Charles University in Prague

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MASTER THESIS

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Microscopic sets and drops in Banach spaces

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Study programme:	Mathematical Analysis
Specialization:	Functional Analysis

Prague 2015

Acknowledgement

Although there are many teachers and friends that deserve my thanks, I would like to name three of them:

docent Ondřej Kalenda for his many helpful consultations;

professor Luboš Pick for proofreading a large part of this work for grammar errors;

and most of all professor Jaroslav Lukeš for his inspirational lectures that sparked my interest in Banach spaces geometry and for his patience and understanding when I was writing this thesis.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date ....

Název práce: Mikroskopické množiny a kapky v Banachových prostorech

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Abstrakt: Nejprve definujeme mikroskopické množiny na reálné ose a zkoumáme jejich vztah k množinám Hausdorffovy a Lebesgueovy míry nula a k množinám první kategorie. V druhé části dokazujeme Ekelandův variační princip a jeho ekvivalenci s větou o okvětních plátcích, Danešovou větou o kapce, Brézis-Browderovou větou, Phelpsovým lemmatem a Caristi-Kirkovou větou. Dále zkoumáme jeho vztah k Bishop-Phelpsově větě. Přitom definujeme pojem kapky jako konvexní obal množiny a bodu. V části třetí dokazujeme, že vlastnost kapky je v jistém smyslu ekvivalentní reflexivitě. Prostor má vlastnost kapky, pokud kapku z Danešovy věty lze najít i v obecnějším případě, než zaručuje věta samotná. Dále tuto vlastnost charakterizujeme pomocí aproximativní kompaktnosti. V poslední části se zabýváme mikroskopickou vlastností kapky, která je oproti původní vlastnosti kapky méně přísná. Zjistíme však, že tyto dva pojmy jsou pro jisté množiny v reflexivních prostorech ekvivalentní.

Klíčová slova: Banachovy prostory, mikroskopické množiny, Danešova věta o kapce, vlastnost kapky

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Abstract: First we define microscopic sets on the real axis and study their relation to the sets of Hausdorff and Lebesgue measure zero and the sets of first category. In the second part, we prove the Ekeland's variational principle and its equivalence with the the Daneš's drop theorem, the Brézis-Browder's theorem, the Phelps' lemma and the Caristi-Kirks's theorem. Furthermore, we discuss its relation to the Bishop-Phelps' theorem. Doing so we define the notion of a drop as the convex hull of a set and a point. In the third part we prove that the drop property equals reflexivity in some sense. A space has the drop property if it is possible to find the drop from the Daneš's theorem even in a more general case than the theorem itself guarantees. Furthermore, we characterize this property using the approximative compactness. Last, we study the microscopic drop property that is more relaxed than the original drop property. We find out that those two notions are for certain sets in reflexive spaces equivalent.

Keywords: Banach spaces, microscopic sets, Daneš's drop theorem, drop property

Contents

1 Preliminaries

Throughout, X will always be a real Banach space. Unless otherwise stated, B is the closed unit ball in X; where confusion might arise, this ball will be denoted with B_X. The unit sphere will be labelled S_X. $B(x, r)$ denotes a closed ball with centre $x \in X$ and radius $r > 0$.

The convex hull of points $x_1, ..., x_n$ will be denoted by $\mathrm{conv}(x_1, ..., x_n)$. The linear hull will be labelled $\mathrm{span}(x_1, ..., x_n)$.

The expression $K(B(z, r), x)$ describes the smallest convex cone with its edge at x that includes the ball $B(z, r)$, that means,

$$K(B(z,r), x) = \{t(y - x) + x : y \in B(z, r), t > 0\}.$$

For $A \subset \mathbb{R}$, $|A|$ will denote the Lebesgue measure of A.

2 Microscopic sets

In this introductory section we define the microscopic sets on the real axis and present several results that show the connection of this notion to other characteristics of "smallness" of a set. We will find out that in comparison with sets of Lebesgue measure 0 and Hausdorff dimension 0, the microscopic sets are "smaller" - more precisely, each microscopic set has Lebesgue measure zero and Hausdorff dimension zero but the converse is not true. However, there exists a microscopic set that is not of the first Baire category and vice versa.

Unless stated otherwise, the results of this section are due to J. Appell [1].

Definition 1. *A set $M \subset \mathbb{R}$ is called* ***microscopic*** *if for any $\varepsilon > 0$ there exists a sequence of intervals I_n such that*

$$M \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } |I_n| < \varepsilon^n \text{ for every } n \in \mathbb{N}. \tag{1}$$

This property is invariant with regard to translation: if it is possible to cover a set M with intervals of the form (a_n, b_n), then the set $M_t := \{x + t : x \in M\}$ can be covered by intervals of the form $(a_n + t, b_n + t)$ for any $t \in \mathbb{R}$.

Example 2.1. *Any countable set is microscopic.*

Proof. We order the set in a sequence $(x_n)_{n\in\mathbf{N}}$. Choose $\varepsilon > 0$. For all natural n, set $I_n := (x_n - \frac{\varepsilon^n}{2}; x_n + \frac{\varepsilon^n}{2})$. Those intervals cover M and their length is for any natural number bounded by ε^n. □

Theorem 2.2. *Any microscopic set has Lebesgue measure 0.*

Proof. Let M be microscopic. Choose $\varepsilon > 0$ and find intervals I_n covering M according to the definition. Their Lebesgue measure fulfills

$$|M| \leq \sum_{n=1}^{\infty} |I_n| \leq \sum_{n=1}^{\infty} \varepsilon^n = \frac{\varepsilon}{1-\varepsilon}$$

which for $\varepsilon \to 0$ tends to zero and so M has to be a nullset. □

As a consequence, whenever a set M containts an interval, then it cannot be microscopic. Similarly, open sets are never microscopic.

Theorem 2.3. *Every microscopic set has the Hausdorff dimension 0.*

Proof. Let us recall first that the Hausdorff dimension of a set M is given by

$$\dim_H(M) := \inf\{\alpha > 0 : \mathcal{H}^\alpha(M) = 0\},$$

where

$$\mathcal{H}^\alpha(M) := \lim_{\delta\to 0} \inf\{\sum_{n=1}^{\infty} (\text{diam} M_n)^\alpha : M \subseteq \bigcup_{n=1}^{\infty} M_n,\ \text{diam} M_n < \delta\}$$

is the Hausdorff measure with respect to $\alpha \geq 0$.

We want to show that $\mathcal{H}^\alpha(M) = 0$ for any $\alpha > 0$. Because M is microscopic, for $\delta > 0$ given there exists a sequence of intervals I_n such that M is covered by their union and $|I_n| \leq \delta^n < \delta$ (we are assuming without loss of generality that $\delta < 1$). Those intervals are considered in the computation of the Hausdorff measure and they fulfil

$$\sum_{n=1}^{\infty}(\mathrm{diam} I_n)^\alpha \leq \sum_{n=1}^{\infty}(\delta^n)^\alpha = \sum_{n=1}^{\infty}(\delta^\alpha)^n = \frac{\delta^\alpha}{1-\delta^\alpha} \to 0$$

for $\delta \to 0$, and so $\mathcal{H}^\alpha(M) = 0$. □

Theorem 2.4. *There exists a set that is not microscopic despite being of the first category and having Lebesgue measure 0.*

Proof. The Cantor set on the interval $[0,1]$ has Hausdorff dimension $\frac{\ln 2}{\ln 3}$ [4] and therefore cannot be microscopic. At the same time it is a set of the first category and Lebesgue measure zero. □

Theorem 2.5. *[3] There exists a set that is not microscopic despite having Hausdorff dimension 0.*

Proof. Our example will be a certain kind of the Cantor set. Choose a constant $c \geq 3$. We start with the interval $[0,1]$. In the first step we cut from its middle the open segment of the length $1 - 2c^{-1}$. Two closed intervals will be created, each of the length c^{-1}, from whose centres we will in the second step cut open intervals of the length $c^{-1} - 2c^{-4}$. In the n-th step, we will then have 2^n closed intervals of the length c^{-n^2} and we will be cutting from the middle of each one of them a segment of the measure $c^{-n^2} - 2c^{-(n+1)^2}$. The set we will end up with at the end of this construction will be labelled N.

First we choose $\alpha > 0$ and compute the Hausdorff measure of the set N with the coefficient α. Because N is an intersection of a nested sequence of sets, it is for each natural n a subset of those intervals that are remaining in the appropriate step; there are 2^n such intervals and they are of length c^{-n^2}. The Hausdorff measure can thus be estimated as

$$\mathcal{H}^\alpha(N) \leq 2^n(c^{-n^2})^\alpha.$$

However, this expression decreases as n grows: Because $c > 1$ and $\alpha > 0$ is a fixed number, we have $(c^\alpha)^n \to \infty$ for $n \to \infty$. Thus there exists a natural number m such that $(c^\alpha)^m > 2$. For a sufficiently large n it is then possible to estimate

$$2^n(c^{-n^2})^\alpha = \frac{2^n}{(c^\alpha)^{n^2}} \leq (\frac{2}{c^{\alpha m}})^n \to 0.$$

The Hausdorff measure of the set N is thus zero for every positive α and so its Hausdorff dimension is zero as well.

Now we show that N is not microscopic. To this end, we choose $\varepsilon := c^{-4}$ and a sequence of intervals I_k such that $|I_k| \leq \varepsilon^k$ and show that there exists some point in N not covered by any I_k.

For each n, find all intervals from the $(n-1)$-th step of the construction of N that intersect any I_k where $k < \frac{(n+1)^2}{4}$ and denote their number a_n. As there exists no natural number fulfilling $k < \frac{n^2}{4}$ for $n = 1$, we have $a_0 = 0$.

Now we infer a relationship between a_{n-1} and a_n. We split the intervals I_k, $k < \frac{(n+1^2)}{4}$ in two groups - the intervals where $k < \frac{n^2}{4}$ go in the first one and in the second group we put the ones where $\frac{n^2}{4} \leq k < \frac{(n+1)^2}{4}$.

The intervals from the first group intersect a_{n-1} intervals from the $(n-2)$-th step from the definition of a_{n-1}. In the $(n-1)$-th step, each of those intervals splits in two. Therefore, the intervals from the first group cannot intersect more than $2a_{n-1}$ intervals from the $(n-1)$-th step.

For the other group, we notice that after the $(n-1)$-th step, we have 2^n intervals and that the distance between them is at least the segment we have cut out in the $(n-1)$-th step, which is $c^{-(n-1)^2} - 2c^{-n^2}$. However, we can estimate the length of the I_k from this group as

$$|I_k| \leq \varepsilon^k \leq \varepsilon^{\frac{n^2}{4}} = c^{-n^2} < c^{-(n-1)^2} - 2c^{-n^2}. \tag{2}$$

The gap between each two intervals that are remaining after the $(n-1)$-th step is therefore bigger than the length of any line segment I_k from the given set; this means that each I_k can intersect no more than one of the 2^n intervals from the $(n-1)$-th step. The number of I_k in the second group is estimated by n:

$$\frac{(n+1)^2}{4} - \frac{n^2}{4} = \frac{2n+1}{4} < n.$$

Thus, the intervals from the second group can only intersect n intervals from the $(n-1)$-th step at maximum. Together, we get the recursion formula

$$a_n \leq 2a_{n-1} + n.$$

Using induction and the already known fact that $a_0 = 0$, we infer

$$a_n \leq 2^n - n.$$

This means that after the $(n-1)$-th step, we have 2^n intervals in total but at most $2^n - n$ of them are intersected by some of the respektive I_k. As all the uncovered intervals are disjoint and their number is finite, their union is a closed set; in addition, this union is a subset of all unions of the uncovered intervals that were left out in the previous steps. This means that we have obtained a sequence of nested closed sets. Because $[0,1]$ is a compact space, the intersection of those sets is nonempty. This intersection is the element of the set N that is not covered by the union of our intervals I_k. □

Theorem 2.6. *There exists a set M of the first category and N microscopic such that $[0,1] = M \bigcup N$.*

Proof. We begin with the interval $[0,1]$ and order the rational numbers contained in it in a sequence $(r_n)_{n\in\mathbf{N}}$. For each pair of natural numbers j, k we then define an interval

$$I_{j,k} := [0,1] \cap (r_j - 2^{-j-k}; r_j + 2^{-j-k})$$

and set

$$M := [0,1]\backslash N \text{ where } N := \bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty} I_{j,k}.$$

We claim that M is of first category while N is microscopic. Once we show this, we will know that N is the desired microscopic residual set.

We start by writing M as

$$M = [0,1] \cap N^c = [0,1] \cap (\bigcup_{k=1}^{\infty}\bigcap_{j=1}^{\infty} I_{j,k}^c).$$

Now we show that $\bigcap_{j=1}^{\infty} I_{j,k}^c$ is a nowhere dense set for any k. First, $I_{j,k}^c$ is closed as the complement of an open interval in $[0,1]$. Therefore, $\bigcap_{j=1}^{\infty} I_{j,k}^c$ is closed as well.

Suppose for contradiction there exists a point x in the interior of this set. Then, from the openness of the interior, a neighbourhood U of this point would be contained there as well:

$$U \subseteq \text{int}(\bigcap_{j=1}^{\infty} I_{j,k}^c) \subset \bigcap_{j=1}^{\infty} I_{j,k}^c.$$

In each open interval, we can find a rational number. However, we have defined the sets M and N in such a way that all rational numbers fall in N, and because $N \cap U = \emptyset$, we get a contradiction. Therefore, no such point x can exist, the interior is empty and $\cap_{j=1}^{\infty} I_{j,k}^c$ is nowhere dense. Thus, M is a countable union of nowhere dense sets, that is, a set of first category.

Now we show that N is microscopic. Choose $\varepsilon > 0$ and for each $j \in \mathbb{N}$ find $k(j)$ such that $|I_{j,k(j)}| \leq \varepsilon^j$. This is possible because

$$|I_{j,k}| = |(r_j - 2^{-k-j}; r_j + 2^{-k-j})| = 2^{-k-j+1},$$

which can be made arbitrarily small for k large enough.

The intervals then cover N:

$$N = \bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty} I_{j,k} \subset \bigcup_{j=1}^{\infty} I_{j,k(j)}.$$

$\square$

Observe that the set N created in the proof cannot be of the first category: if it was, then the whole interval would be a union of two sets of the first category which is in contradiction with the Baire category theorem. Also, the set N is uncountable because countable sets are of the first category.

To conclude this section we will try to describe the behavior of the system of microscopic sets.

Theorem 2.7. *[2] The set of all microscopic sets $\mathcal{M}$ is a σ-ideal.*

Proof. We will verify the properties that define a σ-ideal.

(1) The empty set is surely contained in $\mathcal{M}$ as it is the subset of any interval.

(2) If $A \subset B$ and B is microscopic, then A is microscopic as well as any covering of B is at the same time a covering of A.

(3) If (A_n) is a sequence of microscopic sets, then even their union is microscopic. To see this, denote

$$\bigcup_{n=1}^{\infty} A_n =: A$$

and choose $\varepsilon > 0$. Denote $\varepsilon_k := \varepsilon^{2^k}$. Each of the sets A_k can be covered with intervals I_n^k of the lengths $\varepsilon_k^n = \varepsilon^{2^k n}$. We order them in a sequence in the following way: Each natural number can be uniquely represented as a product of a power of two and an odd number, this follows from the prime factorization. The map

$$\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N},\ \psi(n,k) := 2^{k-1}(2n-1)$$

is thus bijective, which means that for any natural number m we can find exactly one pair (n,k) such that $\psi(n,k) = m$. Using this decomposition, set $J_m := I_n^k$. Because ψ is a bijection, each interval I_n^k corresponds with some J_m. Thus,

$$A \subset \bigcup_n \bigcup_k I_n^k = \bigcup_m J_m.$$

In addition,

$$|J_m| = |I_n^k| \leq \varepsilon_k^n = \varepsilon^{2^{k-1}2n} < \varepsilon^{2^{k-1}(2n-1)} = \varepsilon^m.$$

It follows that A is microscopic. $\square$

3 Daneš's drop theorem and its equivalents

In this section, we prove the equivalence of several important theorems. In doing so, we introduce the second important concept of this thesis, namely the drop. In addition, we will provide a direct proof of the Ekeland's variational principle as it is the one that is probably known the best from the equivalent statements mentioned here.

Later, we discuss the relation to the Bishop-Phelps theorem. In this case, however, the equivalence will only be proved under an additional assumption.

3.1 The Ekeland's variational principle

The Ekeland's variational principle is an important theorem in the minimization theory. In general, we do not know if a function has a minimum, even if it is continuous, bounded from below and identically ∞ everywhere outside of a bounded set. However, we can find a point that is "nearly a minimum" - a point $a \in X$ such that maybe $f(a) > f(x)$ for some $x \in X$ but at the same time, $f(a) \leq f(x) + k(a, x)$ everywhere for some correction term $k(a, x)$. The Ekeland's theorem provides a useful version of such a "nearly a minimum". [5]

Theorem 3.1 (Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a lower-semicontinuous function which is bounded from below and not identically ∞. Choose $\varepsilon > 0$ and $\lambda > 0$. Then, for every $x_0 \in X$ fulfilling $f(x_0) \leq \inf f + \varepsilon$, there exists a point $a \in X$ fulfilling:*

1) $f(a) < f(x) + \frac{\varepsilon}{\lambda} d(x, a)$ for every $x \in X$ which is not equal a,

2) $f(a) \leq f(x_0)$ and

3) $d(x_0, a) \leq \lambda$.

Before we start with the proof, we will introduce the following partial ordering on $X \times \mathbb{R}$:

$$(x_1, t_1) \prec (x_2, t_2) \text{ if } (t_2 - t_1) + \alpha d(x_1, x_2) \leq 0,$$

where $\alpha > 0$ is given. This relation fulfils:
Reflexivity: $(x, a) \prec (x, a)$ since $(a - a) + \alpha d(x, x) = 0$.
Antisymetry: If $(x, a) \prec (y, b)$ and $(y, b) \prec (x, a)$, then

$$0 \geq (b - a) + \alpha d(x, y) + (a - b) + \alpha d(y, x) = 2\alpha d(x, y)$$

which implies $x = y$. This in turn means that $b - a \leq 0$ and $a - b \leq 0$ and so $a = b$ and $(x, a) = (y, b)$.
Transitivity: If $(x, a) \prec (y, b) \prec (z, c)$, then

$$(c - a) + \alpha d(x, z) \leq (c - b) + \alpha d(z, y) + (b - a) + \alpha d(x, y) \leq 0$$

and so $(x, a) \prec (z, c)$.

Furthermore, for any $(x, a) \in X \times \mathbb{R}$, the set $\{(y, b) : (y, b) \succ (x, a)\}$ is closed in $X \times \mathbb{R}$: if $(y_n, b_n) \to (y, b)$, then $b_n - a \to b - a$ and $d(y_n, x) \to d(y, x)$ and so $(b - a) + \alpha d(y, x) \leq 0$.

Furthermore, this relation has under some condition a maximal element on any closed subset of $V \times \mathbb{R}$:

Lemma 3.2. *Let $S \subset X \times \mathbb{R}$ be closed and let there exist $m \in \mathbb{R}$ such that for any $(x,t) \in S$ we have $t \geq m$. Then, for any $(x_1, t_1) \in S$ there exists $(\bar{x}, \bar{a}) \in S$ which is under the above defined ordering maximal on S and greater than (x_1, t_1).*

Proof. We start with (x_1, t_1) and define inductively a sequence (x_n, t_n) in S. With (x_n, t_n) already known, set

$$\begin{aligned} S_n &:= \{(x,t) \in S : (x,t) \succ (x_n, t_n)\} \text{ and} \\ m_n &:= \inf\{t \in \mathbb{R} : (x,t) \in S_n \text{ for some } x \in X\}. \end{aligned}$$

We choose as (x_{n+1}, t_{n+1}) an arbitrary point of S_n that fulfils $t_{n+1} \leq \frac{1}{2}(t_n + m_n)$; such a point exists as $t_n \geq m_n$ and from the definition of m_n, we can find $(x,t) \in S_n$ with t arbitrarily close to m_n.

It holds $S_{n+1} \subset S_n$ for every natural n, all the sets S_n are closed from the remark above this lemma and they are nonempty as $x_n \in S_n$. Furthermore, $\operatorname{diam} S_n \to 0$:

Since $m_{n+1} \geq m_n$, from the definition of t_{n+1} it follows

$$t_{n+1} - m_{n+1} \leq \frac{1}{2}(t_n + m_n) - m_{n+1} \leq \frac{1}{2}(t_n - m_n)$$

and using induction, we get

$$|t_{n+1} - m_{n+1}| \leq \frac{1}{2^n}|t_1 - m|.$$

For a general point $(x,t) \in S_{n+1}$, we know $(x,t) \succ (x_{n+1}, t_{n+1})$. Using the definition of an ordering, it follows that $t - t_{n+1} \leq -\alpha d(x_{n+1}, x) \leq 0$ and so $t_{n+1} \geq t \geq m_{n+1}$. Thus,

$$|t_{n+1} - t| \leq |t_{n+1} - m_{n+1}| \leq \frac{1}{2^n}|t_1 - m|.$$

Furthermore, again from the same definition,

$$d(x_{n+1}, x) \leq \frac{1}{\alpha}|t_{n+1} - t| \leq \frac{1}{\alpha}(\frac{1}{2^n}|t_1 - m|)$$

which proves that $\operatorname{diam} S_n \to 0$ as n grows. Since $X \times \mathbb{R}$ is a complete metric space, from the Cantor's intersection theorem there exists a point $(a,s) \in \bigcap_n S_n$. This is the maximal element we are searching for:

Because $(a,s) \in S_1$, we know that $(a,s) \succ (x_1, t_1)$. Suppose there exists $(x,t) \in S$ such that $(x,t) \succ (a,s)$. Since the relation is transitive and $(a,s) \succ (x_n, t_n)$ for every n, we get $(x,t) \succ (x_n, t_n)$ and $(x,t) \in S_n$ for every $n \in \mathbb{N}$. From the Cantor's theorem, however, we know that the intersection point is unique. Thus, $(x,t) = (a,s)$ and (a,s) is indeed maximal. □

Proof of Ekeland's theorem. In order to use our lemma, take S to be the epigraph of f, that is, $S := \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$. Because f is lower semicontinuous, S is closed. Take $\alpha := \frac{\varepsilon}{\lambda}$ and $(x_1, t_1) := (x_0, f(x_0))$. The above lemma yields a maximal element $(a,s) \in S$. From the definition of S, we know $(a, f(a)) \succ (a,t)$ for every $t \geq f(a)$; thus, from maximality, $s = f(a)$.

Choose $x \in X$. Since $(x, f(x)) \in S$, we know from the maximality of $(a, f(a))$ that

$$f(a) - f(x) + \frac{\varepsilon}{\lambda} d(a, x) \leq 0.$$

However, from the proof of the above lemma, $(a, f(a))$ is unique in the sense that whenever $(x, t) \succ (a, f(a))$, it follows $(x, t) = (a, f(a))$. Thus, for any $x \neq a$, we have

$$f(a) - f(x) + \frac{\varepsilon}{\lambda} d(a, x) < 0$$

which is inequality 1) from the formulation of Ekeland's theorem. Moreover, from $(a, f(a)) \succ (x_0, f(x_0))$, we get

$$f(a) \leq f(x_0) - \frac{\varepsilon}{\lambda} d(a, x_0) \leq f(x_0)$$

which shows inequality 2). Finally, from the choice of x_0 we have $f(a) \geq f(x_0) - \varepsilon$. Thus,

$$\frac{\varepsilon}{\lambda} d(a, x_0) \leq f(x_0) - f(a) \leq \varepsilon$$

which yields inequality 3). □

While the above formulation is the most common, there are other equivalent formulations of the Ekeland's variational principle. For us, the following will be useful:

Theorem 3.3 (Simple Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a lower-semicontinuous function which is bounded from below and not identically ∞. Then there exists a point $a \in X$ fulfilling*

$$f(a) < f(x) + d(x, a) \text{ for every } x \in X \setminus \{a\}. \tag{3}$$

Proof. Choose $\varepsilon = \lambda > 0$ arbitrary and find any point $x_0 \in X$ that satisfies $f(x_0) < \inf(f) + \varepsilon$. The existence of a follows from the version of Ekelands's principle above (Theorem 3.1). □

Theorem 3.4 (Altered Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a lower-semicontinuous function which is bounded from below and not identically ∞. Then, for every $\varepsilon > 0$ and every $x_0 \in X$, there exists a point $a \in X$ fulfilling:*

1) $f(a) < f(x) + \varepsilon d(x, a)$ for every $x \in X$ which is not equal a

2) $f(a) \leq f(x_0) - \varepsilon d(x_0, a)$.

Proof. We will prove this statement from the Simple Ekeland's variational principle.

As preparation, note that if $d(x, y)$ is a metric on X, then $\varepsilon d(x, y)$ is a metric which keeps the completeness of X - it is easy to check that $\varepsilon d(x, y)$ fulfils all requirements from the definition of a metric. Therefore, whenever the conditions of Theorem 3.1 are met for a space (X, d), we can apply it to the space $(X, \varepsilon d)$ instead. This way, the theorem yields a point $a \in X$ fulfilling

$$f(a) < f(x) + \varepsilon d(x, a) \text{ whenever } x \neq a.$$

Later in this proof, we will use this without further notice.

From the altered Ekeland's theorem, we are given a lower semi-continous function $f : X \to \mathbb{R} \cup \{\infty\}$ and a point $x_0 \in X$. Define

$$C := \{x \in X : f(x) \le f(x_0) - \varepsilon d(x_0, x)\}.$$

Because for every lower semi-continuous function ϕ, the level-sets $\{\phi(x) \le t\}$ are closed for any $t \in \mathbb{R}$ and because $f(x) + \varepsilon d(x_0, x)$ is a lower semi-continuous function, C is closed.

Define $g : X \to \mathbb{R} \cup \{\infty\}$ as follows:

$$g(x) = f(x) \text{ on } C;$$
$$g(x) = \infty \text{ on } X \setminus C.$$

For any $t \in \mathbb{R}$, we have

$$\{g(x) \le t\} = \{f(x) \le t\} \cap C.$$

Since the sets on the right hand side of this equality are both closed, the level-sets of g are closed which means g is lower semi-continous. It is also bounded from below by the bound of f. Therefore, we can use the simple Ekeland's variational principle and obtain $a \in X$ fulfilling

$$g(a) < g(x) + \varepsilon d(a, x) \text{ whenever } a \ne x. \tag{4}$$

Since this property cannot be held by a point in which the functional value is ∞, we have $a \in C$. Thus, from definition of C we have

$$f(a) \le f(x_0) - \varepsilon d(x_0, a).$$

Now we want to prove that $f(a) < f(x) + \varepsilon d(x, a)$ for any $x \in X$. If $x \in C$, we have $f(x) = g(x)$ and the inequality follows from (4). If on the other hand $x \notin C$, we can use the definition of C, the (already proved) condition 2) of the theorem and the triangle inequality to compute

$$f(x) > f(x_0) - \varepsilon d(x_0, x) \ge f(a) + \varepsilon d(x_0, a) - \varepsilon d(x_0, x) \ge f(a) - \varepsilon d(x, a).$$

Thus, a is a minimum in the sense of Ekeland's theorem. □

Finally, we will show the usual version of Ekeland's variational principle (Theorem 3.1) from the altered version (Theorem 3.4).

Theorem 3.4 implies Theorem 3.1. Let $\varepsilon > 0$ and $\lambda > 0$ be given, together with $x_0 \in X$ fulfilling $f(x_0) < \inf_X(f) + \varepsilon$. Using the altered Ekeland's theorem with the coefficient $\frac{\varepsilon}{\lambda}$, we obtain $a \in X$ such that

$$1)\ f(a) < f(x) + \frac{\varepsilon}{\lambda} d(x, a) \text{ for every } x \ne a \text{ and}$$
$$2)\ f(a) \le f(x_0) - \frac{\varepsilon}{\lambda} d(a, x_0) \le f(x_0).$$

This proves the first two requirements from the statement we want to show. From the definition of x_0, however, we know that $f(x_0) - f(a) \le \varepsilon$. Thus, the inequality 2) above can be written as

$$\frac{\varepsilon}{\lambda} d(a, x_0) \le f(x_0) - f(a) \le \varepsilon \tag{5}$$

and therefore, $d(a, x_0) \le \lambda$ which proves the requirement 3) and concludes the proof. □

3.2 The Flower petal theorem

Definition 2. *Let* (X, d) *be a metric space,* $a, b \in X$ *and* $\gamma > 0$*. The* ***petal*** $P_\gamma(a, b)$ *is the set*

$$P_\gamma(a, b) := \{x \in X : \gamma d(x, a) + d(x, b) \leq d(a, b)\}.$$

Note that for $\gamma > 1$, we have $P_\gamma(a, b) = \{a\}$. If $\gamma = 1$, the petal is the line segment $\{a + tb : 0 \leq t \leq 1\}$ and for $0 < \gamma < 1$, it is a set with nonempty interior containing the aforementioned line segment.

Theorem 3.5 (The Flower petal theorem)**.** *Let* A *be a complete subset of a metric space* (X, d)*. Let* $x_0 \in A$ *and* $b \in X \setminus A$*. Then for every* $\gamma > 0$ *there exists* $a \in A \cap P_\gamma(x_0, b)$ *such that*

$$P_\gamma(a, b) \cap A = \{a\}.$$

Statement 3.6. *The altered Ekeland's variational principle imlies the Flower petal theorem.*

Proof. Let A be a closed subset of the complete metric space X; on A, we will use the metric induced from X. Choose x_0, γ and b as in the Flower petal theorem. Define

$$f : A \to \mathbb{R},\ f(x) := d(x, b).$$

This function is continuous and bounded from below by 0. Furthermore, as A is a closed subset of a complete metric space, it is a complete metric space itself. Therefore, the altered Ekeland's variational principle (Theorem 3.4) can be used to obtain a point $a \in A$ satisfying

$$f(a) < f(x) + \gamma d(a, x) \text{ for any } x \in A$$

and

$$f(a) \leq f(x_0) - \gamma d(a, x_0).$$

If we translate the second equation using the definition of f, we get

$$d(a, b) \leq d(x_0, b) - \gamma d(a, x_0),$$

which by definition means that $a \in A \cap P_\gamma(x_0, b)$.

Choose now a point $x \in A \setminus \{a\}$. From the first equation, we know that

$$d(a, b) < d(x, b) + \gamma d(a, x)$$

which means that this point cannot lie in the petal $P_\gamma(a, b)$ as this contradicts its definition. Therefore, $A \cap P_\gamma(a, b) = \{a\}$. □

3.3 The Daneš's drop theorem

Definition 3. *Let X be a Banach space, $B \subset X$ its closed bounded convex subset and $x \in X$ a point. The* ***drop*** *$D(B,x)$ is the convex hull of $\{x\} \cup B$:*

$$D(B,x) = \{(1-t)x + t(y-x) : t \in [0,1], y \in B\}.$$

While the definition would make sense even for general B, the above assumptions make the drop a bounded, closed and convex set which is easier to work with. Those assumptions are necessary to preserve the closedness of the drop:

1) Consider $B \subset \mathbb{R}^2$, $B = \{(x,y) : x > 0, y \geq \frac{1}{x}\}$ and $z = (0,0)$. Then, B is closed, convex but not bounded and $\operatorname{conv}(B,\{z\}) = (0,\infty) \times (0,\infty)$ which is not closed.
2) Consider $B \subset \mathbb{R}^2$, $B = (-1,1) \times \{0\}$ convex, bounded but not closed and $z = (0,1)$. Then, $\operatorname{conv}(B,\{z\})$ does not contain the point $(-1,0)$ and is therefore not closed.
3) Consider $B \subset l^2$, $B = \{\frac{1}{n}e_1 \pm e_{n+1} : n \in \mathbb{N}\}$ where e_n are the canonical basis vectors and set $z = e_1$. Then, B is closed, bounded but not convex and $\operatorname{conv}(B,z) = \operatorname{conv}(B)$ is not closed, because $0 \notin B$ even though $\frac{1}{n}e_1 \in B$ for every n.

Lemma 3.7. *Let $C \subset X$ be convex, closed and bounded and $x \notin C$. Then, the drop $D(C,x)$ is closed, convex and bounded.*

Proof. The drop is convex and bounded as a convex hull of two bounded sets.

For closedness, choose an arbitrary convergent sequence $p_k \in D(C,x)$ and denote the limit point p. From the construction of a drop, we can write

$$p_k = (1-\lambda_k)x + \lambda_k q_k$$

where $q_k \in C$. The coefficients λ_k form a sequence in the compact interval $[0,1]$, which means that there exists a convergent subsequence - assume without loss of generality that $\lambda_k \to \lambda$. As a consequence, $(1-\lambda_k)x$ is a convergent sequence and that means that $\lambda_k q_k = p_k - (1-\lambda_k)x$ converges as well because it is a difference of two convergent sequences.

Assume $\lambda = 0$. Then, $p_k \to x \in D(C,x)$. On the other hand, if $\lambda > 0$, q_k is a convergent sequence. Since C is a closed set, it follows $q_k \to q \in C$. Then, p can be written as $p = \lambda x + (1-\lambda)q$ and so $p \in D(C,x)$. □

The drop theorem comes in two versions, a simpler one and a generalised one.

Theorem 3.8 (The Daneš's drop theorem)**.** *Let X be a Banach space, $C \subset X$ a nonempty closed subset and $z_0 \in X \setminus C$. Choose $\rho > 0$, $r > 0$ and $R > 0$ such that $0 < r < R = \operatorname{dist}(z_0, C) < \rho$. Then there exists a point $a \in C$ fulfilling $\|a - z_0\| \leq \rho$ and $D(B(z_0,r),a) \cap C = \{a\}$.*

Theorem 3.9 (The generalized Daneš's drop theorem)**.** *Let X be a Banach space, $C \subset X$ a nonempty closed subset and $x_0 \in C$. Let $B \subset X$ be another subset of X that is nonempty, bounded, closed and convex and fulfils $\operatorname{dist}(C,B) > 0$. Then there exists a point $a \in C \cap D(B,x_0)$ such that $C \cap D(B,a) = \{a\}$.*

In essence, the generalized version works for any closed, convex and bounded set B with a positive distance from C while the ordinary version can only be used on balls. In addition, the generalized version gives a stricter requirement on the position of a - it not only sets a maximal distance from z_0 but also specifies that the point a lies in some cone. The proof that the generalized theorem implies the ordinary one is very easy as the sets from the ordinary theorem satisfy the requirements of the generalized one. The converse direction is discussed below.

Statement 3.10. *The ordinary Daneš's drop theorem implies the generalized Daneš's drop theorem.*

Proof. Choose a nonempty closed set $C \subset X$ and a nonempty closed convex bounded set B with a positive distance from C. For any given $x_0 \in C$, the set $D(B, x_0) \cap C$ is bounded and closed and for any $a \in D(B, x_0) \cap C$ it holds that $D(B, a) \cap D(B, x_0) = D(B, a) \cap C$ - this follows from $D(B, a) \subset D(B, x_0)$. Thus, we can assume without loss of generality that C is bounded and contained in the drop $D(B, x_0)$.

Observe that if we take

$$B_1 := \{x \in X : \operatorname{dist}(x, B) \leq \frac{\operatorname{dist}(B, C)}{2}\},$$

we create a set that will be nonempty, closed, convex and bounded as the properties of B carry over. In addition, it will have nonempty interior and the distance of B_1 from C will still be positive. Finally, $D(B, a) \subset D(B_1, a)$ for every $a \in X$, which means that if we find a point $a \in C$ fulfilling $\{a\} = C \cap D(B_1, a)$, we will immediatelly know that also $\{a\} = C \cap D(B, a)$. Thus, we can from now on assume without loss of generality that B has nonempty interior.

Assume for now that there exists a ball $B_0 \subset X$ such that $B \subset B_0$ and $\operatorname{dist}(B_0, C) > 0$. Then, from the ordinary Daneš's theorem, there exists $a \in C$ fulfilling $D(B_0, a) \cap C = \{a\}$. Since $D(B, a) \subset D(B_0, a)$, the statement of the generalized theorem follows.

Now, not every convex set can be enveloped by a ball in this way. We will solve this by going over to an equivalent norm. Such a norm preserves the topology on X. Therefore, in the new norm, C will still be a closed set with a positive distance from the closed convex bounded and nonempty set B and we will find a point a such that $D(B_0, a) = \{a\}$. As the convex hull of two sets does not depend on the norm, this a will be the desired solution. Thus, the task is now to find a suitable equivalent norm that allows to enclose the given set B in a ball.

Define $K := \{x_0 + r(b - x_0) : b \in B, r > 0\}$ as the cone generated by the set B. Find $R > 0$ such that $B \cup C \subset B_R(x_0)$ and $\operatorname{dist}(C, B_R(x_0)^C) > \operatorname{dist}(B, C)$ (here, $B_R(x_0)^C$ denotes the complement of the ball with center x_0 and radius R); such a number exists since both B and C are bounded. Because K is an unbounded set, we can find $u \in K$ such that $u \notin B_{4R}(x_0)$ and $B_R(u) \subset K$. Then, if we set $s := \frac{x_0+u}{2}$, we will know that $s \notin B_{2R}(x_0)$.

Denote $B_2 := \{2s - b : b \in B\}$ the point reflection of B across the point s. Since $B \subset B_R(x_0)$ and the image of $B_R(x_0)$ under this reflection is $B_R(u)$ from construction of u, we know $B_2 \subset B_R(u) \subset K$. In addition, since $u \notin B_{4R}(x_0)$ and $C \subset B_R(x_0)$, it follows $\operatorname{dist}(B_2, C) \geq R > \operatorname{dist}(C, B)$.

Moving forward, we set $B_3 := \text{conv}(B \cup B_2)$. This way we produce a convex bounded set with nonempty interior. In the following, we will want to show that $\text{dist}(B_3, C) > 0$.

Set

$$M := \{x_0 + t(b - x_0) : b \in B, t \geq 1\}.$$

Obviously, $M \subset K$. Take two points $x_0 + t_i(b_i - x_0) \in M, i = 1, 2$. Then, for their convex combination given by $0 \leq \tau \leq 1$, we can compute

$$\begin{aligned} \tau(x_0 + t_1(b_1 - x_0)) + (1 - \tau)(x_0 + t_2(b_2 - x_0)) = \\ = x_0 + \tau t_1 b_1 + (1 - \tau)t_2 b_2 - \tau t_1 x_0 - (1 - \tau)t_2 x_0 = \\ = x_0 + (\tau t_1 b_1 + (1 - \tau)t_2 b_2) - (\tau t_1 + (1 - \tau)t_2)x_0 = \\ x_0 + (\tau t_1 + (1 - \tau)t_2)(b - x_0), \end{aligned}$$

where

$$b = (\tau t_1 b_1 + (1 - \tau)t_2 b_2)/(\tau t_1 + (1 - \tau)t_2).$$

Here, $\tau t_1 + (1 - \tau)t_2 \geq 1$ as it is a convex combination of two numbers greater or equal 1. Furthermore, $b \in B$ as it is a convex combination of b_1 and b_2. Together we can conclude that M is convex.

Clearly, $B \in M$. Furthermore, the whole set $\{x_0 + t(b - x_0) : t \leq 1\} = D(B, x_0)$ is from construction contained in the ball $B(x_0, R)$ because the ball contains B and x_0 and is convex. In addition, $C \subset B(x_0, R)$ but $\text{dist}(B_2, C) > R$. From this we can infer that $B_2 \in K \setminus B(x_0, R) \subset M$. From convexity of M it then follows $B_3 \subset M$.

Now we claim that $\text{dist}(B, C) = \text{dist}(M, C)$. Once we have it, from $B_3 \subset M$ it will follow $\text{dist}(B_3, C) \geq \text{dist}(M, C) = \text{dist}(B, C) > 0$.

The inequality $\text{dist}(B, C) \geq \text{dist}(M, C)$ is obvious as $B \subset M$. For the converse inequality, choose $c \in C$ and $m \in M$. From definition of M, there exist $b_1 \in B$ and $t_1 \geq 1$ such that $m = x_0 + t_1(b_1 - x_0)$. On the other hand, from $C \subset D(B, x_0)$ and $\text{dist}(C, B) > 0$ we can find $b_2 \in B$ and $t_2 < 1$ fulfilling $c = x_0 + t_2(b_2 - x_0)$. Since the line defined by m and b_1 and the line defined by c and b_2 intersect in x_0, the four points m, c, b_1 and b_2 lie in the same plane and form a quadrangle. Denote $q \in X$ the intersection of the diagonals of this quadrangle, that is, of the line segments $[m, c]$ and $[b_1, b_2]$. This intersection exists inside the quadrangle because $\|m - x_0\| \geq \|b_1 - x_0\|$ and $\|c - x_0\| < \|b_2 - x_0\|$. As it is a point of $[m, c]$, it follows $\|q - c\| \leq \|m - c\|$. Finally, because q is a point of $[b_1, b_2]$ and both b_1 and b_2 lie in the convex set B, we have $q \in B$ and so $\|q - c\| \geq \text{dist}(B, C)$. Together we get $\|c - m\| \geq \text{dist}(C, B)$. As this is true for an arbitrary pair $c \in C$ and $m \in M$, we conclude $\text{dist}(B, C) \leq \text{dist}(M, C)$.

Finally, we set $B_4 := cl(B_3)$ the closure of B_3. This set is closed, bounded, convex, has a positive distance from C and s is its interior point. Because the drop theorem is invariant under translation (by which we mean: if a fulfils the statement of the theorem for x_0, B and C, then $a - s$ will be the point yielded by the same theorem used on $x_0 - s, B - s$ and $C - s$), we can without loss of generality assume $s = 0$. This makes B_4 into a balanced set. Last but not least, as 0 is an interior point of B_4, the set is absorbing.

Let $p \in X^*$, $p(x) := \inf\{\lambda \in \mathbb{R} : x \in \lambda B_4\}$ be the Minkowski functional over B_4. From Lemma 3.11 below, it follows that p is a norm.

What is more, this new norm is equivalent to the old one: As B_4 is bounded, it is contained in some $\|.\|$-ball $B_r := \{x \in X : \|x\| \leq r\}$. This means that any point $x \in B_4$, that is, any point with $p(x) \leq 1$, satisfies $\|x\| \leq r$. Because both norms are linear, it follows $p(x) \leq r\|x\|$ for every $x \in X$. Conversely, we know that 0 is an interior point of B_4 and so that there exists $\epsilon > 0$ such that $B_\epsilon := \{x \in X : \|x\| \leq \epsilon\} \subset B_4$. This means that whenever $\|x\| \leq \epsilon$, it follows $p(x) \leq 1$ and therefore for any $x \in X$, $\|x\| \leq \frac{1}{\epsilon} p(x)$.

Because $\text{dist}(B_4, C) > 0$, there exists $\kappa > 0$ such that $(1+\kappa)B_3 \cap C = \emptyset$. Thus, the ball $\{x \in X : p(x) \leq 1 + \kappa\}$ does not intersect C and the known version of the Daneš's drop theorem can be applied as described at the beginning of this proof. □

Lemma 3.11. *Let $D \subset X$ be closed, convex, bounded and absorbing in the norm topology. Then the Minkowski functional $p(x) := \inf\{\lambda \in \mathbb{R} : x \in \lambda D\}$ is a norm.*

Proof. First, $\lambda 0 = 0 \in D$ for any $\lambda \in \mathbb{R}$ which means $p(0) = 0$. On the other hand, if $x \neq 0$, there exists neighbourhood of 0 which separates 0 and x as the norm topology is Hausdorf, let us call it V. Because D is bounded, there exists $\lambda > 0$ such that $\lambda D \subset V$ and so $p(x) \geq \lambda > 0$.

Moreover, for $\alpha > 0$, $p(\alpha x) = \inf\{\lambda \in \mathbb{R} : \alpha x \in \lambda D\} = \alpha \inf\{\lambda \in \mathbb{R} : x \in D\} = \alpha p(x)$.

Finally, if $x \in lD$ and $y \in kD$, then $x + y \in (k + l)D$ and so

$$p(x + y) = \inf\{\lambda \in \mathbb{R} : x + y \in \lambda D\} \leq k + l$$

for every $k > p(x)$,$l > p(y)$ which means

$$p(x + y) \leq p(x) + p(y).$$

□

Statement 3.12. *The Flower petal theorem implies the Daneš's drop theorem.*

Proof. [7] According to the assumptions of the drop theorem, choose a nonempty closed set C and a $z_0 \in X \setminus C$ with $d := \text{dist}(z_0, C) > 0$ and denote $B := B(z_0, r)$ the ball centered at z_0 for some $r < \text{dist}(C, z_0)$. Choose an arbitrary $x_0 \in C$ and set $A := C \cap D(B, x_0)$ and $\gamma := \frac{d-r}{d+r}$. As A is a closed subset of the complete space X, we can use the Flower petal theorem to obtain a point $a \in A \cap P_\gamma(x_0, z_0)$ such that $P_\gamma(a, z_0) \cap A = \{a\}$. We claim that a is the point we seek in the drop theorem.

The fact that $a \in C$ is obvious as $a \in A = C \cap D(B, x_0)$. In addition, since our a lies in $A = C \cap D(B, x_0)$ we know that $D(B, a) \subset D(B, x_0)$.

Now we want to show that $D(B, a) \subset P_\gamma(a, z_0)$. As a preparation, for a positive real number t we compute

$$\begin{aligned} \frac{d-r}{d+r} &\leq \frac{t-r}{t+r} && \iff \\ (t+r)(d-r) &\leq (t-r)(d+r) && \iff \\ -tr + rd &\leq tr - rd && \iff \\ d &\leq t. \end{aligned}$$

This in particular means that

$$\gamma = \frac{d-r}{d+r} \leq \frac{t-r}{t+r}$$

if we set $t := \|a - z_0\|$. This number is surely greater or equal d because a is a point of C.

First, choose a $z \in B(z_0, r)$. From the fact that this point fulfils

$$\|x - a\| \leq \|a - z_0\| + r \text{ and}$$

$$\|x - z_0\| \leq r$$

and from the above computation, we can conclude the following estimate:

$$\gamma\|a - z\| + \|z - z_0\| \leq \frac{t-r}{t+r}\|z - a\| + \|z - z_0\| \leq \frac{t-r}{t+r}(\|a - z_0\| + r) + r = t.$$

A general point x lying in $D(B, a)$ can be expressed as a convex combination of a point of B and a:

$$x = \alpha a + (1 - \alpha)z, \text{ where } z \in B(z_0, r) \text{ and } \alpha \in [0, 1].$$

Using this, we can finally compute

$$\begin{aligned}\gamma\|x - a\| + \|x - z_0\| &= \gamma\|\alpha a + (1-\alpha)z - a\| + \|\alpha a + (1-\alpha)z - z_0\| \leq \\ &\leq \alpha\|a - z_0\| + (1-\alpha)(\gamma\|z - a\| + \|z - z_0\|) \leq \\ &\leq \alpha\|a - z_0\| + (1-\alpha)\|a - z_0\| = \|a - z_0\|\end{aligned}$$

which, according to the definition, states that $x \in P_\gamma(a, z_0)$.

Finally, we can show that a is the only intersection of C and $D(B, a)$:

$$D(B, a) \cap C \subset D(B, a) \cap (C \cap D(B, x_0)) \subset P_\gamma(a, z_0) \cap A = \{a\}.$$

□

3.4 The Brézis-Browder's theorem

Theorem 3.13 (The Brézis-Browder theorem). *Let C be a nonempty closed subset of a Banach space X and $z_0 \in X \setminus C$. Let $0 < r < \operatorname{dist}(z_0, C)$ and $x_0 \in C$. Denote*

$$K := \{tx : x \in B(z_0 - x_0, r), t > 0\} \tag{6}$$

the convex cone generated by the ball $B(z_0 - x_0, r)$. Then there exists a point $a \in C \cap (x_0 + K)$ such that

$$C \cap (a + K) \cap B(a, \delta) = \{a\}$$

for any $0 < \delta < \operatorname{dist}(z_0, C) - r$.

For the proof of this statement, we will use the following [8]:

Lemma 3.14. *Let X be a Banach space, $B \subset X$ convex nonempty, $s_0 \in X$ arbitrary and $s \in D(B, s_0)$. Find $\alpha \in [0, 1]$ and $u \in B$ such that*

$$s = \alpha s_0 + (1 - \alpha)u.$$

Then the following holds true:

1) For any $t \in [0, \alpha]$ we have $s + t(D(B, s_0) - s_0) \subset D(B, s) \subset D(B, s_0)$.

2) Assume in addition that $\operatorname{dist}(s_0, B) > 0$ and set $t_0 = \frac{\operatorname{dist}(s,B)}{\operatorname{dist}(s_0,B)}$. Then $t_0 \leq \alpha$ and for any $0 \leq t_1 \leq t_2 \leq 1$ we get

$$\begin{aligned}&(s + K(B - s_0)) \ \cap \ B(s, t_1 \operatorname{dist}(s, B)) \subset \\ &s + t_2 t_0 (D(B, s_0) - s_0) \subset D(B, s) \subset D(B, s_0).\end{aligned}$$

Here, the expression $K(B - s_0)$ denotes the convex cone generated by the set $B - s_0$: $K(B - s_0) = \{tx \in X : t > 0, x \in B - s_0\}$.

Proof. 1) For a $t \in [0, \alpha]$, choose an arbitrary $x \in s + t(D(B, s_0) - s_0)$. Then for some $\beta \in [0, 1]$ and $z \in B$ we can write

$$x = s + t(bs_0 + (1 - b)z - s_0) = s + t(1 - b)(z - s_0).$$

If $\alpha = 0$, then x lies in $B \subset D(B, s)$ and there is nothing to show.

Assume from now on that $\alpha > 0$. Then, using $s = \alpha s_0 + (1 - \alpha)u$, we can write $s_0 = \alpha^{-1}s - \alpha^{-1}(1 - \alpha)u$ which then yields

$$\begin{aligned}x &= s + t(1 - b)(z - \alpha^{-1}s + \alpha^{-1}(1 - \alpha)u) \\ &= s + \alpha^{-1}t(1 - b)(\alpha z - s + (1 - \alpha)u) \\ &= (1 - \alpha^{-1}t(1 - b))s + \alpha^{-1}t(1 - b)(\alpha z + (1 - \alpha)u) \\ &= (1 - c)s + cw,\end{aligned}$$

where $c = \alpha^{-1}t(1 - b)$ and $w = \alpha z + (1 - \alpha)u$. Since both z and u lie in the convex set B, w is a point of B, too. Moreover, because t was chosen such that $t \leq \alpha$, we know $0 \leq c \leq 1 - b \leq 1$. This together with $s \in B$ means that x is a convex combination of s and a point lying in B and therefore, $x \in D(B, s)$. This proves the first inclusion in the lemma.

For the second one, just observe that $s \in D(B, s_0)$, $D \subset D(B, s_0)$ and the drop $D(B, s_0)$ is convex.

2) First we notice that the cone $K(B - s_0) = \{tx \in X : t \in \mathbb{R}^+, x \in B - s_0\}$ does not change if multiplied by a positive constant. Together with $t_1 \leq t_2 \leq 1$ this means

$$\begin{aligned}(s + K(B - s_0)) \cap B(s, t_1 \operatorname{dist}(s, B)) &= s + t_1(K(B - s_0) \cap B(0, \operatorname{dist}(s, B))) \\ &\subset s + t_2(K(B - s_0) \cap B(0, \operatorname{dist}(s, B))) \\ &= (s + K(B - s_0)) \cap B(s, t_2 \operatorname{dist}(s, B)).\end{aligned}$$

In addition, since B is convex, $D(B, s_0) - s_0$ is a convex set containing the origin. Together with $0 \leq t_2 \leq 1$, we get

$$s + t_2 t_0 (D(B, s_0) - s_0) \subset s + t_0 (D(B, s_0) - s_0).$$

Therefore, it suffices to show that

$$\begin{aligned}(s + K(B - s_0)) \cap B(s, t_2 \operatorname{dist}(s, B)) \subset s + t_0 t_2 (D(B, s_0) - s_0) \text{ and} \\ s + t_0(D(B, s_0) - s_0) \subset D(B, s).\end{aligned} \tag{7}$$

To this end, fix $x \in (s + K(B - s_0)) \cap B(s, t_2 \operatorname{dist}(s, B))$. Because this x is a point of the cone $s + K(B - s_0)$, there exist $r \geq 0$ and $z \in B$ such that $x = s + r(z - s_0)$.

At the same time, x lies in the ball $B(s, t_2 \operatorname{dist}(s, B))$ and so it holds

$$t_2 \operatorname{dist}(s, B) \geq \|x - s\| = r\|z - s_0\|.$$

From this, using $z \in B$, we can deduce

$$r \leq \frac{t_2 \operatorname{dist}(s, B)}{\|z - s_0\|} \leq \frac{t_2 \operatorname{dist}(s, B)}{\operatorname{dist}(s_0, B)} = t_2 t_0.$$

If it happens that $t_0 = 0$, it means that $\operatorname{dist}(s, B) = 0$. In such a case,

$$B(s, t_2 \operatorname{dist}(s, B)) = \{s\} \subset s + t_0 t_2 (D(B, s_0) - s_0)$$

and the first inclusion in (7) is true.

Assume now $t_0 > 0$ and set $b := 1 - r t_0^{-1} t_2^{-1}$. Then, because $r \leq t_0 t_2$, we know that $b \in [0, 1]$. Thus, because $b s_0 + (1 - b)z$ lies in $D(B, s_0)$ as a convex combination of s_0 and a point of B, the point $s + t_0 t_2 (b s_0 + (1 - b)z - s_0)$ is contained in the set $s + t_0 t_2 (D(B, s_0) - s_0)$. This point, however, is exactly x, as the following computation shows:

$$\begin{aligned}s + t_0 t_2 (b s_0 + (1 - b)z - s_0) = s + t_0 t_2 ((1 - r t_0^{-1} t_2^{-1}) s_0 + r t_0^{-1} t_2^{-1} z - s_0) = \\ = s + t_0 t_2 (r t_0^{-1} t_2^{-1} z - r t_0^{-1} t_2^{-1} s_0) = s + r(z - s_0) = x.\end{aligned}$$

Summed up, we have shown that the statement $x \in s + t_0 t_2 (D(B, s_0) - s_0)$ is true which is the first part of (7).

For the inequality $t_0 \leq \alpha$, we use the convexity of B to compute

$$\begin{aligned}\operatorname{dist}(s, B) = \operatorname{dist}(\alpha s_0 + (1 - \alpha)u, B) \leq \\ \leq \alpha \operatorname{dist}(s_0, B) + (1 - \alpha) \operatorname{dist}(u, B) = \alpha \operatorname{dist}(s_0, B).\end{aligned}$$

With this relation known, we can deduce from the first part of the lemma the inclusion $s + t_0(D(B, s_0) - s_0) \subset D(B, s)$, that is, the remaining part of (7). □

Statement 3.15. *The Daneš's drop theorem implies the Brézis-Browder theorem.*

Proof. [8] From assumptions of the Brézis-Browder theorem, we have a nonempty closed set $C \subset X$, points $x_0 \in C$ and $z_0 \in X \setminus C$ and a real number r so that $0 < r < \operatorname{dist}(z_0, C)$. In addition, we set $K := \{tx : t > 0, x \in B(z_0 - x_0, r)\}$.

Denote $B = B(z_0, r)$. This B is naturally nonempty, convex, closed and bounded. Furthermore, C is another nonempty closed set fulfilling $\operatorname{dist}(C, B) > 0$. Thus, the assumptions of the drop theorem are met and we obtain a point $a \in C \cap D(B, x_0)$ such that $C \cap D(B, a) = \{a\}$.

Now we want to show that

$$C \cap (a + K) \cap B(a, \delta) \subset C \cap D(B, a) = \{a\} \tag{8}$$

for any $\delta > 0$ such that $\delta < \operatorname{dist}(z_0, C) - r$. From part 2) of the above lemma, setting $t_1 = 1$, it follows

$$(a + K) \cap B(a, \operatorname{dist}(a, B)) \subset D(B, a).$$

Using the facts that $B = B(z_0, r)$ and $a \in C \setminus B$, we get

$$d(a, B(z_0, r)) = \|a - z_0\| - r \geq d(z_0, C) - r.$$

This means that $B(a, \operatorname{dist}(z_0, C) - r) \subset B(a, \operatorname{dist}(a, B))$ and from there, we can deduce

$$(a + K) \cap B(a, \operatorname{dist}(z_0, C) - r) \subset D(B, a).$$

Thus, the equation (8) holds and the Brézis-Browder theorem is proved. □

3.5 The Phelps' lemma

The final part of our circle of implications is very similar to the Brézis-Browder theorem - the only important difference is that now, we are working with a cone that is defined using a general convex set instead of a ball.

Theorem 3.16 (The Phelp's lemma)**.** *Let X be a Banach space and $C \subset X$ a nonempty closed set. Let $B \subset X$ be a nonempty closed convex bounded set such that $0 \notin B$. Let K be a cone defined as*

$$K := \{tx : t > 0 \text{ and } x \in B\}.$$

Then, for each $x_0 \in C$ with the property that $C \cap (x_0 + K)$ is bounded, there exists $a \in C \cap (x_0 + K)$ such that $\{a\} = C \cap (a + K)$.

Statement 3.17. *The Brézis-Browder theorem implies the Phelps' lemma.*

Proof. Choose a nonempty closed set $C \subset X$ and a nonempty closed convex bounded set B that does not contain the origin and define K as in the statement of the Phelps' lemma. Choose $x_0 \in C$ such that the set $(x_0 + K) \cap C$ is bounded.

Note that for any $a \in (x_0 + K) \cap C$ it holds that $(a + K) \cap C \subset (x_0 + K) \cap C$. To show this, choose $x \in (a + K)$. From definition of K, there exist $b_1, b_2 \in B$ and $t_1, t_2 \geq 0$ such that $x = a + t_1(b_1 - a)$ and $a = x_0 + t_2(b_2 - x_0)$. Then,

$$\begin{aligned} x &= x_0 + t_2(b_2 - x_0) + t_1(b_1 - (x_0 + t_2(b_2 - x_0))) \\ &= x_0 + t_1 b_1 + (t_2 + t_1 t_2) b_2 - (t_2 + t_1 + t_1 t_2) x_0 \\ &= x_0 + (t_2 + t_1 + t_1 t_2)(b - x_0), \end{aligned}$$

where

$$b = \frac{t_1 b_1 + (t_2 + t_1 t_2) b_2}{t_2 + t_1 + t_1 t_2}.$$

From convexity, $b \in B$ and so $x \in (x_0 + K)$. Thanks to this, we can assume without loss of generality that C is bounded and $C \subset (x_0 + K)$.

Observe that if we take

$$B_1 := \{x \in X : \text{dist}(x, B) \leq \frac{\text{dist}(B, 0)}{2}\},$$

we create a set that will be nonempty, closed, convex and bounded as the properties of B carry over. In addition, it will have nonempty interior and the distance of B_1 from 0 will still be positive. Finally, $(a + K) \subset K(B_1, a)$ for every $a \in X$, which means that if we find a point $a \in C$ fulfilling $\{a\} = C \cap K(B_1, a)$, we will immediatelly know that also $\{a\} = C \cap (a + K)$. Thus, we can from now on assume without loss of generality that B has nonempty interior.

Assume that there exists a ball $B_0 \subset X$ such that $B \subset B_0$ and $\text{dist}(B_0, 0) > 0$. Then, the Brézis-Browder theorem yields $a \in C$ fulfilling $K(B_0, a) \cap C = \{a\}$. Since $(a + K) \subset K(B_0, a)$, the statement of the Phelps' lemma follows.

As in the proof of Statement 3.10, we cannot be sure if we can envelop B in a ball in this way and so we will go over to an equivalent norm. The cone does not depend on the norm and thus, the point a we will obtain from applying the Brézis-Browder theorem after the transition to the new norm will be the desired solution. Thus, we want to find a suitable equivalent norm that allows to enclose the given set B in a ball that does not contain the origin.

The rest of the proof is exactly the same as the proof of 3.10, with the exception that the role of C will be played by the set $\{0\}$. □

To close our circle of implications, we now go back to the point where we started - to the Ekeland's variational principle.

Statement 3.18. *The Phelps' lemma implies the Ekeland's variational principle.*

Proof. From the formulation of the Ekeland's theorem, we are given a point $x_0 \in X$ and a function $f : X \to \mathbb{R} \cup \infty$ that is lower semicontinuous, bounded from below and not identically ∞. The idea is to use the Phelps' lemma on the epigraph of f, that is, on the set $C \subset X \times \mathbb{R}$, $C := \{(x, t) : t \geq f(x)\}$. Note that if we define a norm on $X \times \mathbb{R}$ by

$$\|(x, t)\|_{X \times \mathbb{R}} := \|x\|_X + |t|,$$

it will turn $X \times \mathbb{R}$ into a Banach space and C in a closed set.

Set $B := \{(x, -1) \in X \times \mathbb{R} : \|x\| \leq \frac{\varepsilon}{\lambda}\}$ where λ and ε are given from the Ekeland's theorem. This way, B is nonempty, closed, convex and bounded and generates the cone

$$K := \{(x, t) \in X \times \mathbb{R} : \frac{\epsilon}{\lambda} d(x, 0) + t \leq 0\}.$$

Now we observe that the set $C \cap ((x_0, f(x_0)) + K)$ is bounded. Denote L the lower bound of f. Since

$$C \cap ((x_0, f(x_0)) + K) = \{(x, t) : t \geq f(x) \text{ and } \frac{\varepsilon}{\lambda} d(x, x_0) + t - f(x_0) \leq 0\},$$

from $(x, t) \in C \cap ((x_0, f(x_0)) + K)$ immediately follows that $L \leq t \leq f(x_0)$: if $t < L$, the condition $t \geq f(x)$ cannot be fulfilled, and if $t > f(x_0)$, we have $\frac{\varepsilon}{\lambda} d(x, x_0) + t - f(x_0) > 0$. Furthermore, we get

$$d(x, x_0) \leq \frac{\lambda}{\varepsilon}(f(x_0) - t)$$

and so the set is indeed bounded.

Thus, we can use the Phelps' lemma and obtain $(a, s) \in C \cap ((x_0, f(x_0)) + K)$ such that $C \cap ((a, s) + K) = \{(a, s)\}$. We claim that this a is the point we are seeking in the Ekeland's theorem.

First, since $(a, f(a)) \in C \cap ((a, s) + K)$ for any $s \geq f(a)$, we know $s = f(a)$.

Since for any $x \in C$, $(x, f(x))$ lies in C by definition, it follows

$$(x, f(x)) \notin (a, f(a)) + K = \{(x, t) \in X \times \mathbb{R} : \frac{\epsilon}{\lambda} d(x, a) + t - f(a) \leq 0\}.$$

From this we get $\frac{\epsilon}{\lambda} d(x, a) + f(x) - f(a) > 0$ and so $f(a) < f(x) + \frac{\epsilon}{\lambda} d(x, a)$ which is the condition 1) of the Ekeland's variational principle.

Furthermore, $(a, f(a)) \in (x, f(x)) + K = \{(x, t) : \frac{\epsilon}{\lambda} d(x, x_0) + t - f(x_0) \leq 0\}$ from the Phelps' lemma. This means that $\frac{\epsilon}{\lambda} d(a, x_0) + f(a) - f(x_0) \leq 0$ and so $f(a) \leq f(x)$ which is the condition 2) of the Ekeland's theorem.

Finally, from the definition of x_0 we know that $f(x_0) < f(a) + \epsilon$ (since x_0 is chosen such that $f(x_0) < \inf_X(f) + \epsilon$). Thus, again from the inequality $\frac{\epsilon}{\lambda} d(x_0, a) + f(a) - f(x_0) \leq 0$, we get $d(x_0, a) \leq \frac{\lambda}{\epsilon}(f(x) - f(a)) < \frac{\lambda}{\epsilon}\epsilon = \lambda$. This is the last condition and so the proof is complete. □

3.6 The Caristi-Kirk's theorem

The theorems we have proved in the previous sections are equivalent to some more famous statements. One of them is the following:

Theorem 3.19 (Caristi-Kirk). *Let (X, d) be a complete metric space, f a real lower semicontinuous function on X that is bounded from below and $T : X \to X$ a map such that for any $x \in X$, $d(x, Tx) \leq f(x) - f(Tx)$. Then T has a fixed point.*

Statement 3.20. *The simple Ekeland's variational principle implies the Caristi-Kirk theorem.*

Proof. Let T and f be as in the requirements of the Caristi-Kirk theorem. It is possible to apply the Ekeland's theorem to f; doing so, we obtain a point $x_0 \in X$ satisfying

$$f(x_0) < f(x) + d(x, x_0) \text{ for every } x \in X \setminus \{x_0\}.$$

Assume x_0 is not a fixed point of T and set $x := T(x_0)$. From the above inequality we have

$$f(x_0) - f(T(x_0)) < d(T(x_0), x_0)$$

but, at the same time, the definition of T requires

$$f(x_0) - f(T(x_0)) \geq d(T(x_0), x_0).$$

This is a contradiction and so x_0 has to be a fixed point of T. □

Statement 3.21. *The Caristi-Kirk theorem implies the simple Ekeland's variational principle.*

Proof. We will use a proof by a contradiction. If the Ekeland's theorem is not true then there exists no minimum in the Ekeland's sense. In other words, for any $x \in X$ there exists some $Tx \in X \setminus \{x\}$ such that

$$f(x) \geq f(Tx) + d(x, Tx).$$

This means, however, that T is a map that fulfils the requirement of the Caristi-Kirk theorem and so it has to have a fixed point. As we have constructed T so that it never maps a point on itself, we have found our contradiction. □

3.7 The Bishop-Phelps' theorem

The last theorem that will be presented in this chapter is probably the best known of them all. Unfortunately, here the author was unable to prove the full equivalence - as presented here, the direction from Bishop-Phelps to the others requires an additional assumption.

Theorem 3.22 (Bishop-Phelps)**.** *Let X be a Banach space over the real numbers. Let C be a nonempty bounded closed convex subset of X. Then the functionals that attain their maximum at C are norm-dense in X^*.*

Statement 3.23. *The Bishop-Phelps theorem implies the simple Ekeland's variational principle, under the additional assumption that the function f from the Ekeland's theorem is convex.*

Proof. The idea is to apply the Bishop-Phelps theorem to the epigraph of a lower semicontinuous function. The epigraph of a function f is the set

$$\{(x, \mu) : f(x) \leq \mu\}.$$

First, we will observe that in the space $X \oplus \mathbb{R}$, each continuous linear functional ψ of the norm 1 has a special form. By setting $x = 0$, the functional ψ turns into a linear functional on $\mathbb{R}$ since for $a, b \in \mathbb{R}$, $\psi((0, at + bs)) = a\psi((0, t)) + b\psi((0, s))$ from linearity of ψ. However, continuous linear functionals on $\mathbb{R}$ have always the form $\Phi(t) = \alpha t$ for some $\alpha \in \mathbb{R}$. Similarly, $\psi((x, 0))$ is a linear functional on X and so there exists $\psi' \in X^*$ such that $\psi((x, 0)) = \psi'(x)$ for all $x \in X$. Put together, we get

$$\psi((x, t)) = \psi((x, 0)) + \psi((0, t)) = \psi'(x) + \alpha t.$$

Let f be a convex lower semicontiuous function on X that is bounded from below by a real number L and choose x_0 arbitrary. Denote

$$C := \{(x, \mu) : x \in X, f(x) \leq \mu \leq f(x_0) - d(x, x_0)\}.$$

C is closed, nonempty as $x_0 \in C$ and convex from convexity of f. Furthermore, if $(x, \mu) \in C$, we have $L \leq \mu \leq f(x_0)$ and $d(x, x_0) \leq f(x_0) - L$ and so C is bounded.

Define a functional $\phi \in (X \oplus \mathbb{R})^*$ as

$$\phi((x, t)) = t.$$

Choose $\varepsilon > 0$ and apply the Bishop-Phelps theorem to C. This is the step where we need the convexity of f - without this assumption, C might not be a convex set and so the Bishop-Phelps theorem would be unapplicable.

We will receive a functional ψ that attains its minimum on C. Decompose

$$\psi(x,t) = \psi'(x) + \alpha t.$$

On C, we can rewrite this as $\psi(x, f(x)) = \psi'(x) + \alpha f(x)$. Since $\|\psi\| = 1$ and $\|\psi\| = \|\psi'\| + |\alpha|$, it follows $1 - \varepsilon < \alpha \leq 1$ and $\|\psi'\| = 1 - \alpha$.

Denote x_0 the point at which ψ attains its minimum and assume it is not the point sought in the Ekeland's variational principle, that is, assume there exists $x \in C$ such that $f(x) < f(x_0) - d(x, x_0)$. Then, using that $\|\psi'\| = 1 - \alpha$, it follows that

$$\begin{aligned}\psi(x, f(x)) &= \psi'(x) + \alpha f(x) < \\ &< \psi'(x) + \alpha f(x_0) - \alpha d(x, x_0) \leq \\ &\leq \psi'(x_0) + (1-\alpha)d(x,x_0) + \alpha f(x_0) - \alpha d(x, x_0) = \\ &= \psi'(x_0) + \alpha f(x_0) + (1 - 2\alpha)d(x, x_0) = \\ &= \psi(x_0, f(x_0) + (1 - 2\alpha)d(x, x_0) < \\ &< \psi(x_0, f(x_0)).\end{aligned}$$

The last inequality folows from the fact that $\alpha > 1 - \varepsilon$ which for a small enough ε means that $1 - 2\alpha < 0$.

We have obtained the inequality $\psi(x, f(x)) < \psi(x_0, f(x_0))$ which is a contradiction as $(x_0, f(x_0))$ is the minimum of ψ on C. Thus, x_0 has to be the minimizing point in the sense of the Ekeland's variational principle. □

Statement 3.24. *The Ekeland's variational principle implies the Bishop-Phelps theorem.*

Proof. [10] Let $C \subset X$ be closed, convex and bounded. Choose a functional $f \in X^*$ and $\varepsilon > 0$. We want to find a functional g that attains its maximum on C and fulfils $\|f - g\| \leq \varepsilon$.

Define a new function $\bar{f} : X \to \mathbb{R} \cup \{\infty\}$ by

$$\begin{aligned}&\bar{f}(x) := -f(x) \text{ on } C \text{ and} \\ &\bar{f} = \infty \text{ on } X \setminus C.\end{aligned}$$

This function is lower semicontinuous from continuousness of f and bounded from below from boundedness of C and linearity of f. Thus, from the simple Ekeland's variational principle (Theorem 3.3), there exists $x_0 \in C$ such that $\bar{f}(x_0) \leq \bar{f}(x) + \varepsilon\|x - x_0\|$ for all $x \in X$. Going back to f, this gives us $f(x_0) \geq f(x) - \varepsilon\|x - x_0\|$.

Define the following subsets of $X \oplus \mathbb{R}$:

$$\begin{aligned}K_1 &:= \{(x,t) : x \in C, t \leq f(x)\}, \\ K_2 &:= \{(x,t) : x \in X, t \geq f(x_0) + \epsilon\|x - x_0\|\}.\end{aligned}$$

Notice that for any fixed point $x \in C$, there exists t_1 such that $(x, t_1) \in K_1$ and t_2 such that $(x, t_2) \in K_2$. Also, note that while only points of C can build pairs that would be included in K_1, any x is eligible to form a point of K_2.

We start by proving both these sets are convex. If (x_1, t_1) and (x_2, t_2) are points of K_1 and $0 \leq \lambda \leq 1$, then $\lambda x_1 + (1-\lambda)x_2 \in C$ from convexity of C and

$$\lambda t_1 + (1-\lambda)t_2 \leq \lambda f(x_1) + (1-\lambda)f(x_2) = f(\lambda x_1 + (1-\lambda)x_2)$$

from linearity of f, which together implies that $\lambda(x_1, t_1) + (1-\lambda)(x_2, t_2) \in K_1$. If, on the other hand, (x_1, t_1) and (x_2, t_2) lie in K_2, we get

$$f(x_0)+\varepsilon\|\lambda x_1+(1-\lambda)x_2-x_0\| \leq f(x_0)+\lambda\|x_1-x_0\|+(1-\lambda)\|x_2-x_0\| \leq \lambda t_1+(1-\lambda)t_2$$

from the triangle inequality and so $\lambda(x_1, t_1) + (1-\lambda)(x_2, t_2) \in K_2$.

Since in addition K_2 has nonempty interior whose intersection with K_1 is empty, from the Hahn-Banach separation theorem there exists a non-zero functional that separates K_1 and K_2. This, from the properties of $(X \oplus \mathbb{R})^*$ mentioned in the beginning of the proof of Theorem 3.23, means there exists $\phi \in X^*$ and two real numbers α and β such that

$$\begin{aligned} \phi(x) + \alpha t &\leq \beta \text{ on } K_1, \\ \phi(x) + \alpha t &\geq \beta \text{ on } K_2. \end{aligned} \tag{9}$$

Now we show that α has to be positive. Suppose $\alpha < 0$ and choose $x \in C$. Then, for t huge enough, $(x, t) \in K_2$ but $\phi(x) + \alpha t < \beta$ which contradicts the choice of ϕ. If on the other hand $\alpha = 0$, then for every $x \in X$, $\phi(x) \geq \beta$ which is impossible for a non-zero functional. Thus, $\alpha > 0$ and after renorming, we can assume $\alpha = 1$.

Because $(x_0, f_0) \in K_1 \cap K_2$, it follows from (9) that $\phi(x_0) + f(x_0) = \beta$. Furthermore, for any $x \in C$, the pair $(x, f(x))$ lies in K_1. Together this we get for all $x \in C$

$$\phi(x) + f(x) \leq \beta = \phi(x_0) + f(x_0)$$

and so the functional $\phi + f$ attains its maximum on C in the point x_0.

Furthermore, for an arbitrary $x \in X$ and $t := f(x_0) + \varepsilon\|x - x_0\|$, we have $(x, t) \in K_2$. Thus, again from (9),

$$\begin{aligned} &\phi(x_0) + f(x_0) = \beta \leq \phi(x) + \varepsilon\|x - x_0\| \text{ and so} \\ &\phi(x_0 - x) \leq \varepsilon\|x - x_0\|. \end{aligned}$$

This means that for any $z \in X$, we have $\phi(z) \leq \varepsilon\|z\|$ which implies $\|f-(f+\phi)\| = \|\phi\| \leq \varepsilon$.

Put together, we have shown that the functional $\phi + f$ attains its norm on C and is a good enough approximation of f. With this, the Bishop-Phelps theorem is proved. □

4 The drop property

In the previous chapters we have discussed the drop theorem and several other statements in general Banach spaces. As those results are often very powerful one can ask if their statements can be generalized in some ways.

The first obvious way to do this is to extend them to more general spaces. It turns out that at least in locally convex spaces, those statements make good sense and lead to interesting results. This kind of extension goes beyond the scope of this work, however, an interested reader might consult the paper of Hamel [11], for example.

Another natural way to relax the requirements of the drop theorem is to allow the distance of the two sets in question, the unit ball and the closed set S, to be equal 0. More precisely, we can do it as follows:

Definition 4. *The norm of a Banach space X has the* ***drop property*** *if for any closed set S disjoint with the unit ball B there exists a point $x \in S$ such that $S \cap D(B, x) = \{x\}$.*

A very easy example of a space with the drop property would be $\mathbb{R}^n$ or any finite dimensional Banach space. Since the unit sphere is compact, it has a positive distance from any closed set that does not intersect it and so we know that this space has the drop property from the Daneš's drop theorem.

On the other hand, the space l^1 does not have the drop property. To show this, we will construct a countable set $S = \{f_n\}$ of points that have no convergent subsequence. This way, S will be closed as a discrete set. In addition, having constructed the first n points we will find the $(n+1)$-th one inside the drop $D(B, f_n)$, making it impossible to find a drop such as required in the definition of the drop property.

Set $f_1 := (2, 0, 0, ...)$. For $n \geq 2$, define $f_{n+1} := \frac{1}{2} f_n + \frac{1}{2} e_{n+1}$, where e_k is the standard unit vector. Because f_{n+1} is a convex combination of f_n and $e_{n+1} \in B$, it indeed lies in the drop $D(B, f_n)$.

First we compute the norm of f_n. Because f_n is zero on the only coordinate where e_{n+1} is nonzero, we have

$$\|f_n + e_{n+1}\| = \|f_n\| + \|e_{n+1}\| = \|f_n\| + 1$$

and therefore

$$\|f_{n+1}\| = \|\frac{1}{2} f_n + \frac{1}{2} e_{n+1}\| = \frac{1}{2}(\|f_n\| + 1).$$

Because $\|f_1\| = 2$, it follows that $\|f_n\| > 1$ for all $n \in \mathbb{N}$ and $S \cap B = \emptyset$. In addition, we see that $\|f_n\|$ converges to 1 which means that $\operatorname{dist}(S, B) = 0$.

To show that S has no convergent subsequence we estimate the distance between its points f_n and f_m where it is assumed without loss of generality that $m > n$. Then, however, it follows from the construction that the m-th coordinate of f_n is zero while the m-th coordinate of f_m is $\frac{1}{2}$. Thus,

$$\|f_n - f_m\| \geq \frac{1}{2} \text{ for all } n, m \in \mathbb{N},$$

which makes the convergence for any subsequence impossible. Together this means that l_1 indeed does not have the drop property.

Now when we know there are spaces both with and without the drop property we would like to have some characterizations of this notion. The first uses a notion introduced by Rolewicz [12].

Definition 5. *A sequence $(x_n) \subset X$ such that $\|x_n\| > 1$ is called a* ***stream*** *if $x_{n+1} \in D(B, x_n)$ for each $n \in \mathbb{N}$.*

The assumption that the norms of the points are bigger than one is important: We want the stream to be a sequence that lies outside the unit ball but is nearing its surface in some restricted area. If we abandoned this requirement, any sequence inside the unit ball would be a stream, even sequences with a very wild behavior. As a consequence, the following useful theorem would not be true.

Theorem 4.1. *X has the drop property if and only if each stream has a convergent subsequence.*

Proof. (i) Assume that X does not have the drop property. This means that there exists a closed set S with $S \cap B = \emptyset$ such that for any $x \in S$ there exists $y \in S$, $y \neq x$ fulfilling $y \in D(B, x)$. From the Daneš's drop theorem we know that $\operatorname{dist}(S, B) = 0$.

Fix $x \in S$. We observe that we can choose a $y \in D(B, x) \cap S$ with norm as close to 1 as we like: the distance of the closed set $D(B, x) \cap S$ from B is zero, as if the converse was true, we would be able to use the drop theorem and find a $z \in D(B, x) \cap S$ such that $D(B, z) \cap D(B, x) \cap S = D(B, z) \cap S = \{z\}$ which would be a contradiction to the choice of S.

Knowing this, we can construct a stream $(y_n) \subset S$ such that

$$\operatorname{dist}(B, y_n) < 1 + \frac{1}{n}.$$

Because each stream has a convergent subsequence and S is closed, there exists a limit point $y \in S$ with $\operatorname{dist}(B, y) = 0$ which means that $y \in B$ and therefore, $S \cap B \neq \emptyset$ which is a contradiction to the choice of S.

(ii) Choose a stream $S = \{y_n : n \in \mathbb{N}\}$ that does not contain a convergent subsequence, this means that S is closed. Assume X has the drop property. Then there exists $n \in \mathbb{N}$ such that $S \cap D(B, y_n) = \{y_n\}$. At the same time, however, $y_m \in D(B, y_n)$ for every $m \geq n$, which means that $y_m = y_n$ for all such m. We have found out the stream converges despite we have assumed there is no convergent subsequence; this is a contradiction and so the proof is completed. □

Now we turn our attention to the relation between the drop property and reflexivity. The finding that the drop property implies reflexivity is due to Rolewicz [12]; the opposite direction was later proved by Montesinos [13]. First, however, we have to do some preparatory work.

Definition 6. *For $A \subset X$, the* ***Kuratowski measure of noncompactness*** *$\alpha(A)$ is a nonnegative real number defined as*

$$\alpha(A) := \inf\{r > 0 : A \subset \bigcup A_n \textit{ for some finite family of sets } A_n \subset X \textit{ satisfying } \operatorname{diam}(A_n) \leq r\}.$$

Note that for any closed set we have $\alpha(A) = 0 \Leftrightarrow A$ is compact. Indeed, if $\alpha(A) > 0$, then there exists $\delta > 0$ such that A cannot be covered by finitely many sets of diameter less than δ which means that the covering of A by all such sets has no finite subcovering. On the other hand, if $\alpha(A) = 0$, the set is totally bounded, and since a closed subset of a complete space is complete, we conclude that A is compact.

Lemma 4.2. *Suppose there is a continuous linear functional $f \in X^*$ with $\|f\| = 1$ such that for the sets*

$$G_\varepsilon := \{x \in B_X : |f(x)| \geq 1 - \varepsilon\}$$

the Kuratowski measure of noncompactness does not tend to zero as ε goes to zero, that is, $\gamma := \inf\{\alpha(G_\epsilon) : \epsilon > 0\} > 0$. Then the space X does not have the drop property.

Proof. Choose a positive $\delta < \frac{\gamma}{4}$. We want to construct a sequence (x_n) such that

$$|f(x_n)| > 1 \text{ and}$$
$$\inf\{\|x_n - z\| : z \in \operatorname{span}(x_0, ..., x_{n-1})\} > \delta.$$

First we observe that for any finite-dimensional subspace $L \subset X$ and any $\varepsilon > 0$, there exists a point $x \in G_\varepsilon$ such that $\operatorname{dist}(x, L) \geq \frac{\gamma}{4}$. Suppose this is not the case and choose $\theta > 0$. For any $x \in G_\epsilon$, there exists $y \in L$ such that $\|x - y_x\| < \frac{\gamma}{4}$. From the triangle inequality and the fact that $x \in G_\epsilon \subset B_X$, we know that $\|y_x\| \leq \|y_x - x\| + \|x\| < \frac{\gamma}{4} + 1$.

As L is finite dimensional, the set $(1 + \frac{\gamma}{4})B_L$ is compact. Therefore, there exists a finite θ-net, we will label it $u_1, ..., u_n$. This in particular means that for every x there exists a point u_i fulfilling $\|u_i - y_x\| < \theta$. Together we infer that for any $x \in G_\epsilon$, we have a point u_i satisfying

$$\|x - u_i\| \leq \|x - y_x\| + \|y_x - u_i\| < \frac{\gamma}{4} + \theta.$$

We have thus covered G_ϵ with a finite number of balls with diameter $2(\frac{\gamma}{4} + \theta)$. Therefore,

$$\alpha(G_\epsilon) < 2(\frac{\gamma}{4} + \theta),$$

and because θ can be chosen arbitrarily small,

$$\alpha(G_\epsilon) < \frac{\gamma}{2}.$$

However, this is a contradiction to the definition of γ and so the observation has to be true.

We begin the construction of our desired sequence by choosing an arbitrary $x_0 \in X$ fulfilling $f(x_0) > 1$.

We will proceed by induction. Having constructed x_i for every natural $i \leq n$, we choose $\varepsilon_n > 0$ satisfying

$$f(x_n) - 1 > \varepsilon_n$$

and find $y_{n+1} \in G_{\varepsilon_n}$ such that

$$\inf\{\|y_{n+1} - z\| : z \in \operatorname{span}\{x_0, ..., x_n\}\} > \delta.$$

This is possible thanks to the observation our proof begins with and to the choice of δ. Using the definition of y_{n+1}, we set

$$x_{n+1} := \frac{y_{n+1} + x_n}{2}.$$

This especially means that $x_{n+1} \in D(B, x_n)$ because $G_{\varepsilon_n} \subset B$. In addition, using that $y \in G_\varepsilon$ and the choice of ε_n, we get

$$f(x_{n+1}) = \frac{1}{2}(f(x_n) + f(y_{n+1}) \geq \frac{1}{2}(f(x_n) + (1 - \varepsilon_n)) > 1$$

and

$$\inf\{\|x_{n+1} - z\| : z \in \operatorname{span}(x_0, ..., x_n) = \frac{1}{2}\inf\{\|y_{n+1} - z\| : z \in \operatorname{span}(x_0, ..., x_n)\} > \frac{\delta}{2}.$$

This together means that $\{x_n : n \in \mathbb{N}\}$ is a stream that has no convergent subsequence. Therefore, the norm cannot have the drop property. □

Theorem 4.3. *Any Banach space with the drop property is reflexive.*

Proof. Choose a continuous linear functional f with $\|f\| = 1$. Define

$$G_n := \{x \in B_X : |f(x)| \geq 1 - \frac{1}{n}\}$$

and set

$$\delta_n := \alpha(G_n) + \frac{1}{n} \text{ and } G := \bigcap_{n \in \mathbb{N}} G_n.$$

From Lemma 4.2 we know that the Kuratowski measure of noncompactness of the sets G_n tends to zero as n goes to infinity. Using this, we get $\delta_n \to 0$ for $n \to \infty$.

We want to show that G is nonempty. Because $G = \{x \in B_X : |f(x)| = 1\}$, we will then know that f attains its norm at some point of the unit ball. As f is arbitrary, we will then be able to conclude that X is reflexive from the James' characteristic.

For the proof that G is nonempty, we know that G_1 is covered by a finite number of balls with radius less than some number δ_1, let us call them $C_{1,i}$ where i is an index that goes from 1 to some finite number. Since $G_{n+1} \subset G_n$ for every $n \in \mathbb{N}$, there exists $C_{1,i}$ that intersects every G_n.

Should this not be the case, it would mean that for every $C_{1,i}$ there exists some G_{n_i} such that $G_{n_i} \cap C_{1,i} = \emptyset$. Denote j the maximum of such n_i - this number exists as we are examining only a finite family of sets $C_{1,i}$. Because $G_j \subset G_{n_i}$ for every i, we would have $G_j \cap C_{1,i} = \emptyset$ for all i. On the other hand, from $G_1 \subset \bigcup_i C_{1,i}$ it would follow that $G_1 \cap G_j = \emptyset$ which is impossible.

Using this, we can set $K_1 := G_1 \cap \overline{C_{1,i}}$ for this $C_{1,i}$ intersecting every G_n. This is a closed set that is containted in G_1 and intersects every G_n. In addition, its diameter is smaller than δ_1.

Moving to G_2, we know that $G_2 \subset \bigcup_i C_{2,i}$ for some balls $C_{2,i}$ with radii smaller than δ_2. Using similar arguments as above, there exists some $C_{2,i}$ intersecting K_1 and every G_n for $n > 2$. Set $K_2 := K_1 \cap G_2 \cap \overline{C_{2,i}}$. This is a closed set with diamater smaller than δ_2 intersecting every G_n for $n > 2$.

Iterating this argument, we get a sequence of closed sets K_n such that $K_{n+1} \subset K_n$ and $\operatorname{diam}(K_n) < \delta_n$. Because $\delta_n \to 0$ for $n \to \infty$, it follows from the Cantor intersection theorem that there exists some point $x \in \cap K_n \subset \cap G_n = G$. □

Having proved that drop property implies reflexivity, we will now turn our attention to the converse statement. We start by defining the following notions:

Definition 7. *A norm in X is **locally uniformly rotund (LUR)** if for any sequence $(x_n)_{n=1}^{\infty}$ and any $x \in X$ such that $\|x_n\| \leq 1$, $\|x\| = 1$ and $\lim_{n\to\infty} \|x + x_n\| = 2$ it follows that $x_n \to x$ in norm.*

Definition 8. *A norm in X has the **Radon-Riesz property (RR)** if for any sequence $(x_n)_{n=1}^{\infty}$ in X with $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ we have $x_n \to x$ strongly.*

The terminology is not unified here: according to [15], the above property can be named both Radon-Riesz and Kadets-Klee.

Lemma 4.4. *It is sufficient to test the Radon-Riesz property on a unit ball. More precisely: Suppose we know that for any sequence fulfilling*

$$\|x_n\| \to \|x\| = 1, \|x_n\| \leq 1, x_n \to x \text{ weakly}$$

it follows that $x_n \to x$ in norm. Then this norm already has the Radon-Riesz property.

Proof. Choose an arbitrary (x_n) converging weakly to a point $x \in X$ such that $\|x_n\| \to \|x\|$. If $\|x\| = 0$, the statement is trivially true. Suppose $\|x\| > 0$. Without loss of generality, assume $\|x_n\| \neq 0$ for all $n \in \mathbb{N}$. Then,

$$\|\frac{x_n}{\|x_n\|}\| = \|\frac{x}{\|x\|}\| = 1$$

and for any functional $f \in X^*$,

$$f(\frac{x_n}{\|x_n\|}) = \frac{1}{\|x_n\|} f(x_n) \to \frac{1}{\|x\|} f(x) = f(\frac{x}{\|x\|}).$$

Thus, all assumptions are met and the sequence $\frac{x_n}{\|x_n\|}$ converges to $\frac{x}{\|x\|}$ in norm. As $\|x_n\| \to \|x\|$, we conclude $x_n \to x$ in norm. □

Theorem 4.5. *A locally uniformly rotund norm has the Radon-Riesz property.*

Proof. From the above lemma we know it is sufficient to work on the unit ball. Choose (x_n) that fulfils all assumptions of the Lemma 4.4. First, we can estimate

$$\limsup_{n\to\infty} \|x_n + x\| \leq \lim_{n\to\infty} (\|x_n\| + \|x\|) = 2.$$

Furthermore, from the Hahn-Banach separation theorem there exists a functional $g \in B_{X^*}$ such that $g(x) = 1$. Then, because $x_n \to x$, it follows that

$$\liminf_{n\to\infty} \|x + x_n\| \geq \liminf_{n\to\infty} g(x + x_n) = 2g(x) = 2.$$

Therefore, $\|x_n + x\| \to 2$. This means that all assumptions on the sequence from the (LUR) property are fulfilled and thus, $x_n \to x$ in norm. □

Theorem 4.6. *The drop property implies the Radon-Riesz property.*

Proof. We know from Theorem 4.2 that for any functional $f \in B_{X^*}$ the Kuratowski index of noncompactness of the sets $G_\delta := \{x \in B_X : f(x) > 1 - \delta\}$ fulfils

$$\alpha(G_\delta) \to 0 \text{ as } \delta \to 0.$$

Choose an arbitrary sequence $(x_n) \subset B_X$ such that $x_n \to x_0$ weakly for some $x_0 \in X$ with $\|x_0\| = 1$. From the Hahn-Banach theorem we can find a functional $f \in B_{X^*}$ such that $f(x_0) = 1$. We define the sets G_δ as above with respect to f and find for a given $\varepsilon > 0$ a $\delta > 0$ fulfilling $\alpha(G_\delta) < \varepsilon$.

Since y_n converges weakly to x_0, we can find $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have $y_n \in G_\delta$. Because $\alpha(G_\delta) < \varepsilon$, the set G_δ is covered by a finite number of balls of radius smaller than ε. Thus, there exists some y_n such that $\|y_n - x_0\| < \varepsilon$. By induction, we can therefore find a subsequence $(z_n) \subset (y_n)$ that converges to x_0 strongly.

Since our choice of y_n was arbirary, it follows that $x_n \to x_0$ strongly. Indeed, should this not be the case, we would be able to find a subsequence $(y_n) \subset (x_n)$ such that $\|y_n - x_0\| > \kappa$ for some $\kappa > 0$ and for any $n \in \mathbb{N}$. However, it would be impossible to choose a convergent subsequence of this (y_n).

This concludes the proof for sequences that are subsets of the unit ball. For a general sequence, use Lemma 4.4. □

Lemma 4.7. *Let $(x_1, x_2, ..., x_n)$ be the first n points of a stream, that means, $x_i \notin B$, $x_{i+1} \in D(B, x_i)$. Let $z \in \operatorname{conv}\{x_1, ..., x_{n-1}\}$. Then $z \notin B$ and $x_n \in D(B, z)$.*

Proof. We will proceed by induction. For $n = 2$ the claim is obviously true as in this case, $z = x_1$. For $n = 3$, we can assume without loss of generality that $x_1 \neq x_2$, $z \neq x_i$, $i = 1, 2$. Since points x_2 and x_3 lie in certain drops, we can write the following convex combinations:

$$\begin{aligned} x_2 &= \lambda_2 x_1 + (1 - \lambda_2) b_2, \quad b_2 \in B_X, \lambda_2 \in (0, 1) \\ x_3 &= \lambda_3 x_2 + (1 - \lambda_3) b_3, \quad b_3 \in B_X, \lambda_3 \in (0, 1) \\ z &= \theta x_1 + (1 - \theta) x_2, \qquad \theta \in (0, 1). \end{aligned}$$

Hence,

$$\begin{aligned} z &= \theta x_1 + (1 - \theta)[\lambda_2 x_1 + (1 - \lambda_2) b_2] \\ &= k x_1 + (1 - \theta)(1 - \lambda_2) b_2, \\ &\text{where } k = \theta + (1 - \theta)\lambda_2 \in (\lambda_2, 1). \end{aligned}$$

Therefore,

$$x_1 = \frac{1}{k}[z - (1 - \theta)(1 - \lambda_2) b_2]$$

and, plugging the expressions for x_1 and x_2 in the formula for x_3,

$$\begin{aligned} x_3 &= \lambda_3 x_2 + (1 - \lambda_3) b_3 \\ &= \lambda_3[\lambda_2 x_1 + (1 - \lambda_2) b_2] + (1 - \lambda_3) b_3 \\ &= \lambda_3 \lambda_2 x_1 + \lambda_3 (1 - \lambda_2) b_2 + (1 - \lambda_3) b_3 \\ &= \lambda_3 \lambda_2 \frac{1}{k}[z - (1 - \theta)(1 - \lambda_2) b_2] + \lambda_3 (1 - \lambda_2) b_2 + (1 - \lambda_3) b_3 \\ &= \frac{1}{k} \lambda_3 \lambda_2 z + \frac{1}{k}[k \lambda_3 (1 - \lambda_2) - (1 - \theta)(1 - \lambda_2) \lambda_3 \lambda_2] b_2 + (1 - \lambda_3) b_3. \end{aligned}$$

This means that x_3 is a convex combination of the points z, b_2 and b_3, and thus $x_3 \in D(B,z)$. To prove this, we first verify that the coefficients are non-negative numbers. This is immediately to see for the first and third one. For the second, we compute

$$\frac{1}{k}[k\lambda_3(1-\lambda_2) - (1-\theta)(1-\lambda_2)\lambda_3\lambda_2] \\ = \frac{1}{k}(1-\lambda_2)\lambda_3[k-(1-\theta)\lambda_2]$$

which is a non-negative number since $\lambda_2, \lambda_3 \in [0,1]$, $\theta \in (0,1)$ and $k \in (\lambda_2, 1)$.

Furthermore, by adding the three coefficients, we get

$$\begin{aligned}
&\frac{1}{k}\lambda_3\lambda_2 + \frac{1}{k}[k\lambda_3(1-\lambda_2) - (1-\theta)(1-\lambda_2)\lambda_3\lambda_2] + (1-\lambda_3) \\
&= \frac{1}{k}\lambda_2\lambda_3 + \lambda_3(1-\lambda_2) - \frac{1}{k}(1-\theta)\lambda_2\lambda_3(1-\lambda_2) + 1 - \lambda_3 \\
&= \frac{1}{k}\lambda_2\lambda_3 + \lambda_3 - \lambda_2\lambda_3 - [\frac{1}{k}\lambda_2\lambda_3 - \frac{1}{k}\theta\lambda_2\lambda_3 - \frac{1}{k}\lambda_2^2\lambda_3 + \frac{1}{k}\theta\lambda_2^2\lambda_3] + 1 - \lambda_3 \\
&= 1 - \lambda_2\lambda_3 + \frac{1}{k}\lambda_2\lambda_3[\theta + \lambda_2 - \theta\lambda_2] \\
&= 1 - \lambda_2\lambda_3 + \lambda_2\lambda_3 = 1.
\end{aligned}$$

In addition, if the norm of z were less or equal 1, we would have $\|x_3\| \leq 1$ as x_3 would be a convex combination of points from the unit ball. Since we know that $\|x_3\| > 1$, it follows that $\|z\| > 1$. This completes the proof of the case $n = 3$.

For the general case, let us assume the lemma is true for $n \geq 3$ and choose points $x_1, ..., x_{n+1}$ and $z \in \text{conv}\{x_1, ..., x_n\}$. There exists $u \in \text{conv}(x_{n-1}, x_n)$ such that $z \in \text{conv}(x_1, ..., x_{n-2}, u)$. Since $u \in D(B, x_{n-2})$, it follows from the case $n = 3$ that $u \notin B$ and $x_{n+1} \in D(B,u)$. From the induction hypothesis we can conclude that $z \notin B$ and $x_{n+1} \in D(B,z)$. □

Theorem 4.8. *If $(X, \|.\|)$ is a reflexive Banach space with a norm that has the Radon-Riesz property, then the norm has the drop property.*

Proof. Choose a stream $(x_n)_{n=1}^{\infty}$. We want to show that this stream has a convergent subsequence. Once this is established, the claim will follow from Theorem 4.1.

From definition, $\|x_n\|$ is a non-increasing sequence and so it has a limit. We will discuss separately the cases where the limit is equal to 1 and greater.

Assuming $\|x_n\| \to L > 1$, $\{x_n\}$ is a closed set in X with a positive distance from the unit ball. From the Daneš's drop theorem we can find an $x_m, m \in \mathbb{N}$ such that

$$\{x_m\} = \{x_n : n \in \mathbb{N}\} \cap D(B, x_m).$$

From the definition of a stream it then follows that $x_n = x_m$ for every natural number $n \geq m$. This means that the sequence x_n itself is convergent.

For the other case, assume that $\|x_n\| \to 1$. Because the set $\{x_n : n \in \mathbb{N}\}$ is bounded we can find a weakly convergent subsequence converging to a point

$x_0 \in X$. Again, assume without loss of generality that $x_n \to x_0$ weakly. Because the norm is weakly lower semicontinuous, we know that

$$\|x_0\| \leq \liminf_{n\to\infty} \|x_n\| = 1. \tag{10}$$

On the other hand, for convex sets, the weak and strong closures are the same. Therefore,

$$x_0 \in \overline{\text{conv}\{x_n : n \in \mathbb{N}\}}^{w} = \overline{\text{conv}\{x_n : n \in \mathbb{N}\}}^{\|\cdot\|}.$$

From the previous lemma we know that $\|x_0\| \geq 1$. Together with (10) we deduce $\|x_0\| = 1$.

Summed up, we have found a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x_0$ weakly and $\|x_n\| \to \|x_0\|$. From the Radon-Riesz property we can conclude that $x_n \to x_0$ in the norm topology, too. This means that even in this last remaining case we have for our stream found a convergent subsequence. □

Theorem 4.9. *In any reflexive Banach space there exists an equivalent norm with the drop property.*

Proof. Any reflexive Banach space X is weakly compactly generated. From the Troyanski's renorming theorem (see [14]), there exists an equivalent norm on X which is locally uniformly rotund. We have shown in Theorem 4.5 that this norm also has the Radon-Riesz property. From Theorem 4.8 it then follows that this norm has the drop property. □

The last characteristic of a norm with the drop property uses the following definition:

Definition 9. *A set $M \subset X$ is called **approximately compact** if for any sequence (y_n) in M such that $\|y_n - x\| \to \text{dist}(x, M)$ for some $x \in X$ one can find a convergent subsequence.*

Theorem 4.10. *A norm in a space X has the drop property if and only if each closed, convex and bounded set in X is approximately compact.*

Proof. Assume first the norm has drop property. From Theorem 4.3, the space is reflexive. Choose an $M \subset X$ closed, convex and bounded. Furthermore, choose an arbitrary $x \in X$ and a sequence $(y_n) \subset M$ that minimizes the distance of x from M. We can assume without loss of generality that $x = 0 \notin M$, this means, $\|y_n\| \to \text{dist}(0, M)$. Since the sequence (y_n) is bounded in a reflexive space (X is reflexive because it has the drop property, as proved in Theorem 4.3), it is weakly compact and so there exists a weakly convergent subsequence y_{n_k} converging to some point $y \in M$. From the computation

$$\text{dist}(0, M) \leq \|y\| \leq \text{liminf}\|y_{n_k}\| = \text{dist}(0, M)$$

we get that $\|y_{n_k}\| \to \|y\|$. Since, according to Theorem 4.6, a norm with the drop property has the Radon-Riesz property and we know that $y_{n_k} \to y$ weakly, it follows that $y_{n_k} \to y$ strongly and M is approximately compact.

For the converse direction, assume for a contradiction that there exists a closed set $S \in X$ such that $S \cap B = \emptyset$ but for any point $x \in S$, $S \cap D(B, x) \neq \{x\}$.

Using these assumptions, we can construct a stream $(x_n) \subset S$ such that $\|x_n\| \to 1$. Define $M = \overline{\text{conv}}(x_1, x_2, x_3, ...)$. This set is convex and closed and therefore approximately compact by assumption. Furthermore, the stream (x_n) minimizes the distance of M from the origin. This means that there exists a subsequence $(x_{n_k}) \subset (x_n)$ that converges strongly to some point $z \in M$; we then have $\|x_n\| \to \|z\|$ and $\|x_n\| \to 1$ which means that $z \in B$. However, because $(x_n) \subset S$ and S is closed, z lies in S as well. Therefore, $S \cap B$ is not an empty, which is a contradiction. □

To sum up, we have shown the following characterisations:

Theorem 4.11. *For a Banach space $(X, \|.\|)$, it is equivalent:*
1) $(X, \|.\|)$ has the drop property;
2) X is reflexive and $\|.\|$ has the Radon-Riesz property and
3) Each convex closed and bounded subset of X is approximately compact.

5 Microscopic sets on Banach spaces

In the first chapter we have introduced the notion of a microscopic set on the real line. Here, we will generalize it and explore the connection between microscopic sets and drops.

According to Donnini and Martellotti [16], there are at least two ways how to introduce this notion:

Definition 10. *$M \subset X$ is called* ***microscopic*** *if for any $\varepsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $M \subset \bigcup_{n=1}^{\infty}(x_n + \varepsilon^n B_X)$. $M \subset X$ is called* ***scalarly microscopic*** *if for any functional $f \in X^*$ the set $f(M) \subset \mathbb{R}$ is microscopic.*

Using the properties of microscopic sets on the real line, one can easily find out the following:

Lemma 5.1. *1) Any microscopic set is scalarly microscopic.*
2) If $A \subset B$ and B is scalarly microscopic, then A is scalarly microscopic.
3) If $A = \bigcup_{n=1}^{\infty} A_n$ and A_n are all scalarly microscopic, then A is scalarly microscopic.
4) If M is scalarly microscopic, then $x + M$ and αM are scalarly microscopic for all $x \in X$ and $\alpha \in \mathbb{R}$.
5) Any countable set is scalarly microscopic.

Proof. 1) Choose $f \in X^*$ and let M be microscopic. Since $f(B_{X^*})$ is containted in the closed interval $[-\|f\|, \|f\|]$, we have

$$f(M) \subset f(\bigcup_{n=1}^{\infty}(x_n + \varepsilon^n B_X)) \subset \bigcup_{n=1}^{\infty}[f(x_n) - \varepsilon^n\|f\|, f(x_n) + \varepsilon^n\|f\|].$$

2) We know that $f(B)$ is microscopic in $\mathbb{R}$. Since $f(A) \subset f(B)$ and a subset of a microscopic set is again microscopic, A is scalarly microscopic.
3) It is easy to check that $f(A) = \bigcup_{n=1}^{\infty} f(A_n)$. A countable union of microscopic sets is microscopic from Theorem 2.7.
4) It suffices to shift and rescale the intervals covering the original set.
5) The image of a countable set is countable and each countable subset of the real axis is microscopic. □

As an example, X is not scalarly microscopic as for a functional $f \in X^* \setminus \{0\}$, $f(X) = \mathbb{R}$ which is not microscopic.

Define a segment in a Banach space as the convex hull of two points x and y, $x \neq y$. Such a set is not scalarly microscopic, either: there exists a functional $f \in X^*$ such that $f(x) < f(y)$. Then, $f(\mathrm{conv}(\{x, y\})) = [f(x), f(y)]$ and again, an interval is never microscopic.

It is an interesting question what is the relation between microscopic and scalarly microscopic sets. We have seen in the theorem above that a microscopic set is always scalarly microscopic. Conversely, in a nonseparable Hilbert space, the orthonormal basis is scalarly microscopic but not microscopic: label the basis C and choose $\epsilon < \frac{1}{4}$. Then, if a point of C is covered by a ball with diameter ϵ, the ball does not intersect C in any other point which means that C cannot be covered by a finite number of balls of radii ϵ^n and thus, C is not microscopic. On the other hand, from the Riesz's reprezentation theorem, each continuous functional on H can be reprezented by a point $x \in H$. As C is a basis, x can be written as $x = \sum \lambda_i e_i$ where e_i are the elements of C. Furthermore, as $\|x\| < \infty$, $\lambda_i = 0$ for all but countably many i.

Now we want to examine the image of C under the functional represented by x. Choose $e_i \in C$. Taking the scalar product, $(e_i, x) = (e_i, \sum \lambda_i e_i) = \lambda_i$ from orthonormality. This means that countably many e_i map on a non-zero number λ_i and all remaining on zero. Thus, the image of C is a countable set. Those are always microscopic. Because this holds for an arbitrary functional, we can conclude that C is a scalarly microscopic set.

Before we move on to drops, we will need to introduce the following notion that will be useful in later proofs.

Definition 11. *Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. The* ***induced polygonal*** *$P(x_n)$ is defined as the union of the intervals connecting the adjacent points of the sequence:*

$$P(x_n) = \bigcup_{n=1}^{\infty} \operatorname{conv}(\{x_n, x_{n+1}\}).$$

Let $C \subset X$ be closed, convex and bounded. A sequence $(x_n)_{n\in\mathbb{N}}$ is called a ***stream with basis*** *C if $x_{n+1} \in D(C, x_n)$ but $x_n \notin C$ for all $n \in \mathbb{N}$.*

Lemma 5.2. *Let $C \subset X$ be closed, convex and bounded. Let $(y_n)_{n\in\mathbb{N}}$ be a dyadic stream with the basis C, that is, a stream such that for any $n \in \mathbb{N}$, there exists an $x_n \in C$ fulfilling*

$$y_n = \frac{y_{n-1} + x_n}{2} \text{ and}$$
$$y_1 = \frac{x + x_1}{2} \text{ for some } x \in X \setminus C.$$

Suppose there exists $\delta > 0$ such that $\operatorname{dist}(y_n, \operatorname{span}\{y_1, \ldots, y_{n-1}\}) > \delta$. Then the induced polygonal $P := P(y_n)$ is closed.

Proof. See [16], Lemma 1. □

In the previous chapters, we have studied the drop property for a norm. Here, we will use a slightly more general version and define the drop property of a set. The drop property defined in Definition 4 can be obtained from the new definition of drop property by choosing the unit ball for the set C.

Definition 12. *A closed, bounded, convex set $C \subset X$ has the* ***drop property*** *if for any $F \subset X$ closed and disjoint with C one can find a point $x_0 \in F$ such that*

$$D(C, x_0) \cap F = \{x_0\}.$$

A closed, bounded, convex set $C \subset X$ has the ***microscopic drop property*** *if for any $F \subset X$ closed and disjoint with C one can find a point $x_0 \in F$ such that $D(C, x_0) \cap F$ is scalarly microscopic.*

Obviously, the drop property implies the microscopic drop property. We will prove, however, that in reflexive spaces for some special sets, those two notions are actually the same. For the proof, we will need the following:

Definition 13. *Let $C \subset X$, $\delta > 0$ and $f \in X^*$ a functional that is bounded on C. Denote $M := \sup_{x \in C}\{f(x)\}$. The* ***slice*** *$S(f, C, \delta)$ is a subset of C where the functional almost attains its supremum:*

$$S(f, C, \delta) = \{x \in C : f(x) > M - \delta\}.$$

The set C has the ***property*** *(α) if for each functional $f \in X^* \setminus \{0\}$,*

$$\lim_{\delta \to 0^+} \alpha(S(f, C, \delta)) = 0,$$

where α is the Kuratowski measure of noncompatness.

Theorem 5.3. *Suppose $C \subset X$ is a convex set with the microscopic drop property and choose a stream (x_n) with basis C. Then the induced polygonal is not closed.*

Proof. First, we show that P does not intersect C. Assume there exists $z \in P \cap C$. From the definition of an induced polygonal, we can find $n \in \mathbb{N}$ such that $z \in [x_n, x_{n+1}]$. If $z = x_n$, we get $x_n = z \in P \cap C$ and so the stream (x_n) intersects C which is in contradiction to the definition of a stream with basis C. The case $z = x_{n+1}$ works the same way. If $x_n \neq z \neq x_{n+1}$, we use the fact that $x_{n+1} \in D(C, x_n)$ to obtain a point $c \in C$ such that $x_{n+1} \in [x_n, c]$. Then, however, all the four points x_n, z, x_{n+1} and c lie in the same line and $x_{n+1} \in [z, c]$. Since both z and c are points of the convex set C, it follows $x_{n+1} \in C$ which is again a contradiction to the definition of a stream with basis C. Thus, for any induced polygonal it holds $P \cap C = \emptyset$.

Going now to the statement of the theorem, assume for contradiction that there exists a stream $(x_n)_{n \in \mathbb{N}}$ whose induced polygonal P is closed.

Choose $y \in P$. Then there exists a $k \in \mathbb{N}$ such that y lies in the line segment between x_k and x_{k+1}. Repeating the argument from the previous paragraph, $x_{k+1} \in [x_k, c]$ for some $c \in C$. Since y is a convex combination of x_k and x_{k+1}, it is a point of the same line as x_k, x_{k+1} and c; furthermore, $\|y - c\| \geq \|x_{k+1} - c\|$. This means x_{k+1} can be easily written as a convex combination of y and c. Thus, $x_{k+1} \in D(C, y)$.

Since y and x_{k+1} are containted in $D(C, y)$, it follows from convexity that the whole interval $[y, x_{k+1}]$ lies in $D(C, y) \cap P$. However, this interval is not scalarly microscopic and so the drop cannot be scalarly microscopic as well.

This already means that C cannot have the microscopic drop property. If it was the case, because $P \cap C = \emptyset$ and P is closed, there would exist $y \in P$ such that $D(C, y) \cap P$ would be scalarly microscopic which is according to the previous arguments never the case. This is a contradiction to the theorem statement and therefore, P cannot be closed. □

Theorem 5.4. *If X is reflexive and a closed bounded non-compact set $C \subset X$ has the microscopic drop property, then it fulfils the property (α).*

Proof. Assume C does not have the property (α). Then there exists a functional $f \in X^*$ such that $\inf_{\varepsilon > 0} \alpha(S(f, C, \varepsilon)) > 0$. We find a stream (x_n) such that

$\text{dist}(x_n, \text{span}\{x_1, \dots, x_{n-1}\}) > \frac{\delta}{2}$, where $0 < \delta < \frac{1}{2} \inf_{\varepsilon>0} \alpha(S(f, C, \varepsilon))$ - the construction goes as in Lemma 4.2. Set $P := P(x_n)$. P is closed from Lemma 5.2 which is a contradiction to Theorem 5.3. □

Lemma 5.5. *For arbitrary two sets C and D, the Kuratowski measure of noncompatness fulfils $\alpha(C + D) \leq \alpha(C) + \alpha(D)$.*

Proof. First we observe that if $\text{diam}(U) = k_U$ and $\text{diam}(V) = k_V$, then $\text{diam}(U + V) \leq k_U + k_V$. To see this, choose two points in $U + V$. By the construction of this set, they can be written as $u_1 + v_1$ and $u_2 + v_2$, where $u_1, u_2 \in U$, $v_1, v_2 \in V$. Then,

$$\|(u_1 + v_1) - (u_2 + v_2)\| \leq \|u_1 - u_2\| + \|v_1 - v_2\| \leq k_U + k_V.$$

We know that there exist finite families of sets $\{C_n\}$ and $\{D_m\}$ such that $C \subset \bigcup_n C_n$, $D \subset \bigcup_m D_m$, $\text{diam}(C_n) < \alpha(C)$ for all n and $\text{diam}(D_m) < \alpha(D)$ for every m. The system $\{C_n + D_m\}$ is a finite family of sets with diameter less or equal $\alpha(C) + \alpha(D)$ that covers $C + D$: we can express any given point in $C + D$ as $c + d$, where $c \in C$, $d \in D$. Then, there exist some n, m such that $c \in C_n$, $d \in D_m$ and so $c + d \in C_n + D_m$. □

Theorem 5.6. *If X is reflexive and $C \subset X$ is a closed convex bounded set with nonempty interior and the property (α), then C has the drop property.*

Proof. Since X is reflexive, it has an equivalent norm with the drop property as we have shown in the previous chapter. Both the drop property of C and the property (α) are invariant under isomorphisms, however. Consequently, we can assume without loss of generality that the norm itself already has the drop property.

Assume for contradiction that C does not have the drop property. Then there exists a closed set A, $A \cap C = \emptyset$ such that for any $x \in A$ we can find an $a \in D(C, x) \cap A$, $a \neq x$. Note that from the Daneš's drop theorem, $\text{dist}(A, C) = 0$. Also observe that for such a, the distance from C will be smaller than $\text{dist}(x, C)$. This follows from simple plane geometry: there exist $0 < \lambda < 1$ and $c \in C$ such that $a = \lambda x + (1 - \lambda)c$. Since C is a closed convex set in a Banach space, there exists a point $y \in C$ that minimizes the distance of x from C. Setting $z := \lambda y$, we obtain a triangle with edges c, z, a that is similar to the triangle with edges c, x, y. Thus, $\|a - z\| = \lambda \|x - y\|$ and therefore,

$$\text{dist}(a, C) < \text{dist}(x, C). \tag{11}$$

Choose $x_1 \in A$ arbitrary. We construct a stream $(x_n)_{n \in \mathbb{N}} \subset A$ with basis C such that if we set

$$d_n := \text{dist}(x_n, C) \text{ and}$$
$$d_n' := \inf\{\text{dist}(a, C) : a \in A \cap D(C, x_n)\}$$

we will have

$$d_{n+1} < d_n' + \frac{1}{n}. \tag{12}$$

Because $(x_n)_{n\in\mathbb{N}}$ is a stream, the sequence $(d_n)_{n\in\mathbb{N}}$ is non-increasing; furthermore, it is bounded from below by 0. Therefore, we can set

$$\varepsilon := \lim_{n\to\infty} d_n \geq 0.$$

There are two cases that we will handle separately:

In the first case, assume $\varepsilon = 0$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - x_n\| \to 0$. Set $A_1 = \text{conv}(x_n)$. Because (x_n) is a stream and C is convex, we know that $A_1 \cap C = \emptyset$. Using the fact that C has nonempty interior, we can find a functional $f \in X^* \setminus \{0\}$ that separates A_1 and C:

$$M := \sup\{f(x) : x \in C\} \leq \inf\{f(x) : x \in A_1\}.$$

Because of this, $f(y_n) \to M$. The sequence (y_n) is bounded as a subset of the bounded set C. Thus, because X is reflexive, $(y_n)_{n\in\mathbb{N}}$ has a subsequence that is weakly convergent to a point y - without loss of generality, identify this subsequence with (y_n). Because C is closed, it is also weakly closed and so $y \in C$. From the weak convergence, $f(y) = M$.

In addition, we show that the convergence $y_n \to y$ is actually strong. Choose $\varepsilon > 0$. From the property α, we know that there exists $\delta > 0$ such that all points in $x \in C$ where $f(x) > M - \delta$ can be covered by finitely many sets of diameter less than ε. This is especially the case for y and infinitely many y_n. This means that in any ball with center y and radius ε, there are contained infinitely many y_n and so there exists a subsequence of y_n that converges strongly to y. Since we already know that $y_n \to y$ weakly and since the limit point is unique, however, it follows $y_n \to y$ strongly.

Now that we know that $\|y_n - y\| \to 0$, we use the triangle inequality to find out that $\|x_n - y\| \to 0$. Since A is closed, however, it follows $y \in A \cap C$, which is a contradiction with the choice of A.

In the second case we assume $\varepsilon > 0$. This means that there exists $n \in \mathbb{N}$ such that $\text{dist}(A \cap D(C, x_n), C) > 0$. However, from closedness of C, the set $A \cap D(C, x_n)$ is closed as well. Thus, from the Daneš's drop theorem, there exists $a \in A \cap D(C, x_n)$ such that

$$\{a\} = D(C, a) \cap (A \cap D(C, x_n)).$$

This is a contradiction to the choice of A. □

Summed up, we have proved the following:

Theorem 5.7. *For X reflexive and C non-compact, closed, convex and bounded with non-empty interior, the following statements are equivalent:*
1) C has the drop property.
2) C has the microscopic drop property.
3) C has the property (α).

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