Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Samuel Mokriš

Problém realizace von Neumannovsky regulárních okruhů

Katedra algebry

Vedoucí diplomové práce: Mgr. Pavel Růžička, Ph.D.
Studijní program: Matematika
Studijní obor: Matematické struktury

Praha 2015

Na tomto místě bych rád poděkoval vedoucímu práce za velmi vstřícný přístup a mnoho zajímavých podnětů. Děkuji Mgr. Dominiku Mokrišovi za jazykové a stylistické rady. Zejména bych si přál poděkovat doc. Pavlu Příhodovi, Ph.D., za nesčetné konzultace k tématu práce.

Prohlašuji, že jsem tuto diplomovou práci vypracoval samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

Beru na vědomí, že se na moji práci vztahují práva a povinnosti vyplývající ze zákona č. 121/2000 Sb., autorského zákona v platném znění, zejména skutečnost, že Univerzita Karlova v Praze má právo na uzavření licenční smlouvy o užití této práce jako školního díla podle §60 odst. 1 autorského zákona.

V Praze dne 31.7.2015 Samuel Mokriš

Název práce: Problém realizace von Neumannovsky regulárních okruhů

Autor: Samuel Mokriš

Katedra: Katedra algebry

Vedoucí diplomové práce: Mgr. Pavel Růžička, Ph.D., Katedra algebry

Abstrakt: Každému okruhu R s jednotkou lze přiřadit komutativní monoid $\mathcal{V}(R)$ tříd izomorfismů konečně generovaných pravých projektivních R-modulů. Příslušný monoid je redukovaný s jednotkou, v případě von neumannovsky regulárních okruhů má navíc Rieszovu zjemňovací vlastnost. Práce se zabývá otázkou, za jakých podmínek je naopak redukovaný komutativní zjemňovací monoid s jednotkou realizovatelný jako $\mathcal{V}(R)$ nějakého von neumannovsky regulárního okruhu či dokonce regulární algebry, zejména pro spočetné monoidy. Jsou uvedena dvě možná zobecnění konstrukce $\mathcal{V}(R)$ pro okruhy bez jednotky a je rozebrán vztah mezi nimi. Za tímto účelem jsou rozvíjeny vlastnosti okruhů s lokálními jednotkami a modulů nad takovými okruhy. Dále je v práci předvedena konstrukce leavittovských algeber cest nad orientovanými grafy s násobnými hranami a kontrukce monoidu asociovaného s grafem, který je izomorfní monoidu $\mathcal{V}(R)$ leavittovské algebry cest nad týmž grafem. Tyto metody jsou využity k předvedení, jak realizovat direktní sjednocení konečně generovaných volných komutativních monoidů jako $\mathcal{V}(R)$ pro regularní algebru nad libovolným tělesem. Rovněž je v práci prezentován způsob, jak konstruovat redukované komutativní zjemňovací monoidy, které nejsou realizovatelné jako $\mathcal{V}(R)$ pro regulární algebry nad žádným nespočetným tělesem. Na závěr práce je popsán monoid $\mathcal{V}(R)$ algebry R nad spočetným tělesem sestrojené Chuangem a Leem.

Klíčová slova: von neumannovsky regulární okruh, zjemňovací monoid, leavittovská algebra cest, moritovská ekvivalence okruhů bez jednotky

Title: The realization problem for von Neumann regular rings

Author: Samuel Mokriš

Department: Department of Algebra

Supervisor of the master thesis: Mgr. Pavel Růžička, Ph.D., Department of Algebra

Abstract: With every unital ring R, one can associate the abelian monoid $\mathcal{V}(R)$ of isomorphism classes of finitely generated projective right R-modules. Said monoid is a conical monoid with order-unit. Moreover, for von Neumann regular rings, it satisfies the Riesz refinement property. In the thesis, we deal with the question, under what conditions an abelian conical refinement monoid with order-unit can be realized as $\mathcal{V}(R)$ for some unital von Neumann regular ring or algebra, with emphasis on countable monoids. Two generalizations of the construction of $\mathcal{V}(R)$ to the context of nonunital rings are presented and their interrelation is analyzed. To that end, necessary properties of rings with local units and modules over such rings are developed. Further, the construction of Leavitt path algebras over quivers is presented, as well as the construction of a monoid associated with a quiver that is isomorphic to $\mathcal{V}(R)$ of the Leavitt path algebra over the same quiver. These methods are then used to realize directed unions of finitely generated free abelian monoids as $\mathcal{V}(R)$ of algebras over any given field. A method of constructing abelian conical refinement monoids that are not realizable as $\mathcal{V}(R)$ of regular algebras over any uncountable field is also presented. The thesis is concluded by computation of the monoid $\mathcal{V}(R)$ of an algebra R over a countable field, constructed by Chuang and Lee.

Keywords: von Neumann regular ring, refinement monoid, Leavitt path algebra, nonunital Morita equivalence

Contents

Introduction

With $\mathcal{V}(R)$ denoting the abelian monoid of finitely generated projective modules over a unital ring R, where the monoid operation is induced by direct sums, the following is still an outstanding open problem:

Fundamental open problem ((Goodearl, c1995, p. 254))**.** Which abelian monoids arise as $\mathcal{V}(R)$'s for unital von Neumann regular rings R?

The motivation of the problem is the effort to understand what direct sum decomposition properties hold in proj-R, the category of finitely generated projective R-modules, for unital von Neumann regular rings R. The transition to monoids is based on the premise that some pathological decomposition properties are constructed more easily in the language of monoids rather than von Neumann regular rings. Hence, should one fully aximoatize the monoids arising as $\mathcal{V}(R)$'s in monoid-theoretic terms, then there would be no need to construct rings when demontrating decomposition properties of the category proj-R (with R unital von Neumann regular).

We leave further discussion regarding the realization problem and some open problems related with it to Section 1.2. Given some recent results on nonunital von Neumann regular algebras, we state some variations on the problems to incorporate nonunital rings there. Before doing so, we shall write down all required monoid-related definitions in Section 1.1. The monoid-theoretic terminology used is standard; for convenience, we sum it up in one place nevertheless.

In Chapter 2, we shall discuss nonunital von Neumann regular rings and rings with local units. We will not need any class of nonunital rings more general than rings with local units due to the fact that every von Neumann regular ring has local units (Proposition 2.7). We define the category of unitary modules over a ring with local units and deal with nonunital Morita equivalence. Then we present two equivalent generalizations of the functor $\mathcal{V}(-)$ to the nonunital context and verify some of the functor's properties in detail. Namely, we show that it is continuous (Proposition 2.26), that every ring is mapped by $\mathcal{V}(-)$ to the same monoid as the opposite ring (Proposition 2.27), that the two generalizations are indeed equivalent (Proposition 2.34), and that Morita equivalent rings are mapped to the same monoid (Theorem 2.45).

In Chapter 3, we overview recent results on the so-called Leavitt path algebras that contribute greatly to solving the realization problem, as they yield a class of abelian monoids that can be realized even by von Neumann regular algebras over arbitrary fields. We use the construcion of Leavitt path algebras to realize the additive monoid of nonnegative rational numbers as $\mathcal{V}(-)$ of a nonunital von Neumann regular algebra in a nonstandard fashion (Subsection 3.2.2). We also show how any directed union of finitely generated free abelian monoids can be realized in a similar way.

Then, in Chapter 4, we present a criterion by Goodearl for an abelian monoid to *not* be realizable by von Neumann regular algebras over any uncountable fields (Proposition 4.11). We also present a way of constructing such monoids (Proposition 4.12).

In the final chapter, we compute the monoid $\mathcal{V}(R)$ for the regular unital ring R constructed in Chuang – Lee (1990), as, to our knowledge, this monoid has not been computed anywhere in the literature.

Some conventions

Throughout the thesis, a ring always means an associative ring, but not necessarily with a unit; we denote the category of (possibly nonunital) rings by $\mathscr{Rng}$. We allow the singleton $0 := \{ 0 \}$ as a nonunital ring; it is the zero object in $\mathscr{Rng}$. However, for unital rings (and for fields in particular), we assume that $0 \neq 1$.

A module will always mean a right module, unitary in the sense of Definition 2.8; nevertheless, we shall sometimes add the adjective "right" or "unitary" for emphasis. For a ring I, we denote the category of unitary right I-modules by $\operatorname{Mod}\text{-}I$. For a unital ring R, $\operatorname{proj}\text{-}R$ denotes the category of finitely generated projective right R-modules.

For a ring I, we denote by I^{op} the ring opposite to I, and by $\operatorname{Idemp} I$ the set of all idempotents from I. If $x \in I$, we denote by xI the set $\{ xr \mid r \in I \}$; we will see that if I has local units, then xI coincides with the principal right ideal in I generated by x (Remark 2.6).

For rings and algebras, "regular" will always mean "von Neumann regular" (Definition 2.1). All monoids in the thesis are abelian; hence, whenever we speak of monoids, we mean abelian monoids, and denote the monoid operation by $+$. The category of abelian monoids is denoted by $\mathscr{Mon}$. We use $\mathbb{N}_0$, $\mathbb{Q}^{\geq 0}$ and $\mathbb{R}^{+}$ to denote the additive monoids of nonnegative integers, nonnegative rational numbers and real numbers, respectively. We also use $\mathbb{N}_0$ as a set when we do not need its monoid structure, and we use $\mathbb{N}$ to denote the set of strictly positive integers (i.e., $\mathbb{N} = \mathbb{N}_0 \setminus \{ 0 \}$) and $\mathbb{Z}$ for the set of all integers. For I a ring and $n \in \mathbb{N}$, by $M_n(I)$ we mean the ring of $n \times n$ matrices over I.

Throughout the text, we use "iff" as an abbreviation of "if and only if", and we use "UMP" for the "universal mapping property" (of direct limits, kernels etc.). "SES" stands for "short exact sequence", and we write "w.l.o.g." in place of "without loss of generality".

We use $X \subseteq_{\mathrm{fin}} M$ to denote that X is a finite subset of M, regardless of any algebraic structure on M. We use the symbol $\dot{\cup}$ for disjoint unions.

By $\operatorname{Id}_{\mathscr{C}}$ we denote the identity functor on a category $\mathscr{C}$, while id_M will be the identity morphism of an algebraic object M (monoid, ring, module, algebra). Also, Id will denote the identity matrix, either finite or infinite; whenever we need to specify its size, an appropriate subscript is be added. The symbol $\operatorname{Ker}\varphi$ denotes the module-theoretic kernel of a ring or module homomorphism φ (in particular, it is a submodule of the domain of φ), while $\ker\varphi$ stands for the congruence generated by $\{ (a, b) \mid \varphi a = \varphi b \}$. We use the latter only for monoid homomorphisms.

We use the symbol $\square$ in the following contexts:

- to signify the end of a proof of a claim within another proof (in this context, $\square$ is followed by *Claim* and the number of the claim);
- to signify the end of a proof;
- to signify that a result taken from the literature is provided without proof;
- to signify that we will comment no further on the proof of a statement. This includes observations, direct corollaries of preceding results, and statements proofs of which have been hinted at enough prior to the statements in question.

Whenever we say that a set is countable, we mean it is either finite or countably infinite. If we need to stress that some set is of cardinality $\aleph_0$, we say it is countably infinite. In particular, when discussing quivers, we always imply that they are finite or countably infinite.

Chapter 1

The problem

1.1 Monoid properties used

Given a monoid M, putting $x \leq y$ for $x, y \in M$ iff there exists a $z \in M$ satisfying $x+z=y$ defines a preorder relation on M, called the *algebraic preorder*. An *order-unit* in a monoid M is an element $u \in M$ such that for each $x \in M$, there exists an $n \in \mathbb{N}$ satisfying $x \leq nu$[1] in the algebraic preorder (i.e., for each $x \in M$, there exist a $z \in M$ and an $n \in \mathbb{N}$ such that $x+z=nu$). An *order-ideal* in a monoid M is a submonoid S of M such that whenever $x \in S$ and $y \leq x$ in the algebraic preorder on M, then $y \in S$.

A monoid is *conical* if whenver $x+y=0$, then $x=0=y$.

Observation 1.1. *A finite subdirect product of conical monoids (that is, a submonoid of a direct product of finitely many conical monoids) is a conical monoid as well.* $\square$

A monoid M satisfies the *Riesz refinement property* if whenever $y_1+y_2=x_1+x_2$ is an equality of sums in M, then there exist elements $z_{11}, z_{12}, z_{21}, z_{22} \in M$ such that

$$x_1 = z_{11} + z_{12}, \qquad x_2 = z_{21} + z_{22},$$
$$y_1 = z_{11} + z_{21}, \qquad y_2 = z_{12} + z_{22}.$$

Instead of writing down the above equalities, we use the following matrix notation:

$$\begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & z_{11} & z_{12} \\ x_2 & z_{21} & z_{22} \end{array}. \tag{1.1}$$

We say that (1.1) is a refinement of the equality $y_1+y_2=x_1+x_2$ in M. A monoid satisfying the Riesz refinement property is called a *refinement monoid.*

Observation 1.2. *An order-ideal in a refinement monoid is a refinement monoid.* $\square$

[1] We use multiplication of an element u of an abelian monoid by an element $n \in \mathbb{N}_0$ as a short-hand notation for taking the sum of n copies of u.

A monoid M is *unperforated* if whenever $nx \leq ny$ holds for $x, y \in M$ and $n \in \mathbb{N}$, then $x \leq y$ also holds. A *cancellative monoid* is a monoid M such that whenever $x, y, z \in M$ satisfy $x+z = y+z$, then $x = y$ holds. A *separative monoid* is a monoid M such that whenever $x, y \in M$ satisfy $x + x = x + y = y + y$, then $x = y$ holds.

A monoid M is *stably finite* if for all $x, y \in M$, $x + y = x$ implies that $y = 0$.

1.2 The realization problem and some of its variations

We say that a monoid is *realizable* or *realized* by R if it is isomorphic to the monoid $\mathcal{V}(R)$. In order for a monoid M to be realizable by some unital regular ring R, there are—apart from M being abelian—three known necessary conditions:

(V1) M is conical by Proposition 2.36;

(V2) M is a refinement monoid by Proposition 2.37; and

(V3) M has an order-unit by Observation 2.38.

Hence, a question to ask is whether all conical refinement monoids with order-unit are realizable by unital regular rings. This would be in analogy with (Bergman, 1974, Theorem 6.4), where it is shown that every conical abelian monoid with order-unit is realizable by an algebra over any given field. However, in the context of regular rings, not all monoids satisfying (V1), (V2) and (V3) are realizable: It is proved in Wehrung (1998) that there exists a conical refinement monoid of cardinality $\aleph_2$ with order-unit that cannot be realized by any unital regular ring (Wehrung, 1998, Corollary 2.12 and the paragraph following it). Still, it is of interest what the situation in smaller cardinalities is. Hence, the question is the following:

Open problem 1.3 ((Ara, c2009, Realization problem for von Neumann regular rings))**.** Is every countable conical refinement monoid realizable by a regular ring?

Let us take note that there is no mention of the monoids having order-unit or the rings being unital in Open problem 1.3; we will get back to this in a little while.

As Bergman's realization result actually yields algebras over arbitrary fields, there is a natural variation on Open problem 1.3.

Problem 1.4 ((Ara, c2009, Realization problem for von Neumann regular K-algebras))**.** Let K be a fixed field. Is every countable conical refinement monoid realizable by a von Neumann regular K-algebra?

For uncountable fields, Problem 1.4 has already been answered in the negative, as shown in Proposition 4.16—in fact, we devote Chapter 4 to developing ways of constructing countable conical refinement monoids not realizable by regular algebras over uncountable fields. However, for the monoids constructed in Chapter 4, we do not a priori know whether they can be realized by regular rings or regular algebras over countable fields. For countable fields, Problem 1.4 is still open.

Opposed to results from Chapter 4, by recent results of Ara et al. (2007) and Ara – Brustenga (2007), there is a class of countable conical refinement monoids realizable by unital regular algebras over any given field; we discuss the results in Chapter 3. With a suitable generalization of $\mathcal{V}(I)$ to incorporate nonunital rings I (see Section 2.3), said results include a wider class of monoids that are realizable by (in general nonunital) regular algebras over arbitrary fields. Due to condition (V3), monoids without order-unit cannot be realized by unital rings; nevertheless, there are such monoids that are realizable by nonunital regular algebras, see e.g. Example 3.4. Not including conditions of order-units and unital rings in Open problem 1.3 thus makes the problem more general.

Since results presented in Chapter 3 also include realizations of monoids *with order-units* by nonunital regular algebras, while we do not know whether they are realizable by unital ones (Example 3.18), we state the following variation on Open problem 1.3:

Problem 1.5. Is there a countable conical refinement monoid with order-unit that is realizable by a nonunital regular ring, but not by a unital regular ring?

In Goodearl (c1995), more questions on general properties of $\mathcal{V}(R)$ and proj-R are raised, e.g. whether all $\mathcal{V}(R)$'s are separative. We mention this problem only briefly in connection with Chapter 3; more on recent results on problems from Goodearl (c1995) can be found in the survey Ara (c2009).

Chapter 2

Regular rings and rings with local units

Definitions in generalizations of the standard module theory for modules over nonunital rings vary from author to author. In this place, we develop some of the theory for modules over rings with local units that we need manually.

Definition 2.1. A ring I is *regular* if for every $x \in I$, there exists a $y \in I$ such that $xyx = x$. Such (in general not unique) element y is called a *quasi-inverse* of x.

Remark 2.2. As opposed to unital rings, for a general ring I, the set $xI = \{ xr \mid r \in I \}$ need not contain x as its element. However, this is not the case when I is regular: For, if $x, y \in I$ satisfy $xyx = x$, then $x = x(yx) \in xI$. Since the set xI is closed under addition in I, we see that it is a right ideal in I containing x. The element x cannot be contained in any strictly smaller ideal (inclusion-wise), whence xI is the principal right ideal in I generated by x. Moreover, as $xyxy = xy$, the element xy is an idempotent in I, and it is clear that $xI = xyI$. Thus, in a regular ring, every principal right ideal is generated by an idempotent. Similarly to unital regular rings, this extends to finitely generated right ideals:

Proposition 2.3. *For a regular ring I, each finitely generated right ideal in I is principal, and as such generated by an idempotent.*

Proof. Due to Remark 2.2, we can verbatim use the proof of implication (b)⇒(c) from (Goodearl, 1979, Theorem 1.1). □

Since the definition of regular rings is left-right symmetric, Proposition 2.3 also holds if we replace the word "right" with "left".

Corollary 2.4. *For a unital regular ring R, every finitely generated right ideal in R is a projective R-module.*

Proof. With the finitely generated right ideal expressible as eR by Proposition 2.3, we see that $R = eR \oplus (1 - e)R$. □

Definition 2.5. A ring I is a *ring with local units* provided that for every finite subset X of I, there is an $e \in \operatorname{Idemp} I$ such that X is contained in eIe. Note that $X \subseteq eIe$ iff $ex = x = xe$ holds for all $x \in X$).

Remark 2.6. If x is an element of a ring I with local units, then applying Definition 2.5 to the set $\{ x \} \subseteq I$ yields in particular that $x \in xI$. Similarly to the case of regular rings, we can thus write xI for the principal right ideal generated by x. Also, note that in any ring I, the empty set is contained in the set $0 = 0I0$. Hence, whenever inquiring whether a ring has local units, we only check the defining conditions for nonempty finite subsets of I.

Proposition 2.7. *Every regular ring is a ring with local units.*

Proof. Let I be regular and let $x_1, \ldots, x_n \in I$ with $n \geq 1$. By Proposition 2.3, there is an idempotent g in I satisfying $gI = x_1 I + \ldots x_n I$. Then $gx_i = x_i$ holds for all $i = 1, \ldots, n$. From the version of Proposition 2.3 for left ideals, there is an idempotent f in I satisfying $gf = g$ and $x_i f = x_i$ for all i. Put $e := f + g - fg$; it follows from $gf = g$ and from f and g being idempotent that $e^2 = f^2 + fg - f^2 g + gf + g^2 - gfg - gf^2 - gfg + gfgf = e$. For any i, we have:

$$ex_i = fx_i + gx_i - fgx_i = fx_i + x_i - fx_i = x_i = x_i f + x_i g - x_i fg = x_i e.$$

Thus, given any $X \subseteq_{\text{fin}} I$, there is an idempotent $e \in R$ satisfying $X \subseteq eIe$, as desired. ☐

2.1 Unitary modules over nonunital rings

Once we lose the condition that rings be unital, we shall seek a replacement for the condition that modules be unitary (in the classical sense that $m \cdot 1_R = m$ for all elements m of a right R-module); for, by leaving it out without any replacement, the category of I-modules would become uncomfortably large (e.g., every abelian group with zero multiplication would be an I-module). To obviate this inconvenience, there is a generalization of the classical unitarity condition:

Definition 2.8. A (right) module M over a ring I is *unitary* provided that $MI = M$, that is, for every $m \in M$ there are $m_1, \ldots, m_n \in M$ and $r_1, \ldots, r_n \in I$ such that $m = m_1 r_1 + \cdots + m_n r_n$. By Mod-$I$, we shall denote the category of unitary right I-modules.

Remark 2.9. If R is a unital ring, if M is a unitary R-module in the sense of Definition 2.8 and if $m \in M$, we can write $m = \sum m_j r_j$, so

$$m \cdot 1_R = \left(\sum m_j r_j\right) \cdot 1_R = \sum m_j \cdot (r_j \cdot 1_R) = \sum m_j r_j = m$$

holds. Thus, for unital rings, unitary modules in the sense of Definition 2.8 are precisely those that are unitary in the classical sense.

Lemma 2.10. *If I is a ring with local units, then an I-module M is unitary if and only if for each $m \in M$, there is an idempotent $e \in I$ such that $me = m$.*

Proof. The if-part is clear. For the only-if-part, let $m \in M = MI$. Then $m = \sum m_j r_j$. Since I has local units, we can find an idempotent e satisfying $r_j e = r_j$ for all j. Then,

$$m \cdot e = \left(\sum m_j r_j\right) e = \sum m_j (r_j e) = \sum m_j r_j = m. \qquad \square$$

From now on, unless stated otherwise, by a module we always mean a *unitary* module.

Very much like in the category Mod-R with R a unital ring, we may define some categorical terms in Mod-I with I a ring with local units as concrete modules and morphisms; namely, we define kernels, images, cokernels and the zero module in Mod-I as is standard in Mod-R. It is easy to check that a morphism in Mod-I is injective iff its kernel is zero, and this happens iff the morphism is a monomorphism. We give a proof that epimorphisms are precisely surjective homomorphisms:

Lemma 2.11. *Let I be a ring and $A \xrightarrow{\varphi} B$ a morphism in* Mod-I. *Then φ is an epimorphism in* Mod-I *iff* $\operatorname{Im}\varphi = B$.

Proof. First, suppose that φ is onto. Let us have

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & C \end{tikzcd}$$

in Mod-I with $\alpha\varphi = \beta\varphi$. Then $\alpha|_{\operatorname{Im} f} = \beta|_{\operatorname{Im} f}$; since $\operatorname{Im}\varphi = B$, we conclude that $\alpha = \beta$. We have thus shown that φ is an epimorphism.

For the converse, let φ be an epimorphism in Mod-I. Let $\pi : B \longrightarrow B/\operatorname{Im}\varphi$ be the canonical projection; then π is onto and $\pi\varphi = 0$. We then have

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, bend left, "\pi"] \arrow[r, bend right, "0"'] & B/\operatorname{Im}\varphi \end{tikzcd}$$

in Mod-I with $\pi\varphi = 0 = 0\varphi$. Since φ is an epimorphism, it follows that $\pi = 0$. As π is onto, we conclude that $\operatorname{Im}\varphi = B$. $\square$

Also, Mod-I is an exact abelian category (in the sense of (Mitchell, 1965, §I.15 and §I.20)), and we note that $\varphi : A \longrightarrow B$ is a monomorphism iff the sequence $0 \longrightarrow A \xrightarrow{\varphi} B$ is exact, and φ is an epimorphism iff $A \xrightarrow{\varphi} B \longrightarrow 0$ is exact.

Lemma 2.12. *If I is a ring with local units and $M \in$* Mod-I, *then there is a surjective homomorphism $I^{(A)} \xrightarrow{f} M$ for some set A, i.e., every I-module is the epimorphic image of a direct sum of copies of I.*

Proof. For each $m \in M$, the map $I \xrightarrow[x \mapsto mx]{f_m} M$ is an I-module homomorphism. Since M is unitary, there is, by Lemma 2.10, an idempotent $e \in I$ satisfying $m = me = f_m(e)$, so m is in the image of f_m. Take $f := \bigoplus_{m \in M} f_m : I^{(M)} \longrightarrow M$. $\square$

2.2 Morita equivalence for rings with local units

Definition 2.13. We say that rings I, J are *Morita equivalent* if the categories Mod-I, Mod-J are additively equivalent, i.e., there are additive functors G : Mod-I $\longrightarrow$ Mod-J and H : Mod-J $\longrightarrow$ Mod-I such that GH is naturally equivalent to $\mathrm{Id}_{\text{Mod-}J}$ and HG is naturally equivalent to $\mathrm{Id}_{\text{Mod-}I}$.

Remark 2.14. Our definition of Mod-I, and thus also the definition of Morita equivalence, follows Ánh – Márki (1987). For nonunital rings, some authors use other definitions of Mod-I, which carry over to Morita equivalence meaning equivalence of a different pair of categories—e.g., in Goodearl (2009), Mod-I denotes the category of unitary modules in our sense (albeit they are called "full" instead) that are also "nondegenerate", meaning that in any module, 0 is its only element x satisfying $xI = 0$. Thus, when using results from the literature, we have to take heed of the definitions that the particular author uses.

Proposition 2.15. *Let I, J be rings with local units Morita equivalent via $G : \text{Mod-}I \longrightarrow \text{Mod-}J$ and $H : \text{Mod-}J \longrightarrow \text{Mod-}I$. Then the functors G and H preserve direct sums.*

Proof. Let, say, $(M_\alpha | \alpha \in A)$ be a system of I-modules. Then, using the natural equivalences $HG \simeq \mathrm{Id}_{\text{Mod-}I}$ and $GH \simeq \mathrm{Id}_{\text{Mod-}J}$ and the coproduct UMP of $\bigoplus_{\alpha\in A} M_\alpha$ in Mod-I, one verifies that $G\left(\bigoplus_{\alpha\in A} M_\alpha\right)$ satisfies the UMP for coproduct of the system $(GM_\alpha | \alpha \in A)$ in Mod-J. Thus, $G\left(\bigoplus_{\alpha\in A} M_\alpha\right) \simeq \bigoplus_{\alpha\in A} GM_\alpha$. ☐

In the process of proving Theorem 2.45, we shall need the fact that, in analogy with the unital case, the functors G, H in a Morita equivalence are necessarily exact. Due to Lemma 2.20 (taken from Ánh – Márki (1987)), we only need to be concerned by the tensor functor and by a variation on the Hom functor (for the variation and its justification, see Remark 2.18). Let us begin with the tensor functor:

For I, J rings and M_I, ${}_IN_J$ (unitary) modules, there is a right J-module structure on $M \otimes_I N$ given by $(m \otimes n) \cdot j := m \otimes (n \cdot j)$; this is proved as in (Anderson – Fuller, 1992, Proposition 19.5) for modules over unital rings, as multiplication by 1 does not occur in the proof. We only need to check that $M \otimes_I N$ is, as a J-module, unitary. To that end, note that as an abelian group, $(M \otimes N)J$ is generated by elements $(m \otimes n)j$ with $m \in M$, $n \in N$ and $j \in J$. As N is a unitary J-module, its additive group is generated by elements nj with $n \in N$ and $j \in J$; in turn, the additive group $M \otimes (NJ) = M \otimes N$ is generated by elements $m \otimes (nj)$ $(m \in M, n \in N, j \in J)$. Thus, as additive groups, $M \otimes N$ and $(M \otimes N)J$ are generated by the same sets of elements, whence they coincide.

Hence, with ${}_IN_J$ a unitary bimodule, sending $M \in \text{Mod-}I$ to $M \otimes_I N$ is a map of objects of Mod-I to objects of Mod-J. As in (Anderson – Fuller, 1992, Theorem 19.10), one shows that:

Proposition 2.16. *Let I, J be rings and ${}_IN_J$ a bimodule. Then*

$$- \otimes_I N : \text{Mod-}I \longrightarrow \text{Mod-}J$$

is an additive functor. ☐

Now that we have a well-defined tensor functor between relevant categories, we shall prove that it is an epifunctor:

Lemma 2.17. *The functor $- \otimes N$ of Proposition 2.16 preserves epimorphisms.*

Proof. Let $A_I \xrightarrow{\varphi} B_I \longrightarrow 0$ be exact in Mod-I; we want to prove that then $A \otimes_I N \xrightarrow{\varphi\otimes_I \mathrm{id}_N} B \otimes_I N$ is an epimorphism in Mod-J.

Let $b \in B$, $n \in N$. By surjectivity of φ (Lemma 2.11), $b = \varphi a$ for some $a \in A$. Then

$$b \otimes n = (\varphi a) \otimes n = (\varphi \otimes_I \mathrm{id}_N)(a \otimes n) \in \mathrm{Im}\,\varphi \otimes_I \mathrm{id}_N .$$

Thus, the set $\{\, b \otimes n \in B \otimes_I N \mid b \in B, n \in N \,\}$ generating $B \otimes_I N$ is a subset of $\mathrm{Im}\,\varphi \otimes_I \mathrm{id}_N$, whence $\varphi \otimes_I \mathrm{id}_N$ is onto. Consequently, by Lemma 2.11, the morphism $\varphi \otimes_I \mathrm{id}_N$ is an epimorphism in Mod-J. ☐

Remark 2.18. For $B \in$ Mod-I and $A \in J$-Mod-I a bimodule (unitary both as a left J-module and as a right I-module), the abelian group $\mathrm{Hom}_I(A, B)$ can be given a (not necessarily unitary) right J-module structure in a standard way: For $\varphi \in \mathrm{Hom}_I(A, B)$ and $x \in J$, put $(\varphi \cdot x)a := \varphi(x \cdot a)$ for all $a \in A$. However, the resulting right J-module need not be unitary: As an example, take $J := K^{(\omega)}$ for some field K. We see that J is a nonunital regular algebra with local units. Suppose now that $\mathrm{Hom}_J(J, J)$ is a unitary right J-module. Then, by Lemma 2.10, there is an idempotent $u \in J$ satisfying $\mathrm{id}_J \cdot u = \mathrm{id}_J$. However, since $J = K^{(\omega)}$ is an infinite direct sum, there exists a nonzero $v \in J$ such that $uv = 0$. Then $0 \neq v = \mathrm{id}_J\, v = (\mathrm{id}_J \cdot u)v = \mathrm{id}_J(uv) = uv = 0$, a contradiction. Thus, we have found an example of the J-module $\mathrm{Hom}_I(A, B)$ not being unitary.

To fix this, instead of the additive group $\mathrm{Hom}_I(A, B)$, we shall focus on $\mathrm{Hom}_I(A, B)\, J = \{\sum \varphi \cdot x \mid \varphi \in \mathrm{Hom}_I(A, B), x \in J\}$. Since rings with local units are idempotent, this group—with the J-module structure from $\mathrm{Hom}_I(A, B)$—is a unitary J-module, i.e., an element of Mod-J.

It is easy to check that if $\varphi : B \longrightarrow C$ is a morphism in Mod-I and if $A \in J$-Mod-I, then the assignment

$$\begin{aligned} \mathrm{Hom}_I(A, \varphi)\, J : \mathrm{Hom}_I(A, B)\, J &\longrightarrow \mathrm{Hom}_I(A, C)\, J \\ \psi &\longrightarrow \varphi\psi \end{aligned}$$

is a J-module homomorphism. Clearly, if $\varphi' : C \longrightarrow D$ is another I-homomorphism, then $(\mathrm{Hom}_I(A, \varphi')\, J) \circ (\mathrm{Hom}_I(A, \varphi)\, J) = \mathrm{Hom}_I(A, \varphi'\varphi)\, J$, and $\mathrm{Hom}_I(A, \mathrm{id}_B)\, J = \mathrm{id}_{\mathrm{Hom}_I(B,C)J}$, whence $\mathrm{Hom}_I(A, -)\, J$ is a functor from Mod-I to Mod-J.

Lemma 2.19. *Let I, J be rings with local units and let $A \in J$-Mod-I. Then the functor $\mathrm{Hom}_I(A, -)\, J$: Mod-$I \longrightarrow$ Mod-J is additive and preserves monomorphisms.*

Proof. Additivity is clear from the definition. For preservation of monomorphisms, let us have

$$\begin{tikzcd} & A \arrow[d, "\psi"] & \\ 0 \arrow[r] & B \arrow[r, "\varphi"] & C \end{tikzcd}$$

in Mod-I with the row exact and suppose that $0 = (\mathrm{Hom}_I(A, \varphi)\, J)\psi = \varphi\psi$. Then, by the UMP of the kernel of φ, the morphism ψ factors through $\mathrm{Ker}\,\varphi = 0$, whence $\psi = 0$. Thus, for φ a monomorphism, we have shown that $\mathrm{Hom}_I(A, \varphi)\, J$ is injective. ☐

Now, translating a part of (Ánh – Márki, 1987, Theorem 2.1) into the language of right modules, we have:

Lemma 2.20. *Let I, J be rings with local units Morita equivalent via*

$$\mathrm{Mod}\text{-}I \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Mod}\text{-}J.$$

Set $P := H(J_J)$ and $Q := G(I_I)$. Then $P \in J\text{-}\mathrm{Mod}\text{-}I$, $Q \in I\text{-}\mathrm{Mod}\text{-}J$ and:

(i) $G \simeq \mathrm{Hom}_I(P, -)\, J$, $H \simeq \mathrm{Hom}_J(Q, -)\, I$;

(ii) $G \simeq - \otimes_I Q$, $H \simeq - \otimes_J P$. □

Proposition 2.21. *Let I, J be rings with local units Morita equivalent via*

$$\mathrm{Mod}\text{-}I \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Mod}\text{-}J.$$

Then the functors G and H are exact.

Proof. From part (i) of Lemma 2.20 and from Lemma 2.19, we see that the functors G and H preserve monomorphisms, while from Lemma 2.20(ii) and from Lemma 2.17, it follows that G and H preserve epimorphisms.

Let us have a SES $0 \longrightarrow A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \longrightarrow 0$ in Mod-I; we want to prove that then $0 \longrightarrow GA \xrightarrow{G\alpha} GB \xrightarrow{G\beta} GC \longrightarrow 0$ is a SES in Mod-J. Since $(G\beta)(G\alpha) = G(\beta\alpha) = G0 = 0$ and since G peserves monomorphisms and epimorphisms, we only need to verify that $\mathrm{Ker}\, G\beta \subseteq \mathrm{Im}\, G\alpha$.

Let $x \in \mathrm{Ker}\, G\beta$. Denote by $\iota_x : xJ \longrightarrow GB$ the inclusion map of the J-submodule of GB generated by x into GB. Applying H and the natural isomorphism $\eta : HG \longrightarrow \mathrm{Id}_{\mathrm{Mod}\text{-}I}$, we have the following commutative diagram in Mod-I:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
0 \arrow[r] & HGA \arrow[u, "\eta_A", "\simeq"'] \arrow[r, "HG\alpha"] & HGB \arrow[u, "\eta_B", "\simeq"'] \arrow[r, "HG\beta"] & HGC \arrow[u, "\eta_C", "\simeq"'] \arrow[r] & 0 \\
 & & H(xJ). \arrow[u, "H\iota_x"] & &
\end{tikzcd}$$

Note that since the top row in the above diagram is exact, so is the middle one, as η_A, η_B, η_C are isomorphisms. In particular, $HG\alpha : HGA \longrightarrow HGB$ satisfies the UMP of the kernel of $HG\beta$. Since $x \in \mathrm{Ker}\, G\beta$, we have $(G\beta)\iota_x = 0$, whence also $(HG\beta)H\iota_x = 0$. Thus, by the UMP of the kernel of $HG\alpha$, the morphism $H\iota_x$ factors through $HG\alpha$, i.e., there is a morphism $\xi : H(xJ) \longrightarrow HGA$ satisfying $H\iota_x = (HG\alpha)\xi$.

Applying G and the natural isomorphism $\zeta : GH \longrightarrow \mathrm{Id}_{\mathrm{Mod}\text{-}J}$, we obtain the following commutative diagram in Mod-J:

$$\begin{tikzcd}
 & & GB & & \\
GA \arrow[urr, "G\alpha"] & & & & xJ \arrow[ull, "\iota_x"'] \\
 & & GHGB \arrow[uu, "\zeta_{GB}", "\simeq"'] & & \\
GHGA \arrow[uu, "\zeta_{GA}", "\simeq"'] \arrow[urr, "GH G\alpha"] & & & & GH(xJ). \arrow[uu, "\zeta_{xJ}", "\simeq"'] \arrow[ull, "GH\iota_x"] \arrow[llll, "G\xi"]
\end{tikzcd}$$

Chasing this diagram, we see that

$$\iota_x = \zeta_{GB}(GH\iota_x)\zeta_{xJ}^{-1} = \zeta_{GB}(GHG\alpha)(G\xi)\zeta_{xJ}^{-1} = (G\alpha)\zeta_{GA}(G\xi)\zeta_{xJ}^{-1}.$$

In particular, we have shown that for a general $x \in \operatorname{Ker} G\beta$, we have $x \in \operatorname{Im} \iota_x \subseteq \operatorname{Im} G\alpha$, so $\operatorname{Ker} G\beta \subseteq \operatorname{Im} G\alpha$, which is what was left to be proved. $\square$

2.3 The functor $\mathcal{V}(-)$

In the study of unital regular rings, the monoid $\mathcal{V}(R)$ is defined as the monoid of isomorphism classes of finitely generated projective right R-modules, with the monoid operation defined by $[P] + [Q] := [P \oplus Q]$ (Goodearl, c1995, §5). An equivalent definition is as the monoid of equivalence classes of idempotent infinite matrices over R.[1] The latter definition, which we will informally call the "idempotent picture",[2] is easily generalized for nonunital rings (see Definition 2.24). In Definition 2.28, we present a generalization of the construction via projective modules (the so-called "projective picture") for nonunital rings. We prove in Proposition 2.34 that up to isomorphism, we end up with the same monoid either way. Nevertheless, it is useful to keep both definitions and always use the more convenient one; for example, we use the "idempotent picture" while proving that $\mathcal{V}(-)$ is in fact a functor (moreover, a continuous one, see Proposition 2.26), while we use the "projective picture" to prove that the resulting monoid has the Riesz refinement property (Proposition 2.37).

2.3.1 Definition of $\mathcal{V}(-)$ using idempotents

Let I be a ring. For square matrices $a \in M_n(I)$, $b \in M_k(I)$, we denote by $a \oplus b$ the block sum of a and b, i.e., $a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+k}(I)$. We think of the

[1] This construction is also used in the study of C^*-algebras, see (Blackadar, 1998, Definition 5.1.2).

[2] The terminology of "idempotent" and "projective" picture is taken from Goodearl (2009).

injective map

$$\begin{aligned} M_n(I) &\longrightarrow M_{n+1}(I) \\ a &\longmapsto a \oplus 0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

as of inclusion of rings, and we let $M_\infty(I)$ be the directed union of all $M_n(I)$'s. Hence, elements of $M_\infty(I)$ can be viewed as (countably) infinite square matrices over I with only finitely many nonzero entries. For *idempotent* matrices $e, g \in M_\infty(I)$, we say that e and g are *equivalent* and write $e \sim g$ if there exist $x, y \in M_\infty(I)$ such that $exgye = e$ and $gyexg = g$; clearly, $\sim$ is an equivalence relation on the set of all idempotent elements of $M_\infty(I)$.

In the unital case, the notion of two matrices being equivalent is a way of saying that the images of the endomorphisms of free modules given by the matrices are isomorphic:

Lemma 2.22. *If R is a unital ring, if e, g are idempotent matrices over R with $e \in M_n(R) \subseteq M_\infty(R)$ and $g \in M_m(R) \subseteq M_\infty(R)$ and if $x \in M_\infty(R)$, then left multiplication by exg defines a right R-module homomorphism $gR^m \longrightarrow eR^n$. In particular, if $y \in M_\infty(R)$ and if $exgye = e$ and $gyexg = g$ hold, then the R-modules eR^n and gR^m are isomorphic.*

Proof. Straightforward, using that left multiplication by an element of R is a right R-module homomorphism $R_R \longrightarrow R_R$ (Anderson – Fuller, 1992, Proposition 4.11). □

Observation 2.23. *For a ring homomorphism $\varphi : I \longrightarrow J$ and $n \in \{1, 2, \dots\} \cup \{\infty\}$, replacing entries in matrices over I by their respective images under φ defines a ring homomorphism $M_n(\varphi) : M_n(I) \longrightarrow M_n(J)$. Moreover, the assignment $M_n(-) : \varphi \longmapsto M_n(\varphi)$ is functorial.* □

Definition 2.24. For a ring I, we define $\mathcal{V}(I)$ as the monoid of equivalence classes of idempotents from $M_\infty(I)$ with addition induced from block sums, that is, $[e] + [g] := [e \oplus g]$.

For two rings I, J, observe that $M_\infty(I \times J) \simeq M_\infty(I) \times M_\infty(J)$. Moreover, a pair of idempotent matrices from $M_\infty(I \times J)$ is equivalent iff both the corresponding pairs of matrices over I and J are equivalent. Hence we can state that:

Observation 2.25. *For I and J rings, there is a monoid isomorphism $\mathcal{V}(I \times J) \simeq \mathcal{V}(I) \times \mathcal{V}(J)$.* □

We shall now provide a detailed proof that $\mathcal{V}(-)$ can be also defined on ring homomorphisms in a way that it forms a continuous functor from the category of nonunital rings to the category of abelian monoids:

Proposition 2.26. *$\mathcal{V}(-)$ is a functor from $\mathscr{Ring}$ to $\mathscr{Mon}$ that preserves direct limits.*

Proof. For a ring homomorphism $\varphi : I \longrightarrow J$ and $[e] \in \mathcal{V}(I)$, let us put

$$\mathcal{V}(\varphi)[e] := [M_\infty(\varphi)(e)]. \tag{2.1}$$

Claim 1. *The assignment* $[e] \longmapsto \mathcal{V}(\varphi)[e]$ *is a well-defined map* $\mathcal{V}(\varphi) : \mathcal{V}(I) \longrightarrow \mathcal{V}(J)$.

Proof of Claim. If $e \sim g$ in $M_\infty(I)$, then we have $exgye = e$ and $gyexg = g$ for some $x, y \in M_\infty(I)$. Applying the homomorphism $M_\infty(\varphi)$, we obtain $M_\infty(\varphi)(e) \sim M_\infty(\varphi)(g)$ via $M_\infty(\varphi)(x)$ and $M_\infty(\varphi)(y)$, so the assignment $[e] \longmapsto [M_\infty(\varphi)(e)]$ is independent of the choice of representative of $[e]$. Since $M_\infty(\varphi)$ is a homomorphism, we also see that if e is an idempotent, then so is $M_\infty(\varphi)(e)$, whence $\mathcal{V}(\varphi)[e] \in \mathcal{V}(J)$. □ *Claim 1.*

Observe that the ring homomorphism $M_\infty(\varphi)$ commutes with taking block sums.From the definition of $\mathcal{V}(\varphi)$, we then arrive at

$$\begin{aligned}\mathcal{V}(\varphi)([e]+[g]) &= \mathcal{V}(\varphi)[e \oplus g] = [M_\infty(\varphi)(e \oplus g)] = [M_\infty(\varphi)(e) \oplus M_\infty(\varphi)(g)] \\ &= \mathcal{V}(\varphi)[e] + \mathcal{V}(\varphi)[g]\end{aligned}$$

for any $e, g \in \operatorname{Idemp} M_\infty(I)$. Thus, the map $\mathcal{V}(\varphi) : \mathcal{V}(I) \longrightarrow \mathcal{V}(J)$ above is in fact a monoid homomorphism.

The functoriality of $\mathcal{V}(-)$ follows from $M_\infty(-)$ being functorial.

To establish that $\mathcal{V}(-)$ preserves direct limits, we use the following explicit construction of direct limits in $\mathscr{Ring}$: Let $(I_\alpha | \alpha \in A)$ be a directed system of rings with transition maps $f_\alpha^\beta : I_\alpha \longrightarrow I_\beta$ for $\alpha \leq \beta$. We define a ring I as follows:

- Elements of I: Equivalence classes of elements of the disjoint union $\dot{\bigcup\limits_{\alpha \in A}} I_\alpha$, where $x \in I_\alpha, y \in I_\beta$ are equivalent if and only if there is a $\gamma \geq \alpha, \beta$ satisfying $f_\alpha^\gamma x = f_\beta^\gamma y$.
- Ring operations in I: For $x \in I_\alpha, y \in I_\beta$, there is a $\gamma \geq \alpha, \beta$ in A; we define the product and sum of $[x]$ and $[y]$ in I as $[x] \cdot [y] := [f_\alpha^\gamma(x) \cdot f_\beta^\gamma(y)]$ and $[x] + [y] := [f_\alpha^\gamma(x) + f_\beta^\gamma(y)]$, respectively.

It is routine to check that the operations above are well-defined (independent of the choice of represenatives of $[x]$ and $[y]$, as well as of the choice of γ), that the neutral element in the sum operation is the common class of all 0's in I_α's, that $-[x] = [-x]$ and that I with these operations constitutes a ring. It is also easy to verify that I with the canonical maps

$$\begin{aligned} f_\alpha : I_\alpha &\longrightarrow I \\ x &\longmapsto [x] \end{aligned}$$

satisfies the UMP defining the direct limit of the system $(I_\alpha | \alpha \in A)$ in $\mathscr{Ring}$, whence we can write $I = \varinjlim I_\alpha$.

Suppose now that we have a directed system $(I_\alpha | \alpha \in A)$ in $\mathscr{Ring}$ with direct limit $I = \varinjlim I_\alpha$ constructed as above. We want to prove that $\mathcal{V}(I)$ with the maps $\mathcal{V}(f_\alpha) : \mathcal{V}(I_\alpha) \longrightarrow \mathcal{V}(I)$ satisfy the UMP defining $\varinjlim \mathcal{V}(I_\alpha)$ in $\mathscr{Mon}$. Suppose M is an abelian monoid and suppose $\psi_\alpha : \mathcal{V}(I_\alpha) \longrightarrow M$, $\alpha \in A$, are monoid homorphisms satisfying

$$\psi_\alpha = \psi_\beta \mathcal{V}\left(f_\alpha^\beta\right) \text{ whenever } \alpha \leq \beta; \tag{2.2}$$

we are looking for a morphism $\psi : \mathcal{V}(I) \longrightarrow M$ satisfying $\psi\mathcal{V}(f_\alpha) = \psi_\alpha$ for all $\alpha \in A$, and we want to show that it is unique.

Until the end of the proof, let us write F_α instead of $M_\infty(f_\alpha)$ for any $\alpha \in A$, and F_α^β instead of $M_\infty(f_\alpha^\beta)$ whenever $\alpha \leq \beta$. We thus have $F_\alpha : M_\infty(I_\alpha) \longrightarrow M_\infty(I)$ and $F_\alpha^\beta : M_\infty(I_\alpha) \longrightarrow M_\infty(I_\beta)$ replacing entries of matrices over I_α by their images under f_α and f_α^β, respectively. We will find this short-hand notation particularly useful when writing down (2.3) and its consequences.

Let $e \in \operatorname{Idemp} M_\infty(I)$. Since entries of e are in $I = \varinjlim I_\alpha$ and only finitely many of them are nonzero, there exists a $\beta \in A$ such that all entries of e are in $\operatorname{Im} f_\beta$. Hence, there is a matrix $\hat{e} \in M_\infty(I_\beta)$ such that $F_\beta\hat{e} = e$. However, the matrix $\hat{e}$ need not be idempotent. Still, for any i, j, we have

$$f_\beta((\hat{e}^2)_{ij}) = (e^2)_{ij} = e_{ij}^2 = f_\beta((\hat{e})_{ij})$$

by idempotence of e. Then, by the construction of I, there is a $\gamma_{ij} \geq \beta$ such that $f_\beta^{\gamma_{ij}}((\hat{e}^2)_{ij}) = f_\beta^{\gamma_{ij}}((\hat{e})_{ij})$. Take

$$\gamma := \max\{\, \gamma_{ij} \mid i,j \in \mathbb{N} \text{ such that } (\hat{e})_{ij} \neq 0 \text{ or } (\hat{e}^2)_{ij} \neq 0 \,\};$$

note that we are taking maximum from a finite set. Then

$$F_\beta^\gamma\hat{e} = F_\beta^\gamma(\hat{e}^2) = (F_\beta^\gamma\hat{e})^2$$

holds. Hence, $\tilde{e} := F_\beta^\gamma\hat{e} \in M_\infty(I_\gamma)$ is idempotent, so its equivalence class $[\tilde{e}]$ is an element of $\mathcal{V}(I_\gamma)$. Moreover, since $f_\beta = f_\gamma f_\beta^\gamma$ holds and since $\mathcal{V}(-)$ is a functor, $[\tilde{e}]$ is a preimage of $[e]$ under $\mathcal{V}(f_\gamma) : \mathcal{V}(I_\gamma) \longrightarrow \mathcal{V}(I)$. In order for $\psi\mathcal{V}(f_\gamma) = \psi_\gamma$ to hold, we then have no choice but to put $\psi([e]) = \psi_\gamma([\tilde{e}])$. Hence, once we verify that by iterating this construction for every $[e] \in \mathcal{V}(I)$, we obtain a well-defined homomorphism $\psi : \mathcal{V}(I) \longrightarrow M$ of abelian monoids (Claims 2 through 4), it is clear that ψ will be the *unique* homomorphism from $\mathcal{V}(I)$ to M to satisfy $\psi_\alpha = \psi\mathcal{V}(f_\alpha)$ for all $\alpha \in A$.

Claim 2. *For $e \in \operatorname{Idemp} M_\infty(I)$, if $a, b \in A$ and $a \in \operatorname{Idemp} M_\infty(I_\alpha)$ and $b \in \operatorname{Idemp} M_\infty(I_\beta)$ satisfy $\mathcal{V}(f_\alpha)[a] = [e] = \mathcal{V}(f_\beta)[b]$, then $\psi_\alpha[a] = \psi_\beta[b]$.*

Proof of Claim. From the definition of $\mathcal{V}(-)$ on morphisms, we have

$$[F_\alpha a] = \mathcal{V}(f_\alpha)[a] = \mathcal{V}(f_\beta)[b] = [F_\beta b].$$

Then, by the definition of equivalence of idempotents in $M_\infty(I)$, there are $x, y \in M_\infty(I)$ satisfying

$$\left.\begin{array}{rcl} F_\alpha a &=& (F_\alpha a)x(F_\beta b)y(F_\alpha a) \quad \text{and} \\ F_\beta b &=& (F_\beta b)y(F_\alpha a)x(F_\beta b). \end{array}\right\} \tag{2.3}$$

Since x and y have only finitely many nonzero entries, there are $\gamma \in A$ (w.l.o.g., $\gamma \geq \alpha, \beta$) and $\hat{x}, \hat{y} \in I_\gamma$ such that $x = F_\gamma\hat{x}, y = F_\gamma\hat{y}$. From (2.3), from the equality $f_\alpha = f_\gamma f_\alpha^\gamma$ and from $F_\gamma = M_\infty(f_\gamma)$ being a ring homomorphism, we obtain

$$F_\alpha = (F_\gamma F_\alpha^\gamma a)(F_\gamma\hat{x})(F_\gamma F_\beta^\gamma b)(F_\gamma\hat{y})(F_\gamma F_\alpha^\gamma a) = F_\gamma\big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big) \tag{2.4}$$

and, similarly,

$$F_\alpha b = F_\gamma\big((F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\big). \tag{2.5}$$

In particular, for any i, j, we obtain from (2.4) that

$$(F_\alpha a)_{ij} = f_\alpha(a_{ij}) = \Big(F_\gamma\big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big)\Big)_{ij} = f_\gamma\Big(\big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big)_{ij}\Big)$$

holds in I. Then, from the construction of I, there is a $\zeta_{ij} \geq \gamma$ such that

$$f_\alpha^{\zeta_{ij}}(a_{ij}) = f_\gamma^{\zeta_{ij}}\Big(\big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big)_{ij}\Big)$$

Take

$$\zeta := \max\{\, \zeta_{ij} \mid i, j \in \mathbb{N} \text{ such that } a_{ij} \neq 0 \text{ or } \big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big)_{ij} \neq 0 \,\};$$

then

$$F_\alpha^\zeta a = F_\gamma^\zeta\big((F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\big).$$

hold. Similarly, from (2.5) we derive that

$$F_\beta^\eta b = F_\gamma^\eta\big((F_\beta^\gamma b)\hat{y}(F_\alpha^\gamma a)\hat{x}(F_\beta^\gamma b)\big)$$

holds for some $\eta \geq \gamma$. For $\xi := \max\{\, \zeta, \eta \,\}$, we then have

$$\begin{aligned} F_\alpha^\xi a &= (F_\alpha^\xi a)(F_\gamma^\xi \hat{x})(F_\beta^\xi b)(F_\gamma^\xi y)(F_\alpha^\xi a) \quad \text{and} \\ F_\beta^\xi b &= (F_\beta^\xi b)(F_\gamma^\xi \hat{y})(F_\alpha^\xi a)(F_\gamma^\xi x)(F_\beta^\xi b), \end{aligned}$$

whence we conclude that $\left[F_\alpha^\xi a\right] = \left[F_\beta^\xi b\right]$ holds in $\mathcal{V}(I_\xi)$. Thus:

$$\begin{aligned} \psi_\alpha[a] &= \psi_\xi\left(\mathcal{V}\left(f_\alpha^\xi\right)[a]\right) && \text{by (2.2),} \\ &= \psi_\xi\left[F_\alpha^\xi(a)\right] && \text{by (2.1),} \\ &= \psi_\xi\left[F_\beta^\xi(b)\right] \\ &= \psi_\xi\left(\mathcal{V}\left(f_\beta^\xi\right)[b]\right) && \text{by (2.1),} \\ &= \psi_\beta[b] && \text{by (2.2),} \end{aligned}$$

as desired. □ *Claim 2.*

Claim 3. *For $e, g \in \operatorname{Idemp} M_\infty(I)$ with $[e] = [g]$ in $\mathcal{V}(I)$, if $\beta, \gamma \in A$ and $\tilde{e} \in \operatorname{Idemp} M_\infty(I_\beta)$, $\tilde{g} \in \operatorname{Idemp} M_\infty(I_\gamma)$ satisfy $\mathcal{V}(f_\beta)[\tilde{e}] = [e]$ and $[g] = \mathcal{V}(f_\gamma)[\tilde{g}]$, then $\psi_\beta[\tilde{e}] = \psi_\gamma[\tilde{g}]$.*

Proof of Claim. By the definition of equivalence in $M_\infty(I)$, there are $x, y \in M_\infty(I)$ such that both

$$\begin{aligned} F_\beta\tilde{e} &= (F_\beta\tilde{e})x(F_\gamma\tilde{g})y(F_\beta\tilde{e}) \quad \text{and} \\ F_\gamma\tilde{g} &= (F_\gamma\tilde{g})y(F_\beta\tilde{e})x(F_\gamma\tilde{g}) \end{aligned}$$

hold. As in the proof of Claim 2, we find a $\xi \geq \beta, \gamma$ and $\tilde{x}, \tilde{y} \in M_\infty(I_\xi)$ satisfying

$$\begin{aligned} F_\beta^\xi\tilde{e} &= (F_\beta^\xi\tilde{e})\tilde{x}(F_\gamma^\xi\tilde{g})\tilde{y}(F_\beta^\xi\tilde{e}) \quad \text{and} \\ F_\gamma^\xi\tilde{g} &= (F_\gamma^\xi\tilde{g})\tilde{y}(F_\beta^\xi\tilde{e})\tilde{x}(F_\gamma^\xi\tilde{g}); \end{aligned}$$

thence, $[F^\xi_\beta \tilde{e}] = [F^\xi_\gamma g]$ holds in $\mathcal{V}(I_\xi)$. We conclude that

$$\psi_\beta[\tilde{e}] = \psi_\xi \mathcal{V}\left(f^\xi_\beta\right)[\tilde{e}] = \psi_\beta[F^\xi_\beta \tilde{e}] = \psi_\gamma[F^\xi_\gamma \tilde{g}] = \psi_\xi \mathcal{V}\left(f^\xi_\gamma\right)[\tilde{g}] = \psi_\gamma[\tilde{g}]. \ \square \ \textit{Claim 3.}$$

By Claims 2 a 3, we have a well-defined map $\psi : \mathcal{V}(I) \longrightarrow M$. To conclude the proof of Proposition 2.26, it now only remains to verify that ψ is in fact a morphism in $\mathscr{Mon}$.

Claim 4. *The map $\psi : \mathcal{V}(I) \longrightarrow M$ is a monoid homomorphism.*

Proof of Claim. For any $\alpha \in A$, we have $[F_\alpha(0)] = [0]$, whence $\psi[0] = \psi_\alpha[0] = 0 \in M$. Let now $[e], [g] \in \mathcal{V}(I)$. Again, there are idempotents $\tilde{e}, \tilde{g} \in \operatorname{Idemp} M_\infty(I_\alpha)$ for some $\alpha \in A$ satisfying $[F_\alpha \tilde{e}] = \mathcal{V}(f_\alpha)[\tilde{e}] = [e]$ and $[F_\alpha \tilde{g}] = \mathcal{V}(f_\alpha)[\tilde{g}] = [g]$. Since $\mathcal{V}(f_\alpha)$ is a monoid homomorphism, we have

$$[e \oplus g] = [e] + [g] = \mathcal{V}(f_\alpha)([\tilde{e}] + [\tilde{g}]) = \mathcal{V}(f_\alpha)[\tilde{e} \oplus \tilde{g}].$$

Hence, $\psi[e \oplus g] = \psi_\alpha[\tilde{e} \oplus \tilde{g}] = \psi_\alpha[\tilde{e}] + \psi_\alpha[\tilde{g}] = \psi[e] + \psi[g]$. $\square$ *Claim 4.* $\square$

Next, we shall show that applying the functor $\mathcal{V}(-)$ to either a nonunital ring I or to the ring opposite to I, I^{op}, yields the same monoid:

Proposition 2.27. *Let I be a ring. Then $\mathcal{V}(I) \simeq \mathcal{V}(I^{\mathrm{op}})$.*

Proof. Throughout this proof, we shall denote multiplication in I^{op} by $\cdot_{\mathrm{op}}$, i.e., for $x, y \in I$, $x \cdot_{\mathrm{op}} y := yx$ (with concatenation denoting multiplication in I). For matrices over I, $\cdot_{\mathrm{op}}$ will mean matrix multiplication as matrices over I^{op}. For any matrix m, the symbol $m^\intercal$ will denote the matrix transpose of m, i.e., $(m^\intercal)_{ij} = m_{ji}$ for all i, j.

Claim 1. *The assignment*

$$\begin{aligned} {}^\intercal : M_\infty(I) &\longrightarrow M_\infty(I^{\mathrm{op}}) \\ m &\longmapsto m^\intercal \end{aligned}$$

is a ring antiisomorphism.

Proof of Claim. Let a,$b \in M_\infty(I)$. Then, for any i, j, we have:

$$(ab)_{ij} = \sum_k a_{ik} b_{kj} = \sum_k b_{kj} \cdot_{\mathrm{op}} a_{ik} = \sum_k (b^\intercal)_{jk} \cdot_{\mathrm{op}} (a^\intercal)_{ki} = (b^\intercal \cdot_{\mathrm{op}} a^\intercal)_{ji}.$$

Hence, $(ab)^\intercal = b^\intercal \cdot_{\mathrm{op}} a^\intercal$. As $m \longmapsto m^\intercal$ clearly preserves sums and the zero matrix, it follows that it is an antihomomorphism. Starting with I^{op} instead of I, we also have an antihomomorphism ${}^\intercal : M_\infty(I^{\mathrm{op}}) \longrightarrow M_\infty(I)$. Now that $(m^\intercal)^\intercal = m$ for any $m \in M_\infty(I)$ or $m \in M_\infty(I^{\mathrm{op}})$, we conclude that ${}^\intercal$ is an antiisomorphism.
$\square$ *Claim 1.*

For an idempotent matrix $e \in M_\infty(I)$, we have $e^\intercal \cdot_{\mathrm{op}} e^\intercal = (ee)^\intercal = e^\intercal$, so ${}^\intercal$ maps idempotent matrices to idempotent matrices.

Let $e, g \in \operatorname{Idemp} M_\infty(I)$ be equivalent, i.e., by definition, $e = exgye$ and $g = gyexg$ for some $x, y \in M_\infty(I)$. Then, by Claim 1,

$$\begin{aligned} e^\intercal = (exgye)^\intercal &= e^\intercal \cdot_{\mathrm{op}} y^\intercal \cdot_{\mathrm{op}} g^\intercal \cdot_{\mathrm{op}} x^\intercal \cdot_{\mathrm{op}} e^\intercal, &\quad \text{and} \\ g^\intercal &= g^\intercal \cdot_{\mathrm{op}} x^\intercal \cdot_{\mathrm{op}} e^\intercal \cdot_{\mathrm{op}} y^\intercal \cdot_{\mathrm{op}} g^\intercal, \end{aligned}$$

so $e^\intercal \sim g^\intercal$ as elements of $M_\infty(I^{\mathrm{op}})$. It follows that $^\intercal$ induces a map $[-^\intercal] : \mathcal{V}(I) \longrightarrow \mathcal{V}(I^{\mathrm{op}})$. Since $(e \oplus g)^\intercal = e^\intercal \oplus g^\intercal$ and $0^\intercal = 0$, the map $[-^\intercal]$ is a monoid homomorphism. Finally, the similarly constructed homomorphism $[-^\intercal] : \mathcal{V}(I^{\mathrm{op}}) \longrightarrow \mathcal{V}(I)$ is a two-sided inverse of the above homomorphism, whence these two are actually isomorphisms between $\mathcal{V}(I)$ and $\mathcal{V}(I^{\mathrm{op}})$. ☐

2.3.2 Definition of $\mathcal{V}(-)$ using projective modules

We shall now present a generalization of the classical definition of $\mathcal{V}(R)$ as the monoid of isomorphism classes of finitely generated projective right R-modules for a *unital* ring R into the nonunital setting and prove that the resulting monoid is isomorphic to the monoid of equivalence classes of idempotent matrices.

Definition 2.28. For a ring I and a unital ring R containing I as a two-sided ideal, we put $\mathrm{FP}(I, R) := \{ P \in \mathrm{proj}\text{-}R \mid PI = P \}$, that is, $\mathrm{FP}(I, R)$ is the class of all finitely generated projective right R-modules (unitary in the classical sense) that are unitary as I-modules (with the I-structure defined by restriction of scalars). We define $\mathcal{V}_R(I)$ as the abelian monoid of isomorphism classes of elements of $\mathrm{FP}(I, R)$ (as R-modules), with addition induced from direct sums.

Remark 2.29. For an arbitrary ring I, there always exists a unital ring R containing I as a two-sided ideal; however, such ring R is by no means unique. Possible constructions of R include formally adjoining a unit element to I, see (Faith, 1973, p. 384); or, one can construct the multiplier algebra as in Ara – Perera (2000)[3]; or, starting with a regular ring, it is shown in Fuchs – Halperin (1964) how to embed it as a two-sided ideal in a regular unital ring. Nevertheless, we show that the monoid $\mathcal{V}_R(I)$ is, up to isomorphism, independent of the choice of R, see Proposition 2.34 and Remark 2.35. Once this is established, we will drop the subscript $_R$ from $\mathcal{V}_R(I)$ and denote by $\mathcal{V}(I)$ either of the monoids defined in Definition 2.24 or Definition 2.28.

Until Remark 2.35 inclusive, let us fix I and R (I still being a two-sided ideal in a unital ring R). We assume that I is a ring with local units. For our purposes, this assumption comes at no cost, since all regular rings have local units (Proposition 2.7).

We start the proof by constructing a monoid homomorphism from $\mathcal{V}_R(I)$ to $\mathcal{V}(I)$; the first step is to find a suitable map from $\mathrm{FP}(I, R)$ to $M_\infty(I)$:

Lemma 2.30. *Let $P \in \mathrm{FP}(I, R)$. Then there is an $n \in \mathbb{N}$ and an $e \in \mathrm{Idemp}\, M_n(I)$ satisfying $P \simeq eR^n = eI^n$ as R-modules.*

Proof. Since P is a finitely generated projective R-module, it is isomorphic to a direct summand of R^n for some finite n. Hence, there are morphisms π, ι in $\mathrm{Mod}\text{-}R$ such that the following diagram is commutative and both the row and

[3] Although defined only for semiprime rings in Ara – Perera (2000), the definition works perfectly fine without the requirement of semiprimeness.

the column in it are exact:

$$\begin{tikzcd} 0 \arrow[d] & & \\ P \arrow[d, "\iota"'] \arrow[dr, "\mathrm{id}_P"] & & \\ R^n \arrow[r, "\pi"'] & P \arrow[r] & 0. \end{tikzcd}$$

Since $\pi\iota = \mathrm{id}_P$, the morphism $e := \iota\pi : R^n \longrightarrow R^n$ is idempotent. Since ι is injective and π is surjective, we see that $P \simeq eR^n$ via ι. Identifying $\mathrm{End}_R(R^n)$ with $M_n(R)$ in the standard way (with elements of R^n viewed as *column* vectors and with matrices acting on the *left*, since we are dealing with *right* modules), we view e as an idempotent element of $M_n(R) \subseteq M_\infty(R)$. Then, for any $i \leq n$, the i-th column of e is precisely the element $e(b_i) \in eR^n$, where b_i is the i-th vector in the canonical basis of R^n (i.e., $b_1 = (1, 0, \ldots, 0), b_2 = (0, 1, 0, \ldots, 0), \ldots, b_n = (0, \ldots, 0, 1)$).

We have $eR^n = \iota P \simeq P$; from $PI = P$ and from e being an R-module homomorphism, it follows that $eR^n = (eR^n)I$. Clearly, $eI^n \subseteq eR^n$; on the other hand, since I is an ideal in R, we have $(eR^n)I \subseteq eI^n$. Put together, we have $P \simeq eR^n = eI^n$. It remains to show that e is in fact an element of $M_n(I)$. From $eR^n = eI^n$, we have that $eb_i \in eI^n \subseteq I^n$ for all $i \leq n$. Hence, with columns of e being the images of b_i's, we conclude that all entries of e are elements of I, as desired. $\square$

Remark 2.31. For an idempotent matrix $e \in M_n(I)$, clearly $eI^n \subseteq eR^n$. On the other hand, since I is an ideal in R and since e has entries in I, we have $eR^n \subseteq I^n$, so, from the idempotence of e, we obtain $eR^n = e(eR^n) \subseteq eI^n$. So, as sets, we have $eI^n = eR^n$. Henceforth, we shall write eR^n when viewing this set as an R-module and eI^n when viewing it as an I-module.

Since we assume that I has local units, we see that $eI^n = (eI^n)I$: Indeed, as an element of $eI^n \subseteq I^n$ is an n-tuple $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ with $x_1, \ldots, x_n \in I$, there is an idempotent $u \in I$ sarisfying $\mathbf{x}u = \mathbf{x}$, whence $\mathbf{x} \in (eI^n)I$. We conclude that $eR^n \in \mathrm{FP}\,(I, R)$.

It will be useful to have written down the following fact:

Observation 2.32. *If* $g^2 = g \in M_m(I)$ *and* $n > m$, *then* $gR^m \simeq \begin{pmatrix} g & 0 \\ 0 & 0_{n-m} \end{pmatrix} R^n$.$\square$

Thanks to Lemma 2.30, whenever we deal with (isomorphism classes of) elements of $\mathrm{FP}\,(I, R)$, we can focus on idempotent matrices over I instead. However, we should note that neither the matrix e of Lemma 2.30 nor its size is unique: for example, $(1)R \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R^2$ for any unital ring R. Nevertheless, the matrix is unique up to equivalence:

Lemma 2.33. *If $e \in \operatorname{Idemp} M_n(I)$, $g \in \operatorname{Idemp} M_m(I)$ satisfy $eR^n \simeq gR^m$, then $e \sim g$ as elements of $M_\infty(I)$.*

Proof. By Observation 2.32, we may w.l.o.g. suppose that $n = m$.

Let x, y be mutually inverse isomorphisms between eR^n and gR^n:

$$\begin{tikzcd} eR^n \arrow[r, bend left, "y"] & gR^n. \arrow[l, bend left, "x"] \end{tikzcd}$$

As eR^n and gR^n are both direct summands of R^n, both x, y can be extended to endomorohisms of R^n; hence, view x, y as elements of $M_n(R)$. Since I is an ideal of R and since e, g are matrices over I, we see that ex and gy are—as matrices—elements of $M_n(I)$. Since e and g are idempotent, they act as identity on eR^n and gR^n, respectively. With $xy = \mathrm{id}_{eR^n}$ and $yx = \mathrm{id}_{gR^n}$, we conclude that $e(ex)g(gy)e = (ex)(gy)e = xye = e$ and $g(gy)e(ex)g = g$, whence $e \sim g$. $\square$

For $[P] \in \mathcal{V}_R(I)$, put

$$\varphi[P] := [e] \in \mathcal{V}(I)\,, \tag{2.6}$$

where e is any idempotent element of $M_n(I)$ satisfying $eR^n \simeq P$. By Lemma 2.30, such e always exists; by Lemma 2.33, $\varphi[P]$ is independent of the choice of representative P of $[P]$ and is uniquely determined by $[P]$. Finally, observe that if $e^2 = e \in M_n(I)$, $g^2 = g \in M_m(I)$, then

$$eR^n \oplus gR^m \simeq \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} R^{n+m}, \tag{2.7}$$

so

$$\varphi([eR^n] + [gR^m]) = \varphi[eR^n \oplus gR^m] = [e \oplus g] = [e] + [g] = \varphi[eR^n] + \varphi[gR^m].$$

All put together, $\varphi : \mathcal{V}_R(I) \longrightarrow \mathcal{V}(I)$ is a well-defined monoid homomorphism. Finding its inverse is easier:

For $e \in \operatorname{Idemp} M_\infty(I)$, there is an $n < \infty$ such that $e \in M_n(I)$; by Observation 2.32, $eR^n \simeq eR^m$ for any other $m < \infty$ satisfying $e \in M_m(I)$. Moreover, by Remark 2.31, we have $eR^n \in \mathrm{FP}\,(I, R)$. Hence, assigning the isomorphism class of $[eR^n]$ to e is a well-defined map from $\operatorname{Idemp} M_\infty(I)$ to $\mathcal{V}_R(I)$.

Suppose now that e, g are equivalent idempotent matrices over I satisfying $e \in M_n(I) \subseteq M_\infty(I)$, $g \in M_m(I) \subseteq M_\infty(I)$. Then, by Lemma 2.22, $eR^n \simeq gR^m$ holds, so the elements $[eR^n]$ and $[gR^m]$ of $\mathcal{V}_R(I)$ are equal. With the preceding paragraph, we have thus shown that mapping $[e] \in \mathcal{V}(I)$ to $[eR^n] \in \mathcal{V}_R(I)$ for n large enough is a well-defined map. As above, we see from (2.7) that this map is a monoid homomorphism. It is clearly an inverse of φ, so we can conclude:

Proposition 2.34. *The map $\varphi : \mathcal{V}_R(I) \longrightarrow \mathcal{V}(I)$ defined by (2.6) is an isomorphism in the category $\mathscr{Mon}$.* $\square$

Remark 2.35. As R plays no role in the definition of $\mathcal{V}(I)$ (via idempotent matrices over I), Proposition 2.34 tells us that $\mathcal{V}_R(I)$ is actually independent of the choice of R. Notice that if I is unital to begin with, we can choose $R = I$; then $\mathrm{FP}\,(I, I) = \mathrm{FP}\,(I, R)$ becomes proj-I, and $\mathcal{V}(I)$ is then the classical monoid of isomorphism classes of finitely generated projective I-modules.

From now on, we refrain from using the subscript $_R$ in $\mathcal{V}_R(I)$ and denote either of the isomorphic monoids by $\mathcal{V}(I)$.

2.3.3 Common properties of $\mathcal{V}(-)$'s of regular rings

Proposition 2.36. *Let I be a ring with local units. Then the monoid $\mathcal{V}(I)$ is conical.*

Proof. Let R be a unital ring containing I as a two-sided ideal. For an R-module M, note that $M \simeq 0$ iff $M = 0 = \{0\}$. Thus, if $P, Q \in \mathrm{FP}(I, R)$ satisfy $[P \oplus Q] = [0]$, then $P \oplus Q \simeq 0$, so $P \oplus Q$ is the zero module. As P and Q can be embedded into $P \oplus Q$, it follows that $P = 0 = Q$. $\square$

Proposition 2.37. *Let I be a regular ring. Then $\mathcal{V}(I)$ is a refinement monoid.*

Proof. Let R be a unital regular ring containing I as a two-sided ideal; such R always exists by (Fuchs – Halperin, 1964, Theorem 1). Viewing $\mathcal{V}(R)$ as the monoid of isomorphism classes of modules from $\mathrm{FP}(R, R) = \mathrm{proj}\text{-}R$, the monoid $\mathcal{V}(R)$ is a refinement monoid by (Goodearl, 1979, Theorem 2.8). From (Ara et al., 1998, Proposition 1.4), $\mathcal{V}(I)$ is an order-ideal in $\mathcal{V}(R)$.[4] Thus, by Observation 1.2, $\mathcal{V}(I)$ is a refinement monoid. $\square$

Finally, since every projective module over a *unital* ring is isomorphic to a direct summand of a free module, we have:

Observation 2.38. *If R is a unital ring, then $[R]$ is an order-unit in $\mathcal{V}(R)$.* $\square$

2.3.4 $\mathcal{V}(-)$'s of Morita equivalent rings with local units

For *unital* rings, it follows from the classical constuction of $\mathcal{V}(R)$ as the monoid of isomorphism classes of finitely generated projective R-modules that the monoids $\mathcal{V}(R), \mathcal{V}(S)$ for Morita equivalent unital rings R and S are ismomorphic. We shall now prove a similar statement for Morita equivalent rings with local units, using the "projective picture". We shall make use of the following fact:

Proposition 2.39. *Let I, J be rings with local units with I contained in J as a two-sided ideal. Then, for any $M \in \mathrm{Mod}\text{-}I$, there is a natural unitary J-module structure on M extending the original I-structure. The category $\mathrm{Mod}\text{-}I$ is a full subcategory of $\mathrm{Mod}\text{-}J$.*

Proof. Let $M \in \mathrm{Mod}\text{-}I$. For $m \in M$, by Lemma 2.10, there is an idempotent $u \in I$ satisfying $m = mu$. For $r \in J$, we then have $ur \in I$, since I is an ideal in J; put then $m \cdot r := m \cdot (ur)$.

Claim 1. *The definition of $m \cdot r$ is independent of u.*

Proof of Claim. Let $u, v \in \mathrm{Idemp}\, I$ satisfy $mu = m = mv$. Since I has local units, there is an idempotent $w \in I$ with $uw = u$ and $vw = v$. Then $wr \in I$;

[4] A prerequisite for (Ara et al., 1998, Proposition 1.4) to apply is that R be a unital exchange ring. Thankfully, all unital regular rings are such, as can be checked using (Nicholson, 1977, Theorem 2.1) and (Goodearl, 1979, Theorem 1.7).

using $\cdot$ to denote the I-action on M and concatenation to denote multilpication in J, we then have

$$\begin{aligned} m \cdot (ur) - m \cdot (vr) &= m \cdot ((uw)r) - m \cdot ((vw)r) \\ &= m \cdot (u(wr)) - m \cdot (v(wr)) \\ &= (m \cdot u) \cdot (wr) - (m \cdot v) \cdot (wr) \\ &= m \cdot (wr) - m \cdot (wr) \\ &= 0, \end{aligned}$$

whence $m \cdot (ur) = m \cdot (vr)$. $\square$ *Claim 1.*

By Claim 1, we have a well-defined map

$$\begin{aligned} M \times R &\longrightarrow M \\ (m, r) &\longmapsto m \cdot r. \end{aligned}$$

We want to show that this map defines a J-module structure on M.

- Let $m, m' \in M$ and $r \in J$. Then, since M is a unitary I-module, there are $u, u' \in \operatorname{Idemp} I$ satisfying $mu = m$ and $m'u' = m'$ (Lemma 2.10). Since I has local units, there is an idempotent $v \in I$ such that $uv = u$ and $u'v = u'$. Then $mv = (mu)v = m(uv) = mu = m$, and similarly $m'v = m'$, so $(m + m')v = m + m'$ holds. Hence

$$(m + m') \cdot r = (m + m')(vr) = m(vr) + m'(vr) = m \cdot r + m' \cdot r.$$

- Let $m \in M$, $r, s \in R$ and let $u \in \operatorname{Idemp} I$ satisfy $mu = m$. Then

$$m \cdot (r + s) = m(u(r + s)) = m(ur + us) = m(ur) + m(us) = m \cdot r + m \cdot s.$$

- Let once again $m \in M$, $r, s \in J$ and let $u \in \operatorname{Idemp} I$ satisfy $mu = m$. Since I has local units and since ur is an element of I, there is an idempotent $v \in I$ satisfying $(ur)v = ur$. Then $(m(ur))v = m((ur)v) = m(ur)$ holds, whence

$$m \cdot (rs) = m(u(rs)) = m(((ur)v)s) = m((ur)(vs)) = (m(ur))(vs) = (m \cdot r) \cdot s.$$

Finally, to prove that we have defined a *unitary* J-module structure on M, if $m \in M$, there is—by Lemma 2.10—an idempotent $u \in I$ with $mu = m$; since $u \in I \subseteq J$ and $m \cdot u = m(uu) = mu = m$, Lemma 2.10 yields that M as a J-module is unitary.

We have shown how to embed objects from $\operatorname{Mod-}I$ into $\operatorname{Mod-}J$. For $\operatorname{Mod-}I$ being a subcategory of $\operatorname{Mod-}J$, let $\varphi : M \longrightarrow N$ be a homomorphism of I-modules; we want to show—with the J-structure on M and N defined as above—that φ is also a J-module homomorphism. To that end, let $m \in M$, $r \in J$, $u \in \operatorname{Idemp} I$ and $mu = m$. Then, since φ is an I-module homomorphism, we have $\varphi(m)u = \varphi(mu) = \varphi(m)$. Hence,

$$\varphi(m \cdot r) = \varphi(m(ur)) = \varphi(m)(ur) = \varphi(m) \cdot r.$$

We have thus shown that taking a morphism in $\mathrm{Mod}\text{-}I$, the same map between the underlying sets is also a morphism in $\mathrm{Mod}\text{-}J$. It follows that $\mathrm{Mod}\text{-}I$ is a subcategory of $\mathrm{Mod}\text{-}J$.

We see from the definition of the J-module structure on M that restriction of scalars to I yields the original I-module structure on M; in particular, every morphism in $\mathrm{Mod}\text{-}J$ between I-modules is also a morphism in $\mathrm{Mod}\text{-}I$, so the subcategory $\mathrm{Mod}\text{-}I$ is indeed a full subcategory of $\mathrm{Mod}\text{-}J$. ☐

Remark 2.40. In general, the subcategory $\mathrm{Mod}\text{-}I$ in $\mathrm{Mod}\text{-}J$ of Proposition 2.39 is not dense: For example, if $I = 0$, then $\mathrm{Mod}\text{-}I$, the category of *unitary* I-modules, contains only one isomorphism class of objects, namely, the class of modules isomorphic to 0. If J is any nontrivial ring, then $J_J \in \mathrm{Mod}\text{-}J$ is not isomorohic to 0, so it is not (isomorphic to) a unitary I-module, i.e., it is not (isomorphic to) an object from $\mathrm{Mod}\text{-}I$.

Lemma 2.41. *If I is a ring with local units contained in a unital ring R as a two-sided ideal and if $A \in \mathrm{FP}\,(I, R)$, then, after restriction of scalars:*

(i) A is finitely generated in $\mathrm{Mod}\text{-}I$.

(ii) A is projective in $\mathrm{Mod}\text{-}I$;

Proof. (i): By the definition of $\mathrm{FP}\,(I, R)$, A is finitely generated as an R-module. Hence, there is a finite subset $X \subseteq_{\text{fin}} A$ satisfying $A = \sum_{x\in X} xR$. Clearly, $\sum_x xI \subseteq \sum_x xR = A$ holds. If, on the other hand, $a \in A$, it is necessarily of the form $a = \sum_x xr_x$ with $r_x \in R$. From the definition of $\mathrm{FP}\,(I, R)$, A is a unitary I-module; thus, by Lemma 2.10, there is an $e \in I$ satisfying $a = ae = \left(\sum_x xr_x\right) e = \sum_x x(r_x e)$, which is an element of $\sum_x xI$. We conclude that $A = \sum_{x\in X} xI$, so as an I-module, A is spanned by the finite set X.

(ii): Assume there is a diagram

$$\begin{tikzcd} & A \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \tag{2.8}$$

in $\mathrm{Mod}\text{-}I$ with the row exact. By Proposition 2.39, we have the same diagram in $\mathrm{Mod}\text{-}R$; note that, by Lemma 2.11, exactness of the row in either of the categories is equivalent to the morphism $B \longrightarrow C$ being onto, whence the row in (2.8) is exact in $\mathrm{Mod}\text{-}R$, too. Since A is projective as an R-module, there exists a morphism $A \longrightarrow B$ in $\mathrm{Mod}\text{-}R$ making the following diagram commutative:

$$\begin{tikzcd} & A \arrow[d] \arrow[dl, dashed, "\exists"'] & \\ B \arrow[r] & C \arrow[r] & 0. \end{tikzcd} \tag{2.9}$$

By Proposition 2.39, $\mathrm{Mod}\text{-}I$ is a full subcategory of $\mathrm{Mod}\text{-}R$; thus, (2.9) is also a commutative diagram in $\mathrm{Mod}\text{-}I$. As we have shown for an arbitrary I-module epimorphism $B \longrightarrow C$ that (2.8) can be completed to (2.9) in $\mathrm{Mod}\text{-}I$, we conclude that A is a projective I-module. ☐

Suppose now that I, J are Morita equivalent rings with local units. Then, there are additive functors $G : \text{Mod-}I \longrightarrow \text{Mod-}J$ and $H : \text{Mod-}J \longrightarrow \text{Mod-}I$ and natural isomorphisms $\eta : HG \longrightarrow \mathrm{Id}_{\text{Mod-}I}$ and $\zeta : GH \longrightarrow \mathrm{Id}_{\text{Mod-}J}$. From G and H, we shall derive maps between $\mathrm{FP}(I, R)$ and $\mathrm{FP}(J, S)$ (with I a two-sided ideal in a unital ring R and J a two-sided ideal in a unital ring S) that will induce mutually inverse monoid homomorphisms between $\mathcal{V}(I)$ and $\mathcal{V}(J)$.

Let $A \in \mathrm{FP}(I, R)$. From the definition of $\mathrm{FP}(I, R)$, we have $AI = A$, whence, after restriction of scalars, A is a unitary I-module. By applying Proposition 2.39 to the unitary J-module GA, we obtain $GA \in \text{Mod-}S$. We shall show that as an S-module, GA is finitely generated and projective.

Lemma 2.42. *The J-module GA is finitely generated. In particular, it is also finitely generated as an S-module.*

Proof. By Lemma 2.12, there is an exact sequence $J^{(X)} \xrightarrow{\varphi} GA \longrightarrow 0$ for some set X. Since H preserves direct sums (Proposition 2.15), we have

$$\begin{tikzcd}
(HJ)^{(X)} \arrow[r, "\xi", "\simeq"'] \arrow[rr, bend left, "\alpha := (H\varphi)\circ\xi"] & H\left(J^{(X)}\right) \arrow[r, "H\varphi"] & HGA \arrow[r, "\eta_A", "\simeq"'] & A
\end{tikzcd}$$

in $\text{Mod-}I$, and since H is exact (Proposition 2.21), $H\varphi$ is surjective. Since ξ is an isomorphism, $\alpha := (H\varphi)\xi$ is also surjective. Since A is finitely generated in $\text{Mod-}R$, it is, by Lemma 2.41(i), a finitely generated I-module. From $HGA \simeq A$, we infer that HGA is a finitely generated I-module, too. With $HGA = \mathrm{Im}\left(\bigoplus_{x\in X} \alpha_x\right) = \sum_{x \in X} \mathrm{Im}\,\alpha_x$, where α_x denotes the morphism from the x-th copy of HJ to HGA, and with each $\mathrm{Im}\,\alpha_x$ being an I-submodule of HGA, there is a finite subset $Y \subseteq_{\text{fin}} X$ such that $HGA = \sum_{x\in Y} \mathrm{Im}\,\alpha_x$. Hence, $\bigoplus_Y \alpha_x : (HJ)^{(Y)} \longrightarrow HGA$ is an epimorphism in $\text{Mod-}I$; by exactness of G (Proposition 2.21), we have that $G\left(\bigoplus_Y \alpha_x\right) : G\left((HJ)^{(Y)}\right) \longrightarrow GHGA$ is an epimorphism in $\text{Mod-}J$. With H preserving direct limits, the following composition is a composition of an epimorphism with isomorphisms, whence it is an epimorphism from $J^{(Y)}$ to GA in $\text{Mod-}J$:

$$\begin{tikzcd}
J^{(Y)} \arrow[r, "\eta^{-1}_{J^{(Y)}}", "\simeq"'] \arrow[rrrr, bend right] & GH\left(J^{(Y)}\right) \arrow[r, "\simeq"'] & G\left((HJ)^{(Y)}\right) \arrow[r, "G\left(\bigoplus_Y \alpha_x\right)"] & GHGA \arrow[r, "G(\eta_A)", "\simeq"'] & GA.
\end{tikzcd}$$

With Y being finite, we conclude that GA is a finitely generated J-module, as desired. That it is also finitely generated as an S-module now follows from Proposition 2.39, since a finite set spanning GA as a J-module spans it as an S-module as well. $\square$

Lemma 2.43. *As an S-module, GA is projective.*

Proof. Let us have a diagram

$$\begin{tikzcd}
 & GA \arrow[d, "\varphi"] & \\
M \arrow[r, "\psi"] & N \arrow[r] & 0
\end{tikzcd}$$

in Mod-S with the row exact; we want to find a homomorphism $\overline{\varphi} : GA \longrightarrow M$ satisfying $\psi\overline{\varphi} = \varphi$.

Since GA is unitary, we have, for every $x \in GA$, an idempotent $e_x \in J$ satisfying $xe_x = x$ (Lemma 2.10). Then $\varphi x = \varphi(xe_x) = (\varphi x)e_x$ holds, so $\varphi x \in NJ$. As this holds for any $x \in GA$, we obtain $\operatorname{Im}\varphi \subseteq NJ$. Denoting by ι_N the inclusion map $NJ \subseteq N$, we then have a commutative diagram

$$\begin{tikzcd} GA \arrow[dd, "\varphi"] \arrow[dr, "\varphi'"] & \\ & NJ \arrow[dl, "\iota_N"] \\ N & \end{tikzcd} \tag{2.10}$$

in Mod-S. Similarly, considering the S-submodule MJ of M, we see that $\operatorname{Im}(\psi|_{MJ}) \subseteq NJ$. Hence, we have the following commutative diagram in Mod-S:

$$\begin{tikzcd} MJ \arrow[r, "\psi|_{MJ}"] \arrow[d, "\iota_M"] & NJ \arrow[d, "\iota_N"] \\ M \arrow[r, "\psi"] & N. \end{tikzcd} \tag{2.11}$$

If, on the other hand, $n \in N$ and $e \in \operatorname{Idemp} J$, then, by surjectivity of ψ, there is an $m \in M$ such that $\psi m = n$. Then $ne = (\psi m)e = \psi(me) \in \operatorname{Im}\psi|_{MJ}$. Thus, $\psi|_{MJ} : MJ \longrightarrow NJ$ is surjective, and as such an epimorphism (Lemma 2.11).

By idempotence of rings with local units, MJ and NJ are unitary J-modules; hence, in Mod-J, we have the following diagram with the row exact:

$$\begin{tikzcd} & GA \arrow[d, "\varphi'"] & \\ MJ \arrow[r, "\psi|_{MJ}"] & NJ \arrow[r] & 0. \end{tikzcd}$$

Applying the exact functor H (Proposition 2.21), we obtain

$$\begin{tikzcd} & HGA \arrow[d, "H\varphi'"] & \\ H(MJ) \arrow[r, "H(\psi|_{MJ})"] & H(NJ) \arrow[r] & 0. \end{tikzcd}$$

in Mod-I with the row exact. By Lemma 2.41(ii), A is projective as an I-module, whence, with $\eta : HG \longrightarrow \mathrm{Id}_{\text{Mod-}I}$ a natural isomorphism, there exists an I-homomorphism ξ making the following diagram commutative:

$$\begin{tikzcd} & A \arrow[d, "\eta_A^{-1}", "\simeq"'] \arrow[ddl, "\xi"'] \\ & HGA \arrow[d, "H\varphi'"] \\ H(MJ) \arrow[r, "H(\psi|_{MJ})"] & H(NJ). \end{tikzcd}$$

Applying the functor G, with $\zeta : GH \longrightarrow \mathrm{Id}_{\mathrm{Mod}\text{-}J}$ a natural isomorphism, we obtain the following commutative diagram in Mod-J:

$$\begin{tikzcd}
& GA \arrow[d, "G(\eta_A^{-1})=(G\eta_A)^{-1}", "\simeq"'] \arrow[dl, "G\xi"'] & \\
& GHGA \arrow[r, "\zeta_{GA}", "\simeq"'] \arrow[d, "GH\varphi'"] & GA \arrow[d, "\varphi'"] \\
GH(MJ) \arrow[r, "GH(\psi|_{MJ})"] & GH(NJ) \arrow[r, "\zeta_{NJ}", "\simeq"'] & NJ.
\end{tikzcd} \tag{2.12}$$

Using ζ again, (2.12) yields:

$$\begin{tikzcd}
GA \arrow[r, "(G\eta_A)^{-1}", "\simeq"'] \arrow[d, "G\xi"] & GHGA \arrow[r, "\zeta_{GA}", "\simeq"'] \arrow[d, "GH\varphi'"] & GA \arrow[dd, "\varphi'"] \\
GH(MJ) \arrow[r, "GH(\psi|_{MJ})"] \arrow[d, "\zeta_{MJ}"] & GH(NJ) \arrow[dr, "\zeta_{NJ}"] & \\
MJ \arrow[rr, "\psi|_{MJ}"] & & NJ.
\end{tikzcd} \tag{2.13}$$

Put $\nu := \zeta_{MJ} \circ G\xi \circ G\eta_A \circ \zeta_{GA} : GA \longrightarrow MJ$ (a morphism in Mod-J). From the commutativity of (2.13), we then have $\varphi' = (\psi|_{MJ})\nu$. Transferring to Mod-$S$ (Proposition 2.39) and using commutativity of (2.10) and (2.11), we obtain the following commutative diagram in Mod-S:

$$\begin{tikzcd}
& GA \arrow[dl, "\nu"'] \arrow[d, "\varphi'"] \arrow[dd, bend left=60, "\varphi"] \\
MJ \arrow[r, "\psi|_{MJ}"] \arrow[d, "\iota_M"] & NJ \arrow[d, "\iota_N"] \\
M \arrow[r, "\psi"] & N.
\end{tikzcd}$$

Putting $\overline{\varphi} := \iota_M\nu$ yields the desired factorization of φ through ψ in Mod-S. $\square$

Combining Lemmas 2.42 and 2.43 with Proposition 2.39, we conclude that:

Proposition 2.44. *Let I, J be rings with local units Morita equivalent via*

$$\mathrm{Mod}\text{-}I \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Mod}\text{-}J.$$

Let I be a two-sided ideal in a unital ring R, J a two-sided ideal in a unital ring S and let $A \in \mathrm{FP}(I, R)$. Then $GA \in \mathrm{FP}(J, S)$. $\square$

Theorem 2.45. *Let I, J be rings with local units Morita equivalent via*

$$\mathrm{Mod}\text{-}I \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Mod}\text{-}J.$$

Then $\mathcal{V}(I) \simeq \mathcal{V}(J)$.

Proof. Let R, S be unital rings, R containing I and S containing J, both as two-sided ideals. If $A, B \in \mathrm{FP}(I, R)$ satisfy $A \simeq B$ in $\mathrm{Mod}\text{-}R$, then A and B are also isomorphic in $\mathrm{Mod}\text{-}I$. Then, since G is a functor, the J-modules GA and GB are isomorphic; notice that then they are also isomorphic as S-modules. Thus, $[GA] = [GB]$ holds in $\mathcal{V}(J)$ (due to Proposition 2.44, GA and GB are elements of $\mathrm{FP}(J, S)$, whence it makes sense to consider their S-isomorphism classes as elements of $\mathcal{V}(J)$). It now follows that G induces a map

$$\begin{aligned} \tilde{G} : \mathcal{V}(I) &\longrightarrow \mathcal{V}(J) \\ [A] &\longmapsto [GA]. \end{aligned}$$

Similarly, there is a map $\tilde{H} : \mathcal{V}(J) \longrightarrow \mathcal{V}(I)$ induced by H. Since $HGA \simeq A$ as I-modules holds for any module $A \in \mathrm{FP}(I, R)$, we infer that also $[A] = [HGA]$ in $\mathcal{V}(I)$ (use Proposition 2.39). Thus, $\tilde{H}\tilde{G} = \mathrm{id}_{\mathcal{V}(I)}$. Symmetrically, $\tilde{G}\tilde{H} = \mathrm{id}_{\mathcal{V}(J)}$. To conclude that the monoids $\mathcal{V}(I)$ and $\mathcal{V}(J)$ are isomorphic, it remains to show that $\tilde{G}$ and $\tilde{H}$ are monoid homomorphisms. To that end, notice that for $A, B \in \mathrm{FP}(I, R)$, the direct sum $A \oplus B$ in $\mathrm{Mod}\text{-}R$ is aslo an element of $\mathrm{FP}(I, R)$, and after restriction of scalars, the same module is also the direct sum of A and B in $\mathrm{Mod}\text{-}I$; that $\tilde{G}$ and $\tilde{H}$ are homomorphisms in $\mathscr{Mon}$ now follows from Proposition 2.15. ☐

Chapter 3

Leavitt path algebras

In this place, we overview some results on Leavitt path algebras related with the realization problem.

3.1 Quivers and path algebras

3.1.1 Quivers and their duals

Let us first fix terminology we use later in the chapter. In our introduction of the Leavitt path algebras, we mostly follow the outline of Goodearl (2009).

A *quiver*[1] $E = (E^0, E^1, s, r)$ consists of disjoint sets E^0 and E^1 and maps $s, r : E^1 \longrightarrow E^0$. We always assume that the set E^0 is not empty. We refer to elements of E^0 as *vertices* of E and to elements of E^1 as *edges* (or *arrows*) of E. For an edge e, the vertex $s(e)$ is called the *source* of e and $r(e)$ is called the *range* of e. We then say that e is an edge from $s(e)$ to $r(e)$. We also say that $s(e)$ *emits* e and that $r(e)$ *receives* e. A vertex is called a *receiver* if it receives at least one edge, an *emitter* if it emits at least one edge, and an *infinite emitter* if it emits infinitely many edges. A vertex that is not an emitter is called a *sink*, while it is called a *source*[2] if it is not a receiver. A vertex is called *singular* if it is either a sink or an infinite emitter, and a *regular vertex* is a vertex that is not singular (that is, a vertex $v \in E^0$ is regular iff $0 < |s_E^{-1}(v)| < \infty$). A *loop* is an edge with the same source and range.

If $E = (E^0, E^1, s_E, r_E)$, $F = (F^0, F^1, s_F, r_F)$ are quivers, we say that F is a *subquiver* of E if $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, $s_F = s_E|_F$ and $r_F = r_E|F$.

Since our primary focus is on countable monoids, we restrict our attention to countable quivers only; hence, by a quiver, we will always mean a countable one, that is, with only countably many vertices and edges.

A *path* of *length* n in a quiver is a sequence $e_1, e_2, \ldots, e_n$ of edges satisfying $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n-1$. We shall often write paths in the form of a product[3], that is, $p = e_1 e_2 \cdots e_n$ stands for the path above. We also say that $s(e_1)$

[1] Also called "graph" or "directed graph" in papers cited.

[2] The disticntion between the source of an edge and a source in a quiver will be clear at all times.

[3] Note that in path algebras, the product $e_1 e_2$ for $e_1, e_2 \in E^1$ will be defined even if $r(e_1) \neq s(e_2)$. By saying that $p = e_1 e_2 \cdots e_n$ is a *path*, we imply that the ranges and sources of adjacent edges match.

is the source of p and $r(e_2)$ the range of p and denote these two vertices by $s(p)$ and $r(p)$, respectively. We consider each vertex v to be a path in E, specifically, a path of legth zero; the source, $s(v)$, and range, $r(v)$, of such path is v itself.

A *cycle* in a quiver E is a path p in E of length $n > 0$ satisfying $s(p) = r(p)$; an *acyclic* quiver is a quiver containing no cycles (so an acyclic quiver is such that its only paths with the same source and range are the vertices, i.e., paths of length zero).

We say that a quiver is *row-finite* if it has no infinite emitters.

The *dual* of a quiver E is the quiver[4] $E^* = ((E^*)^0, (E^*)^1, s, r)$ consisting of the same vertices as E (that is, $(E^*)^0 = E^0$) and arrows from E *reversed*, i.e., $(E^*)^1 := \{\, e^* \mid e \in E^1 \,\}$ with $s(e^*) = r(e)$ and $r(e^*) = s(e)$ for all $e \in E^1$. If $p = e_1 e_2 \cdots e_n$ is a path in E, we denote by p^* the corresponding path in E^*: $p^* = e_n^* e_{n-1}^* \cdots e_1^*$.

By the *double* of a quiver E, denoted by $\widehat{E}$, we mean the union of E and its dual, E^*, with the two sets of edges considered disjoint. We call the edges (or paths) from E *real* and edges (paths) from E^* *ghosts*.

For quivers $E = (E^0, E^1, s_E, r_E)$, $F = (F^0, F^1, s_F, r_F)$, a *quiver homomorphism* from E to F is given by two maps, $\varphi^0 : E^0 \longrightarrow F^0$ and $\varphi^1 : E^1 \longrightarrow F^1$, satisfying the following compatibility conditions:

- $s_F(\varphi^1 e) = \varphi^0(s_F(e))$, and
- $r_F(\varphi^1 e) = \varphi^0(r_F(e))$ for all $e \in E^1$.

In other words, a quiver homomorphism $E \longrightarrow F$ is a map $E^0 \,\dot{\cup}\, E^1 \longrightarrow F^0 \,\dot{\cup}\, F^1$ that respects sources and targets. A homomorphism φ is called *complete* if it is injective (both on vertices and on edges) and if it maps $s_E^{-1}(v)$ onto $s_F^{-1}(\varphi^0(v))$ bijectively, whenever $v \in E^0$ is a regular vertex. We say that a subquiver F of E is a *complete subquiver* of E if the inclusion map $F \longrightarrow E$ is a complete quiver homomorphism.

Observation 3.1. *(i) The identity map on a quiver is a complete homomorphism. If $E \longrightarrow F \longrightarrow G$ are complete quiver homomorphisms, then so is their composition. Hence, (countable) quivers as objects and complete quiver homomorphisms as morphisms form a category. We denote this category by $\mathscr{Q}uiv$.*

(ii) If $\varphi : E \longrightarrow F$ is a complete quiver homomorphism, then the quiver $\varphi(E) = \big(\varphi(E^0), \varphi(E^1), s_F|_{\varphi(E^1)}, r_F|_{\varphi(E^1)}\big)$ is a complete subquiver of F. □

We denote the full subcategory of $\mathscr{Q}uiv$ consisting of all countable row-finite quivers and complete homomorphisms between such by $\mathscr{rfQ}uiv$.[5]

Proposition 3.2. *(i) (Goodearl, 2009, Lemma 2.5 (a)) Arbitrary direct limits exist in the category $\mathscr{rfQ}uiv$.*

[4]Some authors use E^* to denote the set of all paths in E; we prefer the notation presented here, with * denoting what will be called "taking ghosts" of edges or paths.

[5]$\mathscr{rfQ}uiv$ is the category $\mathcal{G}$ of Ara et al. (2007), while $\mathscr{Q}uiv$ is the full subcategory of countable quivers of the category **CKGr** from Goodearl (2009).

(ii) *(Ara et al., 2007, Lemma 3.1) Every row-finite quiver is a direct limit in the category $\mathscr{rfQuiv}$ of the directed system of all its finite complete subquivers.*□

Example 3.3 (the infinite rose[6] quiver)**.** We define the "infinite rose quiver" R_∞ as a quiver with one vertex and (countably) infinitely many edges, necessarily loops from the single vertex to itself. We picture R_∞ as

$$^{(\infty)}\circlearrowright \bullet$$

(when we depict quivers graphically, a positive integer n or the infinity sign, ∞, in parentheses near an arrow symbolize that there are n or countably infinitely many arrows, respectively, with the same source and range as the arrow in the picture). Let E be a finite quiver and $\varphi : E \longrightarrow R_\infty$ a complete graph homomorpihsm. Since φ is injective, E has only one vertex, say, $E^0 = \{ v \}$. Suppose that the set E^1 is nonempty. Then, since φ is complete, there is a bijection between $s^{-1}(v) = E^1$ and $(R_\infty)^1$, the infinite set of edges of R_∞. Hence, the quiver E is not finite, a contradiction. Thus, unlike the row-finite case (Proposition 3.2(ii)), R_∞ is not a direct limit of finite graphs in $\mathscr{Quiv}$.

3.1.2 Path algebras

For a quiver E and a field K, we define the *path algebra* of E over K as the K-algebra with basis the set of all paths in E, and with the following multiplication: If p, q are paths in E, we let pq be the concatenation of p and q if $r(p) = s(q)$, and zero otherwise. This in particular means that:

- If $p = e_1 \cdots e_n$ and $q = f_1 \cdots f_m$ are paths of strictly positive length with $r(e_n) = s(f_1)$, then $pq = e_1 \cdots e_n f_1 \cdots f_m$.
- If p is a path, then $s(p)p = p = pr(p)$. In particular, for a vertex v, $v^2 = v$.

We denote the path algebra of E over K by KE.

Example 3.4. Let V_∞ denote the quiver consisting of a countably infinite set of vertices and no edges. Then $KV_\infty \simeq K^{(\omega)}$ is a nonunital regular algebra (we've already seen this algebra in Remark 2.18). We see that the monoid $\mathcal{V}(KV_\infty) \simeq (\mathbb{N}_0)^{(\omega)}$ does not have an order-unit; in particular, it cannot be realized by any *unital* regular ring.

Observe that if the set E^0 is finite, then the sum of all vertices of E is a unit in KE, irrespective of the cardinality of the set E^1. On the other hand, if the set E^0 is infinite, then the algebra KE cannot have a unit: Suppose the contrary, that is, that $1 = \sum_{p \text{ a path in } E} \alpha_p p$ is a unit in KE, with $\alpha_p \in K$. Then, for every vertex $v \in E^0$, we would have

$$v = v \cdot 1 = \sum_{\substack{p \text{ a path in } E, \\ s(p) = v}} \alpha_p p.$$

[6]Recall that a rose is a rose is a rose, even if it is infinite.

Since the set of all paths in E is linearly independent over K, we conclude that $\alpha_v = 1$ for every $v \in E^1$, a contradiction, since only finitely many α_p's can be nonzero. Nevertheless, even for E^0 infinite, we have:

Lemma 3.5. *For a quiver E and a field K, the path algebra KE has local units.*

Proof. Given a finite set P of paths in E (also admitting paths of length zero, i.e., vertices, as elements of P), put $V := \{\, s(p) \mid p \in P \,\} \cup \{\, r(p) \mid p \in P \,\}$. Then, putting $x := \sum_{v \in V} v$, we see that x is an idempotent in KE satisfying $xp = p = px$ for all $p \in P$. □

Definition 3.6. The *Leavitt path algebra* of E over K is the quotient of the path algebra $K\widehat{E}$ (i.e., of the path algebra of the *double* of E over K) modulo the ideal generated by the following elements:

- $e^*e - r(e)$ for every $e \in E^1$;
- e^*e' for all pairs e, e' of distinct edges in E;
- $v - \sum\limits_{\substack{e \in E^1, \\ v = s(e)}} ee^*$ for all vertices v with $0 < |s^{-1}(v)| < \infty$.

We denote said algebra by $L_K(E)$ or $L(E)$.

Dealing with elements of $L(E)$, we will use the same names for elements of $K\widehat{E}$ and their cosets in $L(E)$. Thus, the following relations hold true in $L(E)$:

(CK1) $e^*e' = \begin{cases} r(e) & \text{if } e = e', \\ 0 & \text{otherwise;} \end{cases}$

(CK2) $v = \sum\limits_{\substack{e \in E^1, \\ v = s(e)}} ee^*$ for every $v \in E^0$ that is neither a sink nor an infinite emitter.

Remark 3.7. For a quiver E, applying Lemma 3.5 to the quiver $\widehat{E}$, we have that $K\widehat{E}$ is a ring with local units. As a quotient of $K\widehat{E}$, $L(E)$ is always a ring with local units, and, moreover, it is a unital ring if the set E^0 is finite.

A few words on which elements of $K\widehat{E}$ remain distinct in $L(E)$ are in order: Firstly, (the cosets of) vertices are not only distinct, but also linearly independent:

Lemma 3.8 ((Goodearl, 2009, Lemma 1.5))**.** *Let K be a field and E a quiver. Then the cosets of the vertices from E^0 are K-linearly independent elements of $L_K(E)$.* □

In (Siles Molina, 2008, Lemma 1.1), it is shown that distinct real paths in $L(E)$ are linearly independent; in Goodearl (2009), this has been extended to include ghost paths:

Lemma 3.9 ((Goodearl, 2009, Lemma 1.6))**.** *Then the quotient map $K\widehat{E} \longrightarrow L_K(E)$ restricts to an embedding of the subspace $KE + KE^*$ of $K\widehat{E}$ into $L_K(E)$.* □

In $K\widehat{E}$, any set consisting of distinct real paths, distinct ghost paths and distinct vertices is linearly independent. Thus, Lemma 3.9 tells us that the same holds in $L(E)$.

Regarding K-dimension of KE or $L_K(E)$, we see from the definitions that:

Observation 3.10. *For a field K and a quiver E, the following are equivalent:*

(i) The quiver E is finite *(in the sense that both E^0 and E^1 are finite sets) and acyclic;*

(ii) $\dim_K KE$ *is finite;*

(iii) $\dim_K KE^*$ *is finite.* □

However, if $E^1 \neq \emptyset$, then $\widehat{E}$ contains a cycle, whence the K-dimension of $K\widehat{E}$ is infinite; nevertheless, the necessary and sufficient conditions for $L(E)$ to be finite-dimensional are the same as for KE by a result of Abrams et al. (2007):

Proposition 3.11 ((Abrams et al., 2007, Corollary 3.6))**.** *For E a quiver and K a field, the Leavitt path algebra $L_K(E)$ is a finite-dimensional K-algebra iff E is finite and acyclic.* □

Functoriality of taking Leavitt path algebras

It is shown in (Goodearl, 2009, §2.4) that for a field K, the assignment $E \longmapsto L_K(E)$ can be extended to a functor $L_K(-)$ from $\mathscr{Quiv}$ to the category of K-algebras, and in (Goodearl, 2009, Lemma 2.5(b)) that this functor is continuous. This functor plays a role in the naturality of the isomorphism of Theorem 3.17.

3.1.3 Regularity conditions for Leavitt path algebras

An important question from our perspective is whether Leavitt path algebras can be regular rings, and if so, then under what conditions. A result on this topic is the following:

Theorem 3.12 ((Abrams – Rangaswamy, 2010, Theorem 1))**.** *For a quiver E and a field K, the following are equivalent:*

(i) $L_K(E)$ is a regular ring;

(ii) E is acyclic;

(iii) $L_K(E)$ is locally K-matricial, i.e., it is the direct union of subrings, each of which is isomorphic to a finite direct sum of finite matrix rings over K.

Notice that unlike Propositon 3.11, there is no finiteness condition imposed on E in (ii) in the above theorem. Thus, there also are infinite-dimensional Leavitt path algebras that are regular.

3.2 $\mathcal{V}(-)$ of Leavitt path algebras

3.2.1 The monoid M_E associated with a quiver E

Now that we have that some regular rings can be obtained as Leavitt path algebras, we are interested in what the monoid $\mathcal{V}(-)$ of such rings can look like. Fortunately, it can be described in terms of generators and relations between them based on the quiver E (Theorem 3.17).

Definition 3.13. Let E be a (general) quiver. Let us denote by F_{E^0} the free abelian monoid freely generated by the set E^0 (i.e., by vertices of E; cf. (Burris – Sankappanavar, 2012, Definition 10.5)), and let Λ_E be the congruence on F_{E^0} generated by the relations

$$v \equiv \sum_{e \in s^{-1}(v)} r(e) \text{ for every } \textit{regular} \text{ vertex } v \in E^0. \tag{3.1}$$

We put M_E to be the factor monoid F_{E^0}/Λ_E; we call M_E the monoid *associated with E*. When computing with the monoid M_E, we shall denote the Λ_E-class of a vertex $v \in E^0$ also by v.

Remark 3.14. A monoid associated with a quiver E has a presentation

$$\langle X | \Delta \rangle\,, \tag{3.2}$$

where $X = E^0$ is a countable set and $\Delta = \{\, x = \sum_{y \in X} n_{xy} y \mid x \in X \,\}$, with n_{xy} elements of $\mathbb{N}_0$, all but finitely many of them nonzero for each x. Conversely, any abelian monoid with such presentation is associated with a suitable quiver E: Let $E^0 := X$ be the set of vertices, and for all pairs $x, y \in X$, let there be n_{xy} arrows from x to y in E. Then indeed M_E has presentation (3.2).

Immediately from the definition, we see that we can slightly modify a quiver without affecting the monoid M_E:

Observation 3.15. *Let E, F be quivers such that F can be obtained from E by removing a regular vertex $v \in E^0$ and adding an edge $f_{(e,e')}$ from $s(e)$ to $r(e')$ for each* pair *of edges $e, e' \in E^1$ satisfying $r(e) = v$ and $s(e') = v$; that is, we take a regular vertex $v \in E^0$ and put $F^0 = E^0 \setminus \{\, v \,\}$ and*

$$F^1 = \left(E^1 \setminus (s_E^{-1}(v) \cup r_E^{-1}(v))\right) \dot{\cup} \{\, f_{(e,e')} \mid e \in r_E^{-1}(v), e' \in s_E^{-1}(v) \,\}\,,$$

where $s_F\left(f_{(e,e')}\right) = s_E(e)$ and $r_F\left(f_{(e,e')}\right) = r_E(e')$ for all relevant e, e'. Then

$$\begin{aligned} M_F &\longrightarrow M_E \\ w &\longmapsto w, \qquad\qquad w \in F^0, \end{aligned}$$

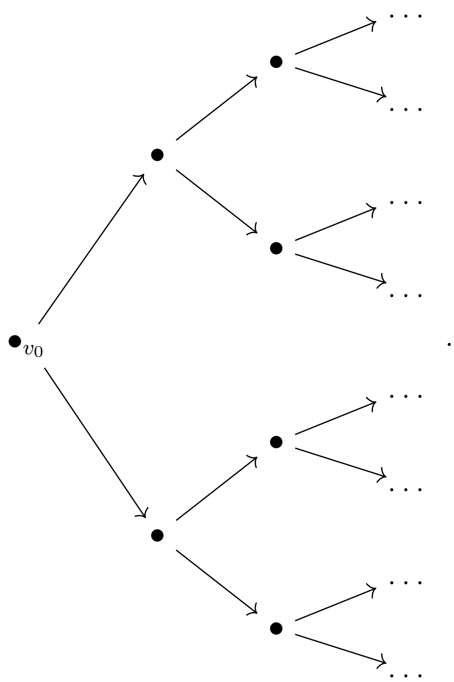
is a monoid isomorphism. In particular, removing a source that is regular from a quiver E does not affect M_E. ◻

For row-finite quivers, the following holds:

Lemma 3.16 ((Ara et al., 2007, Lemma 3.4)). *The assignment $E \longmapsto M_E$ can be extended to a continuous functor from $\mathscr{RQ}uiv$ to $\mathscr{M}on$. With Proposition 3.2* (ii), *it follows that every monoid of the form M_E is a direct limit of finite monoids $M_{E'}$ in the category $\mathscr{RQ}uiv$.* □

Theorem 3.17 ((Ara et al., 2007, Theorem 3.5)). *For a row-finite quiver E, there is a monoid isomorphism $\gamma_E : M_E \longrightarrow \mathcal{V}(L_K(E))$, natural in the sense that if $\varphi : M_E \longrightarrow M_F$ is a morphism in $\mathscr{RQ}uiv$, then $\mathcal{V}(L_K(\varphi))\,\gamma_E = \gamma_F M_\varphi$.* □

Example 3.18 (the binary tree quiver). Consider the infinite binary tree quiver, E, as in the diagram:



Then in the algebraic preorder on M_E, $v \le v_0$ holds for each $v \in E^0$. Thus, for a general element $x = \sum_v n_v v$ of M_E, we have $x \le (\sum_v n_v)\, v_0$. We conclude that v_0 is an order-unit in M_E. Since there are no cycles in E, the Leavitt path algebra $L_K(E)$ is regular by Theorem 3.12. As E is row-finite, the monoid $\mathcal{V}(L_K(E))$ is isomorphic to M_E by Theorem 3.17; hence, $\mathcal{V}(L_K(E))$ has an order-unit. Since the quiver E has infinitely many vertices, the algebra $L_K(E)$ is not unital. Thus, the monoid $\mathcal{V}(L_K(E))$ is an example of a countable conical refinement monoid *with order-unit* realizable by a nonunital regular ring, where we do not know if it is also realizable by a unital ring.

3.2.2 A nonstandard construction of the additive monoid of nonnegative rationals as $\mathcal{V}(A)$ with A a regular Leavitt path algebra

We shall now construct an acyclic row-finite quiver E such that M_E is isomorphic to $\mathbb{Q}^{\ge 0}$, the additive monoid of nonnegative rational numbers. With E acyclic, we will have that for any filed K, the algebra $L_K(E)$ is regular (by Theorem 3.12), and by Theorem 3.17 that $\mathbb{Q}^{\ge 0} \simeq M_E \simeq \mathcal{V}(L_K(E))$.

For $i \in \mathbb{N}$, let p_i denote the i-th prime number, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$ etc.

Theorem 3.19. *Let $E^0 := \{ v_1, v_2, v_3, \dots \} = \{ v_i \mid i \in \mathbb{N} \}$ be a countably infinite set of vertices. Let E be a quiver with vertex set E^0 such that for each n, there are $p_1 p_2 \cdots p_n = \prod_{i=1}^{n} p_i$ arrows from v_n to v_{n+1}, and such that there are no edges from v_n to any v_m except v_{n+1}. Then, for any field K, the Leavitt path algebra $L_K(E)$ is regular and $\mathcal{V}(L_K(E)) \simeq \mathbb{Q}^{\geq 0}$.*

Proof. The quiver E is as in the following diagram, with the number in parentheses above an arrow indicating the number of arrows with the same source and range:

$$\bullet_{v_1} \xrightarrow{(2)} \bullet_{v_2} \xrightarrow{(6)} \bullet_{v_3} \xrightarrow{(30)} \bullet_{v_4} \xrightarrow{(210)} \cdots .$$

Claim 1. *For $m > n$, the equality*

$$v_n = \left(\prod_{j=n}^{m-1} \prod_{i=1}^{j} p_i \right) v_m = (p_1^{m-n} p_2^{m-n} \cdots p_n^{m-n})(p_{n+1}^{m-n-1} p_{n+2}^{m-n-2} \cdots p_{m-2}^{2} p_{m-1}) v_m$$

holds in M_E.

Proof of Claim. By (3.1), $v_n = \sum_{e \in s^{-1}(v_n)} v_{n+1}$ holds in E, so $v_n = k v_{n+1}$, where k is the number of arrows from v_n to v_{n+1} in E. From the definition of E, $k = \prod_{i=1}^{n} p_i$. Proceed by induction. □ *Claim 1.*

It is clear that E is acyclic, so $L_K(E)$ is regular by Theorem 3.12. By Theorem 3.17, we only need to show that the monoids M_E and $\mathbb{Q}^{\geq 0}$ are isomorphic. To that end, we shall construct monoid homomorphisms $M_E \longrightarrow \mathbb{Q}^{\geq 0}$ and $\mathbb{Q}^{\geq 0} \longrightarrow M_E$ that compose to identities on M_E and $\mathbb{Q}^{\geq 0}$, respectively. Let us begin with the one from M_E to $\mathbb{Q}^{\geq 0}$:

With the set E^0 freely generating F_{E^0}, the assingnemnt

$$v_n \longmapsto \prod_{i=1}^{n} p_i^{i-n} = \frac{1}{p_1^{n-1} \cdot p_2^{n-2} \cdot \dots \cdot p_{n-2}^{2} \cdot p_{n-1}} \tag{3.3}$$

for each $n \in \mathbb{N}$ defines a monoid homomorphism $\psi : F_{E^0} \longrightarrow \mathbb{Q}^{\geq 0}$. Notice that then, for each n,

$$\psi(v_{n+1}) = \frac{1}{p_1 p_2 \cdots p_n} \psi(v_n)$$

holds. Thus, since there are $\prod_{i=1}^{n} p_i$ distinct arrows from v_n to v_{n+1} in E and since ψ is a monoid homomorphism, we have

$$\psi(v_n) = \left(\prod_{i=1}^{n} p_i \right) \psi(v_{n+1}) = \sum_{e \in s^{-1}(v_n)} \psi(v_{n+1}) = \psi \left(\sum_{e \in s^{-1}(v_n)} v_{n+1} \right)$$

for each $n \in \mathbb{N}$. Hence, ψ respects the congruence Λ_E, whence (3.3) also defines a monoid homomorphism $\overline{\psi} : M_E \longrightarrow \mathbb{Q}^{\geq 0}$.

For the opposite direction, let us first define a map φ from $\mathbb{N}_0 \times \mathbb{N}$ to M_E. For a $q \in \mathbb{N}$, with $q = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and with $a_n \neq 0$, put

$$j_q := \max \{ a_i + i \mid i = 1, \dots, n \}$$

and

$$\pi_q := \prod_{i=1}^{j_q} p_i^{j_q - a_i - i}. \tag{3.4}$$

From the definition of j_q, the inequality $j_q - a_i - i \geq 0$ holds for all $i \leq j_q$. Also, $j_q - a_{j_q} - j_q = 0$ holds, so we can write $\pi_q = \prod_{i=1}^{j_q-1} p_i^{j_q-a_i-i}$ instead of (3.4). We let $j_1 := 1$ and $\pi_1 := 1$. Now, for any $(p,q) \in \mathbb{N}_0 \times \mathbb{N}$, put

$$\varphi((p,q)) := (p \cdot \pi_q) \cdot v_{j_q}. \tag{3.5}$$

Claim 2. *For $(p,q) \in \mathbb{N}_0 \times \mathbb{N}$ and $m \in \mathbb{N}$, $\varphi((p,q)) = \varphi((mp,mq))$ holds.*

Proof of Claim. For $m = 1$, the assertion is trivial.

Suppose now that m is a prime, that is, $m = p_k$ for some $k \in \mathbb{N}$. If $q = 1$, then $j_{qm} = j_{p_k} = k + 1$ and

$$\pi_{qm} = \pi_{p_k} = \prod_{i=1}^{k-1} p_i^{k+1-i} = p_1^k p_2^{k-1} \cdots p_{k-1}^2; \tag{3.6}$$

hence,

$$\begin{aligned}
\varphi((mp,mq)) &= \varphi((p \cdot p_k, p_k)) \\
&= ((pp_k)\pi_{p_k})v_{k+1} && \text{by (3.5)}, \\
&= \left(p \prod_{i=1}^{k} p_i^{k+1-i}\right) v_{k+1} && \text{by (3.6)}, \\
&= p \left(\prod_{j=1}^{k} \prod_{i=1}^{j} p_i\right) v_{k+1} \\
&= pv_1 && \text{by Claim 1}, \\
&= p\pi_1 v_1 && \text{from } \pi_1 = 1, \\
&= \varphi((p,q)) && \text{from } q = 1 \text{ and (3.5)},
\end{aligned}$$

as asserted.

Let now $q \neq 1$, so we can write $q = p_1^{a_1} \cdots p_n^{a_n}$ with $a_n \neq 0$. Let us divide the situation into three cases, depending on the relationship between j_q and $a_k + k$:

- If $j_q > a_k + k$, then $j_q = j_{mq}$. We see from (3.4) that $\pi_{qm} = \frac{\pi_q}{m}$, whence $\varphi((pm,qm)) = (pm\frac{\pi_q}{m})v_{j_{qm}} = (p\pi_q)v_{j_q} = \varphi((p,q))$.

- If $j_q = a_k + k$, then $j_{qm} = j_q + 1$, so

$$\pi_{qm} = \pi_q \cdot p_1 p_2 \cdots p_{k-1} \cdot p_{k+1} \cdots p_{j_q} = \pi_q \cdot \frac{1}{m} \cdot \prod_{i=1}^{j_q} p_i.$$

Thus,

$$\begin{aligned}
\varphi((pm, qm)) &= \left(pm \frac{\pi_q}{m} \prod_{i=1}^{j_q} p_i \right) v_{j_{qm}} && \text{by (3.5),} \\
&= p\pi_q \left(\prod_{i=1}^{j_q} p_i \right) v_{j_q+1} && \text{from } j_{qm} = j_q + 1, \\
&= p\pi_q v_{j_q} && \text{by Claim 1,} \\
&= \varphi((p, q)).
\end{aligned}$$

- If $j_q < a_k + k$, then we see from the definition of j_q that $k > n$, and that $j_{qm} = k + 1$ holds. Defining $a_i := 0$ for all $n < i < k$ and $a_k := 1$ (so that $qm = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$), we then have:

$$\begin{aligned}
\pi_{qm} &= \prod_{i=1}^{k} p_i^{k+1-a_i-i} = \prod_{i=1}^{k-1} p_i^{k+1-a_i-i} \\
&= \left(\prod_{i=1}^{j_q} \left(\left(p_i^{j_q-a_i-1} \right) \left(p_i^{k+1-j_q} \right) \right) \right) \left(\prod_{i=j_q+1}^{k-1} p_i^{k+1-i} \right) \\
&= \left(\prod_{i=1}^{j_q} p_i^{j_q-a_i-1} \right) \left(\prod_{i=1}^{j_q} p_i^{k+1-j_q} \right) \left(\prod_{i=j_q+1}^{k-1} p_i^{k+1-i} \right) \frac{p_k}{p_k} \\
&= \pi_q \left(\prod_{i=1}^{j_q} p_i^{k+1-j_q} \right) \left(\prod_{i=j_q+1}^{k} p_i^{k+1-i} \right) \frac{1}{p_k} \\
&= \frac{\pi_q}{p_k} \prod_{j=j_q}^{k} \prod_{i=1}^{j} p_i,
\end{aligned}$$

whence, using Claim 1 once again,

$$\varphi((pm, qm)) = pp_k \left(\frac{\pi_q}{p_k} \prod_{j=j_q}^{k} \prod_{i=1}^{j} p_i \right) v_{k+1} = p\pi_q v_{j_q} = \varphi((p, q))$$

holds.

As one of the three cases above must occur, we have proved that for m a prime, $\varphi((pm, qm)) = \varphi((p, q))$ holds. Decomposing a general $m > 1$ to a product of primes, the general assertion of the claim follows. $\square$ *Claim 2.*

It follows from Claim 2 that the assignment $\frac{p}{q} \longmapsto \varphi((p, q))$ is a well-defined map from $\mathbb{Q}^{\geq 0}$ to M_E; let us denote it by $\overline{\varphi}$. Next, we show that this map is in fact a monoid homomorphism. It is clear that $\overline{\varphi}(0) = 0$. Let now $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}^{\geq 0}$;

then

$$\begin{aligned}
\overline{\varphi}\left(\frac{p}{q}+\frac{p'}{q'}\right) &= \overline{\varphi}\left(\frac{pq'+p'q}{qq'}\right) \\
&= (pq'+p'q)\pi_{qq'}v_{j_{qq'}} && \text{by (3.5),} \\
&= pq'\pi_{qq'}v_{j_{qq'}} + p'q\pi_{qq'}v_{j_{qq'}} \\
&= \overline{\varphi}\left(\frac{pq'}{qq'}\right) + \overline{\varphi}\left(\frac{p'q}{qq'}\right) = \overline{\varphi}\left(\frac{p}{q}\right) + \overline{\varphi}\left(\frac{p'}{q'}\right).
\end{aligned}$$

We conclude that $\overline{\varphi} : \mathbb{Q}^{\geq 0} \longrightarrow M_E$ is indeed a monoid homomorphism.

For any $n \in \mathbb{N}$, we have $\overline{\psi}(v_n) = \frac{1}{q}$, where $q = \prod_{i=1}^{n} p_i^{n-i}$ by (3.3). Observe that then $j_q = n$, so $\pi_q = \prod_{i=1}^{n} p_i^{n-(n-i)-i} = 1$, so $\overline{\varphi}\left(\frac{1}{q}\right) = 1 \cdot 1 \cdot v_{j_q} = v_n$; as we have shown that $\overline{\varphi} \circ \overline{\psi}$ maps each of the generators v_n of M_E to itself, we conclude that $\overline{\varphi} \circ \overline{\psi} = \mathrm{id}_{M_E}$.

As for the composition $\overline{\psi} \circ \overline{\varphi}$, for any $p \in \mathbb{N}_0$, we see that $\overline{\psi} \circ \overline{\varphi}\left(\frac{p}{1}\right) = \overline{\psi}(pv_1) = p$. If $\frac{p}{q} \in \mathbb{Q}^{\geq 0}$ with $q = p_1^{a_1} \cdots p_n^{a_n}$ and $a_n \neq 0$, then

$$\begin{aligned}
\overline{\psi}\left(\overline{\varphi}\left(\frac{p}{q}\right)\right) &= \overline{\psi}\left(p\pi_q v_{j_q}\right) = p\pi_q \overline{\psi}(v_{j_q}) \\
&= p\left(\prod_{i=1}^{j_q} p_i^{j_q - a_i - i}\right)\left(\prod_{i=1}^{j_q} p_i^{i-j_q}\right) = p\prod_{i=1}^{j_q} p_i^{-a_i} = \frac{p}{q},
\end{aligned}$$

so $\overline{\psi} \circ \overline{\varphi} = \mathrm{id}_{\mathbb{Q}^{\geq 0}}$. We have shown that $M_E \simeq \mathbb{Q}^{\geq 0}$, as required. ☐

Remark 3.20. With little effort, one can see that in Theorem 3.19, if instead of the sequence $p_1, p_2, \ldots$ going through *all* prime numbers, we only chose *some* (be it finitely or infinitely many), and if we adjusted the quiver E accordingly, a similar proof would yield a regular Leavitt path algebra $L_K(E)$ over an arbitrary field K such that the monoid $\mathcal{V}(L_K(E))$ would be isomorphic to the submonoid of $\mathbb{Q}^{\geq 0}$, consisting only of rational numbers that can be expressed as fractions having only products of powers of the chosen primes in the denominator. Such submonoids of $\mathbb{Q}^{\geq 0}$ have "nonzero refinements" in the sense of Remark 4.13 (for a proof, also see said remark), so they can be used in generalizations of the constructions of Proposition 4.12 presented in Remark 4.13.

For an alternative proof of Theorem 3.19, see Remark 3.34.

3.2.3 Desingularization and $\mathcal{V}(L_K(E))$ for E a general quiver

Example 3.21 (the infinite edges quiver)**.** Let E_∞ denote the "infinite edges quiver", consisting of two vertices v, w, (countably) infinitely many edges from v to w and no other edges, pictured as

$$\bullet_v \xrightarrow{(\infty)} \bullet_w.$$

For this particular quiver, the algebra $L(E_\infty)$ is unital, regular by Theorem 3.12, and isomorphic to the ring $\{\, A + k\,\mathrm{Id} \mid A \in M_\infty(K), k \in K \,\}$ (see (Abrams – Aranda Pino, 2008, Lemma 1.1)). That set aside, a fact of interest to us is that

$L(E_\infty)$ is not isomorphic to any Leavitt path algebra over a row-finite quiver (Abrams – Aranda Pino, 2008, Proposition 5.5). Thus, it witnesses that there are quivers $E \in \mathscr{Quiv}$ whose Leavitt path algebras cannot be realized as Leavitt path algebras of row-finite quivers; as Theorem 3.17 then does not apply to these quivers, we might wonder what the monoids $\mathcal{V}(-)$ of such Leavitt path algebras look like. As we will see in Corollary 3.24, the process of desingularization of a quiver answers this question.

For a quiver E, a *desingularization* of E is a quiver F obtained from E in the following way:

- For every sink v_0 in E, an infinite quiver of the form

$$\bullet_{v_0} \longrightarrow \bullet_{v_1} \longrightarrow \bullet_{v_2} \longrightarrow \bullet_{v_3} \cdots \tag{3.7}$$

  is attached at v_0.

- For every infinite emitter v_0 in E, write $s_E^{-1}(v_0) = \{ e_1, e_2, e_3, \dots \}$; an infinite quiver of the form (3.7) is attached at v_0, and for every $i \in \mathbb{N}$, the edge e_i is removed, while a new edge from v_{i-1} to $r_E(e_i)$ is added.

Remark 3.22. All vertices of a desingularization of a quiver are clearly regular; in particular, a desingularization of a quiver E is a row-finite quiver. We speak of *a* desingularization of E and not of *the* desingularization of E, since, in general, the order in which we "desingularize" the vertices, or even different ordering of the set $s_E^{-1}(v_0)$ for an infinite emitter v_0 may yield different (nonisomorhic) quivers: For example, consider the infinite rose quiver, R_∞, of Example 3.3, and attach to it one new vertex and one new edge e from the original vertex to the new one. The resulting quiver is

$$(\infty) \circlearrowright \bullet_v \xrightarrow{e} \bullet_w .$$

Taking the edge e as "e_1", the desingularization process yields

$$\begin{tikzcd}
\cdots \bullet \arrow[rrr, bend left] & \bullet \arrow[l] \arrow[rr, bend left] & \bullet \arrow[l] \arrow[r, bend left] & \bullet_v \arrow[l] \arrow[r] & \bullet_w \arrow[r] & \bullet \arrow[r] & \bullet \cdots ,
\end{tikzcd}$$

while taking it as "e_2", desingularization yields

$$\begin{tikzcd}
 & & & \bullet_v \arrow[loop above] \arrow[d] & & & \\
\cdots \bullet \arrow[urrr] & \bullet \arrow[l] \arrow[urr] & \bullet \arrow[l] \arrow[ur] & \bullet \arrow[l] \arrow[r] & \bullet_w \arrow[r] & \bullet \arrow[r] & \bullet \cdots .
\end{tikzcd}$$

For our purposes, an important fact about desingularizations is the following:

Theorem 3.23 ((Abrams – Aranda Pino, 2008, Theorem 5.2))**.** *Let K be a field, E an arbitrary quiver and F a desingularization of E. Then the algebras $L_K(E)$ and $L_K(F)$ are Morita equivalent.* $\square$

Let us note here that Theorem 3.23 is proved in Abrams – Aranda Pino (2008) using results from Ánh – Márki (1987); in particular, it works with the same definition of the category Mod-I for I a ring with local units as we do, as we have adopted the definitions of Ánh – Márki (1987). Thus, Theorem 3.23 states that the algebras $L_K(E)$ and $L_K(F)$ are indeed Morita equivalent in our sense. Hence, Theorem 2.45 applies, so we immediately obtain from Theorem 3.23 that:

Corollary 3.24. *For K a field, E a quiver and F a desingularization of E, the monoids $\mathcal{V}(L_K(E))$ and $\mathcal{V}(L_K(F))$ are isomorphic.* □

Even for quivers E such as E_∞ of Example 3.21, whose Leavitt path algebras cannot be realized as Leavitt path algebras over any row-finite quiver, Corollary 3.24 states that the monoid $\mathcal{V}(L_K(E))$ can be computed as $\mathcal{V}(L_K(F))$ for a suitable row-finite quiver F. In particular, Corollary 3.24 together with Theorem 3.17 yield:

Corollary 3.25. *For any countable quiver E and any field K, $M_E \simeq \mathcal{V}(L_K(E))$ holds.* □

3.2.4 A naïve alternative to desingularization

From Definition 3.13, we can derive a simpler alernative to desingularization for finding a row-finite quiver F such that $M_E \simeq M_F$ for an arbitrary quiver E. Let us call it the *crop*[7] of the quiver E.

Definition 3.26. For a quiver E, the *crop* of E is the quiver F obtained from E by removing all edges whose source is an infinite emitter in E.

As only edges emitted by regular vertices play a role in the definition of Λ_E (see (3.1)), we see that:

Observation 3.27. *If F is the crop of a quiver E, then F is a row-finite quiver satisfying $M_E \simeq M_F$.* □

It now follows from Theorem 3.17 that applying the functor $\mathcal{V}(-)$ to E and to its crop, we obtain, up to isomorphism, the same monoid. Apart from being a much simpler method than desingularization (with no need of results such as Theorem 2.45), another advantage of "cropping" over desingularization is that the resulting quiver is unique. However, even in the unital case, "cropping" does not preserve some properties of the Leavitt path algebras that desingularization does, such as $L_K(E)$ being simeple (cf. (Anderson – Fuller, 1992, Proposition 21.8(1)) and Theorem 3.23). An example of this phenomenon is the quiver E with two vertices, v and w, one arrow from v to w, one from w to v and infinitely many loops from v to itself:

$$(\infty) \circlearrowleft \bullet_v \rightleftarrows \bullet_w \; .$$

The crop of E is then the quiver $F = \bullet_v \longleftarrow \bullet_w$. One can use (Abrams – Aranda Pino, 2008, Theorem 3.1) to show that $L_K(E)$ is simple, while $L_K(F)$ is not.

[7] The word "crop" is chosen to be as suggestive as possible, hopefully not interfering with any standard graph-theoretic terminology.

Nevertheless, Observation 3.27 can be used instead of Corollary 3.24 to prove Corollary 3.25.

3.2.5 Properties of $\mathcal{V}(L_K(E))$

General properties of monoids associated with quivers

Either using desingularization or cropping, we have seen in Corollary 3.25 that any monoid associated with a quiver is isomorphic to a monoid associated with a row-finite quiver. Hence, even though originally stated for row-finite quivers, (Ara et al., 2007, Proposition 4.4, Theorem 6.3 and Proposition 6.4) also hold for any countable quiver. We can thus sum these three statements into:

Theorem 3.28. *Let E be a quiver and K a field. Then $M_E \simeq \mathcal{V}(L_K(E))$ is an unperforated separative refinement monoid.* $\square$

Stable finiteness for monoids associated with acyclic quivers

Proposition 3.29 ((Abrams – Aranda Pino, 2006, Proposition 4))**.** *Let E be a row-finite quiver. Then E is acyclic iff $L(E)$ is a union of a chain of finite-dimensional subalgebras.* $\square$

Remark 3.30. In the proof of Proposition 3.29 in Abrams – Aranda Pino (2006), it is shown that for any acyclic row-finite quiver E, it is possible to find a chain of finite complete subquivers $(F_i | i \in \mathbb{N})$ such that $L(E) = \bigcup_{i=1}^{\infty} L(F_i)$. In particular, since each F_i is finite, the subalgebra $L(F_i)$ of $L(E)$ is unital (and finite-dimensional, cf. Theorem 3.12). Hence, we can restate Proposition 3.29 as:

Corollary 3.31. *A row-finite quiver is acyclic iff it is a union of a chain of finite-dimensional subalgebras that are unital.*[8] $\square$

Theorem 3.32. *Let K be a field and A a K-algebra such that A is the union of a chain of finite-dimensional unital subalgebras. Then the monoid $\mathcal{V}(A)$ is stably finite. In particular, for an acyclic row-finite quiver E, the monoid $\mathcal{V}(L_K(E))$ is stably finite.*

Proof. Suppose the contrary, that is, that there are idempotents $e, g \in M_\infty(A)$ satisfying $[g] \neq 0$ (in particular, $g \neq 0$) and $[e] = [e] + [g] = [e \oplus g]$ in $\mathcal{V}(A)$. Then, there are $x, y \in M_\infty(A)$ such that both $ex(e \oplus g)ye = e$ and $(e \oplus g) = (e \oplus g)yex(e \oplus g)$. Since the matrices e, g, x, y have only finitely many nonzero entries and since A is the union of a chain of finite-dimensional unital subalgebras, there is a finite-dimensional unital subalgebra B of A such that e, g, x, y can be viewed as elements of $M_\infty(B)$. However, as $e \in M_n(B)$ and $g \in M_m(B)$ for suitable $m, n \in \mathbb{N}$, we have $(e \oplus g) \in \operatorname{Idemp} M_{n+m}(B)$, so Lemma 2.22 yields that $ex(e \oplus g) : (e \oplus g)B^{n+m} \longrightarrow eB^n$ is a B-module isomorphism. We thus have $eB^n \simeq (e \oplus g)B^{n+m} \simeq eB^n \oplus gB^m$. In particular, the finitely generated B-modules eB^n and $eB^n \oplus gB^m$ are, as vector spaces over K, of the same finite dimension, whence $gB^m = 0$. But that is only possible for $g = 0$, a contradiction.

The assertion for acyclic row-finite quivers follows from Corollary 3.31. $\square$

[8] In general, even for a unital algebra, a finite-dimensional subalgebra in our sense (i.e., a subring closed under multilpication by scalars) need not be unital: Consider e.g. the subalgebra of a Leavitt path algebra generated by a single edge.

3.2.6 Realizing directed unions of free abelian monoids by regular Leavitt path algebras

Directed unions of chains of finitely generated free abelian monoids are a particular example of stably finite monoids; hence, the following proposition is a partial reversal of Theorem 3.32 (cf. also Theorem 3.12 (iii)):

Proposition 3.33. *Each directed union of monoids of the form* $(\mathbb{N}_0)^k$ *(with each* k *a strictly positive integer) is realizable as the monoid* $\mathcal{V}(L_K(E))$ *for some acyclic row-finite quiver* E.

Proof. Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} \cdots$$

be a directed system in $\mathscr{Mon}$, with $M_i = (\mathbb{N}_0)^{k_i}$ for all i and with every φ_i injective. For each i, let $\{ v_{i1}, \ldots, v_{ik_i} \}$ be the canonical generating set of M_i. For vertices of E, take $E^0 := \{ v_{ij} \mid i \in \mathbb{N}, 1 \leq j \leq k_i \}$. From each vertex v_{ij}, we have $\varphi_i(v_{ij}) = \sum_{l=1}^{k_{i+1}} n_{ijl} v_{i+1,l}$ for some $n_{ijl} \in \mathbb{N}_0$; in E, let there be n_{ijl} edges from v_{ij} to $v_{i+1,l}$ and no edges from v_{ij} to any v_{ml} with $m \neq i+1$. We have defined E in such a way that

$$\varphi_i(v_{ij}) = \sum_{e \in s_E^{-1}(v_{ij})} r(e) \tag{3.8}$$

holds for each i, j. We claim that for any field K, $\mathcal{V}(L_K(E)) \simeq \varinjlim M_i$ holds.

By Theorem 3.17, it is sufficient to show that M_E satisfies the UMP of $\varinjlim M_i$. To that end, let $N \in \mathscr{Mon}$, let there be monoid homomorphisms $\psi_i : M_i \longrightarrow N$ satisfying $\psi_i = \psi_{i+1}\varphi_i$ for all i, let $\iota_i : M_i \longrightarrow F_{E^0}$ be the inclusion map, and let $\pi : F_{E^0} \longrightarrow M_E = F_{E^0}/\Lambda_E$ be the canonical projection (cf. Definition 3.13). We are looking for a unique filler $\overline{\phi} : M_E \longrightarrow N$ of the following commutative diagram in $\mathscr{Mon}$:

$$\begin{tikzcd}
 & & & & M_E \arrow[dd, dashed, "\overline{\phi}"', "?"] \\
M_1 \arrow[r, "\varphi_1"] \arrow[urrrr, bend left, "\pi\iota_1"] \arrow[drrrr, bend right, "\psi_1"'] & M_2 \arrow[r, "\varphi_2"] \arrow[urrr, bend left, "\pi\iota_2"] \arrow[drrr, bend right, "\psi_2"'] & M_3 \arrow[r, "\varphi_3"] \arrow[urr, bend left, "\pi\iota_3"] \arrow[drr, bend right, "\psi_3"'] & \cdots & \\
 & & & & N
\end{tikzcd} \tag{3.9}$$

For $\overline{\phi}\pi\iota_i = \psi_i$ to hold for each i, there is no option but

$$\overline{\phi}(\pi v_{ij}) = \psi_i(v_{ij}) \text{ for each } i, j. \tag{3.10}$$

Since the set E^0 freely generates the monoid F_{E^0}, the assignment $\phi(v_{ij}) := \psi_i(v_{ij})$ for each i, j induces a unique monoid homomorphism $\phi : F_{E^0} \longrightarrow N$ (cf. (Burris –

Sankappanavar, 2012, Lemma 10.6)); moreover, $\phi\iota_i = \psi_i$ holds for each i. Observe that then for each $v_{ij} \in E^0$,

$$\begin{aligned}
\phi\iota_{i+1}\left(\sum_{e\in s^{-1}(v_{ij})} r(e)\right) &= \psi_{i+1}\left(\sum_{e\in s^{-1}(v_{ij})} r(e)\right) \\
&= \psi_{i+1}\varphi_i(v_{ij}) && \text{by (3.8)}, \\
&= \psi_i(v_{ij}) = \phi\iota_{i+1}(v_{ij})
\end{aligned}$$

holds. In particular, $\Lambda_E \subseteq \ker\phi$, whence there is a monoid homomorphism $\overline{\phi} : M_E \longrightarrow N$ satisfying $\phi = \overline{\phi}\pi$. We have thus shown that there exists a monoid homomorphism $\overline{\phi}$ making the diagram (3.9) commutative; from (3.10), such homomorphism is already unique. We conclude that the monoid M_E indeed satisfies the UMP defining $\varinjlim M_i$, whence $\varinjlim M_i \simeq M_E \simeq \mathcal{V}(L_K(E))$. $\square$

Remark 3.34. The previous proposition gives an alternative—and, admittedly, a more structural—way of proving Theorem 3.19. More specifically, the proof of Proposition 3.33 explains the origin of the graph E in Theorem 3.19 and of its variations from Remark 3.20. To see this, let $P = \{p_1, p_2, \dots\}$ be a set of primes and consider the directed system

$$\mathbb{N}_0 \xrightarrow{\varphi_1} \mathbb{N}_0 \xrightarrow{\varphi_2} \mathbb{N}_0 \xrightarrow{\varphi_3} \dots, \tag{3.11}$$

where $\varphi_n(1) = \prod_{i=1}^n p_i$ for each n. Then the monoid $\mathbb{Q}_P^{\geq 0}$ is the direct limit of the system (3.11) (with $\mathbb{Q}_P^{\geq 0} = \mathbb{Q}^{\geq 0}$ if P is the set of all prime numbers).

3.3 The regular algebra of a quiver containing a cycle

Thus far, we have presented how the monoid $\mathcal{V}(-)$ of a Leavitt path algebra can be constructed (Theorem 3.17 for row-finite quivers, generalized to any quivers from $\mathscr{Quiv}$ in Corollary 3.25), and we know that for acyclic quivers, the Leavitt path algebra is regular (Theorem 3.12). The Leavitt path algebras for quivers with cycles are not regular; nevertheless, there is a construction realizing M_E as $\mathcal{V}(-)$ of a regular algebra for any row-finite quiver E (including quivers with cycles):

Theorem 3.35 ((Ara – Brustenga, 2007, Theorems 4.4 and 4.2))**.** *If E is a row-finite quiver and K a field, then there is a regular algebra $Q_K(E)$ satisfying $\mathcal{V}(Q_K(E)) \simeq M_E$. If the quiver E is finite, then the algebra $Q_K(E)$ is unital.* $\square$

Remark 3.36. Theorems 4.2 and 4.4 of Ara – Brustenga (2007) are stated for column-finite quivers; it is the result of working with "opposite arrows" than we do. For example, complete quiver homomorphisms are in Ara – Brustenga (2007) defined as monoid homomorphisms f that restrict to bijections between $r^{-1}(v)$ and $r^{-1}(f^0v)$ for each vertex, and the monoid $M(E)$ used in said theorems is, by our definition, the monoid M_{E^*} (recall that E^* is the dual quiver of E).

Due to Theorem 3.35 and Corollary 3.25, we may conclude that:

Theorem 3.37. *Let E be a countable quiver and K a field. Then the monoid M_E associated with E is realizable by a regular K-algebra. If E is finite, then M_E is realizable by a regular unital K-algebra.* $\square$

Example 3.38 (rose with two petals)**.** Let R_2 denote the "rose with two petals" quiver, that is, a quiver with a single vertex and with two edges (loops from the single vertex to itself), pictured as:

$$\circlearrowleft \bullet \circlearrowright .$$

The monoid M_{R_2} is seen to be the monoid $\mathbf{2}$ consisting of two elements, 0 and 1, and with max as the monoid operation (cf. Lemma 4.14). The algebra $L_K(R_2)$ is not regular; nevertheless, by Theorem 3.35, there is a unital regular algebra $Q_K(R_2)$ such that $\mathbf{2} \simeq M_E \simeq \mathcal{V}\,(Q_K(R_2))$.

Chapter 4

On non-realizability by regular algebras over arbitrary fields

By results by Friedrich Wehrung and Kenneth Goodearl, there is a criterion for countable conical refinement monoids (with or without order-unit) to *not* be realizable by regular algebras over any uncountable fields. In Section 4.2, we present a proof of the criterion (Proposition 4.11) based on the presentation in (Ara, c2009, Proposition 4.1) and then we present a way of conrtucting examples fitting the criterion. But before that, we establish some needed properties of regular rings and algebras in Section 4.1

4.1 Stable range 1 and cancellation in $\mathcal{V}(R)$

Lemma 4.1. *For a regular ring R that is unital, if $x, y \in R$ and $\varphi : xR \longrightarrow yR$ is an R-module homomorphism, then there is a $z \in R$ such that $\varphi(w) = zw$ for all $w \in xR$, i.e., φ is in fact left multiplication by an element of R. In particular, if x, y are idempotent, then $\varphi = yzx \cdot -$.*

Proof. As R is regular, there are x', y' satisfying $xx'x = x, yy'y = y$; then $xR = xx'R, yR = yy'R$ and φ can be extended to

$$R_R = xR \oplus (1 - xx')R \xrightarrow{\overline{\varphi}:=\begin{pmatrix}\varphi & 0\\ 0 & 0\end{pmatrix}} yR \oplus (1 - yy')R = R_R.$$

As such, $\overline{\varphi}$ is left multiplication by an element of R (Anderson – Fuller, 1992, Proposition 4.11), hence so is φ. If x and y are idempotent, then $x \cdot - = \mathrm{id}_{xR}$ and $y \cdot - = \mathrm{id}_{yR}$, so $\varphi = (y \cdot -) \circ \varphi \circ (x \cdot -)$. ☐

Lemma 4.2. *Let I be a regular ring and $e, g \in I$ idempotents. If e and g are orthogonal, then $eI \cap gI = 0$. Conversely, if $eI \cap gI = 0$ holds and if I is* unital, *then e and g are orthogonal.*

Proof. Suppose first that $a \in eI \cap gI$; then $a = eb$ for some $b \in I$ and, by idempotence of g, $a = ga$ holds. Thus, if $a \neq 0$, then $0 \neq a = ga = geb$, whence $ge \neq 0$. We conclude that if $eI \cap gI \neq 0$, then e and g cannot be orthogonal.

For the converse, let I be unital and let $eI \cap gI = 0$. Then $gI \subseteq (1 - e)I$; thus, $eg \in egI \subseteq e(1 - e)I = 0I = 0$, so $eg = 0$, and, symmetrically, $ge = 0$. ☐

Definition 4.3. A unital ring has *unit 1-stable range* if for any $a, x, b \in R$ satisfying $ax + b = 1_R$, there is an invertible element $u \in R$ such that $a + bu$ is invertible in R. *Stable range 1* is the same property as unit 1-stable range, except that it is not required that u be invertible (hence, stable range 1 is a weaker property than unit 1-stable range).

A *zero-divisor* in a (general) ring R is an element $x \in R$ such that whenever $xy = 0$ or $yx = 0$ for some $y \in R$, then $y = 0$; a *non-zero-divisor* is an element that is not a zero-divisor.

Lemma 4.4 ((Goodearl – Menal, 1988, Theorem 2.2))**.** *Let R be a unital algebra over an uncountable field, such that all non-zero-divisors in R are invertible. If R contains no uncountable direct sums of nonzero right or left ideals, then R has unit 1-stable range.* ☐

Lemma 4.5 (cf. (Goodearl, 1979, Proposition 4.13 and Theorem 4.14))**.** *Let R be a regular ring with unit and let R have stable range 1. Then, if $A \in \operatorname{proj}\text{-}R$ and $B, C \in \operatorname{Mod}\text{-}R$ satisfy $A \oplus B \simeq A \oplus C$, then $B \simeq C$. In particular, the monoid $\mathcal{V}(R)$ is cancellative.*

Proof. We shall show that if $R \oplus B \simeq R \oplus C$, then $B \simeq C$, as then, by induction, one will obtain that whenever $R^{(n)} \oplus B \simeq R^{(n)} \oplus C$, then $B \simeq C$. Since A is finitely generated and projective, it is (isomorphic to) a direct summand of $R^{(n)}$; adding the complement of A in $R^{(n)}$ to both sides of $A \oplus B \simeq A \oplus C$, we will have $R^{(n)} \oplus B \simeq R^{(n)} \oplus C$.

So, let $\psi : R \oplus C \longrightarrow R \oplus B$ be an isomorphism. By (Anderson – Fuller, 1992, Proposition 4.11), there is a ring isomorphism $R \simeq \operatorname{End} R_R$, whence $\operatorname{End} R_R$ has stable range 1. From the biproduct structure of $R \oplus B$ and $R \oplus C$, there are morphisms $\pi_R, \pi_B, \pi'_R, \iota_R, \iota_B, \iota'_R$ in $\operatorname{Mod}\text{-}R$ as in the diagram

$$\begin{tikzcd}
 & R \oplus B \arrow[dl, bend right, "\pi_R"'] \arrow[dr, bend left, "\pi_B"] & & R \oplus C \arrow[ll, "\simeq", "\psi"'] \arrow[dr, bend left, "\pi'_R"] & \\
R \arrow[ur, bend right, "\iota_R"'] & & B \arrow[ul, bend left, "\iota_B"] & & R \arrow[ul, bend left, "\iota'_R"]
\end{tikzcd}$$

satisfying $\pi_R \iota_R = \mathrm{id}_R$, $\pi'_R \iota'_R = \mathrm{id}_R$, $\iota_R \pi_R + \iota_B \pi_B = \mathrm{id}_{R \oplus B}$ and $\pi_R \iota_B = 0$. Notice that then $R \oplus B = \operatorname{Im} \iota_R \oplus \operatorname{Ker} \pi_R$, $B \simeq \operatorname{Im} \iota_B = \operatorname{Ker} \pi_R$ and $C \simeq \operatorname{Ker} \pi'_R$.

Put $f := \pi'_R \psi^{-1} : R \oplus B \longrightarrow R$ and $g := \psi \iota'_R : R \longrightarrow R \oplus B$. Then:

$$\begin{aligned}
\mathrm{id}_R = \pi'_R \iota'_R = \pi'_R \psi^{-1} \psi \iota'_R = fg = f \, \mathrm{id}_{R \oplus B} \, g &= f(\iota_R \pi_R + \iota_B \pi_B) g \\
&= (f \iota_R)(\pi_R g) + f \iota_B \pi_B g.
\end{aligned}$$

As $f \iota_R$, $\pi_R g$ and $f \iota_B \pi_B g$ are elements of $\operatorname{End} R_R$, there exists—from $\operatorname{End} R_R$ having stable range 1—a $y \in \operatorname{End} R_R$ such that $f \iota_R + f \iota_B \pi_B g y$ is invertible in $\operatorname{End} R_R$ (i.e., an automorphism of R_R). Putting $k := \iota_R + \iota_B \pi_B g y : R \longrightarrow R \oplus B$, we then have that fk is an automorphism of R, whence $R \oplus B = \operatorname{Ker} f \oplus \operatorname{Im} k$. Also, since ψ is an isomorphism, we have $\operatorname{Ker} f = \operatorname{Ker}(\pi'_R \psi^{-1}) = \psi(\operatorname{Ker} \pi'_R) \simeq C$.

Next, observe that $\pi_R k = \pi_R \iota_R + 0 = \mathrm{id}_R$, whence $R \oplus B = \operatorname{Ker} \pi_R \oplus \operatorname{Im} k$. Thus, in the module $R \oplus B$, both $\operatorname{Ker} f$ and $\operatorname{Ker} \pi_B$ are complements of the same submodule, $\operatorname{Im} k$; as such, necessarily $\operatorname{Ker} f \simeq \operatorname{Ker} \pi_B$. With $\operatorname{Ker} f \simeq C$ and $\operatorname{Ker} \pi_B \simeq B$, we conclude that $B \simeq C$; thus, we have shown that if $R \oplus B \simeq R \oplus C$ with B, C any elements of $\operatorname{Mod}\text{-}R$, then $B \simeq C$. ☐

Lemma 4.6. *For a regular unital algebra R over an uncountable field, if R contains no uncountable direct sums of nonzero right or left ideals, then the monoid $\mathcal{V}(R)$ is cancellative.*

Proof. For Lemma 4.4 to apply, we need to show that every non-zero-divisor in R is invertible. To that end, if x is not a zero-divisor, then $xy \neq 0$ holds for all nonzero $y \in R$; thus, the R-module homomorphism $x \cdot - : R_R \longrightarrow xR_R$ is injective. As it clearly is onto, we have $xR \simeq R_R$, whence, by Lemma 4.1, there is an $a \in R$ satisfying $ax = 1$. Symmetrically (via left modules), there is a $b \in R$ such that $xb = 1$. Now $a = a1 = axb = 1b = b$ is a two-sided inverse of x in R, so, indeed, every non-zero-divisor in R is invertible. Hence, Lemma 4.4 applies, so R has unit 1-stable range; in particular, it has stable range 1. That $\mathcal{V}(R)$ is a cancellative monoid now follows from Lemma 4.5. $\square$

Lemma 4.7. *Let R be a regular unital algebra over an uncountable field such that there is a monoid homomorphism $s : \mathcal{V}(R) \longrightarrow \mathbb{R}^+$ satisfying $s([P]) > 0$ for all nonzero $[P] \in \mathcal{V}(R)$. Then R contains no uncountable direct sum of nonzero* right *ideals.*

Proof. Suppose the contrary, that is, that there is a direct sum $\bigoplus_{\alpha \in A} I_\alpha$ of nonzero right ideals in R with the set A uncountable. For each α, choose a nonzero idempotent $e_\alpha \in I_\alpha$ (this is possible by regularity of R: for, each I_α contains a nonzero element x_α; multiplying x_α from the right by its quasi-inverse yields a nonzero idempotent contained in I_α).[1] Then each $e_\alpha R$ is a nonzero element of proj-R, so $s([e_\alpha R])$ is defined and nonzero. Also, since the sum above is direct, $e_\alpha R \cap e_\beta R = 0$ whenever $\alpha \neq \beta$; in particular, the idempotents e_α are pairwise orthogonal (Lemma 4.2).

For each n, put $A_n := \{\, \alpha \in A \mid s([e_\alpha R]) > \frac{s([R])}{n} \,\}$. As $s([e_\alpha R]) > 0$ for each α, every $\alpha \in A$ is contained in some A_n; thus, $A = \bigcup\limits_{n=1}^{\infty} A_n$. Since the set A is uncountable, it cannot be the union of a chain of finite subsets. Hence, there is an $m < \infty$ such that A_m is infinite; in particular, A_m contains at least m distinct elements, say, $\alpha_1, \ldots, \alpha_m$. Then, by orthogonality of the e_α's,

$$R_R = \left(1 - \sum_{i=1}^{m} e_{\alpha_i}\right) R \oplus \bigoplus_{i=1}^{m} e_{\alpha_i} R. \tag{4.1}$$

The R-module $P := (1 - \sum_{i=1}^{m} e_{\alpha_i})\, R$ is, as a principal right ideal in R, a finitely generated projective R-module (Corollary 2.4), whence $[P] \in \mathcal{V}(R)$. From the properties of s, $s([P]) \geq 0$ holds. Thus, since s is a monoid homomorphism, it follows from (4.1) that $s([R]) \geq s([\bigoplus_{i=1}^{m} e_{\alpha_i} R])$. However, from the definition of A_m, the inequality $s([e_{\alpha_i}]) > \frac{s([R])}{m}$ holds for each $i \leq m$. Thus,

$$s([R]) = m \cdot \frac{s([R])}{m} < s\left(\left[\bigoplus_{i=1}^{m} e_{\alpha_i} R\right]\right) \leq s([R]),$$

a contradiction. We conclude that the assumption that there is a direct sum of uncountably many right ideals in R cannot hold. $\square$

[1] Of course, we admit the axiom of choice.

4.2 Monoids not realizable by regular algebras over uncountable fields

Now for the key point of this chapter, first in the unital case (cf. (Ara, c2009, Proposition 4.1, credited to Goodearl)):

Proposition 4.8. *Let R be a regular unital algebra over an uncountable field such that there is a monoid homomorphism $s : \mathcal{V}(R) \longrightarrow \mathbb{R}^+$ satisfying $s([P]) > 0$ for all nonzero $[P] \in \mathcal{V}(R)$. Then $\mathcal{V}(R)$ is cancellative.*

Proof. As $\mathcal{V}(R) \simeq \mathcal{V}(R^{\mathrm{op}})$ (Proposition 2.27), there is also a monoid homomorphism $\mathcal{V}(R^{\mathrm{op}}) \longrightarrow \mathbb{R}^+$ mapping all nonzero elements of $\mathcal{V}(R^{\mathrm{op}})$ to strictly positive real numbers. Applying Lemma 4.7 to both R and R^{op}, we have that R contains no uncountable direct sums of nonzero right or left ideals. Thus, cancellation of $\mathcal{V}(R)$ follows by Lemma 4.6. □

Lemma 4.9. *Let I be a ring and v an indempotent in I. Then $\mathcal{V}(vIv)$ is a submonoid of $\mathcal{V}(I)$.*

Proof. As $vIv \subseteq I$, we have $M_\infty(vIv) \subseteq M_\infty(I)$. Clearly, if two idempotents from $M_\infty(vIv)$ are equivalent as elements of $\operatorname{Idemp} M_\infty(vIv)$, then they are also equivalent as elements of $\operatorname{Idemp} M_\infty(I)$. Hence, mapping the equivalence class of e in $\operatorname{Idemp} M_\infty(vIv)$ to the equivalence class of e in $\operatorname{Idemp} M_\infty(I)$ for every idempotent $e \in M_\infty(vIv)$ is a well-defined map from $\mathcal{V}(vIv)$ to $\mathcal{V}(I)$; one readily sees that this map is a monoid homomorphism. We want to prove that it is injective.

To that end, let $e, g \in \operatorname{Idemp} M_\infty(vIv)$ such that $[e] = [g]$ as elements of $\mathcal{V}(I)$, that is, there are $x, y \in M_\infty(I)$ satisfying $exgye = e$ and $gyexg = g$; we want to show that as elements of $M_\infty(vIv)$, e and g are equivalent. Since all entries of e and g are elements of vIv, so are all entries in exg and gye, i.e., $exg, gye \in M_\infty(vIv)$. But then, by idempotence of e and g, both $e = e(exg)g(gye)e$ and $g = g(gye)e(exg)g$ hold. □

Observation 4.10. *Let K be a field, I a K-algebra and $v \in I$ an idempotent. Then the subring vIv of I is a K-algebra.* □

We are now in position to restate and prove (Ara, c2009, Proposition 4.1), including nonunital algebras:

Proposition 4.11. *Let M be a conical refinement monoid that is* not *cancellative and such that there exists a monoid homomorphism $s : M \longrightarrow \mathbb{R}^+$ satisfying $s(x) > 0$ for all nonzero $x \in M$. Then there is* no *regular algebra I over any uncountable field such that $\mathcal{V}(I) \simeq M$.*[2]

[2]In Ara (c2009), in the proof of the same proposition only with the extra assumptions that M have an order-unit and the condition that s maps said order-unit to 1, it is stated that "Clearly we can assume that R is unital (...)" (with R in the role of our I). In oder to verify the validity of said "clear" assumption, we needed Lemma 4.9, Observation 4.10, and the entire proof of Proposition 4.11 to come. Should the reader see a simpler reasoning—ideally a "clear" one—why it is sufficient to only prove Proposition 4.8 for Proposition 4.11 to hold, I'd persnonally very much like to have it explained to me.

Proof. Suppose the contrary, that is, that there is a regular algebra I over an uncountable field satisfying $\mathcal{V}(I) = M$. Since M is not cancellative, there are idempotent matrices e, g, h over I satisfying $e \oplus h \sim g \oplus h$ such that e and g are not equivalent. By equivalence of $e \oplus h$ with $g \oplus h$, there are $x, y \in M_\infty(I)$ satisfying $(e \oplus h)x(g \oplus h)y(e \oplus h) = e \oplus h$ and $(g \oplus h)y(e \oplus h)x(g \oplus h) = g \oplus h$. As all the matrices in question have only finitely many nonzero entries and since the regular ring I has local units (Proposition 2.7), there is an idempotent $v \in I$ such that e, g, h, x and y are elements of $M_\infty(vIv)$. We then have $[e \oplus h] = [g \oplus h]$, but $[e] \neq [g]$ in $\mathcal{V}(vIv)$, so $\mathcal{V}(vIv)$ is not a cancellative monoid either.

On the other hand, by Lemma 4.9, $\mathcal{V}(vIv)$ is a submonoid of $\mathcal{V}(I) = M$. Hence, the restriction $s|_{\mathcal{V}(vIv)} : \mathcal{V}(vIv) \longrightarrow \mathbb{R}^+$ is a monoid homomorphism, again satisfying $s|_{\mathcal{V}(vIv)}([a]) > 0$ for all nonzero $[a] \in \mathcal{V}(vIv)$. Since, by Observation 4.10, vIv is a regular unital algebra (with unit v) over an uncountable field, and with $s|_{\mathcal{V}(vIv)}$ at hand, the monoid $\mathcal{V}(vIv)$ is cancellative by Proposition 4.8, a contradiction. $\square$

Now that we have a criterion (that is, a sufficient condition) for a conical refinement monoid to not be realizable as a $\mathcal{V}(I)$ for any regular algebra I over an uncountable field, a question to ask is whether there exists a monoid satisfying the assumptions of Proposition 4.11, and if it is possible for such monoid to be countable. We answer both parts of this question in the affirmative in Example 4.15 (with the answer restated, for clarity, in Proposition 4.16). However, instead of argumenting only for the particular case of the monoid $Q_2 = \mathbf{2} \times \mathbb{Q}^{\geq 0} \setminus \{(1,0)\}$ of Example 4.15, we instead prove that there is a more general way to build such examples, starting with any conical refinement monoid that is not cancellative.

Proposition 4.12. *Let A be a conical refinement monoid that is not cancellative. Then the submonoid $C := (A \times \mathbb{Q}^{\geq 0}) \setminus ((A \setminus \{0\}) \times \{0\})$ of $A \times \mathbb{Q}^{\geq 0}$ is a conical refinement monoid that is not cancellative, and there is a monoid homomorphism $s : C \longrightarrow \mathbb{R}^+$ satisfying $s(x) > 0$ for all nonzero $x \in C$. Moreover, if A has an order-unit, then so does C.*

Proof. Firstly, we need to check that the subset C of $A \times \mathbb{Q}^{\geq 0}$ is closed under the monoid operation of $A \times \mathbb{Q}^{\geq 0}$. This follows from $\mathbb{Q}^{\geq 0}$ being a conical monoid: If (a, q) and (a', q') are nonzero elements of C (with $a, a' \in A, q, q' \in \mathbb{Q}$), then, by the definition of C, both $q \geq 0$ and $q' \geq 0$. Hence, $q + q' \geq 0$, so

$$(a, q) + (a', q') = (a + a', q + q') \in C.$$

With $0 = (0, 0) \in C$, we thus have that C is a submonoid of $A \times \mathbb{Q}^{\geq 0}$.

Since A and $\mathbb{Q}^{\geq 0}$ are conical monoids, so is C by Observation 1.1.

For the existence of s, we let s be the composition of the canonical projection $A \times \mathbb{Q}^{\geq 0} \xrightarrow{\pi} \mathbb{Q}^{\geq 0}$ with the two inclusion maps $C \xrightarrow{\subseteq} A \times \mathbb{Q}^{\geq 0}$ and $\mathbb{Q}^{\geq 0} \xrightarrow{\subseteq} \mathbb{R}^+$, as in the following commutative diagram in $\mathscr{Mon}$:

$$\begin{tikzcd} & A \times \mathbb{Q}^{\geq 0} \arrow[r, "\pi"] & \mathbb{Q}^{\geq 0} \arrow[dr, "\subseteq"] & \\ C \arrow[ur, "\subseteq"] \arrow[rrr, "s"] & & & \mathbb{R}^+ \end{tikzcd}$$

(as both the inclusions and the canonical projections are monoid homomorphisms, so is s). Due to the exclusion of all elements of the form $(a, 0)$ except for $0 = (0, 0)$ from C, we see that $s((a, q)) > 0$ whenever $(a, q) \neq 0$.

To show that C is a refinement monoid, observe that $\mathbb{Q}^{\geq 0}$ has "nonzero refinements" in the sense that if $a + b = c + d$ in $\mathbb{Q}^{\geq 0}$ with $a, b, c, d \neq 0$, then there is a refinement in $\mathbb{Q}^{\geq 0}$ with none of the refining elements zero: indeed, supposing w.l.o.g. that $a \leq c$,

$$\begin{array}{c|cc} & a & b \\ \hline c & \frac{a}{2} & c - \frac{a}{2} \\ d & \frac{a}{2} & d - \frac{a}{2} \end{array}$$

is such refinement. Let $a_1 + a_2 = b_1 + b_2$ in A and $p_1 + p_2 = q_1 + q_2$ in $\mathbb{Q}^{\geq 0}$ with $(a_1, p_1), (a_2, p_2), (b_1, q_1), (b_2, q_2) \in C$. Either of the following two cases occurs:

(i) p_1, p_2, q_1, q_2 are all nonzero: Then, there are refinements

$$\begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & r_{11} & r_{12} \\ b_2 & r_{21} & r_{22} \end{array} \text{ in } A \text{ and } \begin{array}{c|cc} & p_1 & p_2 \\ \hline q_1 & s_{11} & s_{12} \\ q_2 & s_{21} & s_{22} \end{array} \text{ in } \mathbb{Q}^{\geq 0}$$

with all $s_{ij} \neq 0$; then

$$\begin{array}{c|cc} & (a_1, p_1) & (a_2, p_2) \\ \hline (b_1, q_1) & (r_{11}, s_{11}) & (r_{12}, s_{12}) \\ (b_2, q_2) & (r_{21}, s_{21}) & (r_{22}, s_{22}) \end{array}$$

is a refinement in C.

(ii) One of the elements of p_1, p_2, q_1, q_2 is zero, say, $p_1 = 0$, then—since $(a_1, p_1) \in C$—necessarily $(a_1, p_1) = 0$; then

$$\begin{array}{c|cc} & 0 & (a_2, p_2) \\ \hline (b_1, q_1) & 0 & (b_1, q_1) \\ (b_2, q_2) & 0 & (b_2, q_2) \end{array}$$

is a refinement in C.

In either case, we found a refinement to $(a_1, p_1) + (a_2, p_2) = (b_1, q_1) + (b_2, q_2)$ in C, so C is indeed a refinement monoid.

Since A is not cancellative, there are elements $a, b, x \in A$ such that $a+x = b+x$ holds while $a \neq 0$. Taking $(a, 1), (b, 1), (x, 1) \in C$, we see that

$$(a, 1) + (x, 1) = (a + x, 2) = (b + x, 2) = (b, 1) + (x, 1)$$

holds, while $(a, 1) \neq (b, 1)$, so C is not a cancellative monoid.

Finally, if $a \in A$ is an order-unit in A, then clearly $(a, 1)$ is an order-unit in C. □

Remark 4.13. It is clear from the proof of Proposition 4.12 that instead of $\mathbb{Q}^{\geq 0}$, we could have used any conical monoid B allowing a monoid homomorphism $\varphi : B \longrightarrow \mathbb{R}^+$ with $\varphi(x) \neq 0$ for all nonzero $x \in B$ such that B has "nonzero refinements" in the sense used in the proof. However, for simplicity, we keep $\mathbb{Q}^{\geq 0}$

in Proposition 4.12 instead, as it is a countable, easy-to-imagine monoid. A particular example of a suitable monoid B is the submonoid $\mathbb{Q}_P^{\geq 0}$ of $\mathbb{Q}^{\geq 0}$ consisting, for a fixed nonempty set P of primes, of rationals expressible as fractions with only products of powers of elements of P in the denominator, as presented in Remark 3.20. To see that $\mathbb{Q}_P^{\geq 0}$ has "nonzero refinements", let $a + b = c + d$ in $\mathbb{Q}_P^{\geq 0}$ with $a, b, c, d \neq 0$ and let $p \in P$ (we assume $P \neq \emptyset$), and w.l.o.g. suppose that $a \leq c$. Then the following is a refinement in $\mathbb{Q}_P^{\geq 0}$ with all entries nonzero:

$$\begin{array}{c|cc} & a & b \\ \hline c & \frac{a}{p} & c - \frac{a}{p} \\ d & \frac{(p-1)a}{p} & d - \frac{(p-1)a}{p} \end{array}.$$

Notice that we can use the inclusion map $\mathbb{Q}_P^{\geq 0} \xrightarrow{\subseteq} \mathbb{R}^+$ as φ.

However, the requirement of the existence of "nonzero refinements" in B is necessary, as we shall show in Remark 4.17. Before that, for construction of examples, the following lemma will come handy:

Lemma 4.14. *For any linearly ordered set X with a least element, the monoid $\boldsymbol{X} := (X, \max)$ is a conical refinement monoid. If the set X has at least two elements, then the monoid $\boldsymbol{X}$ is not cancellative.*

Proof. Clearly, taking maximum of two elements is an associative and commutative binary operation, hence an abelian semigroup operation on X. Let 0 denote the least element of X. Then 0 is the zero element of $\boldsymbol{X}$, since $\max\{0, x\} = x$ for all $x \in X$, so $\boldsymbol{X}$ is a monoid. From 0 being the least element of X, it follows that $\boldsymbol{X}$ is a conical monoid. As for refinement, if $\max\{a, b\} = \max\{c, d\}$ for some $a, b, c, d \in X$, suppose w.l.o.g. that $a = \max\{a, b\} = \max\{c, d\} = c$. We then have the following refinement:

$$\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ d & d & \min\{b, d\} \end{array}.$$

Finally, if $0 \neq x \in X$, then the equality $\max\{0, x\} = \max\{x, x\}$ is a witness to $\boldsymbol{X}$ not being cancellative. ☐

Example 4.15. Taking the linearly ordered two-element set, $2 := \{0, 1\}$ (with $0 < 1$), consider the countable monoid $\mathbf{2} = (\{0, 1\}, \max)$. By Lemma 4.14, it is a conical refinement monoid that is not cancellative. Thus, the monoid $Q_2 := \mathbf{2} \times \mathbb{Q}^{\geq 0} \setminus \{(1, 0)\}$ is, by Proposition 4.12, a conical refinement monoid that is not cancellative and such that there is a monoid homomorphism from Q_2 to $\mathbb{R}^+$ mapping all nonzero elements to strictly positive real numbers. Hence, the monoid Q_2 satisfies the assumptions of Proposition 4.11, so it cannot be realized by any $\mathcal{V}(I)$ with I a regular algebra over an uncountable field. Thus, the monoid $Q_2 = \mathbf{2} \times \mathbb{Q}^{\geq 0} \setminus \{(1, 0)\}$ is a withess to the main result of this chapter, that is:

Proposition 4.16. *There exists a countable conical refinement monoid that is not isomorphic to any $\mathcal{V}(A)$ with A a regular algebra over an uncountable field.* ☐

Remark 4.17. The monoid $\mathbf{2}$ will serve us to show that while generalizing Proposition 4.12 in Remark 4.13, we cannot replace $\mathbb{Q}^{\geq 0}$ with a general conical refinement monoid (with a suitable monoid homomorphism to $\mathbb{R}^+$); we show this on the example of $\mathbb{N}_0$.

In $\mathbb{N}_0$, the only refinement of $1+1 = 1+1$ is, up to order of columns,

$$\begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}.$$

In $\mathbf{2}$, the only refinement of $\max\{1,1\} = \max\{1,0\}$ is

$$\begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}.$$

Hence, the only refinement of $(1,1)+(1,1) = (1,1)+(0,1)$ in $\mathbf{2} \times \mathbb{N}_0$ is

$$\begin{array}{c|cc} & (1,1) & (1,1) \\ \hline (1,1) & (1,1) & (1,0) \\ (0,1) & (0,0) & (0,1) \end{array}. \tag{4.2}$$

(again, up to order of columns). Notice that while both $(1,1)$ and $(0,1)$ are elements of $\mathbf{2} \times \mathbb{N}_0 \setminus \{(1,0)\}$, the element $(1,0)$ is not. As $(1,0)$ is necessary in the refinement (4.2), we conclude that $\mathbf{2} \times \mathbb{N}_0 \setminus \{(1,0)\}$ is *not* a refinement monoid; thus, it not only fails to satisfy the assumptions of Proposition 4.11—in view of Proposition 2.37, it also loses all relevance to us.

Replacing the equality $1+1 = 1+1$ with any equality of sums of nonzero elements that does not have a "nonzero refinement" (provided that there is such) in a general conical refinement monoid, we see that the assumption that the monoid B in Remark 4.13 have "nonzero refinements" is necessary.

Remark 4.18. Another remark about the construction from Propostion 4.12 is in order: In the proof of said proposition, we use the exclusion of all elements of the form $(a,0)$ with $a \neq 0$ from the product $A \times \mathbb{Q}^{\geq 0}$ in order to easily obtain a monoid homomorphism $s : C \longrightarrow \mathbb{R}^+$ with $s(x) > 0$ whenever $x \neq 0$, so that we can apply Proposition 4.11 to C. Let us have a look at the entire product $A \times \mathbb{Q}^{\geq 0}$ instead of C: not only that we then cannot in general ensure the existence of a suitable monoid homomorphism for Proposition 4.11 to apply, but even the assertion of said proposition—that is, that the resulting monoid cannot be realized as $\mathcal{V}(-)$ of any regular algebra over an uncoutable field—could fail, as demonstrated in the following example:

Example 4.19. For R_2 the "rose with two petals" quiver of Example 3.38, there is a unital regular algebra $Q_K(R_2)$ such that $Q_K(R_2) \simeq \mathbf{2}$. Also, $\mathbb{Q}^{\geq 0} \simeq \mathcal{V}(L_K(E))$ for a quiver E as in Theorem 3.19. By Observation 2.25,

$$\mathbf{2} \times \mathbb{Q}^{\geq 0} \simeq \mathcal{V}(Q_K(R_2)) \times \mathcal{V}(L_K(E)) \simeq \mathcal{V}(Q_K(R_2) \times L_K(E)),$$

and the algebra $Q_K(R_2) \times L_K(E)$ is, as a direct product of two regular K-algebras, also a regular algebra over K. Hence, the monoid $\mathbf{2} \times \mathbb{Q}^{\geq 0}$ can be realized as $\mathcal{V}(I)$ with I a regular algebra over *any given field*.

Note that for monoids from Proposition 4.12, we only know that they cannot be realized by regular algebras over *uncountable* fields, thus witnessing the negative answer to Problem 1.4 for K uncountable. However, we do not know of any means to prove either their realizability or non-realizability by regular rings (or even regular algebras over countable fields).

Chapter 5

The monoid $\mathcal{V}(R)$ of the Chuang-Lee ring R

5.1 The Chuang-Lee ring R

In this section, we will reconstruct the ring R constructed in Chuang – Lee (1990), and in the following two, we shall compute its monoid $\mathcal{V}(R)$. Let us fix some notation for this chapter first:

Let K be a *countable* field, $K[t]$ the ring of polynomials over K in an indeterminate t, and $K(t)$ the quotient field of $K[t]$. Let us define a valuation ∂ on $K(t)$ as follows: Put $\partial 0 := +\infty$, and if t divides neither $f(t)$ nor $g(t)$ and if $n \in \mathbb{Z}$, put $\partial t^n \frac{f(t)}{g(t)} := n$.

Let $V := \{\, r \in K(t) \mid \partial r \geq 0 \,\}$. Note that, as a subring of $K(t)$, the set V is a vector space over K. Next, observe that for $n \in \mathbb{N}_0$, we have $t^n V = \{\, r \mid \partial r \geq n \,\}$; subsets—subspaces, in fact—of V of this form play a key role in what is to come.

Fixing some more notation, let $E := \operatorname{End}_K V$ and let S denote the subset of E consisting of all $x \in E$ such that there exists a $\varphi x \in K(t)$ and an $n \in \mathbb{N}_0$ satisfying $(x - \varphi x)t^n V = 0$. Informally speaking, we thus let S consist of all K-endomorphisms of V that act as multiplication by an element of $K(t)$ on all elements of V with sufficiently large valuation ∂.

Along with verifying other properties of S, let us justify the name of the element φx of $K(t)$ by the following lemma:

Lemma 5.1. *(i) For each $x \in S$, the element $\varphi x \in K(t)$ is unique and does not depend on n.*

(ii) The set S is an K-subalgebra of V, and $\varphi : S \longrightarrow K(t)$ is a surjective homomorphism of K-algebras.

Proof. (i) Suppose $(x - b)t^k V = 0$ for some $b \in K(t)$ and $k \in \mathbb{N}_0$. Then

$$(x - \varphi x)t^{n+k} = 0 = (x - b)t^{n+k},$$

whence $(\varphi x - b)t^{n+k} = 0$. As $t^{n+k} \neq 0$, we conclude that $\varphi x = b$.

(ii) It follows from (i) that $\varphi : S \longrightarrow K(t)$ is a well-defined map. Observe that whenever $x, y \in S$, then $x - y \in S$ with $\varphi(x - y) = \varphi x - \varphi y$. Furthermore, suppose that $(x - \varphi x)t^n V = 0$ and $(y - \varphi y)t^k V = 0$ with $k, n \in \mathbb{N}_0$. Then clearly

$x(y-\varphi y)t^k V = 0$. Since $\varphi y \in K(t)$, we have $(\varphi y)t^m \in V$ for sufficiently large $m \in \mathbb{N}_0$, and thus

$$\begin{aligned}(xy-(\varphi x)(\varphi y))t^{m+n+k}V &= x(y-\varphi y)t^{m+n+k}V + (x-\varphi x)(\varphi y)t^{m+n+k}V \\ &\subseteq x(y-\varphi y)t^k V + (x-\varphi x)t^n V = 0.\end{aligned}$$

It follows that φ is a homomorphism of K-algebras. Finally, to show that φ is onto, consider $r \in K(t)$. As with φy above, multiplying r by a sufficient power of t makes its valuation non-negative, that is, we have $rt^m V \in V$ for some m. Multiplication by r is then an K-homomorphism from the subspace $t^m V$ of V into V; as such, it can be extended to an endomorphism, say, x, of V. Then $(x-r)t^m V = 0$, whence $r = \varphi x$. □

We shall need a basis of V that is "well-behaved" with respect to the valuation ∂:

Lemma 5.2. *There is a basis* $\{\, v_i \mid i \in \mathbb{N}_0 \,\}$ *of* V *over* K *such that* $\partial v_i = i$ *holds for all* $i \in \mathbb{N}_0$.

Proof. As a subring of $K(t)$, the ring V is a vector space over K. Since K is countable, so are the rings $K[t]$, $K(t)$ and V. Thus, the dimension of V over K is also at most countable. With $1, t, t^2, \ldots$ being a linearly independent subset[1] of V, the dimension of V must be infinite.

Let now $\{\, u_i \mid i \in \mathbb{N}_0 \,\}$ be a basis of V. Inductively, we construct a strictly increasing sequence $B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \ldots$ of linearly independent subsets of V satisfying the following for every $i \in \mathbb{N}_0$:

(i) the set B_i spans the same subspace of V as $\{\, u_0, \ldots, u_i \,\}$ does; and

(ii) whenever $v \neq w$ in B_i, then $\partial v \neq \partial w$.

Put $B_0 := \{\, u_0 \,\}$. Now, if ∂u_i is different from all ∂w with $w \in B_{i-1}$, put $B_i := B_{i-1} \cup \{\, u_i \,\}$; if on the other hand $\partial u_i = \partial w_1 = k$ for some (necessarily unique) $w_1 \in B_{i-1}$, we need to "alter" the element u_i before adding it to the set B_{i-1}: By the definition of ∂, we have

$$u_i = t^k \frac{f_u}{g_u} \text{ and } w_1 = t^k \frac{f_w}{g_w}$$

for some f_u, g_u, f_w, g_w not divisible by t in $K[t]$. Hence, $f_u g_w = tp + \alpha$ and $g_u f_w = tq + \beta$ for some $p, q \in K[t]$ and $\alpha, \beta \in F$. Putting $\lambda_1 := \frac{\alpha}{\beta}$, we obtain

$$\begin{aligned}\frac{u_n}{w_1} - \lambda_1 &= \frac{f_u g_w}{g_u f_w} - \frac{\alpha}{\beta} \\ &= \frac{\beta f_u g_w - \alpha g_u f_w}{\beta g_u f_w} \\ &= \frac{\beta(tp+\alpha) - \alpha\,(tq+\beta)}{\beta g_u f_w} \\ &= t\frac{\beta p - \alpha q}{\beta g_u f_w} \in tV,\end{aligned}$$

[1] Clearly, $\partial t^i = i$ for all $i \in \mathbb{N}_0$.

whence $\partial\left(\frac{u_i}{w_1}-\lambda_1\right)\geq 1$. Thus,

$$\partial(u_i-\lambda_1 w_1)=\partial w_1+\partial\left(\frac{u_i}{w_1}-\lambda_1\right)>\partial w_1.$$

If now $\partial(u_i-\lambda_1 w_1)=\partial w_2$ with $w_2\in B_{i-1}$, by repeating the same process, we find a $\lambda_2\in F$ such that

$$\partial(u_i-\lambda_1 w_1-\lambda_2 w_2)>\partial w_2>\partial w_1.$$

Since the valuations obtained in this process increase with every iteration and since B_{i-1} is finite, we end up—after finitely many, say, k, steps—with an element

$$u:=u_i-\sum_{j=1}^{k}\lambda_j w_j\in V,$$

where $\lambda_j\in K$ and $w_j\in B_{i-1}$ for all j, such that ∂u is different from ∂w for all $w\in B_{i-1}$. Put $B_i:=B_{i-1}\cup\{\,u\,\}$.

That B_i spans the same subspace of V as $\{\,u_0,\dots,u_i\,\}$ does is obvious from the construction of B_i.

Put $B:=\bigcup B_i$; then B is a base of V satisfying $\partial v\neq\partial w$ for any two distinct elements v,w of B. After a suitable reordering of B, we obtain $B=\{\,v_i\mid i\in\mathbb{N}_0\,\}$ with $\partial v_0<\partial v_1<\partial v_2<\dots$.

For $v=\sum_{i=k}^{n}\alpha_i v_i\in V$ with $\alpha_i\in K$ and $\alpha_k\neq 0$, we observe that $\partial v=\partial v_k$. With $1,t,t^2,\dots$ being in the span of B (that is, in V), every $i\in\mathbb{N}_0$ thus appears as ∂v_j for some $j\in\mathbb{N}_0$. Minding the ordering of the ∂v_i's above, we conclude that $\partial v_i=i$ for every i. ☐

Let us *fix* a basis $\{\,v_i\mid i\in\mathbb{N}_0\,\}$ of V as in Lemma 5.2, that is, with $\partial v_i=i$ for all i. Observe that then for any $i\in\mathbb{N}_0$, the set $\{\,v_j\mid j\geq i\,\}$ forms a basis of t^iV. By π_i, we shall denote the projection of V on its "first $i+1$ coordinates"[2] in the above basis, that is, the endomorphism of V defined as $\pi_i(v_j)=v_j$ for $j\leq i$ and $\pi_i(v_j)=0$ for $j>i$. Note that with this notation, $(1-\pi_i)V=t^{i+1}V$. The matrix of π_i is block-diagonal of the form $\begin{pmatrix}\mathrm{Id}_{i+1} & 0\\ 0 & 0\end{pmatrix}$.

It will be convenient to characterise the classical notions of row-finiteness and of column-finiteness in the following way:

Observation 5.3. *(i) The matrix of $x\in E$ is* row-finite *(i.e., each of its rows has only finitely many non-zero entries) if and only if for any $i\in\mathbb{N}_0$, there exists a $j\in\mathbb{N}_0$ such that $\pi_i x(1-\pi_j)=0$.*

(ii) The matrix of $x\in E$ is column-finite *(each of its columns has only finitely many non-zero entries) if and only if for any $i\in\mathbb{N}_0$, there exists a $j\in\mathbb{N}_0$ such that $(1-\pi_j)x\pi_i=0$.* ☐

Clearly, the matrix of any endomorphism of V is column-finite. We note that matrices of elements of S are also necessarily row-finite:

[2] The fact that the rank of π_i is $i+1$, as well as indexing matrix entries and terms in sequences below starting from 0 instead of 1, can admittedly get confusing. We choose this notation not to shift indices used in Chuang – Lee (1990) and to keep indices related to the valuation ∂.

Lemma 5.4. *If $x \in S$, then the matrix of x is row-finite.*

Proof. Since $x \in S$, there is an $n \in \mathbb{N}_0$ such that $(x - \varphi x)t^n V = 0$. Given $i \in \mathbb{N}_0$, there is also an $m \in \mathbb{N}_0$ satisfying $(\varphi x)t^m V \subseteq t^{i+1}V$. Taking $j := \max\{m,n\}$, we see that $(x - \varphi x)t^j V = xt^j V \subseteq t^{i+1}V$, whence $\pi_i x(1 - \pi_j) = 0$. $\square$

We put $W := S \times \prod_{k=0}^{\infty} \pi_k E \pi_k$. Notice that we can view $\pi_k E \pi_k$ as a subset of $M_{k+1}(K) \subseteq M_\infty(K)$, so $W \subseteq M_\infty(K) \times \prod_{k=0}^{\infty} M_k(K)$.

Let R consist of all elements $\boldsymbol{w} = (w_S, w_0, w_1, \dots) \in W$ satifying both:

(i) for any $m \geq 0$, there is an $n \geq 0$ such that $w_k \pi_m = w_S \pi_m$ for all $k \geq n$, and

(ii) for any $m \geq 0$, there is an $n \geq 0$ such that $\pi_m w_k = \pi_m w_S$ for all $k \geq n$.

Viewing w_S and all the w_i's as infinite matrices, the translation of the above conditions is

(i) for any $m \in \mathbb{N}_0$, the first m columns of w_i are the same as in w_S for all but finitely many i's, and

(ii) for any $m \in \mathbb{N}_0$, the first m rows of w_i are the same as in w_S for all but finitely many i's.

It is shown in (Chuang – Lee, 1990, pp.18-19) that S is a regular unital ring and that R is a regular unital K-algebra.

5.2 Idempotents of R

Before computing the monoid $\mathcal{V}(R)$, we shall find necessary and sufficient conditions for principal right ideals of R to be isomorphic; the conditions are stated in Proposition 5.15. In Section 5.3, we will see that—thanks to Proposition 5.16—we will not need to work with any larger R-modules and still be able to compute the monoid $\mathcal{V}(R)$.

Lemma 5.5. *For an idempotent $e \in S$, either $\varphi e = 0$ or $\varphi e = 1$ holds.*

Proof. Since $e \in S$, there is an n such that $(e - \varphi e)t^n V = 0$. For any $k \geq n$, we then have $ev_k = (\varphi e)v_k$. Applying e, we obtain

$$ev_k = (\varphi e)ev_k. \tag{5.1}$$

Now, if $\varphi e \neq 1$, (5.1) yields $ev_k = 0$. Since this holds for all $k \geq n$, we obtain $(e - 0)t^n V = 0$, whence $\varphi e = 0$. $\square$

Observation 5.6. *For an idempotent $e \in S$, if $\varphi e = 0$, then the matrix of e is block diagonal of the form*

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix},$$

while if $\varphi e = 1$, then the matrix of e is

$$\begin{pmatrix} X & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix}.$$

In either case, X is an idempotent square matrix of finite size. $\square$

Lemma 5.7. *Let* $\boldsymbol{w} = \left(\begin{pmatrix} X & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, w_0, w_1, \dots\right)$ *be an idempotent element of* R*, where* $X \in M_n(K)$*. Then there is an idempotent*

$$\boldsymbol{w}' = \left(\begin{pmatrix} 0_{n-\operatorname{rank} X} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, w_0', w_1', \dots\right) \in R$$

such that $\boldsymbol{w}R \simeq \boldsymbol{w}'R$*.*

Proof. As X is an idempotent matrix from $M_n(K)$, there exists an invertible matrix $A \in GL(n,K)$ such that

$$AXA^{-1} = \begin{pmatrix} 0_{n-\operatorname{rank} X} & 0 \\ 0 & \mathrm{Id}_{\operatorname{rank} X} \end{pmatrix}.$$

Put $a_S := \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix}$, for $i < n$, put $a_i := \mathrm{Id}_{i+1}$, and for $i \geq n$, put $a_i := \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id}_{i+1-n} \end{pmatrix}$. Then $\boldsymbol{a} := (a_S, a_0, a_1, \dots) \in R$ is invertible in R: its inverse, $\boldsymbol{a}^{-1} \in R$, has its respective terms $\begin{pmatrix} A & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix}$, Id_{i+1} and $\begin{pmatrix} A^{-1} & 0 \\ 0 & \mathrm{Id}_{i+1-n} \end{pmatrix}$.

One readily checks that $\boldsymbol{w}\boldsymbol{a}^{-1}\boldsymbol{w}'\boldsymbol{w}'\boldsymbol{a}\boldsymbol{w} = \boldsymbol{w}$ and $\boldsymbol{w}'\boldsymbol{a}\boldsymbol{w}\boldsymbol{w}\boldsymbol{a}^{-1}\boldsymbol{w}' = \boldsymbol{w}'$, so we have mutually inverse R-isomorphisms

$$\begin{tikzcd} \boldsymbol{w}R \arrow[r, bend left, "\boldsymbol{w}'\boldsymbol{a}\boldsymbol{w}"] & \boldsymbol{w}'R \arrow[l, bend left, "\boldsymbol{w}\boldsymbol{a}^{-1}\boldsymbol{w}'"] \end{tikzcd}$$

(Lemma 2.22). ☐

Lemma 5.8. *If* $\boldsymbol{w} = (w_S, w_0, w_1, \dots)$ *is an idempotent in* R *with* $w_S = \mathrm{Id}$*, then* $\boldsymbol{w}R \simeq \boldsymbol{u}R$*, where* $\boldsymbol{u} = (u_S, u_0, u_1, \dots)$*,* $u_S = \mathrm{Id}$ *and* $u_i = \begin{pmatrix} \mathrm{Id}_{\operatorname{rank} w_i} & 0 \\ 0 & 0_{i+1-\operatorname{rank} w_i} \end{pmatrix}$*.*[3]

Proof. For any i, find the greatest $n_i \in \mathbb{N}_0$ such that $w_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & W_i \end{pmatrix}$ for some matrix W_i. Since the matrix W_i is idempotent, there is an invertible matrix A_i of appropriate size satisfying $A_i W_i A_i^{-1} = \begin{pmatrix} \mathrm{Id}_{\operatorname{rank} W_i} & 0 \\ 0 & 0 \end{pmatrix}$. With $a_S = \mathrm{Id}$, we have $\boldsymbol{a} \in R$, since we have $\lim\limits_{i\to\infty} n_i = \infty$[4] due to $\boldsymbol{w}$ being an element of R. Then, as in Lemma 5.7, $\boldsymbol{u}\boldsymbol{a}\boldsymbol{w}$ and $\boldsymbol{w}\boldsymbol{a}^{-1}\boldsymbol{u}$ are the desired isomorphism and its inverse. ☐

Lemma 5.9. *If* $\boldsymbol{w} = (w_S, w_0, w_1, \dots)$ *is an idempotent in* R *with* $w_S = \begin{pmatrix} 0_n & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix}$*, then* $\boldsymbol{w}R \simeq \boldsymbol{u}R$*, where* $\boldsymbol{u} = (u_S, u_0, u_1, \dots)$*,* $u_S = w_S$*,* $u_i = \begin{pmatrix} \mathrm{Id}_{\operatorname{rank} w_i} & 0 \\ 0 & 0 \end{pmatrix}$ *for finitely many* i*'s and* $u_i = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id}_{\operatorname{rank} w_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *for the remaining* i*'s.*

[3]Note that if $\operatorname{rank} w_i = i+1$, then the zero-block in u_i is of size zero, while if $\operatorname{rank} w_i = 0$, then the identity-block in u_i is of size zero. Henceforth, we leave similar situations without further comment.

[4]We write $\lim\limits_{i\to\infty} n_i = \infty$ to state that for each $l \in \mathbb{N}$, $n_i \geq l$ holds for all but finitely many i's.

Proof. Find i_0 such that whenever $i \geq i_0$, the first n rows and columns of w_i are zero. For $i > i_0$, let $n_i \geq 0$ be the greatest integer such that $w_i = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id}_{n_i} & 0 \\ 0 & 0 & W_i \end{pmatrix}$ holds for some W_i. For $i < i_0$, let a_i be an invertible matrix satisfying $a_i w_i a_i^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$, and for $i \geq i_0$, take $a_i = \begin{pmatrix} \mathrm{Id}_{n+n_i} & 0 \\ 0 & A_i \end{pmatrix}$, where A_i is invertible and satisfies $A_i W_i A_i^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$. As in the proof of Lemma 5.8, $\boldsymbol{a} \in R$ is invertible in R and $\boldsymbol{uaw}$ and $\boldsymbol{wa}^{-1}\boldsymbol{u}$ are the sought-after isomorphisms. ☐

Lemma 5.10. *If $\boldsymbol{w} = (w_S, w_0, w_1, \dots)$, $\boldsymbol{u} = (u_S, u_0, u_1, \dots)$ are idempotents in R, if $w_S = \mathrm{Id}$ and $u_S = \begin{pmatrix} 0_n & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix}$ and if $\operatorname{rank} w_i = \operatorname{rank} u_i$ for all i, then $\boldsymbol{w}R \simeq \boldsymbol{u}R$.*

Proof. By Lemma 5.9, we may assume that $w_i = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$ for all i, $u_i = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$ for finitely many i's and $u_i = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all remaining i's.

Consider the two following endomorphisms of V:

- $\alpha : v_i \longmapsto t^n v_i$ for all i;
- $\beta : v_i \longmapsto \begin{cases} t^{-n} v_i & \text{for } i \geq n, \text{ and} \\ 0 & \text{for } i < n. \end{cases}$

By the definition of ∂, $\partial \alpha v_i = i + n$ for all i and $\partial \beta v_i = i - n$ whenever $i \geq n$. Thus, viewed as matrices over K, we have

$$\alpha = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

with the first n rows zero and with A a lower triangular matrix and

$$\beta = \begin{pmatrix} 0 & B \end{pmatrix}$$

with the first n columns zero and B a lower triangular matrix. Notice that $\alpha\beta = u_S$ and $\beta\alpha = w_S$; in particular, $B = A^{-1}$. Let A_i and B_i be the $i \times i$ upper-left corners of A and B, respectively. Since both A and $B = A^{-1}$ are lower triangular, we see that

$$A_i^{-1} = B_i \tag{5.2}$$

holds for all i.

Let $n_i := \operatorname{rank} u_i = \operatorname{rank} w_i$. For $i < i_0$, let $c_i = d_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix}$ (as then $w_i = u_i$, we need not alter the i-th coordinate when looking for an isomorphism between $\boldsymbol{w}R$ and $\boldsymbol{u}R$); then $u_i c_i w_i w_i d_i u_i = u_i$ and $w_i d_i u_i u_i c_i w_i = w_i$. Let now $i \geq i_0$. Let c_i be the matrix with first n rows zero, A_{n_i} in the next n_i rows and first n_i columns, and the rest zero, i.e., $c_i = \begin{pmatrix} 0 & 0 & 0 \\ A_{n_i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that the blocks of this matrix that are necessarily square are the top-right one (size n), the middle-left one (size n_i), and the bottom-middle one (size $i+1-n_i-n$). Also notice that multiplication from the left by $u_i = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id}_{n_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ preserves the middle-row blocks in c_i (while killing the rest; however, as the rest is already zero, it is preserved as well) and multiplication from the right by $w_i = \begin{pmatrix} Idn_i & 0 \\ 0 & 0 \end{pmatrix}$ preserves the first-column blocks in c_i (again, killing the rest). Thus, we see that

$$u_i c_i w_i = c_i.$$

Similarly, defining d_i as the matrix with the first n columns zero, B_{n_i} in the next n_i columns and first n_i rows, and the rest zero, i.e., of the form $d_i = \begin{pmatrix} 0 & B_{n_i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$w_i d_i u_i = d_i.$$

Now that B_{n_i} is in fact $A_{n_i}^{-1}$ by (5.2), we conclude that

$$u_i c_i w_i w_i d_i u_i = c_i d_i = \begin{pmatrix} 0 & 0 & 0 \\ A_{n_i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & B_{n_i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id}_{n_i} & 0 \\ 0 & 0 & 0 \end{pmatrix} = u_i$$

and

$$w_i d_i u_i u_i c_i w_i = d_i c_i = \begin{pmatrix} 0 & B_{n_i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ A_{n_i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix} = w_i.$$

Hence we have $u_i c_i w_i w_i d_i u_i = u_i$ and $w_i d_i u_i u_i c_i w_i = w_i$ for all i. Defining $\boldsymbol{c} := (\alpha, c_0, c_1, \dots)$ and $\boldsymbol{d} := (\beta, d_0, d_1, \dots)$, we now see that $\boldsymbol{ucwwdu} = \boldsymbol{u}$ and $\boldsymbol{wduucw} = \boldsymbol{w}$. Thence, to prove that $\boldsymbol{ucw}$ and $\boldsymbol{wdu}$ are mutually inverse R-isomorphisms of $\boldsymbol{u}R$ and $\boldsymbol{v}R$, it only remains to verify that $\boldsymbol{c}$ and $\boldsymbol{d}$ are elements of R.

Since the matrix of α is lower triangular, we see that for every $i \geq i_0$, the first $n+n_i$ rows of c_i and α coincide; similarly, the first n_i rows of β and d_i coincide. Hence we have $\pi_{n+n_i-1} c_i = \pi_{n+n_i-1}\alpha$ and $\pi_{n_i-1}\beta = \pi_{n_i-1} d_i$, while $\lim_{i\to\infty} n_i = \infty$ holds due to $\boldsymbol{w} \in R$. As for columns, since α and β are column-finite, there is for every m a k_m such that whenever $k \geq k_m$, then the first $m+1$ columns of α and c_k coincide—so $\alpha\pi_m = c_k\pi_m$ for every $k \geq k_m$—and that the first $m+1$ columns of β and d_i coincide, so $\beta\pi_m = d_k\pi_m$ whenever $k \geq k_m$. Hence $\alpha, \beta \in R$, which concludes the proof. □

Proposition 5.11. *Let $\boldsymbol{u} = (u_S, u_0, u_1, \dots)$, $\boldsymbol{v} = (v_S, v_0, v_1, \dots)$ be idempotents in R with u_S, v_S of infinite rank. Then $\boldsymbol{u}R \simeq \boldsymbol{v}R$ if and only if* $\operatorname{rank} u_i = \operatorname{rank} v_i$ *holds for all i.*

Proof. The i-th component of an R-isomorphism is always a $\pi_i E \pi_i$-isomorphism; the only-if-part follows. For the if-part, due to Lemmas 5.7 and 5.9, we may assume that $u_S = \begin{pmatrix} 0_n & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$, $u_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix}$ for finitely many i's and $u_i = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & \mathrm{Id}_{n_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for the remaining i's, and, similarly, that $v_S = \begin{pmatrix} 0_m & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$, $v_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix}$ for finitely many i's and $v_i = \begin{pmatrix} 0_m & 0 & 0 \\ 0 & \mathrm{Id}_{n_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for the rest. Lemma 5.10 then asserts that both $\boldsymbol{u}R$ and $\boldsymbol{v}R$ are isomorphic to $\boldsymbol{w}R$, where $\boldsymbol{w} = (w_S, w_0, w_1, \dots)$ with $w_S = \mathrm{Id}$ and $w_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix}$ for all i. □

We shall now search for an analogue of Propostion 5.11 for idempotents of R with the S-coordinate of finite rank.

Lemma 5.12. *Let $\boldsymbol{w} = (w_S, w_0, w_1, \dots)$ be an idempotent in R with $w_S = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$. Then $\boldsymbol{w}R$ is isomorphic to $\boldsymbol{u}R$ for some $\boldsymbol{u} = (u_S, u_0, u_1, \dots) \in \mathrm{Idemp}\, R$ with $u_S = \begin{pmatrix} \mathrm{Id}_{\mathrm{rank}\, X} & 0 \\ 0 & 0 \end{pmatrix}$.*

Proof. Suppose $X \in M_n(K)$. Since X is idempotent, there is a matrix $A \in GL(n, K)$ satisfying $AXA^{-1} = \begin{pmatrix} \mathrm{Id}_{\mathrm{rank}\, X} & 0 \\ 0 & 0 \end{pmatrix}$. Since $\boldsymbol{w} \in R$, there is an $i_0 \geq n$ such that whenever $i \geq i_0$, $w_i = \begin{pmatrix} X & 0 \\ 0 & W_i \end{pmatrix}$ holds for some matrix $W_i \in M_{i+1-n}(K)$. Let $a_S := \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$, $a_i := \mathrm{Id}_{i+1}$ for all $i < i_0$ and $a_i := \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ for all $i \geq i_0$. Then clearly $\boldsymbol{a} = (a_S, a_0, a_1, \dots)$ is an invertible element of R, and we may put $\boldsymbol{u} := \boldsymbol{a}\boldsymbol{w}\boldsymbol{a}^{-1}$. □

Lemma 5.13. *In Lemma 5.12, one can find $\boldsymbol{u}$ such that $u_i = \begin{pmatrix} \mathrm{Id}_{\mathrm{rank}\, w_i} & 0 \\ 0 & 0_{i+1-\mathrm{rank}\, w_i} \end{pmatrix}$ for finitely many i's and $u_i = \begin{pmatrix} \mathrm{Id}_{\mathrm{rank}\, X} & 0 & 0 \\ 0 & 0_{n-\mathrm{rank}\, X} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$ for the remaining i's.*

Proof. In the proof of Lemma 5.12), the matrices w_i are—for $i \geq i_0$—in fact of the form $\begin{pmatrix} X & 0 & 0 \\ 0 & 0_{m_i} & 0 \\ 0 & 0 & Y_i \end{pmatrix}$ for some m_i and some idempotent matrices $Y_i \in M_{i+1-n-m_i}(K)$. Since $\boldsymbol{w} \in R$, notice that $\lim_{i\to\infty} m_i = \infty$. Since the Y_i's are idempotent, there are invertible matrices $B_i \in GL(i+1-n-m_i, K)$ such that $B_i Y_i B_i^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathrm{rank}\, Y_i} \end{pmatrix}$. Take then again $a_S := \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ and $a_i := \mathrm{Id}$ for $i < i_0$, and for $i \geq i_0$, take $a_i := \begin{pmatrix} A & 0 & 0 \\ 0 & \mathrm{Id}_{m_i} & 0 \\ 0 & 0 & B_i \end{pmatrix}$. The rest of the proof is the same as for Lemma 5.12. □

Note that in the above lemma, due to $\boldsymbol{u}$ being an element of R, the size of the zero-block in the middle of the matrices u_i gradually increases with increasing i; this explains the nature of condition (M3a).

Lemma 5.13, together with the same reasoning as in the only-if-part of Proposition 5.11, easily translates into the following:

Proposition 5.14. *For* $\boldsymbol{v} = (v_S, v_0, v_1, \dots), \boldsymbol{w} = (w_S, w_0, w_1, \dots) \in \operatorname{Idemp} R$ *with* $\operatorname{rank} v_S = \operatorname{rank} w_S < \infty$*, the modules* $\boldsymbol{v}R$ *and* $\boldsymbol{w}R$ *are isomorphic if and only if* $\operatorname{rank} v_i = \operatorname{rank} w_i$ *for all* i. □

Finally, combining Propositions 5.11 and 5.14 immediately yields:

Proposition 5.15. *For* $\boldsymbol{v} = (v_S, v_0, v_1, \dots), \boldsymbol{w} = (w_S, w_0, w_1, \dots) \in \operatorname{Idemp} R$*, the modules* $\boldsymbol{v}R$ *and* $\boldsymbol{w}R$ *are isomorphic if and only if both* $\operatorname{rank} v_S = \operatorname{rank} w_S$ *and* $\operatorname{rank} v_i = \operatorname{rank} w_i$ *for all* i. □

5.3 The monoid $\mathcal{V}(R)$ of the Chuang-Lee ring R

Let us define the binary relation $\equiv$ on $\mathbb{N}_0 \times \mathbb{N}_0$ as $(n, m) \equiv (n', m')$ if either $(n, m) = (n', m')$ or $m = m' > 0$. Note that $\equiv$ is a congruence on the monoid $\mathbb{N}_0 \times \mathbb{N}_0$. Let us define M_S as the factormonoid $(\mathbb{N}_0 \times \mathbb{N}_0) / \equiv$ and denote the congruence class of (n,m) by $[n,m]$.

Informally, the monoid M_S can be viewed as a copy of the semigroup $\mathbb{N}$ "above" a copy of the monoid $\mathbb{N}_0$. Elements of the form $[n,0]$ are thought of as being "downstairs" (in the copy of $\mathbb{N}_0$, with its ususual addition), while elements $[0,m]$ are "upstairs" (in the copy of $\mathbb{N}$ with the usual addition), and adding an element from downstairs to one from upstairs does not change the element from upstairs.

It is easy to verify that M_S is a refinement monoid.

Put $N := M_S \times (\mathbb{N}_0)^{\omega}$. Then, taking refinements component-wise, N is a refinement monoid. Let M be the submonoid of N consisting of all sequences $([r_d, r_u], r_0, r_1, \dots) \in N$ satisfying:

(M1) there exists an $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}_0$, $r_i \leq n(i+1)$ holds, and:

(M2) if $r_u > 0$, then $\lim\limits_{i\to\infty} r_i = \infty$;

(M3) if $r_u = 0$, then both:

- (a) $\lim\limits_{i\to\infty} n(i+1) - r_i = \infty$ for the same n as in condition (M1), and
- (b) there exists an $i_0 \in \mathbb{N}_0$ such that for all $i \geq i_0$, the inequality $r_i \geq r_d$ holds.

We will show that then $M \simeq \mathcal{V}(R)$, using the following:

Proposition 5.16 (Růžička, 2011, Lemma 4.4)**.** *Let* R *be a regular ring, let* M *be a refinement monoid, and let* $f : R \longrightarrow M$ *be a map satisfying:*

(i) $aR \simeq bR$ *if and only if* $f(a) = f(b)$*, for all* $a, b \in R$*.*

(ii) If $x + y = f(c)$ for an idempotent $c \in R$ and $x, y \in M$, then there are orthogonal idempotents $a, b \in R$ such that $f(a) = x$, $f(b) = y$, and $a+b = c$.

(iii) $f(1)$ is an order-unit in M.

Then $\mathcal{V}(R) \simeq M$. □

In order to do so, we first need to show that the monoid M satisfies the refinement property.

Lemma 5.17. *The monoid M is a refinement monoid.*

Proof. Suppose we have elements

$$\boldsymbol{r}^j = \left(\left[r_d^j, r_u^j\right], r_0^j, r_1^j, \dots\right)$$

of M with $j \in \{1, 2, 3, 4\}$ and with

$$\boldsymbol{r}^1 + \boldsymbol{r}^2 = \boldsymbol{r}^3 + \boldsymbol{r}^4 = \boldsymbol{s} = ([s_d, s_u], s_0, s_1, \dots). \tag{5.3}$$

We search for a refinement of these sums depending on the M_S-component of $\boldsymbol{s}$:

Case 1: $s_u = 0$. Then clearly $0 = r_u^j$ for all j. From condition (M3) for $\boldsymbol{r}^j$, find a common i_0 such that for all $i \geq i_0$ and all j, $r_i^j \geq r_d^j$ holds. For all j, let

$$\begin{aligned} s_d^j &:= r_d^j, \\ s_u^j &:= 0, \\ s_i^j &:= 0 && \text{for } i < i_0, \text{ and} \\ s_i^j &:= r_d^j && \text{for } i \geq i_0. \end{aligned}$$

As we have $r_d^1 + r_d^2 = r_d^3 + r_d^4$ in $\mathbb{N}_0$, there is a refinement:

$$\begin{array}{c|cc} & r_d^1 & r_d^2 \\ \hline r_d^3 & \alpha_d & \beta_d \\ r_d^4 & \gamma_d & \delta_d \end{array},$$

i.e., there are $\alpha_d, \beta_d, \gamma_d, \delta_d \in \mathbb{N}_0$ such that $\alpha_d + \beta_d = r_d^3$, $\alpha_d + \gamma_d = r_d^1$, $\gamma_d + \delta_d = r_d^4$ and $\beta_d + \delta_d = r_d^2$. For $i < i_0$, find refinements of the i-th coordinate of the two sums in (5.3):

$$\begin{array}{c|cc} & r_i^1 & r_i^2 \\ \hline r_i^3 & \alpha_i & \beta_i \\ r_i^4 & \gamma_i & \delta_i \end{array}.$$

For $i \geq i_0$, find first a refinement of the of the sum $(r_i^1 - r_d^1) + (r_i^2 - r_d^2) = (r_i^3 - r_d^3) + (r_i^4 - r_d^4)$ in $\mathbb{N}_0$:

$$\begin{array}{c|cc} & r_i^1 - r_d^1 & r_i^2 - r_d^2 \\ \hline r_i^3 - r_d^3 & \alpha_i' & \beta_i' \\ r_i^4 - r_d^4 & \gamma_i' & \delta_i' \end{array}.$$

Then, put $\alpha_i := \alpha_i' + \alpha_d$, $\beta_i := \beta_i' + \beta_d$, $\gamma_i := \gamma_i' + \gamma_d$, $\delta_i := \delta_i' + \delta_d$, and

$$\begin{aligned} \boldsymbol{\alpha} &:= ([\alpha_d, 0], \alpha_0, \alpha_1, \dots), \\ \boldsymbol{\beta} &:= ([\beta_d, 0], \beta_0, \beta_1, \dots), \\ \boldsymbol{\gamma} &:= ([\gamma_d, 0], \gamma_0, \gamma_1, \dots), \\ \boldsymbol{\delta} &:= ([\delta_d, 0], \delta_0, \delta_1, \dots). \end{aligned}$$

We have found a refinement

$$\begin{array}{c|cc} & \boldsymbol{r}^1 & \boldsymbol{r}^2 \\ \hline \boldsymbol{r}^3 & \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{r}^4 & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{array}$$

in the monoid N; we claim that this refinement is in fact in M: We shall only verify that, say, $\boldsymbol{\alpha} \in M$, as the remaining elements behave similarly. Condition (M1) holds, since for each $i \in \mathbb{N}_0$, $\alpha_i \leq r_i^1$. Condition (M2) is satisfied automatically, as $\alpha_u = 0$. As for (M3), from $\alpha_i \leq r_i^1$, we have that $n(i+1) - r_i^1 \leq n(i+1) - \alpha_i$, whence (M3a) holds. Finally, for $i \geq i_0$ (with the i_0 we fixed above), $\alpha_i = \alpha_i' + \alpha_d \geq \alpha_d$, so (M3b) is satisfied as well.

Case 2: $s_u > 0$. We will still check the refinement property separately for three different cases:

Case 2a: $r_u^1 = 0$ and $r_u^3 = 0$ (and hence $r_u^2 = r_u^4 > 0$): w.l.o.g., suppose that $r_d^1 \leq r_d^3$. Find i_0 such that for all $i \geq i_0$, both $r_i^1 \geq r_d^1$ and $r_i^3 \geq r_d^3$ hold (from (M3)) and also that $r_i^2 \geq r_d^3 - r_d^1$ (from (M2)). Put

$$\boldsymbol{\alpha}' := \left([r_d^1, 0], \overbrace{0, \ldots, 0}^{\alpha_0', \ldots, \alpha_{i_0-1}'}, \underbrace{r_d^1, r_d^1, \ldots}_{\alpha_{i_0}', \alpha_{i_0+1}', \ldots} \right), \text{ and}$$

$$\boldsymbol{\beta}' := \left([r_d^3 - r_d^1, 0], \overbrace{0, \ldots, 0}^{\beta_0', \ldots, \beta_{i_0-1}'}, \underbrace{r_d^3 - r_d^1, r_d^3 - r_d^1, \ldots}_{\beta_{i_0}', \beta_{i_0+1}', \ldots} \right).$$

We shall now find a suitable refinement

$$\begin{array}{r|cc} & \boldsymbol{r}^1 - \boldsymbol{\alpha}' & \boldsymbol{r}^2 - \boldsymbol{\beta}' \\ \hline \boldsymbol{r}^3 - \boldsymbol{\alpha}' - \boldsymbol{\beta}' & \boldsymbol{\alpha}'' & \boldsymbol{\beta}'' \\ \boldsymbol{r}^4 & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{array}$$

component-wise as follows: There is no choice but $[0,0] = [\alpha_d'', \alpha_u''] = [\beta_d'', \beta_u''] = [\gamma_d, \gamma_u]$ and $[\delta_d, \delta_u] = [0, r_u^2]$. For $i \in \mathbb{N}_0$, consider the refinement

$$\begin{array}{r|cc} & r_i^1 - \alpha_i' & r_i^2 - \beta_i' \\ \hline r_i^3 - \alpha_i' - \beta_i' & \alpha_i'' & \beta_i'' \\ r_i^4 & \gamma_i & \delta_i \end{array}$$

(in $\mathbb{N}_0$) with $\alpha_i'' = \min\{ r_i^1 - \alpha_i', r_i^3 - \alpha_i' - \beta_i' \}$. Then

$$\begin{array}{c|cc} & \boldsymbol{r}^1 & \boldsymbol{r}^2 \\ \hline \boldsymbol{r}^3 & \boldsymbol{\alpha}' + \boldsymbol{\alpha}'' & \boldsymbol{\beta}' + \boldsymbol{\beta}'' \\ \boldsymbol{r}^4 & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{array}$$

is a refinement in M: (M2) is satisfied for $\boldsymbol{\delta}$, since for every i, either $\delta_i = r_i^4$ or $\delta_i = r_i^2 - \beta_i'$. Since γ_d, α_d'' and β_d'' are all zero and by the choice of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$, we see that $\boldsymbol{\alpha}' + \boldsymbol{\alpha}''$, $\boldsymbol{\beta}' + \boldsymbol{\beta}''$ and $\boldsymbol{\gamma}$ satisfy (M3b); the remaining conditions are verified as above.

Case 2b: $r^3_u = 0$, while r^1_u, r^2_u and r^4_u are nonzero. Find i_0 such that for all $i > i_0$, both $r^1_i \geq r^1_d$ and $r^3_i \geq r^1_d$. With

$$\boldsymbol{\alpha}' := \left([r^1_d,0]\,, \overbrace{0,\ldots,0}^{\alpha'_0,\ldots,\alpha'_{i_0-1}}\,, \underbrace{r^1_d, r^1_d, \ldots}_{\alpha'_{i_0},\alpha'_{i_0+1},\ldots} \right) \quad \text{and}$$

$$\boldsymbol{\alpha}'' := \left([0,0], \min\{\, r^1_0 - \alpha'_0, r^3_0 - \alpha'_0 \,\}, \{\, r^1_1 - \alpha'_1, r^3_1 - \alpha'_1 \,\}, \ldots \right),$$

the resulting refinement

$$\begin{array}{c|cc} & \boldsymbol{r}^1 & \boldsymbol{r}^2 \\ \hline \boldsymbol{r}^3 & \boldsymbol{\alpha}' + \boldsymbol{\alpha}'' & \boldsymbol{\beta} \\ \boldsymbol{r}^4 & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{array}$$

is a again in M.

Case 2c: All r^j_u are nonzero (and thus all r^j_d are irrelevant). From (M2) for all r^j, find for each $l \in \mathbb{N}_0$ an $i_l \in \mathbb{N}_0$ such that whenever $i \geq i_l$, the inequality $r^j_i \geq l$ holds for all j. Put $[\zeta_d,\zeta_u] := [0,1]$ and for all i, $\zeta_i := \max\{\, l \mid i \geq i_l \,\}$. Find a refinement

$$\begin{array}{c|cc} & \boldsymbol{r}^1 - \boldsymbol{\zeta} & \boldsymbol{r}^2 - \boldsymbol{\zeta} \\ \hline \boldsymbol{r}^3 - \boldsymbol{\zeta} & \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{r}^4 - \boldsymbol{\zeta} & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{array}$$

in N. Since all $\boldsymbol{r}^j$ satisfy (M2), so does $\boldsymbol{\zeta}$, and we coclude that

$$\begin{array}{c|cc} & \boldsymbol{r}^1 & \boldsymbol{r}^2 \\ \hline \boldsymbol{r}^3 & \boldsymbol{\alpha} + \boldsymbol{\zeta} & \boldsymbol{\beta} + \boldsymbol{\zeta} \\ \boldsymbol{r}^4 & \boldsymbol{\gamma} + \boldsymbol{\zeta} & \boldsymbol{\delta} + \boldsymbol{\zeta} \end{array}$$

is a refinement in M. □

Let us now define the map $f : R \longrightarrow M$: Let $\boldsymbol{w} = (w_S, w_0, w_1, \ldots) \in R$. For all i, put $f(\boldsymbol{w})_i := \operatorname{rank} w_i$, and

$$[f(\boldsymbol{w})_d, f(\boldsymbol{w})_u] := \begin{cases} [\operatorname{rank} w_S, 0] & \text{if the rank of } w_S \text{ is finite,} \\ [0,1] & \text{otherwise.} \end{cases}$$

Lemma 5.18. *The assignment f above is a well-defined map from R to M.*

Proof. Since the ranks of each w_i is at most $i+1$, we have $f(\boldsymbol{w}) \in N$ and satisfying (M1) (with $n = 1$). Condition (M2) is of interest only if the rank of w_S is infinite; then for any given $a \in \mathbb{N}$, there is an m such that $w_S \pi_m$ is at least of rank a; by the definition of R, there is a k' such that for all $n \geq k'$, $w_n \pi_m = w_S \pi_m$. Since w_S is column-finite, ther is a k'' such that all the non-zero entries of $w_S \pi_m$ are actually in $\pi_{k''} w_S \pi_m$. Thus, whenever $n \geq \max\{\, k', k'' \,\}$, the rank of w_n is at least a. We have thus shown that $\lim_{i\to\infty} \operatorname{rank} w_i = \infty$, whence (M2) is satisfied. As for (M3), suppose that the rank of w_S is finite. Then there is an m such that $w_S = \pi_m w_S$ (indeed, there are only finitely many linearly independent columns in w_S and w_S is column-finite, whence there are only finitely many non-zero rows in w_S). By the definition of R, there is an i_0 such that for all $i \geq i_0$, $\pi_m w_S = \pi_m w_i$. Then

$$\operatorname{rank} w_i \geq \operatorname{rank} \pi_m w_i = \operatorname{rank} \pi_m w_S = \operatorname{rank} w_S,$$

whence $f(\boldsymbol{w})_i \geq f(\boldsymbol{w})_d$ for all $i \geq i_0$ and $f(\boldsymbol{w})$ satisfies (M3b). As for (M3a), given $a \in \mathbb{N}$, find a $k \geq m+a$ such that for all $i \geq k$, the equality $\pi_{m+a}w_S = \pi_{m+a}w_i$ holds. Since $i \geq k \geq m+a$ and $(1-\pi_m)w_S = 0$, we coclude that w_i has at most $i+1-a$ nonzero rows. Thus, $\operatorname{rank} w_i \leq i+1-a$, whence $a \geq (i+1) - f(\boldsymbol{w})_i$. We have thus shown that $\lim_{i\to\infty}(i+1) - f(\boldsymbol{w})_i = \infty$, as desired. $\square$

Since an R-module isomorphism is necessarily an isomorphism in each of its components (i.e., an S-isomorphism in the S-component and a $\pi_i E \pi_i$-isomorphism in the i-component for every i), we observe that:

Observation 5.19. *If $\boldsymbol{a}, \boldsymbol{b} \in R$ satisfy $\boldsymbol{a}R \simeq \boldsymbol{b}R$, then $f(\boldsymbol{a}) = f(\boldsymbol{b})$.* $\square$

With the map f defined, we shall now proceed with verifying the conditions of Proposition 5.16: Verifying that condition (i) is satisfied is what Subsection 5.2 was dedicated to, namely, Proposition 5.15 translates into the desired condition. Condition (iii) is checked easily:

Lemma 5.20. *$f(1) = ([0,1], 1, 2, 3, \dots)$ is an order-unit in M.*

Proof. Let $\boldsymbol{r} = ([r_d, r_u], r_0, r_1, \dots) \in M$. Suppose first that $r_u = 0$. From (M1) and (M3), there is an n such that $r_i \leq n(i+1)$ for all i and that

$$\lim_{i\to\infty} n(i+1) - r_i = \infty.$$

Putting $[s_d,s_u] := [0,n]$ and $s_i = n(i+1) - r_i$, we see that $\boldsymbol{s} = ([s_d, s_u], s_0, s_1, \dots) \in M$ and that $\boldsymbol{s} + \boldsymbol{r} = nf(1)$. Let now $r_u > 0$; then, again, there is an n with $r_i \leq n(i+1)$ for all i. Put $m := \max\{r_u, n\} + 1$, $[s_d,s_u] := [0, m - r_u]$ and $s_i := (i+1)m - r_i$. Due to the choice of m, we see that

$$s_i = (i+1)m - r_i \geq (i+1)m - (i+1)n \geq i+1$$

holds for all i; thus, $\boldsymbol{s} = ([s_d, s_u], s_0, s_1, \dots)$ satisfies condition (M2), and so $\boldsymbol{s} \in M$. Clearly, $\boldsymbol{s} + \boldsymbol{r} = mf(1)$. $\square$

Now for the remainder, which is condition (ii) of Proposition 5.16: we check this first for the special forms of idempotents of R that we find in Lemmas 5.8 and 5.13, and then proceed with the general statement of Lemma 5.22.

Lemma 5.21. *Let $\boldsymbol{x} = ([x_d, x_u], x_0, x_1, \dots), \boldsymbol{y} = ([y_d, y_u], y_0, y_1, \dots) \in M$ and $\boldsymbol{c} = (c_S, c_0, c_1, \dots) \in \operatorname{Idemp} R$ satisfy $f(\boldsymbol{c}) = \boldsymbol{x} + \boldsymbol{y}$. Furthermore, let either*

(i) $c_S = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & 0 \end{pmatrix}$ *(with n finite),* $c_i = \begin{pmatrix} Idn_i & 0 \\ 0 & 0 \end{pmatrix}$ *for i's satisfying $x_i < x_d$ or $y_i < y_d$, and* $c_i = \begin{pmatrix} \mathrm{Id}_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{n_i - n} \end{pmatrix}$ *for the rest, or*

(ii) $c_S = \mathrm{Id}$*, and* $c_i = \begin{pmatrix} \mathrm{Id}_{n_i} & 0 \\ 0 & 0 \end{pmatrix}$ *for all i.*

Then there are orthogonal idempotents $\boldsymbol{a}, \boldsymbol{b} \in R$ such that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$, $f(\boldsymbol{a}) = \boldsymbol{x}$ and $f(\boldsymbol{b}) = \boldsymbol{y}$.

Proof. **Case** (i)**.** We have $f(\boldsymbol{c}) = ([n,0], n_0, n_1, \dots)$, whence x_u and y_u are both zero, and we see that $n = x_d + y_d$ and $n_i = x_i + y_i$ for all i. From condition (M3b) for both $\boldsymbol{x}$ and $\boldsymbol{y}$, we see that both $x_i \geq x_d$ and $y_i \geq y_d$ whenever c_i is of the form $\begin{pmatrix} \mathrm{Id}_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{n_i-n} \end{pmatrix}$ with $n_i > n$; in particular, we then have $n_i - n \geq x_i - x_d$ and $n_i - n \geq y_i - y_d$. Put $a_S := \begin{pmatrix} \mathrm{Id}_{x_d} & 0 \\ 0 & 0 \end{pmatrix}$, $b_S := \begin{pmatrix} 0_{x_d} & 0 & 0 \\ 0 & \mathrm{Id}_{y_d} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For i satisfying $x_i < x_d$ or $y_i < y_d$, put $a_i := \begin{pmatrix} \mathrm{Id}_{x_i} & 0 \\ 0 & 0 \end{pmatrix}$ and $b_i := \begin{pmatrix} 0_{x_i} & 0 & 0 \\ 0 & \mathrm{Id}_{y_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and for all the other i's, put $a_i := \begin{pmatrix} \mathrm{Id}_{x_d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{x_i} & 0 \\ 0 & 0 & 0 & 0_{y_i} \end{pmatrix}$ and $b_i := \begin{pmatrix} 0_{x_d} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_{y_d} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{y_i} \end{pmatrix}$. Then $\boldsymbol{a} := (a_S, a_0, a_1, \dots)$ and $\boldsymbol{b} := (b_S, b_0, b_1, \dots)$ are elements of R, since so is $\boldsymbol{c}$, and we see that $\boldsymbol{a}$ and $\boldsymbol{b}$ are in fact orthogonal idempotents satisfying $f(\boldsymbol{a}) = x$, $f(\boldsymbol{b}) = y$ and $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$.

Case (ii)**.** We have $f(\boldsymbol{c}) = ([0,1], n_0, n_1, \dots)$, so suppose w.l.o.g. that $[y_d, y_u] = [0,1]$ and $[x_d, x_u] = [x_d,0]$ with $x_d \geq 0$. Let $a_S := \begin{pmatrix} \mathrm{Id}_{x_d} & 0 \\ 0 & 0 \end{pmatrix}$ and $b_S := \begin{pmatrix} 0_{x_d} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$. From condition (M3b) for $\boldsymbol{x}$, we have $x_i < x_d$ for only finitely many i's; for these, put $a_i := \begin{pmatrix} \mathrm{Id}_{x_i} & 0 \\ 0 & 0 \end{pmatrix}$ and $b_i := \begin{pmatrix} 0_{x_i} & 0 & 0 \\ 0 & \mathrm{Id}_{y_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For the rest, let $a_i := \begin{pmatrix} \mathrm{Id}_{x_d} & 0 & 0 & 0 \\ 0 & 0_{y_i} & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{x_i-x_d} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $b_i := \begin{pmatrix} 0_{x_d} & 0 & 0 \\ 0 & \mathrm{Id}_{y_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is obvious that $\boldsymbol{a} = (a_S, a_0, a_1, \dots)$ and $\boldsymbol{b} = (b_S, b_0, b_1, \dots)$ are orthogonal idempotents and that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$. It remains to verify that $\boldsymbol{a}, \boldsymbol{b} \in R$; but for both $\boldsymbol{a}$ and $\boldsymbol{b}$, this follows from $\lim\limits_{i \to \infty} y_i = \infty$ (condition (M2) for $\boldsymbol{y}$). ☐

Lemma 5.22. *If $f(\boldsymbol{w}) = \boldsymbol{x} + \boldsymbol{y}$ with $\boldsymbol{w} \in \operatorname{Idemp} R$ and $\boldsymbol{x}, \boldsymbol{y} \in M$, then there are orthogonal idempotents $\boldsymbol{u}, \boldsymbol{v}$ in R satisfying $f(\boldsymbol{u}) = x$, $f(\boldsymbol{v}) = y$ and $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{w}$.*

Proof. If $\operatorname{rank} w_S$ is infinite, then, by Proposition 5.11, there is an R-module isomorphism $\psi : \boldsymbol{c}R \longrightarrow \boldsymbol{w}R$, where $\boldsymbol{c}$ is as in Lemma 5.21 (ii); if, on the other hand, $\operatorname{rank} w_S = n$ is finite, then, by Proposition 5.14, there is an isomorphism $\psi : \boldsymbol{c}R \longrightarrow \boldsymbol{w}R$ with $\boldsymbol{c}$ as in Lemma 5.21 (i). In both cases, $f(\boldsymbol{w}) = f(\boldsymbol{c})$. Notice that in the latter case, the fact that $\boldsymbol{c} \in R$ is checked in the proof of Lemma 5.13.

Apply Lemma 5.21 to obtain orthogonal idempotents $\boldsymbol{a}, \boldsymbol{b} \in R$ with $f(\boldsymbol{a}) = \boldsymbol{x}$, $f(\boldsymbol{b}) = \boldsymbol{y}$ and $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$. With Observation 5.19 on mind, we infer that we can take $\boldsymbol{u} := \psi(\boldsymbol{a})$ and $\boldsymbol{v} := \psi(\boldsymbol{b})$. ☐

We may now conclude that the monoid M defined at the beginning of this section is isomorphic to the monoid $\mathcal{V}(R)$:

Proposition 5.23. $\mathcal{V}(R) \simeq M$.

Proof. By Lemmas 5.17, 5.22 and 5.20 and by Proposition 5.15, the sufficiency conditions of Proposition 5.16 are satisfied. $\square$

Proposition 5.24. *The monoid M, and thus also $\mathcal{V}(R)$, is stably finite, separative and not cancellative.*

Proof. In M_S, the elements $[1,0]$ and $[0,0]$ are distinct, while $[1,1]=[0,1]$. Thus, the summand on both sides of the equality

$$([1,0],1,1,1,\dots)+([0,1],1,2,3,\dots)=([0,0],1,1,1,\dots)+([0,1],1,2,3,\dots)$$

cannot be cancelled out.

Suppose now that $0 \neq \boldsymbol{x} \in M$. If $[x_d,x_u]=0$, we have from $\boldsymbol{x} \neq 0$ that $x_i \neq 0$ for some $i \in \mathbb{N}_0$. If, on the other hand, $[x_d,x_u] \neq 0$, then $x_i > 0$ for some $i \in \mathbb{N}_0$ due to (M2) or (M3b). In either case, for any $\boldsymbol{y} \in M$, $y_i + x_i \neq y_i$ holds, so $\boldsymbol{x}+\boldsymbol{y} \neq \boldsymbol{y}$.

Finally, if $[a_d,a_u],[b_d,b_u]$ are elements of M_S satisfying $2[a_d,a_u]=2[b_d,b_u]$, then, from the definition of $\equiv$, either:

- $2a_u = 2b_u > 0$; this occurs iff $a_u = b_u > 0$. Or,
- $2a_u = 2b_u = 0$, then also $2a_d = 2b_d$. Consequently, $a_d = b_d$.

In either case, $[a_d,a_u]=[b_d,b_u]$ holds. Thus, M_E is a separative monoid. Since each $\mathbb{N}_0$ is separative as well, it follows that the monoids $M_S \times (\mathbb{N}_0)^\omega$ and M are separative. $\square$

Bibliography

ABRAMS, G. – ARANDA PINO, G. The Leavitt path algebras of arbitrary graphs. *Houston Journal of Mathematics*. 2008, 34, s. 423–442.

ABRAMS, G. – ARANDA PINO, G. Purely infinite simple Leavitt path algebras. *Journal of Pure and Applied Algebra*. 2006, 207, s. 553–563.

ABRAMS, G. – RANGASWAMY, K. M. Regularity conditions for arbitrary Leavitt path algebras. *Algebras and Representation Theory*. 2010, 13, s. 319–334.

ABRAMS, G. – ARANDA PINO, G. – SILES MOLINA, M. Finite-dimensional Leavitt path algebras. *Journal of Pure and Applied Algebra*. 2007, 209, s. 753–762.

ANDERSON, F. W. – FULLER, K. R. *Rings and categories of modules*. New York : Springer-Verlag, 2nd edition, 1992, viii, 376 s. ISBN 0-387-97845-3.

ÁNH, P. N. – MÁRKI, L. Morita equivalence for rings without identity. *Tsukuba journal of mathematics*. 1987, 11, s. 1–16.

ARA, P. – GOODEARL, K. R. – O'MEARA, K. C. Separative cancellation for projective modules over exchange rings. *Israel journal of mathematics*. 1998, 105, s. 105–137.

ARA, P. The realization problem for von Neumann regular rings. In MARUBAYASHI, H. (Ed.) *Ring theory 2007 : proceedings of the fifth China-Japan-Korea International Conference, Tokyo, Japan, 10-15 September 2007*, s. 21–37, Singapore, c2009. World Scientific. ISBN 978-981-281-832-4, xxix, 283 s.

ARA, P. – BRUSTENGA, M. The regular algebra of a quiver. *Journal of Algebra*. 2007, 309, s. 207–235.

ARA, P. – PERERA, F. Multipliers of von Neumann regular rings. *Communications in algebra*. 2000, 28, s. 3359–3385.

ARA, P. – MORENO, M. A. – PARDO, E. Nonstable K-theory for graph algebras. *Algebras and Represenation Theory*. 2007, 10, s. 157–178.

BERGMAN, G. M. Coproducts and some universal ring constructions. *Transactions of the American Mathematical Society*. 1974, 200, s. 33–88.

BLACKADAR, B. *K-theory for operator algebras.* Cambridge : Cambridge University Press, 2nd edition, 1998, xx, 300 s. ISBN 0-521-63532-2.

BURRIS, S. – SANKAPPANAVAR, H. P. *A course in universal algebra* [online]. 2012. Dostupné z: `www.math.uwaterloo.ca/~snburris/htdocs/UALG/univ-algebra2012.pdf`.

CHUANG, C.-L. – LEE, P.-H. On regular subdirect products of simple artinian rings. *Pacific Journal of Mathematics.* 1990, 142, s. 17–21.

FAITH, C. C. *Algebra: Rings, modules and categories I.* Berlin : Springer-Verlag, 1973, xxii, 565 s. ISBN 3-540-05551-7.

FUCHS, L. – HALPERIN, I. On the imbedding of a regular ring in a regular ring with identity. *Fundamenta Mathematicae.* 1964, 54, s. 285–290.

GOODEARL, K. R. *Von Neumann regular rings.* London : Pitman Publishing, 1979, xvii, 369 s. ISBN 0-273-08400-3.

GOODEARL, K. R. Von Neumann regular rings and direct sum decomposition problems. In FACCHINI, A. – MENINI, C. (Ed.) *Abelian groups and modules : proceedings of the Padova conference, Padova, Italy, June 23-July 1, 1994*, s. 249–255, Dordrecht, c1995. Kluwer Academic Publishers. ISBN 0-7923-3756-5, 517 s.

GOODEARL, K. R. Leavitt path algebras and direct limits. *Contemporary Mathematics.* 2009, 480, s. 165–187.

GOODEARL, K. R. – MENAL, P. Stable range one for rings with many units. *Journal of Pure and Applied Algebra.* 1988, 54, s. 261–287.

MITCHELL, B. *Theory of categories.* New York : Academic Press, 1965, xi, 273 s.

NICHOLSON, W. K. Lifting idempotents and exchange rings. *Transactions of the American Mathematical Society.* 1977, 229, s. 269–278.

RŮŽIČKA, P. Computing $\mathcal{V}(R)$ of regular rings. *Preprint.* 2011.

SILES MOLINA, M. Algebras of quotients of path algebras. *Journal of Algebra.* 2008, 319, s. 5265–5278.

WEHRUNG, F. Non-measurability properties of interpolation vector spaces. *Israel journal of Mathematics.* 1998, 103, s. 177–206.

List of abbreviations

Abbreviation	Meaning
w.l.o.g.	without loss of generality
iff.............	if and only if
UMP	universal mapping property
SES	short exact sequence