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Referee report on Dissertation Thesis

entitled

On existence and regularity of solutions to perturbed systems of Stokes type

written by

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This thesis represents a worth contribution to the mathematical analysis of partial differential equations (PDEs in short) describing steady flows of incompressible fluids. Two types of systems are analyzed: *(i)* a generalized Stokes system where the Laplace operator is replaced by a linear second order elliptic operator $-div(AD(v))$, the pressure gradient is replaced by a suitable operator of the first order $B\nabla p$, and non-homogeneous constraint is put on the velocity divergence, and *(ii)* a stationary Navier-Stokes-like system with the viscosity depending on the shear rate and the pressure. The theme of the thesis is very timely, as it carries on very recent (mostly existence) results analyzing the time-dependent incompressible fluid models with non-constant viscosities (depending on the shear rate, the pressure, the temprerature, etc.), the models that are used in various areas of natural sciences. The fact that the considered systems have been analyzed rarely (if at all) has opened to the author a space to learn plenty of deep methods and technically complicated tools of the regularity theory ($W^{k,2}$-regularity, Hölder regularity, partial regularity), and he then has applied and extended these techniques to new systems of PDEs. The author devoted a remarkable attention to the validity of the results up to the boundary (a topic that is frequently omitted in the literature).

Regarding the system *(i)* studied in Chapter 1, the author first establishes the existence and uniqueness results for the case B constant, then continues to prove $W^{k,2} \times W^{k-1,2}$-regularity for the velocity and the pressure (near a smoothly curved boundary), finally he derives Caccioppoli's estimates (up to the boundary) in order to establish $C^{0,\alpha}$-regularity for the velocity gradient and the pressure.

Some questions to the author concerning Chapter 1:

(1) In Section 2, the assumptions on A and B are more general than those used in Section 1. What can be said on the existence and uniqueness for such more general systems?
(2) Why is not an analogy of Theorem 1.6 valid (or written down) up to the boundary?

Chapter 2 focuses on the proof of partial regularity for **the system** *(ii)*, restricting to Hilbert structures, considering however the case near the flat boundary. Interestingly, the system studied in Chapter 1 comes out as the so-called "blow up" system.

Some questions to the author concerning Chapter 2:

(1) According to (2.25), the result in Lemma 2.1. should be stated with the power $\frac{1}{R}$ and not $\frac{1}{R^2}$. Please provide the explanation.

(2) What is your opinion regarding full $C^{0,\alpha}$-regularity for the velocity gradient and the pressure for two-dimensional flows (i.e. if $d = 2$)?

(3) In Preliminaries you mention as an open problem the question of regularity for three-dimensional flows to the evolutionary Navier-Stokes equations. In the thesis you deal however with stationary Navier-Stokes-like systems, and it is well known that the stationary Navier-Stokes equations in three dimensions have smooth solution for smooth data. Could you briefly explain if the full regularity for the system considered in Chapter 3 is indeed an open problem? (In Preliminaries, you refer to some articles treating partial regularity for flows driven by the evolutionary Navier-Stokes system. To your records let me mention two additional recent papers of interest written by Fang-Hua Lin, and by Alexis Vasseur.)

Certainly, the thesis could be written in a better way. Technical parts could be explained more transparently (similar proofs should be more unified; surprisingly, it seems that the author does not refer in Chapter 2 to results established in Chapter 1; relation between Morrey and Hölder spaces should be mentioned in Preliminaries); some symbols have multiple definition, others are not defined on the first occurrence; many mathematical symbols are not typeset in a mathematical mode, there is a number of misprints.

Despite these drawbacks, the thesis brings new valuable results (some of them were already published, others should be submitted for publication) on regularity theory for novel systems of incompressible fluid mechanics. Nguyen Duc Huy has proved that he is able to learn sophisticated analytical methods, handle complicated mathematical tools and achieve interesting results on his own. As a consequence, to my opinion, Nguyen Duc Huy deserves to receive Ph.D. degree and I am pleased to recommend it.

Prague, August 8, 2006