the matrix of the pseudoscalar fields ϕ. Let us use only the terms up to $\mathcal{O}(\phi^4)$. The relevant expansion of the basic building block, including its hermitian conjugation, is then

$$u \simeq \mathbb{1} + \frac{i}{\sqrt{2}F}\phi - \frac{1}{4F^2}\phi^2 - \frac{i}{12\sqrt{2}F^3}\phi^3 + \mathcal{O}(\phi^4)\,, \tag{B.1}$$

$$u^\dagger \simeq \mathbb{1} - \frac{i}{\sqrt{2}F}\phi - \frac{1}{4F^2}\phi^2 + \frac{i}{12\sqrt{2}F^3}\phi^3 + \mathcal{O}(\phi^4)\,, \tag{B.2}$$

where it is obvious that $(\phi^2)^\dagger \equiv \phi^2$ and $(\phi^3)^\dagger \equiv \phi^3$ etc. Using basic manipulations with matrices, such as

$$\partial_\mu \phi^2 = \{\partial_\mu\phi, \phi\}\,, \tag{B.3}$$

$$\partial_\mu \phi^3 = \{\partial_\mu\phi, \phi^2\} + \phi(\partial_\mu\phi)\phi\,, \tag{B.4}$$

allow us to obtain derivatives of (B.1):

$$\begin{aligned}\partial_\mu u \simeq & \frac{i}{\sqrt{2}F}\partial_\mu\phi - \frac{1}{4F^2}\{\partial_\mu\phi, \phi\} - \frac{i}{12\sqrt{2}F^3}\{\partial_\mu\phi, \phi^2\} \\ & - \frac{i}{12\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi + \mathcal{O}(\phi^4)\end{aligned} \tag{B.5}$$

and similarly for (B.2)

$$\begin{aligned}\partial_\mu u^\dagger \simeq & - \frac{i}{\sqrt{2}F}\partial_\mu\phi - \frac{1}{4F^2}\{\partial_\mu\phi, \phi\} + \frac{i}{12\sqrt{2}F^3}\{\partial_\mu\phi, \phi^2\} \\ & + \frac{i}{12\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi + \mathcal{O}(\phi^4)\,.\end{aligned} \tag{B.6}$$

The chiral operator (2.12) couples the vector and axial-vector sources together with the pseudoscalar fields. Its expansion has the form

$$\begin{aligned}u_\mu \simeq & - \frac{\sqrt{2}}{F}\partial_\mu\phi + 2a_\mu - \frac{i\sqrt{2}}{F}[\phi, v_\mu] + \frac{1}{F^2}\phi a_\mu \phi - \frac{1}{2F^2}\{\phi^2, a_\mu\} \\ & - \frac{1}{3\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi + \frac{1}{6\sqrt{2}F^3}\{\partial_\mu\phi, \phi^2\} - \frac{i}{2\sqrt{2}F^3}\phi[\phi, v_\mu]\phi \\ & + \frac{i}{6\sqrt{2}F^3}[\phi^3, v_\mu] + \mathcal{O}(\phi^4)\,.\end{aligned} \tag{B.7}$$

Indeed, it is easy to show that (B.7) still holds the relation $u_\mu = u_\mu^\dagger$ (see 2.12). The only possibly dangerous terms are the commutators, excluding them, we have left only the hermitian single terms and symmetrical hermitian anticommutators. However, it is easy to show that the mentioned hermiticity holds, because of the sign change of the imaginary units and the reordering of the terms in the commutators, that leave the whole term itself the same as was before. In detail, we can write:

$$\left(-\frac{i\sqrt{2}}{F}[\phi, v_\mu]\right)^\dagger = \frac{i\sqrt{2}}{F}[v_\mu, \phi] = -\frac{i\sqrt{2}}{F}[\phi, v_\mu]\,, \tag{B.8}$$

$$\left(\frac{i}{6\sqrt{2}F^3}[\phi^3, v_\mu]\right)^\dagger = -\frac{i}{6\sqrt{2}F^3}[v_\mu, \phi^3] = \frac{i}{6\sqrt{2}F^3}[\phi^3, v_\mu]\,, \tag{B.9}$$

$$\left(-\frac{i}{2\sqrt{2}F^3}\phi[\phi, v_\mu]\phi\right)^\dagger = \frac{i}{2\sqrt{2}F^3}\phi[v_\mu, \phi]\phi = -\frac{i}{2\sqrt{2}F^3}\phi[\phi, v_\mu]\phi\,. \tag{B.10}$$

The chiral operators (2.13) couple the scalar and pseudoscalar external sources together with the pseudoscalar fields. The expansions of the operators have the forms

$$\begin{aligned}\chi_- \simeq & 4iB_0 p - \frac{2\sqrt{2}iB_0}{F}\{\phi, s\} - \frac{2iB_0}{F^2}\phi p \phi - \frac{iB_0}{F^2}\{\phi^2, p\} \\ & + \frac{iB_0}{\sqrt{2}F^3}\phi\{\phi, s\}\phi + \frac{iB_0}{3\sqrt{2}F^3}\{\phi^3, s\} + \mathcal{O}(\phi^4)\end{aligned} \tag{B.11}$$

and

$$\begin{aligned}\chi_+ \simeq & 4B_0 s + \frac{2\sqrt{2}B_0}{F}\{\phi, p\} - \frac{2B_0}{F^2}\phi s \phi - \frac{B_0}{F^2}\{\phi^2, s\} \\ & - \frac{B_0}{\sqrt{2}F^3}\phi\{\phi, p\}\phi - \frac{B_0}{3\sqrt{2}F^3}\{\phi^3, p\} + \mathcal{O}(\phi^4)\,.\end{aligned} \tag{B.12}$$

The left and right non-Abelian field-strength tensors (2.20), (2.21) occur in the following combinations:

$$F_R^{\mu\nu} + F_L^{\mu\nu} = 2(\partial^\mu v^\nu - \partial^\nu v^\mu) - 2i[v^\mu, v^\nu] - 2i[a^\mu, a^\nu]\,, \tag{B.13}$$

$$F_R^{\mu\nu} - F_L^{\mu\nu} = 2(\partial^\mu a^\nu - \partial^\nu a^\mu) - 2i[v^\mu, a^\nu] - 2i[a^\mu, v^\nu]\,. \tag{B.14}$$

Using (B.13) and (B.14) one can extract the structure of the operators (2.18) that couple the vector and axial-vector external sources together with the pseudoscalar fields. The results are complicated and as follows:

$$\begin{aligned}f_-^{\mu\nu} \simeq & -2(\partial^\mu a^\nu - \partial^\nu a^\mu) + 2i[v^\mu, a^\nu] + 2i[a^\mu, v^\nu] + \frac{\sqrt{2}}{F}\big[\phi, [v^\mu, v^\nu]\big] \\ & + \frac{\sqrt{2}}{F}\big[\phi, [a^\mu, a^\nu]\big] + \frac{i\sqrt{2}}{F}[\phi, \partial^\mu v^\nu - \partial^\nu v^\mu] + \frac{i}{F^2}\phi[v^\mu, a^\nu]\phi \\ & + \frac{i}{F^2}\phi[a^\mu, v^\nu]\phi - \frac{1}{F^2}\phi(\partial^\mu a^\nu - \partial^\nu a^\mu)\phi - \frac{i}{2F^2}\big\{\phi^2, [v^\mu, a^\nu]\big\} \\ & - \frac{i}{2F^2}\big\{\phi^2, [a^\mu, v^\nu]\big\} + \frac{1}{2F^2}\{\phi^2, \partial^\mu a^\nu - \partial^\nu a^\mu\} \\ & + \frac{i}{2\sqrt{2}F^3}\phi[\phi, \partial^\mu v^\nu - \partial^\nu v^\mu]\phi + \frac{1}{2\sqrt{2}F^3}\phi\big[\phi, [v^\mu, v^\nu]\big]\phi \\ & + \frac{1}{2\sqrt{2}F^3}\phi\big[\phi, [a^\mu, a^\nu]\big]\phi - \frac{i}{6\sqrt{2}F^3}[\phi^3, \partial^\mu v^\nu - \partial^\nu v^\mu] \\ & - \frac{1}{6\sqrt{2}F^3}\big[\phi^3, [v^\mu, v^\nu]\big] - \frac{1}{6\sqrt{2}F^3}\big[\phi^3, [a^\mu, a^\nu]\big] + \mathcal{O}(\phi^4)\end{aligned} \tag{B.15}$$

and

$$
\begin{aligned}
f_+^{\mu\nu} \simeq\, & 2(\partial^\mu v^\nu - \partial^\nu v^\mu) - 2i[v^\mu, v^\nu] - 2i[a^\mu, a^\nu] - \frac{i\sqrt{2}}{F}[\phi, \partial^\mu a^\nu - \partial^\nu a^\mu] \\
& - \frac{\sqrt{2}}{F}\big[\phi, [v^\mu, a^\nu]\big] - \frac{\sqrt{2}}{F}\big[\phi, [a^\mu, v^\nu]\big] + \frac{1}{F^2}\phi(\partial^\mu v^\nu - \partial^\nu v^\mu)\phi \\
& - \frac{i}{F^2}\phi[v^\mu, v^\nu]\phi - \frac{i}{F^2}\phi[a^\mu, a^\nu]\phi - \frac{1}{2F^2}\{\phi^2, \partial^\mu v^\nu - \partial^\nu v^\mu\} \\
& + \frac{i}{2F^2}\big\{\phi^2, [v^\mu, v^\nu]\big\} + \frac{i}{2F^2}\big\{\phi^2, [a^\mu, a^\nu]\big\} \\
& - \frac{i}{2\sqrt{2}F^3}\phi[\phi, \partial^\mu a^\nu - \partial^\nu a^\mu]\phi - \frac{1}{2\sqrt{2}F^3}\phi\big[\phi, [v^\mu, a^\nu]\big]\phi \\
& - \frac{1}{2\sqrt{2}F^3}\phi\big[\phi, [a^\mu, v^\nu]\big]\phi + \frac{i}{6\sqrt{2}F^3}[\phi^3, \partial^\mu a^\nu - \partial^\nu a^\mu] \\
& + \frac{1}{6\sqrt{2}F^3}\big[\phi^3, [v^\mu, a^\nu]\big] + \frac{1}{6\sqrt{2}F^3}\big[\phi^3, [a^\mu, v^\nu]\big] + \mathcal{O}(\phi^4)\,.
\end{aligned}
\tag{B.16}
$$

The chiral connection (2.23) has the following form of the expansion:

$$
\begin{aligned}
\Gamma_\mu \simeq & - iv_\mu - \frac{1}{\sqrt{2}F}[\phi, a_\mu] - \frac{i}{2F^2}\phi v_\mu \phi - \frac{1}{4F^2}[\partial_\mu\phi, \phi] + \frac{i}{4F^2}\{\phi^2, v_\mu\} \\
& - \frac{1}{4\sqrt{2}F^3}\phi[\phi, a_\mu]\phi + \frac{1}{12\sqrt{2}F^3}[\phi^3, a_\mu] + \mathcal{O}(\phi^4)\,,
\end{aligned}
\tag{B.17}
$$

which allows us to determine the form for the covariant derivative (2.22) of an arbitrary operator X:

$$
\begin{aligned}
\nabla_\mu X \simeq\, & \partial_\mu X - i[v_\mu, X] - \frac{1}{\sqrt{2}F}\big[[\phi, a_\mu], X\big] - \frac{i}{2F^2}[\phi v_\mu \phi, X] \\
& - \frac{1}{4F^2}\big[[\partial_\mu\phi, \phi], X\big] + \frac{i}{4F^2}\big[\{\phi^2, v_\mu\}, X\big] \\
& - \frac{1}{4\sqrt{2}F^3}\big[\phi[\phi, a_\mu]\phi, X\big] + \frac{1}{12\sqrt{2}F^3}\big[[\phi^3, a_\mu], X\big] + \mathcal{O}(\phi^4)\,.
\end{aligned}
\tag{B.18}
$$

Finally, the results obtained above enable us to identify the expansion of the operator (2.19) with the most complicated structure. Since the structure is very complicated, we will show here the result only up to $\mathcal{O}(\phi^3)$:

$$
\begin{aligned}
h_{\mu\nu} \simeq & - \frac{\sqrt{2}}{F}\partial_\mu\partial_\nu\phi + 2\partial_\mu a_\nu - \frac{i\sqrt{2}}{F}[\partial_\mu\phi, v_\nu] - \frac{i\sqrt{2}}{F}[\phi, \partial_\mu v_\nu] \\
& - \frac{1}{2F^2}\{\phi^2, \partial_\mu a_\nu\} - \frac{1}{2F^2}\big\{\{\partial_\mu\phi, \phi\}, a_\nu\big\} + \frac{1}{F^2}(\partial_\mu\phi)a_\nu\phi \\
& + \frac{1}{F^2}\phi(\partial_\mu a_\nu)\phi + \frac{1}{F^2}\phi a_\nu(\partial_\mu\phi) + \frac{i}{2F^2}\Big[\{\phi^2, v_\mu\} - 2\phi v_\mu\phi + i[\partial_\mu\phi, \phi], a_\nu\Big] \\
& - i\Big[v_\mu, -\frac{\sqrt{2}}{F}\partial_\nu\phi + 2a_\nu - \frac{i\sqrt{2}}{F}[\phi, v_\nu] + \frac{1}{F^2}\phi a_\nu\phi - \frac{1}{2F^2}\{\phi^2, a_\nu\}\Big] \\
& - \frac{1}{\sqrt{2}F}\Big[[\phi, a_\nu], -\frac{\sqrt{2}}{F}\partial_\nu\phi + 2a_\nu - \frac{i\sqrt{2}}{F}[\phi, v_\nu]\Big] + \mathcal{O}(\phi^3) \\
& + (\mu \leftrightarrow \nu)\,.
\end{aligned}
\tag{B.19}
$$

Long story short, we now know the expansions of the basic building blocks into the terms of vector, axial-vector, scalar and pseudoscalar fields. The results of this chapter are summarized below where we state the content of the building blocks in term of pseudoscalar fields and external sources.

$$u \sim \phi, \phi^2, \phi^3 \ldots , \tag{B.20}$$
$$u_\mu \sim a, \phi, v\phi, a\phi^2, \phi^3, v\phi^3 \ldots , \tag{B.21}$$
$$h_{\mu\nu} \sim a, \phi, v\phi, a\phi^2, \phi^3, v\phi^3 \ldots , \tag{B.22}$$
$$f_-^{\mu\nu} \sim a, av, v\phi, v^2\phi, a^2\phi, a\phi^2, av\phi^2, v\phi^3, a^2\phi^3, v^2\phi^3 \ldots , \tag{B.23}$$
$$f_+^{\mu\nu} \sim v, a^2, v^2, a\phi, av\phi, v\phi^2, a^2\phi^2, v^2\phi^2, a\phi^3, av\phi^3 \ldots , \tag{B.24}$$
$$\chi \sim s, p , \tag{B.25}$$
$$\chi_- \sim p, s\phi, p\phi^2, s\phi^3 \ldots , \tag{B.26}$$
$$\chi_+ \sim s, p\phi, s\phi^2, p\phi^3 \ldots \tag{B.27}$$

Chiral operators for η' exchanges

Taking the η' meson into account requires a modification of the basic building block (2.5) into [1]

$$\widetilde{u} = \exp\left(\frac{i}{\sqrt{2}F}\phi^0 T^0\right) u , \tag{B.28}$$

where

$$T^0 = \frac{1}{\sqrt{N_F}} \mathbb{1} \tag{B.29}$$

is the ninth Gell-Mann matrix with N_F being the number of flavours (see also (A.10), where we explicitly used $N_F = 6$). Therefore, we have an expression for the pseudoscalar singlet in the form

$$\phi^0 = -iF\sqrt{\frac{2}{N_F}} \ln(\det \widetilde{u}) . \tag{B.30}$$

The construction of the new chiral building blocks requires an application of (B.28) and usual external sources introduced in Chapters 1 and 2. Here we will show only the results:

$$\widetilde{u}_\mu = u_\mu - \frac{\sqrt{2}}{F}(D_\mu \phi^0) T^0 , \tag{B.31}$$
$$\widetilde{\chi}_\pm = \exp\left(-\frac{i\sqrt{2}}{F}\phi^0 T^0\right) u^\dagger \chi u^\dagger \pm \exp\left(\frac{i\sqrt{2}}{F}\phi^0 T^0\right) u \chi^\dagger u \tag{B.32}$$
$$= \chi_\pm - \frac{i}{F}\sqrt{\frac{2}{N_F}}\phi^0 \chi_\mp + \ldots , \tag{B.33}$$
$$\langle l_\mu \rangle = l_\mu^0 \sqrt{\frac{N_F}{2}} \tag{B.34}$$

etc., where the covariant derivative (in fact invariant) of ϕ^0 is defined as

$$D_\mu \phi^0 = \partial_\mu \phi^0 - 2F a_\mu^0 . \tag{B.35}$$

For detailed discussion see [1].

C. Spin one particle formalisms

The spin one particles can be described in the means of a quntum field theory in two main formalisms. Either we consider the particles to be described by the vector field formalism or by the antisymmetric tensor fields formalism. Here we present the most important properties of the mentioned formalism.

C.1 Vector field formalism

Let us consider a spin one field with a mass m. In the vector field formalism, the free field Lagrangian can be written as [8], [41], [42], [43]

$$\mathcal{L}_V = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \,, \tag{C.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{C.2}$$

Easily, using the Euler-Lagrangian equation, the classical equation of motion reads

$$\partial^2 A_\mu + m A_\mu - \partial_\mu(\partial \cdot A) = 0 \,. \tag{C.3}$$

Considering the Coulomb gauge condition,

$$\partial \cdot A = 0 \,, \tag{C.4}$$

we get

$$(\partial^2 + m^2)A_\mu = 0 \,. \tag{C.5}$$

The solution to the equation above can be guessed as a Fourier transform in the form

$$A_\mu = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E}} \left[B_\mu(\boldsymbol{p},\lambda)e^{ip\cdot x} + B^*_\mu(\boldsymbol{p},\lambda)e^{-ip\cdot x}\right] , \tag{C.6}$$

where it is necessary to sum over all polarizations, i.e. $\lambda = \{-1, 0, +1\}$.

In a quantum field theory it is required to change operators for functions. In this case, (C.6) hence has the form

$$A_\mu = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E}} \left[\varepsilon_\mu(\boldsymbol{p},\lambda)a(\boldsymbol{p},\lambda)e^{ip\cdot x} + \varepsilon^*_\mu(\boldsymbol{p},\lambda)a^\dagger(\boldsymbol{p},\lambda)e^{-ip\cdot x}\right] , \tag{C.7}$$

where we have the following properties for the polarization vector ε_μ:

$$\sum_\lambda \varepsilon_\mu(\boldsymbol{p},\lambda)\varepsilon^*_\nu(\boldsymbol{p},\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \,, \tag{C.8}$$

$$\varepsilon_\mu(\boldsymbol{p},\lambda)\varepsilon^\mu(\boldsymbol{p},\lambda') = -\delta_{\lambda\lambda'} \,, \tag{C.9}$$

$$p_\mu \varepsilon^\mu(\boldsymbol{p},\lambda) = 0 \,, \tag{C.10}$$

and for the creation $a^\dagger(\boldsymbol{p},\lambda)$ and annihilation $a(\boldsymbol{p},\lambda)$ operators:

$$\left[a(\boldsymbol{p},\lambda), a(\boldsymbol{p'},\lambda')\right] = 0 \,, \tag{C.11}$$

$$\left[a(\boldsymbol{p},\lambda), a^\dagger(\boldsymbol{p'},\lambda')\right] = \delta^3(\boldsymbol{p'} - \boldsymbol{p})\delta_{\lambda\lambda'} \,, \tag{C.12}$$

$$\left[a^\dagger(\boldsymbol{p},\lambda), a^\dagger(\boldsymbol{p'},\lambda')\right] = 0 \,. \tag{C.13}$$

The vacuum expectation value of the time ordered product of two vector fields (C.7) is called the propagator of the field, i.e. [8], [41], [42], [43]

$$i\Delta_F^V(x-y)_{\mu\nu} \equiv \langle 0 \big| \mathrm{T}[A_\mu(x)A_\nu(y)] \big| 0 \rangle \,. \tag{C.14}$$

The covariant part of the result has the form

$$i\Delta_F^V(x-y)_{\mu\nu} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} i\Delta_F(p)_{\mu\nu} e^{-ip\cdot(x-y)} \,, \tag{C.15}$$

where

$$i\Delta_F(p)_{\mu\nu} = -\frac{i}{p^2 - m^2 + i\epsilon}\left(g_{\mu\nu} - \frac{p_\mu p \nu}{m^2}\right) \tag{C.16}$$

is the vector propagator in the momentum representation. Here we tacitly assume the "Feynman epsilon" $i\epsilon$ to be infinitesimally close to $i0$.

C.2 Antisymmetric tensor field formalism

Equivalently as in the case above, let us consider a field with a mass m. In the antisymmetric tensor field formalism, the free field Lagrangian can be written as [8], [41], [42], [43]

$$\mathcal{L}_T = -\frac{1}{2} W_\mu W^\mu + \frac{1}{4} m^2 R_{\mu\nu} R^{\mu\nu} \,, \tag{C.17}$$

where

$$W_\mu = \partial^\alpha R_{\alpha\mu} \,. \tag{C.18}$$

Classical equation of motion now reads

$$\partial_\mu \partial^\alpha R_{\alpha\nu} - \partial_\nu \partial^\alpha R_{\alpha\mu} + m^2 R_{\mu\nu} = 0 \,, \tag{C.19}$$

that implies

$$(\partial^2 + m^2)(m\, \partial^\alpha R_{\alpha\mu}) = 0 \,, \tag{C.20}$$

where the derivation of the field is multiplied by the mass due to the correct dimension of the field. A general solution of the equation above is

$$R_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \frac{\mathrm{d}^3 \boldsymbol{p}}{\sqrt{2E}} \left[A_{\mu\nu}(\boldsymbol{p},\lambda) a(\boldsymbol{p},\lambda) e^{ip\cdot x} + B_{\mu\nu}(\boldsymbol{p},\lambda) a^\dagger(\boldsymbol{p},\lambda) e^{-ip\cdot x} \right] . \tag{C.21}$$

using the identities,

$$ip^\mu A_{\mu\nu} = m\varepsilon^\mu(\boldsymbol{p},\lambda) \,, \tag{C.22}$$

$$-ip^\mu B_{\mu\nu} = m\varepsilon^{\mu\,*}(\boldsymbol{p},\lambda) \,, \tag{C.23}$$

obtained by applying the derivative in the momentum space, we can write the solution in the form

$$R_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \frac{i}{m} \sum_\lambda \frac{\mathrm{d}^3 \boldsymbol{p}}{\sqrt{2E}} \Big[\big(p_\nu \varepsilon_\mu(\boldsymbol{p},\lambda) - p_\mu \varepsilon_\nu(\boldsymbol{p},\lambda)\big) a(\boldsymbol{p},\lambda) e^{ip\cdot x} \tag{C.24}$$

$$+ \big(p_\nu \varepsilon^*_\mu(\boldsymbol{p},\lambda) - p_\mu \varepsilon^*_\nu(\boldsymbol{p},\lambda)\big) a^\dagger(\boldsymbol{p},\lambda) e^{-ip\cdot x} \Big] \,.$$

The propagator of the field is now defined as [8], [41], [42], [43]

$$i\Delta_F^T(x-y)_{\alpha\beta\mu\nu} \equiv \langle 0 \big| \mathrm{T}[R_{\alpha\beta}(x) R_{\mu\nu}(y)] \big| 0 \rangle. \tag{C.25}$$

A direct calculation gives the covariant part

$$i\Delta_F^T(x-y)_{\alpha\beta\mu\nu} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} i\Delta_F(p)_{\alpha\beta\mu\nu} e^{-ip\cdot(x-y)}\,, \tag{C.26}$$

where

$$\begin{aligned} i\Delta_F^T(p)_{\alpha\beta\mu\nu} &= \\ &= -\frac{i}{m^2(p^2-m^2+i\epsilon)}\left[g_{\alpha\mu}g_{\beta\nu}(m^2-p^2) + g_{\alpha\mu}p_\beta p_\nu - g_{\alpha\nu}p_\beta p_\mu\right] - (\mu \leftrightarrow \nu)\,. \end{aligned} \tag{C.27}$$

is the tensor propagator in the momentum representation.

D. Feynman rules of χPT and RχT vertices

In this chapter we show the Feynman rules, corresponding to the Lagrangians we have used in this paper.

Having some same currents, forming general Green functions, it forces us to include all possible permutations of the Lorentz and the group indices of that current. This naturally leads to presence of the terms in Lagrangians that differ from one another only by the indices. To shorten the writing of such similar terms, we present the following notation that is helpful in given cases.

$$\sum_{(a,b)} \equiv (a \leftrightarrow b)\,, \tag{D.1}$$

$$\sum_{(a,b,c)} \equiv (a \leftrightarrow b) + (a \leftrightarrow c) + (b \leftrightarrow c)\,, \tag{D.2}$$

$$\sum_{(a,b)} \sum_{(c,d)} \equiv (a \leftrightarrow b) \wedge (c \leftrightarrow d)\,, \tag{D.3}$$

$$\sum_{(a,b,c,d)} \equiv (a \leftrightarrow b) + (a \leftrightarrow c) + (a \leftrightarrow d) + (b \leftrightarrow c) + (b \leftrightarrow d) + (c \leftrightarrow d)\,. \tag{D.4}$$

Obviously, in the descending order, the previous notations (D.1)-(D.4) are needed in the cases of the $VVA, AAA, VVPP$ and $VVVV$ Green functions.

Now, let us define the projectors that one uses to calculate the Feynman rules.

projection	projector
$\phi^a, p^a, s^a \to \phi^b, p^b, s^b$	δ_a^b
$v_\mu^a, a_\mu^a \to v_\nu^b, a_\nu^b$	$g_\nu^\mu \delta_a^b$
$R^a \to R^b$	δ_a^b
$R_\mu^a \to R_\nu^b$	$g_\nu^\mu \delta_a^b$
$R_{\mu\nu}^a \to R_{\alpha\beta}^b$	$\frac{1}{2}(g_\alpha^\mu g_\beta^\nu - g_\alpha^\nu g_\beta^\mu)\delta_a^b$

Table D.1: Projectors.

Regarding the derivatives, coupled to the fields, let us remember that we consider all 4-momenta as ingoing into vertices. In that case, the Feynman rule for the field with derivative is $-i$ multiplied by the component of that 4-momentum with the Lorentz index same as the Lorentz index carried by the derivative. A more detailed explanation can be found in the examples below.

Examples

To make the calculations as clear as possible, we will present illustrative examples of calculations of the Feynman rules from the corresponding Lagrangians.

Vertex ϕp This vertex comes from the lowest Lagrangian of the χPT and contributes by the term

$$\mathcal{L}_\chi^{(2)} = \frac{B_0 F}{\sqrt{2}} \langle \{\phi, p\} \rangle\,. \tag{D.5}$$

Since we have only one pseudoscalar external field and one pseudoscalar field, we can rewrite those terms in the means of the individual components p_a and ϕ_b, i.e.

$$\mathcal{L}_{\chi}^{(2)} = B_0 F \langle \{\phi_b T^b, p_a T^a\} \rangle = B_0 F p_a \phi_b \langle \{T^b, T^a\} \rangle \,, \tag{D.6}$$

where we used the substitutions

$$p = p_a T^a \,, \tag{D.7}$$

$$\phi = \sqrt{2}\phi_b T^b \,. \tag{D.8}$$

Carrying out the trace over group indices we finally found the Lagrangian in the form

$$\mathcal{L}_{\chi}^{(2)} = B_0 F p_a \phi_b \delta^{ab} \,. \tag{D.9}$$

Let us assume that we want to keep the group indices of the fields in the Feynman graph the same as the indices we used in the Lagrangian. To coincide with this assumption, to obtain the corresponding Feynman rule, one needs to apply the projectors we introduced in the previous subsection. Let us rewrite the Lagrangian with the changed group indices a' and b' for the correct use of the projectors so we could have the Feynman diagrams with the indices a and b. Then, we have

$$\mathcal{L}_{\chi}^{(2)} = B_0 F p_{a'} \phi_{b'} \delta^{a'b'} \tag{D.10}$$

and by applying the corresponding projectors we find the Feynman rule to be

$$V^{ab} = i B_0 F \delta^{a}_{a'} \delta^{b}_{b'} \delta^{a'b'} = i B_0 F \delta^{ab} \,, \tag{D.11}$$

where we obviously added the imaginary unit due to the transition from the Lagrangian to the S-matrix.

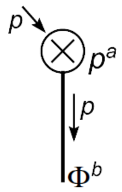


Figure D.1: Vertex ϕp.

Vertex $Sa\phi$ Next we will examine the vertex that is formed by the lowest resonance Lagrangian (2.71) and couples together scalar resonance with the axial-vector external source and pseudoscalar field. The relevant part is

$$\mathcal{L}_{S}^{(4)} = -\frac{2\sqrt{2}c_d}{F} \langle S \{\partial_\mu \phi, a^\mu\} \rangle \,. \tag{D.12}$$

In this case we follow the procedure from the previous case until we get

$$\mathcal{L}_{S}^{(4)} = -\frac{2\sqrt{2}c_d}{F} d^{abc} a_a^\mu S_b (\partial_\mu \phi_c) \,. \tag{D.13}$$

Now again, let us assume that we need to keep all the indices the same as on the Feynman diagram. To allow this, we will rename the indices and apply the suitable projectors. Also, it is crucial to notice that here we deal with the derivative coupled to the pseudoscalar field, so we will also get 4-momentum of the pseudoscalar in the Feynman rule. It is obvious that the result is

$$V_{\mu}^{abc} = i \left(\frac{-2\sqrt{2}c_d}{F} \right) d^{a'b'c'} g_{\mu}^{\mu'} \delta^{a}_{a'} \delta^{b}_{b'} (-i r_{\mu'}) \delta^{c}_{c'} = -\frac{2\sqrt{2}c_d}{F} d^{abc} r_\mu \,. \tag{D.14}$$

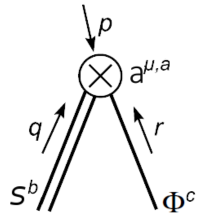


Figure D.2: Vertex $Sa\phi$.

Vertex VVP For the last example we will take a look at the vertex that consists of two vector and one pseudoscalar resonances. The relevant Lagrangian is

$$\mathcal{L}^{VVP} = \kappa^{VVP} \langle V^{\mu\nu} V^{\alpha\beta} P \rangle \varepsilon_{\mu\nu\alpha\beta} \,. \tag{D.15}$$

We perform the same steps as in previous calculations but we need to take into account that now we have two vector currents,

$$V^{\mu\nu} = \sqrt{2} V_a^{\mu\nu} T^a + \sqrt{2} V_b^{\mu\nu} T^b \,, \tag{D.16}$$

each one of them carries its 4-momentum. Let us assign 4-momentum p to the $V_a^{\mu\nu}$ component of the vector current and 4-momentum q to $V_b^{\mu\nu}$. The assignment of the 4-momenta depends only on the group indices. The pseudoscalar resonance carries 4-momentum r. i.e.

$$\mathcal{L}^{VVP} = 2\sqrt{2}\kappa^{VVP} \langle (V_a^{\mu\nu} T^a + V_b^{\mu\nu} T^b)(V_a^{\alpha\beta} T^a + V_b^{\alpha\beta} T^b) P_c T^c \rangle \varepsilon_{\mu\nu\alpha\beta} \,. \tag{D.17}$$

Now, the key part is to realize that only interference terms make up the interaction, i.e. we are interested only in the combination of the fields that carry different group indices. The terms with two same indices do not represent relevant parts of the vertex, therefore we do not consider them at all. Keeping that in mind we find

$$\mathcal{L}^{VVP} = 2\sqrt{2}\kappa^{VVP} \langle (V_a^{\mu\nu} V_b^{\alpha\beta} T^a T^b + V_b^{\mu\nu} V_a^{\alpha\beta} T^b T^a) P_c T^c \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.18}$$

$$= \frac{\kappa^{VVP}}{\sqrt{2}} \sum_{(a,b)} h^{abc} V_a^{\mu\nu} V_b^{\alpha\beta} P_c \varepsilon_{\mu\nu\alpha\beta} \,. \tag{D.19}$$

The last part of the calculation represents using the projectors. To make the calculation as clear as possible, we will carry out the full procedure:

$$V_{\mu\nu}^{abc} = i2\sqrt{2}\kappa^{VVP} \Bigg[\frac{1}{2}(g_\mu^{\mu'} g_\nu^{\nu'} - g_\mu^{\nu'} g_\nu^{\mu'}) \delta_{a'}^a \frac{1}{2}(g_\alpha^{\alpha'} g_\beta^{\beta'} - g_\alpha^{\beta'} g_\beta^{\alpha'}) \delta_{b'}^b \delta_{c'}^c h^{a'b'c'} \tag{D.20}$$
$$+ \frac{1}{2}(g_\alpha^{\mu'} g_\beta^{\nu'} - g_\alpha^{\nu'} g_\beta^{\mu'}) \delta_{b'}^b \frac{1}{2}(g_\mu^{\alpha'} g_\nu^{\beta'} - g_\mu^{\beta'} g_\nu^{\alpha'}) \delta_{a'}^a \delta_{c'}^c h^{b'a'c'} \Bigg] \varepsilon_{\mu'\nu'\alpha'\beta'} \,.$$

After some algebra, we get the result

$$V_{\mu\nu\alpha\beta}^{abc} = i\sqrt{2}\kappa^{VVP} d^{abc} \varepsilon_{\mu\nu\alpha\beta} \,. \tag{D.21}$$

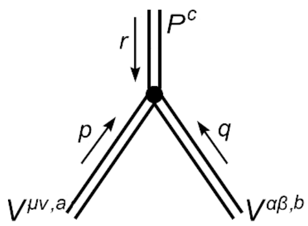


Figure D.3: Vertex VVP.

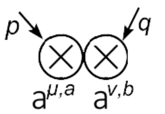
D.1 χPT vertices

D.1.1 Vertices up to $\mathcal{O}(p^2)$

Let us start with the lowest Lagrangian (2.16) of χPT. Considering only the expansion of the chiral operators that can build up three-point vertices at most, we can extract the Lagrangians that describe the following types of vertices:

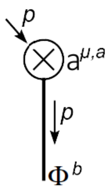
$$\mathcal{L}_\chi^{(2)} \sim a^2, a\phi, av\phi, v\phi^2, p\phi, s\phi^2 \dots \tag{D.22}$$

Vertex a^2



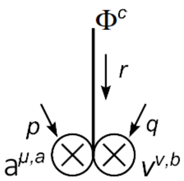
$$\mathcal{L}_\chi^{(2)} = F^2 \langle a_\mu a^\mu \rangle \,, \tag{D.23}$$
$$V_{\mu\nu}^{ab} = iF^2 g_{\mu\nu} \delta^{ab} \,. \tag{D.24}$$

Vertex $a\phi$



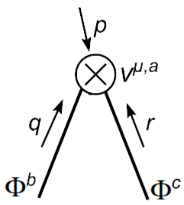
$$\mathcal{L}_\chi^{(2)} = -\frac{F}{\sqrt{2}} \langle \{ \partial_\mu \phi, a^\mu \} \rangle \,, \tag{D.25}$$
$$V_\mu^{ab} = F p_\mu \delta^{ab} \,. \tag{D.26}$$

Vertex $av\phi$



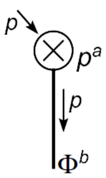
$$\mathcal{L}_\chi^{(2)} = -\frac{iF}{\sqrt{2}} \big\langle \big\{ [\phi, v^\mu], a_\mu \big\} \big\rangle \,, \tag{D.27}$$
$$V_{\mu\nu}^{abc} = -iF g_{\mu\nu} f^{abc} \,. \tag{D.28}$$

Vertex $v\phi^2$



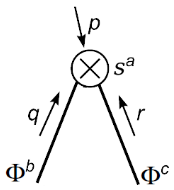
$$\mathcal{L}_\chi^{(2)} = \frac{i}{2} \big\langle \big\{ \partial^\mu \phi, [\phi, v_\mu] \big\} \big\rangle \,, \tag{D.29}$$
$$V_\mu^{abc} = -(q - r)_\mu f^{abc} \,. \tag{D.30}$$

Vertex $p\phi$



$$\mathcal{L}_\chi^{(2)} = \frac{B_0 F}{\sqrt{2}} \langle \{ \phi, p \} \rangle \,, \tag{D.31}$$
$$V^{ab} = iB_0 F \delta^{ab} \,. \tag{D.32}$$

Vertex $s\phi^2$



$$\mathcal{L}_\chi^{(2)} = -\frac{B_0}{2}\langle \phi s \phi\rangle - \frac{B_0}{4}\langle\{\phi^2, s\}\rangle\,, \tag{D.33}$$

$$V^{abc} = -iB_0 d^{abc}\,. \tag{D.34}$$

D.1.2 Vertices up to $\mathcal{O}(p^4)$

Since we do not consider the χPT Lagrangian in the next-to-leading order (2.17), we take into account only the anoumalous Wess-Zumino-Witten Lagrangian (2.40). For the purpose of the calculation of the WZW Lagrangian, we need to expand the definitions (2.42) and (2.43):

$$L_\mu \simeq \ell_\mu + \frac{i}{\sqrt{2}F}[\phi, \ell_\mu] + \frac{1}{2F^2}\phi\ell_\mu\phi - \frac{1}{4F^2}\{\phi^2, \ell_\mu\} + \frac{i}{4\sqrt{2}F^3}\phi[\phi, \ell_\mu]\phi - \frac{i}{12\sqrt{2}F^3}[\phi^3, \ell_\mu] + \mathcal{O}(\phi^4)\,, \tag{D.35}$$

$$L_{\mu\nu} \simeq \partial_\mu\ell_\nu + \frac{i}{\sqrt{2}F}[\phi, \partial_\mu\ell_\nu] + \frac{1}{2F^2}\phi(\partial_\mu\ell_\nu)\phi - \frac{1}{4F^2}\{\phi^2, \partial_\mu\ell_\nu\} + \frac{i}{4\sqrt{2}F^3}\phi[\phi, \partial_\mu\ell_\nu]\phi - \frac{i}{12\sqrt{2}F^3}[\phi^3, \partial_\mu\ell_\nu] + \mathcal{O}(\phi^4)\,, \tag{D.36}$$

$$R_\mu \simeq r_\mu - \frac{i}{\sqrt{2}F}[\phi, r_\mu] + \frac{1}{2F^2}\phi r_\mu\phi - \frac{1}{4F^2}\{\phi^2, r_\mu\} - \frac{i}{4\sqrt{2}F^3}\phi[\phi, r_\mu]\phi + \frac{i}{12\sqrt{2}F^3}[\phi^3, r_\mu] + \mathcal{O}(\phi^4)\,, \tag{D.37}$$

$$R_{\mu\nu} \simeq \partial_\mu r_\nu - \frac{i}{\sqrt{2}F}[\phi, \partial_\mu r_\nu] + \frac{1}{2F^2}\phi(\partial_\mu r_\nu)\phi - \frac{1}{4F^2}\{\phi^2, \partial_\mu r_\nu\} - \frac{i}{4\sqrt{2}F^3}\phi[\phi, \partial_\mu r_\nu]\phi + \frac{i}{12\sqrt{2}F^3}[\phi^3, \partial_\mu r_\nu] + \mathcal{O}(\phi^4)\,. \tag{D.38}$$

Next, we will also use the definition (2.44). It is easy to obtain the appropriate expansions in terms of pseudoscalar fields:

$$\sigma_\mu \simeq \frac{i\sqrt{2}}{F}\partial_\mu\phi + \frac{i}{3\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi - \frac{i}{6\sqrt{2}F^3}\{\partial_\mu\phi, \phi^2\} + \mathcal{O}(\phi^4)\,, \tag{D.39}$$

$$\sigma_\mu^\dagger \simeq -\frac{i\sqrt{2}}{F}\partial_\mu\phi - \frac{i}{3\sqrt{2}F^3}\phi(\partial_\mu\phi)\phi + \frac{i}{6\sqrt{2}F^3}\{\partial_\mu\phi, \phi^2\} + \mathcal{O}(\phi^4)\,. \tag{D.40}$$

Obviously, in the special case of $u = \mathbb{1}$ we have

$$L_\mu = \ell_\mu\,, \qquad L_{\mu\nu} = \partial_\mu\ell_\nu \tag{D.41}$$

$$R_\mu = r_\mu\,, \qquad R_{\mu\nu} = \partial_\mu r_\nu \tag{D.42}$$

and

$$\sigma_\mu = \sigma_\mu^\dagger = \mathbb{0}\,. \tag{D.43}$$

Let us now move forward to the calculation of (2.41). Let us rewrite the definition, together with the special case of $u = \mathbb{1}$. We have

$$
\begin{aligned}
W_{\mu\nu\alpha\beta}(u,\ell,r) = \quad & L_\mu L_\nu L_\alpha R_\beta + \frac{1}{4} L_\mu R_\nu L_\alpha R_\beta + i L_{\mu\nu} L_\alpha R_\beta + i R_{\mu\nu} L_\alpha R_\beta \\
& - i\sigma_\mu L_\nu R_\alpha L_\beta + \sigma_\mu R_{\nu\alpha} L_\beta - \sigma_\mu \sigma_\nu R_\alpha L_\beta + \sigma_\mu L_\nu L_{\alpha\beta} \\
& + \sigma_\mu L_{\nu\alpha} L_\beta - i\sigma_\mu L_\nu L_\alpha L_\beta + \frac{1}{2} \sigma_\mu L_\nu \sigma_\alpha L_\beta - i\sigma_\mu \sigma_\nu \sigma_\alpha L_\beta \\
& - R_\mu R_\nu R_\alpha L_\beta - \frac{1}{4} R_\mu L_\nu R_\alpha L_\beta - i R_{\mu\nu} R_\alpha L_\beta - i L_{\mu\nu} R_\alpha L_\beta \\
& + i\sigma_\mu^\dagger R_\nu L_\alpha R_\beta - \sigma_\mu^\dagger L_{\nu\alpha} R_\beta + \sigma_\mu^\dagger \sigma_\nu^\dagger L_\alpha R_\beta - \sigma_\mu^\dagger R_\nu R_{\alpha\beta} \\
& - \sigma_\mu^\dagger R_{\nu\alpha} R_\beta + i\sigma_\mu^\dagger R_\nu R_\alpha R_\beta - \frac{1}{2} \sigma_\mu^\dagger R_\nu \sigma_\alpha^\dagger R_\beta + i\sigma_\mu^\dagger \sigma_\nu^\dagger \sigma_\alpha^\dagger R_\beta \,,
\end{aligned} \tag{D.44}
$$

$$
\begin{aligned}
W_{\mu\nu\alpha\beta}(\mathbb{1},\ell,r) = \quad & \ell_\mu \ell_\nu \ell_\alpha r_\beta + \frac{1}{4} \ell_\mu r_\nu \ell_\alpha r_\beta + i(\partial_\mu \ell_\nu) \ell_\alpha r_\beta + i(\partial_\mu r_\nu) \ell_\alpha r_\beta \\
& - r_\mu r_\nu r_\alpha \ell_\beta - \frac{1}{4} r_\mu \ell_\nu r_\alpha \ell_\beta - i(\partial_\mu r_\nu) r_\alpha \ell_\beta - i(\partial_\mu \ell_\nu) r_\alpha \ell_\beta \,.
\end{aligned} \tag{D.45}
$$

To be able to extract the interaction terms, one must deal with a lot of terms, many of them eventually cancel each other out. For completeness, we will show all individual terms that are present in (D.44).

$$
\begin{aligned}
L_\mu L_\nu L_\alpha R_\beta \simeq & - a_\mu a_\nu a_\alpha a_\beta + a_\mu a_\nu v_\alpha a_\beta - a_\mu a_\nu a_\alpha v_\beta + a_\mu a_\nu v_\alpha v_\beta \\
& + v_\mu a_\nu a_\alpha a_\beta - v_\mu a_\nu v_\alpha a_\beta + v_\mu a_\nu a_\alpha v_\beta - v_\mu a_\nu v_\alpha v_\beta \\
& + a_\mu v_\nu a_\alpha a_\beta - a_\mu v_\nu v_\alpha a_\beta + a_\mu v_\nu a_\alpha v_\beta - a_\mu v_\nu v_\alpha v_\beta \\
& - v_\mu v_\nu a_\alpha a_\beta + v_\mu v_\nu v_\alpha a_\beta - v_\mu v_\nu a_\alpha v_\beta + v_\mu v_\nu v_\alpha v_\beta \,,
\end{aligned} \tag{D.46}
$$

$$
\begin{aligned}
L_\mu R_\nu L_\alpha R_\beta \simeq \quad & a_\mu a_\nu a_\alpha a_\beta - a_\mu a_\nu v_\alpha a_\beta + a_\mu a_\nu a_\alpha v_\beta - a_\mu a_\nu v_\alpha v_\beta \\
& - v_\mu a_\nu a_\alpha a_\beta + v_\mu a_\nu v_\alpha a_\beta - v_\mu a_\nu a_\alpha v_\beta + v_\mu a_\nu v_\alpha v_\beta \\
& + a_\mu v_\nu a_\alpha a_\beta - a_\mu v_\nu v_\alpha a_\beta + a_\mu v_\nu a_\alpha v_\beta - a_\mu v_\nu v_\alpha v_\beta \\
& - v_\mu v_\nu a_\alpha a_\beta + v_\mu v_\nu v_\alpha a_\beta - v_\mu v_\nu a_\alpha v_\beta + v_\mu v_\nu v_\alpha v_\beta \,,
\end{aligned} \tag{D.47}
$$

$$
\begin{aligned}
i L_{\mu\nu} L_\alpha R_\beta \simeq \quad & i(\partial_\mu a_\nu) a_\alpha a_\beta - i(\partial_\mu a_\nu) v_\alpha a_\beta + i(\partial_\mu a_\nu) a_\alpha v_\beta - i(\partial_\mu a_\nu) v_\alpha v_\beta \\
& - i(\partial_\mu v_\nu) a_\alpha a_\beta + i(\partial_\mu v_\nu) v_\alpha a_\beta - i(\partial_\mu v_\nu) a_\alpha v_\beta + i(\partial_\mu v_\nu) v_\alpha v_\beta \,,
\end{aligned} \tag{D.48}
$$

$$
\begin{aligned}
i R_{\mu\nu} L_\alpha R_\beta \simeq & - i(\partial_\mu a_\nu) a_\alpha a_\beta + i(\partial_\mu a_\nu) v_\alpha a_\beta - i(\partial_\mu a_\nu) a_\alpha v_\beta + i(\partial_\mu a_\nu) v_\alpha v_\beta \\
& - i(\partial_\mu v_\nu) a_\alpha a_\beta + i(\partial_\mu v_\nu) v_\alpha a_\beta - i(\partial_\mu v_\nu) a_\alpha v_\beta + i(\partial_\mu v_\nu) v_\alpha v_\beta \,,
\end{aligned} \tag{D.49}
$$

$$
\begin{aligned}
-i\sigma_\mu L_\nu R_\alpha L_\beta \simeq \quad & \frac{\sqrt{2}}{F}(\partial_\mu \phi) a_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu \phi) a_\nu v_\alpha a_\beta - \frac{\sqrt{2}}{F}(\partial_\mu \phi) a_\nu a_\alpha v_\beta \\
& - \frac{\sqrt{2}}{F}(\partial_\mu \phi) a_\nu v_\alpha v_\beta - \frac{\sqrt{2}}{F}(\partial_\mu \phi) v_\nu a_\alpha a_\beta - \frac{\sqrt{2}}{F}(\partial_\mu \phi) v_\nu v_\alpha a_\beta \\
& + \frac{\sqrt{2}}{F}(\partial_\mu \phi) v_\nu a_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu \phi) v_\nu v_\alpha v_\beta \,,
\end{aligned} \tag{D.50}
$$

$$
\begin{aligned}
\sigma_\mu R_{\nu\alpha} L_\beta \simeq & - \frac{i\sqrt{2}}{F}(\partial_\mu \phi)(\partial_\nu a_\alpha) a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu \phi)(\partial_\nu a_\alpha) v_\beta \\
& - \frac{i\sqrt{2}}{F}(\partial_\mu \phi)(\partial_\nu v_\alpha) a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu \phi)(\partial_\nu v_\alpha) v_\beta \,,
\end{aligned} \tag{D.51}
$$

$$
\begin{aligned}
-\sigma_\mu \sigma_\nu R_\alpha L_\beta \simeq & - \frac{2}{F^2}(\partial_\mu \phi)(\partial_\nu \phi) a_\alpha a_\beta - \frac{2}{F^2}(\partial_\mu \phi)(\partial_\nu \phi) v_\alpha a_\beta \\
& + \frac{2}{F^2}(\partial_\mu \phi)(\partial_\nu \phi) a_\alpha v_\beta + \frac{2}{F^2}(\partial_\mu \phi)(\partial_\nu \phi) v_\alpha v_\beta \,,
\end{aligned} \tag{D.52}
$$

$$\begin{aligned}
\sigma_\mu L_\nu L_{\alpha\beta} \simeq{}& \frac{i\sqrt{2}}{F}(\partial_\mu\phi)a_\nu(\partial_\alpha a_\beta) - \frac{i\sqrt{2}}{F}(\partial_\mu\phi)v_\nu(\partial_\alpha a_\beta) \\
&- \frac{i\sqrt{2}}{F}(\partial_\mu\phi)a_\nu(\partial_\alpha v_\beta) + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)v_\nu(\partial_\alpha v_\beta)\,, \qquad (D.53)
\end{aligned}$$

$$\begin{aligned}
\sigma_\mu L_{\nu\alpha} L_\beta \simeq{}& \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)a_\beta - \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)v_\beta \\
&- \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\,, \qquad (D.54)
\end{aligned}$$

$$\begin{aligned}
-i\sigma_\mu L_\nu L_\alpha L_\beta \simeq{}& -\frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha v_\beta \qquad (D.55) \\
&- \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha a_\beta - \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha a_\beta \\
&- \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
\sigma_\mu L_\nu \sigma_\alpha L_\beta \simeq{}& -\frac{2}{F^2}(\partial_\mu\phi)a_\nu(\partial_\alpha\phi)a_\beta + \frac{2}{F^2}(\partial_\mu\phi)a_\nu(\partial_\alpha\phi)v_\beta \qquad (D.56) \\
&+ \frac{2}{F^2}(\partial_\mu\phi)v_\nu(\partial_\alpha\phi)a_\beta - \frac{2}{F^2}(\partial_\mu\phi)v_\nu(\partial_\alpha\phi)v_\beta\,,
\end{aligned}$$

$$-i\sigma_\mu\sigma_\nu\sigma_\alpha L_\beta \simeq \frac{2\sqrt{2}}{F^3}(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\alpha\phi)a_\beta - \frac{2\sqrt{2}}{F^3}(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\alpha\phi)v_\beta\,. \qquad (D.57)$$

Identically, the terms that arrise from the deduction of the $L \leftrightarrow R$ interchange gives the individual contributions:

$$\begin{aligned}
-R_\mu R_\nu R_\alpha L_\beta \simeq{}& a_\mu a_\nu a_\alpha a_\beta + a_\mu a_\nu v_\alpha a_\beta - a_\mu a_\nu a_\alpha v_\beta - a_\mu a_\nu v_\alpha v_\beta \qquad (D.58) \\
&+ v_\mu a_\nu a_\alpha a_\beta + v_\mu a_\nu v_\alpha a_\beta - v_\mu a_\nu a_\alpha v_\beta - v_\mu a_\nu v_\alpha v_\beta \\
&+ a_\mu v_\nu a_\alpha a_\beta + a_\mu v_\nu v_\alpha a_\beta - a_\mu v_\nu a_\alpha v_\beta - a_\mu v_\nu v_\alpha v_\beta \\
&+ v_\mu v_\nu a_\alpha a_\beta + v_\mu v_\nu v_\alpha a_\beta - v_\mu v_\nu a_\alpha v_\beta - v_\mu v_\nu v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
-R_\mu L_\nu R_\alpha L_\beta \simeq{}& - a_\mu a_\nu a_\alpha a_\beta - a_\mu a_\nu v_\alpha a_\beta + a_\mu a_\nu a_\alpha v_\beta + a_\mu a_\nu v_\alpha v_\beta \qquad (D.59) \\
&- v_\mu a_\nu a_\alpha a_\beta - v_\mu a_\nu v_\alpha a_\beta + v_\mu a_\nu a_\alpha v_\beta + v_\mu a_\nu v_\alpha v_\beta \\
&+ a_\mu v_\nu a_\alpha a_\beta + a_\mu v_\nu v_\alpha a_\beta - a_\mu v_\nu a_\alpha v_\beta - a_\mu v_\nu v_\alpha v_\beta \\
&+ v_\mu v_\nu a_\alpha a_\beta + v_\mu v_\nu v_\alpha a_\beta - v_\mu v_\nu a_\alpha v_\beta - v_\mu v_\nu v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
-iR_{\mu\nu} R_\alpha L_\beta \simeq{}& i(\partial_\mu a_\nu)a_\alpha a_\beta + i(\partial_\mu a_\nu)v_\alpha a_\beta - i(\partial_\mu a_\nu)a_\alpha v_\beta - i(\partial_\mu a_\nu)v_\alpha v_\beta \qquad (D.60) \\
&+ i(\partial_\mu v_\nu)a_\alpha a_\beta + i(\partial_\mu v_\nu)v_\alpha a_\beta - i(\partial_\mu v_\nu)a_\alpha v_\beta - i(\partial_\mu v_\nu)v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
-iL_{\mu\nu} R_\alpha L_\beta \simeq{}& - i(\partial_\mu a_\nu)a_\alpha a_\beta - i(\partial_\mu a_\nu)v_\alpha a_\beta + i(\partial_\mu a_\nu)a_\alpha v_\beta + i(\partial_\mu a_\nu)v_\alpha v_\beta \qquad (D.61) \\
&+ i(\partial_\mu v_\nu)a_\alpha a_\beta + i(\partial_\mu v_\nu)v_\alpha a_\beta - i(\partial_\mu v_\nu)a_\alpha v_\beta - i(\partial_\mu v_\nu)v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
i\sigma_\mu^\dagger R_\nu L_\alpha R_\beta \simeq{}& -\frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha a_\beta - \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha v_\beta \qquad (D.62) \\
&+ \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha v_\beta - \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha a_\beta \\
&- \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
-\sigma_\mu^\dagger L_{\nu\alpha} R_\beta \simeq{}& -\frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)a_\beta - \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)v_\beta \qquad (D.63) \\
&+ \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\,,
\end{aligned}$$

$$\begin{aligned}
\sigma_\mu^\dagger \sigma_\nu^\dagger L_\alpha R_\beta \simeq\ & \frac{2}{F^2}(\partial_\mu\phi)(\partial_\nu\phi)a_\alpha a_\beta - \frac{2}{F^2}(\partial_\mu\phi)(\partial_\nu\phi)v_\alpha a_\beta \\
&+ \frac{2}{F^2}(\partial_\mu\phi)(\partial_\nu\phi)a_\alpha v_\beta - \frac{2}{F^2}(\partial_\mu\phi)(\partial_\nu\phi)v_\alpha v_\beta\,,
\end{aligned} \tag{D.64}$$

$$\begin{aligned}
-\sigma_\mu^\dagger R_\nu R_{\alpha\beta} \simeq\ & \frac{i\sqrt{2}}{F}(\partial_\mu\phi)a_\nu(\partial_\alpha a_\beta) + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)v_\nu(\partial_\alpha a_\beta) \\
&+ \frac{i\sqrt{2}}{F}(\partial_\mu\phi)a_\nu(\partial_\alpha v_\beta) + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)v_\nu(\partial_\alpha v_\beta)\,,
\end{aligned} \tag{D.65}$$

$$\begin{aligned}
-\sigma_\mu^\dagger R_{\nu\alpha} R_\beta \simeq\ & \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu a_\alpha)v_\beta \\
&+ \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)a_\beta + \frac{i\sqrt{2}}{F}(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\,,
\end{aligned} \tag{D.66}$$

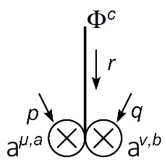
$$\begin{aligned}
i\sigma_\mu^\dagger R_\nu R_\alpha R_\beta \simeq\ & \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu a_\alpha v_\beta \\
&+ \frac{\sqrt{2}}{F}(\partial_\mu\phi)a_\nu v_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha a_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha a_\beta \\
&+ \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu a_\alpha v_\beta + \frac{\sqrt{2}}{F}(\partial_\mu\phi)v_\nu v_\alpha v_\beta\,,
\end{aligned} \tag{D.67}$$

$$\begin{aligned}
-\sigma_\mu^\dagger R_\nu \sigma_\alpha^\dagger R_\beta \simeq\ & \frac{2}{F^2}(\partial_\mu\phi)a_\nu(\partial_\alpha\phi)a_\beta + \frac{2}{F^2}(\partial_\mu\phi)a_\nu(\partial_\alpha\phi)v_\beta \\
&+ \frac{2}{F^2}(\partial_\mu\phi)v_\nu(\partial_\alpha\phi)a_\beta + \frac{2}{F^2}(\partial_\mu\phi)v_\nu(\partial_\alpha\phi)v_\beta\,,
\end{aligned} \tag{D.68}$$

$$i\sigma_\mu^\dagger \sigma_\nu^\dagger \sigma_\alpha^\dagger R_\beta \simeq -\frac{2\sqrt{2}}{F^3}(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\alpha\phi)a_\beta - \frac{2\sqrt{2}}{F^3}(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\alpha\phi)v_\beta\,. \tag{D.69}$$

As we have mentioned earlier, many terms will drop out but a lot of terms will survive at the same time. For our purposes it will be sufficient to present only the terms that form interaction vertices consisted of two external sources and one pseudoscalar. Then, we have only two simple contributions.

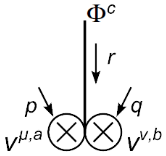
Vertex $a^2\phi$



$$\mathcal{L} = \frac{N_C}{12\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)a_\nu(\partial_\alpha a_\beta)\rangle\varepsilon^{\mu\nu\alpha\beta}\,, \tag{D.70}$$

$$V_{\mu\nu}^{abc} = -i\frac{N_C}{24\pi^2 F}d^{abc}\varepsilon_{\mu\nu(p)(q)}\,. \tag{D.71}$$

Vertex $v^2\phi$



$$\begin{aligned}
\mathcal{L} =\ & \frac{N_C}{12\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)v_\nu(\partial_\alpha v_\beta)\rangle\varepsilon^{\mu\nu\alpha\beta} \\
&+ \frac{N_C}{6\sqrt{2}\pi^2 F}\langle(\partial_\mu\phi)(\partial_\nu v_\alpha)v_\beta\rangle\varepsilon^{\mu\nu\alpha\beta}\,,
\end{aligned} \tag{D.72}$$

$$V_{\mu\nu}^{abc} = -\,i\frac{N_C}{8\pi^2 F}d^{abc}\varepsilon_{\mu\nu(p)(q)}\,. \tag{D.73}$$

D.2 RχT vertices

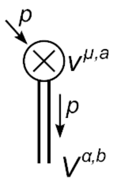
D.2.1 Vector formalism up to $\mathcal{O}(p^6)$

The Lagrangians (2.60)-(2.61) in the vector formalism have the following three-point structures:

$$\mathcal{L}^V \sim V(v, \phi^2, a\phi, a^2, p\phi, ap, av, v\phi \ldots)\,, \tag{D.74}$$

$$\mathcal{L}^A \sim A(a, v\phi, va, a\phi, a^2 \ldots)\,. \tag{D.75}$$

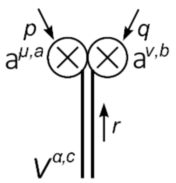
Vertex Vv



$$\mathcal{L} = -\frac{f_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu} f_+^{\mu\nu}\rangle\,, \tag{D.76}$$

$$V^{ab}_{\mu\alpha} = -if_V\delta^{ab}(p^2 g_{\alpha\mu} - p_\alpha p_\mu)\,. \tag{D.77}$$

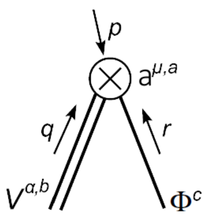
Vertex Va^2



$$\mathcal{L} = -\frac{ig_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu}[u^\mu, u^\nu]\rangle + i\alpha_V\langle \widehat{V}_\mu [u_\nu, f_-^{\mu\nu}]\rangle\,, \tag{D.78}$$

$$\begin{aligned} V^{abc}_{\mu\nu\alpha} = {} & 2f^{abc}\big[\sqrt{2}\alpha_V g_{\mu\nu}(q-p)_\alpha \\ & + g_{\alpha\mu}(\sqrt{2}p_\nu\alpha_V - g_V r_\nu) + g_{\alpha\nu}(g_V r_\mu - \sqrt{2}q_\mu\alpha_V)\big]\,. \end{aligned} \tag{D.79}$$

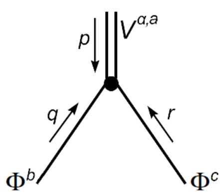
Vertex $Va\phi$



$$\mathcal{L} = -\frac{ig_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu}[u^\mu, u^\nu]\rangle + i\alpha_V\langle \widehat{V}_\mu [u_\nu, f_-^{\mu\nu}]\rangle\,, \tag{D.80}$$

$$\begin{aligned} V^{abc}_{\mu\alpha} = {} & -\frac{2i}{F}f^{abc}\Big[g_{\alpha\mu}\big(\sqrt{2}\alpha_V(p\cdot r) - g_V(q\cdot r)\big) \\ & + g_V q_\mu r_\alpha - \sqrt{2}p_\alpha r_\mu\alpha_V\Big]\,. \end{aligned} \tag{D.81}$$

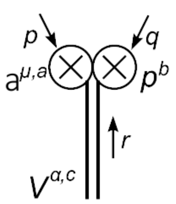
Vertex $V\phi^2$



$$\mathcal{L} = -\frac{ig_V}{2\sqrt{2}}\langle \widehat{V}_{\mu\nu}[u^\mu, u^\nu]\rangle\,, \tag{D.82}$$

$$V^{abc}_{\alpha} = -\frac{2g_V}{F^2}f^{abc}\big[(p\cdot q)r_\alpha - (p\cdot r)q_\alpha\big]\,. \tag{D.83}$$

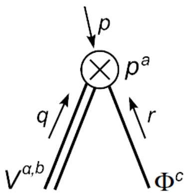
Vertex Vap



$$\mathcal{L} = \beta_V\langle \widehat{V}_\mu[u^\mu, \chi_-]\rangle\,, \tag{D.84}$$

$$V^{abc}_{\mu\alpha} = -4i\sqrt{2}B_0\beta_V f^{abc} g_{\alpha\mu}\,. \tag{D.85}$$

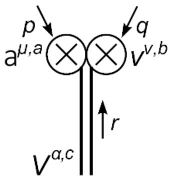
Vertex $V\phi p$



$$\mathcal{L} = \beta_V \langle \widehat{V}_\mu [u^\mu, \chi_-] \rangle \,, \tag{D.86}$$

$$V_\alpha^{abc} = \frac{4\sqrt{2} B_0 \beta_V}{F} f^{abc} r_\alpha \,. \tag{D.87}$$

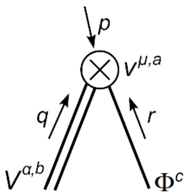
Vertex Vav



$$\mathcal{L} = h_V \langle \widehat{V}^\mu \{u^\nu, f_+^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.88}$$

$$V_{\mu\nu\alpha}^{abc} = -4\sqrt{2} h_V d^{abc} \varepsilon_{\alpha\mu\nu(q)} \,. \tag{D.89}$$

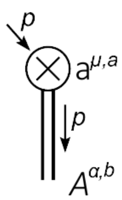
Vertex $Vv\phi$



$$\mathcal{L} = h_V \langle \widehat{V}^\mu \{u^\nu, f_+^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.90}$$

$$V_{\mu\alpha}^{abc} = -\frac{4i\sqrt{2} h_V}{F} d^{abc} \varepsilon_{\alpha\mu(p)(r)} \,. \tag{D.91}$$

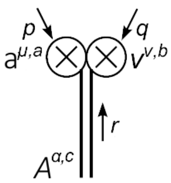
Vertex Aa



$$\mathcal{L} = -\frac{f_A}{2\sqrt{2}} \langle \widehat{A}_{\mu\nu} f_-^{\mu\nu} \rangle \,, \tag{D.92}$$

$$V_{\mu\alpha}^{ab} = i f_A \delta^{ab} (p^2 g_{\alpha\mu} - p_\alpha p_\mu) \,. \tag{D.93}$$

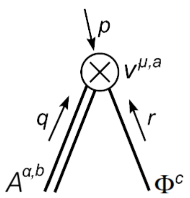
Vertex Aav



$$\mathcal{L} = i\alpha_A \langle \widehat{A}_\mu [u_\nu, f_+^{\mu\nu}] \rangle \,, \tag{D.94}$$

$$V_{\mu\nu\alpha}^{abc} = -2\sqrt{2} \alpha_V f^{abc} (q_\alpha g_{\mu\nu} - q_\mu g_{\alpha\nu}) \,. \tag{D.95}$$

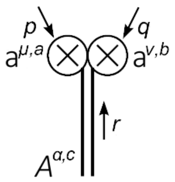
Vertex $Av\phi$



$$\mathcal{L} = i\alpha_A \langle \widehat{A}_\mu [u_\nu, f_+^{\mu\nu}] \rangle \,, \tag{D.96}$$

$$V_{\mu\alpha}^{abc} = -\frac{2i\sqrt{2} \alpha_V}{F} f^{abc} \big[p_\alpha r_\mu - g_{\alpha\mu} (p \cdot r) \big] \,. \tag{D.97}$$

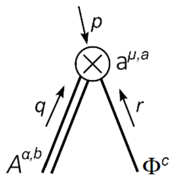
Vertex Aa^2



$$\mathcal{L} = h_A \langle \widehat{A}^\mu \{u^\nu, f_-^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.98}$$

$$V^{abc}_{\mu\nu\alpha} = -4\sqrt{2} h_A d^{abc} (\varepsilon_{\alpha\mu\nu(p)} - \varepsilon_{\alpha\mu\nu(q)})\,. \tag{D.99}$$

Vertex $Aa\phi$



$$\mathcal{L} = h_A \langle \widehat{A}^\mu \{u^\nu, f_-^{\alpha\beta}\} \rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.100}$$

$$V^{abc}_{\mu\alpha} = \frac{4i\sqrt{2} h_A}{F} d^{abc} \varepsilon_{\alpha\mu(p)(r)}\,. \tag{D.101}$$

D.2.2 Antisymmetric tensor formalism up to $\mathcal{O}(p^4)$

Now, we present Feynman rules for the lowest resonance Lagrangian (2.68) that has the following three-point structures:

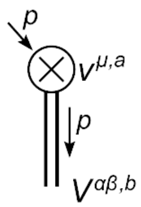
$$\mathcal{L}_V^{(4)} \sim V(v, \phi^2, a^2, v^2, a\phi \ldots)\,, \tag{D.102}$$

$$\mathcal{L}_A^{(4)} \sim A(a, av, v\phi \ldots)\,, \tag{D.103}$$

$$\mathcal{L}_S^{(4)} \sim S(s, p\phi, \phi^2, a^2, a\phi \ldots)\,, \tag{D.104}$$

$$\mathcal{L}_P^{(4)} \sim P(p, s\phi \ldots)\,. \tag{D.105}$$

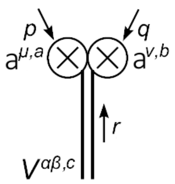
Vertex Vv



$$\mathcal{L}_V^{(4)} = \frac{F_V}{\sqrt{2}} \big\langle V_{\mu\nu} (\partial^\mu v^\nu - \partial^\nu v^\mu) \big\rangle\,, \tag{D.106}$$

$$V^{ab}_{\mu\alpha\beta} = \frac{F_V}{2} \delta^{ab} (p_\alpha g_{\beta\mu} - p_\beta g_{\alpha\mu})\,. \tag{D.107}$$

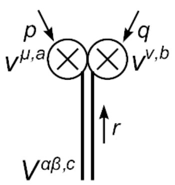
Vertex Va^2



$$\mathcal{L}_V^{(4)} = -\frac{iF_V}{\sqrt{2}} \big\langle V_{\mu\nu} [a^\mu, a^\nu] \big\rangle + i\sqrt{2} G_V \big\langle V_{\mu\nu} [a^\mu, a^\nu] \big\rangle\,, \tag{D.108}$$

$$V^{abc}_{\mu\nu\alpha\beta} = i\left(G_V - \frac{F_V}{2} \right) f^{abc} (g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu})\,. \tag{D.109}$$

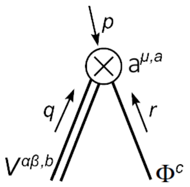
Vertex Vv^2



$$\mathcal{L}_V^{(4)} = -\frac{iF_V}{\sqrt{2}} \big\langle V_{\mu\nu} [v^\mu, v^\nu] \big\rangle\,, \tag{D.110}$$

$$V^{abc}_{\mu\nu\alpha\beta} = -\frac{iF_V}{2} f^{abc} (g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu})\,. \tag{D.111}$$

Vertex $Va\phi$

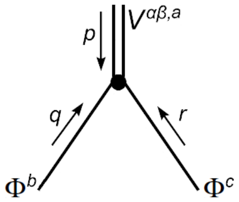


$$\mathcal{L}_V^{(4)} = -\frac{iF_V}{2F}\big\langle V_{\mu\nu}[\phi, \partial^\mu a^\nu - \partial^\nu a^\mu]\big\rangle \qquad \text{(D.112)}$$

$$- \frac{iG_V}{F}\big\langle V_{\mu\nu}[\partial^\mu \phi, a^\nu]\big\rangle - \frac{iG_V}{F}\big\langle V_{\mu\nu}[a^\mu, \partial^\nu \phi]\big\rangle\,,$$

$$V^{abc}_{\mu\alpha\beta} = \frac{f^{abc}}{2F}\Big[F_V(p_\alpha g_{\beta\mu} - p_\beta g_{\alpha\mu}) + 2G_V(r_\alpha g_{\beta\mu} - r_\beta g_{\alpha\mu})\Big]\,. \qquad \text{(D.113)}$$

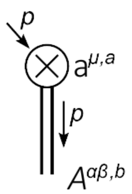
Vertex $V\phi^2$



$$\mathcal{L}_V^{(4)} = \frac{iG_V}{\sqrt{2}F^2}\big\langle V_{\mu\nu}[\partial^\mu \phi, \partial^\nu \phi]\big\rangle\,, \qquad \text{(D.114)}$$

$$V^{abc}_{\alpha\beta} = -\frac{iG_V}{F^2} f^{abc}(q_\beta r_\alpha - q_\alpha r_\beta)\,. \qquad \text{(D.115)}$$

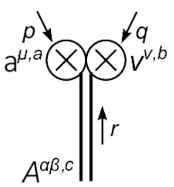
Vertex Aa



$$\mathcal{L}_A^{(4)} = -\frac{F_A}{\sqrt{2}}\big\langle A_{\mu\nu}(\partial^\mu a^\nu - \partial^\nu a^\mu)\big\rangle\,, \qquad \text{(D.116)}$$

$$V^{ab}_{\mu\alpha\beta} = -\frac{F_A}{2}\delta^{ab}(p_\alpha g_{\beta\mu} - p_\beta g_{\alpha\mu})\,. \qquad \text{(D.117)}$$

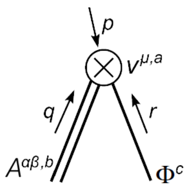
Vertex Aav



$$\mathcal{L}_A^{(4)} = \frac{iF_A}{\sqrt{2}}\big\langle A_{\mu\nu}[v^\mu, a^\nu]\big\rangle + \frac{iF_A}{\sqrt{2}}\big\langle A_{\mu\nu}[a^\mu, v^\nu]\big\rangle\,, \qquad \text{(D.118)}$$

$$V^{abc}_{\mu\nu\alpha\beta} = \frac{iF_A}{2} f^{abc}(g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu})\,. \qquad \text{(D.119)}$$

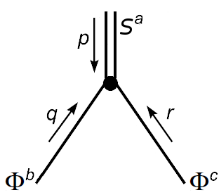
Vertex $Av\phi$



$$\mathcal{L}_A^{(4)} = \frac{iF_A}{2F}\big\langle A_{\mu\nu}[\phi, \partial^\mu v^\nu - \partial^\nu v^\mu]\big\rangle\,, \qquad \text{(D.120)}$$

$$V^{abc}_{\mu\alpha\beta} = -\frac{F_A}{2F} f^{abc}(p_\alpha g_{\beta\mu} - p_\beta g_{\alpha\mu})\,. \qquad \text{(D.121)}$$

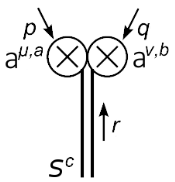
Vertex $S\phi^2$



$$\mathcal{L}_S^{(4)} = \frac{2c_d}{F^2}\langle S(\partial_\mu \phi)(\partial^\mu \phi)\rangle\,, \qquad \text{(D.122)}$$

$$V^{abc} = -\frac{2i\sqrt{2}c_d}{F^2} d^{abc}(q \cdot r)\,. \qquad \text{(D.123)}$$

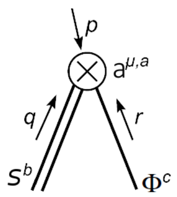
Vertex Sa^2



$$\mathcal{L}_S^{(4)} = 4c_d\langle S a^\mu a_\mu \rangle\,, \tag{D.124}$$

$$V_{\mu\nu}^{abc} = 2i\sqrt{2}c_d d^{abc} g_{\mu\nu}\,. \tag{D.125}$$

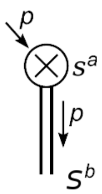
Vertex $Sa\phi$



$$\mathcal{L}_S^{(4)} = -\frac{2\sqrt{2}c_d}{F}\langle S\{\partial_\mu\phi, a^\mu\}\rangle\,, \tag{D.126}$$

$$V_\mu^{abc} = -\frac{2\sqrt{2}c_d}{F} d^{abc} r_\mu\,. \tag{D.127}$$

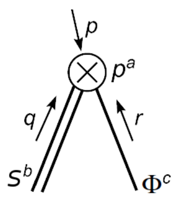
Vertex Ss



$$\mathcal{L}_S^{(4)} = 4B_0 c_m\langle S s\rangle\,, \tag{D.128}$$

$$V^{ab} = 2i\sqrt{2}B_0 c_m \delta^{ab}\,. \tag{D.129}$$

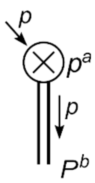
Vertex $Sp\phi$



$$\mathcal{L}_S^{(4)} = \frac{2\sqrt{2}B_0 c_m}{F}\langle S\{\phi, p\}\rangle\,, \tag{D.130}$$

$$V^{abc} = \frac{2i\sqrt{2}B_0 c_m}{F} d^{abc}\,. \tag{D.131}$$

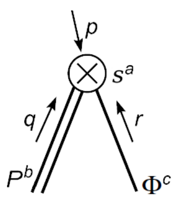
Vertex Pp



$$\mathcal{L}_P^{(4)} = -4B_0 d_m\langle P p\rangle\,, \tag{D.132}$$

$$V^{ab} = -2i\sqrt{2}B_0 d_m \delta^{ab}\,. \tag{D.133}$$

Vertex $Ps\phi$



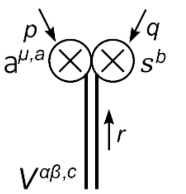
$$\mathcal{L}_P^{(4)} = \frac{2\sqrt{2}B_0 d_m}{F}\langle P\{\phi, s\}\rangle\,, \tag{D.134}$$

$$V^{abc} = \frac{2i\sqrt{2}B_0 d_m}{F} d^{abc}\,. \tag{D.135}$$

D.2.3 Antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

Here we show the Feynman rules of appropriate Lagrangians consisted of the monomials shown in tables 2.3-2.5. We consider only the three-point structures, i.e. all the chiral operators in the Lagrangians below are considered to be expanded only to the linear terms.

Vertex $\boldsymbol{Vsa}$



$$\mathcal{L}_4^V = i\kappa_4^V \langle [V^{\mu\nu}, \nabla^\alpha \chi_+] u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.136}$$

$$V^{abc}_{\mu\alpha\beta} = -4\sqrt{2} B_0 \kappa_4^V f^{abc} \varepsilon_{\alpha\beta\mu(q)} \,, \tag{D.137}$$

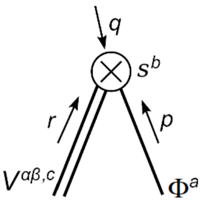
$$\mathcal{L}_{15}^V = i\kappa_{15}^V \langle V^{\mu\nu} [f_-^{\alpha\beta}, \chi_+] \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.138}$$

$$V^{abc}_{\mu\alpha\beta} = -8\sqrt{2} B_0 \kappa_{15}^V f^{abc} \varepsilon_{\alpha\beta\mu(p)} \,, \tag{D.139}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta} = -4\sqrt{2} B_0 f^{abc} (2\kappa_{15}^V \varepsilon_{\alpha\beta\mu(p)} + \kappa_4^V \varepsilon_{\alpha\beta\mu(q)}) \,. \tag{D.140}$$

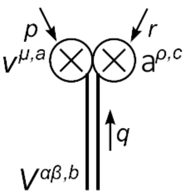
Vertex $\boldsymbol{Vs\phi}$



$$\mathcal{L}_4^V = i\kappa_4^V \langle [V^{\mu\nu}, \nabla^\alpha \chi_+] u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.141}$$

$$V^{abc}_{\alpha\beta} = -\frac{4i\sqrt{2} B_0 \kappa_4^V}{F} f^{abc} \varepsilon_{\alpha\beta(p)(q)} \,. \tag{D.142}$$

Vertex $\boldsymbol{Vva}$



$$\mathcal{L}_{11}^V = \kappa_{11}^V \langle V^{\mu\nu} \{ f_+^{\alpha\rho}, f_-^{\beta\sigma} \} \rangle g_{\rho\sigma} \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.143}$$

$$\begin{aligned} V^{abc}_{\mu\rho\alpha\beta} = -2i\sqrt{2}\kappa_{11}^V d^{abc} \big[&- g_{\mu\rho} \varepsilon_{\alpha\beta(p)(r)} - r_\mu \varepsilon_{\alpha\beta\rho(p)} \\ &+ p_\rho \varepsilon_{\alpha\beta\mu(r)} - (p\cdot r) \varepsilon_{\alpha\beta\mu\rho} \big] \,, \end{aligned} \tag{D.144}$$

$$\mathcal{L}_{12}^V = \kappa_{12}^V \langle V^{\mu\nu} \{ f_+^{\alpha\rho}, h^{\beta\sigma} \} \rangle g_{\rho\sigma} \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.145}$$

$$\begin{aligned} V^{abc}_{\mu\rho\alpha\beta} = 2i\sqrt{2}\kappa_{12}^V d^{abc} \big[&- g_{\mu\rho} \varepsilon_{\alpha\beta(p)(r)} + r_\mu \varepsilon_{\alpha\beta\rho(p)} \\ &+ p_\rho \varepsilon_{\alpha\beta\mu(r)} + (p\cdot r) \varepsilon_{\alpha\beta\mu\rho} \big] \,, \end{aligned} \tag{D.146}$$

$$\mathcal{L}_{16}^V = \kappa_{16}^V \langle V^{\mu\nu} \{ \nabla^\alpha f_+^{\beta\sigma}, u_\sigma \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.147}$$

$$V^{abc}_{\mu\rho\alpha\beta} = -2i\sqrt{2} p_\rho \kappa_{16}^V d^{abc} \varepsilon_{\alpha\beta\mu(p)} \,, \tag{D.148}$$

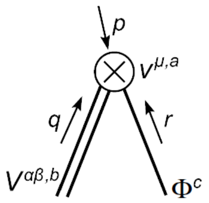
$$\mathcal{L}_{17}^V = \kappa_{17}^V \langle V^{\mu\nu} \{ \nabla_\sigma f_+^{\alpha\sigma}, u^\beta \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.149}$$

$$V^{abc}_{\mu\rho\alpha\beta} = 2i\sqrt{2}\kappa_{17}^V d^{abc} (p_\mu \varepsilon_{\alpha\beta\rho(p)} + p^2 \varepsilon_{\alpha\beta\mu\rho}) \,, \tag{D.150}$$

i.e. all together

$$\begin{aligned} V^{abc}_{\mu\rho\alpha\beta} = -2i\sqrt{2} d^{abc} \Big[&- p^2 \kappa_{17}^V \varepsilon_{\alpha\beta\mu\rho} + (\kappa_{11}^V - \kappa_{12}^V)(p_\rho \varepsilon_{\alpha\beta\mu(r)} - g_{\mu\rho} \varepsilon_{\alpha\beta(p)(r)}) \\ &- \big[\kappa_{17}^V p_\mu + (\kappa_{11}^V + \kappa_{12}^V) r_\mu \big] \varepsilon_{\alpha\beta\rho(p)} + \kappa_{16}^V p_\rho \varepsilon_{\alpha\beta\mu(p)} \\ &- (\kappa_{11}^V + \kappa_{12}^V)(p\cdot r) \varepsilon_{\alpha\beta\mu\rho} \Big] \,. \end{aligned} \tag{D.151}$$

Vertex $Vv\phi$



$$\mathcal{L}_{12}^V = \kappa_{12}^V \langle V^{\mu\nu} \{ f_+^{\alpha\rho}, h^{\beta\sigma} \} \rangle g_{\rho\sigma} \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.152}$$

$$V_{\mu\alpha\beta}^{abc} = \frac{4\sqrt{2}\kappa_{12}^V}{F} d^{abc} \big[r_\mu \varepsilon_{\alpha\beta(p)(r)} - (p\cdot r) \varepsilon_{\alpha\beta\mu(r)} \big] \,, \tag{D.153}$$

$$\mathcal{L}_{16}^V = \kappa_{16}^V \langle V^{\mu\nu} \{ \nabla^\alpha f_+^{\beta\sigma}, u_\sigma \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.154}$$

$$V_{\mu\alpha\beta}^{abc} = \frac{2\sqrt{2}\kappa_{16}^V}{F} d^{abc} (p\cdot r) \varepsilon_{\alpha\beta\mu(p)} \,, \tag{D.155}$$

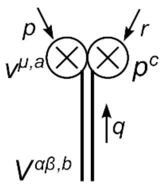
$$\mathcal{L}_{17}^V = \kappa_{17}^V \langle V^{\mu\nu} \{ \nabla_\sigma f_+^{\alpha\sigma}, u^\beta \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.156}$$

$$V_{\mu\alpha\beta}^{abc} = \frac{2\sqrt{2}\kappa_{17}^V}{F} d^{abc} \big(p_\mu \varepsilon_{\alpha\beta(p)(r)} - p^2 \varepsilon_{\alpha\beta\mu(r)} \big) \,, \tag{D.157}$$

i.e. all together

$$\begin{aligned} V_{\mu\alpha\beta}^{abc} = -\frac{2\sqrt{2}}{F} d^{abc} \Big[&- (\kappa_{17}^V p_\mu + 2\kappa_{12}^V r_\mu) \varepsilon_{\alpha\beta(p)(r)} + 2\kappa_{12}^V (p\cdot r) \varepsilon_{\alpha\beta\mu(r)} \\ &+ p^2 \kappa_{17}^V \varepsilon_{\alpha\beta\mu(r)} - \kappa_{16}^V (p\cdot r) \varepsilon_{\alpha\beta\mu(p)} \Big] \,. \end{aligned} \tag{D.158}$$

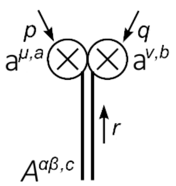
Vertex Vvp



$$\mathcal{L}_{14}^V = i\kappa_{14}^V \langle V^{\mu\nu} \{ f_+^{\alpha\beta}, \chi_- \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.159}$$

$$V_{\mu\alpha\beta}^{abc} = 8\sqrt{2} B_0 \kappa_{14}^V d^{abc} \varepsilon_{\alpha\beta\mu(p)} \,. \tag{D.160}$$

Vertex Aa^2



$$\mathcal{L}_{3}^A = \kappa_{3}^A \langle A^{\mu\nu} \{ \nabla^\alpha h^{\beta\sigma}, u_\sigma \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.161}$$

$$V_{\mu\nu\alpha\beta}^{abc} = 2i\sqrt{2}\kappa_3^A d^{abc} \big(p_\nu \varepsilon_{\alpha\beta\mu(p)} + q_\mu \varepsilon_{\alpha\beta\nu(q)} \big) \,, \tag{D.162}$$

$$\mathcal{L}_{8}^A = \kappa_{8}^A \langle A^{\mu\nu} \{ f_-^{\alpha\sigma}, h^{\beta\sigma} \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.163}$$

$$V_{\mu\nu\alpha\beta}^{abc} = -4i\sqrt{2}\kappa_8^A d^{abc} \big(q_\mu \varepsilon_{\alpha\beta\nu(p)} + p_\nu \varepsilon_{\alpha\beta\mu(q)} \big) \,, \tag{D.164}$$

$$\mathcal{L}_{15}^A = \kappa_{15}^A \langle A^{\mu\nu} \{ \nabla^\alpha f_-^{\beta\sigma}, u_\sigma \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.165}$$

$$V_{\mu\nu\alpha\beta}^{abc} = 2i\sqrt{2}\kappa_{15}^A d^{abc} \big(p_\nu \varepsilon_{\alpha\beta\mu(p)} + q_\mu \varepsilon_{\alpha\beta\nu(q)} \big) \,, \tag{D.166}$$

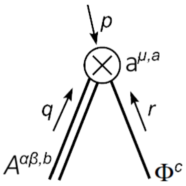
$$\mathcal{L}_{16}^A = \kappa_{16}^A \langle A^{\mu\nu} \{ \nabla_\sigma f_-^{\alpha\sigma}, u^\beta \} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.167}$$

$$\begin{aligned} V_{\mu\nu\alpha\beta}^{abc} = 2i\sqrt{2}\kappa_{16}^A d^{abc} \big(&- p_\mu \varepsilon_{\alpha\beta\nu(p)} - p^2 \varepsilon_{\alpha\beta\mu\nu} \\ &- q_\nu \varepsilon_{\alpha\beta\mu(q)} + q^2 \varepsilon_{\alpha\beta\mu\nu} \big) \,, \end{aligned} \tag{D.168}$$

i.e. all together

$$\begin{aligned} V_{\mu\nu\alpha\beta}^{abc} = -2i\sqrt{2} d^{abc} \Big[&- (\kappa_3^A + \kappa_{15}^A)(p_\nu \varepsilon_{\alpha\beta\mu(p)} + q_\mu \varepsilon_{\alpha\beta\nu(q)}) + \kappa_{16}^A p_\mu \varepsilon_{\alpha\beta\nu(p)} \\ &+ \kappa_{16}^A (p^2 - q^2) \varepsilon_{\alpha\beta\mu\nu} + \kappa_{16}^A q_\nu \varepsilon_{\alpha\beta\mu(q)} + 2\kappa_8^A q_\mu \varepsilon_{\alpha\beta\nu(p)} \\ &+ 2\kappa_8^A p_\nu \varepsilon_{\alpha\beta\mu(q)} \Big] \,. \end{aligned} \tag{D.169}$$

Vertex $Aa\phi$



$$\mathcal{L}_3^A = \kappa_3^A \langle A^{\mu\nu}\{\nabla^\alpha h^{\beta\sigma}, u_\sigma\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.170}$$

$$V^{abc}_{\mu\alpha\beta} = -\frac{2\sqrt{2}\kappa_3^A}{F} d^{abc}(p\cdot r)\varepsilon_{\alpha\beta\mu(p)}\,, \tag{D.171}$$

$$\mathcal{L}_8^A = \kappa_8^A \langle A^{\mu\nu}\{f_-^{\alpha\sigma}, h^{\beta\sigma}\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.172}$$

$$V^{abc}_{\mu\alpha\beta} = -\frac{4\sqrt{2}\kappa_8^A}{F} d^{abc}\big[r_\mu \varepsilon_{\alpha\beta(p)(r)} - (p\cdot r)\varepsilon_{\alpha\beta\mu(r)}\big]\,, \tag{D.173}$$

$$\mathcal{L}_{15}^A = \kappa_{15}^A \langle A^{\mu\nu}\{\nabla^\alpha f_-^{\beta\sigma}, u_\sigma\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.174}$$

$$V^{abc}_{\mu\alpha\beta} = -\frac{2\sqrt{2}\kappa_{15}^A}{F} d^{abc}(p\cdot r)\varepsilon_{\alpha\beta\mu(p)}\,, \tag{D.175}$$

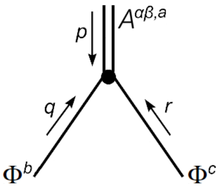
$$\mathcal{L}_{16}^A = \kappa_{16}^A \langle A^{\mu\nu}\{\nabla_\sigma f_-^{\alpha\sigma}, u^\beta\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.176}$$

$$V^{abc}_{\mu\alpha\beta} = -\frac{2\sqrt{2}\kappa_{16}^A}{F} d^{abc}(p_\mu \varepsilon_{\alpha\beta(p)(r)} - p^2 \varepsilon_{\alpha\beta\mu(r)})\,, \tag{D.177}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta} = \frac{2\sqrt{2}}{F} d^{abc}\Big[- \varepsilon_{\alpha\beta(p)(r)}(\kappa_{16}^A p_\mu + 2\kappa_8^A r_\mu) - (\kappa_{15}^A + \kappa_3^A)(p\cdot r)\varepsilon_{\alpha\beta\mu(p)} + \varepsilon_{\alpha\beta\mu(r)}\big[p^2\kappa_{16}^A + 2\kappa_8^A(p\cdot r)\big]\Big]\,. \tag{D.178}$$

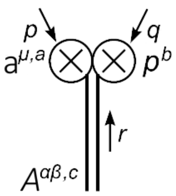
Vertex $A\phi^2$



$$\mathcal{L}_3^A = \kappa_3^A \langle A^{\mu\nu}\{\nabla^\alpha h^{\beta\sigma}, u_\sigma\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.179}$$

$$V^{abc}_{\alpha\beta} = 0\,. \tag{D.180}$$

Vertex Aap



$$\mathcal{L}_{11}^A = i\kappa_{11}^A \langle A^{\mu\nu}\{f_-^{\alpha\beta}, \chi_-\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.181}$$

$$V^{abc}_{\mu\alpha\beta} = -8\sqrt{2}B_0\kappa_{11}^A d^{abc}\varepsilon_{\alpha\beta\mu(p)}\,, \tag{D.182}$$

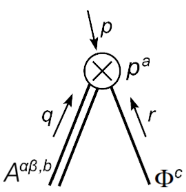
$$\mathcal{L}_{12}^A = i\kappa_{12}^A \langle A^{\mu\nu}\{\nabla^\alpha \chi_-, u^\beta\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.183}$$

$$V^{abc}_{\mu\alpha\beta} = 4\sqrt{2}B_0\kappa_{12}^A d^{abc}\varepsilon_{\alpha\beta\mu(q)}\,, \tag{D.184}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta} = -4\sqrt{2}B_0 d^{abc}(2\kappa_{11}^A \varepsilon_{\alpha\beta\mu(p)} - \kappa_{12}^A \varepsilon_{\alpha\beta\mu(q)})\,. \tag{D.185}$$

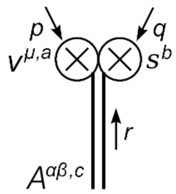
Vertex $A\phi p$



$$\mathcal{L}_{12}^A = i\kappa_{12}^A \langle A^{\mu\nu}\{\nabla^\alpha \chi_-, u^\beta\}\rangle \varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.186}$$

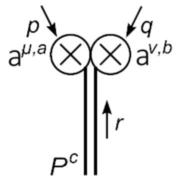
$$V^{abc}_{\alpha\beta} = -\frac{4i\sqrt{2}B_0\kappa_{12}^A}{F} d^{abc}\varepsilon_{\alpha\beta(p)(r)}\,. \tag{D.187}$$

Vertex Avs



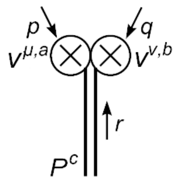
$$\mathcal{L}_{14}^{A} = i\kappa_{14}^{A}\langle A^{\mu\nu}[f_{+}^{\alpha\beta}, \chi_{+}]\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.188}$$
$$V_{\mu\alpha\beta}^{abc} = 8\sqrt{2}B_0\kappa_{14}^{A}f^{abc}\varepsilon_{\alpha\beta\mu(p)}\,. \tag{D.189}$$

Vertex Pa^2



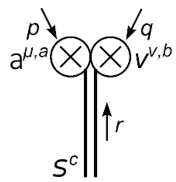
$$\mathcal{L}_{1}^{P} = \kappa_{1}^{P}\langle P\{f_{-}^{\mu\nu}, f_{-}^{\alpha\beta}\}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.190}$$
$$V_{\mu\nu}^{abc} = 16i\sqrt{2}\kappa_{1}^{P}d^{abc}\varepsilon_{\mu\nu(p)(q)}\,. \tag{D.191}$$

Vertex Pv^2



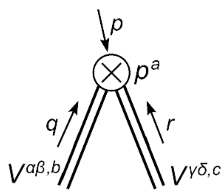
$$\mathcal{L}_{5}^{P} = \kappa_{5}^{P}\langle P\{f_{+}^{\mu\nu}, f_{+}^{\alpha\beta}\}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.192}$$
$$V_{\mu\nu}^{abc} = 16i\sqrt{2}\kappa_{5}^{P}d^{abc}\varepsilon_{\mu\nu(p)(q)}\,. \tag{D.193}$$

Vertex Sav



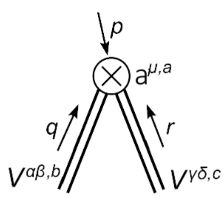
$$\mathcal{L}_{2}^{S} = i\kappa_{2}^{S}\langle S[f_{+}^{\mu\nu}, f_{-}^{\alpha\beta}]\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.194}$$
$$V_{\mu\nu}^{abc} = -8i\sqrt{2}\kappa_{2}^{S}f^{abc}\varepsilon_{\mu\nu(p)(q)}\,. \tag{D.195}$$

Vertex VVp



$$\mathcal{L}_{2}^{VV} = i\kappa_{2}^{VV}\langle\{V^{\mu\nu}, V^{\alpha\beta}\}\chi_{-}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.196}$$
$$V_{\alpha\beta\gamma\delta}^{abc} = -8iB_0\kappa_{2}^{VV}d^{abc}\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{D.197}$$

Vertex VVa

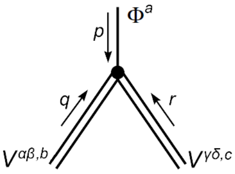


$$\mathcal{L}_{3}^{VV} = \kappa_{3}^{VV}\langle\{\nabla_{\sigma}V^{\mu\nu}, V^{\alpha\sigma}\}u^{\beta}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.198}$$
$$V_{\mu\alpha\beta\gamma\delta}^{abc} = \kappa_{3}^{VV}d^{abc}\big(-q_{\gamma}\varepsilon_{\alpha\beta\delta\mu} + q_{\delta}\varepsilon_{\alpha\beta\gamma\mu} - r_{\alpha}\varepsilon_{\beta\gamma\delta\mu} + r_{\beta}\varepsilon_{\alpha\gamma\delta\mu}\big)\,, \tag{D.199}$$
$$\mathcal{L}_{4}^{VV} = \kappa_{4}^{VV}\langle\{\nabla^{\beta}V^{\mu\nu}, V^{\alpha\sigma}\}u_{\sigma}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.200}$$
$$V_{\mu\alpha\beta\gamma\delta}^{abc} = \kappa_{4}^{VV}d^{abc}\big(-g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(q)} + g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(q)} - g_{\alpha\mu}\varepsilon_{\beta\gamma\delta(r)} + g_{\beta\mu}\varepsilon_{\alpha\gamma\delta(r)}\big)\,, \tag{D.201}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta\gamma\delta} = -\kappa^{VV}_3 d^{abc}(q_\gamma \varepsilon_{\alpha\beta\delta\mu} - q_\delta \varepsilon_{\alpha\beta\gamma\mu} + r_\alpha \varepsilon_{\beta\gamma\delta\mu} - r_\beta \varepsilon_{\alpha\gamma\delta\mu}) - \kappa^{VV}_4 d^{abc}\big(g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(q)} - g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(q)} + g_{\alpha\mu}\varepsilon_{\beta\gamma\delta(r)} - g_{\beta\mu}\varepsilon_{\alpha\gamma\delta(r)}\big)\,. \tag{D.202}$$

Vertex $VV\phi$



$$\mathcal{L}^{VV}_3 = \kappa^{VV}_3 \langle\{\nabla_\sigma V^{\mu\nu}, V^{\alpha\sigma}\}u^\beta\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.203}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = \frac{i\kappa^{VV}_3}{F} d^{abc}\big(-q_\gamma \varepsilon_{\alpha\beta\delta(p)} + q_\delta \varepsilon_{\alpha\beta\gamma(p)} - r_\alpha \varepsilon_{\beta\gamma\delta(p)} + r_\beta \varepsilon_{\alpha\gamma\delta(p)}\big)\,, \tag{D.204}$$

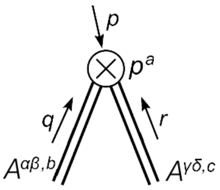
$$\mathcal{L}^{VV}_4 = \kappa^{VV}_4 \langle\{\nabla^\beta V^{\mu\nu}, V^{\alpha\sigma}\}u_\sigma\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.205}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = \frac{i\kappa^{VV}_4}{F} d^{abc}\big(-p_\gamma \varepsilon_{\alpha\beta\delta(q)} + p_\delta \varepsilon_{\alpha\beta\gamma(q)} - p_\alpha \varepsilon_{\beta\gamma\delta(r)} + p_\beta \varepsilon_{\alpha\gamma\delta(r)}\big)\,, \tag{D.206}$$

i.e. all together

$$V^{abc}_{\alpha\beta\gamma\delta} = -\frac{i\kappa^{VV}_3}{F} d^{abc}(q_\gamma \varepsilon_{\alpha\beta\delta(p)} - q_\delta \varepsilon_{\alpha\beta\gamma(p)} + r_\alpha \varepsilon_{\beta\gamma\delta(p)} - r_\beta \varepsilon_{\alpha\gamma\delta(p)}) - \frac{i\kappa^{VV}_4}{F} d^{abc}(p_\gamma \varepsilon_{\alpha\beta\delta(q)} - p_\delta \varepsilon_{\alpha\beta\gamma(q)} + p_\alpha \varepsilon_{\beta\gamma\delta(r)} - p_\beta \varepsilon_{\alpha\gamma\delta(r)})\,. \tag{D.207}$$

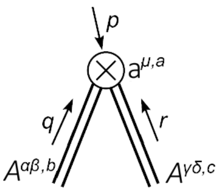
Vertex AAp



$$\mathcal{L}^{AA}_2 = i\kappa^{AA}_2 \langle\{A^{\mu\nu}, A^{\alpha\beta}\}\chi_-\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.208}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = -8iB_0\kappa^{AA}_2 d^{abc}\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{D.209}$$

Vertex AAa



$$\mathcal{L}^{AA}_3 = \kappa^{AA}_3 \langle\{\nabla_\sigma A^{\mu\nu}, A^{\alpha\sigma}\}u^\beta\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.210}$$

$$V^{abc}_{\mu\alpha\beta\gamma\delta} = \kappa^{AA}_3 d^{abc}(-q_\gamma \varepsilon_{\alpha\beta\delta\mu} + q_\delta \varepsilon_{\alpha\beta\gamma\mu} - r_\alpha \varepsilon_{\beta\gamma\delta\mu} + r_\beta \varepsilon_{\alpha\gamma\delta\mu})\,, \tag{D.211}$$

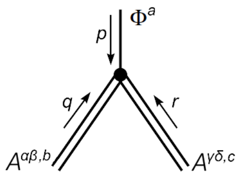
$$\mathcal{L}^{AA}_4 = \kappa^{AA}_4 \langle\{\nabla^\beta A^{\mu\nu}, A^{\alpha\sigma}\}u_\sigma\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.212}$$

$$V^{abc}_{\mu\alpha\beta\gamma\delta} = \kappa^{AA}_4 d^{abc}\big(-g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(q)} + g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(q)} - g_{\alpha\mu}\varepsilon_{\beta\gamma\delta(r)} + g_{\beta\mu}\varepsilon_{\alpha\gamma\delta(r)}\big)\,, \tag{D.213}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta\gamma\delta} = -\kappa^{AA}_3 d^{abc}(q_\gamma \varepsilon_{\alpha\beta\delta\mu} - q_\delta \varepsilon_{\alpha\beta\gamma\mu} + r_\alpha \varepsilon_{\beta\gamma\delta\mu} - r_\beta \varepsilon_{\alpha\gamma\delta\mu}) - \kappa^{AA}_4 d^{abc}\big(g_{\gamma\mu}\varepsilon_{\alpha\beta\delta(q)} - g_{\delta\mu}\varepsilon_{\alpha\beta\gamma(q)} + g_{\alpha\mu}\varepsilon_{\beta\gamma\delta(r)} - g_{\beta\mu}\varepsilon_{\alpha\gamma\delta(r)}\big)\,. \tag{D.214}$$

Vertex $AA\phi$



$$\mathcal{L}_3^{AA} = \kappa_3^{AA} \langle \{ \nabla_\sigma A^{\mu\nu}, A^{\alpha\sigma} \} u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.215}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = \frac{i\kappa_3^{AA}}{F} d^{abc} \big(-q_\gamma \varepsilon_{\alpha\beta\delta(p)} + q_\delta \varepsilon_{\alpha\beta\gamma(p)} - r_\alpha \varepsilon_{\beta\gamma\delta(p)} + r_\beta \varepsilon_{\alpha\gamma\delta(p)} \big) \,, \tag{D.216}$$

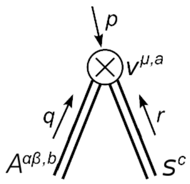
$$\mathcal{L}_4^{AA} = \kappa_4^{AA} \langle \{ \nabla^\beta A^{\mu\nu}, A^{\alpha\sigma} \} u_\sigma \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.217}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = \frac{i\kappa_4^{AA}}{F} d^{abc} \big(-p_\gamma \varepsilon_{\alpha\beta\delta(q)} + p_\delta \varepsilon_{\alpha\beta\gamma(q)} - p_\alpha \varepsilon_{\beta\gamma\delta(r)} + p_\beta \varepsilon_{\alpha\gamma\delta(r)} \big) \,, \tag{D.218}$$

i.e. all together

$$V^{abc}_{\alpha\beta\gamma\delta} = -\frac{i\kappa_3^{AA}}{F} d^{abc} \big(q_\gamma \varepsilon_{\alpha\beta\delta(p)} - q_\delta \varepsilon_{\alpha\beta\gamma(p)} + r_\alpha \varepsilon_{\beta\gamma\delta(p)} - r_\beta \varepsilon_{\alpha\gamma\delta(p)} \big) - \frac{i\kappa_4^{AA}}{F} d^{abc} \big(p_\gamma \varepsilon_{\alpha\beta\delta(q)} - p_\delta \varepsilon_{\alpha\beta\gamma(q)} + p_\alpha \varepsilon_{\beta\gamma\delta(r)} - p_\beta \varepsilon_{\alpha\gamma\delta(r)} \big) \,. \tag{D.219}$$

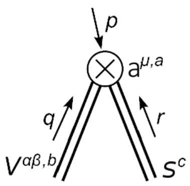
Vertex SAv



$$\mathcal{L}_1^{SA} = i\kappa_1^{SA} \langle [A^{\mu\nu}, S] f_+^{\alpha\beta} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.220}$$

$$V^{abc}_{\mu\alpha\beta} = 4\kappa_1^{SA} f^{abc} \varepsilon_{\alpha\beta\mu(p)} \,. \tag{D.221}$$

Vertex SVa



$$\mathcal{L}_1^{SV} = i\kappa_1^{SV} \langle [V^{\mu\nu}, S] f_-^{\alpha\beta} \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.222}$$

$$V^{abc}_{\mu\alpha\beta} = -4\kappa_1^{SV} f^{abc} \varepsilon_{\alpha\beta\mu(p)} \,, \tag{D.223}$$

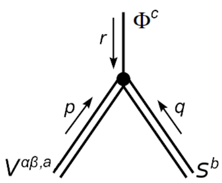
$$\mathcal{L}_2^{SV} = i\kappa_2^{SV} \langle [V^{\mu\nu}, \nabla^\alpha S] u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.224}$$

$$V^{abc}_{\mu\alpha\beta} = 2\kappa_2^{SV} f^{abc} \varepsilon_{\alpha\beta\mu(r)} \,, \tag{D.225}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta} = -2 f^{abc} \big(2\kappa_1^{SV} \varepsilon_{\alpha\beta\mu(p)} - \kappa_2^{SV} \varepsilon_{\alpha\beta\mu(r)} \big) \,. \tag{D.226}$$

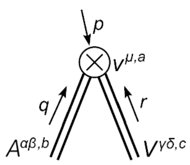
Vertex $SV\phi$



$$\mathcal{L}_2^{SV} = i\kappa_2^{SV} \langle [V^{\mu\nu}, \nabla^\alpha S] u^\beta \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.227}$$

$$V^{abc}_{\alpha\beta} = -\frac{2i\kappa_2^{SV}}{F} f^{abc} \varepsilon_{\alpha\beta(q)(r)} \,. \tag{D.228}$$

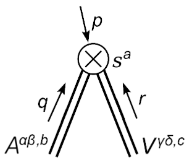
Vertex VAv



$$\mathcal{L}_5^{VA} = \kappa_5^{VA} \langle \{ V^{\mu\nu}, A^{\alpha\rho} \} f_+^{\beta\sigma} \rangle g_{\rho\sigma} \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.229}$$

$$V^{abc}_{\mu\alpha\beta\gamma\delta} = \kappa_5^{VA} d^{abc} \big(-g_{\alpha\mu} \varepsilon_{\beta\gamma\delta(p)} + g_{\beta\mu} \varepsilon_{\alpha\gamma\delta(p)} + p_\alpha \varepsilon_{\beta\gamma\delta\mu} - p_\beta \varepsilon_{\alpha\gamma\delta\mu} \big) \,. \tag{D.230}$$

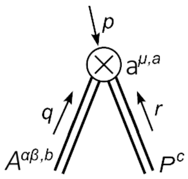
Vertex $\boldsymbol{VAs}$



$$\mathcal{L}_6^{VA} = i\kappa_6^{VA}\langle[V^{\mu\nu}, A^{\alpha\beta}]\chi_+\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.231}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = 4iB_0\kappa_6^{VA}f^{abc}\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{D.232}$$

Vertex $\boldsymbol{PAa}$



$$\mathcal{L}_1^{PA} = \kappa_1^{PA}\langle\{A^{\mu\nu}, P\}f_-^{\alpha\beta}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.233}$$

$$V^{abc}_{\mu\alpha\beta} = 4\kappa_1^{PA}d^{abc}\varepsilon_{\alpha\beta\mu(p)}\,, \tag{D.234}$$

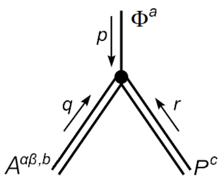
$$\mathcal{L}_2^{PA} = \kappa_2^{PA}\langle\{A^{\mu\nu}, \nabla^\alpha P\}u^\beta\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.235}$$

$$V^{abc}_{\mu\alpha\beta} = -2\kappa_2^{PA}d^{abc}\varepsilon_{\alpha\beta\mu(r)}\,, \tag{D.236}$$

i.e. all together

$$V^{abc}_{\mu\alpha\beta} = 2d^{abc}\big(2\kappa_1^{PA}\varepsilon_{\alpha\beta\mu(p)} - \kappa_2^{PA}\varepsilon_{\alpha\beta\mu(r)}\big)\,. \tag{D.237}$$

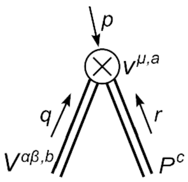
Vertex $\boldsymbol{PA\phi}$



$$\mathcal{L}_2^{PA} = \kappa_2^{PA}\langle\{A^{\mu\nu}, \nabla^\alpha P\}u^\beta\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.238}$$

$$V^{abc}_{\alpha\beta} = -\frac{2i\kappa_2^{PA}}{F}d^{abc}\varepsilon_{\alpha\beta(p)(r)}\,. \tag{D.239}$$

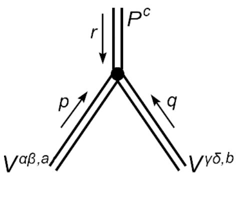
Vertex $\boldsymbol{PVv}$



$$\mathcal{L}_3^{PV} = \kappa_3^{PV}\langle\{V^{\mu\nu}, P\}f_+^{\alpha\beta}\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.240}$$

$$V^{abc}_{\mu\alpha\beta} = -4\kappa_3^{PV}d^{abc}\varepsilon_{\alpha\beta\mu(p)}\,. \tag{D.241}$$

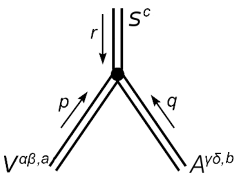
Vertex $\boldsymbol{VVP}$



$$\mathcal{L}^{VVP} = \kappa^{VVP}\langle V^{\mu\nu}V^{\alpha\beta}P\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.242}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = i\sqrt{2}\kappa^{VVP}d^{abc}\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{D.243}$$

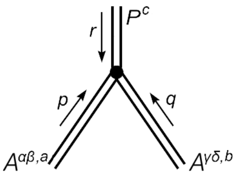
Vertex $\boldsymbol{VAS}$



$$\mathcal{L}^{VAS} = i\kappa^{VAS}\langle[V^{\mu\nu}, A^{\alpha\beta}]S\rangle\varepsilon_{\mu\nu\alpha\beta}\,, \tag{D.244}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = -i\sqrt{2}\kappa^{VAS}f^{abc}\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{D.245}$$

Vertex AAP



$$\mathcal{L}^{AAP} = \kappa^{AAP} \langle A^{\mu\nu} A^{\alpha\beta} P \rangle \varepsilon_{\mu\nu\alpha\beta} \,, \tag{D.246}$$

$$V^{abc}_{\alpha\beta\gamma\delta} = i\sqrt{2}\kappa^{AAP} d^{abc} \varepsilon_{\alpha\beta\gamma\delta} \,. \tag{D.247}$$

E. Algorithm in FeynCalc

Here we will describe the construction of the algorithm 'Mercury' in Feyncalc, that we have developed and used, and explain how it may be run by the reader to check the results obtained in this thesis.

First of all, let us recall that FeynCalc is a package that is possible to instal into Wolfram Mathematica software which allows us to do algebraic calculations in quantum field theory easily. The FeynCalc can be obtained from the web page [44] and manually installed or simply substituted by putting

```
Import["http://www.feyncalc.org/install.m"]
```

in the Mathematica notebook and executing this command. After restarting the Kernel, the package is succesfully ready to use.

Now, let us assume that we already have the package at our disposal. If so, we need to load the FeynCalc by the command

```
<< HighEnergyPhysics‘FeynCalc‘
```

To work with FeynCalc correctly, one has to use the given notation. Before getting any further, we will introduce here some designations that FeynCalc work with.

E.1 Implementation of fields and chiral operators

In what follows we present the source code of the Mercury algorithm[3]. We simply follow the structure of the code contained in the Mathematica files attached to this thesis but we will enhance the source code for a discussion to get a better look at the subject involved. Let us start with the definitions of the pseudoscalar fields nad external sources. This part of the source code is contained in the file

```
01-mercury-definitions.nb
```

which must be run first to ensure we load all definitions to work with.

We have defined the components of pseudoscalar fields, pseudoscalar and scalar external sources, together with their derivations (three, at most), in a following way.

```
1 Pseudoscalar[a_]:=QuantumField[GaugeField,SUNIndex[a]];

2 PseudoscalarDer[μ_,a_]:=QuantumField[PartialD[μ,GaugeField,
  SUNIndex[a]];

3 PseudoscalarDerDer[μ_,ν_,a_]:=QuantumField[PartialD[μ,PartialD[ν],
  GaugeField,SUNIndex[a]];

4 PseudoscalarDerDerDer[ς_,μ_,ν_,a_]:=QuantumField[PartialD[ς],
  PartialD[μ],PartialD[ν],GaugeField,SUNIndex[a]];

5 ScalarSource[a_]:=QuantumField[GaugeField,SUNIndex[a]];

6 ScalarSourceDer[α_,a_]:= QuantumField[PartialD[α],GaugeField,
  SUNIndex[a]];
```

[3] A reader should be aware of the spacing used in this text. Since the individual lines of the source codes are usually long but the lenght of the row in this text is not, we simply break the line when needed and continue on the next line without any unique meaning. For simplicity we use numbering of the lines for a reader's comfort. Every new definitions or commands start at the newly numbered line.

```
7 PseudoscalarSource[a_]:= QuantumField[GaugeField,SUNIndex[a]];
```

```
8 PseudoscalarSourceDer[α_,a_]:= QuantumField[PartialD[α],
  GaugeField,SUNIndex[a]];
```

Similarly as in previous steps we have also defined the components of vector and axial-vector external sources. The only difference is in the presence of `GaugeField` that is coupled directly to the components of the sources.

```
9 VectorSource[μ_,a_]:= QuantumField[GaugeField,LorentzIndex[μ],
  SUNIndex[a]];
```

```
10 AxialSource[μ_,a_]:= QuantumField[GaugeField,LorentzIndex[μ],
   SUNIndex[a]];
```

```
11 VectorSourceDer[α_,μ_,a_]:= QuantumField[PartialD[α],GaugeField,
   LorentzIndex[μ],SUNIndex[a]];
```

```
12 VectorSourceDerDer[α_,β_,μ_,a_]:= QuantumField[PartialD[α],
   PartialD[β],GaugeField,LorentzIndex[μ],SUNIndex[a]];
```

```
13 AxialSourceDer[α_,μ_,a_]:= QuantumField[PartialD[α],GaugeField,
   LorentzIndex[μ],SUNIndex[a]];
```

```
14 AxialSourceDerDer[α_,β_,μ_,a_]:= QuantumField[PartialD[α],
   PartialD[β],GaugeField,LorentzIndex[μ],SUNIndex[a]];
```

Now we can define the components of scalar and pseudoscalar resonances.

```
15 ScalarResonance[a_]:= QuantumField[GaugeField,SUNIndex[a]];
```

```
16 ScalarResonanceDer[α_,a_]:= QuantumField[PartialD[α],GaugeField,
   SUNIndex[a]];
```

```
17 PseudoscalarResonance[a_]:= QuantumField[GaugeField,SUNIndex[a]];
```

```
18 PseudoscalarResonanceDer[α_,a_]:= QuantumField[PartialD[α],
   GaugeField,SUNIndex[a]];
```

Similarly as above, the definitions of vector and axial-vector resonances in the antisymmetric tensor formalism are following.

```
19 VectorResonance[μ_,ν_,a_]:= QuantumField[GaugeField,
   LorentzIndex[μ],LorentzIndex[ν],SUNIndex[a]];
```

```
20 VectorResonanceDer[ρ_,μ_,ν_,a_]:= QuantumField[PartialD[ρ],
   GaugeField,LorentzIndex[μ],LorentzIndex[ν],SUNIndex[a]];
```

```
21 AxialResonance[μ_,ν_,a_]:= QuantumField[GaugeField,LorentzIndex[μ],
   LorentzIndex[ν],SUNIndex[a]];
```

```
22 AxialResonanceDer[ρ_,μ_,ν_,a_]:= QuantumField[PartialD[ρ],
   GaugeField,LorentzIndex[μ],LorentzIndex[ν],SUNIndex[a]];
```

The resonances in the vector formalism takes the form

```
23 AxialResonanceVecForm[μ_,a_]:=QuantumField[GaugeField,
   LorentzIndex[μ],SUNIndex[a]];
```

```
24 VectorResonanceVecForm[μ_,a_]:=QuantumField[GaugeField,
   LorentzIndex[μ],SUNIndex[a]];

25 AxialResonanceVecFormDer[ρ_,μ_,a_]:=QuantumField[PartialD[ρ],
   GaugeField,LorentzIndex[μ],SUNIndex[a]];

26 VectorResonanceVecFormDer[ρ_,μ_,a_]:=QuantumField[PartialD[ρ],
   GaugeField,LorentzIndex[μ],SUNIndex[a]];
```

Now it is possible to construct all kinds of fields and the chiral operators. Simply, only by using `FeynRule` we can exchange a given object for its Feynman rule as we have explained in Appendix D. We now present the implementation of the chiral operators expanded only into their first term. We also define their first derivatives. The designation of the functions fully coincides with their real meaning.

```
27 uAxial[μ_,a_,ν_,b_,p_]:=2 (-I)FeynRule[AxialSource[μ,a],
   {AxialSource[ν,b][p]}]SUNT[SUNIndex[a]];

28 uPseudoscalar[μ_,a_,b_,p_]:=-2/F (-I)FeynRule[PseudoscalarDer[μ,a],
   {Pseudoscalar[b][p]}]SUNT[SUNIndex[a]];

29 hAxial[μ_,ν_,a_,α_,b_,p_]:=2((-I)FeynRule[AxialSourceDer[μ,ν,a],
   {AxialSource[α,b][p]}]+(-I)FeynRule[AxialSourceDer[ν,μ,a],
   {AxialSource[α,b][p]}])SUNT[SUNIndex[a]];

30 hPseudoscalar[μ_,ν_,a_,b_,p_]:=-4/F (-I)FeynRule[
   PseudoscalarDerDer[μ,ν,a],{Pseudoscalar[b][p]}]SUNT[SUNIndex[a]];

31 hDerAxial[ς_,μ_,ν_,a_,α_,b_,p_]:=2((-I)FeynRule[
   AxialSourceDerDer[ς,μ,ν,a],{AxialSource[α,b][p]}]+(-I)FeynRule[
   AxialSourceDerDer[ς,ν,μ,a],{AxialSource[α,b][p]}])
   SUNT[SUNIndex[a]];

32 hDerPseudoscalar[ς_,μ_,ν_,a_,b_,p_]:=-4/F (-I)FeynRule[
   PseudoscalarDerDerDer[ς,μ,ν,a],{Pseudoscalar[b][p]}]
   SUNT[SUNIndex[a]];

33 f_-[μ_,ν_,a_,α_,b_,p_]:=-2((-I)FeynRule[AxialSourceDer[μ,ν,a],
   AxialSource[α,b][p]]-(-I)FeynRule[AxialSourceDer[ν,μ,a],
   AxialSource[α,b][p]])SUNT[SUNIndex[a]];

34 fDer_-[ς_,μ_,ν_,a_,α_,b_,p_]:=-2((-I)FeynRule[
   AxialSourceDerDer[ς,μ,ν,a],{AxialSource[α,b][p]}]-(-I)FeynRule[
   AxialSourceDerDer[ς,ν,μ,a],{AxialSource[α,b][p]}])
   SUNT[SUNIndex[a]];

35 f_+[μ_,ν_,a_,α_,b_,p_]:=2((-I)FeynRule[VectorSourceDer[μ,ν,a],
   {VectorSource[α,b][p]}]-(-I)FeynRule[VectorSourceDer[ν,μ,a],
   {VectorSource[α,b][p]}])SUNT[SUNIndex[a]];

36 fDer_+[ς_,μ_,ν_,a_,α_,b_,p_]:=2((-I)FeynRule[
   VectorSourceDerDer[ς,μ,ν,a],{VectorSource[α,b][p]}]-(-I)FeynRule[
   VectorSourceDerDer[ς,ν,μ,a],{VectorSource[α,b][p]}])
   SUNT[SUNIndex[a]];

37 χ_-[a_,b_,p_]:=4I B_0(-I)FeynRule[PseudoscalarSource[a],
   {PseudoscalarSource[b][p]}]SUNT[SUNIndex[a]];
```

```
38 χDer-[α_,a_,b_,p_]:=4I B0(-I)FeynRule[
   PseudoscalarSourceDer[α,a],{PseudoscalarSource[b][p]}]
   SUNT[SUNIndex[a]];

39 χ+[a_,b_,p_]:=4B0(-I)FeynRule[ScalarSource[a],{ScalarSource[b][p]}]
   SUNT[SUNIndex[a]];

40 χDer+[α_,a_,b_,p_]:=4B0(-I)FeynRule[ScalarSourceDer[α,a],
   {ScalarSource[b][p]}]SUNT[SUNIndex[a]];
```

For our convenience, we also introduce definitions for the pseudoscalars and individual external sources:

```
41 axial[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[AxialSource[μ,a],
   {AxialSource[ν,b][p]}]SUNT[SUNIndex[a]];

42 axialDer[μ_,ν_,a_,α_,b_,p_]:=(-I)FeynRule[AxialSourceDer[μ,ν,a],
   {AxialSource[α,b][p]}]SUNT[SUNIndex[a]];

43 vector[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[AxialSource[μ,a],
   {AxialSource[ν,b][p]}]SUNT[SUNIndex[a]];

44 vectorDer[μ_,ν_,a_,α_,b_,p_]:=(-I)FeynRule[AxialSourceDer[μ,ν,a],
   {AxialSource[α,b][p]}]SUNT[SUNIndex[a]];

45 phi[a_,b_,p_]:=(-I)FeynRule[Pseudoscalar[a],{Pseudoscalar[b][p]}]
   SUNT[SUNIndex[a]]Sqrt[2];

46 phiDer[μ_,a_,b_,p_]:=(-I)FeynRule[PseudoscalarDer[μ,a],
   {Pseudoscalar[b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

47 pSource[a_,b_,p_]:=(-I)FeynRule[PseudoscalarSource[a],
   {PseudoscalarSource[b][p]}]SUNT[SUNIndex[a]];

48 pSourceDer[μ_,a_,b_,p_]:=(-I)FeynRule[PseudoscalarSourceDer[μ,a],
   {PseudoscalarSource[b][p]}]SUNT[SUNIndex[a]];

49 sSource[a_,b_,p_]:=(-I)FeynRule[ScalarSource[a],
   {ScalarSource[b][p]}]SUNT[SUNIndex[a]];

50 sSourceDer[μ_,a_,b_,p_]:=(-I)FeynRule[ScalarSourceDer[μ,a],
   {ScalarSource[b][p]}]SUNT[SUNIndex[a]];
```

Resonances in the antisymmetric tensor formalism are defined as follows.

```
51 VectorFRule[μ_,ν_,a_,α_,β_,b_,p_]:=(-I) 1/2 (FeynRule[
   VectorResonance[μ,ν,a],VectorResonance[α,β,b][p]]-FeynRule[
   VectorResonance[ν,μ,a],VectorResonance[α,β,b][p]]);

52 VectorFRuleDer[ρ_,μ_,ν_,a_,α_,β_,b_,p_]:=(-I) 1/2 (FeynRule[
   VectorResonanceDer[ρ, μ,ν,a],VectorResonance[α,β,b][p]]-FeynRule[
   VectorResonanceDer[ρ, ν,μ,a],VectorResonance[α,β,b][p]]);

53 AxialFRule[μ_,ν_,a_,α_,β_,b_,p_]:=(-I) 1/2 (FeynRule[
   AxialResonance[μ,ν,a],AxialResonance[α,β,b][p]]-FeynRule[
   AxialResonance[ν,μ,a],AxialResonance[α,β,b][p]]);
```

```
AxialFRuleDer[ρ_,μ_,ν_,a_,α_,β_,b_,p_]:=(-I) 1/2 (FeynRule[
AxialResonanceDer[ρ, μ,ν,a],AxialResonance[α,β,b][p]]-FeynRule[
AxialResonanceDer[ρ, ν,μ,a],AxialResonance[α,β,b][p]]);

VectorR[μ_,ν_,a_,α_,β_,b_,p_]:=VectorFRule[μ,ν,a,α,β,b,p]
SUNT[SUNIndex[a]]Sqrt[2];

VectorRDer[ρ_,μ_,ν_,a_,α_,β_,b_,p_]:=VectorFRuleDer[ρ,μ,ν,a,α,β,b,p]
SUNT[SUNIndex[a]]Sqrt[2];

AxialR[μ_,ν_,a_,α_,β_,b_,p_]:=AxialFRule[μ,ν,a,α,β,b,p]
SUNT[SUNIndex[a]]Sqrt[2];

AxialRDer[ρ_,μ_,ν_,a_,α_,β_,b_,p_]:=AxialFRuleDer[ρ,μ,ν,a,α,β,b,p]
SUNT[SUNIndex[a]]Sqrt[2];

ScalarR[a_,b_,p_]:=(-I)FeynRule[ScalarResonance[a],
{ScalarResonance[b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

ScalarRDer[α_,a_,b_,p_]:=(-I)FeynRule[ScalarResonanceDer[α,a],
{ScalarResonance[b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

PseudoscalarR[a_,b_,p_]:=(-I)FeynRule[PseudoscalarResonance[a],
{PseudoscalarResonance[b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

PseudoscalarRDer[α_,a_,b_,p_]:=(-I)FeynRule[
PseudoscalarResonanceDer[α,a],{PseudoscalarResonance[b][p]}]
SUNT[SUNIndex[a]]Sqrt[2];
```

On the other hand, resonances in the vector formalism take the forms

```
AxialResonanceVecFormFRule[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[
AxialSource[μ,a],{AxialSource[ν,b][p]}];

VectorResonanceVecFormFRule[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[
VectorSource[μ,a],{VectorSource[ν,b][p]}];

AxialResonanceVecFormDerFRule[ρ_,μ_,a_,ν_,b_,p_]:=(-I)FeynRule[
AxialSourceDer[ρ,μ,a],{AxialSource[ν,b][p]}];

VectorResonanceVecFormDerFRule[
ρ_,μ_,a_,ν_,b_,p_]:=(-I)FeynRule[VectorSourceDer[ρ,μ,a],
{VectorSource[ν,b][p]}];

VectorR1VecForm[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[VectorSource[μ,a],
{VectorSource[ν,b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

AxialR1VecForm[μ_,a_,ν_,b_,p_]:=(-I)FeynRule[AxialSource[μ,a],
{AxialSource[ν,b][p]}]SUNT[SUNIndex[a]]Sqrt[2];

VectorR2VecForm[μ_,ν_,a_,α_,b_,p_]:=(-I)(FeynRule[
VectorSourceDer[μ,ν,a],{VectorSource[α,b][p]}]-FeynRule[
VectorSourceDer[ν, μ,a],{VectorSource[α,b][p]}])SUNT[SUNIndex[a]]
Sqrt[2];

AxialR2VecForm[μ_,ν_,a_,α_,b_,p_]:=(-I)(FeynRule[
AxialSourceDer[μ,ν,a],{AxialSource[α, b][p]}]-FeynRule[
AxialSourceDer[ν,μ,a],{AxialSource[α,b][p]}])SUNT[SUNIndex[a]]
Sqrt[2];
```

To summarize this section, we present here the table in which the commands that one explicitly uses in our code are shown.

Command	Chiral operator
`uAxial[μ,a,ν,b,p]`	(2.28)
`uPseudoscalar[μ,a,b,p]`	(2.28)
`hAxial[μ,ν,a,α,b,p]`	(2.29)
`hPseudoscalar[μ,ν,a,b,p]`	(2.29)
`hDerAxial[ρ,μ,ν,a,α,b,p]`	
`hDerPseudoscalar[ρ,μ,ν,a,b,p]`	
$\texttt{f}_-$`[μ,ν,a,α,b,p]`	(2.33)
$\texttt{fDer}_-$`[ρ,μ,ν,a,α,b,p]`	
$\texttt{f}_+$`[μ,ν,a,α,b,p]`	(2.32)
$\texttt{fDer}_+$`[ρ,μ,ν,a,α,b,p]`	
χ_-`[a,b,p]`	(2.31)
$\chi\texttt{Der}_-$`[α,a,b,p]`	
χ_+`[a,b,p]`	(2.30)
$\chi\texttt{Der}_+$`[α,a,b,p]`	

Command	External source
`axial[μ,a,ν,b,p]`	(1.37)
`axialDer[μ,ν,a,α,b,p]`	
`vector[μ,a,ν,b,p]`	(1.37)
`vectorDer[μ,ν,a,α,b,p]`	
`phi[a,b,p]`	(2.6)
`phiDer[μ,a,b,p]`	
`pSource[a,b,p]`	(1.37)
`pSourceDer[μ,a,b,p]`	
`sSource[a,b,p]`	(1.37)
`sSourceDer[μ,a,b,p]`	

Command	Resonance field
`VectorR[μ,ν,a,α,β,b,p]`	(1.27)
`VectorRDer[ρ,μ,ν,a,α,β,b,p]`	
`AxialR[μ,ν,a,α,β,b,p]`	(1.28)
`AxialRDer[ρ,μ,ν,a,α,β,b,p]`	
`ScalarR[a,b,p]`	(1.31)
`ScalarRDer[α,a,b,p]`	
`PseudoscalarR[a,b,p]`	(1.32)
`PseudoscalarRDer[α,a,b,p]`	
`VectorR1VecForm[μ,a,α,b,p]`	(2.57)
`VectorR2VecForm[ρ,μ,a,α,b,p]`	(2.58)
`AxialR1VecForm[μ,a,α,b,p]`	(2.57)
`AxialR2VecForm[ρ,μ,a,α,b,p]`	(2.58)

Table E.1: Chiral operators, pseudoscalar fields, external sources and resonances defined in the Mercury code.

E.2 Implementation of propagators

The definitions for the propagators are as follows.

```
71 PropagatorVectorFormVectorR[p_,μ_,ν_,a_,b_]:=
   -I/(ScalarProduct[p]-M_V^2)
   (MetricTensor[μ,ν]-(FourVector[p,μ]FourVector[p,ν])/M_V^2)
   SUNDelta[SUNIndex[a],SUNIndex[b]];

72 PropagatorVectorFormAxialR[p_,μ_,ν_,a_,b_]:=
   -I/(ScalarProduct[p]-M_A^2)
   (MetricTensor[μ,ν]-(FourVector[p,μ]FourVector[p,ν])/M_A^2)
   SUNDelta[SUNIndex[a],SUNIndex[b]];

73 PropagatorTensorFormVectorR[p_,μ_,ν_,α_,β_,a_,b_]:=
   -I/(M_V^2(ScalarProduct[p]-M_V^2))
   ((M_V^2-ScalarProduct[p])MetricTensor[μ,α]MetricTensor[ν,β]
   +MetricTensor[μ,α]FourVector[p,ν]FourVector[p,β]
   -MetricTensor[μ,β]FourVector[p,ν]FourVector[p,α]
   -(M_V^2-ScalarProduct[p])MetricTensor[ν,α]MetricTensor[μ,β]
   -MetricTensor[ν,α]FourVector[p,μ]FourVector[p,β]
   +MetricTensor[ν,β]FourVector[p,μ]FourVector[p,α])
   SUNDelta[SUNIndex[a],SUNIndex[b]];

74 PropagatorTensorFormAxialR[p_,μ_,ν_,α_,β_,a_,b_]:=
   -I/(M_A^2(ScalarProduct[p]-M_A^2))
   ((M_A^2-ScalarProduct[p])MetricTensor[μ,α]MetricTensor[ν,β]
   +MetricTensor[μ,α]FourVector[p,ν]FourVector[p,β]
   -MetricTensor[μ,β]FourVector[p,ν]FourVector[p,α]
   -(M_A^2-ScalarProduct[p])MetricTensor[ν,α]MetricTensor[μ,β]
   -MetricTensor[ν,α]FourVector[p,μ]FourVector[p,β]
   +MetricTensor[ν,β]FourVector[p,μ]FourVector[p,α])
   SUNDelta[SUNIndex[a],SUNIndex[b]];

75 PropagatorPseudoscalar[p_,a_,b_]:=I/ScalarProduct[p]
   SUNDelta[SUNIndex[a],SUNIndex[b]];

76 PropagatorPseudoscalarR[p_,a_,b_]:=I/(ScalarProduct[p]-M_P^2)
   SUNDelta[SUNIndex[a],SUNIndex[b]];
```

As we have already used subdiagrams in Chapters 4-7, we now present definitions for them in our code.

```
77 SubDiag1[p_,μ_,a_,b_]:=(I F)/ScalarProduct[p] FourVector[p,μ]
   SUNDelta[SUNIndex[a],SUNIndex[b]];

78 SubDiag2[p_,μ_,α_,β_,a_,b_]:=(I F_A)/(ScalarProduct[p]-M_A^2)
   (FourVector[p,α]MetricTensor[μ,β]-FourVector[p,β]MetricTensor[μ,α])
   SUNDelta[SUNIndex[a],SUNIndex[b]];

79 SubDiag2VecForm[p_,μ_,α_,a_,b_]:=f_A(-Pair[LorentzIndex[α],
   Momentum[p]]Pair[LorentzIndex[μ],Momentum[p]]+Pair[LorentzIndex[α],
   LorentzIndex[μ]]Pair[Momentum[p],Momentum[p]])SUNDelta[SUNIndex[a],
   SUNIndex[b]])/(Pair[Momentum[p],Momentum[p]]-M_A^2);
```

```
80 SubDiag3[p_,μ_,α_,β_,a_,b_]:=(-I F_V)/(ScalarProduct[p]-M_V^2)
   (FourVector[p,α]MetricTensor[μ,β]-FourVector[p,β]MetricTensor[μ,α])
   SUNDelta[SUNIndex[a],SUNIndex[b]];

81 SubDiag3VecForm[p_,μ_,α_,a_,b_]:=f_V(Pair[LorentzIndex[α],
   Momentum[p]]Pair[LorentzIndex[μ],Momentum[p]]-Pair[LorentzIndex[α],
   LorentzIndex[μ]]Pair[Momentum[p],Momentum[p]])SUNDelta[SUNIndex[a],
   SUNIndex[b]])/(Pair[Momentum[p],Momentum[p]]-M_V^2);

82 SubDiag4[p_,a_,b_]:=(-F B_0)/ScalarProduct[p]
   SUNDelta[SUNIndex[a],SUNIndex[b]];

83 SubDiag5[p_,a_,b_]:=(2Sqrt[2]B_0 d_m)/(ScalarProduct[p]-M_P^2)
   SUNDelta[SUNIndex[a],SUNIndex[b]];

84 SubDiag6[p_,a_,b_]:=(-2Sqrt[2]B_0 c_m)/(ScalarProduct[p]-M_S^2)
   SUNDelta[SUNIndex[a],SUNIndex[b]];
```

At the end of this section, we present a table that consists of the propagators and subdiagrams defined in the code.

Command	Vertex
PropagatorTensorFormVectorR[p,μ,ν,α,β,a,b]	(4.33)
PropagatorTensorFormAxialR[p,μ,ν,α,β,a,b]	(4.33)
PropagatorVectorFormVectorR[p,μ,ν,a,b]	(6.13)
PropagatorVectorFormAxialR[p,μ,ν,a,b]	(6.13)
PropagatorPseudoscalar[p,a,b]	(4.34)
PropagatorPseudoscalarR[p,a,b]	(6.11)
PropagatorScalarR[p,a,b]	(6.12)

Table E.2: Propagators defined in the Mercury algorithm.

Command	Vertex
SubDiag1[p,μ,a,b]	(4.48)
SubDiag2[p,μ,α,β,a,b]	(4.50)
SubDiag2VecForm[p,μ,α,a,b]	
SubDiag3[p,μ,α,β,a,b]	(4.52)
SubDiag3VecForm[p,μ,α,a,b]	(6.14)
SubDiag4[p,a,b]	(6.15)
SubDiag5[p,a,b]	(6.16)
SubDiag6[p,a,b]	

Table E.3: Subdiagrams defined in the Mercury algorithm.

E.3 Procedures and rules

Finally, we also present the procedures and rules that we have used, mostly for a simpler form of strictures generated by the traces of products of the Gell-Mann matrices. These definitions are based on the identities (A.15) and (A.16)-(A.17).

```
85 DFJacobiRule=
   SUND[SUNIndex[a_],SUNIndex[b_],SUNIndex[k_]]
   SUNF[SUNIndex[k_],SUNIndex[c_],SUNIndex[l_]]:>
   -SUND[SUNIndex[b],SUNIndex[c],SUNIndex[k]]
   SUNF[SUNIndex[k],SUNIndex[a],SUNIndex[l]]
   -SUND[SUNIndex[c],SUNIndex[a],SUNIndex[k]]
   SUNF[SUNIndex[k],SUNIndex[b],SUNIndex[l]];
```

```
86 DDSumRule=
   SUND[SUNIndex[a_],SUNIndex[b_],SUNIndex[k_]]
   SUND[SUNIndex[k_],SUNIndex[c_],SUNIndex[l_]]:>
   1/3 (SUNDelta[SUNIndex[a],SUNIndex[c]]
   SUNDelta[SUNIndex[b],SUNIndex[l]]
   +SUNDelta[SUNIndex[a],SUNIndex[l]]
   SUNDelta[SUNIndex[b],SUNIndex[c]]
   -SUNDelta[SUNIndex[a],SUNIndex[b]]
   SUNDelta[SUNIndex[c],SUNIndex[l]]
   +SUNF[SUNIndex[a],SUNIndex[c],SUNIndex[k]]
   SUNF[SUNIndex[b],SUNIndex[l],SUNIndex[k]]
   +SUNF[SUNIndex[a],SUNIndex[l],SUNIndex[k]]
   SUNF[SUNIndex[b],SUNIndex[c],SUNIndex[k]]);
```

```
87 FFSumRule=
   SUNF[SUNIndex[a_],SUNIndex[b_],SUNIndex[k_]]
   SUNF[SUNIndex[k_],SUNIndex[c_],SUNIndex[l_]]:>
   2/3 (SUNDelta[SUNIndex[a],SUNIndex[c]]
   SUNDelta[SUNIndex[b],SUNIndex[l]]
   -SUNDelta[SUNIndex[a],SUNIndex[l]]
   SUNDelta[SUNIndex[b],SUNIndex[c]])
   +SUND[SUNIndex[a],SUNIndex[c],SUNIndex[k]]
   SUND[SUNIndex[b],SUNIndex[l],SUNIndex[k]]
   -SUND[SUNIndex[a],SUNIndex[l],SUNIndex[k]]
   SUND[SUNIndex[b],SUNIndex[c],SUNIndex[k]];
```

The following rule represents the expressions (3.6)-(3.8):

```
88 Epsilon3Rule=
   Eps[LorentzIndex[α_],LorentzIndex[β_],Momentum[p],Momentum[r]]:>
   -Eps[LorentzIndex[α],LorentzIndex[β],Momentum[p],Momentum[q]],
   Eps[LorentzIndex[α_],LorentzIndex[β_],Momentum[q],Momentum[r]]:>
   Eps[LorentzIndex[α],LorentzIndex[β],Momentum[p],Momentum[q]],
   Eps[LorentzIndex[α_],Momentum[p],Momentum[q],Momentum[r]]:>0;
```

E.4 Vertex functions

For comfort, we have defined the following vertex functions that already represent Feynman rules for the relevant vertices. The following functions are defines in the file

```
02-mercury-vertices.nb
```

An important fact to mention is that all vertex functions below are defined in the full agreement with the designation of the vertices shown in Appendix D. Also, when it is necessary to explicitly emphasize the demand on the 4-momenta conservation, we use $-p-q$ instead of r.

Command	Vertex
`ChPTp2Vertex1[μ,a,p,ν,b,q]`	(D.24)
`ChPTp2Vertex2[μ,a,p,b,-p]`	(D.26)
`ChPTp2Vertex3[μ,a,p,ν,b,q,c,-p-q]`	(D.28)
`ChPTp2Vertex4[μ,a,p,b,q,c,r]`	(D.30)
`ChPTp2Vertex8[a,p,b,-p]`	(D.32)
`ChPTp2Vertex9[a,p,b,q,c,r]`	(D.34)

Table E.4: Vertex functions for the χPT vertices up to $\mathcal{O}(p^2)$.

Command	Vertex
`ChPTp4WZWVertex1[μ,a,p,ν,b,q,c,r]`	(D.71)
`ChPTp4WZWVertex2[μ,a,p,ν,b,q,c,r]`	(D.73)

Table E.5: Vertex functions for the Wess-Zumino-Witten χPT vertices up to $\mathcal{O}(p^4)$.

Command	Vertex
`RchTp6VectorVertex1[μ,a,p,α,b,-p]`	(D.77)
`RchTp6VectorVertex2[μ,a,p,ν,b,q,α,c,r]`	(D.79)
`RchTp6VectorVertex3[μ,a,p,α,b,q,c,r]`	(D.81)
`RchTp6VectorVertex4[α,a,p,b,q,c,r]`	(D.83)
`RchTp6VectorVertex5[μ,a,p,b,q,α,c,r]`	(D.85)
`RchTp6VectorVertex6[a,p,α,b,q,c,r]`	(D.87)
`RchTp6VectorVertex11[μ,a,p,ν,b,q,α,c,r]`	(D.89)
`RchTp6VectorVertex12[μ,a,p,α,b,q,c,r]`	(D.91)
`RchTp6VectorVertex13[μ,a,p,α,b,-p]`	(D.93)
`RchTp6VectorVertex14[μ,a,p,ν,b,q,α,c,r]`	(D.95)
`RchTp6VectorVertex15[μ,a,p,α,b,q,c,r]`	(D.97)
`RchTp6VectorVertex20[μ,a,p,ν,b,q,α,c,r]`	(D.99)
`RchTp6VectorVertex21[μ,a,p,α,b,q,c,r]`	(D.101)

Table E.6: Vertex functions for the RχT vertices in the vector formalism up to $\mathcal{O}(p^6)$.

Command	Vertex
`RchTp4TensorVertex1[μ,a,p,α,β,b,-p]`	(D.107)
`RchTp4TensorVertex2[μ,a,p,ν,b,q,α,β,c,-p-q]`	(D.109)
`RchTp4TensorVertex3[μ,a,p,ν,b,q,α,β,c,-p-q]`	(D.111)
`RchTp4TensorVertex4[μ,a,p,α,β,b,q,c,r]`	(D.113)
`RchTp4TensorVertex7[α,β,a,p,b,q,c,r]`	(D.115)

RchTp4TensorVertex8[μ,a,p,α,β,b,-p]	(D.117)
RchTp4TensorVertex9[μ,a,p,ν,b,q,α,β,c,-p-q]	(D.119)
RchTp4TensorVertex10[μ,a,p,α,β,b,q,c,r]	(D.121)
RchTp4TensorVertex14[a,p,b,q,c,r]	(D.123)
RchTp4TensorVertex15[μ,a,p,ν,b,q,c,r]	(D.125)
RchTp4TensorVertex16[μ,a,p,b,q,c,r]	(D.127)
RchTp4TensorVertex19[a,p,b,-p]	(D.129)
RchTp4TensorVertex20[a,p,b,q,c,r]	(D.131)
RchTp4TensorVertex22[a,p,b,-p]	(D.133)
RchTp4TensorVertex23[a,p,b,q,c,r]	(D.135)

Table E.7: Vertex functions for the RχT vertices in the antisymmetric tensor formalism up to $\mathcal{O}(p^4)$.

Command	Vertex
RchTp6TensorVertex5[μ,a,p,b,q,α,β,c,r]	(D.140)
RchTp6TensorVertex6[a,p,b,q,α,β,c,r]	(D.142)
RchTp6TensorVertex10[μ,a,p,α,β,b,q,ρ,c,r]	(D.151)
RchTp6TensorVertex11[μ,a,p,α,β,b,q,c,r]	(D.158)
RchTp6TensorVertex12[μ,a,p,α,β,b,q,c,r]	(D.160)
RchTp6TensorVertex18[μ,a,p,ν,b,q,α,β,c,r]	(D.169)
RchTp6TensorVertex19[μ,a,p,α,β,b,q,c,r]	(D.178)
RchTp6TensorVertex20[α,β,a,p,b,q,c,r]	(D.180)
RchTp6TensorVertex24[μ,a,p,b,q,α,β,c,r]	(D.185)
RchTp6TensorVertex25[a,p,α,β,b,q,c,r]	(D.187)
RchTp6TensorVertex29[μ,a,p,b,q,α,β,c,r]	(D.189)
RchTp6TensorVertex30[μ,a,p,ν,b,q,c,r]	(D.191)
RchTp6TensorVertex39[μ,a,p,ν,b,q,c,r]	(D.193)
RchTp6TensorVertex43[μ,a,p,ν,b,q,c,r]	(D.195)
RchTp6TensorVertex44[a,p,α,β,b,q,γ,δ,c,r]	(D.197)
RchTp6TensorVertex45[μ,a,p,α,β,b,q,γ,δ,c,r]	(D.202)
RchTp6TensorVertex46[a,p,α,β,b,q,γ,δ,c,r]	(D.207)
RchTp6TensorVertex47[a,p,α,β,b,q,γ,δ,c,r]	(D.209)
RchTp6TensorVertex48[μ,a,p,α,β,b,q,γ,δ,c,r]	(D.214)
RchTp6TensorVertex49[a,p,α,β,b,q,γ,δ,c,r]	(D.219)
RchTp6TensorVertex50[μ,a,p,α,β,b,q,c,r]	(D.221)
RchTp6TensorVertex54[μ,a,p,α,β,b,q,c,r]	(D.226)
RchTp6TensorVertex55[α,β,a,p,b,q,c,r]	(D.228)
RchTp6TensorVertex59[μ,a,p,α,β,b,q,γ,δ,c,r]	(D.230)
RchTp6TensorVertex60[a,p,α,β,b,q,γ,δ,c,r]	(D.232)
RchTp6TensorVertex61[μ,a,p,α,β,b,q,c,r]	(D.237)
RchTp6TensorVertex62[a,p,α,β,b,q,c,r]	(D.239)
RchTp6TensorVertex66[μ,a,p,α,β,b,q,c,r]	(D.241)
RchTp6TensorVertex67[α,β,a,p,γ,δ,b,q,c,r]	(D.243)
RchTp6TensorVertex68[α,β,a,p,γ,δ,b,q,c,r]	(D.245)
RchTp6TensorVertex69[α,β,a,p,γ,δ,b,q,c,r]	(D.247)

Table E.8: Vertex functions for the RχT vertices in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$.

E.5 Source code for Green functions

To conclude this chapter, we show here the source code for the calculations of all five nontrivial Green functions in the odd-intrinsic parity sector. The individual files can be found in the attachment:

```
03-vvp-rcht-tensor-p6.nb

04-vas-rcht-tensor-p6.nb

05-aap-rcht-tensor-p6.nb

06-vva-rcht-tensor-p6.nb

07-aaa-rcht-tensor-p6.nb
```

VVP Green function in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

```
1 I SUNSimplify[Contract[RchTp6TensorVertex12[μ,a,p,α,β,d,q,c,r]
  SubDiag3[q,ν,α,β,b,d]]]

2 I SUNSimplify[Contract[RchTp6TensorVertex12[ν,b,q,α,β,d,p,c,r]
  SubDiag3[p,μ,α,β,a,d]]]

3 I SUNSimplify[Contract[RchTp6TensorVertex39[μ,a,p,ν,b,q,d,r]
  SubDiag5[r,c,d]]]

4 I SUNSimplify[Contract[RchTp6TensorVertex11[μ,a,p,α,β,d,q,e,r]
  SubDiag3[q,ν,α,β,b,d]SubDiag4[r,c,e]]]

5 I SUNSimplify[Contract[RchTp6TensorVertex11[ν,b,q,α,β,d,p,e,r]
  SubDiag3[p,μ,α,β,a,d]SubDiag4[r,c,e]]]

6 I SUNSimplify[Contract[RchTp6TensorVertex66[μ,a,p,α,β,d,q,e,r]
  SubDiag3[q,ν,α,β,b,d]SubDiag5[r,c,e]]]

7 I SUNSimplify[Contract[RchTp6TensorVertex66[ν,b,q,α,β,d,p,e,r]
  SubDiag3[p,μ,α,β,a,d]SubDiag5[r,c,e]]]

8 I SUNSimplify[Contract[RchTp6TensorVertex44[c,r,α,β,d,p,γ,δ,e,q]
  SubDiag3[q,ν,γ,δ,b,e]SubDiag3[p,μ,α,β,a,d]]]

9 I SUNSimplify[Contract[RchTp6TensorVertex46[f,r,α,β,d,p,γ,δ,e,q]
  SubDiag3[p,μ,α,β,a,d]SubDiag3[q,ν,γ,δ,b,e]SubDiag4[r,c,f]]]

10 I SUNSimplify[Contract[RchTp6TensorVertex67[α,β,d,p,γ,δ,e,q,f,r]
  SubDiag3[p,μ,α,β,a,d]SubDiag3[q,ν,γ,δ,b,e]SubDiag5[r,c,f]]]
```

VAS Green function in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

```
1 I SUNSimplify[Contract[RchTp6TensorVertex5[ν,b,q,c,r,α,β,d,p]
  SubDiag3[p,μ,α,β,a,d]]]

2 I SUNSimplify[Contract[RchTp6TensorVertex29[μ,a,p,c,r,α,β,d,q]
  SubDiag2[q,ν,α,β,b,d]]]

3 I SUNSimplify[Contract[RchTp6TensorVertex43[ν,b,q,μ,a,p,d,r]
  SubDiag6[r,c,d]]]
```

4 I SUNSimplify[Contract[RchTp6TensorVertex6[c,r,e,q,α,β,d,p]
SubDiag1[q,ν,b,e]SubDiag3[p,μ,α,β,a,d]]]

5 I SUNSimplify[Contract[RchTp6TensorVertex50[μ,a,p,α,β,d,q,e,r]
SubDiag2[q,ν,α,β,d,b]SubDiag6[r,c,e]]]

6 I SUNSimplify[Contract[RchTp6TensorVertex54[ν,b,q,α,β,d,p,e,r]
SubDiag3[p,μ,α,β,a,d]SubDiag6[r,c,e]]]

7 I SUNSimplify[Contract[RchTp6TensorVertex60[c,r,γ,δ,e,q,α,β,d,p]
SubDiag2[q,ν,γ,δ,b,e]SubDiag3[p,μ,α,β,a,d]]]

8 I SUNSimplify[Contract[RchTp6TensorVertex55[α,β,d,p,f,r,e,q]
SubDiag1[q,ν,b,e]SubDiag3[p,μ,α,β,a,d]SubDiag6[r,c,f]]]

9 I SUNSimplify[Contract[RchTp6TensorVertex68[α,β,d,p,γ,δ,e,q,f,r]
SubDiag2[q,ν,γ,δ,b,e]SubDiag3[p,μ,α,β,a,d]SubDiag6[r,c,f]]]

AAP Green function in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

1 I SUNSimplify[Contract[RchTp6TensorVertex24[μ,a,p,c,r,α,β,d,q]
SubDiag2[q,ν,α,β,b,d]]]

2 I SUNSimplify[Contract[RchTp6TensorVertex24[ν,b,q,c,r,α,β,d,p]
SubDiag2[p,μ,α,β,a,d]]]

3 I SUNSimplify[Contract[RchTp6TensorVertex30[μ,a,p,ν,b,q,d,r]
SubDiag5[r,c,d]]]

4 I SUNSimplify[Contract[RchTp6TensorVertex19[μ,a,p,α,β,d,q,e,r]
SubDiag2[q,ν,α,β,b,d]SubDiag4[r,c,e]]]

5 I SUNSimplify[Contract[RchTp6TensorVertex19[ν,b,q,α,β,d,p,e,r]
SubDiag2[p,μ,α,β,a,d]SubDiag4[r,c,e]]]

6 I SUNSimplify[Contract[RchTp6TensorVertex25[c,r,α,β,d,p,e,q]
SubDiag1[q,ν,b,e]SubDiag2[p,μ,α,β,a,d]]]

7 I SUNSimplify[Contract[RchTp6TensorVertex25[c,r,α,β,d,q,e,p]
SubDiag1[p,μ,a,e]SubDiag2[q,ν,α,β,b,d]]]

8 I SUNSimplify[Contract[RchTp6TensorVertex61[μ,a,p,α,β,d,q,e,r]
SubDiag2[q,ν,α,β,b,d]SubDiag5[r,c,e]]]

9 I SUNSimplify[Contract[RchTp6TensorVertex61[ν,b,q,α,β,d,p,e,r]
SubDiag2[p,μ,α,β,a,d]SubDiag5[r,c,e]]]

10 I SUNSimplify[Contract[RchTp6TensorVertex47[c,r,α,β,d,p,γ,δ,e,q]
SubDiag2[p,μ,α,β,a,d]SubDiag2[q,ν,γ,δ,b,e]]]

11 I SUNSimplify[Contract[RchTp6TensorVertex20[α,β,d,p,f,r,e,q]
SubDiag1[q,ν,b,e]SubDiag2[p,μ,α,β,a,d]SubDiag4[r,c,f]]]

12 I SUNSimplify[Contract[RchTp6TensorVertex20[α,β,d,q,f,r,e,p]
SubDiag1[p,μ,a,e]SubDiag2[q,ν,α,β,b,d]SubDiag4[r,c,f]]]

13 I SUNSimplify[Contract[RchTp6TensorVertex62[d,p,α,β,e,q,f,r]
SubDiag1[p,μ,a,d]SubDiag2[q,ν,α,β,b,e]SubDiag5[r,c,f]]]

14 I SUNSimplify[Contract[RchTp6TensorVertex62[d, q,α,β,e,p,f,r] SubDiag1[q,ν,b,d]SubDiag2[p,μ,α,β,a,e]SubDiag5[r,c,f]]]

15 I SUNSimplify[Contract[RchTp6TensorVertex49[f,r,α,β,d,p,γ,δ,e,q] SubDiag2[p,μ,α,β,a,d]SubDiag2[q,ν,γ,δ,b,e]SubDiag4[r,c,f]]]

16 I SUNSimplify[Contract[RchTp6TensorVertex69[α,β,d,p,γ,δ,e,q,f,r] SubDiag2[p,μ,α,β,a,d]SubDiag2[q,ν,γ,δ,b,e]SubDiag5[r,c,f]]]

VVA Green function in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

1 SUNSimplify[Contract[RchTp6TensorVertex10[μ,a,p,α,β,d,q,ρ,c,r] SubDiag3[q,ν,α,β,b,d]]]

2 SUNSimplify[Contract[RchTp6TensorVertex10[ν,b,q,α,β,d,p,ρ,c,r] SubDiag3[p,μ,α,β,a,d]]]

3 SUNSimplify[Contract[RchTp6TensorVertex11[μ,a,p,α,β,b,q,e,r] SubDiag1[r,ρ,c,e]SubDiag3[q,ν,α,β,b,d]]]

4 SUNSimplify[Contract[RchTp6TensorVertex11[ν,b,q,α,β,d,p,e,r] SubDiag1[r,ρ,c,e]SubDiag3[p,μ,α,β,a,d]]]

5 SUNSimplify[Contract[RchTp6TensorVertex45[ρ,c,r,α,β,d,p,γ,δ,e,q] SubDiag3[p,μ,α,β,a,d]SubDiag3[q,ν,γ,δ,b,e]]]

6 SUNSimplify[Contract[RchTp6TensorVertex46[f,r,α,β,d,p,γ,δ,e,q] SubDiag1[r,ρ,c,f]SubDiag3[p,μ,α,β,a,d]SubDiag3[q,ν,γ,δ,b,e]]]

7 SUNSimplify[Contract[RchTp6TensorVertex59[μ,a,p,α,β,d,q,γ,δ,e,r] SubDiag2[r,ρ,γ,δ,c,e]SubDiag3[q,ν,α,β,b,d]]]

8 SUNSimplify[Contract[RchTp6TensorVertex59[ν,b,q,α,β,d,p,γ,δ,e,r] SubDiag2[r,ρ,γ,δ,c,e]SubDiag3[p,μ,α,β,a,d]]]

AAA Green function in the antisymmetric tensor formalism up to $\mathcal{O}(p^6)$

1 SUNSimplify[Contract[RchTp6TensorVertex18[μ,a,p,ν,b,q,α,β,d,r] SubDiag2[r,ρ,α,β,c,d]]]

2 SUNSimplify[Contract[RchTp6TensorVertex18[μ,a,p,ρ,c,r,α,β,d,q] SubDiag2[q,ν,α,β,b,d]]]

3 SUNSimplify[Contract[RchTp6TensorVertex18[ν,b,q,ρ,c,r,α,β,d,p] SubDiag2[p,μ,α,β,a,d]]]

4 SUNSimplify[Contract[RchTp6TensorVertex19[μ,a,p,α,β,d,q,e,r] SubDiag1[r,ρ,c,e]SubDiag2[q,ν,α,β,b,d]]]

5 SUNSimplify[Contract[RchTp6TensorVertex19[μ,a,p,α,β,d,r,e,q] SubDiag1[q,ν,b,e]SubDiag2[r,ρ,α,β,c,d]]]

6 SUNSimplify[Contract[RchTp6TensorVertex19[ν,b,q,α,β,d,p,e,r] SubDiag1[r,ρ,c,e]SubDiag2[p,μ,α,β,a,d]]]

7 SUNSimplify[Contract[RchTp6TensorVertex19[ν,b,q,α,β,d,r,e,p] SubDiag1[p,μ,a,e]SubDiag2[r,ρ,α,β,c,d]]]

```
8  SUNSimplify[Contract[RchTp6TensorVertex19[ρ,c,r,α,β,d,p,e,q]
   SubDiag1[q,ν,b,e]SubDiag2[p,μ,α,β,a,d]]]

9  SUNSimplify[Contract[RchTp6TensorVertex19[ρ,c,r,α,β,d,q,e,p]
   SubDiag1[p,μ,a,e]SubDiag2[q,ν,α,β,b,d]]]

10 SUNSimplify[Contract[RchTp6TensorVertex48[μ,a,p,α,β,d,q,γ,δ,e,r]
   SubDiag2[q,ν,α,β,b,d]SubDiag2[r,ρ,γ,δ,c,e]]]

11 SUNSimplify[Contract[RchTp6TensorVertex48[ν,b,q,α,β,d,p,γ,δ,e,r]
   SubDiag2[p,μ,α,β,a,d]SubDiag2[r,ρ,γ,δ,c,e]]]

12 SUNSimplify[Contract[RchTp6TensorVertex48[ρ,c,r,α,β,d,p,γ,δ,e,q]
   SubDiag2[p,μ,α,β,a,d]SubDiag2[q,ν,γ,δ,b,e]]]

13 SUNSimplify[Contract[RchTp6TensorVertex49[f,p,α,β,d,q,γ,δ,e,r]
   SubDiag1[p,μ,a,f]SubDiag2[q,ν,α,β,b,d]SubDiag2[r,ρ,γ,δ,c,e]]]

14 SUNSimplify[Contract[RchTp6TensorVertex49[f,q,α,β,d,p,γ,δ,e,r]
   SubDiag1[q,ν,b,f]SubDiag2[p,μ,α,β,a,d]SubDiag2[r,ρ,γ,δ,c,e]]]

15 SUNSimplify[Contract[RchTp6TensorVertex49[f,r,α,β,d,p,γ,δ,e,q]
   SubDiag1[r,ρ,c,f]SubDiag2[p,μ,α,β,a,d]SubDiag2[q,ν,γ,δ,b,e]]]
```

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List of Abbreviations

QCD	Quantum chromodynamics
χPT	Chiral perturbation theory
OPE	Operator product expansion
RχT	Resonance chiral theory
WZW	Wess-Zumino-Witten
HLbL	Hadronic light-by-light
etc.	et cetera
i.e.	id est
c.f.	confer, compare

Attachments

The attached CD-ROM includes the following files:

1. This thesis in the PDF format.
2. The ZIP file of the Wolfram Mathematica 8 notebooks with the algorithm Mercury. The specifical notebooks are:
   - `01-mercury-definitions.nb`
   - `02-mercury-vertices.nb`
   - `03-vvp-rcht-tensor-p6.nb`
   - `04-vas-rcht-tensor-p6.nb`
   - `05-aap-rcht-tensor-p6.nb`
   - `06-vva-rcht-tensor-p6.nb`
   - `07-aaa-rcht-tensor-p6.nb`