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MASTER THESIS

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S-matrix and homological perturbation lemma

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Title: S-matrix and homological perturbation lemma

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Abstract: Loop homotopy Lie algebras, which appear in closed string field theory, are a generalization of homotopy Lie algebras. For a loop homotopy Lie algebra, we transfer its structure on its homology and prove that the transferred structure is again a loop homotopy algebra. Moreover, we show that the homological perturbation lemma can be regarded as a path integral, integrating out the degrees of freedom which are not in the homology. The transferred action then can be interpreted as an effective action in the Batalin-Vilkovisky formalism. A review of necessary results from Batalin-Vilkovisky formalism and homotopy algebras is included as well.

Keywords: homotopy algebras, minimal models, homological perturbation theory, Batalin-Vilkovisky formalism, S-matrix

Contents

Introduction

In this thesis, we describe a connection between *BV formalism*, *homological perturbation lemma* and *loop homotopy algebras*. The Batalin-Vilkovisky formalism [1] is a powerful method in quantum field theory for quantizing gauge theories. A central identity to BV formalism is the *quantum master equation*

$$2\hbar\Delta S + \{S, S\} = 0$$

for an action S, which is a condition on the gauge invariance of the theory. Among other situations, a BV action was constructed for closed string field theory in [2] by Zwiebach.

It was already known that the tree level of a closed string field theory has a structure (its interaction vertices, to be precise) of a L_∞ algebra, an example of homotopy algebra. In the general case, the algebraic structure was named *loop homotopy Lie algebra* by Markl in [3]. The BV formalism, which can be defined for a loop homotopy algebra, then encodes its main axioms in the quantum master equation constructed from the algebra operations.

A homological perturbation lemma is a computational tool for transferring differentials along homotopy equivalences of chain complexes. For homotopy algebras, it was used for construction of *minimal models* (see references in [4]), new homotopy algebras defined on a homology of the original algebra.

One of the aims of this thesis was to repeat a similar transfer of algebra structure on the homology for a loop homotopy Lie algebra. However, because these algebras are intricately connected with physics, the homological perturbation lemma should have a physical counterpart. This turns out to be a construction of *effective action*, using the path integral to integrate out the degrees of freedom not in the homology. This gives a striking interpretation of the various formulas that appear in homological perturbation lemma – they are just summarizing the Feynman diagram expansion of path integral.

This physical interpretation of homological perturbation theory appears e.g. in works of P. Mnev [5], K. Costello and O. Gwilliam [6, 7] and H. Kajiura [4] and notably in a presentation C. Albert's [8] from Cargèse conference 2009.

Note to the reader: Apart from the construction we now described, we also tried to work out part of the underlying theory. This means that the thesis does not take the shortest path to the results in chapter 4. For a more streamlined reading, most of the first chapter and first two sections from the second chapter can be safely omitted.

The first chapter introduces (classical) homotopy algebras in the language of coalgebras. Part of the motivation for writing it was the aim to collect the useful results and explain some factors for the two most common homotopy algebras, A_∞ and L_∞. It is the most mathematically minded chapter of all.

For a physicist, the second chapter is a better place to start. Here, we discuss the physical origin and geometry behind BV formalism. The sections 2.4.1, 2.4.2 are important in establishing the formalism used in later chapters.

The third chapter uses the formalism of previous two chapter to motivate and define loop homotopy algebras. We also recast them into a convention useful to us.

In the last chapter, we recall the homological perturbation lemma and use previous results to build a special deformation retract out of a loop homotopy algebra. Applying the perturbation lemma, we obtain new differential on a homology. We prove that it indeed defines a new loop homotopy algebra, by directly analysing the transferred differential. We also present a heuristic argument connecting the transferred algebra structure with the effective action.

We conclude the thesis by some remarks of speculative character on the possible further work.

Notation

Notation

$\triangle$ and $\diamond$	denote the end of definition and example, respectively.
$V^{\#}$	is a linear dual of V, for graded vector spaces see appendix A. All vector spaces are assumed to be finite-dimensional, all graded vector spaces are degree-wise finite-dimensional.
d, Q	usually denote differentials. We will work in the cohomological convention, that is with differentials of degree $+1$.
$\mathbb{S}_n$	is the permutation group of n elements.
$\mathbb{1}$	is used to denote identity morphisms.
$\mathbb{k}$	is a field of characteristic 0, we always have real or complex numbers on mind.
$\circ, \circ_i$	$\circ$ is a composition of maps, $f \circ_j g$ is defined as $$f \circ \left(\mathbb{1}^{\otimes(j-1)} \otimes g \otimes \mathbb{1}^{\otimes(n-j)}\right)$$ for f with n arguments.
$\frac{\partial_R}{\partial \phi}, \frac{\partial_L}{\partial \phi}$	are the right and left derivative.
$a_i \phi^i$	means $\sum_i a_i \phi^i$, i.e. we use the Einstein summation convention.

Abbreviations

RHS and LHS	are short for *right hand side* and *left hand side.*
BV	stands for Batalin-Vilkovisky.
BRST	stands for Becchi, Rouet, Stora and Tyutin.

1. Homotopy algebras

In this chapter, we introduce A_∞ and L_∞ algebras together with the necessary machinery. The most direct way to define these *homotopy algebras* is to introduce multilinear operations $V^{\otimes i} \to V$ on a vector space V that generalize the operations from differential graded associative and Lie algebras. These operations have to satisfy some conditions, which will generalize the associativity and Jacobi identity.

These identities might look mysterious at first, but their origin becomes clear when we look at them through a classical constructions of algebra: the bar complex and the Chevalley-Eilenberg complex. We take this chance to work out some of the technical details involved in these constructions, e.g. the coalgebra coderivations.

The history of homotopy algebras goes back to 1963, when the A_∞ algebra was defined by J. D. Stasheff in his thesis [9, 10]. The term L_∞ algebra was defined in [11]. For more history and prehistory, see a recent lecture by Stasheff [12] and also books [13, part I] [14, section 13.2.17].

For a physicist, homotopy algebras appear in different areas, notably in the structure of closed string field theory. See the paper by Lada and Stasheff [11] and the nLab page [15] for more.

To choose among many applications in mathematics, let us mention the deformation quantization of Poisson manifolds by Kontsevich, which uses a L_∞ morphism between two Lie algebras [16] (see also the chapter 8 of lecture notes [17]).

This chapter follows three main sources: H. Kajiura's thesis and a paper with J. Stasheff [4, 18], whose notation we adopt, the book by Loday and Vallette [14], which is more comprehensive, but chooses different notation, and lecture notes for a course by M. Markl [17] taken by M. Doubek and P. Zima. The first few sections are intended to be mostly self-contained, later we will collect relevant results from the literature.

1.1 The associative and Lie algebras

An associative algebra on a vector space V is given by a linear product $\mu : V \otimes V \to V$ that is associative:

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \,.$$

Similarly, Lie algebra on $\mathfrak{g}$ is a linear, antisymmetric bracket $[\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \,.$$

One direct way to generalize these algebras is to replace vector spaces with graded vector spaces and to introduce a differential. For us, a good motivation for this comes from physics, where *fermionic fields* anticommute. We will take the vector spaces to be $\mathbb{Z}$-graded, see the appendix A and section 2.4.1 for more details.

Definition 1.1. A *dg associative algebra* on a graded vector space V is given by two linear maps:

- Multiplication, degree 0 map $\mu : V \otimes V \to V$ that is associative

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \,.$$

- Differential, degree $+1$ map $\mathrm{d} : V \to V$ that squares to zero and is a derivation of the multiplication

$$\mathrm{d}\mu(a, b) = \mu(\mathrm{d}a, b) + (-1)^{|a|}\mu(a, \mathrm{d}b) \,.$$

Here $a, b, c \in V$. △

Definition 1.2. A *dg Lie algebra* on a graded vector space $\mathfrak{g}$ is given by two linear maps:

- Bracket, degree 0 map $[\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that is graded antisymmetric
$$[a, b] = -(-1)^{|a||b|}[b, a]$$
and satisfies the graded Jacobi identity
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]] \,.$$
- Differential, degree $+1$ map $\mathrm{d} : \mathfrak{g} \to \mathfrak{g}$ that squares to zero and is a derivation of the bracket
$$\mathrm{d}[a, b] = [\mathrm{d}a, b] + (-1)^{|a|}[a, \mathrm{d}b] \,.$$

Here $a, b, c \in \mathfrak{g}$. △

Both of these definitions are, of course, reasonably simple. They can, however, be further simplified and put on a common ground. Omitting the details, we will consider the map $d + [\,]$ as a *coderivation* on a certain coalgebra, the three properties from the definition 1.2 are all encoded in a simple equation

$$(d + [\,])^2 = 0 \,.$$

Similar construction also works for associative algebra. Our goal for now is to make this construction explicit, which will allow for direct generalization:

Here, the coderivation includes two maps, taking one and two arguments. Taking a general coderivation will lead us to the notion of a homotopy algebra, and the square-zero property of coderivation will provide the defining axioms of this algebra.

1.2 Coalgebras

There are two meanings to the statement "coalgebras are dual to algebras". First, taking a linear dual $^{\#}$ of an algebra on V leads to a coassociative coalgebra on $V^{\#}$ with coproduct $\Delta : V^{\#} \to V^{\#} \otimes V^{\#}$ given by the dual of the product μ. [1]

The second meaning is a categorical one: the definition of a coalgebra can be obtained by "reversing all arrows" in the definition of an associative algebra. For example, the associativity of a product μ can be encoded in the commutativity of the following diagram:

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{\mu \otimes \mathbb{1}} & V \otimes V \\ {\scriptstyle \mathbb{1} \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ V \otimes V & \xrightarrow{\quad \mu \quad} & V \end{array}$$

[1] This statement only holds for (degree-wise) finite-dimensional V. For infinite dimensional V, $V^{\#} \otimes V^{\#}$ is not isomorphic to $(V \otimes V)^{\#}$, there is only inclusion $i : V^{\#} \otimes V^{\#} \hookrightarrow (V \otimes V)^{\#}$ given by

$$i(\varphi \otimes \psi)(a \otimes b) = (-1)^{|a||\psi|}\varphi(a)\psi(b) \,,$$

where $\varphi, \psi \in V^{\#}$, $a, b \in V$. Therefore, for a coalgebra on C with coproduct $\Delta : C \to C \otimes C$, we can define product $\mu_C : C^{\#} \otimes C^{\#} \to C^{\#}$ by composition $\mu_C = \Delta^{\#} \circ i$. It can be shown that this product inherits the associativity from the coassociativity of Δ, see e. g. [14, section 1.2.2].

Therefore, the coassociativity of a coproduct Δ will come from the diagram:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathbb{1}} \\ C\otimes C & \xrightarrow{\mathbb{1}\otimes\Delta} & C\otimes C\otimes C \end{array}$$

An unit of an associative algebra is an element of the vector space $1_V \in V$ satisfying $\mu(1_V, a) = a = \mu(a, 1_V)$. This can be equivalently taken to be a linear map from the field $\Bbbk$ to the vector space, sending the unit of the field to the unit of the algebra: $\varepsilon(1_{\Bbbk}) = 1_V$. Again, we can draw a diagram

$$\begin{array}{ccccc} \Bbbk\otimes V & \xrightarrow{\varepsilon\otimes\mathbb{1}} & V\otimes V & \xleftarrow{\mathbb{1}\otimes\varepsilon} & V\otimes\Bbbk \\ & \searrow^{\sim} & \downarrow{\scriptstyle\mu} & {}^{\sim}\swarrow & \\ & & V & & \end{array}$$

and reverse all the arrows. A counit is then only a map $\eta : C \to \Bbbk$ such that following diagram commutes

$$\begin{array}{ccccc} & & C & & \\ & {}^{\sim}\swarrow & \downarrow{\scriptstyle\Delta} & \searrow^{\sim} & \\ \Bbbk\otimes C & \xleftarrow[\eta\otimes\mathbb{1}]{} & C\otimes C & \xrightarrow[\mathbb{1}\otimes\eta]{} & C\otimes\Bbbk \end{array}$$

In the following definition, we now collect these axioms for coalgebras and define a few more properties we will need. See also section 1.2 of [14].

Definition 1.3. *Coalgebra* on a graded vector space C is given by a linear, degree 0 map $\Delta : C \to C \otimes C$ called *coproduct*, which is coassociative

$$(\Delta\otimes\mathbb{1})\circ\Delta = (\mathbb{1}\otimes\Delta)\circ\Delta\,.$$

Coalgebra is called *counital* if it has a degree 0 counit $\eta : C \to \Bbbk$ such that

$$(\eta\otimes\mathbb{1})\circ\Delta = \mathbb{1} = (\mathbb{1}\otimes\eta)\circ\Delta\,,$$

where the isomorphisms $\Bbbk\otimes C \sim C \sim C\otimes\Bbbk$ are omitted.

Coalgebra is called *cocommutative* if $\Delta = \tau\circ\Delta$, where τ is a *switching map* for the graded vector spaces, given by $\tau(a\otimes b) = (-1)^{|a||b|} b\otimes a$.

A *morphism of coalgebras* is a degree 0 map of graded vector spaces commuting with the coproduct: if we have coalgebras (C, Δ) and (C', Δ'), the morphism is a map $F : C \to C'$ satisfying

$$(F\otimes F)\circ\Delta = \Delta'\circ F\,,$$

which comes from the diagram for morphism of associative algebras $f : (V, \mu) \to (V', \mu')$

$$\begin{array}{ccc} V\otimes V & \xrightarrow{\mu} & V \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ V'\otimes V' & \xrightarrow{\mu'} & V' \end{array} \qquad \text{reverse} \to \qquad \begin{array}{ccc} C & \xrightarrow{F} & C' \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta'} \\ C\otimes C & \xrightarrow{F\otimes F} & C'\otimes C' \end{array}$$

For counital coalgebras with counits η and η' respectively, we also want

$$\eta'\circ F = \eta\,.$$

Finally, a *coderivation* m of degree k of a coalgebra (C, Δ) is a degree k linear map $m : C \to C$ satisfying the dual version of the Leibniz rule

$$\Delta\circ m = (m\otimes\mathbb{1} + \mathbb{1}\otimes m)\circ\Delta\,.$$

△

Remark: Note that the definition of coalgebra was motivated by considering the dual of associative algebra, but this isn't the coalgebra we will associate with associative algebras. Instead, we need to consider a tensor coalgebra TV and its coderivations.

We will also make us of the following lemma, which serves as a nice illustration of the notions we just defined.

Lemma 1.4. *A coderivation m of counital coalgebra (C, Δ, η) satisfies*

$$\eta \circ m = 0\,.$$

Proof. Again, without writing the isomorphism $\Bbbk \otimes C \sim C$, we can write

$$\begin{aligned} m &= (\eta \otimes \mathbb{1}) \circ \Delta \circ m \\ &= (\eta \otimes \mathbb{1}) \circ (m \otimes \mathbb{1} + \mathbb{1} \otimes m) \circ \Delta \\ &= [(\eta \circ m) \otimes \mathbb{1}] \circ \Delta + (\mathbb{1} \otimes m) \circ (\eta \otimes \mathbb{1}) \circ \Delta \\ &= [(\eta \circ m) \otimes \mathbb{1}] \circ \Delta + m\,, \end{aligned}$$

i.e. we get

$$[(\eta \circ m) \otimes \mathbb{1}] \circ \Delta = 0\,.$$

Now we act with $\mathbb{1} \otimes \eta$ on this equation, which gives us

$$\begin{aligned} 0 &= (\mathbb{1} \otimes \eta) \circ [(\eta \circ m) \otimes \mathbb{1}] \circ \Delta \\ &= (\mathbb{1} \otimes \eta) \circ (\eta \otimes \mathbb{1}) \circ (m \otimes \mathbb{1}) \circ \Delta \\ &= (\eta \otimes \mathbb{1}) \circ (m \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \eta) \circ \Delta \\ &= \eta \circ m\,. \end{aligned}$$

□

1.2.1 Cofree coalgebras

In this section, we introduce the the coalgebras TV and $\operatorname{Sym} V$, to describe associative and Lie algebras. We start by giving their explicit definition.

However, these tensor coalgebras are also characterized by an universal property, i.e. they are *cofree coalgebras* in some category: more on this in the next subsection. Note that although this universal property help us to get explicit formulas for morphisms and coderivations, this language is not vital to our application.

Definition 1.5. For a graded vector space V, we will define coproducts on two vector spaces

$$TV_{\geq 1} \equiv \bigoplus_{k \geq 1} V^{\otimes k}\,, \quad TV \equiv \Bbbk \oplus TV_{\geq 1}\,.$$

The coproduct on $TV_{\geq 1}$ goes as $TV_{\geq 1} \to TV_{\geq 1} \otimes TV_{\geq 1}$, so conceptually there are two different tensor products $\otimes$. For clarity, we will sometimes denote the tensor product $\tilde{\otimes}$ when writing elements of $TV_{\geq 1}$ and TV, i.e. $(v_1 \tilde{\otimes} v_2) \otimes v_3 \in TV \otimes TV$.

We call the coproduct on $TV_{\geq 1}$ the *reduced tensor coproduct* and denote it as $\bar{\Delta} : TV_{\geq 1} \to TV_{\geq 1} \otimes TV_{\geq 1}$. For $v_i \in V$, it is defined as

$$\bar{\Delta}(v_1 \tilde{\otimes} v_2 \tilde{\otimes} \ldots \tilde{\otimes} v_n) \equiv \sum_{k=1}^{n-1} (v_1 \tilde{\otimes} \ldots \tilde{\otimes} v_k) \otimes (v_{k+1} \tilde{\otimes} \ldots \tilde{\otimes} v_n)\,.$$

The *tensor coproduct* $\Delta : TV \to TV \otimes TV$ is defined as

$$\Delta(V) \equiv 1_{\Bbbk} \otimes V + \bar{\Delta}(V) + V \otimes 1_{\Bbbk} \text{ for } V \in TV_{\geq 1} \text{ and } \Delta(1_{\Bbbk}) \equiv 1_{\Bbbk} \otimes 1_{\Bbbk}\,. \tag{1.1}$$

The unit for this coproduct is $\eta : TV \to \Bbbk$, the projection on the $\Bbbk$ component of TV. The counital coalgebra (TV, Δ, η) is called the *tensor coalgebra*, the coalgebra $(TV_{\geq 1}, \bar{\Delta})$ is called the *reduced tensor coalgebra*

The coassociativity of these coproducts follows from the associativity of the tensor product $\otimes$.

Note that the equation 1.1 can be concisely written using the projection η as

$$\Delta(V) = 1_{\Bbbk} \otimes V + \bar{\Delta}(V) + V \otimes 1_{\Bbbk} - \eta(V)\, 1_{\Bbbk} \otimes 1_{\Bbbk} \,. \tag{1.2}$$

△

Other important construction is obtained by symmetrizing all the tensor products. Note that we will take $\mathrm{Sym}^n(V)$ to be the space of $\mathbb{S}_n$ invariants, i.e. $\mathrm{Sym}^n(V) \in V^{\otimes n}$. See the appendix A for more details.

Definition 1.6. For a vector space V, define two vector spaces

$$\mathrm{Sym}\, V_{\geq 1} \equiv \bigoplus_{k \geq 1} \mathrm{Sym}^k(V)\,, \quad \mathrm{Sym}\, V \equiv \Bbbk \oplus \mathrm{Sym}\, V_{\geq 1}\,.$$

The *reduced coproduct* on $\mathrm{Sym}_{\geq 1}$ is defined as

$$\bar{\Delta}(v_1 \odot \cdots \odot v_n) \equiv \sum_{i=1}^{n-1} \frac{1}{i!(n-i)!} \sum_{\sigma \in \mathbb{S}_n} \epsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)})\,.$$

Here, $\epsilon(\sigma)$ is the Koszul sign given by permuting the graded elements v_i. We used the right $\mathbb{S}$ action only for brevity, the result is the same as if we used σ^{-1}. The combinatorial factor accounts for the permutations permuting symmetric elements: the first i and last $n-i$ symmetric tensors are already symmetric. Using *unshuffles* (see A), this can be written as

$$\bar{\Delta}(v_1 \odot \cdots \odot v_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathrm{Unsh}(i, n-i)} \epsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)})\,.$$

We can define $\Delta(V)$ as for TV, i. e.

$$\Delta(V) \equiv 1_{\Bbbk} \otimes V + \bar{\Delta}(V) + V \otimes 1_{\Bbbk} \text{ for } V \in \mathrm{Sym}\, V_{\geq 1} \text{ and } \Delta(1_{\Bbbk}) \equiv 1_{\Bbbk} \otimes 1_{\Bbbk} \tag{1.3}$$

and the counit as a projection on $\Bbbk$. Also the condensed formula for coproduct 1.2 holds.

Together, we have two coalgebras, which are now cocommutative: the symmetric tensor coalgebra $(\mathrm{Sym}\, V, \Delta, \eta)$, which is a cocommutative counital coalgebra, and the reduced tensor coalgebra $(\mathrm{Sym}\, V_{\geq 1}, \bar{\Delta})$, which is a cocommutative coalgebra. △

1.2.2 Morphisms of cofree coalgebras

The tensor coalgebra TV we just constructed are *cofree* in the subcategory category of *conilpotent* coalgebras, meaning any coalgebra morphism from conilpotent coalgebra C to TV is uniquely determined by the vector space morphism $C \to V$. Similarly, $\mathrm{Sym}\, V$ is cofree in the subcategory of cocommutative conilpotent coalgebras.

Categorically, this means that the *cofree coalgebra* is a functor from the category of graded vector spaces to the category of conilpotent coalgebras which is *right adjoint* to the forgetfull functor that takes a coalgebra to its underlying graded vector space (see [13, section II.3.7]).

Definition 1.7. Counital coalgebra (C, Δ, η) is called *coaugmented* if there is an element $1 \in C$ such $\Delta 1 = 1 \otimes 1$ and $\eta(1) = 1_{\Bbbk}$. For coaugmented coalgebra, we can write $C = \bar{C} \oplus \Bbbk 1$, see [14, section 1.2.1].

Coaugmented coalgebra is called *conilpotent* [14] or sometimes connected [19] if $C = \bigcup_{r=0}^{\infty} F_r C$ where $F_r C$ is a *conilpotent filtration* defined as

$$
\begin{aligned}
&F_0 C \equiv \Bbbk 1\,,\\
&F_r C \equiv \{x \in C \mid \bar{\Delta} x \equiv \Delta x - x \otimes 1 - 1 \otimes x \in F_{r-1} C \otimes F_{r-1} C\}\,.
\end{aligned}
$$

$\triangle$

Both TV and $\operatorname{Sym} V$ are conilpotent, $1_{\Bbbk}$ being the coaugmentation. The filtration $F_r TV$ or $F_r \operatorname{Sym} V$ is given by all tensors with weight r or less. These cofree coalgebras are now specified by this universal condition on their morphisms:

Lemma 1.8. *For (C, Δ) a conilpotent coalgebra, any morphism of graded vector spaces $\varphi : C \to V$ such that $\varphi(1) = 0$ extends to a coalgebra morphism $\tilde{\varphi} : C \to TV$ satisfying $\varphi = \operatorname{proj}_V \circ \tilde{\varphi}$.*

If C is a cocommutative conilpotent coalgebra, then any morphism of graded vector spaces $\psi : C \to V$ such that $\psi(1) = 0$ extends to cocommutative coalgebra morphism $\tilde{\psi} : C \to \operatorname{Sym} V$ satisfying $\psi = \operatorname{proj}_V \circ \tilde{\psi}$. We also write this in diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{\exists! \tilde{\varphi}} & TV \\
& {\scriptstyle \forall \varphi} \searrow & \big\downarrow {\scriptstyle \operatorname{proj}_V} \\
& & V
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & \xrightarrow{\exists! \tilde{\psi}} & \operatorname{Sym} V \\
& {\scriptstyle \forall \psi} \searrow & \big\downarrow {\scriptstyle \operatorname{proj}_V} \\
& & V
\end{array}
$$

Proof. See e.g. [19, appendix B, prop 4.1] for proof. For us, the explicit isomorphism from graded vector space morphism to coalgebra morphism is interesting, so we give the formulas. See the example 1.9 to see how one arrives to such formulas.

- In the TV case, the coalgebra morphism is given by

$$\tilde{\varphi}(1) = 1_{\Bbbk} \in TV$$

  and

$$\tilde{\varphi}(x) = \operatorname{concat}\left(\sum_{n=1}^{\infty} \varphi^{\otimes n} \Delta^{n-1}(x)\right),\ x \in \bar{C}\,.$$

  Here Δ^0 is the identity on C and Δ^n is any of the equal (co)bracketings of coproducts on C, e.g. $\Delta^{n+1} = (\Delta \otimes \mathbb{1}^{\otimes n}) \circ \Delta^n$. The concatenation replaces all the tensor products $\otimes$ with the tensor products $\tilde{\otimes}$ in TV. The sum is guaranteed to end, owing to the conilpotency of the coalgebra and to the condition $\varphi(1) = 0$.

  This formula also appears in [14, proposition 1.2.7].

- For symmetric coalgebras, $\Delta^n(x)$ is symmetric, and so is the result after acting with $\psi^{\otimes n}$ and concatenation. This symmetric tensor then can be projected on $\operatorname{Sym} V$. Again,

$$\tilde{\psi}(1) = 1_{\Bbbk} \in \operatorname{Sym} V$$

and[2]

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \frac{1}{n!}\boldsymbol{\sigma} \circ \mathrm{concat}\left(\psi^{\otimes n}\Delta^{n-1}(x)\right), \; x \in \bar{C}\,.$$

Note that we can write the result using the symmetric product $\odot$ in $\mathrm{Sym}\, V$, more on this in a moment.

Note that since $\varphi(1) = 0$, we can use just $\bar{\Delta}$ in these formulas. □

For us, the only important example of this lemma is $C = TU$ for some graded vector space U. Then we can decompose the morphism $\varphi : TU \to V$ to morphisms $\varphi_n = \varphi \circ \mathrm{proj}_{U^{\otimes n}} : U^{\otimes n} \to V$ and write the induced coalgebra morphism by one of the so-called *rake formulas*[3]

$$\tilde{\varphi}(u_1 \tilde{\otimes} \dots \tilde{\otimes} u_n) = \sum_{k=1}^{n} \sum_{r_1+\dots+r_k=n} \varphi_{r_1}(u_1, \dots, u_{r_1}) \tilde{\otimes} \dots \tilde{\otimes} \varphi_{r_k}(u_{r_1+\dots+r_{k-1}+1}, \dots, u_n)\,. \tag{1.4}$$

Here, we take $r_i \geq 1$ and φ_0 is zero because of the condition $\varphi(1) = 0$.

For the symmetric tensor coalgebra morphism $\psi : \mathrm{Sym}\, U \to V$, we can write the symmetric tensor $\tilde{\psi}(x)$ using the symmetric product $\odot$. The explicit formula is (see [20, proposition A.2])

$$\begin{aligned} \tilde{\psi}(u_1 \odot \dots \odot u_n) = &\sum_{k=1}^{n} \sum_{r_1+\dots+r_k=n} \sum_{\sigma \in \mathrm{Unsh}(r_1,\dots,r_k)} \\ &\frac{\epsilon(\sigma)}{k!} \psi_{r_1}(u_{\sigma(1)}, \dots, u_{\sigma(r_1)}) \odot \dots \odot \psi_{r_k}(u_{\sigma(r_1+\dots+r_{k-1}+1)}, \dots, u_{\sigma(n)})\,. \end{aligned} \tag{1.5}$$

Again, ψ_0 is taken to be 0. By $\psi_r(u_1, \dots, u_r)$ we mean $\psi_r(u_1 \odot \dots \odot u_r)$.

Note the combinatorial factor that appears here; it accounts for overcounting when summing over the unshuffles and the partitions r_i. For example, if $r_i \neq r_j$, then two partitions $\dots, r_i, \dots, r_j, \dots$ and $\dots, r_j, \dots, r_i, \dots$ give the same contribution. If $r_i = r_j$, then the overcounting is due to unshuffles: exchanging the vectors from ψ_{r_i} and ψ_{r_j} gives a different unshuffle, but the same contribution to the sum. Note that this factor is independent from the representation of symmetric powers (i.e. choice of inclusion $\mathrm{Sym}\, V \to TV$).

Remark: The morphism $TU \to TV$ uniquely specifies a morphism $TU_{\geq 1} \to TV_{\geq 1}$, which is given by restriction. This is also true for symmetric tensor coalgebras.

Example 1.9. In this example, we sketch a procedure for obtaining morphisms of cofree coalgebras from their projections. Let's start with a morphism $\varphi : TU \to V$, which we want to extend to a coalgebra morphism $\tilde{\varphi} : TU \to TV$. To illustrate the importance of the conilpotency condition, we will place no further conditions on the morphism φ or $\tilde{\varphi}$ apart from

$$\Delta_{TV} \circ \tilde{\varphi} = (\tilde{\varphi} \otimes \tilde{\varphi}) \circ \Delta_{TU}\,, \tag{1.6}$$

[2] Here, an additional factor $1/n!$ is there because after concat, without the factor $1/n!$ we used in the definition of $\odot$. For example,

$$\bar{\Delta}^2(v_1 \odot v_2 \odot v_3) = v_1 \otimes v_2 \otimes v_3 + \text{ other permutations of (123)}$$

for even v_i. After applying $\psi^{\otimes^3}$, we are left with $3!\,\psi(v_1) \odot \psi(v_2) \odot \psi(v_3)$.

[3] This is a mnemonic for remembering the formula: φ_n corresponds to rake with n spikes, which pair with n vectors. All the vectors have to be covered with all possible rake sizes. We learned this terminology from Martin Doubek.

specifically it doesn't have to commute with counits.

Now we repeatedly use this equation to determine the morphism $\tilde{\varphi}$. We choose an ansatz

$$\tilde{\varphi}(1_{\mathbb{k}}) = \alpha 1_{\mathbb{k}} + \Phi_0 \,,$$

where $\Phi_0 \in TV$ contains $\varphi_0(1_{\mathbb{k}})$ and then only terms of weight at least 2, since we know that the projection on V is equal to φ_0. Equation 1.6 gives us

$$\Delta_{TV}(\alpha 1_{\mathbb{k}} + \Phi_0) = (\alpha 1_{\mathbb{k}} + \Phi_0) \otimes (\alpha 1_{\mathbb{k}} + \Phi_0) \,,$$

or

$$\alpha 1_{\mathbb{k}} \otimes 1_{\mathbb{k}} + 1_{\mathbb{k}} \otimes \Phi_0 + \Phi_0 \otimes 1_{\mathbb{k}} + \bar{\Delta}_{TV}\Phi_0 = \alpha^2 1_{\mathbb{k}} \otimes 1_{\mathbb{k}} + \alpha 1_{\mathbb{k}} \otimes \Phi_0 + \alpha \Phi_0 \otimes 1_{\mathbb{k}} + \Phi_0 \otimes \Phi_0 \,.$$

We see that $\alpha = 1_{\mathbb{k}}$ or $\alpha = 0_{\mathbb{k}}$ (but we will see that $\alpha = 0_{\mathbb{k}}$ corresponds to zero morphism). In both cases, we have

$$\bar{\Delta}_{TV}\Phi_0 = \Phi_0 \otimes \Phi_0 \,.$$

One solution is $\Phi_0 = 0$. We can also solve this equation perturbatively. If we have $\varphi_0 = \mathrm{proj}_V(\Phi_0)$, then in the order $V \otimes V$ we have

$$\mathrm{proj}_{V\otimes V}\, \bar{\Delta}_{TV}\Phi_0 = \varphi_0 \otimes \varphi_0 \,,$$

i.e. $\Phi_0 = \varphi_0 + \varphi_0 \tilde{\otimes} \varphi_0 + \ldots$. Iterating, we get

$$\Phi_0 = \exp(\varphi_0) - 1_{\mathbb{k}} \equiv \sum_{k=1}^{\infty} \varphi_0^{\tilde{\otimes} k}, \quad \text{or} \quad \tilde{\varphi}(1_{\mathbb{k}}) = \exp(\varphi_0) \,.$$

Of course, this is not an element of TV, but rather its completion. This is another reason why the cofree object in the category of coalgebras is a subspace of completed tensor algebra. [4] If we work only with TV, the only viable option is $\Phi_0 = 0$.

Now we look at $\tilde{\varphi}(u)$ with $u \in U$. Again, we choose an ansatz

$$\tilde{\varphi}(u) = \beta 1_{\mathbb{k}} + \Phi_1 \,,$$

where Φ_1 starts with $\varphi_1(u)$. The equation 1.6 gives us

$$\begin{aligned}
&\beta 1_{\mathbb{k}} \otimes 1_{\mathbb{k}} + 1_{\mathbb{k}} \otimes \Phi_1 + \Phi_1 \otimes 1_{\mathbb{k}} + \bar{\Delta}_{TV}\Phi_1 \\
&= (\alpha 1_{\mathbb{k}} + \Phi_0) \otimes (\beta 1_{\mathbb{k}} + \Phi_1) + (\beta 1_{\mathbb{k}} + \Phi_1) \otimes (\alpha 1_{\mathbb{k}} + \Phi_0) \\
&= 2\alpha\beta 1_{\mathbb{k}} \otimes 1_{\mathbb{k}} + \alpha 1_{\mathbb{k}} \otimes \Phi_1 + \alpha \Phi_1 \otimes 1_{\mathbb{k}} + \beta 1_{\mathbb{k}} \otimes \Phi_0 + \beta \Phi_0 \otimes 1_{\mathbb{k}} + \Phi_1 \otimes \Phi_0 + \Phi_0 \otimes \Phi_1 \,.
\end{aligned}$$

We see that for $\alpha = 0_{\mathbb{k}}$ we also need $\beta = 0_{\mathbb{k}}$ and $\Phi_1 = 0$, which corresponds to the zero morphism. For $\alpha = 1_{\mathbb{k}}$, we also need $\beta = 0_{\mathbb{k}}$ and we have a condition

$$\bar{\Delta}_{TV}\Phi_1 = \Phi_1 \otimes \Phi_0 + \Phi_0 \otimes \Phi_1 \,,$$

which can again be recursively solved and gives

$$\Phi_1 = \exp(\varphi_0) \tilde{\otimes} \varphi_1(u) \tilde{\otimes} \exp(\varphi_0) \,.$$

[4] Note that H. Kajiura [4, equation 2.5] gives the formula for cofree coalgebra morphism *with* this exponential factor. However, there is no exp() term after the last morphism component, which, we believe, is an omission.

Similar procedure can be done for symmetric tensor coalgebras. If we use the assumption $\eta_{\mathrm{Sym}\,V} \circ \tilde{\psi} = \eta_{\mathrm{Sym}\,U}$ from beginning, we can omit the terms like $\beta 1_{\Bbbk}$ from the ansatz. Then, for example for two even vectors $u_1 \odot u_2$, we set

$$\tilde{\psi}(u_1 \odot u_2) = \psi_2(u_1, u_2) + \Psi_2$$

and get from equation 1.6

$$\begin{aligned} 1_{\Bbbk} \otimes \psi_2(u_1, u_2) + \psi_2(u_1, u_2) \otimes 1_{\Bbbk} + 1_{\Bbbk} \otimes \Psi_2 + \Psi_2 \otimes 1_{\Bbbk} + \bar{\Delta}_{\mathrm{Sym}\,V} \Psi_2 = \\ \psi_2(u_1, u_2) \otimes 1_{\Bbbk} + \Psi_2 \otimes 1_{\Bbbk} + 1_{\Bbbk} \otimes \psi_2(u_1, u_2) + 1_{\Bbbk} \otimes \Psi_2 \\ + \psi_1(u_1) \otimes \psi_1(u_2) + \psi_1(u_2) \otimes \psi_1(u_1) \,. \end{aligned}$$

This reduces to

$$\bar{\Delta}_{\mathrm{Sym}\,V} \Psi_2 = \psi_1(u_1) \otimes \psi_1(u_2) + \psi_1(u_2) \otimes \psi_1(u_1) \,,$$

or

$$\tilde{\psi}(u_1, u_2) = \psi_2(u_1, u_2) + \psi_1(u_1) \odot \psi_1(u_2) \,.$$

This illustrates the formula 1.5 and the combinatorial factor, which would counter the sum over two unshuffles in $\mathrm{Unsh}(1,1)$, the identity and the switch permutation. $\diamond$

1.2.3 Coderivation of cofree coalgebras

Now we turn our attention to the coderivation. For a tensor algebra TV, a derivation is completely specified by its action on V, the Leibniz rule tells us how to extend the derivation on higher products. A dual statements is also true: a coderivation is completely specified only by the V component of its result. This holds for symmetric tensor coalgebras, too.

We will denote the space of all coderivation of coalgebra C of degree r by $\mathrm{CoDer}^r(C)$.

Lemma 1.10. *Coderivations of degree r on TV are in 1-1 correspondence with degree r linear maps $TV_{\geq 1} \to V$*

$$\mathrm{CoDer}^r(TV) \cong \mathrm{Lin}^r(TV, V) \,.$$

Coderivations of degree r on $\mathrm{Sym}(V)$ *are in 1-1 correspondence with degree r linear maps* $\mathrm{Sym}\,V_{\geq 1} \to V$

$$\mathrm{CoDer}^r(\mathrm{Sym}\,V) \cong \mathrm{Lin}^r(\mathrm{Sym}\,V, V) \,.$$

Proof. We refer to [19, corollary 4.4] for a proof. Here D. Quillen proves this claim by describing coderivations on C as coalgebra morphisms $C \oplus C \to C$ that are equal to identity on the second summand.

We are, however, interested in explicit formulas (which can also be checked explicitly to establish this lemma). For the TV case, see the lecture notes [17, proposition 4.19], but it has a formula only for coderivations without the constant term. See also [14, proposition 1.2.9] and [4, section 2.1][5]. The $\mathrm{Sym}\,V$ case discussed in [21, lemma 2.4], for formulas see e.g. [20, theorem A.3].

The direction of the isomorphism from coderivations to maps of vector spaces is given just by projection $m \mapsto \mathrm{proj}_V \circ m$. In the other direction, the isomorphism is given by another *rake formula*:

[5] Here, the formula is meant to work also for m_0, but the sum in the formula for $\mathfrak{m}_k$ should go to $n-k+1$ in their notation. This can be easily seen from cases like $k=1$ and $n=1$: the sum would be empty for $n-k$.

- For the tensor coalgebra TV, a map $m : TV \to V$ defines a coderivation $\tilde{m}$ by

$$\tilde{m} = \text{concat} \circ \left(\sum_{n=0}^{\infty} \sum_{i=0}^{n} \text{proj}_V^{\otimes i} \otimes m \otimes \text{proj}_V^{\otimes(n-i)} \right) \circ \Delta^n \,. \tag{1.7}$$

  This can be expanded using the components of m defined as

$$m_s \equiv m \circ \text{proj}_{V^{\otimes s}} \,.$$

  Then we can write the coderivation $\tilde{m}$ as

$$\tilde{m} \circ \text{proj}_{V^{\otimes n}} = \sum_{s=0}^{n} \sum_{i=1}^{n-s+1} \mathbb{1}^{\otimes i} \otimes m_s \otimes \mathbb{1}^{\otimes(n-s-i)} \,, \tag{1.8}$$

  where, for $s = 0$, we use the isomorphism inserting $\Bbbk$ into $V^{\otimes n}$ product. Acting on vectors, this is

$$\tilde{m}(v_1, \dots, v_n) = \sum_{s \geq 1} \sum_{i=0}^{n-s} (-1)^{|m|(|v_1|+\dots+|v_i|)} v_1 \tilde{\otimes} \dots \tilde{\otimes} v_i \tilde{\otimes} m_s(v_{i+1}, \dots, v_{i+s}) \tilde{\otimes} v_{i+s+1} \tilde{\otimes} \dots \tilde{\otimes} v_n \,. \tag{1.9}$$

- For the symmetric tensor coalgebra $\text{Sym}\, V$, a map $l : \text{Sym}\, V \to V$ defines a coderivation $\tilde{l}$ by the same formula as for the TV case, if we view $\text{Sym}\, V$ as a subspace[6] of TV

$$\tilde{l} = \sum_{n=0}^{\infty} \frac{1}{n!} \text{concat} \circ \left(\sum_{i=0}^{n} \text{proj}_V^{\otimes i} \otimes l \otimes \text{proj}_V^{\otimes(n-i)} \right) \circ \Delta^n \,. \tag{1.10}$$

  If we rewrite this to use the symmetric product, then we sum only over the unshuffles

$$\tilde{l}(v_1, \dots, v_n) = \sum_{s=0}^{n} \sum_{\sigma \in \text{Unsh}(s, n-s)} \epsilon(\sigma) l_s(v_{\sigma(1)}, \dots, v_{\sigma(s)}) \odot v_{\sigma(s+1)} \odot \dots \odot v_{\sigma(n)} \,.$$

  Recall that $l_r(v_1, \dots, v_r)$ is just a shorthand for $l_r(v_1 \odot \dots \odot v_r)$.

□

Remark: Note that the formula for coderivations with nonzero constant component m_0 or l_0 follow from the case usually discussed in the sources. This is thanks to the linearity of the isomorphism, adding a constant term $\Bbbk \to V$ to a linear map $TV_{\geq} \to V$ also specifies a coderivation. An explicit form of the isomorphism acting on maps $\Bbbk \to V$ can be easily found, see the following example. This remark was inspired by the MathOverflow discussion [22].

Example 1.11. We already know that $\text{proj}_{\Bbbk} \circ \tilde{m}(x)$ is zero for any $x \in TV$, see lemma 1.4.

Also, a constant component $m_0(1_{\Bbbk})$ is just a vector of V, because the dual Leibniz rule gives us

$$\Delta \tilde{m}(1_{\Bbbk}) = \tilde{m}(1_{\Bbbk}) \otimes 1_{\Bbbk} + 1_{\Bbbk} \otimes \tilde{m}(1_{\Bbbk}) \,,$$

which can only be true for $\tilde{m}(1_{\Bbbk}) \in V$. We will denote this vector simply by m_0.

[6] Again, there is a factor $1/n!$ as for morphisms.

Let us now look at $\tilde{m}(v)$, where $v \in V$. Here we get

$$\begin{aligned}\Delta\tilde{m}(v) &= (\tilde{m}\otimes\mathbb{1}+\mathbb{1}\otimes\tilde{m})(1_{\mathbb{k}}\otimes v + v\otimes 1_{\mathbb{k}})\\ &= m_0\otimes v + \tilde{m}(v)\otimes 1_{\mathbb{k}} + 1_{\mathbb{k}}\otimes\tilde{m}(v) + (-1)^{|v||m|} v\otimes m_0\,,\end{aligned}$$

or

$$\bar{\Delta}\tilde{m}(v) = m_0\otimes v + (-1)^{|v||m|} v\otimes m_0\,.$$

Thus, since we know the V component of TV to be $m_1(v)$, we get

$$\tilde{m}(v) = m_1(v) + m_0\tilde{\otimes}v + (-1)^{|v||m|}v\tilde{\otimes}m_0\,.$$

A similar calculation for $\tilde{m}(v_1, v_2)$ gives

$$\begin{aligned}\tilde{m}(v_1,v_2) = m_2(v_1,v_2) + m_1(v_1)\tilde{\otimes}v_2 + (-1)^{|v_1||m|}v_1\tilde{\otimes}m_1(v_2)\\ +m_0\tilde{\otimes}v_1\tilde{\otimes}v_2 + (-1)^{|v_1||m|}v_1\tilde{\otimes}m_0\tilde{\otimes}v_2 + (-1)^{(|v_1|+|v_2|)|m|}v_1\tilde{\otimes}v_2\tilde{\otimes}m_0\,.\end{aligned}$$

◇

Following simple lemmas are useful to keep in mind for the next section, where we study codifferentials.

Lemma 1.12. *If m is a coderivation of odd degree k on a coalgebra C, $m\circ m$ is a graded coderivation of degree $2k$ on C.*

Proof. We only need to check the Leibniz rule for m^2:

$$\begin{aligned}\Delta\circ m\circ m &= (m\otimes\mathbb{1}+\mathbb{1}\otimes m)\circ\Delta\circ m = (m\otimes\mathbb{1}+\mathbb{1}\otimes m)\circ(m\otimes\mathbb{1}+\mathbb{1}\otimes m)\circ\Delta\\ &= (m^2\otimes\mathbb{1} + m\otimes m + (-1)^{|m|^2} m\otimes m + \mathbb{1}\otimes m^2)\circ\Delta\,,\end{aligned}$$

which give the Leibniz rule for m^2 for odd $|m|$. □

Since m^2 is also a coderivation, it is completely determined by its projection on V, and furthermore if m has only first r nonzero components $m_0, \dots, m_r$, then m^2 will have nonzero components only up to $2r-1$.

Lemma 1.13. *If m is a coderivation of odd degree such that $m_i = 0$ for $i > r$, then the components $(m^2)_i \equiv \mathrm{proj}_V \circ m^2 \circ \mathrm{proj}_{V^{\otimes i}}$ are zero for $i > 2r-1$.*

Proof. This is immediate by using the explicit formulas for coderivations and projecting on V: first coderivation decreases the weight of a tensor by at most $r-1$, and the second coderivation needs a weight at most r to have nonzero projection on V. □

1.3 Algebras as coderivations

Now we utilize the cofree coalgebras to describe associative and Lie algebras.

Example 1.14. For an associative algebra (V, μ) on *non-graded* vector space V, its *bar complex* is defined as (see e.g. [14, section 2.2])

$$V^{\otimes(k+1)} \xrightarrow{\ \mathrm{d}\ } V^{\otimes k} \xrightarrow{\ \mathrm{d}\ } V^{\otimes(k-1)} \qquad \dots \qquad V \xrightarrow{\ \mathrm{d}\ } \mathbb{k}\,.$$

Where $\mathrm{d} : V^{\otimes k+1} \to V^{\otimes k}$ is defined as

$$\mathrm{d}(v_0\otimes\dots v_k) = \sum_{i=0}^{k-1}(-1)^i v_0\otimes\cdots\otimes v_{i-1}\otimes\mu(v_i,v_{i+1})\otimes v_{i+2}\otimes\cdots\otimes v_k \qquad (1.11)$$

and $\mathrm{d} = 0$ on V.

This indeed is a differential, i.e. $\mathrm{d}^2 = 0$. Let us illustrate it for $\mathrm{d} : V^{\otimes 4} \to V^{\otimes 3}$

$$\begin{aligned}\mathrm{d}^2[v_0 \otimes v_1 \otimes v_2 \otimes v_3] &= \mathrm{d}[\mu(v_0, v_1) \otimes v_2 \otimes v_3 - v_0 \otimes \mu(v_1, v_2) \otimes v_3 + v_0 \otimes v_1 \otimes \mu(v_2, v_3)] \\ &= \mu(\mu(v_0, v_1), v_2) \otimes v_3 - \mu(v_0, v_1) \otimes \mu(v_2, v_3) \\ &- \mu(v_0, \mu(v_1, v_2)) \otimes v_3 + v_0 \otimes \mu(\mu(v_1, v_2), v_3) \\ &+ \mu(v_0, v_1) \otimes \mu(v_2, v_3) - v_0 \otimes \mu(v_1, \mu(v_2, v_3)) \\ &= 0\,.\end{aligned}$$

Here, the two important principles of the calculation show up: cancellations due to the signs (which are opposite because μ removes one vector) and the necessary use of associativity.

The form of d should be familiar: it is similar to a coderivation with only component $V^{\otimes 2} \to V$. There are, however some additional signs. To account for these, we can look at μ acting as a coderivation on the tensor coalgebra built from a shifted[7] space ${\downarrow}V$ by defining

$$\mu_{\downarrow V}({\downarrow}v, {\downarrow}w) \equiv {\downarrow}\mu(v, w)\,.$$

The vector space ${\downarrow}V$ is now concentrated in degree -1 and $\mu_{\downarrow V} : T{\downarrow}V \to T{\downarrow}V$ has degree 1, so commuting it with v_i of degree -1 gives a correct sign, coming from the Koszul rule.

We can go further from this. First thing is to take V to be graded vector space. This construction still gives a differential, only the sign in the formula 1.11 has a sign given by $(-1)^{i+|v_0|+\dots+|v_{i-1}|}$ in ith summand. In this case, we have to define

$$\mu_{\downarrow V}({\downarrow}v, {\downarrow}w) \equiv (-1)^{|v|}\ {\downarrow}\mu(v, w)\,,$$

which was not important for $|v| = 0$ case we considered before.

If we take a dg associative algebra (V, μ, d_V), then $\mu_{\downarrow V}^2 = 0$ as a coderivation, but that's not all. We can take $\mathrm{d}_V + \mu$ as a coderivation on $T{\downarrow}V$ with two nonzero components, and it will still be a differential. Again, we do this by defining $\mu_{\downarrow V}$ as before and

$$\mathrm{d}_{\downarrow V}({\downarrow}v) = {\downarrow}\mathrm{d}_V v\,.$$

It suffices to check $(\mathrm{d}_{\downarrow V} + \mu_{\downarrow V})^2 = 0$ on V, $V^{\otimes 2}$ and $V^{\otimes 3}$ only, thanks to lemmas 1.12 and 1.13. On $v \in V$, this is

$$(\mathrm{d}_{\downarrow V} + \mu_{\downarrow V})^2({\downarrow}v) = (\mathrm{d}_{\downarrow V} + \mu_{\downarrow V})({\downarrow}\mathrm{d}_V(v)) = {\downarrow}\mathrm{d}_V^2(v) = 0\,,$$

on $v \otimes w \in V^{\otimes 2}$, this is

$$\begin{aligned}\mathrm{proj}_{\downarrow V}(\mathrm{d}_{\downarrow V} + \mu_{\downarrow V})^2({\downarrow}v \otimes {\downarrow}w) &= \mathrm{proj}_{\downarrow V}(\mathrm{d}_{\downarrow V} + \mu_{\downarrow V}) \\ &[{\downarrow}\mathrm{d}_V(v) \otimes {\downarrow}w + (-1)^{|v|+1}\ {\downarrow}v \otimes {\downarrow}\mathrm{d}_V w + (-1)^{|v|}\ {\downarrow}\mu(v, w)] \\ &= (-1)^{|v|+1}\ {\downarrow}\mu(\mathrm{d}_V(v) \otimes w) + (-1)^{|v|+1+|v|}\ {\downarrow}\mu(v \otimes \mathrm{d}_V w) + (-1)^{|v|}\ {\downarrow}\mathrm{d}_V \mu(v, w)\,.\end{aligned}$$

This gives

$$\mathrm{d}_V \mu(v, w) = \mu(\mathrm{d}_V(v) \otimes w) + (-1)^{|v|}\mu(v \otimes \mathrm{d}_V w)\,.$$

exactly the Leibniz rule for μ. Similarly, on $V^{\otimes 3}$, one would obtain the (graded) associativity. ◇

[7] The shift is down, because we want μ to have the same degree as a potential differential on V, which we conventionally take 1.

Example 1.15. Similar procedure for Lie algebras, usually introduced in the dual setting, is known as a *Chevalley-Eilenberg complex* (with trivial representation). We won't go into details, since the procedure is, with appropriate changes, the same. The main difference is that we define the coderivation on $\mathrm{Sym}(\downarrow\mathfrak{g})$, or in the dual setting, derivation on $\mathrm{Sym}(\uparrow(\mathfrak{g}^{\#}))$. It is this symmetrization that causes the Jacobi identity to appear instead of associativity. This is the same mechanism that makes a commutator of an associative product into a Lie bracket. For more details, see [14, section 13.2.8]. $\diamond$

1.4 Homotopy algebras

As we repeatedly declared, recasting the algebras in this way gives us a direct way to generalize them to their homotopy versions. This is done by allowing more components of the coderivation to be nonzero. Again, we can use TV or $\mathrm{Sym}\,V$, which then gives generalizations of associative and Lie algebras, respectively.

Definition 1.16. A *strongly homotopy associative algebra*, or an A_∞ algebra, on a graded vector space V, is given by a degree 1 coderivation m on TV satisfying $\mathrm{proj}_{\Bbbk} \circ m = 0$ and $m \circ m = 0$, i.e. by a codifferential with $m_0 = 0$. We denote this algebra by (V, m).

If m_0 is not zero, this structure is called *weak* or *curved* A_∞ algebra. $\triangle$

Definition 1.17. A *strongly homotopy Lie algebra*, or a L_∞ algebra, on a graded vector space V, is given by a degree 1 coderivation l on $\mathrm{Sym}\,V$ satisfying $\mathrm{proj}_{\Bbbk} \circ l = 0$ and $l \circ l = 0$, i.e. by a codifferential with $l_0 = 0$. We denote this algebra by (V, l).

If l_0 is not zero, this structure is called *weak* or *curved* L_∞ algebra. $\triangle$

The word *strongly* is often omitted in both cases. The ∞ refers to the (possibly) infinite number of nonzero components of the coderivations. Kajiura [4, definition 2.5] uses *weak* but *curved* seems to be more widespread, see e.g. [23] and references therein. We will be working mostly with the non-curved case, but a *loop homotopy algebra* uses θ, a *derivation of order 2* with nonzero component θ_0.

We can expand the square-zero condition for both of these algebras using the formula from the proof of 1.10. Note that only the vector component of the result needs to be zero, c.f. lemma 1.12. For the associative case, we get

$$\sum_{\substack{k+l=n+1\\0\leq j\leq k-1}} (-1)^{|v_1|+\cdots+|v_j|} m_k(v_1, \dots, v_j, m_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_n) = 0 \tag{1.12}$$

for all $n \geq 1$ and any vectors $v_i \in V$. Here, m_i are degree one maps $V^{\otimes i} \to V$. This can be shortened to

$$\sum_{\substack{k+l=n+1\\1\leq j\leq k}} m_k \circ_j m_l = 0\,, \tag{1.13}$$

again for all $n \geq 1$.

For an L_∞ algebra, this is

$$\sum_{k+l=n+1} \sum_{\sigma\in \mathrm{Unsh}(l,n-l)} \varepsilon(\sigma) l_k(l_l(v_{\sigma(1)}, \dots, v_{\sigma(l)}), v_{\sigma(l+1)}, \dots, v_{\sigma(n)}) \tag{1.14}$$

for $n \geq 1$. The components l_i are graded symmetric maps $V^{\otimes i} \to V$ of degree 1. Again, this can be shortened, by denoting the right action of a permutation σ by σ^r we get

$$\sum_{k+l=n+1} \sum_{\sigma\in \mathrm{Unsh}(l,n-l)} (l_k \circ_1 l_l) \circ \sigma^r = 0\,. \tag{1.15}$$

In both cases, we only consider $k \geq 1$ and $l \geq 1$, but this formula generalizes directly to the curved case.

Example 1.18. For the associative case, the equation 1.13 for low n gives

$$m_1 \circ_1 m_1 = m_1^2 = 0\,,$$
$$m_1 \circ_1 m_2 + m_2 \circ_1 m_1 + m_2 \circ_2 m_1 = m_1 \circ m_2 + m_2 \circ (m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1) = 0\,.$$

From the bar construction, we know that after a shift to $\uparrow V$, these two equations imply that (shifted) m_1 is a differential and a derivative of (shifted) m_2. The identity for $n = 3$ can be written as

$$m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_1) + m_2 \circ (m_2 \otimes \mathbb{1} + \mathbb{1} \otimes m_2) = 0\,.$$

After shifts, the last term gives the associator $\mu(\mu \otimes \mathbb{1} - \mathbb{1} \otimes \mu)$. Therefore, the multiplication m_2 is no longer associative! The form of the associator

$$-m_1 \circ m_3 - m_3 \circ (m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_1)$$

explains the word *homotopy* in *homotopy algebras*: the associator is homotopic to zero, the homotopy being given by m_3.

This is more explicit if we look at cohomology of V with respect to $Q \equiv m_1$. We can define a product m_2 on this cohomology: the formula $Q \circ m_2 = -m_2 \circ (Q \otimes \mathbb{1} + \mathbb{1} \otimes Q)$ tells us that m_2 of cocycles is a cocycle. Moreover, for two cocycles c_1 and c_2, the result does not change if we add coboundaries Qa_1 and Qa_2 to them:

$$m_2(c_1 + Qa_1, c_2 + Qa_2) = m_2(c_1, c_2) + m_2(c_1, Qa_2) + m_2(Qa_1, c_2) + m_2(Qa_1, Qa_2)\,.$$

The last three terms are all exact, because they are all of a form $m_2(\text{coboundary, cocycle})$ (or the other order of arguments):

$$m_2(Qa, c) = -Qm_2(a, c) - (-1)^{|a|} m_2(a, Qc) = -Qm_2(a, c)\,.$$

By a similar argument, m_2 is associative on cohomology: one only needs to notice that associator takes cocycles to a coboundary given by $-Q \circ m_3$.

For a L_∞ algebra, the first two equations give again a differential l_1 that is a derivative of a symmetric bracket l_2: remember that the dg Lie algebra is recovered after a shift. Similarly as before, the Jacobi identity will not be satisfied, the Jacobiator will be again proportional to terms like $l_1 \circ l_3 + l_3 \circ l_1$. The same argument for l_1 holds: l_2 defines graded Lie algebra on the cohomology of l_1.

Note that for curved algebras, we have a element $m_0 \in V$ that is annihilated by m_1. However, m_1 is no longer a coderivation, but it satisfies (this is $\text{proj}_V \circ m \circ m = 0$ applied on $v \in V$)

$$m_1(m_1(v)) + m_2(m_0, v) + (-1)^{|v|} m_2(v, m_0) = 0\,.$$

◇

1.4.1 Morphisms

A morphism of homotopy algebras is, again, very simple to describe in the coalgebra language. We write all the definitions for the associative case first.

Definition 1.19. *Morphism* of A_∞ algebras (V, m) and (V', m') is a degree 0 coalgebra morphism $\mathcal{F} : TV \to TV'$ satisfying $\mathcal{F} \circ \eta = 0$ and commuting with the coderivations

$$\mathcal{F} \circ m = m' \circ \mathcal{F} .$$

$\triangle$

The condition $\mathcal{F} \circ \eta = 0$ just says that $\mathcal{F}$ has no component $\mathcal{F}_0$. This allow us to use cofreeness of the tensor coalgebra TV: the morphism $\mathcal{F}$ is completely determined by its projection on V', which we can further decompose into maps $f_i : V^{\otimes i} \to V'$ defined as

$$f_i = \mathrm{proj}_{V'} \circ \mathcal{F} \circ \mathrm{proj}_{V^{\otimes i}} .$$

The condition $\mathcal{F} \circ m = m' \circ \mathcal{F}$ can be then written, using lemma 1.8 , as (compare with [4, definition 2.7])

$$\sum_{i=1\dots n} \sum_{k_1+\dots+k_i=n} m'_i \circ (f_{k_1} \otimes \dots \otimes f_{k_i}) = \sum_{\substack{k+l=n+1 \\ j=1\dots k}} f_k \circ_j m_l ,$$

which should hold for all $n \geq 1$ and $k_i \geq 1$. Note that both sides of equation $F \circ m = m' \circ F$ are (a mild generalizations of) coderivations, so equality of V components is sufficient. For $n = 1$, this is just

$$m'_1 \circ f_1 = f_1 \circ_1 m_1 ,$$

i.e. f_1 is a chain map. This now allows us to define isomorphisms and quasi-isomorphisms using f_1.

Definition 1.20. A morphism $\mathcal{F} : (V, m) \to (V', m')$ of A_∞ algebras is called *isomorphims* if its component f_1 is an isomorphism of vector spaces V and V'.

If f_1 is only a quasi-isomorphism, i.e. map inducing isomorphism on cohomologies of m_1 and m'_1, then the morphism $\mathcal{F}$ is also called an *quasi-isomorphism.* $\triangle$

This definition of isomorphism agrees with the categorical notion of isomorphism: it has an inverse isomorphism (see e.g. [14, section 10.4.1]).

For the L_∞ algebras, the situation is analogous: let's just state the definitions

Definition 1.21. *Morphism* of L_∞ algebras (V, l) and (V', l') is a degree 0 coalgebra morphism $\mathcal{F} : \mathrm{Sym}(V) \to \mathrm{Sym}(V')$ satisfying $\mathcal{F} \circ \eta = 0$ and commuting with the coderivations

$$\mathcal{F} \circ l = l' \circ \mathcal{F} .$$

Such morphism is called *isomorphism* or *quasi-isomorphism* if its component $f_1 = \mathrm{proj}_{V'} \circ \mathcal{F} \circ \mathrm{proj}_V$ is an isomorphism or a quasi-isomorphism, respectively. $\triangle$

The component-wise condition for L_∞ morphism can be written as

$$\sum_{\substack{i=1\dots n \\ k_1+\dots+k_i=n}} \frac{1}{i!} \sum_{\sigma \in \mathrm{Unsh}(k_1,\dots,k_i)} l'_i \circ (f_{k_1} \otimes \dots \otimes f_{k_i}) \circ \sigma^r = \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Unsh}(k,l)} (f_k \circ_1 l_l) \circ \sigma^r .$$

See [18, definition 6] or [14, proposition 10.2.13][8]

Again, these morphisms generalize morphisms of associative or Lie algebras, as can be easily checked.

[8] Which is, however, missing the factor $1/i!$.

1.4.2 Minimal model

Minimal model theorems are a set of results describing homotopy algebras up to a quasi-isomorphism. We will follow Kajiura's terminology, distinguishing between the minimal model theorem and a decomposition theorem [4, section 5]. The minimal model theorem states that the cohomology of a homotopy algebra has a nontrivial homotopy algebra structure quasi-isomorphic to the original homotopy algebra. We saw that the induced product is associative, but the higher products will still be nontrivial, if one wants to construct the quasi-isomorphism.

Decomposition theorem is a more general statement and the minimal model theorem is a consequence. To describe it, we need to define what a minimal and linear contractible homotopy algebra, see ibid. Note that we don't specify if (V, m) is A_∞ or L_∞ algebra.

Definition 1.22. Homotopy algebra (V, m) is *minimal* if its linear component m_1 is zero, i.e. if a coderivation m satisfies $m \circ \mathrm{proj}_V = 0$.

Homotopy algebra (V, m) is *linear contractible* (*trivial acyclic* in the terminology of [14, section 10.4]) if only the component m_1 is nonzero and the cohomology with respect to m_1 is trivial. △

Now, the decomposition theorem is

Theorem 1.23. *Any (A_∞ or L_∞) homotopy algebra is isomorphic to the direct sum of a minimal and linear contractible homotopy algebra.*

See [4, theorem 5.4] for the A_∞ case and [16, section 4.5.1], which mentions the L_∞ case. Both of these (and more) follow from an operadic description, see [14, theorem 10.4.5].

The direct sum of a linear contractible $(V_{\mathrm{lc}}, m_{\mathrm{lc}})$ and minimal algebra $(V_{\mathrm{m}}, m_{\mathrm{m}})$ is a homotopy algebra on $V_{\mathrm{lc}} \oplus V_{\mathrm{m}}$ with structure operations of the linear contractible being zero on arguments from V_{m} and vice versa.

The minimal model theorem is concerned only with the linear contractible part:

Theorem 1.24. *For a homotopy algebra (V, m), the inclusion $i : H(V) \to V$ can be extended to a quasi-isomorphim of homotopy algebras, where the minimal homotopy algebra $(H(V), m')$ on the cohomology extends the regular (associative or Lie) version of the algebra.*

See [14, theorem 10.3.16] and [4, section 5] for more details.

1.4.3 Cyclic homotopy algebras

Homotopy algebras often appear in physics, where one usually also has a bilinear form. This is captured in the definition of *cyclic* homotopy algebra, which uses an *odd symplectic structure.*

Definition 1.25. An *odd symplectic structure* on a vector space V is a nondegenerate graded skewsymmetric bilinear map $V \otimes V \to \mathbb{k}$ of degree -1. This degree makes it honestly skewsymmetric, i.e. $\omega(v_1, v_2) = -\omega(v_2, v_1)$.

A homotopy algebra (V, m) is called cyclic if there is a odd symplectic structure on V such that a product

$$\begin{aligned} V^{\otimes n} &\to \mathbb{k}\,, \\ v_1 \otimes \cdots \otimes v_n &\mapsto (-1)^{|v|_1} \omega(v_1, m_{n-1}(v_2, \ldots, v_n))\,, \end{aligned} \tag{1.16}$$

is graded cyclic for every $n \geq 2$. Note that this definition also applies to the Lie case. △

This is the same as [4, definition 2.11], which uses condition

$$\omega(v_1, m_{n-1}(v_2,\dots,v_n)) = (-1)^{|v_2|}\omega(v_2, m_{n-1}(v_3,\dots,v_n,v_1)) .$$

This is nonzero only for $\sum_{i=1}^n |v_i| = 0$, which means that the cyclic permutation τ_{cycl}

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto (-1)^{|v_1|\sum_{i=2}^n |v_i|} v_2 \otimes \cdots \otimes v_n \otimes v_1$$

actually has the sign equal to $(-1)^{|v_1|}$. Using the Koszul sign when commuting vectors with m, we get

$$\begin{aligned}\omega(v_1, m_{n-1}(v_2,\dots,v_n)) &= (-1)^{|v_1|}\omega \circ (\mathbb{1} \otimes m_{n-1})(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\\ &= (-1)^{|v_1|}\omega \circ (\mathbb{1} \otimes m_{n-1}) \circ \tau_{\mathrm{cycl}}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\\ &= \omega \circ (\mathbb{1} \otimes m_{n-1})(v_2 \otimes \cdots \otimes v_n \otimes v_1)\\ &= (-1)^{|v_2|}\omega(v_2, m_{n-1}(v_3,\dots,v_n,v_1)) ,\end{aligned}$$

where the second equality follows from the cyclic symmetry. The first cyclicity condition is just a compatibility condition of the differential with the symplectic form

$$\omega(m_1 \otimes \mathbb{1}) + \omega(\mathbb{1} \otimes m_1) = 0 . \tag{1.17}$$

There is a notion of a morphism of cyclic algebras.

Definition 1.26. A homotopy algebra morphism $F : (V, m, \omega) \to (V', m', \omega')$ of cyclic homotopy algebras is called cyclic homotopy algebra morphism if

$$\omega' \circ (f_1 \otimes f_1) = \omega$$

and if for $n \geq 3$

$$\sum_{\substack{k,l\geq 1\\ k+l=n}} \omega' \circ (f_k \otimes f_l) = 0 .$$

△

This definition is somewhat unsatisfactory for us: the first condition states that f_1 is a symplectomorphism, which implies that the vector space V' is always (degreewise) higher dimensional than V.

See H. Kajiura [4, section 2.3] for more details, the cyclic L_∞ algebra is introduced e.g. in [24].

1.4.4 Dual description

Of course, doing calculations with coalgebras is not particularly intuitive, c.f. sections 1.2.1 and 1.2.2. If we dualize everything (and e.g. assume the degree-wise finite dimensionality of our vector spaces or a suitable topology; we will largely ignore such issues), we can talk about derivations on (symmetric) associative tensor algebras.

Then we have to allow for infinite sums to appear in the tensor algebra, since the dual of $m : TV \to V$ can have nonzero components in all tensor powers. Thus, we have to consider

$$\hat{T}V^{\#} = \prod_{n=0}^{\infty} \left(V^{\#}\right)^{\otimes n} ,$$

where we denote by $^{\#}$ the graded dual vector space. This kind of algebra is often interpreted as an algebra of function on a *formal graded noncommutative manifold.* If the vector space has a basis $\mathbf{e}_i$ and the dual basis of $V^\#$ is denoted ϕ^i, element of TV can be written as

$$\sum_{k=1}^{\infty} a_{i_1 \dots i_k} \phi^{i_1} \dots \phi^{i_k} ,$$

where we omit the tensor products and use the Einstein summation convention. This is also the reason why ϕ^i are often called coordinates – they are the coordinates on the formal manifold V. See M. Kontsevich's deformation quantization of Poisson manifolds [16] and also [4, section 3].

Similar description is also available for morphisms of coalgebras, we refer to [4, section 3]. For cyclic coalgebras, we will, in addition, have a non-degenerate bilinear form. It will allow us to raise and lower the indices and encode the codifferential in an element of $\hat{T}V$ or $\mathcal{F}V$, with the $m^2 = 0$ equivalent to the *classical master equation.*

1.4.5 Koszul duality and operads

We end this chapter by sketching a generalization of homotopy algebras, using operads. We will completely omit the discussion of various shifts that are present (e.g. because operad *Ass* uses the usual convention of degree 0 product).

We have seen that homotopy versions of algebras can be obtained using a variant of (co)bar construction. This can be extended to algebras that can be described by operads (satisfying some conditions). For example, the algebras over the operadic bar construction $\mathbf{D}(Ass)$ for the associative operad *Ass* are exactly the A_∞ algebras.

To get L_∞ operad, one needs to start with the operad *Comm* of commutative algebras. This is an example of *Koszul duality*: the Koszul dual of *Comm* is $Comm^! \equiv Lie$, because the algebras over the cobar complex of *Comm* are homotopy versions of Lie algebras.

For associative algebras, the situation is just $Ass^! = Ass$. Because $Lie^! = Comm$, the operad describing C_∞ algebras would be $\mathbf{D}(Lie)$.

The tensor coalgebras get into the story as an alternative description of these operadic homotopy algebras. For a *coalgebra* $\mathcal{C}$, we can define a cofree $\mathcal{C}$-coalgebra $\mathcal{C}(V)$ on a vector space V. Taking for example dual of *Ass* as the coalgebra, we get the tensor coalgebra, for dual of *Comm* we get the symmetric tensor coalgebra. Coderivations of these coalgebras are again uniquely determined by their projections on V.

Using this fact, one can show another characterizations of algebras over $\mathbf{D}\mathcal{P}$, for an operad $\mathcal{P}$. Namely, the codifferentials on cofree $\mathcal{P}^{\mathsf{i}}$-coalgebra $\mathcal{P}^{\mathsf{i}}(V)$ are in one to one correspondence with $\mathbf{D}(\mathcal{P})$ algebras ($\mathcal{P}^{\mathsf{i}}$ is a Koszul dual cooperad of operad $\mathcal{P}$, up to shifts.)

For more than a mere sketch, we refer to the monograph [13, section I.1.13] and to the chapter 10 (and necessary results from previous chapters) of [14].

2. Batalin-Vilkovisky formalism

The Batalin-Vilkovisky formalism (BV henceforth) will allow us to connect quantum field theory and homotopy algebras. This chapter aims to explain its origin from physics, describe its geometrical interpretation and collect necessary results. The connection to the homotopy algebras is then made clear in chapter 3.

The quantum field part of the explanation follows closely the first chapter of textbook [25] by S. Weinberg. See also [26] for a thorough physical treatment.

2.1 BV formalism and path integral

A basic physical object we will be studying is the path integral. As is well known, the path integral itself is not well defined, but allows a diagrammatical description for its expansion in powers of coupling constants. Naive application of these *Feynman rules* doesn't give a right answer in the case of gauge theories (i.e. the computed S-matrix is not unitary).

Faddeev–Poppov procedure and BRST incorporate the gauge invariance by adding more fields to the theory: ghosts c, antighosts b (with *ghost numbers* equal to 1 and -1, respectively) and also Lautrup–Nakanishi field h. We will assume that the statistics of a field is equal to its ghost number $\bmod 2$. This allows us to mention only one grading, the ghost grading (see also section 2.4.1).

Intuitively, the additional ghosts and antighosts cancel the superficial integration over the gauge redundant degrees of freedom; technically, they are used to "lift" a determinant factor to the exponential, so that we can treat it using the standard diagrammatic techniques. They are called ghosts because they don't obey the spin-statistics theorem, i.e. usually they are anticommuting (that is fermionic) scalars.

The BRST formalism, a predecessor of BV, introduces a nilpotent BRST transformation s. One then shows that the actions and observables belong to the cohomology classes of s. Choosing a gauge then reduces to choosing a representative of a cohomology class of the action.

BV formalism introduces new field $\chi^\ddagger$ for every field χ of BRST, called *antifield*. These antifields have opposite statistics and ghost number equal such that $\operatorname{gh}\chi + \operatorname{gh}\chi^\ddagger = -1$. In contrast to BRST, we do not integrate over these new fields, but rather fix a subspace of fields and antifields to integrate over.

Let us illustrate it on the case of Yang Mills theory. After BRST, one arrives at an action that can be written in the form

$$S_{\mathrm{BRST}}[\phi, b, c, h] = S_{\mathrm{YM}}[\phi] + s\Psi[\phi, b, c, h]\,.$$

Here, ϕ is just the original YM gauge field, b and c are ghosts and antighosts and h is the Launtrup-Nakanishi field. The operator s is the nilpotent BRST differential. The fermionic functional Ψ with $\operatorname{gh}\Psi = -1$ is called a gauge fixing fermion. It can be shown that the physical matrix elements (i.e. matrix elements of gauge invariant observables between physical states) are independent of the choice of Ψ. Moreover, thanks to gauge invariance of S_{YM}, $sS_{\mathrm{BRST}} = 0$. The path integral, computing the expectation value of observable $X[\phi]$ is in this case

$$\langle X\rangle = \int \mathcal{D}\phi\mathcal{D}b\mathcal{D}c\mathcal{D}h\; X[\phi]e^{S_{\mathrm{YM}}[\phi]+s\Psi[\phi,b,c,h]}\,.$$

Now, in BV formalism, we add the antifields $\chi^\ddagger$ (note that antighost b and antifield for ghost $c^\ddagger$ are two different fields) and extend the Yang Mills action as

$$S[\chi, \chi^\ddagger] = S_{\mathrm{YM}}[\phi] + (s\chi)^n \chi^\ddagger_n \,,$$

where the index n represents the discrete (representation vector space indices) and the continuous (spacetime) degrees of freedom. Integration and sum is implied by the Einstein summation convention, e.g.

$$(s\chi)^n \chi^\ddagger_n \equiv \sum_i \int \mathrm{d}x^4 (s\chi)^i(x) \chi^\ddagger_i(x) \,.$$

This action now satisfies a *classical master equation*

$$\frac{\delta_R S}{\delta \chi^\ddagger_n} \frac{\delta_L S}{\delta \chi^n} = 0 \,,$$

because

$$\frac{\delta_R S}{\delta \chi^\ddagger_n} = (s\chi)^n$$

and the master equation gives

$$(s\phi)^n \frac{\delta_L S_{\mathrm{YM}}}{\delta \phi^n} + (s\chi)^m \frac{\delta_L (s\chi)^n}{\delta \chi^m} \chi^\ddagger_m = 0 \,.$$

The term without antifields is just sS_{YM}, while the term with antifields gives $s^2\chi^m$. The first term is zero by gauge invariance of S_{YM} and the second by nilpotency of s

When integrating, we prescribe a definite value to the antifields by the formula[1]

$$\chi^\ddagger_n = \frac{\delta \Psi[\chi]}{\delta \chi^n} \,,$$

which gives an action

$$S\left[\chi, \frac{\delta \Psi[\chi]}{\delta \chi^n}\right] = S_{\mathrm{YM}}[\phi] + (s\chi^n) \frac{\delta \Psi[\chi]}{\delta \chi^n} = S_{\mathrm{YM}}[\phi] + s\Psi[\chi] \,,$$

exactly as the BRST procedure.

This action can be generalized to theories where the action is now no longer linear in antifields. We construct the general action such that it is bosonic and has ghost number 0. Again, we integrate by setting $\chi^\ddagger_n = \delta\Psi[\chi]/\delta\chi^n$ and we want the result to be independent of the gauge fixing fermion $\Psi[\chi]$ (with ghost number -1), which generalizes the gauge invariance of the Yang Mills theory.

How do we ensure this gauge invariance? Choosing a general functional $H[\chi, \chi^\ddagger]$, we want the path integral

$$\int \mathcal{D}\chi \, H\left[\chi, \frac{\delta \Psi[\chi]}{\delta \chi}\right] \tag{2.1}$$

to be independent of $\Psi[\chi]$. If we deform $\Psi[\chi]$ by a small functional $\epsilon[\chi]$, the above integral 2.1 changes (to the first order) by

$$\int \mathcal{D}\chi \, \frac{\delta_R H}{\delta \chi^\ddagger_n} \frac{\delta_L \epsilon}{\delta \chi^n} \,,$$

[1] We do not specify if the functional derivative is left or right, since both cases are the same:

$$\frac{\delta_R \Psi}{\delta \chi^n} = (-1)^{(|\Psi| - |\chi^n|)|\chi^n|} \frac{\delta_L \Psi}{\delta \chi^n} \,,$$

where the sign factor is +1 because $|\Psi| = 1$.

where we write the right derivative of H because the other term from the chain rule comes to the right of it. The derivative of ϵ can be left or right, since ϵ is fermionic. Using integration by parts, we obtain

$$\int \mathcal{D}\chi\, (-1)^{|H||\chi^n|+1} \frac{\delta_L}{\delta\chi^n} \frac{\delta_R H}{\delta\chi^\ddagger_n} \epsilon\,.$$

Since ϵ is arbitrary, the invariance of the integral 2.1 is equivalent to the condition

$$\Delta H \equiv (-1)^{|H||\chi^n|+1} \frac{\delta_L}{\delta\chi^n} \frac{\delta_R H}{\delta\chi^\ddagger_n} = 0\,. \tag{2.2}$$

Note that this Δ is *different* from operator Δ_{W} of [25, equation 15.9.34], for bosonic H, this Δ_{W} is minus the Δ from equation 2.2.[2]

Usually, we deal with H in the form $e^{\alpha S[\chi,\chi^\ddagger]}$, where the factor α is usually $\pm i$, $i/\hbar$ or $-1/\hbar$. A simple calculation shows that (for bosonic action)

$$\Delta e^{\alpha S[\chi,\chi^\ddagger]} = \alpha^2 e^{\alpha S[\chi,\chi^\ddagger]} \left(\{S,S\} + \frac{2}{\alpha}\Delta S \right) \stackrel{!}{=} 0\,,$$

where we used *antibracket*

$$\{F,G\} \equiv \frac{\delta_R F}{\delta\chi^n}\frac{\delta_L G}{\delta\chi^\ddagger_n} - \frac{\delta_R F}{\delta\chi^\ddagger_n}\frac{\delta_L G}{\delta\chi^n}\,. \tag{2.3}$$

Thus, if we want a gauge invariant theory, the action has to satisfy the *quantum master equation*

$$\{S,S\} - 2\hbar\Delta S = 0\,, \tag{2.4}$$

where we have chosen $\alpha = -1/\hbar$.

Moreover, if we take an bosonic observable $\mathcal{O}[\chi,\chi^\ddagger]$, the expectation value is gauge independent if

$$0 = \Delta(\mathcal{O}[\chi,\chi^\ddagger] e^{\alpha S[\chi,\chi^\ddagger]}) = (\Delta\mathcal{O} + \alpha\{\mathcal{O},S\})\, e^{\alpha S[\chi,\chi^\ddagger]}\,, \tag{2.5}$$

[2] The Δ_{W} from [25], defined as

$$\Delta_{\mathrm{W}} \equiv \frac{\delta_R}{\delta\chi^\ddagger_n}\frac{\delta_L}{\delta\chi^n} = \frac{\delta_L}{\delta\chi^n}\frac{\delta_R}{\delta\chi^\ddagger_n}\,,$$

is problematic in few aspects, mostly in the overall minus sign: the master equation [25, equation 15.9.35] should actually read $(S,S) + 2i\Delta S = 0$. This is, however, nonstandard, and authors usually choose a different Δ, which is S. Weinberg's $-\Delta_{\mathrm{W}}$ on bosonic functionals: we e.g. Henneaux, Teitelboim [26, section 15.5.3].

In the original reference by Batalin, Vilkovisky [1, equation 16], the Δ is defined with left derivative for the antifield and right derivative for the field. See also physics SE discussion at [27], where the difference between S. Weinberg's and the original Δ is discussed.

Generally, when working with BV algebras, one needs additional sign factor $(-1)^{|H||\chi_n|}$ to make Δ and the bracket compatible: see section 2.3. Henneaux and Teitelboim have this sign, but their Δ_{HT} behaves like a right derivative [26, equation 18.5a]:

$$\Delta_{\mathrm{HT}}(FG) = F(\Delta_{\mathrm{HT}}G) + (-1)^{|G|}(\Delta_{\mathrm{HT}}F)G + (-1)^{|G|}\{F,G\}\,.$$

In the section 2.3 will take our Δ_{our} to be

$$\Delta_{\mathrm{our}}H = (-1)^{|H|}\Delta_{\mathrm{HT}}H\,,$$

so that Δ_{our} acts "from the left".

One of the possible sources of confusion might be the derivation of the master equation: there are multiple choices one can make for the left/right derivatives and at the end arrives at $\Delta H = 0$, which admits additional sign factors like -1 or $(-1)^{|H|}$. Furthermore, H is usually e^S, which is bosonic.

where we used the quantum master equation $\Delta e^{\alpha S} = 0$. Usually, the observables do not depend on antifields and we get a condition

$$\{\mathcal{O}, S\} = 0\,.$$

2.2 Geometrical interpretation

The geometric interpretation of BV formalism was started by a short paper by E. Witten [28], where he interpreted the antifields as a multivector fields on the manifold of fields. More complete treatment in the context of supermanifolds was given by A. Schwarz in well-known [29] and later by his student H. Khudaverdian [30], who used semidensities as natural integration objects on Lagrangian submanifolds.

The basic idea presented in [28] is that although for finite dimensional manifolds, the differential forms and multivector (i.e. antisymmetric vector) fields on a manifold are (non-canonically) isomorphic, this is no longer true for infinite dimension. Specifically, one does not have a volume form to integrate against, but in the multivector picture, the volume form corresponds to functions, a more manageable object.

Back in the finite dimension, the isomorphism between forms and multivectors can be defined by taking a volume form α and giving the isomorphism by i_α, which takes k-dimensional multivector fields to $(n-k)$-dimensional forms, where n is the manifold dimension. Under this isomorphism, the exterior derivative becomes

$$\Delta = \sum_i \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x^i}\,,$$

where ξ_i are vector fields $\partial/\partial x^i$. This is (up to signs) the BV operator Δ from previous section.

The condition $\Delta e^S = 0$ then says that the volume form $e^S \alpha$ is closed, which is trivial for ordinary manifolds. However, in field theory, some of the fields are fermionic and the relevant manifold is supermanifold. This is the situation considered in the Schwarz's paper [29], whose results for BV geometry we will now present.

An SP-manifold M is a (n,n)-dimensional manifold with compatible P and S structures. The P structure can be specified by a closed odd nondegenerate form ω – an odd symplectic form. The S structure is given by a volume element ρ, meaning that it transforms with a *Berezinian* (see section 2.2.1). These two structures are compatible if one can find Darboux coordinates for the form where ρ is equal to 1.

The odd symplectic form allow us to define a bracket in coordinates z^i

$$\{F, G\} = \frac{\partial_R F}{\partial z^i} \omega^{ij} \frac{\partial_L G}{\partial z^j}\,,$$

where ω^{ij} is the inverse matrix to the coordinates expression of the odd symplectic form ω. Furthermore, we can define operator Δ on functions as

$$\Delta H = \operatorname{div} K_H\,,$$

where div is the divergence associated with the density ρ and K_H is the hamiltonian vector field $K_H^i = \omega^{ij} \frac{\partial_L H}{\partial z^j}$. In the Darboux coordinates with x^i even and ξ_i odd and with $\rho = 1$, we get[3]

$$\Delta = \sum_i \frac{\partial_R}{\partial x^a} \frac{\partial_L}{\partial \xi_a}\,.$$

[3] This operator is defined without the signs, as in Weinberg [25], and suffers from the same problems as Δ_{W} from [25] – see previous section.

Here, $\Delta^2 = 0$ and for general (i.e. not necessarily compatible) ω and ρ, the condition $\Delta^2 = 0$ implies the compatibility of ρ and ω (see [29, theorem 5]).

This supermanifold M should be viewed as finite dimensional version of the space of fields and antifields with ω giving their pairing. Thus, we don't want to integrate over the whole manifold, but over its *Lagrangian* submanifolds. These are manifolds L with $\omega_L = 0$, e.g. specified by $\xi_i = \cdots = \xi_k = 0$ and $x^{k+1} = \cdots = x^n = 0$ in Darboux coordinates. On such submanifold, we can integrate over a volume form $\mathrm{d}x^1 \ldots \mathrm{d}x^k \mathrm{d}\xi_{k+1} \ldots \mathrm{d}\xi_n$, in general coordinates the volume form on L would be proportional to $\sqrt{\rho}$.

Schwarz proves two important theorems which explain the results known from the field theory.

Theorem 2.1 ([29, theorems 1, 2])**.** *Let L_0 and L_1 be closed oriented Lagrangian submanifolds of an orientable SP-manifold M. If the two manifolds can be connected with smooth family of Lagrangian submanifolds L_t for $0 \le t \le 1$, then*

$$\int_{L_0} H \mathrm{d}\lambda_0 = \int_{L_1} H \mathrm{d}\lambda_1 \tag{2.6}$$

for every function H satisfying $\Delta H = 0$. Moreover, if $H = \Delta K$, then for every closed Lagrangian manifold L

$$\int_L H \mathrm{d}\lambda = 0\,. \tag{2.7}$$

Proof. See [29], we will only sketch the main ideas in the following. □

A canonical example of a P-manifold is the cotangent bundle ΠT^*N. This is a manifold, locally with points from local coordinates of oriented manifold N in even degree and the covectors in odd degree.

Equivalently, the algebra of functions $C^\infty(\Pi T^*N)$ on this supermanifold is the algebra of multivector fields on N like $f(x)\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$: the coefficient functions $f(x)$ are dual to the points and vectors ξ_i dual to forms.[4] The coordinate functions are even x^i and odd ξ_i and we can define a canonical P-structure on ΠT^*N, given by pairing these two coordinates

$$\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}\xi_i\,.$$

However, there is no canonical S-structure and it needs to be added by hand.

Schwarz does this by looking at forms, for which we have a natural notion of integration over an orientable manifold. Similar to ΠT^*N, we can look at a supermanifold ΠTN, which has algebra of functions $C^\infty(\Pi T^*N)$, which is the exterior algebra of N. If we now choose a volume form α on n, we have an isomorphism between forms and multivector fields, or between functions on ΠTN and on ΠT^*N. This isomorphism F takes a differential form $\omega(x, \eta)$ (η are $\mathrm{d}x$) and outputs $\tilde{\omega}(x, \xi)$. It can be explicitly written as

$$\tilde{\omega}(x, \eta) \equiv F(\omega) = \int e^{\sum_i \xi_i \eta^i} \omega(x, \eta) \alpha(x) \mathrm{d}^n \eta\,.$$

This isomorphism F then takes the exterior derivative d to the operator Δ, i.e.

$$F(\mathrm{d}\omega) = \Delta F(\omega)\,,$$

where the volume element that specifies Δ is defined as

$$\rho(x, \xi) \mathrm{d}^n x\, \mathrm{d}^n \xi = \alpha^{-2}(x) \mathrm{d}^n x\, \mathrm{d}^n \xi\,.$$

[4] We would like to stress the swap that happens here: T^*N has a function algebra of multivector fields, not forms.

This is exactly the connection of forms and multivector fields explained by Witten, and it gives an SP-manifold structure to ΠT^*N. The proof of theorem 2.1 for SP-manifolds of this form is now immediate from following lemma

Lemma 2.2 ([29, lemma 3]). *If ω is a form on N and K is a closed oriented submanifold of N, then*

$$\int_K \omega = \int_{L_K} F(\omega)\mathrm{d}\lambda \tag{2.8}$$

where L_K is a Lagrangian submanifold of ΠT^*N corresponding to K. This submanifold is specified the submanifold K (with dimension k) and the covectors orthogonal to K ($n-k$ dimensional kernel of the restriction map).

It can be furthermore shown the general case can be reduced to this special case [29, theorems 3, 4 etc.].

2.2.1 Semidensities and Berezinian

More natural (although less familiar) integration objects in the previous context would be *semidensities*, which transform with a square root of the Berezinian of the transformation. We have already seen some evidence for this: $\sqrt{\rho}$ as an induced volume form on the Lagrangian submanifold or α^{-2} for the volume form on ΠT^*N.

This remark is based on H. Khudaverdian's interesting lecture notes from Białowieża 2011 [31].

Illustration using the notion of *Berezinian* can shed some light on the need of the square root. For a invertible matrix of linear transformation of (n,n) written in block matrix form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the Berezinian is defined as (see e.g. [32, equation 2.2.20])

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D}.$$

If we look at a coordinate transformation preserving Darboux coordinates (P-transformation in the language of [29]) of ΠT^*N given by

$$x^i = x(x'),$$
$$\xi_i = \frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j} = \frac{\partial x'^j}{\partial x^i}\xi'_j,$$

the Berezinian of this transformation is

$$\mathrm{Ber}\left(\frac{\partial x, \xi}{\partial x', \xi'}\right) \equiv \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial \xi}{\partial x'} \\ \frac{\partial x}{\partial \xi'} & \frac{\partial \xi}{\partial \xi'} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial x'}\frac{\partial^2 x}{\partial x' \partial x'}\xi' \\ 0 & \frac{\partial x'}{\partial x} \end{pmatrix} = \det\left(\frac{\partial x}{\partial x'}\right)^2.$$

We see that Berezinian transforms as square of the Jacobian. Since we are generalizing the integration over the Lagrangian submanifold N of ΠT^*N, we want our volume element to transform with the inverse square root of Berezinian, to get the transformation behaviour of volume forms. This explains the square root of volume element for Lagrangian submanifolds and also the formula $\rho = \alpha^{-2}$.

Note: Doing the same analysis for the supermanifold ΠTN, one arrives at a Berezinian

$$\mathrm{Ber}\left(\frac{\partial x, \eta}{\partial x', \eta'}\right) \equiv \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial \eta}{\partial x'} \\ \frac{\partial x}{\partial \eta'} & \frac{\partial \eta}{\partial \eta'} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial^2 x}{\partial x' \partial x'}\eta' \\ 0 & \frac{\partial x}{\partial x'} \end{pmatrix} = 1.$$

This means that on ΠTN, we have a *canonical Berezin integration* on supermanifold ΠTN: the integral

$$\int_{\Pi TN} \omega(x,\eta)\mathrm{d}^n x \mathrm{d}^n \eta = \int_{\Pi TN} \omega(x',\eta')\mathrm{d}^n x' \mathrm{d}^n \eta'$$

for transformation transforming η^i as $\mathrm{d}x^i$.

This is not surprising: integral of a function $\omega(x,\eta)$ on ΠTN should be equal to integral of corresponding differential form $\omega(x,\mathrm{d}x)$ over N, which is just the ordinary integral over manifolds. Indeed, these notions coincide: the Berezin integral over the odd variables η just isolates the top component of the ω. This can be also viewed as a justification of the definition of the Berezin integral.

2.3 BV algebras

We have seen multiple version of an operator Δ similar to

$$\Delta H = (-1)^{|H||\chi^n|+1} \frac{\delta_L}{\delta \chi^n} \frac{\delta_R H}{\delta \chi^\ddagger_n}\,, \tag{2.9}$$

and brackets

$$\{F,G\} = \frac{\delta_R F}{\delta \chi^n} \frac{\delta_L G}{\delta \chi^\ddagger_n} - \frac{\delta_R F}{\delta \chi^\ddagger_n} \frac{\delta_L G}{\delta \chi^n}\,. \tag{2.10}$$

The Δ operator is a second order differential operator, which means it does not satisfy the Leibniz rule. The failure of the Leibniz rule is precisely the bracket, as can be easily checked

$$\Delta(FG) = F\Delta G + (-1)^{|G|}(\Delta F)G + (-1)^{|G|}\{F,G\}\,. \tag{2.11}$$

Furthermore, Δ squares to 0, the bracket has a graded symmetry

$$\{F,G\} = -(-1)^{(|F|+1)(|G|+1)}\{G,F\}$$

and is a graded derivative in each of its arguments. These are a basis of definition *BV algebra*, but the signs in the equation 2.11 are usually taken to be different, so that Δ is a left, not right derivative. This corresponds to redefinition

$$\Delta H \to \Delta_{\mathrm{our}} H = (-1)^{|H|}\Delta H\,.$$

This gives

$$\Delta_{\mathrm{our}} H = (-1)^{|H||\chi^n|+1+|H|} \frac{\delta_L}{\delta \chi^n} \frac{\delta_R H}{\delta \chi^\ddagger_n} = (-1)^{|H||\chi^n|} \frac{\delta_R}{\delta \chi^n} \frac{\delta_L H}{\delta \chi^\ddagger_n}\,. \tag{2.12}$$

With these signs, we can define BV algebras (see e.g. [33, section 4]).

Definition 2.3. A *BV algebra* is a graded commutative associative algebra on graded vector space V with a bracket $\{,\} : V^{\otimes 2} \to V$ of degree 1 that satisfies

$$\{F,G\} = -(-1)^{(|F|+1)(|G|+1)}\{G,F\}\,, \tag{2.13}$$

$$\{F,\{G,H\}\} = \{\{F,G\},H\} + (-1)^{(|F|+1)(|G|+1)}\{G,\{F,H\}\}\,, \tag{2.14}$$

$$\{F,GH\} = \{F,G\}H + (-1)^{(|F|+1)|G|}G\{F,H\} \tag{2.15}$$

and a square zero operator $\Delta : V \to V$ of degree one such that

$$\Delta(FG) = (\Delta F)G + (-1)^{|F|}F\Delta G + (-1)^{|F|}\{F,G\}\,. \tag{2.16}$$

△

If the vector space V comes with a differential d, we usually also postulate its compatibility with Δ as in $\Delta \mathrm{d} + \mathrm{d}\Delta = 0$. For algebras with unit 1, we will take $\Delta(1) = 0$.

Since the bracket can be defined using Δ, one can define a BV algebra without mentioning it. The Poisson and Jacobi identities of the bracket are then encoded in so-called seven-term identity for Δ

$$\begin{aligned}\Delta(FGH) =&\Delta(FG)H + (-1)^{|G||H|}\Delta(FH)G + (-1)^{|F|(|G|+|H|)}\Delta(GH)F \\ &- \Delta(F)GH - (-1)^{|F|}F\Delta(G)H - (-1)^{|F|+|G|}FG\Delta(H)\,.\end{aligned} \tag{2.17}$$

In the following, we will also use a compatibility between Δ and $\{,\}$ which can be derived from $\Delta^2(FG) = 0$

$$\Delta\{F, G\} = \{\Delta F, G\} + (-1)^{|F|+1}\{F, \Delta G\}\,. \tag{2.18}$$

2.4 Flat BV geometry on $\mathrm{Sym}\, V^{\#}$

For us, a most important example of a *BV* algebra will be a BV algebra associated with a graded vector space with odd symplectic form, as defined in definition 1.25. This is in fact a flat variant of BV algebra on a *SP* supermanifold of A. Schwarz: the *P* structure is given by the odd symplectic form and the *S* structure is just the constant Lebesgue measure on the vector space.

The definition of the BV Δ operator will be different from the one from [29] by signs, see also discussions in previous sections. Note that we now work with graded vector spaces, not super vector spaces.

The following construction appears for example in P. Mnev's [5, section 4.1] and in [34]. We start by recalling some notions useful for discussion of symplectic vector spaces from the book by McDuff and Salamon [35, chapter 2].

Definition 2.4. For ω a symplectic form on V, we define a *symplectic complement* of a subspace $W \subset V$ to be the set of vectors

$$W^{\omega} \equiv \{v \in V \mid \omega(v, w) = 0,\ \forall w \in W\}.$$

This complements satisfies $\dim W + \dim W^{\omega} = \dim V$ and $W^{\omega\omega} = W$, see [35, lemma 2.2].

We can classify subspaces by the behaviour of their complement: we call $W \subset V$
isotropic if $W \subset W^{\omega}$,
coisotropic if $W^{\omega} \subset W$,
symplectic if $W \cap W^{\omega} = \{\mathbf{0}\}$,
Lagrangian or *maximally isotropic* if $W = W^{\omega}$.

$\triangle$

Let us take a graded vector space V and a symplectic form ω of degree -1. At first, we want to define ω on the dual vector space. We can view ω as a map $\natural : V \to V^{\#}$. The $\omega^{\#}$ on the dual vector space can be then defined as

$$\omega^{\#}(\alpha, \beta) = \omega(\natural^{-1}(\alpha), \natural^{-1}(\beta))\,. \tag{2.19}$$

We will describe everything in coordinates: choose a basis $\mathbf{e}_i$ of V and a dual basis ϕ^i of $V^{\#}$. The odd symplectic form in this coordinates is an antisymmetric matrix

$$\omega_{ij} \equiv \omega(\mathbf{e}_i, \mathbf{e}_j)$$

with block structure: ω_{ij} is nonzero only if $|\mathbf{e}_i| + |\mathbf{e}_j| = -|\omega| = 1$. The matrix of $\omega^{\#}$ is exactly the inverse matrix of ω_{ij}, which we will denote ω^{ij}. It is again antisymmetric and has a similar block structure: ω^{ij} is nonzero only for $|\mathbf{e}_i| + |\mathbf{e}_j| = 1$, which can be seen from

$$\sum_j \omega_{ij}\omega^{jk} = \delta_i^k \,,$$

or from equation 2.19.

In the dual basis, this means that ω^{ij} is nonzero only for $|\phi^i| + |\phi^j| = -1$. We define a BV operator $\Delta : \mathcal{F}(V) \to \mathcal{F}(V)$ as

$$\Delta(F) = \frac{1}{2}\sum_{i,j}(-1)^{|\phi^i|}\omega^{ij}\frac{\partial_L}{\partial\phi^i}\frac{\partial_L}{\partial\phi^j}F \,. \tag{2.20}$$

Since $|\phi^i| + |\phi^j| = -1$ for nonzero ω^{ij}, this operator removes covectors of total degree -1 and therefore $|\Delta| = 1$. The factor $(-1)^{|\phi^i|}$ counters the skew-symmetry of ω^{ij}, because without it we would contract it with symmetric $\frac{\partial_L}{\partial\phi^i}\frac{\partial_L}{\partial\phi^j}$. The factor $\frac{1}{2}$ is there exactly because of this symmetry. twice.

Corresponding BV bracket is defined by

$$\Delta(FG) = (\Delta F)G + (-1)^{|F|}F\Delta G + (-1)^{|F|}\{F, G\}$$

and can be written as

$$\{F, G\} = \sum_{i,j}\omega^{ij}\frac{\partial_R F}{\partial\phi^i}\frac{\partial_L G}{\partial\phi^j} \,.$$

These two operations together form a BV algebra in the sense of definition 2.3. This can be checked by direct computation, or by comparing with previous examples: see the next section 2.4.1.

2.4.1 Fields and antifields

The graded vector space V is a "model" for the space of all field and antifield histories. The grading is the *ghost number* and since we will consider only bosonic theories, the fields with even ghost numbers are always bosonic and fields with odd ghost number are always fermionic. This holds even for antifields, since antifield has a opposite statistics *and* ghost number (modulo 2) as the corresponding field.

The form ω is then just the pairing of fields and antifields. Note that in physics, the field histories χ and the dual elements, which we use to write functionals of these histories (e.g. action $S[\chi, \chi^\ddagger]$) are denoted by the same letter. For us, it is probably best to think of χ and $\chi^\ddagger$ as the duals. This means that our ω has exactly the correct degree, $+1$ on duals, since we defined $\mathrm{gh}\,\chi + \mathrm{gh}\,\chi^\ddagger = -1$.

To connect this conventions with previous section, we have to specify which elements of V are fields and which are antifields. The decomposition is done by splitting the vector space V to two Lagrangian subspaces. We will denote dual bases of these two subspaces by χ^i and $\bar{\chi}^j$; and since these two are Lagrangian, we have

$$\omega^{\#}(\chi^i, \bar{\chi}^j) \equiv \Omega^{ij} \,,$$
$$\omega^{\#}(\chi^i, \chi^j) = \omega(\bar{\chi}^i, \bar{\chi}^j) = 0 \,,$$

where we have denoted by Ω the matrix of $\omega^{\#}$ in this basis χ, $\bar{\chi}$.

These χ are the fields, but $\bar{\chi}$ are not antifields yet. We define the antifields by[5]

$$\chi^\ddagger_k = \Omega_{jk}\bar{\chi}^j \,.$$

The derivative with respect to this antifield can be expressed as

$$\frac{\partial_L}{\partial\bar{\chi}^i} = \frac{\partial\chi^\ddagger_k}{\partial\bar{\chi}^i}\frac{\partial_L}{\partial\chi^\ddagger_k} = \Omega_{ik}\frac{\partial_L}{\partial\chi^\ddagger_k} \,.$$

If we look at a definition of Δ in this basis, it decomposes into two sums

$$\begin{aligned}\Delta(F) &= \frac{1}{2}\sum_{i,j}(-1)^{|\phi^i|}\Omega^{ij}\frac{\partial_L}{\partial\phi^i}\frac{\partial_L}{\partial\phi^j}F \\ &= \frac{1}{2}\sum_{i,j}(-1)^{|\chi^i|}\Omega^{ij}\frac{\partial_L}{\partial\chi^i}\frac{\partial_L}{\partial\bar{\chi}^j}F + \frac{1}{2}\sum_{i,j}(-1)^{|\bar{\chi}^i|}\Omega^{ij}\frac{\partial_L}{\partial\bar{\chi}^i}\frac{\partial_L}{\partial\chi^j}F \,.\end{aligned}$$

Using $|\chi^i| + |\bar{\chi}^j| = -1$ and antisymmetry of Ω, we can write the second term as

$$\begin{aligned}\frac{1}{2}\sum_{i,j}(-1)^{|\bar{\chi}^i|}\Omega^{ij}\frac{\partial_L}{\partial\bar{\chi}^i}\frac{\partial_L}{\partial\chi^j}F &= \frac{1}{2}\sum_{i,j}(-1)^{-1-|\chi^j|+1+|\chi^i||\bar{\chi}^j|}\Omega^{ji}\frac{\partial_L}{\partial\chi^j}\frac{\partial_L}{\partial\bar{\chi}^i}F \\ &= \frac{1}{2}\sum_{i,j}(-1)^{|\chi^j|}\Omega^{ji}\frac{\partial_L}{\partial\chi^j}\frac{\partial_L}{\partial\bar{\chi}^i}F \,,\end{aligned}$$

which is the same as the first term. Using the definition of antifields, we get

$$\Delta(F) = \sum_{i,j}(-1)^{|\chi^i|}\Omega^{ij}\frac{\partial_L}{\partial\chi^i}\frac{\partial_L}{\partial\bar{\chi}^j}F = \sum_{i,j}(-1)^{|\chi^i|}\Omega^{ij}\Omega_{jk}\frac{\partial_L}{\partial\chi^i}\frac{\partial_L}{\partial\chi^\ddagger_k}F \,.$$

Two inverse matrices meet here (the sum is only over j corresponding to antifields, but that is only nonzero part of the sum $\Omega^{ij}\Omega_{jk}$). Changing the left field derivative to the right one, we get

$$\Delta(F) = \sum_{i}(-1)^{|\chi^i|}\frac{\partial_L}{\partial\chi^i}\frac{\partial_L}{\partial\chi^\ddagger_i}F = \sum_{i}(-1)^{|\chi^i||F|}\frac{\partial_R}{\partial\chi^i}\frac{\partial_L}{\partial\chi^\ddagger_i}F \,,$$

which is the convention from equation 2.12.

Analogical calculation for bracket gives

$$\{F,G\} = \frac{\partial_R F}{\partial\chi^i}\frac{\partial_L G}{\partial\chi^\ddagger_i} - \frac{\partial_R F}{\partial\chi^\ddagger_i}\frac{\partial_L G}{\partial\chi^i} \,, \tag{2.21}$$

which is the expression we know from equation 2.10.

2.4.2 Master equation and $\hbar$

To write down master equation with explicit $\hbar$, we add $\hbar$ as a formal parameter. This means enlarging our functions $\mathcal{F}(V)$ to $\mathcal{F}(V)[[\hbar]] \equiv \mathcal{F}(V) \otimes \Bbbk[[\hbar]]$, i.e. tensoring with formal power series in $\hbar$. We will, however, abuse the notation slightly and write $\hbar$ as if it were another commutative variable.

[5] This means

$$\omega^\#(\chi^i, \chi^\ddagger_j) = \delta^i_j \,.$$

Then, we can write the master equation

$$2\hbar\Delta S + \{S, S\} = 0\,.$$

Note that S_{class} the classical part of S, i.e. sum of terms with $\hbar^0$, satisfies the *classical master equation*

$$\{S_{\text{class}}, S_{\text{class}}\} = 0\,.$$

Moreover, taking only the quadratic part of this equation and assuming that S has no linear terms, we get that S_0, the quadratic part of S_{class} (which corresponds to the kinetic term) also satisfies classical master equation

$$\{S_0, S_0\} = 0\,.$$

Solutions of master equation give us yet another differential, given by twisting Δ by $e^{S/\hbar}$, i.e. taking a map $F \mapsto e^{-S/\hbar}\Delta(Fe^{S/\hbar})$. This always is a square zero map, but it has no constant component only for S which is a solution of master equation. Direct calculation gives

$$e^{-S/\hbar}\Delta(Fe^{S/\hbar}) = \Delta F + \frac{1}{\hbar}\{S, F\}\,.$$

We will move the constant $\hbar$ to the Δ term and define

$$T_S F \equiv \hbar\Delta F + \{S, F\}\,.$$

We can also compute T_S^2 directly, which gives

$$T_S^2(F) = \hbar\Delta(\hbar\Delta F + \{S, F\}) + \{S, \hbar\Delta F + \{S, F\}\} = \{\hbar\Delta S + \frac{1}{2}\{S, S\}, F\}\,.$$

We see that T_S^2 is zero iff $\hbar\Delta S + \frac{1}{2}\{S, S\}$ is constant. However, because Δ and $\{\}$ have degree 1, $\hbar\Delta S + \frac{1}{2}\{S, S\}$ would have to be degree 1 constant, so it has to be 0. Thus, the condition T_S^2 is equivalent to the quantum master equation for S.

Expressing the BV algebra operations with antifields, we can write the master equation as

$$\frac{\partial_R S}{\partial\chi^i}\frac{\partial_L S}{\partial\chi_i^\ddagger} + \hbar\frac{\partial_R}{\partial\chi^i}\frac{\partial_L}{\partial\chi_i^\ddagger}S = 0\,, \tag{2.22}$$

which is useful when comparing with [2].

2.4.3 Differential on V

In our application, the graded vector space V will usually be equipped with a differential, which we will now denote $\mathcal{Q}$. This differential is also compatible with ω, in a sense that

$$\omega(\mathcal{Q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{Q}) = 0\,,$$

i.e. $\mathcal{Q}$ is self-adjoint with respect to ω. We are now going to consider a cohomology with respect to this differential $\mathcal{Q}$, which we denote H_Q. With the projection operator $[\,] : \operatorname{Ker}\mathcal{Q} \to H_Q$, we can define a symplectic form on this cohomology as

$$\tilde{\omega}([v], [w]) \equiv \omega(v, w)\,. \tag{2.23}$$

This does not depend on the representants v and w: adding exact terms gives

$$\tilde{\omega}([v + \mathcal{Q}x], [w + \mathcal{Q}y]) = \omega(v + \mathcal{Q}x, w + \mathcal{Q}y) = \omega(v, w) + \omega(v, \mathcal{Q}y) + \omega(\mathcal{Q}x, w) + \omega(\mathcal{Q}x, \mathcal{Q}y)\,.$$

Here, the last three terms are zero, since v and w are closed and we can move Q between the arguments. Thus, we get

$$\tilde{\omega}([v + Qx], [w + Qy]) = \omega(v, w) = \tilde{\omega}([v], [w])\,.$$

This $\tilde{\omega}$ is again antisymmetric and has degree -1, but is it non-degenerate? To prove that it is, we will use the fact that homology is in fact a *symplectic reduction*, i.e. a quotient of coisotropic subspace by its symplectic complement. We cite the following from [35].

Lemma 2.5 ([35, lemma 2.7.i]). *Let (V, ω) be a symplectic vector space and $W \subset V$ be a coisotropic subspace. Then the quotient $V' = W/W^{\omega}$ carries a natural symplectic structure $\tilde{\omega}$, defined as $\tilde{\omega}([v], [w]) = \omega(v, w)$, where $[\,]$ is the quotient map.*

Note that this lemma also applies for our case, we can just forget about the grading on our vector space.

Starting with $\operatorname{Ker} Q$, let's look at its symplectic complement. For all $v \in \operatorname{Ker} Q$, the expression $\omega(v, x)$ is zero for all $x \in \operatorname{Im} Q$. Thus $\operatorname{Im} Q$ is a subspace of $(\operatorname{Ker} Q)^{\omega}$, but since it has the right dimension, i.e. $\dim \operatorname{Ker} Q + \dim \operatorname{Im} Q = \dim V$, it has to be the symplectic complement. Thus, we see that $\operatorname{Ker} Q$ is a coisotropic subspace and

$$\operatorname{Ker} Q/(\operatorname{Ker} Q)^{\omega} = \operatorname{Ker} Q/\operatorname{Im} Q = H_Q$$

inherits a nondegenerate symplectic form, defined exactly as we did in equation 2.23.

With $\tilde{\omega}$, we can define a BV algebra on the functions on homology $\mathcal{F}(H_Q)$, which we will denote Δ' and $\{\}'$.

Similarly, if we decompose the vector space V as $V' \oplus V''$ such that $(V')^{\omega} = V''$, we get two *symplectic* subspaces of V, both with non-degenerate symplectic forms. This means we can define a BV algebra on both of this spaces, with operations Δ', $\{\}'$ and Δ'', $\{\}''$ respectively. Moreover, the operator Δ on the total vector space is a sum of these two operators

$$\Delta = \Delta' + \Delta''$$

and similarly for brackets

$$\{\} = \{\}' + \{\}''\,.$$

This is because ω does not pair vectors (or covectors) from V' and V'', and therefore the sums in definitions of Δ and $\{\}$ split into two parts, giving BV Laplacians on V' and V''.

If we choose this decomposition such that V' will be isomorphic to H_Q, with the isomorphism respecting the structures, [6] then we can view the algebra of functions $\mathcal{F}(H_Q)$ as a part of the algebra $\mathcal{F}(V)$, with a decomposition of the BV algebra. This will be a starting point for a construction of *special deformation retract* between $\mathcal{F}(V)$ and $\mathcal{F}(H_Q)$ in the following chapters.

[6] If we choose a injective map $i : H_Q \to V$ such that its a symplectomorphism and a chain map, we immediately have $V = \operatorname{Im} i \oplus (\operatorname{Im} i)^{\omega}$. This is because the symplectic form ω on $\operatorname{Im} i$ is non-degenerate, and thus no vector from $(\operatorname{Im} i)^{\omega}$ can also be in $\operatorname{Im} i$. It is a general fact that $\dim V = \dim \operatorname{Im} i + \dim(\operatorname{Im} i)^{\omega}$, so the disjointedness of these two subspaces gives the direct sum decomposition.

3. Loop homotopy algebras

Loop homotopy algebras are a generalization of cyclic homotopy algebras, admitting maps with *genus* higher than 0. They were first defined by M. Markl in [3] by axiomatizing the algebraic properties of string products of closed string field theory [2]. We will only describe the generalization of cyclic L_∞ algebra, the loop homotopy Lie algebra.

We will be following the two original sources we mentioned, [2] and [3].

3.1 Closed string field theory

In [2], B. Zwiebach constructed *string products* and *string functions*. These operations take multiple elements of a Hilbert space $\mathcal{H}_{\text{rel}}$ (a subspace of the Hilbert space $\mathcal{H}$ of the whole theory) and output another state in $\mathcal{H}_{\text{rel}}$ and an complex number, respectively.

The string function, or a *string field vertex* is defined first, as an integral over a space of surfaces associated to an interaction vertex. Such vertex is specified by the n, the number of incoming strings and the genus g. For states $B_1, \dots, B_n$, and a surface of genus g, Zwiebach constructs a form (for details, see [2, sections 7.3, 7.4]) and integrates it over the mentioned space of surfaces, which gives a number denoted by

$$\{B_1, \dots, B_n\}_g \,. \tag{3.1}$$

This product is graded commutative and nonzero only on fields with total degree [1] $-2n$.

The Hilbert space is also equipped with a bilinear inner product, denoted

$$\langle B_1, B_2 \rangle \,.$$

On the subspace $\mathcal{H}_{\text{rel}}$ of fields B, this product is nondegenerate, symmetric and has degree -5. The nondegeneracy means it can be used to "lift one of the indices" of the multilinear string function. This is done by choosing a two bases $\{\Phi_s\}$ and $\{\Phi^r\}$ of $\mathcal{H}_{\text{rel}}$ that satisfy

$$\langle \Phi_s, \Phi^r \rangle = (-1)^{|\Phi_r|} \delta_s^r \,.$$

Then we can define the string products $[\dots]_g$ as

$$[B_1, \dots, B_n]_g = \sum_t (-1)^{|\Phi_t|} \Phi^t \{\Phi_t, B_1, \dots, B_n\}_g \,.$$

The inverse of this relation is given by

$$\{B_0, B_1, \dots, B_n\}_g = \langle B_0, [B_1, \dots, B_n]_g \rangle \,. \tag{3.2}$$

These brackets then have a degree $3 - 2n$ and are graded symmetric. Moreover, as a consequence of the the symmetry of the string functions, the brackets satisfy additional property: M. Markl expresses this by saying that element [3, eq. 7]

$$\sum_s \Phi_s \otimes [\Phi^s, B_1, \dots, B_{n-1}]_g \in \mathcal{H}_{rel}^{\otimes 2}$$

is antisymmetric with respect to the switching morphism.

[1] Regarding the degrees, the Hilbert space has double grading, $\mathbb{Z}$-grading by *ghost number* and $\mathbb{Z}_2$-grading by *Grassmanality*. Because the strings are bosonic, the Grassmanality is equal to ghost number modulo 2 (see [ZwiebachBV, see the paragraph *Grassmanality* in section 3.1]), which is the situation we considered in 2.4.1.

This is, however, not all the structure these operations have. Because the string vertices have to generate the complete moduli space of Riemann surfaces, Zwiebach uses geometrical recursion relations for the decomposition of surfaces to prove the *main identity* for the string products [2, eq. 4.13]

$$0 = \sum_{\substack{g_1+g_2=g\\ k+l=n}} \sum_{\sigma\in\mathrm{Unsh}(l,k)} \varepsilon(\sigma)[B_{\sigma(1)},\ldots,B_{\sigma(l)},[B_{\sigma(l+1)},\ldots,B_{\sigma(l+k)}]_{g_2}]_{g_1} + \frac{1}{2}\sum_s (-1)^{|\Psi_s|}[\Psi_s,\Psi^s,B_1,\ldots,B_n]_{g-1}\,, \tag{3.3}$$

which should be satisfied for all $n \geq 0$ and $g \geq 0$, for which we set $[\ldots]_{-1}$ equal to 0.

The string functions can be used to define an action S. We take a string field Ψ and define

$$S(\Psi) = \frac{1}{\kappa^2}\sum_{g\geq 0}(\hbar\kappa^2)^g \sum_{n\geq 0}\frac{\kappa^n}{n!}\{\Psi,\ldots,\Psi\}_g\,. \tag{3.4}$$

where the Ψ is inserted n times into the string function $\{\ldots\}_g$ with n arguments. We see that the parameter $\hbar$ just tracks the genus of the string function and κ is equal to $2(g-1)+n$. This κ is the *closed string field coupling constant*, which we will ignore. As a consequence of the main identity, this action then satisfies a *quantum master equation* [2, equation 4.69]

$$\frac{\partial_R S}{\partial\psi^s}\frac{\partial_L S}{\partial\psi^*_s} + \hbar\frac{\partial_R}{\partial\psi^s}\frac{\partial_L S}{\partial\psi^*_s} = 0\,,$$

where the *fields* ψ^s and *antifields* ψ^*_s correspond to regraded elements of dual Hilbert space (see [2, equation 3.33] and also [4, definition 3.2]).

3.2 Loop homotopy Lie algebras

The genus zero products of this theory have a structure L_∞, as was already know before Zwiebach (see [2, section 4.5]). Furthermore, with the bilinear form, we get a cyclic L_∞ algebra.

In [3], Markl coined a name *loop homotopy Lie algebra* for the full structure with $g \neq 0$ operations. To relate it to the usual definition of L_∞ algebra, which in turn is usually defined to agree with the dg Lie algebra, one has to shift the degree: instead of looking at $\mathcal{H}_{\mathrm{rel}}$, we consider $U \equiv \mathbf{r}(\downarrow \mathcal{H}_{\mathrm{rel}})$. On this vector space, operations $l_n^g : U^{\otimes n} \to U$ are defined with a Koszul sign coming from commuting graded elements and the operator $\downarrow$, which has degree 1:

$$l_n^g \equiv (-1)^{(n-1)|v_1|+(n-2)|v_2|+\cdots+v_{n-1}}\mathrm{r}\downarrow[\uparrow\mathbf{r}(v_1),\ldots,\uparrow\mathbf{r}(v_n)]_g \tag{3.5}$$

for $v_i \in U$. This shift and sign then makes these l_n^g antisymmetric, with degree $n-2$.

On U, we define a graded symmetric (which is, due to its degree, symmetric) form B, which comes from the inner product $\langle -,- \rangle$. Here, Markl does not use a "Koszul" sign, but defines

$$B(u,v) \equiv \langle\uparrow\mathbf{r}(u),\uparrow\mathbf{r}(v)\rangle\,.$$

The following definition is almost verbatim [3, definition 2.1].

Definition 3.1. A loop homotopy Lie algebra consists of

1. A $\mathbb{Z}$-graded vector space U,
2. a graded symmetric nondegenerate bilinear degree 3 form $B : U\otimes U \to \Bbbk$

3. a set $\{l_n^g\}_{n,g\geq 0}$ of degree $n-2$ multilinear graded antisymmetric operations $l_n^g : U^{\otimes n} \to U$,

satisfying these two conditions:

1. For any $n, g \geq 0$, for vectors $v_1, \ldots, v_n \in U$, we have the *main identity*

$$0 = \sum_{\substack{k,l,g_1,g_2\geq 0\\ k+l=n+1\\ g_1+g_2=g}} \sum_{\sigma\in \mathrm{unsh}(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}) \\ +\frac{1}{2}\sum_s (-1)^{h_s+n} l_{n+2}^{g-1}(h_s, h^s, v_1, \ldots, v_n)\,.$$

   The map l_n^g with $g=-1$ is taken to be 0. Vectors $\{h_s\}$ and $\{h^s\}$ form two bases of the vector space dual in the sense of the form B

$$B(h^s, h_t) = \delta_t^s\,.$$

2. For any $n, g \geq 0$ and $v_1, \ldots, v_{n-1} \in U$, the element

$$\sum_s (-1)^{n+1} h_s \otimes l_n^g(h^s, v_1, \ldots, v_{n-1}) \in U \otimes U \tag{3.6}$$

   is symmetric (with respect to the action of the switching map τ on $U\otimes U$).

△

One can easily check that the $g=0$ part of the main identity is a defining equation of L_∞ algebra. To get agreement with our convention, we have to shift the degrees and look at a reflection of suspension of U, $\mathbf{r}\uparrow U = \downdownarrows \mathcal{H}_{\mathrm{rel}}$. We will discuss this construction in section 3.3.

Note that as is, this definition also allows for the curved variant of homotopy algebras, with $n=0$. We will only discuss the case with $l_0^0 = 0$, which makes l_1^0 a differential.

3.2.1 Coalgebra description

As for coalgebras, there is a description of loop homotopy Lie algebras as coderivations. To account for higher genus operations, the symmetric coalgebra is tensored with polynomials in a formal variable t. The bilinear form is encoded in a *degree 2 coderivation.*

Coderivations of order r are dual notions to the derivations of order r. Order 1 derivations satisfy the Leibniz rule, order 1 coderivations the dual version of it. Order 2 derivations satisfy the seven-term identity (see section 2.3), and order 2 coderivation θ satisfies [3]

$$(\Delta\otimes \mathbb{1})\circ\Delta\circ\theta - (\mathbb{1}+\sigma_{231}+\sigma_{312})\circ(\Delta\otimes\mathbb{1})\circ(\theta\otimes\mathbb{1})\circ\Delta \\ +(\mathbb{1}+\sigma_{231}+\sigma_{312})\circ(\theta\otimes\mathbb{1}\otimes\mathbb{1})\circ(\Delta\otimes\mathbb{1})\circ\Delta = 0\,,$$

where σ_{ijk} is the right action of permutation given by $\sigma(1)=i$, $\sigma(2)=j$ and $\sigma(3)=k$. A coderivation like this is determined by the U and $\mathrm{Sym}^2(U)$ components of its result [3, proposition 3.7].

Because the definition of [3] uses the ordinary dg Lie algebra grading convention, the coderivations act on a shifted space $W \equiv \uparrow U$. Suspending the bases, we define an element $y \equiv \uparrow h_s \otimes h^s$, which is symmetric. Together with the formal variable t, we work with a space $\operatorname{Sym} W[t]$. The coderivation θ of order 2 is specified by two projections

$$\text{proj}_W \circ \theta = 0$$

and

$$\text{proj}_{\text{Sym}^2 W} \circ \theta(w) = \frac{1}{2} t y \text{ for } w = 1 \in \Bbbk \cdot t^0 \text{ and } 0 \text{ otherwise },$$

i.e. its only nonzero component is the constant one [3, equation 21]. This means that

$$\theta(w_1 \dots w_n) = \frac{1}{2} t y', w_1 \dots w_n \,.$$

Loop homotopy Lie algebra is then given by an order 1 coderivation of degree -1.

Theorem 3.2 ([3, theorem 4.2])**.** *There is a one-to-one correspondence between loop homotopy Lie algebra structures on the and degree -1 coderivations δ on* $\operatorname{Sym} W[t]$ *of order 1 such that*

$$(\delta + \theta)^2 = 0 \,.$$

3.2.2 Loop homotopy algebras and operads

In section 1.4.5, we explained how general homotopy algebras arise as algebras over operadic bar construction over Koszul dual operad. For loop homotopy algebras, a corresponding construction is a *Feynman transform*, which takes a *modular operad* and outputs a *twisted modular operad* defined on graphs decorated with the original modular operad (hence the name *Feynman*).

Loop homotopy Lie algebra is then an algebra over a Feynman transform of $\text{Mod}(\mathit{Comm})$, a modular completion of the operad *Comm* [3, theorem 5.5]. This suggests a generalization for arbitrary operad $\mathcal{P}$: loop homotopy $\mathcal{P}$-algebra is an algebra over the Feynman transform of $\text{Mod}(\mathcal{P}^!)$ [3, definition 6.1].

Barannikov [36] proved that algebras over modular operads are in one-to-one correspondence with solutions of a generalized master equation, where the BV algebra is defined in terms of the modular operad. In [37], this formalism is used to describe these algebras for physically relevant operads. The $\mathcal{P} = \mathit{Lie}$ case gives loop homotopy Lie algebras, as described by Zwiebach [2]. For operad *Ass*, [37] obtains the axioms and noncommutative master equation of loop homotopy associative algebra, or *quantum A_∞ algebra*. A special case for topological open strings of this was described by M. Herbst [38].

We remark that the connection of these algebras over modular operads to the regular operads *Comm* and *Ass* is not automatic, but has to be proven for every case. In other words, it is natural to define modular operads related to closed/open strings, called *quantum closed* and *quantum open* in [37], but showing that these are indeed the Feynman transforms of the modular envelopes of *Comm* and *Ass* is not trivial. For open-closed strings, the situation is not as simple, as the $g = 0$ part of the quantum open-closed operad is not even a cyclic operad (see [37, section VI.E] for more details).

3.3 Loop homotopy Lie algebras and master equation

We have seen that on $V \equiv \downdownarrows \mathcal{H}_{\text{rel}}$, the string products have degree 1. Moreover, the string functions $\{\ \}_g$ have degree 0 and the symmetric bilinear form $\langle , \rangle$ has product -1. If we, however, define

$$\omega(v, w) \equiv (-1)^{|v|} \langle \upuparrows v, \upuparrows w \rangle \,, \tag{3.7}$$

we have

$$\omega(w,v) = (-1)^{|w|}\langle \Uparrow w, \Uparrow v\rangle = (-1)^{|w|}\langle \Uparrow v, \Uparrow w\rangle = (-1)^{|w|+|v|}\omega(v,w) = -\omega(v,w)\,,$$

i.e. an odd symplectic form on V.

We also denote

$$s_n^g(v_1,\ldots,v_n) \equiv \{\Uparrow v_1,\ldots,\Uparrow v_n\}_g$$

and

$$\lambda_n^g(v_1,\ldots,v_n) \equiv \Downarrow [\Uparrow v_1,\ldots,\Uparrow v_n]_g\,.$$

the shifted versions of string functions and string products. They are related by (see also equation 1.16)

$$s_{n+1}^g(v_0,\ldots v_n) = (-1)^{|v_0|}\omega(v_0, \lambda_n^g(v_1,\ldots,v_n))\,.$$

The shifted string functions s_n^g can now be considered as degree 0 elements of $\mathcal{F}(V)$. Adding the $\hbar$ as a formal parameter (see section 2.4.2), we can also consider the action

$$S(v) \equiv \sum_{n,g\geq 0} \frac{\hbar^g}{n!}\{\Uparrow v,\ldots,\Uparrow v\}_g = \sum_{n,g\geq 0}\frac{\hbar^g}{n!} s_n^g(v,\ldots,v)$$

as a degree 0 element of $\mathcal{F}(V)[[\hbar]]$.

The master equation for this action

$$2\hbar\Delta S + \{S,S\} = 0$$

is in fact equivalent to the main identity 3.3. Zwiebach proves this, see [2, equation 4.69]. Note that the factorial factors from the action S correspond to the factorials one gets in the main identity 3.3 from the sum over unshuffles, because all arguments are the same.

Because s_n^g are graded symmetric and have degree 0, we have

$$s_2^g(v,w) = (-1)^{|w|}s_2^g(w,v)\,,$$

which implies

$$(-1)^{|v|}\omega(v,\lambda_1^g(w)) = \omega(w,\lambda_1^g(v))\,.$$

If we denote Q the differential λ_1^0, we can write this as

$$\omega(Qv,w) + (-1)^{|v|}\omega(v,Qw) = \omega\circ(Q\otimes \mathbb{1} + \mathbb{1}\otimes Q)(v,w) = 0\,.$$

Another useful identity is the relation between the action and the multilinear operations λ_n^g via the bracket. We can view $\{S,-\}$ as a left derivative of degree 1, which means it is completely specified by its action on covectors ϕ^k

$$\{S,\phi^k\} = \sum_{g,n}\frac{\hbar^g}{n!}\{s_n^g,\phi_k\}\,.$$

Looking at just the component s_n^g, we have (see A for the convention regarding duals)

$$\begin{aligned}\{s_n^g,\phi^k\} = \frac{\partial_R s_n^g}{\partial\phi^i}\omega^{ik} &= n s_n^g(-,\ldots,-,\mathbf{e}_i)\omega^{ik}\\ &= (-1)^{|\mathbf{e}_i|}n\omega^{ik}s^{g_n}(\mathbf{e}_i,-,\ldots,-)\\ &= n\omega^{ik}\omega(e_i,\lambda_{n-1}^g(-,\ldots,-))\\ &= n\omega^{ik}\omega_{il}\phi^l\circ\lambda_{n-1}^g\\ &= -n\phi^k\circ\lambda_{n-1}^g\\ &= (-1)^{|\phi^k|}n\,(\lambda_{n-1}^g)^{\#}(\phi^k)\,.\end{aligned}$$

This allows us to convert between the action and the loop homotopy algebra operations. A special case of this formula is for $n = 2$ and $g = 0$. This is the quadratic $\hbar^0$ component of the action, for which we have

$$\{S_0, -\} = \{\frac{1}{2}s_2^0, -\} = Q^{\#} \circ (-1)^{|-|} \equiv \hat{Q}^{\#} \tag{3.8}$$

If we extend the dual of Q on $\mathcal{F}(V)[[\hbar]]$ using the Leibniz rule, this equality holds also on this space.

We introduced $\hat{Q}^{\#}$

$$\hat{Q}^{\#}(\phi^k) = (-1)^{|\phi^k|} Q^{\#}(\phi^k)\,.$$

This is just for our convenience, so that we can write $\{S_0, -\} = \hat{Q}^{\#}$ on $V^{\#}$. After extending this as a derivative, we will use notation $\{S_0, -\} = \hat{Q}$.

4. Effective action and minimal model

In this chapter, we will construct a loop homotopy algebra on the homology of the vector space V. We will present two approaches to do so, heuristic path integral and a use of homological perturbation lemma. We will also (heuristically) prove their equivalence.

Underlying ideas for some of these constructions can be found in P. Mnev's [5] and H. Kajiura's [4].

4.1 Homological perturbation lemma for $\mathcal{F}(V)$

Homological perturbation lemma is a useful tool of homological algebra, giving explicit formulas for differential transferred from one chain complex to another using a *homotopy equivalence* between them. This is exactly the situation we have: we will transfer the differential given by the solution of master equation to the homology with respect to Q.

Very nice exposition to the homological perturbation lemma is a paper by Crainic [39]. We will now recall its statement to fix notation.

Definition 4.1. A *homotopy equivalence* is given by the following data

$$h \circlearrowright (V, \mathrm{d}_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, \mathrm{d}_W) \ , \tag{4.1}$$

where (V, d_V) and (W, d_W) are chain complexes, p, i are quasi-isomorphisms and

$$i \circ p = 1 + \mathrm{d}_V \circ h + h \circ \mathrm{d}_V \, . \tag{4.2}$$

Homotopy equivalence for which $p \circ i = \mathbb{1}$ on W is called *deformation retract* and a deformation retract for which the following *annihilation conditions*

$$h \circ i = 0, \; p \circ h = 0, \; h \circ h = 0 \, ,$$

hold is called *special deformation retract*, or SDR.

△

The equation 4.2 tells us that on homology, the maps $i \circ p$ is homotopic to identity, with homotopy given by h. The annihilation conditions give a decomposition of V into three subspaces, given by projectors $i \circ p$, $-\mathrm{d}_V \circ h$ and $-h \circ \mathrm{d}_V$.

We can now state the perturbation lemma.

Lemma 4.2. *[39, lemmas 2.4, 3.2] Given homotopy equivalence as in equation 4.1 and a map $\delta : V \to V$ such that $|\mathrm{d}_V| = |\delta|$ and $(\mathrm{d}_V + \delta)^2 = 0$, and such that $(1 - \delta h)$ is invertible, there is a new homotopy equivalence (with the same vector spaces)*

$$h' \circlearrowright (V, \mathrm{d}_V + \delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (W, \mathrm{d}'_W) \ , \tag{4.3}$$

where the maps are given by

$$h' \equiv h + h(1-\delta h)^{-1}\delta h\,, \tag{4.4}$$

$$p' \equiv p + p(1-\delta h)^{-1}\delta h\,, \tag{4.5}$$

$$i' \equiv i + h(1-\delta h)^{-1}\delta i\,, \tag{4.6}$$

$$\mathrm{d}'_W \equiv \mathrm{d}_W + p(1-\delta h)^{-1}\delta i\,. \tag{4.7}$$

Moreover, if the initial data were a special deformation retract, the perturbed homotopy equivalence is also a special deformation retract.

Proof. Proof is done by explicitly checking the homotopy equivalence conditions for the perturbed maps, see [39]. □

4.1.1 Decomposition of odd symplectic space

In previous chapter, we have seen that associated with a loop homotopy algebra is a vector space V with odd symplectic structure ω and a solution of quantum master equation S. Specially, this gives us a differential Q that is compatible with ω

$$\omega \circ (Q \otimes \mathbb{1} + \mathbb{1} \otimes Q) = 0\,.$$

We want to construct a special deformation retract between this space V and H_Q, the homology with respect to Q. This amounts to choosing a decomposition V into subspaces as

$$V = \tilde{H}_Q \oplus \operatorname{Im} Q \oplus C\,,$$

where $\tilde{H}_Q \oplus \operatorname{Im} Q = \operatorname{Ker} Q$. For simplicity, we will take $\tilde{H}_Q$ to be the cohomology with respect to Q and denote it H_Q. We can do this because $\tilde{H}_Q$ and H_Q are isomorphic by an isomorphism that is also a chain map and a symplectomorphism.

The differential Q is then an isomorphism of vector space $\operatorname{Im} Q$ and C, so the homotopy h is minus of its inverse. The projector onto homology p is now an isomorphism between $\tilde{H}_Q$ and H_Q and i will be its inverse. The differential on H_Q is zero.

To work with this decomposition on the BV algebra on $\mathcal{F}(V)$, we want to put additional conditions to this decomposition. Specifically, we will also ask for the compatibility of h with the odd symplectic form, namely

$$\omega \circ (h \otimes \mathbb{1} - \mathbb{1} \otimes h) = 0\,.$$

Existence of such decomposition follows from [40, theorem 2.7], where it is called *harmonious Hodge decomposition*. From definition 2.1 ibid., we see that such harmonious decomposition for (V, Q, ω) is given by a operator $s : V \to V$ such that

$$s^2 = 0\,,$$
$$\omega(sv, w) = (-1)^{|v|}\omega(v, sw)\,,$$
$$sQs = s\,,$$
$$QsQ = Q\,.$$

Then we define (following proposition 2.5 ibid.)

$$\tilde{H}_Q \equiv \operatorname{Im} t \equiv \operatorname{Im}(\mathbb{1} - sQ - Qs)\,,$$
$$C \equiv \operatorname{Im} s\,,$$

which gives

$$V = \tilde{H}_Q \oplus \operatorname{Im} Q \oplus C$$

with $\tilde{H}_Q \simeq H_Q$ and $\tilde{H}_Q^{\omega} = \operatorname{Im} Q \oplus C$ (this is [40, proposition 2.5]).

Finally, we discuss how the symplectic form pairs the subspaces:

- The subspace $\tilde{H}_Q$ is a symplectic subspace, meaning that the symplectic form on it is nondegenerate. Equivalently, its symplectic complement is $\operatorname{Im} \mathcal{Q} \oplus C$. Thus, we have a situation described in section 2.4.3 and the BV algebra decomposes into operations on $\mathcal{F}(\tilde{H}_Q)$ and on $\mathcal{F}(\operatorname{Im} \mathcal{Q} \oplus C)$. We denote these by Δ', $\{\}'$ and Δ'', $\{\}''$, respectively.
- The subspace $\operatorname{Im} \mathcal{Q}$ has a symplectic complement $(\operatorname{Im} \mathcal{Q})^\omega = \operatorname{Ker} \mathcal{Q} = \operatorname{Im} \mathcal{Q} \oplus \tilde{H}_Q$, thanks to the compatibility with $\mathcal{Q}$. This means that for $v \in \operatorname{Im} \mathcal{Q}$, the expression $\omega(v, w)$ is nozero only for $w \in C$.
- For C, let's choose a nonzero vector $hv \in C = \operatorname{Im} h$. Then

  $$\omega(hv, w) = \pm\omega(v, hw)$$

  and we see that $\operatorname{Ker} h \subset C^\omega$. However, since it has the right dimension, we know they are equal $C^\omega = \operatorname{Ker} h = \tilde{H}_Q \oplus C$. Together with the previous point, we see that on $\operatorname{Im} \mathcal{Q} \oplus C$, the nondegenerate form ω pairs only vectors between C and $\operatorname{Im} \mathcal{Q}$.

To get a deformation retract, set $h = -s$, denote $p : V \to H_Q$ the projection onto a direct summand and $i : H_Q \to V$ its inverse, the inclusion. This gives $i \circ p = t = \mathbb{1} - s\mathcal{Q} - \mathcal{Q}s = \mathbb{1} + h\mathcal{Q} + \mathcal{Q}h$. The annihilation conditions are met because the vector space is decomposed as a direct sum. Thus we have a special deformation retract

$$h \circlearrowright (V, \mathcal{Q}) \underset{i}{\overset{p}{\rightleftarrows}} (H_Q, 0) \ . \tag{4.8}$$

4.1.2 Decomposition of $\mathcal{F}(V)$

The next step is a construction of deformation retract between $\mathcal{F}(V)$ and $\mathcal{F}(H_Q)$. First, we have to dualize the SDR 4.8

$$h^\# \circlearrowright (V^\#, \mathcal{Q}^\#) \underset{p^\#}{\overset{i^\#}{\rightleftarrows}} (H_Q^\#, 0) \ . \tag{4.9}$$

Since the dualization switches the projection and the inclusion, all the conditions on special deformation retract remain satisfied. For example, the equation $i \circ p = \mathbb{1} + \mathcal{Q} \circ h + h \circ \mathcal{Q}$ becomes

$$p^\# \circ i^\# = \mathbb{1} + \mathcal{Q}^\# \circ h^\# + h^\# \circ \mathcal{Q}^\# \,, \tag{4.10}$$

which is what we want, because $p^\#$ is the new inclusion and vice versa.

Next, we replace $\mathcal{Q}^\#$ by $\hat{\mathcal{Q}}^\#$ (recall equation 3.8), so that we can write the differential as $\{S_0, -\}$. This can be done, because the new operator differs only by a sign on homogeneous elements. It means its kernel and image are the same and it the maps $p^\#$ and $i^\#$ are still chain maps. To satisfy the equation 4.10, we also redefine $h^\#$. Since

$$\hat{\mathcal{Q}}^\# \equiv \mathcal{Q}^\# \circ (-1)^{|-|} \,,$$

we define

$$\hat{h}^\# \equiv (-1)^{|-|} \circ h^\# \,.$$

Finally, we extend to symmetric power algebras. We use capital letters to denote maps in this case:

$$\hat{H} \circlearrowright (\mathcal{F} V, \hat{Q}) \underset{I}{\overset{P}{\rightleftarrows}} (\mathcal{F}(H_Q), 0) \ . \tag{4.11}$$

All these operators are linear, so we will only define them on monomials of degree n. For the projection and inclusion, the extension is given by the tensor powers of the dual operators. For $\hat{Q} = \{S_0, -\}$, we use the Leibniz rule. The extension of the homology is more involved and sometimes goes by the name *tensor trick*.

- The differential $\hat{Q}$ extended with graded Leibniz rule. With Koszul sign convention, we can write this as $\hat{Q} = \sum_{j=0}^{n-1} \mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes (n-j-1)}$.
- Since i and p both have different domain and codomain, the most natural way to define them is $P = (i^{\#})^{\otimes n}$ and $I = (p^{\#})^{\otimes n}$.
- We will define $\hat{H}$ drawing an inspiration from [41, section XI.5], where they extend the deformation retract (*contracting homotopy* in their terminology) on the second tensor power by $h_2 = h \otimes \mathbb{1} + ip \otimes h$. Going to third tensor power, one would get $h_3 = h \otimes \mathbb{1} \otimes \mathbb{1} + ip \otimes (h \otimes \mathbb{1} + ip \otimes h)$. The pattern is that there is always one operator h, and the rest is filled with $\mathbb{1}$ and ip. Thus, we define

$$\hat{H}^{(i)} \equiv \mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)}$$

and

$$\hat{H}_{\mathrm{u}} \equiv \sum_{i=0}^{n-1} \hat{H}^{(i)} \,.$$

Since we want the result of $\hat{H}$ to be symmetric, we also have to symmetrize, so we define

$$\hat{H} \equiv \sum_{\sigma \in \mathbb{S}_n} \frac{1}{n!} \sigma^r \circ \hat{H}_{\mathrm{u}} = \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} \,.$$

Where we used the projector $\boldsymbol{\sigma}_n = \sum_{\sigma \in \mathbb{S}_n} \sigma^r$. Similar formula also appears in [42, theorem 1.4].

Now, we compute $\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q}$ on monomial of degree n. At first, note that because $\hat{H} = \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}}$, this is equal to

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = \hat{Q} \circ \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} + \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} \circ \hat{Q}$$

and using the fact that the differential commutes with the symmetric action, we get

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = \boldsymbol{\sigma}_n \circ (\hat{Q} \circ \hat{H}_{\mathrm{u}} + \hat{H}_{\mathrm{u}} \circ \hat{Q}) \,.$$

Because $\hat{H}_{\mathrm{u}}$ is a sum of operators $\hat{H}^{(i)}$, we calculate $\hat{Q} \circ \hat{H}^{(i)} + \hat{H}^{(i)} \circ \hat{Q}$. Even further, only taking $\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes (n-j-1)}$ from $\hat{Q}$, we get

$$\begin{aligned} &\left(\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes (n-j-1)}\right) \circ \left(\mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)}\right) \\ +&\left(\mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)}\right) \circ \left(\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes (n-j-1)}\right) \,. \end{aligned}$$

If $i \neq j$, in one of the terms $\hat{h}^{\#}$ commutes with $\hat{Q}^{\#}$ and in the other term it does not, so these terms subtract. If $i = j$, we get

$$\begin{aligned} &\mathbb{1}^{\otimes i} \otimes (\hat{Q}^{\#} \circ \hat{h}^{\#} + \hat{h}^{\#} \circ \hat{Q}^{\#}) \otimes (p^{\#} \circ i^{\#})^{n-i-1} \\ &= \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#} - \mathbb{1}) \otimes (p^{\#} \circ i^{\#})^{n-i-1} \\ &= \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i)} - \mathbb{1}^{\otimes (i+1)} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)} \,. \end{aligned}$$

We can therefore write

$$\hat{Q} \circ \hat{H}^{(i)} + \hat{H}^{(i)} \circ \hat{Q} = \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i)} - \mathbb{1}^{\otimes (i+1)} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)} \,,$$

since there is exactly one term with $i = j$ in $\hat{Q}$. This is very good, because summing these terms from $i = 0$ to $n-1$ results in a telescopic cancellation, leaving only

$$\hat{Q} \circ \hat{H}_{\mathrm{u}} + \hat{H}_{\mathrm{u}} \circ \hat{Q} = (p^{\#} \circ i^{\#})^{\otimes n} - \mathbb{1}^{\otimes n} .$$

This is symmetric, so for

$$\hat{H} = \frac{1}{n!}\boldsymbol{\sigma}_n \circ \left(\sum_{i=0}^{n-1} \mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes (n-i-1)} \right)$$

we have

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = (p^{\#} \circ i^{\#})^{\otimes n} - \mathbb{1}^{\otimes n} = IP - \mathbb{1} ,$$

where the last identity morphism is on $\mathcal{F}(V)$.

For a special deformation retract, the identities $\hat{H} \circ I = 0$ and $P \circ \hat{H} = 0$ are obviously satisfied. The identity

$$\hat{H}^2 = 0$$

follows from $h^2 = 0$, from the Koszul sign -1 when from commuting operators $\hat{h}^{\#}$, and from the annihilation conditions for $\hat{h}^{\#}$, $p^{\#}$ and $i^{\#}$.

4.1.3 A special basis

For the practical purposes of calculations, we choose bases of the three subspaces of V. We will denote the basis elements of H_Q to be a_i, the basis of $\operatorname{Im} \mathsf{Q}$ to be b_i and the basis of C to be c_i. We also diagonalize the isomorphisms h and Q of $\operatorname{Im} d$ and C, so that

$$\mathsf{Q}(c_i) = -b_i \quad \text{and} \quad h(b_i) = c_i .$$

The corresponding dual elements of $V^{\#}$ are denoted with $\alpha^i \in H_Q^{\#}$, $\beta^i \in (\operatorname{Im} \mathsf{Q})^{\#}$ and $\gamma^i \in C^{\#}$. The dual morphisms $\mathsf{Q}^{\#}$ and $h^{\#}$ exchange their domain and codomain, so that

$$\hat{\mathsf{Q}}^{\#}(\beta^i) = \gamma^i \quad \text{and} \quad \hat{h}^{\#}(\gamma^i) = \beta^i .$$

This is because $\mathsf{Q}^{\#}(\beta^i) = (-1)^{|\beta^i|}(-1)^{|\beta^i|+1}\beta^i \circ \mathsf{Q}$, which is equal to δ^i_j on c_j and similarly for $\hat{h}^{\#}$.

We formally can write these maps on $V^{\#}$ as

$$\hat{\mathsf{Q}}^{\#} = \sum_i \gamma^i \frac{\partial}{\partial \beta^i}$$

and

$$\hat{h}^{\#} = \sum_i \beta^i \frac{\partial}{\partial \gamma^i} .$$

We denote the elements of $F(V)$ by the same symbols as for $V^{\#}$. With the partial derivative obeying the Leibniz rule, we can concisely write

$$Q = \sum_i \gamma^i \frac{\partial_L}{\partial \beta^i} ,$$

because the Leibniz rule is implied in the left derivative.

For P, we have $p(a_i) = a_i$ and $i(a_i) = a_i$. Thus, the operator P is identity on all monomials composed of covectors α^i, and zero on monomials containing at least one β or γ. The I is formally an identity, but goes from $\mathcal{F}(H_Q)$ to $\mathcal{F}(V)$.

One can notice that the symmetrization of $\hat{H}$ looks like the Leibniz rule, but with $p^\# \circ i^\#$ instead of $\mathbb{1}$. However, these are projectors, so in the basis we just defined they are either zero on a element, or leave it unchanged. The symmetrization determines the number of unchanged elements, which after nontrivial manipulation gives a combinatorial factor

$$\hat{H} = \frac{1}{\hat{\eta}} \sum_i \beta^i \frac{\partial_L}{\partial \gamma^i} ,$$

where $\hat{\eta}$ is the number of βs and γs in the monomial $\hat{H}$ is acting on. For example,

$$\begin{aligned}
\hat{H}(\alpha\beta^i\gamma^j) &= \frac{1}{2}(-1)^{|\alpha|+|\beta^i|}\alpha\beta^i\beta^j \\
\hat{Q}\circ\hat{H}(\alpha\beta^i\gamma^j) &= \frac{1}{2}(-1)^{|\beta^i|}\alpha\gamma^i\beta^j + \frac{1}{2}\alpha\beta^i\gamma^j \\
\hat{Q}(\alpha\beta^i\gamma^j) &= (-1)^{|\alpha|}\alpha\gamma^i\gamma^j \\
\hat{H}\circ\hat{Q}(\alpha\beta^i\gamma^j) &= \frac{1}{2}\alpha\beta^i\gamma^j + \frac{1}{2}(-1)^{|\gamma^i|}\alpha\gamma^i\beta^j .
\end{aligned}$$

and because the degree of β^i and γ^i are opposite, we get

$$(\hat{H}\circ\hat{Q} + \hat{Q}\circ\hat{H})(\alpha\beta^i\gamma^j) = \alpha\beta^i\gamma^j$$

The origin of the factor $\frac{1}{\hat{\eta}}$ can be easily seen from computing $\hat{H}\circ\hat{Q} + \hat{Q}\circ\hat{H}$

$$\begin{aligned}
\hat{H}\circ\hat{Q} + \hat{Q}\circ\hat{H} &= \frac{1}{\hat{\eta}}\sum_i \beta^i\frac{\partial_L}{\partial\gamma^i}\sum_j \gamma^j\frac{\partial_L}{\partial\beta^j} + \frac{1}{\hat{\eta}}\sum_i \beta^i\frac{\partial_L}{\partial\gamma^i}\sum_j \gamma^j\frac{\partial_L}{\partial\beta^j} \\
&= \frac{1}{\hat{\eta}}\sum_i\left(\beta^i\frac{\partial_L}{\partial\beta^i} + \gamma^i\frac{\partial_L}{\partial\gamma^i}\right)
\end{aligned}$$

This is a projector on variables β and γ extended like a derivative, which is why we have to divide by their total number.

4.2 Perturbing by master action

Recall from section 2.4.2 that both $\hat{Q} = \{S_0, -\}$ and $T_S = \hbar\Delta + \{S, -\}$ are differentials. We constructed a special deformation retract from the first differential using $\{S_0, -\}$ as the differential on $\mathcal{F}(V)[[\hbar]]$. It is natural to consider the full differential T_S as

$$T_S = \{S_0, -\} + \hbar\Delta + \{S - S_0, -\} \equiv \hat{Q} + \delta ,$$

i.e. to view the full differential as a deformation of $\{S_0, -\}$.

Is this perturbation small? Yes, because $\hbar\Delta + \{S - S_0, -\}$ either adds a power of $\hbar$, or adds the polynomial degree, because we removed S_0, which is quadratic and has no $\hbar$. In the homological perturbation lemma, we need to compute $(1 - \delta\hat{H})^{-1}\delta$. Expanding

$$(1 - \delta\hat{H})^{-1}\delta = \delta + \delta\hat{H}\delta + \delta\hat{H}\delta\hat{H}\delta + \dots$$

we see that for fixed power of $\hbar$ and polynomial degree, we get contribution from a finite number of terms.

The homological perturbation lemma gives us a special deformation retract

$$\hat{H}' \circlearrowright \left(\mathcal{F}(V)[[\hbar]], \hat{Q} + \delta\right) \underset{I'}{\overset{P'}{\rightleftarrows}} \left(\mathcal{F}(H_Q)[[\hbar]], D'\right) . \tag{4.12}$$

Let us look at the transferred differential D'. By lemma 4.2, we have

$$D' = P(1 - \delta\hat{H})^{-1}\delta I = P(\delta + \delta\hat{H}\delta + \dots)\,.$$

The first term of this expansion is $P(\hbar\Delta + \{S - S_0, -\})I$. Evaluating this on a function $F \in \mathcal{F}(H_Q)[[\hbar]]$, we get

$$P\hbar\Delta F + P\{S - S_0, F\}\,,$$

where we do not write the inclusion I, thinking of $\alpha \in H_Q^{\#}$ and $\alpha \in V^{\#}$ as same objects.

In first term, only Δ' survives, since there are no coordinates β and γ for it to act.

The second term, where we denote $S' \equiv S - S_0$, also has only the primed operation $\{,\}'$. Furthermore, all the terms of S' with variables β and γ survive this primed bracket $\{,\}'$ and are annihilated by P. Thus, only the terms of S' without β and γ can play a role, which we can write as

$$D'(F) = \hbar\Delta' F + \{P(S'), F\}' + \dots\,.$$

We conjecture that this is in fact a general form of this differential, i.e.

$$D'(F) = \hbar\Delta' F + \{W, F\}' \tag{4.13}$$

for some $W \in \mathcal{F}(H_Q)[[\hbar]]$. This then implies, because $(D')^2 = 0$, that

$$2\hbar\Delta' W + \{W, W\}' = 0\,, \tag{4.14}$$

i.e. *W would determine a loop homotopy algebra on H_Q*. We will show that D' indeed has a form $\hbar\Delta' + \{W, -\}$, but first we make it seem plausible using BV formalism.

4.2.1 Effective action W

There is in fact a physical construction leading to this W, given by integrating out the variables β and γ. Such integration is physically motivated by the construction of physical states, which are the homology of the differential $\hat{Q}$. The states in $\operatorname{Im} Q$ are then called trivial and states in C, that is not in $\operatorname{Ker} Q$, are called unphysical. Integrating out these degrees of freedom results in an *effective action* that is a functional of the physical states.

We therefore define the *effective action* $W \in \mathcal{F}(H_Q)[[\hbar]]$ as (see [34] for similar calculation)

$$e^{W/\hbar} \equiv \int_{L''} e^{S/\hbar}\,. \tag{4.15}$$

This is the BV formalism integral, which means we have to integrate over Lagrangian submanifold L'' in the space $\operatorname{Im} Q \oplus C$. Note that W depends on the choice L'', but we don't denote it.

This effective action satisfies the master equation in BV algebra on $\mathcal{F}(H_Q)[[\hbar]]$, which can be easily proven

$$\Delta' e^{W/\hbar} = \Delta' \int_{L''} e^{S/\hbar} = \int_{L''} \Delta' e^{S/\hbar} = \int_{L''} (\Delta - \Delta'') e^{S/\hbar}\,.$$

Here, we moved Δ' under the integral because Δ' and the integral act on different variables. This is equal to zero, because $\Delta e^{S/\hbar} = 0$ by master equation and $\int_{L''} \Delta''(\dots) = 0$ by the theorem 2.1.

We also define a normalized *effective observable*, given by integrating a functional $F \in \mathcal{F}(V)[[\hbar]]$ with weight $e^{S/\hbar}$

$$P_{L''}(F) \equiv \frac{\int_{L''} F e^{S/\hbar}}{\int_{L''} e^{S/\hbar}} = e^{-W/\hbar} \int_{L''} F e^{S/\hbar}\,,$$

and we aim to find a Lagrangian subspace L'' such that this map is the projector P' from equation 4.12. At first, we show that it is a chain map between the natural differentials on our spaces,

$$T_S = \hbar e^{-S/\hbar}\Delta(- \cdot e^{S/\hbar}) = \hbar\Delta + \{S, -\}\,,$$

acting on $\mathcal{F}(V)[[\hbar]]$, and

$$T_W = \hbar e^{-W/\hbar}\Delta(- \cdot e^{W/\hbar}) = \hbar\Delta' + \{W, -\}'\,,$$

acting on $\mathcal{F}(H_Q)[[\hbar]]$. To see that $P_{L''}$ is a chain map, we compute

$$\begin{aligned} T_W \circ P_{L''}(F) &= \hbar e^{-W/\hbar}\Delta'\left(P_{L''}(F)\cdot e^{W/\hbar}\right)\\ &= \hbar e^{-W/\hbar}\Delta'\left(\int_{L''} F e^{S/\hbar}\right)\\ &= \hbar e^{-W/\hbar}\int_{L''}\Delta'\left(F e^{S/\hbar}\right)\\ &= \hbar e^{-W/\hbar}\int_{L''}(\Delta-\Delta'')\left(F e^{S/\hbar}\right)\\ &= \hbar e^{-W/\hbar}\int_{L''}\Delta\left(F e^{S/\hbar}\right)\\ &= \hbar e^{-W/\hbar}\int_{L''} e^{S/\hbar}\cdot e^{-S/\hbar}\cdot\Delta\left(F e^{S/\hbar}\right)\\ &= e^{-W/\hbar}\int_{L''} e^{S/\hbar}\cdot T_S(F)\\ &= P_{L''}\circ T_S(F)\,. \end{aligned} \tag{4.16}$$

Now we show how $P_{L''}$ and P' relate, starting from an equation

$$I'\circ P' - \mathbb{1} = H'\circ T_S + T_S\circ H'\,,$$

which holds because 4.12 is a SDR. We evaluate this on $F \in \mathcal{F}(V)[[\hbar]]$ and integrate with weight $e^{S/\hbar}$, obtaining

$$\int_{L''} I'\circ P'(F)\cdot e^{S/\hbar} - \int_{L''} F e^{S/\hbar} = \int_{L''} H'\circ T_S(F)\cdot e^{S/\hbar} + \int_{L''} T_S\circ H'(F)\cdot e^{S/\hbar}\,.$$

We know that $\hat{H} = 1/\hat{\eta}\beta^i\frac{\partial_L}{\partial\gamma^i}$, meaning that either $\hat{H}(G)$ leaves at least one covector β in $\hat{H}(G)$, or $\hat{H}(G) = 0$. Furthermore, we know $H' = \hat{H}\circ(\mathbb{1} + (1-\delta\hat{H})^{-1}\delta\hat{H})$, so also H' leaves either nothing or at least one covector β^i. If we thus choose $L'' = C \subset \operatorname{Ker}\beta^i$ for all i, the term

$$\int_{L''=C} H'\circ T_S(F)\cdot e^{S/\hbar}$$

is zero. The term

$$\int_C T_S\circ H'(F)\cdot e^{S/\hbar}$$

can be written, using similar steps as in equation 4.16, as

$$\begin{aligned} \int_C T_S(H'(F))\cdot e^{S/\hbar} &= e^{W/\hbar}P_C[T_S(H'(F))]\\ &= e^{W/\hbar}T_W[P_C(H'(F))] = \hbar\Delta'\int_C H'(F) = 0\,. \end{aligned}$$

Thus, we have

$$\int_C I' \circ P'(F) \cdot e^{S/\hbar} = \int_C F e^{S/\hbar} .$$

We know that $I' = I + \hat{H} \circ (1 - \delta \circ \hat{H})^{-1} \circ I$ and integral of $\hat{H} \circ (1 - \delta \circ \hat{H})^{-1} \circ I$ is zero, so we have

$$\int_C I \circ P'(F) \cdot e^{S/\hbar} = \int_C F e^{S/\hbar} .$$

The function $I \circ P'(F)$ is only a function on H_Q, which means we can take it out of the integral, which integrates the variables β and γ

$$P'(F) \int_C e^{S/\hbar} = \int_C F \cdot e^{S/\hbar} .$$

Here, we don't write I on the LHS, because we take take $\int_C$ as taking values in $\mathcal{F}(H_Q)[[\hbar]]$. Moving $\int_C e^{S/\hbar} = e^{W/\hbar}$ on the RHS, we get

$$P'(F) = P_C(F) . \tag{4.17}$$

The projection P' intertwines differentials T_S and D', projection P_C intertwines T_S and T_W, but because these two projections are equal, we have

$$T_W \circ P_C = P_C \circ T_S = P' \circ T_S = D' \circ P' .$$

At last, precomposing with I', for which we have $P' \circ I' = \mathbb{1}$, we get

$$T_W = D' ,$$

which holds if we choose the Lagrangian subspace of $\operatorname{Im}_Q \oplus C$. Of course, other Lagrangian subspaces give us different W which again solve a master equation. However, this change of W is only by a gauge transformation (in the space of master actions) – see [5, section 4.3, statement 6].

This computation should be viewed as a heuristic argument for considering the homological perturbation lemma for constructing new loop homotopy algebras. Mainly, we have not shown that the path integral exists even in this finite-dimensional context.

4.2.2 Perturbed differential D'

Coming back to the perturbation lemma, we want to continue studying the formula for the perturbed differential

$$D' = P \circ (1 - \delta \circ \hat{H})^{-1} \circ \delta \circ I = P \circ (\delta + \delta \circ \hat{H} \circ \delta + \dots) \circ I ,$$

where

$$\delta = \hbar\Delta + \{S', -\} .$$

We have already seen that the first term of expansion of D' gives $\hbar\Delta' + \{P(S'), -\}'$. In this section, we show that considering all terms, we get a differential in the correct form to define a loop homotopy algebra on H_Q

Theorem 4.3. *There is a function $W \in \mathcal{F}(H_Q)[[\hbar]]$ such that*

$$D' = \hbar\Delta' + \{W, -\}' .$$

Proof. A typical term in the expansion of D' looks like

$$P \circ \delta \circ \hat{H} \circ \cdots \circ \hat{H} \circ \delta \circ I\,.$$

This term is nonzero only if, when applying it on a function $F \in \mathcal{F}(H_Q)$, we get an expression *without* any variables β or γ, because there is the projection P at the end.

Clearly, the rightmost δ operator, which acts only on a function only with variables α, reduces to

$$\delta \circ I(F) = \hbar\Delta' I(F) + \{S', I(F)\}'\,,$$

because the double-primed operations are zero on F. We will abuse the notation and write F for $I(F)$, so we write

$$\delta \circ I(F) = \hbar\Delta' F + \{S', F\}'\,.$$

Note that this holds even without projecting out on the homology (one could say *off-shell*).

How could the statement that $D'F = \hbar\Delta' F + \{W, F\}'$ go wrong? Roughly, the operators acting on F could be derivatives of order higher than one. Of course, one such order 2 derivative, $\hbar\Delta'$, is there, but there can be no other if we want to write their action as a bracket with W. We now show that there are no such higher order derivatives.

Lemma 4.4. *There are functions $W_i \in \mathcal{F}(H_Q)[[\hbar]]$ such that we can write the perturbed differential D as*

$$D'F = \hbar\Delta' F + W_i\,\omega_\alpha^{ij}\frac{\partial_L F}{\partial \alpha^j}\,, \tag{4.18}$$

where ω_α^{ij} is ω^{ij} for indices i, j corresponding to α.

Proof. (of the lemma) The proof is based on tracking the number of covectors β:

- The operator $\hat{H} = -\frac{1}{\hat{\eta}}\beta^i \frac{\partial_L}{\partial\gamma^i}$ removes one γ and adds one β.
- The operator Δ'' removes one β *and* one γ.
- The bracket with S' either removes β or γ, but can add more terms. Specifically, only $\{S', -\}''$ can remove β or γ.

We see that $\hat{H}$ always adds one β (or the result is zero) and δ might or might not remove a β, which means that together they can only increase the number of covectors β. In a string of operators

$$P \circ \delta \circ \hat{H} \circ \cdots \circ \delta \circ \hat{H} \circ \delta(F)\,, \tag{4.19}$$

the first δ is only acting on the primed coordinates. Then the $\hat{H}$ adds one vector β. The next δ must remove this vector β, because if it didn't, the number of vectors β in this term would be at least one. Such term would be projected to zero with P.

So, denoting $\delta = \delta' + \delta''$ the decomposition of δ to primed and double-primed operators, we have that the leftmost δ is always δ' and the other operators δ are δ'' in every nonzero term of equation 4.19.

Thus, in the string of operators

$$P \circ \delta'' \circ \hat{H} \circ \cdots \circ \delta'' \circ \hat{H} \circ \delta'(F)\,,$$

the first operator δ' differentiates F, but the others never act on covectors α. The operator δ' contains $\hbar\Delta'$, but all terms starting with $\hbar\Delta' F$ followed by $\hat{H}$ are zero, and only the term $P \circ \hbar\Delta' \circ I(F)$ survives.

For terms starting with $\{S', F\}'$, we have $(\delta''\hat{H})^n$ acting on them. The operator $\delta'' \circ \hat{H}$ only affects the double-primed coordinates and therefore, they only act on S' in $\{S', F\}'$, determining the factors W_i:

$$P \circ (\delta'' \circ \hat{H})^n(\{S', F\}') = \left[P \circ (\delta'' \circ \hat{H})^n \left(\frac{\partial_R S}{\partial \alpha^i}\right)\right] \omega_\alpha^{ij} \frac{\partial_L F}{\partial \alpha^j} . \tag{4.20}$$

□

This result also gives some restrictions on S': we see that only terms of S' with no β dependence can appear in W_i. This corresponds to the choice of the Lagrangian subspace as $L'' = C$. Moreover, the number of variables γ in S' is also limited: in a string of operators 4.19, every $\delta'' \circ H$ removes two γ. The actions S' thus have to have $2n$ covectors γ in term $P \circ (\delta'' \circ \hat{H})^n \circ \delta \circ I$.

For example, looking at $P \circ \delta'' \circ \hat{H} \circ \delta(F)$, we get

$$P \circ \delta'' \circ \hat{H} \circ \delta(F) = P\left(\hbar\Delta'' H(\{S', F\}') + \{S', H(\{S', F\}')\}''\right) .$$

Here, the term with Δ'' can be written as

$$P\{\hbar\Delta'' H(S'), F\}' = \{\hbar P \Delta'' H(S'), F\}' ,$$

because $\Delta'' \circ H$ has an degree 0 and does not interact with the α-derivative of $\{,\}'$. Thus, only the part of S' with two operators γ and no β contributes.

The second term $P\{S', \hat{H}(\{S', F\}')\}''$ has exactly one γ in each of the actions S', so that one is removed by $\hat{H}$ and the other by $\{\}''$.

What remains to be shown is that we can write W_i as

$$W_i = \frac{\partial_R W}{\partial \alpha^i} ,$$

for all i, so that

$$W_i \, \omega_\alpha^{ij} \frac{\partial_L F}{\partial \alpha^j} = \{W, F\}' .$$

We start by considering the relation

$$D'^2(F) = 0 .$$

Using the equation 4.18, we get (using a shorthand $W_i \, \omega_\alpha^{ij} \equiv W^j$)

$$\begin{aligned} D'^2(F) &= \hbar\Delta'\hbar\Delta'(F) + \hbar\Delta'\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) \\ &\quad + W^j \frac{\partial_L}{\partial \alpha^j}\left(\hbar\Delta' F\right) + W^i \frac{\partial_L}{\partial \alpha^i}\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) . \end{aligned} \tag{4.21}$$

The first term of RHS is 0, because $\Delta'^2 = 0$. We can write the second term as

$$\hbar\Delta'\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) = \hbar\Delta'\left(W^j\right) \frac{\partial_L F}{\partial \alpha^j} + (-1)^{|W^j|} W^j \hbar\Delta'\left(\frac{\partial_L F}{\partial \alpha^j}\right) + (-1)^{|W^j|}\{W^j, \frac{\partial_L F}{\partial \alpha^j}\}' .$$

Because D' has degree 1, we have $|W^i| - |\alpha^i| = 1$. If we commute Δ' with $\frac{\partial_L}{\partial \alpha^j}$ in the middle term, we get

$$(-1)^{|W^j|} W^j \hbar\Delta'\left(\frac{\partial_L F}{\partial \alpha^j}\right) = -W^j \frac{\partial_L}{\partial \alpha^j} \hbar\Delta' F ,$$

which cancels with the third term in RHS of equation 4.21. In the last term of 4.21, we have

$$W^i \frac{\partial_L}{\partial\alpha^i}\left(W^j \frac{\partial_L F}{\partial\alpha^j}\right) = (-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial\alpha^i}\frac{\partial_L}{\partial\alpha^j} + W^i \frac{\partial_L W^j}{\partial\alpha^i}\frac{\partial_L F}{\partial\alpha^j}\,.$$

Using the standard procedure, we show that the first term is zero

$$\begin{aligned}(-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial\alpha^i}\frac{\partial_L}{\partial\alpha^j} &= (-1)^{|W^i||\alpha^j|+|W^i||W^j|+|\alpha^i||\alpha^j|} W^j W^i \frac{\partial_L}{\partial\alpha^j}\frac{\partial_L}{\partial\alpha^i}\\ &= (-1)^{|W^j||\alpha^i|+|W^j||W^i|+|\alpha^j||\alpha^i|} W^i W^j \frac{\partial_L}{\partial\alpha^i}\frac{\partial_L}{\partial\alpha^j}\\ &= -(-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial\alpha^i}\frac{\partial_L}{\partial\alpha^j}\end{aligned}$$

after using $|W^i| - |\alpha^i| = 1$.

Expanding the bracket $\{\}'$, we finally get

$$\begin{aligned}D'^2(F) &= \hbar\Delta'\left(W^j\right)\frac{\partial_L F}{\partial\alpha^j} + (-1)^{|W^j|}\{W^j, \frac{\partial_L F}{\partial\alpha^j}\}' + W^i \frac{\partial_L W^j}{\partial\alpha^i}\frac{\partial_L F}{\partial\alpha^j}\\ &= (-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}\frac{\partial_L}{\partial\alpha^l}\frac{\partial_L F}{\partial\alpha^j} + \hbar\Delta'\left(W^j\right)\frac{\partial_L F}{\partial\alpha^j} + W^i \frac{\partial_L W^j}{\partial\alpha^i}\frac{\partial_L F}{\partial\alpha^j}\,.\end{aligned}$$

Because this identity holds for all F, we have two identities, for the order 1 and order 2 differential operators. We will see that these two identities encode the two identities from the definition of loop homotopy algebra.

The second order differential operator has to be zero

$$(-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}\frac{\partial_L}{\partial\alpha^l}\frac{\partial_L F}{\partial\alpha^j} = 0\,,$$

which means that a (graded) symmetric part of $(-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}$ with respect to indices l, j is zero:

$$0 = (-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl} + (-1)^{|\alpha^l||\alpha^j|}(-1)^{|W^l|}\frac{\partial_R W^l}{\partial\alpha^k}\omega_\alpha^{kj}\,.$$

Reverting the substitution $W_i\,\omega_\alpha^{ij} \equiv W^j$, we can write

$$0 = (-1)^{|W^j|}\frac{\partial_R W_i}{\partial\alpha^k}\omega_\alpha^{ij}\omega_\alpha^{kl} + (-1)^{|\alpha^l||\alpha^j|}(-1)^{|W^l|}\frac{\partial_R W_m}{\partial\alpha^k}\omega_\alpha^{ml}\omega_\alpha^{kj}\,.$$

Now we contract this with $\omega_{\alpha,jp}\,\omega_{\alpha,lq}$, the inverse matrices to ω_α^{ij}. Using the degree of ω and $|W^i| - |\alpha^i| = 1$, the sign factors give

$$\frac{\partial_R W_p}{\partial\alpha^q} = (-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R W_q}{\partial\alpha^p}\,. \tag{4.22}$$

This is the necessary condition for W to exist, because if it does, we have

$$\frac{\partial_R W_p}{\partial\alpha^q} = \frac{\partial_R}{\partial\alpha^q}\frac{\partial_R W}{\partial\alpha^p} = (-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R}{\partial\alpha^p}\frac{\partial_R W}{\partial\alpha^q} = (-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R W_q}{\partial\alpha^p}\,.$$

Special case of relation 4.22 is following for $p = q$

$$\frac{\partial_R W_p}{\partial\alpha^p} = (-1)^{|\alpha^p|}\frac{\partial_R W_p}{\partial\alpha^p}\,, \tag{4.23}$$

which for fermionic α^p implies that W_p *contains no* α^p. This is important for us, because now we can integrate W^p.

Lemma 4.5. *A function W defined as*

$$W = \sum_i \left(\int_0^1 \mathrm{d}t \, W_i|_{\alpha \to t\alpha} \right) \alpha^i \tag{4.24}$$

satisfies

$$\frac{\partial_R W}{\partial \alpha^k} = W_k \, .$$

This is just application of the homotopy operator for differential forms to the graded context – see e.g. [43].

Proof. (of the lemma) Directly calculating, we get

$$\begin{aligned}
\frac{\partial_R W}{\partial \alpha^k} &= \int_0^1 \mathrm{d}t \, W_k|_{\alpha \to t\alpha} + \sum_i \left(\int_0^1 \mathrm{d}t \, (-1)^{|\alpha^i||\alpha^k|} \frac{\partial_R W_i}{\partial \alpha^k}|_{\alpha \to t\alpha} \cdot t \right) \alpha^i \\
&= \int_0^1 \mathrm{d}t \, W_k|_{\alpha \to t\alpha} + \sum_i \left(\int_0^1 \mathrm{d}t \, \frac{\partial_R W_k}{\partial \alpha^i}|_{\alpha \to t\alpha} \alpha^i \cdot t \right) \\
&= \int_0^1 \mathrm{d}t \, W_k|_{\alpha \to t\alpha} + \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \left(W_k|_{\alpha \to t\alpha} \right) \cdot t \\
&= \int_0^1 \mathrm{d}t \, W_k|_{\alpha \to t\alpha} + [W_k|_{\alpha \to t\alpha} \cdot t]_{t=0}^{t=1} - \int_0^1 \mathrm{d}t \, W_k|_{\alpha \to t\alpha} \\
&= W_k \, .
\end{aligned}$$

□

Because we proved that $D' = \hbar \Delta' + \{W, -\}'$ and $D'^2 = 0$, we get (see section 2.4.2)

$$2\hbar \Delta' W + \{W, W\}' = 0,$$

which means that W determines a loop homotopy Lie algebra.

From W_i, we constructed the function W directly. An alternative proof would be defining loop homotopy algebra operations using W_k. If we decompose $W_k \frac{\partial_L}{\partial \alpha^k}$ into terms of homogeneous weight and $\hbar$ power, by dualizing them we get the loop homotopy algebra operations $\tilde{\lambda}_n^g : H_Q^{\otimes n} \to H_Q$. If we then construct

$$\tilde{s}_{n+1}^g \equiv \omega \circ (\mathbb{1} \otimes \tilde{\lambda}_n^g) \, ,$$

the graded symmetry of these $\tilde{s}$ follows from equation 4.22. Then, we can use them to construct the action W as in section 3.3, which is a solution of a quantum master equation and thus determines a loop homotopy algebra on H_Q. □

4.3 Final remarks and further directions

We would now like to sketch the possible future directions of this work. On the one hand, there is some work from the operadic side, of which is this calculation a special case. On the other hand, there are more ways to use the homological perturbation lemma – recall section 2.4.2 and the possible differentials.

4.3.1 Gauge transformations

We did our calculation with a specific choice of decomposition of V. The result should be independent of such noncanonical choice, and following [5] and [4], we expect that change in the induction data should change the action by a BV gauge transformation. In the path integral formalism, this should correspond to a different Lagrangian subspace. Note that this should also give a "gauge transformed" version of the perturbation lemma.

4.3.2 Feynman diagrams

We know that the effective action W can be computed using the path integral. Writing

$$W = \hbar \ln \int_C e^{S_0/\hbar} e^{S'/\hbar},$$

the perturbative series obtained this way would consist of vertices coming from $S'/\hbar$ paired by edges coming from the inverse of S_0, the propagator. The kinetic term is $S_0 = \frac{1}{2}\omega(\mathbb{1} \otimes Q)$, thus its inverse pairs two variables γ using h (inverse of Q). We see that this corresponds to the operator $(1 - \delta \circ \hat{H})^{-1} \circ \delta$ from the perturbation lemma.

4.3.3 Minimal model

We constructed a nontrivial loop homotopy Lie algebra on a homology of a given loop homotopy Lie algebra. We have encountered similar situation before, for minimal models of homotopy algebras. A definition of a loop homotopy algebra morphism hasn't explicitly appeared in literature yet, but we can postulate that the transferred action W together with the perturbed chain maps P' and I' are a minimal model. Note that dualizing the map $P' = P + P(1 - \delta\hat{H})^{-1}\delta\hat{H}$, one gets $i + h(\dots)$, which is a symplectomorphism, as one would expect.

Definition 4.6. A minimal model of an loop homotopy Lie algebra is given by the transferred action W and the perturbed morphisms P', I' of homological perturbation lemma. △

In the general case of algebras over modular operads, the minimal model was found by a similar diagrammatic expansion by Chuang and Lazarev [44].

4.3.4 Path integral

We were able to relate the path integral to the homological perturbation lemma. It would be interesting to formalize the properties of the path integral we used in section 4.2.1. Because we also have an explicit formula for the integral from the perturbation lemma, it should be possible to prove these properties.

A. On signs, degrees and factors

In this appendix, we summarize some conventions we use and also collect useful results connected with these conventions.

A.1 Graded vector spaces

Throughout the thesis, we use graded vector spaces V_n, see e.g. [14, section 1.5]. Specifically, we use Koszul sign convention, e.g

$$(\phi \otimes \psi)(v \otimes w) \equiv (-1)^{|\psi||v|}\phi(v) \otimes \psi(w)\,.$$

Sometimes, we use $(-1)^{|-|}$ for the operator

$$v \mapsto (-1)^{|v|}v\,.$$

We define a dual of a graded vector space by $(V^{\#})_n = (V_{-n})^{\#}$. The transpose (or dual) of a map $\phi : V \to W$ is defined (for $w \in W^{\#}$) as

$$\phi^{\#}(w) = (-1)^{|w||\phi|+1} w \circ \phi\,,$$

Note that the transpose map has the same degree as the original one.

The grading of will be also referred to as a *ghost number*.

On the (symmetric) tensor powers of V, there is another grading, by the tensor power. We will call this grading *weight*.

A.1.1 Symmetric action

On n-fold tensor products of graded vector spaces we usually use the right $\mathbb{S}_n$ action, defined as

$$\sigma^r(v_1 \otimes \cdots \otimes v_n) = \epsilon(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\,. \tag{A.1}$$

The sign $\epsilon(\sigma)$ is defined as the sign coming from commuting graded elements. If all v_i have even degree, it is 1. We write $\chi(\sigma)$ for $\epsilon(\sigma)$ multiplied by the sign of the permutation. We choose the right action mostly because it is shorter to write $v_{\sigma(i)}$ than $v_{\sigma^{-1}(i)}$. We also define a projector

$$\boldsymbol{\sigma}_n \equiv \sum_{\sigma \in \mathbb{S}_n} \frac{1}{n!}\sigma^r\,.$$

Unshuffles $\sigma \in \text{unsh}(l, n-l)$ are permutations in $\mathbb{S}_n$ for which $\sigma(1) < \cdots < \sigma(l)$ and $\sigma(l+1) < \cdots < \sigma(n)$. Unshuffles from $\text{unsh}(i_1, i_2, \ldots, i_k)$ with $\sum_m i_m = n$ are defined analogously.

A.1.2 Suspensions etc

By a *suspension* of a graded vector space V, we mean a graded vector space $(\uparrow V)_n = V_{n-1}$, which takes $v \in V_n$ into $\uparrow v \in (\uparrow V)_{n+1}$, i.e. it increases the degree by 1. Similarly, desuspension of V is $(\downarrow V)_n = V_{n+1}$. A reflection of V is defined as $(\mathbf{r}V)_n = V_{-n}$. We have $\mathbf{r} \circ \uparrow = \downarrow \circ \mathbf{r}$ and $\mathbf{r} \circ \downarrow = \uparrow \circ \mathbf{r}$.

For map $\phi : V^{\otimes n} \to W$, we define maps

$$\begin{aligned}
\phi^{\uparrow} : (\uparrow V)^{\otimes n} \to W &\text{ as } \quad \phi^{\uparrow} \equiv \phi \circ {\downarrow}^{\otimes n} , \\
\phi^{\downarrow} : (\downarrow V)^{\otimes n} \to W &\text{ as } \quad \phi^{\downarrow} \equiv \phi \circ {\uparrow}^{\otimes n} , \\
\phi_{\uparrow} : V^{\otimes n} \to (\uparrow W) &\text{ as } \quad \phi_{\uparrow} \equiv \uparrow \circ \phi , \\
\phi_{\downarrow} : V^{\otimes n} \to (\downarrow W) &\text{ as } \quad \phi_{\downarrow} \equiv \downarrow \circ \phi .
\end{aligned}$$

Note that for the first two cases, we imply a Koszul sign when commuting vectors with $\uparrow$. What happens with degrees of these maps?

- For $\phi^{\uparrow}$, the degree *decreases* by n, because now the map takes vectors of higher degree to produce the same degree in results. Similarly, desuspending the vector space V *increases* the degree of ϕ by n.
- For suspending and desuspending W, we simply have $|\phi_{\uparrow}| = |\phi| + 1$ and $|\phi_{\downarrow}| = |\phi| - 1$.
- A special case that will be useful is when $\phi : V^{\otimes n} \to V$ and we suspend or desuspend the arguments and results at the same time. In the case of suspension, the degree of ϕ decreases by $n - 1$, for desuspension it increases by $n - 1$.
- Another special case is a map $\omega : V^{\otimes n} \to \Bbbk$. Since $\Bbbk$ as a graded vector space is concentrated in degree 0, ω is nonzero only on vectors of total degree $-|\omega|$. Again, suspending V decreases the degree of ω by n and vice versa.

For general map $\phi : V^{\otimes n} \to W$, the reflection of only arguments or results would not give a map of definite degree. For $\phi : V^{\otimes n} \to V$, we can define

$$\phi_{\mathbf{r}} \equiv \mathbf{r} \circ \phi \circ \mathbf{r}^{\otimes n} .$$

which has degree equal to $-|\phi|$. Similarly, for $\omega : V^{\otimes n} \to \Bbbk$, we can define

$$\omega_{\mathbf{r}} \equiv \omega \circ \mathbf{r}^{\otimes n} ,$$

which again has a degree opposite to that of ω. Of course, there is no Koszul sign for commuting with $\mathbf{r}$, since it has no well-defined degree.

A.2 Symmetric powers

We choose to represent the symmetric tensor power $\mathrm{Sym}^n(V)$ as the subspace of $V^{\otimes n}$ of tensors which are graded symmetric, i.e. for which $T = \sigma^r T$ for all $\sigma \in \mathbb{S}_n$. We define the an associative symmetric product $\odot$ as (see [45, section 4.5])

$$v_1 \odot \cdots \odot v_n = \boldsymbol{\sigma}[v_1 \otimes \cdots \otimes v_n] .$$

With this convention

$$V \odot W = \boldsymbol{\sigma}(V \otimes W) .$$

We also sometimes omit $\odot$ and write

$$v_1 v_2 \equiv v_1 \odot v_2 .$$

A.2.1 Pairing with dual symmetric powers

For $v_1 \odot \cdots \odot v_n \in \mathrm{Sym}^n V$ and $\phi^1 \odot \cdots \odot \phi^n \in \mathrm{Sym}^n V^{\#}$, we define their pairing as

$$\phi^1 \odot \cdots \odot \phi^n (v_1 \odot \cdots \odot v_n) \equiv \sum_{\sigma \in \mathbb{S}_n} \epsilon(\sigma) \phi^1(v_{\sigma(1)}) \ldots \phi^n(v_{\sigma(n)}) ,$$

i.e. without signs from commuting ϕ and v.

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