CHARLES UNIVERSITY
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ON EXISTENCE AND REGULARITY OF SOLUTIONS TO PERTURBED SYSTEMS OF STOKES TYPE

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Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with boundary $\partial\Omega$. We study the smoothness of stationary flows of incompressible fluids with viscosities that depend on the shear rate and the pressure. The typical example we have in mind is

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}\ \mathrm{T}(\mathrm{Du}, \mathrm{p}) + (\mathrm{u} \cdot \nabla)\mathrm{u} + \nabla \mathrm{p} = f \ \text{ in } I \times \Omega, \\ \mathrm{div}\ \mathrm{u} = 0 \ \text{ in } I \times \Omega. \end{aligned} \tag{1}$$

accompanied by initial and boundary conditions.
Here u stands for velocity, $Du = \frac{1}{2}(\nabla u + \nabla^T u)$, p for pressure, f for external body forces and $T(Du, p)$ for stress tensor.

We assume that

- T is continuously differentiable on $\mathbb{R}^{d^2+1}$ and

$$T_{ij}(\xi, \tau) = \nu(|\xi|, \tau)\xi_{ij}, \ i, j = 1, \cdots, d. \tag{2}$$

- There are λ_0, λ_1, $\nu_0 > 0$ such that for any $\tau \in \mathbb{R}$, symmetric $d \times d$ matrix ξ, it holds

$$\begin{aligned} \lambda_0|\xi|^2 \le \sum_{i,j,k,l=1}^{d} \frac{\partial T_{ij}}{\partial \xi_{kl}}(\xi, \tau)\xi_{ij}\xi_{kl} \le \lambda_1|\xi|^2, \\ \sum_{i,j=1}^{d} \frac{\partial T_{ij}}{\partial \tau}(\xi, \tau) \le \nu_0. \end{aligned} \tag{3}$$

Then the global-in-time existence of weak solutions to (1) was proved in [8], [14] and [20] if f, the boundary $\partial\Omega$ and initial data satisfy natural conditions and ν_0 is small enough with respect to λ_0.
In our stationary case, if f satisfies natural conditions on $\partial\Omega$ and ν_0 is small enough with respect to λ_0, λ_1, there is a pair $u \in W_0^{1,2}(\Omega)$, $p \in L^2(\Omega)$ solving problem (1) and satisfying homogeneous Dirichlet boundary conditions

$$u = 0 \ \text{ on } \ \partial\Omega. \tag{4}$$

The smoothness of u and p is a more delicate problem. As we deal with a system of nonlinear elliptic PDEs we can not expect full regularity in space dimensions $d \geq 3$. When proving partial regularity results for such model we come to so-called "blow up" system which has the form

$$\begin{aligned} -\text{div } (A\nabla w) + B\nabla q = g \text{ in } \Omega, \\ \text{div } w = 0 \text{ in } \Omega \end{aligned} \tag{5}$$

with a $d^2 \times d^2$ matrix $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{d}$ and a $d \times d$ matrix $B = (B_{ij})_{i,j=1}^{d}$ given by

$$A_{ik}^{jl} = \frac{\partial T_{ij}}{\partial \xi_{kl}}(a, b), \ i, j, k, l = 1, \cdots, d;$$

$$B_{ij} = \delta_{ij} - \frac{\partial T_{ij}}{\partial \tau}(a, b), \ i, j = 1, \cdots, d$$

where

$$a = \lim_{R \to 0+} \frac{1}{\text{meas } (B_R(x_0))} \int_{B_R(x_0)} Du \, dx,$$

$$b = \lim_{R \to 0+} \frac{1}{\text{meas } (B_R(x_0))} \int_{B_R(x_0)} p \, dx.$$

Saying differently, behaviour of solutions to (5) predicts behaviour of solution to (1) in regular points x_0.

In Chapter 1 we study problem (5) which we call generalization of the linear Stokes problem. Namely, we investigate the following problem: For given $f = (f_1, \cdots, f_d) : \Omega \longrightarrow \mathbb{R}^d$ and $g : \Omega \longrightarrow \mathbb{R}$, $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{d} : \Omega \longrightarrow \mathbb{R}^{d^2 \times d^2}$ and $B = (B_{ij})_{i,j=1}^{d}$ a $d \times d$ matrix we look for $u = (u_1, \cdots, u_d) : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ solving

$$\begin{aligned} -\text{div } (A\nabla u) + B\nabla p = f \text{ in } \Omega, \\ \text{div } u = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6}$$

Here the generalization of classical Stokes problem consists in two points: instead of Laplace operator we consider a general second order elliptic operator in divergence form and instead of gradient of p we consider a class of general first order linear operators . The new feature of system (6) compared with classical Stokes system lies in the fact that operators div u and $B\nabla p$ (for $B \neq E$) do not act as adjoint operators in suitable Banach spaces. While existence of weak solutions to (6) with $B = E$ was extensively studied in (see e.g. [4], [9], [25]), both existence and smoothness properties of solutions to system (6)- as far as we know - have been studied only in [15].

We divide Chapter 1 into three section. In Section 1 we present existence and uniqueness results for constant matrix B. In addition, we illustrate the type of generalized linear Stokes system satisfying our assumptions by several examples. In Section 2 we show the regularity of solutions u, p in $W^{k,2}(\Omega)$ under natural conditions on f, g, A, B, Ω. Schauder estimates of solutions based on a suitable form of Caccioppoli's inequality both in the interior of domain Ω and up to the flat boundary are given in Section 3.

Proof of the regularity of solutions to the problem (1) is very complicated. It is so even in the case when $\nu \equiv 1$ and the viscosities of the problem (1) collapse to the classical constant Navier-Stokes viscosity as the smoothness of solutions to NSE in 3D is a famous open problem For the history of partial regularity of solutions to NSE we refer the reader to the papers [6], [18], [23]. In our case when $d = 2, 3$ and the viscosity depends on shear and pressure, the interior partial regularity of stationary solutions to (1) was studied in [19] for f satisfying natural conditions, growth $m \in (\frac{3d}{d+2}, 2)$ and ν_0 small enough with respect to λ_0, λ_1.

In the Chapter 2 we extend the results of [19] to study the partial regularity of solutions to problem (1) up to the boundary. For simplicity, we consider only $m = 2$ and $\Omega = B_1^+(0) := B_1(0) \cap \{x \in \mathbb{R}^d;\ x_d > 0\}$. The results are obtained for ν_0 small enough with respect to λ_0, λ_1. For the case of solutions $u \in W_0^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0))$ with supp u, supp p $\subset B_1^*(0) := B_1(0) \cap \{x \in \mathbb{R}^d;\ x_d \geq 0\}$ we prove first that $u \in W^{2,2}(B_1^+(0))^d$, $p \in W^{1,2}(B_1^+(0))$. Therefore, we can define an energy

$$E^{u,p}(x, R) = \frac{1}{R^{\frac{d-2}{2}}} \|\nabla^2 u\|_{2, B_R^+(x)} + \frac{1}{R^{\frac{d-2}{2}}} \|\nabla p\|_{2, B_R^+(x)} + R^\alpha,$$

where $x \in B_1^*(0)$, $R \in (0, \text{dist}\,(x, \partial B_1^+(0) \backslash \Gamma))$ and $0 < \alpha < 1$.
Next we establish the estimates for decay of the energy in auxiliary linear systems. Then we present the lemma for decay of energy of system (1) at any regular points $x \in B_1^*(0)$. As a consequence we obtain that if $\bar{x} \in \Gamma$ is such that

$$\liminf_{R \longrightarrow 0^+} E^{u,p}(\bar{x}, R) = 0 \text{ and } \limsup_{R \longrightarrow 0^+} \left(|(u)_{B_R^+(\bar{x})}| + |(\nabla u)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}|\right) < \infty,$$

then there exists $\delta > 0$ such that ∇u and p are Hölder continuous in $B_\delta^+(\bar{x})$. Finally, we show that the $d - 2 + \epsilon$-Hausdorff measure of all singular points in Γ is zero for any $\epsilon > 0$.

Preliminaries

In this section, we introduce notations, definitions and also recall some well-known results used later.

Begin with some definitions and notations. Let Ω be a domain in $\mathbb{R}^d$ $(d \geq 2)$. For $1 \leq q \leq \infty$, $k \in \mathbb{N}$; $L^q(\Omega)$ and $W^{k,q}(\Omega)$ denote Lebesgue and Sobolev spaces.
The norm of $u \in L^q(\Omega)$ is denoted by

$$\|u\|_q = \|u\|_{q,\Omega} := \left(\int_\Omega |\, u \,|^q \, dx\right)^{1/q}. \tag{0.1}$$

The norm of $u \in W^{k,q}(\Omega)$ is defined as

$$\|u\|_{k,q} = \|u\|_{k,q;\Omega} := \left(\int_\Omega \sum_{|\alpha|\leq k} |\, D^\alpha u \,|^q \; dx\right)^{1/q}. \tag{0.2}$$

As usual, $W_0^{k,q}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{k,q}(\Omega)$. We denote by $W^{-1,q'}(\Omega)$ the dual space to $W_0^{1,q}(\Omega)$ where $\frac{1}{q'} + \frac{1}{q} = 1$. If $f \in W^{-1,q'}(\Omega)$, $v \in W_0^{1,q}(\Omega)$ we use the notation $[f, v]$ for the value of the functional f at v.

Set $W^{k,q}(\Omega)^m := W^{k,q}(\Omega, \mathbb{R}^m) = [W^{k,q}(\Omega)]^m$ with norm

$$\|u\|_{k,q} = \|u\|_{k,q;\Omega} = \|(u_1, \ldots, u_m)\|_{k,q;\Omega} := (\sum_{j=1}^m \|u_j\|_{k,q}^q)^{1/q}. \tag{0.3}$$

In a similar way we obtain vector valued Banach spaces $W_0^{1,q}(\Omega)^m$, $L^q(\Omega)^m$ and $W^{-1,q'}(\Omega)^m$ (which denotes the dual space to $W_0^{1,q}(\Omega)^m$). We will also use the symbol $\|u\|_{-1,q'} = \|u\|_{-1,q';\Omega}$ to denote norms of $u \in W^{-1,q'}(\Omega)$ or $u \in W^{-1,q'}(\Omega)^m$.

The space $W_{0,\mathrm{div}}^{1,2}(\Omega)$ is determined by the condition

$$W_{0,\mathrm{div}}^{1,2}(\Omega) := \left\{u \in W_0^{1,2}(\Omega)^d ;\ \mathrm{div}\ \mathrm{u} = 0\right\} \tag{0.4}$$

where the equation div $u = 0$ is satisfied in distribution sense. $W_{0,\mathrm{div}}^{1,2}(\Omega)$ is a closed subspace of $W_0^{1,2}(\Omega)^d$ and thus it is a Hilbert space with scalar product induced from $W_0^{1,2}(\Omega)^d$.

Let $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $f \in W^{k,q}(\Omega)$, $u = (u_1, \cdots, u_d) \in W^{k,q}(\Omega)^d$. We will use notations

$$D_j f := \frac{\partial f}{\partial x_j},\ D_j^2 f := \frac{\partial^2 f}{\partial x_j^2},\ \nabla f := (D_j f)_{j=1}^d,\ \nabla^2 f := (D_j D_l f)_{j,l=1}^d$$

$$\nabla u := (D_j u_i)_{i,j=1}^d,\ Du = (D_{ij}u)_{i,j=1}^d := (\frac{1}{2}(D_j u_i + D_i u_j))_{i,j=1}^d$$

$$x = (x', x_d),\ x' = (x_1, \cdots, x_{d-1}),\ u = (u', u_d),\ u' = (u_1, \cdots, u_{d-1}),$$

$$\nabla = (\nabla', D_d),\ \nabla' = (D_1, \cdots, D_{d-1}),\ D_j u := (D_j u_1 \cdots, D_j u_d).$$

$$B_r(x_0) := \left\{x \in \mathbb{R}^d; |x - x_0| < r\right\},\ B_r^+(x_0) := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d > 0\},$$

$$B_r^*(x_0) := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d \geq 0\},\ x_0 \in \mathbb{R}^d,\ r > 0;$$

$$B_r := B_r(0),\ B_r' := \{y' \in \mathbb{R}^{d-1};\ |y'| < r\},$$

$$\Gamma_r := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d = 0\},\ r > 0,\ \Gamma := \Gamma_1,\ \Gamma_\infty := \{x \in \mathbb{R}^d; x_d = 0\}.$$

For an integrable function u we denote by $(u)_\Omega$ an integral mean value of u over Ω, i.e. $(u)_\Omega := \frac{1}{\text{meas } \Omega} \int_\Omega u(x)dx$. For $\Omega = B_{R_0}(x_0)$ the symbol $(u)_\Omega$ will be abbreviated to $(u)_{x_0, R_0}$.

If $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $y = (y_1, \cdots, y_m) \in \mathbb{R}^d$, and M, N are $d \times d$ matrices we use the notations

$$x.y := \sum_{i=1}^d x_i y_i,\ x \otimes y := (x_i y_j)_{i,j=1}^d,$$

$$Mx.y := \sum_{i,j=1}^d M_{ij} x_i y_j,\ M.N := \sum_{i,j=1}^d M_{ij} N_{ij}, \tag{0.5}$$

while for $d^2 \times d^2$ matrix A and $u, v \in W^{k,q}(\Omega)^d$ we write

$$A\nabla u : \nabla v := \sum_{\alpha,\beta,i,j=1}^d A_{ij}^{\alpha\beta} \frac{\partial u_i}{\partial x_\alpha} \frac{\partial v_j}{\partial x_\beta}.$$

$$ADu : Dv := \sum_{k,l,i,j=1}^d A_{ij}^{kl} D_{kl} u D_{ij} v \tag{0.6}$$

For points $x \in \mathbb{R}^d$ as well as for matrices $M = (M_{ij})_{i,j=1}^m$ we write

$$|x| := (\sum_{i=1}^d |x_i|^2)^{\frac{1}{2}},\ |M| := (\sum_{i,j=1}^m |M_{ij}|^2)^{\frac{1}{2}}. \tag{0.7}$$

In the sequel, we recall local description of boundary $\partial\Omega$ which allows us

to define domains with smooth boundary (see [1] , [21]).
Given $x_0 \in \mathbb{R}^d$, $r > 0, \beta > 0$, a local coordinate system centered in x_0 with coordinates $y = (y', y_d)$ and a real continuous function $h : B'_r \longmapsto \mathbb{R}$ we denote

$$U_{r,\beta,h}(x_0) := \{(y', y_d) \in \mathbb{R}^d; h(y') - \beta < y_d < h(y') + \beta, |y'| < r\}. \qquad (0.8)$$

Let Ω be a domain in $\mathbb{R}^d$ $(d \geq 2)$ with boundary $\partial\Omega$. Then Ω is called a Lipschitz domain, iff for each $x_0 \in \partial\Omega$, there exists a local coordinate system in x_0, constants $r > 0, \beta > o$, and a Lipschitz continuous function h : $B'_r \longmapsto \mathbb{R}$ with the following properties

$$\begin{aligned} U_{r,\beta,h}(x_0) \cap \partial\Omega &= \{(y', y_d); y_d = h(y'), |y'| < r\}, \\ U^+_{r,\beta,h}(x_0) := U_{r,\beta,h}(x_0) \cap \Omega &= \{(y', y_d)\ h(y') < y_d < h(y') + \beta, |y'| < r\}, \qquad (0.9) \\ U^-_{r,\beta,h}(x_0) := U_{r,\beta,h}(x_0) \cap (\mathbb{R}^d \setminus \Omega) &= \{(y', y_d)\ h(y') - \beta < y_d < h(y'), |y'| < r\}. \end{aligned}$$

For $k \in \mathbb{N}$ the domain Ω is called a C^k -domain, iff for each $x_0 \in \partial\Omega$, function h describing the boundary in (0.9) belongs to $C^k(\overline{B'}_r)$.

If Ω is a bounded C^k -domain then for all $\gamma > 0$ we find $x_1, \cdots, x_m \in \partial\Omega$, $h_j := h_{x_j}$, $r_j := r_{x_j}, B'_j = B'_{r_j}$, $U_j := U_{r_j,\beta_j,h_j}(x_j)$, $j = 1, \cdots, m$ with the properties (0.9) such that $\partial\Omega \subset \cup_{j=1}^m U_j$.
Moreover,

$$h_j \in C^k(\overline{B'}_j) \text{ and } \|h_j\|_{C^1(\overline{B'}_j)} \leq \gamma \text{ for } j = 1, \cdots, m. \qquad (0.10)$$

Partition of unity gives existence of functions $\varphi_j \in C_0^\infty(\mathbb{R}^d)$, $j = 1, \cdots, m$; a sequence of open balls $B_k \subset\subset \Omega$, $k = 1, \cdots, l$ and a sequence of functions $\psi_k \in C_0^\infty(\mathbb{R}^d), k = 1, \cdots, l$ with the following properties

$$\begin{aligned} &\operatorname{supp} \varphi_j \subset U_j, 0 \leq \varphi_j \leq 1, \ j = 1, \cdots, m; \\ &\operatorname{supp} \psi_k \subset B_k, 0 \leq \psi_k \leq 1, \ k = 1, \cdots, l; \qquad (0.11) \\ &\overline{\Omega} \subset (\cup_{k=1}^l B_k) \cup (\cup_{j=1}^m U_j); \sum_{k=1}^l \psi_k(x) + \sum_{j=1}^m \varphi_j(x) = 1 \text{ for all } x \in \overline{\Omega}. \end{aligned}$$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $\mu > 0$, $k \in \mathbb{N}$, $0 < \alpha \leq 1$; $L^{2,\mu}(\Omega)$, $\mathcal{L}^{2,\mu}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ denote Morrey, Campanato and Hölder spaces with norms $\|u\|_{L^{2,\mu}(\Omega)}$, $\|u\|_{\mathcal{L}^{2,\mu}(\Omega)} = \|u\|_2 + [u]_{2,\mu;\Omega}$ and $\|u\|_{C^{k,\alpha}(\overline{\Omega})}$ respectively, where

$$\begin{aligned} \|u\|_{L^{2,\mu}(\Omega)} &:= [\sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\mu} \int_{\Omega\cap B_\rho(x)} |u|^2\, dx]^{1/2}, \\ [u]_{2,\mu;\Omega} &:= [\sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\mu} \int_{\Omega\cap B_\rho(x)} |u - (u)_{\Omega\cap B_\rho(x)}|^2\, dx]^{1/2}, \qquad (0.12) \\ \|u\|_{C^{k,\alpha}(\overline{\Omega})} &:= \max_{0\leq|\alpha|\leq k} [\sup_{x,y\in\Omega, x\neq y} \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\alpha}] + \|u\|_{C^k(\overline{\Omega})}. \end{aligned}$$

where α is an d-tuple of nonnegative integers α_i, $|\alpha| = \sum_{i=1}^{d} \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

Now we recall the results on solvability of equations

$$\mathrm{div}\, v = g, \ \nabla p = f.$$

Lemma 0.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, let Ω_0 be a nonemp- -ty subdomain of Ω and let $1 < q < \infty$, $q' = \frac{q}{q-1}$. Then it holds:*
a) There is a constant $C = C(q, \Omega) > 0$ such that for each $g \in L^q(\Omega)$ with $\int_\Omega g\, dx = 0$ there exists at least one $v \in W_0^{1,q}(\Omega)^d$ satisfying

$$\mathrm{div}\, v = g \ in\ \Omega, \ \|\nabla v\|_q \le C\|g\|_q. \tag{0.13}$$

b) There is a constant $C = C(q, \Omega, \Omega_0) > 0$ such that for each $f \in W^{-1,q}(\Omega)^d$ satisfying condition $[f, v] = 0$ for all $v \in W_{0,\mathrm{div}}^{1,q'}(\Omega)$ there exists a unique $p \in L^q(\Omega)$ satisfying

$$\nabla p = f \ in\ \Omega, \ \int_{\Omega_0} p\, dx = 0 \ and\ \|p\|_q \le C\|f\|_{-1,q}. \tag{0.14}$$

See [24], Chapter 2, Lemma 2.1.1, 2.2.2.
Remark. It is easy to see that C does not depend on translations and rotations of the couple Ω, Ω_0. By scaling argument we deduce that for $\Omega = \Omega_0 = B_R(x_0)$ or $\Omega = \Omega_0 = B_R^+(x_0)(x_0 \in \Gamma_\infty)$ the constant C does not depend on R. In this case we denote C_{div} as the infimum of the constants C from the inequality (0.13) Next, we recall a well-known facts verified easily by iterations.

Lemma 0.2. *Let $f(t)$ be a nonnegative bounded function defined in $[\tau_0, \tau_1]$ where $\tau_0 \le 0$. Suppose that for $\tau_0 \le t < s \le \tau_1$ we have*

$$f(t) \le [A(s-t)^{-\alpha} + B] + \theta f(s), \tag{0.17}$$

where A, B, α, θ are nonnegative constants with $0 \le \theta < 1$. Then for all $\tau_0 \le t < s \le \tau_1$ we have

$$f(t) \le C[A(s-t)^{-\alpha} + B], \tag{0.18}$$

where C is a constant depending on α and θ.

See [11], Chapter 3, Lemma 3.1.

Lemma 0.3. *Let $\Phi(\rho)$ be a nonnegative and nondecreasing function on $(0, R_0]$. Suppose that there are nonnegative constants A, B, α, β with $\alpha > \beta$ so that*

$$\Phi(\rho) \le A[(\frac{\rho}{R})^\alpha + \varepsilon]\Phi(R) + BR^\beta \ for\ all\ 0 < \rho < R \le R_0. \tag{0.19}$$

Then there exists positive $\varepsilon_0 = \varepsilon_0(\alpha, \beta, A)$ such that the following holds : If (0.19) is true for some $\varepsilon < \varepsilon_0$ then

$$\Phi(\rho) \leq C(\alpha, \beta, A)[(\frac{\rho}{R})^\alpha \Phi(R) + B\rho^\beta] \tag{0.20}$$

See [12], page 51, Section III. 2.

In connection with estimates of full gradient and symmetric gradient we will use Korn's inequality

Lemma 0.4. *(Korn's inequality) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then for every $u \in W_0^{1,2}(\Omega)$ it holds*

$$\|\nabla u\|_{2;\Omega} \leq \sqrt{2}\|Du\|_{2;\Omega}. \tag{0.21}$$

See [22], Chapter 6, Section 6.3, Theorem 3.1.

Due to the relation

$$\frac{\partial^2 u_i}{\partial u_j \partial u_k} = \frac{\partial D_{ik} u}{\partial x_j} + \frac{\partial D_{ij} u}{\partial x_k} - \frac{\partial D_{jk} u}{\partial x_i}$$

we have lemma

Lemma 0.5. *Let $u \in W^{2,q}(\Omega)$, then it holds*

$$\|\nabla^2 u\|_q \leq 3\|D(\nabla u)\|_q. \tag{0.22}$$

Denote $H^\alpha(\Sigma)$ Hausdorff α-dimensional measure of Σ. We recall the well-known results about estimates of Hausdorff measure of the singular set.

Lemma 0.6. *Let $f \in L^1_{loc}(\Omega)$, $0 \leq \alpha < d$. Set*

$$\Sigma^\alpha := \{x \in \Omega;\ \limsup_{\rho \to 0}\ \rho^{-\alpha} \int_{B_\rho(x)\cap\Omega} |f(y)|\, dy > 0\}.$$

Then

$$H^\alpha(\Sigma^\alpha) = 0. \tag{0.23}$$

In particular, if $u \in W^{1,q}_{loc}(\Omega)$, $q \leq d$ and

$$\Sigma := \{x \in \Omega;\ \limsup_{\rho \to 0}\ \rho^{q-d} \int_{B_\rho(x)\cap\Omega} |\nabla u(y)|^q\, dy > 0\},$$

then

$$H^{d-q}(\Sigma) = 0. \tag{0.24}$$

See [11], Chapter 4, Theorem 2.2.

Lemma 0.7. *Let $u \in W^{1,q}_{loc}(\Omega)$, $q \le d$ and*

$$\Sigma' := \{x \in \Omega;\ \limsup_{\rho \to 0} |(u)_{x,\rho}| = \infty\},$$

then for any $\epsilon > 0$ we have

$$H^{d-q+\epsilon}(\Sigma') = 0. \tag{0.25}$$

See [11], Chapter 4, Theorem 2.1.

We conclude this part by providing a simple result from the linear algebra about a positive definite matrix.

Lemma 0.8. *Let M be $d \times d$ matrix : $\Omega \to \mathbb{R}^d \times \mathbb{R}^d$, $M \in L^\infty(\Omega)$. If M is a positive definite i.e. there exists a constant $\lambda > 0$ such that*

$$\sum_{i,j=1}^{d} M_{ij}\xi_i\xi_j \ge \lambda|\xi|^2 \ \ in\ \Omega\ \ for\ all\ \ \xi \in \mathbb{R}^d., \tag{0.26}$$

then M is regular and an inequality

$$|\det \mathrm{M}| \ge \mathrm{C} \tag{0.27}$$

holds with a constant $C = C(\lambda, d) > 0$.

Proof. Let a $\xi \in \mathbb{R}^d$ satisfies equations $M\xi = 0$. Then

$$0 = M\xi.\xi \ge \lambda|\xi|^2 \ge 0,$$

thus $\xi = 0$. It implies that M is regular.
Next, we have

$$|M^{-1}| = \sqrt{d} \sup_{\eta \in \mathbb{R}^d, |\eta|=1} |M^{-1}\eta|.$$

Choose in (0.26) $\xi = M^{-1}\eta$, then

$$M\xi.\xi = \eta.M^{-1}\eta \ge \lambda|M^{-1}\eta|^2$$

at the same time

$$M\xi.\xi = \eta.M^{-1}\eta \le |\eta||M^{-1}\eta|.$$

Thus

$$\lambda|M^{-1}\eta|^2 \le |\eta||M^{-1}\eta| \ \text{ or } \ |M^{-1}\eta| \le \frac{|\eta|}{\lambda}.$$

Therefore,

$$|M^{-1}| \le \frac{\sqrt{d}}{\lambda}.$$

As $|M^{-1}| \le \frac{\sqrt{d}}{\lambda}$, we deduce $\det|M^{-1}| \le C$ with a constant $C = C(\lambda, d) > 0$. Consequently, $|\det M| \ge \frac{1}{C}$. The lemma is proved. □

Chapter 1

Linear problems

1. Existence of solutions

Let $\Omega \subset \mathbb{R}^d$, $(d \geq 2)$, be a bounded Lipschitz domain with boundary $\partial\Omega$. We will prove the existence and uniqueness of solutions to the generalized linear Stokes system.

First, we consider system

$$\begin{aligned} -\text{div}\,(A\nabla u) + B\nabla p &= f \ \text{ in } \Omega, \\ \text{div}\, u &= 0 \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $f = (f_1, \cdots, f_d) : \Omega \to \mathbb{R}^d, A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^d : \Omega \to \mathbb{R}^{d^2\times d^2}$, $B = (B_{ij})_{i,j=1}^d : \Omega \to \mathbb{R}^{d\times d}$ are given quantities and $u = (u_1, \cdots, u_d) : \Omega \to \mathbb{R}^d, p : \Omega \to \mathbb{R}$ are unknown functions. In applications f has the meaning of density of external forces, $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^d$ can be interpreted as generalized viscosity of a fluid with unknown velocity field $u = (u_1, \cdots, u_d)$ and pressure p.

Definition 1.1. *Let $f \in W^{-1,2}(\Omega)^d$. Then pair $(u,p) \in W^{1,2}_{0,\text{div}}(\Omega) \times L^2(\Omega)$ is called a weak solution to system (1.1) if and only if*

$$-\text{div}\,(A\nabla u) + B\nabla p = f \ \textit{in}\ \Omega \tag{1.2}$$

holds in the sense of distributions i.e. if

$$\sum_{\alpha,\beta,i,j=1}^{d} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_j D_\alpha v_i \, dx - \sum_{i,j=1}^{d} \int_\Omega p D_j(B_{ij} v_i)\, dx = [f, v] \tag{1.3}$$

holds for all $v \in W_0^{1,2}(\Omega)^d$.

Remark. If B is constant and regular then $(u,p) \in W^{1,2}_{0,\text{div}}(\Omega) \times L^2(\Omega)$ is a weak solution to equations (1.1) in the sense of distribution iff (u,p) is a weak solution to equations

$$-\text{div}\,(B^{-1}A\nabla u) + \nabla p = B^{-1}f \tag{1.4}$$

where we have denoted by $B^{-1}A$ a $d^2 \times d^2$ matrix C with

$$C^{\alpha\beta}_{ij} = \sum_{k=1}^{d} (B^{-1})_{ik} A^{\alpha\beta}_{kj} \text{ for } i,j,\alpha,\beta = 1,...,d.$$

Thus 1.4 means that

$$\sum_{\alpha,\beta,i,j=1}^{d} \int_\Omega \sum_{k=1}^{d} (B^{-1})_{ik} A^{\alpha\beta}_{kj} D_\beta u_j D_\alpha v_i \, dx - \sum_{i=1}^{d} \int_\Omega p \, D_i v_i \, dx = [B^{-1}f, v] \tag{1.5}$$

holds for all $v \in W^{1,2}_0(\Omega)^d$.

We assume throughout this section that A, B satisfy the following conditions

- $$B \text{ is a constant regular matrix.} \tag{1.6}$$
- $A^{\alpha\beta}_{ij}$ belong to $L^\infty(\Omega)$ and there is a positive Λ_A such that
$$\text{ess sup}|A^{\alpha\beta}_{ij}| \le \Lambda_A \text{ for all } i,j,\alpha,\beta = 1,\cdots,d. \tag{1.7}$$
- $B^{-1}A$ generates elliptic (generally nonsymetric) bilinear form a on $W^{1,2}_0(\Omega)^d$ where
$$a(u,v) := \int_\Omega (B^{-1}A\nabla u) : \nabla v \, dx$$
for $u, v \in W^{1,2}_{0,\text{div}}(\Omega)$ and there exists a $\lambda > 0$ such that
$$a(v,v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A^{\alpha\beta}_{kj} \, D^\alpha v_i \, D^\beta v_j \, dx \ge \lambda ||\nabla v||_2^2 \tag{1.8}$$
for all $v \in W^{1,2}_0(\Omega)^d$.

Under the above assumptions, we prove the existence and uniqueness of a weak solution (u,p) of system (1.1) for every right hand side $f \in W^{-1,2}(\Omega)^d$.

Theorem 1.1. *Let the assumptions* (1.6), (1.7), (1.8) *be in force and* Ω *be a bounded Lipschitz domain, let* Ω_0 *be a nonempty subdomain of* Ω. *Suppose that* $f \in W^{-1,2}(\Omega)^d$. *Then there exists unique pair* $(u,p) \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega)$ *satisfying* $\int_{\Omega_0} p\, dx = 0$ *and solving system* (1.1).
Moreover, the inequality

$$\|u\|_{1,2} + \|p\|_2 \leq C\|f\|_{-1,2} \tag{1.9}$$

holds with a constant $C = C(\Lambda_A, \lambda, |B|, \Omega, \Omega_0) > 0$.

Proof. It is obvious that $a(u,v)$ is a bilinear form on $W^{1,2}_{0,\mathrm{div}}(\Omega)$ and

$$|a(u,v)| \leq C\|\nabla u\|_2\|\nabla v\|_2 \leq C\|u\|_{1,2}\|v\|_{1,2}$$

for all $u, v \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ with a constant $C = C(\Lambda_A, |B|, \Omega) > 0$.
By assumption (1.8) and Poincaré's inequality we have

$$a(u,u) \geq \lambda\|\nabla u\|_2^2 \geq \frac{\lambda}{C}\|u\|_{1,2}^2$$

for all $u \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ with a constant $C = C(\Omega) > 0$.
Applying the Lax-Milgram theorem, we conclude the existence and uniqueness of $u \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ satisfying

$$\int_\Omega (B^{-1}A)\nabla u : \nabla v\, dx = [B^{-1}f, v] \text{ for all } v \in W^{1,2}_{0,\mathrm{div}}(\Omega) \tag{1.10}$$

By (1.8), we obtain

$$\lambda\|\nabla u\|_2^2 \leq \int_\Omega (B^{-1}A)\nabla u : \nabla v\, dx = [B^{-1}f, v] \leq C\|f\|_{-1,2}\|u\|_{1,2}$$

so that

$$\|u\|_{1,2} \leq C\|f\|_{-1,2} \tag{1.11}$$

with $C = C(\Lambda_A, \lambda, |B|, \Omega) > 0$.

Now we show the existence of a pressure p. Consider a functional $G : W_0^{1,2}(\Omega)^d \longrightarrow \mathbb{R}$ defined by

$$[G, v] := [B^{-1}f + \mathrm{div}(B^{-1}A\nabla u), v] = [B^{-1}f, v] - \int_\Omega (B^{-1}A)\nabla u : \nabla v\, dx.$$

From (1.10) we have [G,v] $= 0$ for all $v \in W^{1,2}_{0,\mathrm{div}}(\Omega)$.
Due to (1.11), it is easily seen that

$$|[G,v]| \leq \|B^{-1}f\|_{-1,2}\|v\|_{1,2} + \Lambda_A|B^{-1}|\|u\|_{1,2}\|v\|_{1,2} \leq C\|f\|_{-1,2}\|v\|_{1,2}$$

for all $v \in W_0^{1,2}(\Omega)$, where a constant $C = C(\Lambda_A, \lambda, |B|, \Omega) > 0$. Therefore, lemma 0.1 guarantees existence and uniqueness of $p \in L^2(\Omega)$ with $\nabla p = G$ and $\int_{\Omega_0} p\ dx = 0$. It implies that (u,p) is a weak solution of system (1.1). Moreover, we have

$$\|p\|_2 \leq C\|G\|_{-1,2} \leq C\|f\|_{-1,2} \tag{1.12}$$

with a constant $C = C(\Lambda_A, \lambda, |B|, \Omega, \Omega_0) > 0$. From (1.11), (1.12), the inequality (1.9) follows.

To prove the uniqueness of (u,p), we suppose that $(\tilde{u}, \tilde{p}) \in W_{0,\text{div}}^{1,2}(\Omega) \times L^2(\Omega)$ is another pair solving (1.1). We see that

$$\int_\Omega (B^{-1}A)\nabla(u - \tilde{u}) : \nabla v\ dx = 0 \text{ for all } v \in W_{0,\text{div}}^{1,2}(\Omega).$$

Setting $v := u - \tilde{u}$, we obtain

$$0 = \int_\Omega (B^{-1}A)\nabla(u - \tilde{u}) : \nabla(u - \tilde{u})\ dx \geq \lambda\|\nabla(u - \tilde{u})\|_2.$$

It implies $\|\nabla(u - \tilde{u})\|_2 = 0$ and, as $u, \tilde{u} \in W_{0,\text{div}}^{1,2}(\Omega)$, also $u = \tilde{u}$.
Of course, the uniqueness of p follows from the above proof, when applying Lemma 0.1. □

Next, we will use Theorem 1.1 and solve a more general system

$$\begin{aligned} -\text{div}\ (A\nabla u) + B\nabla p &= f \text{ in } \Omega, \\ \text{div}\ u &= g \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.13}$$

Theorem 1.2. *Let assumptions* (1.6), (1.7), (1.8) *be in force and* Ω *be a bounded Lipschitz domain, let* Ω_0 *be a nonempty subdomain of* Ω. *Suppose that* $f \in W^{-1,2}(\Omega)^d$, $g \in L^2(\Omega)$ *such that* $\int_\Omega g\ dx = 0$. *Then there exists unique pair* $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ *that solves system* (1.10) *with the following properties :*
u has a decomposition $u = u_0 + u_1$ *with* $u_0 \in W_{0,\text{div}}^{1,2}(\Omega)$, $u_1 \in W_0^{1,2}(\Omega)^d$ *and* $\text{div}\ u_1 = g$ *in* Ω ; $\int_{\Omega_0} p\ dx = 0$.
Moreover, u,p satisfy inequality

$$\|u\|_{1,2} + \|p\|_2 \leq C\,(\|f\|_{-1,2} + \|g\|_2) \tag{1.14}$$

with a constant $C = C(\Lambda_A, \lambda, |B|, \Omega, \Omega_0) > 0$.

Proof. According to Lemma 0.1, we can choose $u_1 \in W_0^{1,2}(\Omega)^d$ satisfying $\text{div}\ u_1 = g$ and

$$\|\nabla u_1\| \leq C\|g\|_2. \tag{1.15}$$

with a constant $C = C(\Omega) > 0$.
Then, using Theorem 1.1, we find a unique pair $(u_0, p) \in W^{1,2}_{0,\text{div}}(\Omega) \times L^2(\Omega)$ satisfying $\int_{\Omega_0} p\, dx = 0$ and $-\text{div}\,(A\nabla u_0) + B\nabla p = f + \text{div}\,(A\nabla u_1)$. If we put $u := u_0 + u_1$, then pair (u,p) solves system (1.10).
From Theorem 1.1 and the inequalities (1.6) and (1.12) we have estimates

$$
\begin{aligned}
\|u\|_{1,2} + \|p\|_2 &\leq \|u_0\|_{1,2} + \|p\|_2 + \|u_1\|_{1,2} \\
&\leq C\,(\|f\|_{-1,2} + \Lambda\|u_1\|_{1,2}) + \|u_1\|_{1,2} \\
&\leq C\,(\|f\|_{-1,2} + \|g\|_2))
\end{aligned}
\tag{1.16}
$$

with a constant $C = C(\Lambda_A, \lambda, |B|, \Omega, \Omega_0) > 0$.

To prove uniqueness we suppose that $(\tilde{u}, \tilde{p})$ is another pair solving system (1.13) and $\tilde{u}$ has a decomposition $\tilde{u} = \tilde{u}_0 + \tilde{u}_1$ where $\tilde{u}_0 \in W^{1,2}_{0,\text{div}}(\Omega)$, $\tilde{u}_1 \in W^{1,2}_0(\Omega)$, div $\tilde{u}_1 = g$; $\int_{\Omega_0} p\, dx = 0$. Then $u_1 - \tilde{u}_1 \in W^{1,2}_0(\Omega)^d$ and div $(u_1 - \tilde{u}_1) = 0$. Therefore $(u - \tilde{u}, p - \tilde{p})$ is a solution to (1.1) as div $(u - \tilde{u}) = 0$, $f = 0$, $\int_{\Omega_0} (p - \tilde{p})\, dx = 0$. The uniqueness result established in Theorem 1.1 implies that $u = \tilde{u}$, $p = \tilde{p}$. □

Examples. To illustrate the type of systems we have in mind we show some examples that satisfy conditions (1.6), (1.7), (1.8).

Proposition 1.1. *(A elliptic, B near to identity) Suppose that*

- $A^{\alpha\beta}_{ij}$ *belong to* $L^\infty(\Omega)$ *and there is a positive* Λ_A *such that*

$$\textit{ess sup}|A^{\alpha\beta}_{ij}| \leq \Lambda_A \textit{ for all } i, j, \alpha, \beta = 1, \cdots, d.$$

- *A generates an elliptic bilinear form a on* $W^{1,2}_0(\Omega)^d$ *i.e. there is a positive constant* λ_A *such that*

$$a(v, v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A^{\alpha\beta}_{ij}\, D^\alpha v_i\, D^\beta v_j\, dx \geq \lambda_A \|\nabla v\|_2^2$$

for all $v \in W^{1,2}_0(\Omega)^d$.

- *B is a constant* $d \times d$ *matrix such that*

$$\zeta = |B - E| < \frac{\lambda_A}{\lambda_A + d^4 \Lambda_A} \tag{1.17}$$

where E is the identity $d \times d$ *matrix.*

Then conditions (1.6), (1.7), (1.8) *hold.*

Proof. We need only to check condition (1.8).
We have $B = E - (E - B)$ and because of the assumption (1.17), B is regular and $B^{-1} = \sum_{l=0}^{\infty}(E - B)^l$. Thus, the condition (1.6), (1.7) are satisfied. We have

$$\int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta}\, D^\alpha v_i\, D^\beta v_j\, dx =$$

$$= \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta}\, D^\alpha v_i\, D^\beta v_j\, dx + \sum_{l=1}^{\infty} \int_\Omega ((E-B)^l A)\nabla v : \nabla v\, dx$$

$$\geq \lambda_A \|\nabla v\|_2^2 - \sum_{l=1}^{\infty} |(B-E)^l| \|A\|_\infty \|\nabla v\|_2^2$$

$$\geq \|\nabla v\|_2^2 (\lambda_A - \frac{\zeta}{1-\zeta} \Lambda_A d^4)$$

$$\geq \epsilon \|\nabla v\|_2^2 \text{ for all } v \in W_0^{1,2}(\Omega)^d$$

where positive ϵ is so small that $\zeta < \frac{\lambda_A - \epsilon}{\Lambda_A d^4 + \lambda_A - \epsilon} < \frac{\lambda_A}{\Lambda_A d^4 + \lambda_A}$. The condition (1.8) is satisfied with $\lambda = \epsilon$.
Remark. Note that A is elliptic for example if

$$a(u,v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta}\, D^\alpha u_i\, D^\beta v_j\, dx$$

and there is a positive λ_A such that

$$\sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geq \lambda_A |\xi|^2 \text{ for all } \xi \in R^{d\times d},$$

or A is constant and

$$a(u,v) = \int_\Omega \sum_{i,j,k,l=1}^{d} A_{ij}^{kl}\, D^{ij} u\, D^{kl} v\, dx$$

where $D^{ij} u = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the symmetric part of ∇u and there is a positive λ_A such that

$$\sum_{i,j,k,l=1}^{d} A_{ij}^{kl} \eta_{ij} \eta_{kl} \geq \lambda_A |\eta|^2 \text{ for all symmetric } \eta \in R^{d\times d}.$$

□

Proposition 1.2. *(Laplace operator on the diagonal, B positive definite) Suppose that* div $(A\nabla v)$ *is Laplace operator on* v_j *in* j *- th equation ,* $j = 1, \cdots, d$ *i.e.*

$$A_{ij}^{\alpha\beta} = \delta_{\alpha\beta}\delta_{ij} \; for \; all \; i, j, \alpha, \beta = 1, \cdots .d$$

and B *is a constant, self adjoint and positive definite matrix.*
Then conditions (1.6), (1.7), (1.8) *are satisfied.*

Remark. Under the hypotheses of Proposition 1.2, system (1.10) takes the form

$$\begin{aligned} -\Delta u + B\nabla p &= f \quad \text{in } \Omega, \\ \text{div } u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.18}$$

Proof. It is easy to check conditions (1.6), (1.7). Now we prove that condition (1.8) is satisfied .

As B is self adjoint and positive definite also B^{-1} is self adjoint positive definite i.e. there exists a constant $\lambda_{B^{-1}} > 0$ such that

$$\sum_{i,j} (B^{-1})_{ij}\xi_i\xi_j \geq \lambda_{B^{-1}}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d.$$

Hence we have

$$\begin{aligned} \sum_{i,j,\alpha,\beta}^{d} \left[\sum_{k=1}^{d} (B^{-1})_{ik} \, A_{kj}^{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j &= \sum_{i,j,\alpha,\beta}^{d} \left[\sum_{k=1}^{d} (B^{-1})_{ik} \, \delta_{kj} \, \delta_{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j \\ &= \sum_{i,j,\alpha,\beta}^{d} \left[(B^{-1})_{ij} \, \delta_{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j \\ &= \sum_{i,j,\alpha}^{d} \left[(B^{-1})_{ij} \; \right] \xi_\alpha^i \, \xi_\alpha^j \\ &\geq \lambda_{B^{-1}}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d. \end{aligned}$$

Thus (1.8) is satisfied and it completes the proof. □

Counterexample 1.3. If B is not regular it is easily seen that system (1.13) need not have in general any solution u, reason being that the system is overdetermined.

If, for example, A is Laplace operator on the diagonal, $B = 0$, $d = 2$, $\Omega := (0, \pi) \times (0, \pi)$, $f = (2 \sin x_1 \sin x_2 \; , \; 0)$, u should satisfy

$$\begin{aligned} -\Delta u &= f \; \text{ in } \Omega, \\ \text{div} u &= 0 \; \text{ in } \Omega, \\ u &= 0 \; \text{ on } \partial\Omega. \end{aligned} \tag{1.19}$$

By elementary calculation, the system

$$-\triangle u = f \ \text{ in } \Omega,$$
$$u = 0 \ \text{ on } \partial\Omega$$

has a unique solution $u = (\sin x_1 \sin x_2, 0)$. This solution doesn't satisfy equation div $u = 0$ in Ω (div $u = \cos x_1 \sin x_2$). Therefore, system (1.19) does not have any solution.

2. Hilbert space regularity

Our purpose is to investigate regularity of solutions to a generalized Stokes system

$$\begin{aligned} -\text{div }(A\nabla u) + B\nabla p &= f \ \text{ in } \Omega, \\ \text{div } u &= g \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned} \tag{1.20}$$

in $W^{k+2,2}(\Omega)^d \times W^{k+1,2}(\Omega)$ for $k \in \mathbb{N}$.
Here A is a $d^2 \times d^2$ matrix and B is a $d \times d$ matrix of sufficiently smooth functions. We assume throughout this section that A,B satisfy the following conditions:

- $$B \ \text{ is regular,} \tag{1.21}$$
- $B^{-1}A$ satisfies uniformly strong ellipticity condition i.e. there exists a positive λ so that

$$\sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2 \text{ in } \Omega \text{ for all } \xi \in R^{d\times d}. \tag{1.22}$$

Under the assumption (1.21) and assuming that $A_{ij}^{\alpha\beta}$, $B_{ij} \in C^{0,1}(\overline{\Omega})$ for all $i, j, \alpha, \beta = 1, \cdots, d$; system (1.20) can be transformed to

$$\begin{aligned} -\text{div }(\bar{A}\nabla u) + \nabla p &= \bar{f} \ \text{ in } \Omega, \\ \text{div } u &= g \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1.23}$$

where $\bar{f} = (\bar{f})_{i=1}^d := \left((B^{-1}f)_i - (\sum_{\alpha,\beta,j,k=1}^d D_\alpha (B^{-1})_{ik} A_{kj}^{\alpha\beta} D_\beta u_j)\right)_{i=1}^d$,
$\bar{A} := B^{-1}A$.

We show that any solution pair $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ under natural conditions on f, g, A, B, Ω satisfies $u \in W^{k+2,2}(\Omega)^d$ and $p \in W^{k+1,2}(\Omega)$ for all $k \in \mathbb{N}$.

Theorem 1.3. *Let $k \in \mathbb{N}$, Ω be a bounded C^{k+2} -domain in $R^d, (d \geq 2)$ $f \in W^{k,2}(\Omega)^d$, $g \in W^{k+1,2}(\Omega)$, $A, B \in C^{k,1}(\overline{\Omega})$, fulfilling* (1.21), (1.22). *Let $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution of system* (1.20). *Then we have*

$$u \in W^{k+2,2}(\Omega)^d, \; p \in W^{k+1,2}(\Omega), \tag{1.24}$$

and inequality

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq C\left(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_{1,2} + \|p\|_2\right) \tag{1.25}$$

holds with a constant $C = C(\lambda, d, \|A\|_{C^{k,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{k,1}(\overline{\Omega})}, \Omega) > 0$.

Proof. We shall prove Theorem 1.3 for $k = 0$ and indicate how the proof can be continued by induction for $k \in \mathbb{N}$.

Let k $= 0$. In Lemmas 1.1-1.3 we prove the assertion under auxiliary assumptions on supports of u and p. Using the decomposition of Ω, partition of unity and these results we shall complete the proof of Theorem 1.3 for Ω.

Lemma 1.1. *Let the assumptions of* Theorem 1.3 *be satisfied for $k = 0$, $x_0 \in \Omega$, $R_0 \in (0, \text{dist}\,(\text{x}_0, \partial\Omega))$. Let $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution of system* (1.20) *such that* supp u, supp $p \subset B_{R_0}(x_0)$.
Then we get

$$u \in W^{2,2}(\Omega)^d, \; p \in W^{1,2}(\Omega), \tag{1.26}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2\right) \tag{1.27}$$

with a constant $C = C(\lambda, R_0, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Proof. By assumptions of Theorem 1.3, it is easily seen that $\bar{A}$ defines an elliptic bilinear form, $\bar{A} \in C^{0,1}(\overline{\Omega})$, $\bar{f} \in L^2(\Omega)$ and we have

$$\|\bar{f}\| \leq C(\|f\|_2 + \|\nabla u\|_2). \tag{1.28}$$

Denote by e_s the unit vector in the x_s direction $(s = 1, \cdots, d)$. Set the difference quotient in the x_s direction $\triangle_{\delta,s} u = \frac{1}{\delta}[u(x + \delta e_s) - u(x)]$ with $0 < |\delta| < \text{dist}(\partial\Omega, B_{R_0}(x_0))$, $s = 1, \cdots, d$.
Since (u,p) solve the system (1.20) we have

$$\begin{aligned} -\text{div}\,[\bar{A}(x + \delta e_s)\nabla u(x + \delta e_s)] + \nabla p(x + \delta e_s) &= \bar{f}(x + \delta e_s), \\ \text{div}\, u(x + \delta e_s) &= g(x + \delta e_s) \end{aligned}$$

on $B_{R_0}(x_0)$. Subtracting (1.23) from those equations and then dividing by δ, we obtain

$$\begin{aligned} -\text{div } \left[\bar{A}(x)\nabla(\triangle_{\delta,s}u(x))\right] + \nabla(\triangle_{\delta,s}p(x)) = \\ \triangle_{\delta,s}\bar{f}(x) + \text{div } [\triangle_{\delta,s}(\bar{A}(x))\nabla u(x+\delta e_s)], \\ \text{div } (\triangle_{\delta,s}u(x)) = \triangle_{\delta,s}g(x) \text{ on } B_{R_0}(x_0). \end{aligned} \tag{1.29}$$

Thanks to Lemma 0.1 the quotient $\triangle_{\delta,s}g(x)$ can be written in the form div $G_{\delta,s}$ with some $G_{\delta,s} \in W_0^{1,2}(\Omega)^d$ such that $\|\nabla G_{\delta,s}\|_2 \leq C\|\triangle_{\delta,s}g\|_2$ with C not depending on δ and s.
We observe that $\triangle_{\delta,s}\bar{f}$ can be written in the form div $F_{\delta,s}$ in the sense of distributions with some $F_{\delta,s} \in L^2(\Omega)^{d^2}$ such that $\|F_{\delta,s}\|_2 \leq C\|\bar{f}\|_2$ with C not depending on δ and s (see (2.27), Chapter 2 concerning this property).
Using definition of weak solutions to the system (1.29) and setting $v = (\triangle_{\delta,s}u - G_{\delta,s}) \in W_{0,\text{div}}^{1,2}(\Omega)$ we get

$$\begin{aligned} &\int_\Omega \bar{A}(x)\,[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)] : [\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\ dx \\ &\qquad + \int_\Omega \bar{A}(x)[\nabla G_{\delta,s}(x)] : [\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\ dx \\ &\qquad\qquad = - \int_\Omega F_{\delta,s}.[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\ dx \\ &\qquad\qquad - \int_\Omega \triangle_{\delta,s}\bar{A}(x)[\nabla u(x+\delta e_s)] : [\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\ dx \end{aligned}$$

Assumptions of Lemma 1.1, (1.22) then lead to

$$\begin{aligned} &\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2^2 \leq \\ &\quad \frac{1}{\lambda}\int_\Omega \bar{A}(x)\,[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)] : [\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\ dx \\ &\quad \leq \frac{C}{\lambda}(\|\bar{A}\|_{C^0(\overline{\Omega})}\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}\|\nabla u\|_2)\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2 \end{aligned}$$

where we estimated $\triangle_{\delta,s}\bar{A}$ by $\|\bar{A}\|_{C^{0,1}(\overline{\Omega})}$. It follows

$$\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2 \leq \frac{C}{\lambda}(\|\bar{A}\|_{C^0(\overline{\Omega})}\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}\|\nabla u\|_2)$$

and

$$\begin{aligned} \|\nabla\triangle_{\delta,s}u\|_2 &\leq \|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2 + \|\nabla G_{\delta,s}\|_2 \\ &\leq C(\|\bar{A}\|_{C^{0,1}(\overline{\Omega})})(\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\nabla u\|_2) \end{aligned} \tag{1.30}$$

with a constant $C(\lambda, \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$ not depending on δ, s.
As supp $p \subset B_{R_0}(x_0)$ and δ is small, we have $\int_\Omega \triangle_{\delta,s}p\ dx = 0$. Applying

Lemma 0.1 to system (1.29) and using inequality (1.30) we conclude that $\triangle_{\delta,s}p \in L^2(\Omega)$ for all $s = 1, \cdots, d$ and we have the estimate

$$\|\triangle_{\delta,s}p\|_2 \leq C(\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\nabla u\|_2) \tag{1.31}$$

with a constant $C = C(\lambda, R_0, \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$ not depending on δ, s.
If we let δ tend to zero in inequalities (1.30), (1.31), we deduce that $D_s\nabla u \in L^2(\Omega)^{d^2}$, $D_s p \in L^2(\Omega)$ for all $s = 1, \cdots, d$ and we have the estimate

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2\right)$$

with a constant $C = C(\lambda, R_0, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})})$. Lemma 1.1 is proved. □

Remarks. Lemma 1.1 holds under slightly weaker ellipticity assumptions satisfied by examples of the previous section.

The next lemma deals with estimates near the flat boundary. It is given here to explain the main ideas of the proof and will not be explicitly used later in this chapter.

For $R_0 > 0,\ \beta_0 > 0,\ x_0 = (x', 0) \in \partial\Omega$ denote

$$\begin{aligned}
\Gamma_{R_0} &= \{x = (x', 0) \in \mathbb{R}^d;\ x' \in B'_{R_0}\},\\
U^+_{R_0,\beta_0} &= \{x = (x', x_d) \in \mathbb{R}^d;\ x' \in B'_{R_0}; 0 < x_d < \beta_0\},\\
U_{R_0,\beta_0} &= \{x = (x', x_d) \in \mathbb{R}^d;\ x' \in B'_{R_0}; |x_d| < \beta_0\}.
\end{aligned}$$

Lemma 1.2. *Let the assumptions of* Theorem 1.3 *be satisfied for* $k = 0$, R_0, β_0 *be positive,* $\Gamma_{R_0} \subset \partial\Omega$, $U^+_{2R_0,2\beta_0} \subset \Omega$. *Let* $(u, p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ *be a weak solution of system* (1.20) *such that*

$$\text{supp}\ u,\ \text{supp}\ p \subset U_{R_0,\beta_0}(x_0) \cap \overline{\Omega}$$

where $U_{R_0,\beta_0}(x_0) := \left\{x \in R^d; |x' - x_0'| < R_0, |x_d| < \beta_0\right\}$ *for* $R_0 > 0, \beta_0 > 0$. *Then it holds*

$$u \in W^{2,2}(\Omega)^d,\ p \in W^{1,2}(\Omega), \tag{1.32}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|\nabla g\|_2 + |\nabla u\|_2\right) \tag{1.33}$$

with a constant $C = C(\lambda, R_0, \beta_0, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Proof. By the same way as in lemma 1.1 we get

$$D_s\nabla u \in L^2(\Omega)^{d^2},\ D_s p \in L^2(\Omega) \text{ for all } s = 1, \cdots, d-1,$$

and we have estimate

$$\|\nabla'\nabla u\|_2 + \|\nabla' p\|_2 \le C\left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2\right) \tag{1.34}$$

with a constant $C = C(\lambda, R_0, \beta_0, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$.
Using the structure of the system (1.23) we have

$$\sum_{j=1}^{d-1} \bar{A}^{dd}_{ij} D^2_d u_j = G_i \ , \ i = 1, \cdots, d-1, \tag{1.35}$$

$$\sum_{j=1}^{d} \bar{A}^{dd}_{dj} D^2_d u_j = G_d + D_d p, \tag{1.36}$$

$$D^2_d u_d = D_d g - \sum_{j=1}^{d-1} D_d D_j u_j, \tag{1.37}$$

where

$$G_i = D_i p - \bar{f}_i - \sum_{\alpha+\beta<2d} \bar{A}^{\alpha\beta}_{ij} D_\alpha D_\beta u_j -$$

$$- \sum_{\alpha,\beta,j=1}^{d} (D_\alpha \bar{A}^{\alpha\beta}_{ij}) D_\beta u_j - \bar{A}^{dd}_{id} D^2_d u_d, i = 1, \cdots, d-1, \tag{1.38}$$

$$G_d = -\bar{f}_d - \sum_{\alpha+\beta<2d} \bar{A}^{\alpha\beta}_{dj} D_\alpha D_\beta u_j - \sum_{\alpha,\beta,j=1}^{d} D_\alpha(\bar{A}^{\alpha\beta}_{dj}) D_\beta u_j. \tag{1.39}$$

(1.34) implies that $D_j D_d u_j \in L^2(\Omega)$, $j = 1, \cdots, d-1$. The assumption $g \in W^{1,2}(\Omega)$ and equation (1.37) give $D^2_d u_d \in L^2(\Omega)$. Since $f \in L^2(\Omega)^d, \nabla' p \in L^2(\Omega)^{d-1}, D'\nabla u \in L^2(\Omega)^{(d-1)d}$, (1.38), (1.39) show that $G_i \in L^2(\Omega)$, $i = 1, \cdots, d$.

System (1.35) consists of $(d-1)$ linear equations, matrix $\bar{A}^{dd}_{ij}$, $i, j = 1, \cdots, d-1$ is regular and its inverse is bounded by $\frac{1}{\lambda}$. Therefore, the L^2-integrability of $D^2_d u_j, j = 1, \cdots, d-1$ follows from L^2-integrability of G_i, $i = 1, \cdots, d-1$. Since $\bar{A} \in C^{0,1}(\overline{\Omega})$, we conclude that $D^2_d u_j \in L^2(\Omega)$, for $j = 1, \cdots, d-1$, and obtain that

$$\|D^2_d u_j\|_2 \le C \sum_{i=1}^{d-1} \|G_i\|_2 \text{ for j} = 1, \cdots, \text{d} - 1. \tag{1.40}$$

Finally, since $\bar{f}_d$, $D^2_d u_j \in L^2(\Omega)$ $j = 1 \cdots, d$, $\bar{A} \in C^{0,1}(\overline{\Omega})$, it follows from the equation (1.36) that $D_d p \in L^2(\Omega)$ and (1.34), (1.37), (1.38), (1.39), (1.40), (1.36), we have the following estimates

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \le C\left(\|f\|_2 + \|\nabla g\|_2 + |\nabla u\|_2\right)$$

with a constant $C = C(\lambda, R_0, \beta_0, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$. The lemma is proved. □

Lemma 1.3. *Let the assumptions of* Theorem 1.3 *be satisfied for* $k = 0$, $R_0 > 0$, $\beta_0 > 0$, $x_0 \in \partial\Omega$. *Let* $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ *be a weak solution of system* (1.20), *and* $\operatorname{supp} u, \operatorname{supp} p \subset U_{R_0,\beta_0,h}(x_0) \cap \overline{\Omega}$. *Then there exists a constant* $K > 0$ *(given in* (1.50)*) so that for*

$$\|h\|_{C^1(\overline{B'}_{R_0})} \leq K, \tag{1.41}$$

it holds

$$u \in W^{2,2}(\Omega)^d, \; p \in W^{1,2}(\Omega), \tag{1.42}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\,(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2) \tag{1.43}$$

with a constant $C = C(\lambda, d, R_0, \beta_0, K, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Proof. In order to reduce Lemma 1.3 to be previous case we use the transformation to new coordinates

$$y = \Phi(x) := (x', x_d - h(x')), \; x \in U_{R_0,\beta_0,h}(x_0). \tag{1.44}$$

We see that Φ is one-to-one mapping of $U_{R_0,\beta_0,h}(x_0)$ on $U_{R_0,\beta_0}(y_0)$, where $y_0 = \Phi(x_0)$. Next, define $\hat{u}, \hat{p}, \hat{f}, \hat{g}, \hat{A}$ by

$$\begin{aligned}\hat{u}(y) &:= u(\Phi^{-1}(y)) = u(x), \; \hat{p}(y) := p(\Phi^{-1}(y)) = p(x); \; \hat{f}(y) := \bar{f}(\Phi^{-1}(y)) \\ &= \bar{f}(x), \; \hat{g}(y) := g(\Phi^{-1}(y)) = g(x), \; \hat{A}(y) := \bar{A}(\Phi^{-1}(y)) = \bar{A}(x).\end{aligned} \tag{1.45}$$

We have also $u(x) = \hat{u}(\Phi(x))$ so that

$$D_\beta u(x) = D_\beta \hat{u}(y) - D_d\hat{u}(y) D_\beta h(y'), \; \beta = 1, \cdots, d-1, \; D_d u(x) = D_d \hat{u}(y)$$

and correspondingly for $p, g, \bar{f}, \bar{A}$. An elementary calculation transforms (1.20) to a new system

$$\begin{aligned}-\operatorname{div}(\tilde{A}\nabla\hat{u}) + \nabla\hat{p} &= \hat{f} + T - (D_d H_1)p + D_d(H_1 p), \\ \operatorname{div}\hat{u} &= \hat{g} + H_2\end{aligned} \tag{1.46}$$

where a $d^2 \times d^2$ matrix $\tilde{A} := (\tilde{A}_{ij}^{\alpha\beta})_{i,j,\alpha,\beta=1}^d$; a vector H_1, a function H_2 are given by

$$\tilde{A}_{ij}^{\alpha\beta} := \hat{A}_{ij}^{\alpha\beta}, \text{ if } \alpha, \beta < d; \; \tilde{A}_{ij}^{\alpha d} := \hat{A}_{ij}^{\alpha d} - \sum_{\beta=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\beta h, \text{ if } \alpha < \mathrm{d};$$

$$\tilde{A}_{ij}^{d\beta} := \hat{A}_{ij}^{d\beta} - \sum_{\alpha=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\alpha h, \text{ if } \beta < \mathrm{d};$$

$$\tilde{A}_{ij}^{dd} := \hat{A}_{ij}^{dd} - \sum_{\alpha=1}^{d-1} \hat{A}_{ij}^{\alpha d} D_\alpha h - \sum_{\beta=1}^{d-1} \hat{A}_{ij}^{d\beta} D_\beta h + \sum_{\alpha,\beta=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\alpha h D_\beta h;$$

for $i, j = 1, \cdots, d$.

$$H_1 := (D_1 h, D_2 h, \cdots, D_{d-1} h, 0); \quad H_2 := \sum_{j=1}^{d-1} D_d \hat{u}_j D_j h;$$

$T := (T_i)_{i=1}^d$ with

$$T_i := \sum_{\alpha,\beta,j=1}^{d} [D_\alpha \hat{A}_{ij}^{\alpha\beta} - (1-\delta_{\alpha d}) D_d \hat{A}_{ij}^{\alpha\beta} D_\alpha h][D_\beta \hat{u}_j - (1-\delta_{\beta d}) D_d \hat{u}_j D_\beta h]$$
$$- \sum_{\alpha,\beta,j=1}^{d} D_\alpha \tilde{A}_{ij}^{\alpha\beta} D_\beta \hat{u}_j.$$

The assumption (1.22) and assumptions of Lemma 1.3 imply that there exists a constant $K_1 > 0$ such that if $\|h\|_{C^1(\overline{B'}_{R_0})} \le K_1$, then

$$\sum_{\alpha,\beta,i,j=1}^{d} \tilde{A}_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \ge \frac{\lambda}{2} |\xi|^2 \tag{1.47}$$

for all $\xi \in \mathbb{R}^{d^2}$.
Thus

$$\Theta := \det \begin{pmatrix} \tilde{A}_{ij}^{dd} & D_i h \\ \tilde{A}_{dj}^{dd} & -1 \end{pmatrix}_{i,j=1}^{d-1} = -\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1}$$
$$+ \sum_{k=1}^{d-1} (-1)^{k+d} D_k h \det \left(\tilde{A}_{ij}^{dd}\right)_{i=1;j=1}^{i=d, i\neq k; j=d-1}$$

and
$$|\Theta| \ge |\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1}| - \|h\|_{C^1(\overline{B'}_{R_0})} C(\|\bar{A}\|_{C^0(\overline{\Omega})}, d)$$

with constant $C(\|\bar{A}\|_{C^0(\overline{\Omega})}, d) > 0$.
If (1.47) holds, then Lemma 0.8 shows that there exists constant $C(\lambda, d) > 0$ such that $|\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1}| > C(\lambda, d)$.
Therefore there exists constant $K_2 \in (0,1)$ such that (1.47) holds, Θ is uniformly bounded away from zero and $\sum_{j=1}^{d-1} |D_j h| < 1$ for all $h \in C^1(\overline{B'}_{R_0})$ such that $\|h\|_{C^1(\overline{B'}_{R_0})} \le K_2$.

Using (1.44), (1.45) and the assumptions of Lemma 1.3, we have

$$\text{supp } \hat{u}, \text{ supp } \hat{f} \subset U_{R_0, \beta_0}(y_0) \cap \overline{\mathbb{R}}_+^d.$$

In a similar way as in Lemma 1.1, we get estimate of $\triangle_{\delta,s} \nabla \hat{u}, \triangle_{\delta,s} \hat{p}, \;\; s =$

$1, \cdots, d-1$ and the inequality

$$\begin{aligned}
&\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}} \le C[\|\triangle_{\delta,s}\hat{g}\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|h\|_{C^1(\overline{B'}_{R_0})}\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|h\|_{C^2(\overline{B'}_{R_0})}\|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|\hat{f}\|_{2;U^+_{R_0,\beta_0}} + \|T\|_{2;U^+_{R_0,\beta_0}} + \|h\|_{C^1(\overline{B'}_{R_0})}\|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|h\|_{C^2(\overline{B'}_{R_0})}\|\hat{p}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}}] \qquad (1.48)
\end{aligned}$$

with some constant $C = C(\lambda, d, R_0, \beta_0, \|\tilde{A}\|_{C^{0,1}(\overline{U^+_{R_0,\beta_0}})}) > 0$.
Here we used the estimates

$$\begin{aligned}
&\|\triangle_{\delta,s}(D_i hp)\|_{2;U^+_{R_0,\beta_0}} \le \|h\|_{C^1(\overline{B'}_{R_0})}\|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}} + C\|h\|_{C^2(\overline{B'}_{R_0})}\|\hat{p}\|_{2;U^+_{R_0,\beta_0}}), \\
&\|\triangle_{\delta,s}(D_i h D_d\hat{u}_j)\|_{2;U^+_{R_0,\beta_0}} \le \|h\|_{C^1(\overline{B'}_{R_0})}\|\triangle_{\delta,s}D_d\hat{u}_j\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + C\|h\|_{C^2(\overline{B'}_{R_0})}\|D_d\hat{u}_j\|_{2;U^+_{R_0,\beta_0}}).
\end{aligned}$$

As $\|h\|_{C^1(\overline{B'}_{R_0})} \le K_2 \in (0,1)$, there exists a constant $C > 0$ which does not depend on h, δ such that $\|\tilde{A}\|_{C^{0,1}(\overline{U^+_{R_0,\beta_0}})} \le C\|\bar{A}\|_{C^{0,1}(\overline{\Omega})}$. Therefore we have estimate

$$\begin{aligned}
&\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}} \le C[\|\triangle_{\delta,s}\hat{g}\|_{2;U^+_{R_0,\beta_0}} \\
&+ \|h\|_{C^2(\overline{B'}_{R_0})}\|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}}) + \|\hat{f}\|_{2;U^+_{R_0,\beta_0}} + \|T\|_{2;U^+_{R_0,\beta_0}} + \|h\|_{C^2(\overline{B'}_{R_0})}\|\hat{p}\|_{2;U^+_{R_0,\beta_0}} + \\
&\|\tilde{A}\|_{C^{0,1}(\overline{U^+_{R_0,\beta_0}})}\|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|h\|_{C^1(\overline{B'}_{R_0})}(\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}})] \\
&\qquad (1.49)
\end{aligned}$$

with a constant $C(\lambda, d, R_0, \beta_0, \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Next, we can choose

$$K = \min(\mathrm{K}_2, \frac{1}{2\mathrm{C}}). \qquad (1.50)$$

Then we have

$$\begin{aligned}
&\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\triangle_{\delta,s}\hat{p}\|_{2;U^+_{R_0,\beta_0}} \le C(\|\triangle_{\delta,s}\hat{g}\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|h\|_{C^2(\overline{B'}_{R_0})}\|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\hat{f}\|_{2;U^+_{R_0,\beta_0}} + \|T\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|h\|_{C^2(\overline{B'}_{R_0})}\|\hat{p}\|_{2;U^+_{R_0,\beta_0}} + \|\tilde{A}\|_{C^{0,1}(\overline{U^+_{R_0,\beta_0}})}\|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}}) \qquad (1.51)
\end{aligned}$$

with a constant $C(\lambda, d, R_0, \beta_0, \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$ that does not depend on s and δ. If we let $\delta \to 0$ in inequalities (1.45), we deduce that $D_s\nabla\hat{u} \in L^2(U^+_{R_0,\beta_0})^{d^2}$, $D_s\hat{p} \in L^2(U^+_{R_0,\beta_0})$ for all $s = 1, \cdots, d-1$ and we have estimate

$$\begin{aligned}
&\|\nabla'\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla'\hat{p}\|_{2;U^+_{R_0,\beta_0}} < C(\|\hat{f}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla\hat{g}\|_{2;U^+_{R_0,\beta_0}} \\
&\quad + \|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}}) + \|\hat{p}\|_{2;U^+_{R_0,\beta_0}}) \qquad (1.52)
\end{aligned}$$

with a constant $C(\lambda, d, R_0, \beta_0, K, \|\tilde{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Adopting arguments of Lemma 1.1 we obtain that all second derivatives of $\hat{u}$ and first derivatives of $\hat{p}$ exist in $U^+_{R_0,\beta_0}$ and we have estimates of their L_2-norms in any strict subdomain. From the first part of this proof we have estimates of $\nabla'\nabla\hat{u}$ and $\nabla'\hat{p}$ up to the boundary of $U^+_{R_0,\beta_0}$. For getting estimates of the remaining terms $D^2_d\hat{u}_j$, $D_d\hat{p}$ up to the boundary we use system (1.46). From the system (1.46), we have

$$\sum_{j=1}^{d-1} \tilde{A}^{dd}_{ij} D^2_d \hat{u}_j + D_i h D_d \hat{p} = \hat{G}_i \ , \ i = 1, \cdots, d-1; \tag{1.53}$$

$$\sum_{j=1}^{d-1} \tilde{A}^{dd}_{dj} D^2_d \hat{u}_j - D_d \hat{p} = \hat{G}_d; \tag{1.54}$$

$$(1 - \sum_{j=1}^{d-1} D_j h) D^2_d \hat{u}_d = D_d \hat{g} - \sum_{j=1}^{d-1} D_d D_j \hat{u}_j \tag{1.55}$$

where

$$\begin{aligned} \hat{G}_i = D_i\hat{p} - \hat{f}_i - (T)_i - \sum_{\alpha+\beta<2d} \tilde{A}^{\alpha\beta}_{ij} D_\alpha D_\beta \hat{u}_j - \\ - \sum_{\alpha,\beta,j} D_\alpha \tilde{A}^{\alpha\beta}_{ij} D_\beta \hat{u}_j - \tilde{A}^{dd}_{id} D^2_d \hat{u}_d, i = 1, \cdots, d-1; \end{aligned} \tag{1.56}$$

$$\hat{G}_d = -\hat{f}_d - (T)_d - \sum_{\alpha+\beta<2d} \tilde{A}^{\alpha\beta}_{dj} D_\alpha D_\beta \hat{u}_j - \tilde{A}^{dd}_{dd} D^2_d \hat{u}_d - \sum_{\alpha,\beta,j} D_\alpha \tilde{A}^{\alpha\beta}_{dj} D_\beta \hat{u}_j. \tag{1.57}$$

As $D_j D_d \hat{u}_j \in L^2(U^+_{R_0,\beta_0})$, $j = 1, \cdots, d-1$, $\hat{g} \in W^{1,2}(U^+_{R_0,\beta_0})$ it follows from equation (1.55) that $D^2_d\hat{u}_d \in L^2(U^+_{R_0,\beta_0})$. Since $\hat{f} \in L^2(U^+_{R_0,\beta_0})^d$, $\nabla'\hat{p} \in L^2(U^+_{R_0,\beta_0})^{d-1}$, $D'\nabla\hat{u} \in L^2(U^+_{R_0,\beta_0})^{(d-1)d}$, $T \in L^2(U^+_{R_0,\beta_0})^d$, (1.56), (1.57) shows that $\hat{G}_i \in L^2(U^+_{R_0,\beta_0})$, $i = 1, \cdots, d$.

System (1.53)-(1.54) is linear with d equations, the determinant of the matrix $\begin{pmatrix} \tilde{A}^{dd}_{ij} & D_i h \\ \tilde{A}^{dd}_{dj} & -1 \end{pmatrix}_{i,j=1}^{d-1}$ is bounded away from zero (thanks to $\|h\|_{C^1(\overline{B'}_{R_0})} \leq K_2$). Therefore, we can express $D^2_d\hat{u}_j, j = 1, \cdots, d-1$ and $D_d\hat{p}$ from right hand sides $\hat{G}_i$, $i = 1, \cdots, d$ and coefficients from $\tilde{A}$, Dh. As $\tilde{A} \in C^{0,1}(\overline{U^+_{R_0,\beta_0}})$, $\hat{G}_i \in L^2(U^+_{R_0,\beta_0})$, $i = 1, \cdots, d$, we deduce that $D^2_d\hat{u}_j \in L^2(U^+_{R_0,\beta_0})$, for $j = 1, \cdots, d-1$, $D_d\hat{p} \in L^2(U^+_{R_0,\beta_0})$ and estimate

$$\begin{aligned} \|\nabla^2\hat{u}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla\hat{p}\|_{2;U^+_{R_0,\beta_0}} < C(\|\hat{f}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla\hat{g}\|_{2;U^+_{R_0,\beta_0}} + \|\nabla\hat{u}\|_{2;U^+_{R_0,\beta_0}} \\ + \|\hat{p}\|_{2;U^+_{R_0,\beta_0}}) \end{aligned}$$

holds with a constant $C(\lambda, d, R_0, \beta_0, \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}) > 0$.

Going back to the original coordinates x we obtain the inequality

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right)$$

with a constant $C = C(\lambda, d, R_0, \beta_0, K, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}) > 0$.
It implies (1.43). The lemma is proved. □

Return to the proof of Theorem 1.3 on Ω. As Ω is a bounded C^2 - domain in $\mathbb{R}^d$ we will use decomposition of Ω described in Preliminaries. Therefore, there is a finite number m of points $x_{j_1} \in \partial\Omega$, constants $R_{j_1}, \beta_{j_1} > 0$, functions $h_{j_1} \in C^2(\overline{B'}_{R_{j_1}})$ satisfying assumption (1.41) in Lemma 1.3 and functions $\varphi_{j_1} \in C_0^\infty(\mathbb{R}^d)$, $j_1 = 1, \cdots, m$ and, moreover, a finite number l of balls $B_{k_1} \subset\subset \Omega$ and functions $\psi_{k_1} \in C_0^\infty(\mathbb{R}^d)$, $k_1 = 1, \cdots, l$ so that

$$\text{supp}\, \varphi_{j_1} \subset U_{R_{j_1}, \beta_{j_1}, h_{j_1}}(x_{j_1}),\ j_1 = 1, \cdots, m,\ \text{supp}\, \psi_{k_1} \subset B_{k_1},\ k_1 = 1, \cdots, l;$$

$$\Sigma_{j_1=1}^m \varphi_{j_1}(x) + \Sigma_{k_1=1}^l \psi_{k_1}(x) = 1, \text{ for all } x \in \overline{\Omega};$$

$$0 \leq \varphi_{j_1}(x),\ \psi_{k_1}(x) \leq 1 \text{ for all } x \in \mathbb{R}^d,\ j_1 = 1, \cdots, m,\ k_1 = 1, \cdots, l.$$

Clearly, it is enough to prove that $\|\varphi_{j_1}\nabla^2 u\|_2 < \infty$, $\|\varphi_{j_1}\nabla p\|_2 < \infty$, $j_1 = 1, \cdots, m$; $\|\psi_{k_1}\nabla^2 u\|_2 < \infty$, $\|\psi_{k_1}\nabla p\|_2 < \infty$, $k_1 = 1, \cdots, l$.
We multiply both sides of system (1.23) by φ_{j_1} and get

$$\begin{aligned} -\text{div}\, [\bar{A}\nabla(\varphi_{j_1} u)] + \nabla(\varphi_{j_1} p) &= \tilde{f}, \\ \text{div}(\varphi_{j_1} u) &= \tilde{g} \end{aligned} \tag{1.58}$$

where $\tilde{f}$, $\tilde{g}$ are given by

$$\tilde{f}_i := \varphi_{j_1}\bar{f}_i + D_i\varphi_{j_1} p - \sum_{\alpha,\beta,j=1}^{d} \bar{A}_{i,j}^{\alpha,\beta}[D_\alpha\varphi_{j_1} D_\beta u_j + (D_\alpha D_\beta \varphi_{j_1}) u_j$$

$$+ D_\alpha u_j D_\beta \varphi_{j_1}] + \sum_{\alpha,\beta,j=1}^{d} (D_\alpha \bar{A}_{i,j}^{\alpha,\beta})(D_\beta\varphi_{j_1}) u_j,\ i = 1, \cdots, d; \tag{1.59}$$

$$\tilde{g} := \varphi_{j_1} g + (\nabla\varphi_{j_1}) u. \tag{1.60}$$

It follows from (1.59) and (1.60) that

$$\|\tilde{f}\|_2 \leq \|\bar{f}\|_2 + C\left(\|p\|_2 + \|u\|_{1,2}\right) \text{ and } \|\tilde{g}\|_{1,2} \leq \|g\|_{1,2} + C\|u\|_{1,2}.$$

It is clear that supp $\varphi_{j_1} u$, supp $\varphi_{j_1} p \subset U_{R_{j_1}, \beta_{j_1}, h_{j_1}}(x_{j_1}) \cap \overline{\Omega}$ and we can apply Lemma 1.3. We obtain

$$\begin{aligned} &\|\nabla^2(\varphi_{j_1} u)\|_2 + \|\nabla(\varphi_{j_1} p)\|_2 \leq C\left(\|\tilde{f}\|_2 + \|\tilde{g}\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right) \\ &\leq C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right). \end{aligned}$$

Relations $D_iD_j(\varphi_{j_1}u) = \varphi_{j_1}(D_iD_ju) + (D_iD_j\varphi_{j_1})u + (D_i\varphi_{j_1})(D_ju)$ $+(D_j\varphi_{j_1})(D_iu)$ and $D_j(\varphi_{j_1}p) = \varphi_{j_1}D_jp + D_j\varphi_{j_1}p$ with $i,j = 1,\cdots,d$ imply also that the inequality

$$\|\varphi_{j_1}\nabla^2 u\|_2 + \|\varphi_{j_1}\nabla p\|_2 \leq C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right) \tag{1.61}$$

holds with a constant $C = C(\lambda, d, \|A\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \Omega) > 0$, for all φ_{j_1} with $j_1 = 1,\cdots,m$.

In a similar way, applying Lemma 1.1 we conclude that

$$\|\psi_{k_1}\nabla^2 u\|_2 + \|\psi_{k_1}\nabla p\|_2 \leq C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right) \tag{1.62}$$

for all ψ_{k_1}, $k_1 = 1,\cdots,l$ with a constant $C = C(\lambda, d, \|A\|_{C^{0,1}(\overline{\Omega})},$ $\|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \Omega) > 0$. This yields (1.25). The theorem is proved for k=0.

In the next step when k=1 we realize that on any strictly embedded domain Ω', $(\triangle_{\delta,s}u, \triangle_{\delta,s}p)$ solve system (1.29) for δ small enough and $s = 1,\cdots,d$. Thus an analogy of Lemma 1.1 gives estimates independent of δ. Letting $\delta \to 0$ guarantees existence of all third derivatives of u and second derivatives of p with L^2 estimates in Ω'. An analogy of Lemma 1.2 can be proved then for $(\triangle_{\delta,s}u, \triangle_{\delta,s}p)$ with δ small enough and $s = 1,\cdots,d-1$ as these difference quotients solve system (1.29) and satisfy zero boundary condition on the flat part of the boundary. Limit procedure for $\delta \to 0$ implies estimates of $\frac{\partial \nabla^2 u}{\partial x_s}$ for $s = 1,...,d-1$. Using ellipticity condition we calculate from the equation differentiated with respect to x_d estimate of the last term $\frac{\partial^3 u}{\partial x_d^3}$. After proving the "flattening of the boundary" as in Lemma 1.3 we can repeat the above proof and we have the conclusion of Theorem 1.3 for k=1. The general case follows by induction. □

3. Schauder regularity

In this section, we study Schauder type estimates for solutions to system

$$\begin{aligned} -\text{div}\,(A\nabla u) + B\nabla p &= \text{div}\, F \ \text{ in } \Omega, \\ \text{div}\, u &= g \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned} \tag{1.63}$$

where A and B are matrices of sufficiently smooth functions, $F : \Omega \longrightarrow \mathbb{R}^{d^2}$ and $g : \Omega \longrightarrow \mathbb{R}^d$. We assume throughout this section that A,B satisfy the following conditions

- $$\begin{aligned} A_{ij}^{\alpha,\beta} &\in L^\infty(\Omega) \ \text{ for all } i,j,\alpha,\beta = 1,\cdots,d; \\ B \text{ is regular on } \Omega,\ B &\in C^{0,1}(\overline{\Omega})^{d^2} \text{ and } B^{-1} \in W^{1,\infty}(\Omega)^{d^2} \end{aligned} \tag{1.64}$$

- $B^{-1}A$ satisfies strong Legendre-Hadamard ellipticity condition i.e. there exists a positive λ so that

$$\sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda |\xi|^2 |\eta|^2 \tag{1.65}$$

in Ω for all $\xi, \eta \in \mathbb{R}^d$.

Under the assumption (1.64), system (1.63) can be transformed to

$$\begin{aligned} -\text{div}\,(\bar{A}\nabla u) + H\nabla u + \nabla p &= B^{-1}\text{div}\,F \ \text{ in } \Omega, \\ \text{div}\,u &= g \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned} \tag{1.66}$$

where
$\bar{A} := B^{-1}A$, $H = \left(H_{ij}^\beta\right)_{\beta,i,j=1}^d := \left(\sum_{\alpha,k=1}^d D_\alpha (B^{-1})_{ik} A_{kj}^{\alpha\beta}\right)_{\beta,i,j=1}^d$,
$H\nabla u = \left(\sum_{\beta,j=1}^d H_{ij}^\beta D_\beta u_j\right)_{i=1}^d$.

3.1. Schauder regularity in the interior domain

First we show a local energy estimate - Caccioppoli's inequality.

Theorem 1.4. *(Caccioppoli's inequality). Suppose (1.64), (1.8). Let $(u,p) \in W_{loc}^{1,2}(\Omega)^d \times L_{loc}^2(\Omega)$ be a weak solution of system (1.66). Then there is a positive constant C such that for all $x_0 \in \Omega$ and all ρ, R with $0 < \rho < R < \text{dist}(x_0, \partial\Omega)$, we have*

$$\|\nabla u\|_{2,B_\rho(x_0)}^2 \leq C\Big[\frac{1}{(R-\rho)^2}\|(u-\nu)\|_{2,B_R(x_0)}^2 + \|F\|_{2,B_R(x_0)}^2 + \|g\|_{2,B_R(x_0)}^2\Big] \tag{1.67}$$

$$\|p - (p)_{x_0,R}\|_{2,B_R(x_0)}^2 \leq C\big[\|\nabla u\|_{2,B_R(x_0)}^2 + \|F\|_{2,B_R(x_0)}^2\big] \tag{1.68}$$

where $\nu \in \mathbb{R}^d$ is an arbitrary constant vector and constant C depends on d, $\frac{\|\bar{A}\|_\infty}{\lambda}$ and $\|B^{-1}\|_{1,\infty}$.

Proof. Let u, p be a weak solution of system (1.63) and $\nu \in \mathbb{R}^d$. Choose a test function $\varphi_1 = \eta^2(u-\nu)$, where $\eta \in C_0^\infty(R^d)$ is a cut-off function :

$$\text{supp}\,\eta \subset B_R(x_0), 0 \leq \eta \leq 1 \text{ on } \mathrm{B_R(x_0)},\ \eta \equiv 1 \text{ on } \mathrm{B_\rho(x_0)};\ |\nabla\eta| \leq \frac{\mathrm{C}}{\mathrm{R}-\rho}.$$

Then we have

$$\begin{aligned} \int_\Omega \eta^2 \bar{A}\nabla u : \nabla u\, dx = &- \int_\Omega 2\bar{A}_{ij}^{\alpha\beta} D_\beta u_j (u_i - \nu_i)\eta D_\alpha \eta + \eta^2 (H\nabla u).(u-\nu) \\ &-(p - p_{x_0,R})2\eta\nabla\eta.(u-\nu) - (p - p_{x_0,R})\eta^2 g - F\nabla(B^{-1}\eta^2(u-\nu))\, dx, \end{aligned}$$

where $F\ \nabla(B^{-1}\eta^2(u-\nu)) = \sum_{i,j,k=1}^d F_{jk}\frac{\partial}{\partial x_k}(B_{ij}^{-1}\eta^2(u_i-\nu_i))$.
Moreover,

$$\lambda\|\nabla(\eta(u-\nu))\|^2_{2,B_R(x_0)} \le \int_\Omega \bar{A}\nabla(\eta(u-\nu)) : \nabla(\eta(u-\nu))\,dx$$
$$= \int_\Omega \eta^2 \bar{A}\nabla u : \nabla u\,dx + \int_\Omega \{\bar{A}(\nabla\eta\otimes(u-\nu)) : \nabla(\eta(u-\nu))$$
$$+\bar{A}\ \nabla(\eta(u-\nu)) : (\nabla\eta\otimes(u-\nu)) + \bar{A}(\nabla\eta\otimes(u-\nu)) : (\nabla\eta\otimes(u-\nu))\}\,dx.$$

Therefore

$$\lambda\|\eta\nabla u\|^2_{2,B_R(x_0)} \le C\|\bar{A}\|_\infty(\|\eta\nabla u\|_{2,B_R(x_0)}\|\nabla\eta\otimes(u-\nu)\|_{2,B_R(x_0)}$$
$$+\|\nabla\eta\otimes(u-\nu)\|^2_{2,B_R(x_0)}) + \|H\|_\infty\|\eta\nabla u\|_{2,B_R(x_0)}\|\eta(u-\nu)\|_{2,B_R(x_0)}$$
$$+2\|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}\|\nabla\eta\otimes(u-\nu)\|_{2,B_R(x_0)} + \|\eta(p-p_{x_0,R})\|_{2,B_R(x_0)}$$
$$\|g\|_{2,B_R(x_0)} + C(\|\eta\nabla u\|_{2,B_R(x_0)} + \|\nabla\eta\otimes(u-\nu)\|_{2,B_R(x_0)})\|F\|_{2,B_R(x_0)}$$

and then by Young's inequality we have estimate

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le \varepsilon\|p-p_{x_0,R}\|^2_{2,B_R(x_0)}$$
$$+ C(\varepsilon)[\frac{1}{(R-\rho)^2}\|u-\nu\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}]. \quad (1.69)$$

For the estimate of pressure term we use Lemma 0.1 for $\Omega = \Omega_0 = B_R(x_0)$ and from system (1.66) we obtain

$$\|p-p_{x_0,R}\|^2_{2,B_R(x_0)} \le C(\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)}). \quad (1.70)$$

Substituting (1.70) into (1.69), we conclude that

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le \varepsilon\|\nabla u\|^2_{2,B_R(x_0)} + C[\frac{1}{(R-\rho)^2}\|u-\nu\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)}$$
$$+ \|g\|^2_{2,B_R(x_0)}]. \quad (1.71)$$

By application of Lemma 0.2, we obtain the inequality (1.67) and Theorem 1.4 is proved. □

Now, we consider systems with constant coefficients

$$\begin{aligned} -\mathrm{div}\,(A_0\nabla u) + \nabla p &= 0 \text{ in } \Omega, \\ \mathrm{div}\,u &= 0 \text{ in } \Omega, \end{aligned} \quad (1.72)$$

where A_0 is a $d^2 \times d^2$ constant matrix.
In a standard way (see for instance first theorem in III.2 in [12]), we have first estimates for the Hölder continuity by a following proposition.

Proposition 1.3. *Suppose that A_0 is a constant matrix satisfying strong Legendre-Hadamard condition (see* (1.65)*). Then there is a positive constant C such that for any weak solution $(u,p) \in W^{1,2}_{loc}(\Omega)^d \times L^2_{loc}(\Omega)$ of system* (1.72)*, for all $x_0 \in \Omega$ and all ρ, R with $0 < \rho \le R < dist(x_0, \partial\Omega)$, the following two estimates are valid:*

$$\|u\|^2_{2,B_\rho(x_0)} \le C(\tfrac{\rho}{R})^d \|u\|^2_{2,B_R(x_0)} \tag{1.73}$$
$$\|u - u_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C(\tfrac{\rho}{R})^{d+2} \|u - u_{x_0,\rho}\|^2_{2,B_R(x_0)}. \tag{1.74}$$

Constant C depend on $\frac{|A_0|}{\lambda}$, d and dist *$(x_0, \partial\Omega)$.*

Proof. See [13], Part I, Proposition 1.9. □

Next, we state the first regularity theorem in Morrey spaces.

Theorem 1.5. *Suppose assumptions* (1.64), (1.65) *be satisfied, $A \in C(\overline{\Omega})^{d^4}$, $F \in L^{2,\mu}(\Omega)^{d^2}$, $g \in L^{2,\mu}(\Omega)$ with $0 < \mu < d$ and $(u,p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ be a weak solution of system* (1.63)*. Then $\nabla u \in L^{2,\mu}_{loc}(\Omega)^{d^2}$, $p \in L^{2,\mu}_{loc}(\Omega)$, and for all $\tilde{\Omega} \subset\subset \Omega$ we have the estimates*

$$\|\nabla u\|_{L^{2,\mu}(\tilde{\Omega})^{d^2}} + \|p\|_{L^{2,\mu}(\tilde{\Omega})} \le C\left(\|\nabla u\|_{2,\Omega} + \|F\|_{L^{2,\mu}(\Omega)^{d^2}} + \|g\|_{L^{2,\mu}(\Omega)}\right) \tag{1.75}$$

with a constant $C = C(\lambda, \mu, d, \|A\|_{C(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{1,\infty}, \text{dist } (\tilde{\Omega}, \Omega),$ diam $\Omega) > 0$.

Proof. Fix $x_0 \in \Omega$ and $R <$ dist $(x_0, \partial\Omega)$. Let v, q be the weak solution to system

$$-\text{div } (\bar{A}(x_0)\nabla v) + \nabla q = 0 \text{ in } B_R(x_0),$$
$$\text{div } v = 0, \ u - v \in W^{1,2}_0(B_R(x_0)).$$

The existence of such a solution is guaranteed by Lax-Milgram theorem. By Proposition 1.3 we get that if $0 < \rho < R$ then

$$\|\nabla v\|^2_{2,B_\rho(x_0)} \le C(\frac{\rho}{R})^d \|\nabla v\|^2_{2,B_R(x_0)}.$$

Set $w = u - v$, then

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \le C\left((\frac{\rho}{R})^d \|\nabla u\|^2_{2,B_R(x_0)} + \|\nabla w\|^2_{2,B_R(x_0)}\right) \tag{1.76}$$

It is easily seen that $w \in W^{1,2}_0(B_R(x_0))^d$ satisfies

$$\int_{B_R(x_0)} \bar{A}(x_0)\nabla w : \nabla\varphi \, dx = \int_{B_R(x_0)} \{[\bar{A}(x_0) - \bar{A}(x)]\nabla u : \nabla\varphi - (H\nabla u).\varphi$$
$$+ F\ \nabla(B^{-1}\varphi)\} \, dx + \int_{B_R(x_0)} [(p - (p)_{x_0,R}) - (q - (q)_{x_0,R})]\text{div } \varphi \, dx,$$
$$\text{for all } \varphi \in W^{1,2}_0(B_R(x_0))^d; \quad \text{div } w = g. \tag{1.77}$$

By choosing $\varphi = w$, using the assumptions of Theorem 1.8 and Poincaré's inequality, we get

$$\|\nabla w\|^2_{2,B_R(x_0)} \leq C[(\omega^2(R) + R^2 H_c^2)\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + |\int_{B_R(x_0)} [(p - (p)_{x_0,R}) - (q - (q)_{x_0,R})]g \, dx|] \tag{1.78}$$

where

$$\omega(R) = \max_{i,j,\alpha,\beta}[\sup_{B_R(x_0)} |\bar{A}_{ij}^{\alpha\beta}(x) - \bar{A}_{ij}^{\alpha\beta}(z_0)|], \; H_c = \|H\|_\infty.$$

On the other hand, from system (1.77) applying Lemma 0.1 we have

$$\|(p-q) - (p-q)_{x_0,R}\|_{2,B_R(x_0)} \leq C[\|\nabla w\|_{2,B_R(x_0)} + (\omega(R) + RH_c)\|\nabla u\|_{2,B_R(x_0)} + \|F\|_{2,B_R(x_0)}].$$

Therefore

$$\begin{aligned}
&|\int_{B_R(x_0)} [(p - (p)_{x_0,R}) - (q - (q)_{x_0,R})]g \, dx| \\
&\qquad \leq \|(p-q) - (p-q)_{x_0,R}\|_{2,B_R(x_0)}\|g\|_{2,B_R(x_0)} \\
&\leq C[\|\nabla w\|_{2,B_R(x_0)} + (\omega(R) + RH_c)\|\nabla u\|_{2,B_R(x_0)} + \|F\|_{2,B_R(x_0)}]\|g\|_{2,B_R(x_0)} \\
&\leq \varepsilon\|\nabla w\|^2_{2,B_R(x_0)} + C(\varepsilon)[(\omega^2(R) + R^2H_c^2)\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} \\
&\qquad + \|g\|^2_{2,B_R(x_0)}]
\end{aligned} \tag{1.79}$$

with $R \leq R_1$ small enough and not dependent on z_0.
From the inequalities (1.78), (1.79), we obtain a following estimate

$$\|\nabla w\|^2_{2,B_\rho(x_0)} \leq C\{[(\omega^2(R) + R^2H_c^2]\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}\}.$$

Thanks to (1.76), it implies

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \leq C\{[(\frac{\rho}{R})^d + \omega^2(R) + R^2H_c^2]\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{2,B_R(x_0)} + \|g\|^2_{2,B_R(x_0)}\}.$$

Since $\bar{A}, H \in C^0(\overline{\Omega})$ there exists $R_0 < \text{dist}(x_0, \partial\Omega)$ such that for $0 < \rho \leq R \leq R_0$, we have

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \leq C[(\frac{\rho}{R})^d + \epsilon]\|\nabla u\|^2_{2,B_R(x_0)} + CR^\mu(\|F\|^2_{L^{2,\mu}(\Omega)} + \|g\|^2_{L^{2,\mu}(\Omega)}).$$

Thus applying the Lemma 0.3 with $\epsilon < \epsilon_0$ for $0 < \rho \leq R \leq R_0$ we have

$$\|\nabla u\|^2_{2,B_\rho(x_0)} \leq C(R^{-\mu}\|\nabla u\|^2_{2,B_R(x_0)} + \|F\|^2_{L^{2,\mu}(\Omega)} + \|g\|^2_{L^{2,\mu}(\Omega)})\rho^\mu, \tag{1.80}$$

$\nabla u \in L^{2,\mu}_{loc}(\Omega)^{d^2}$ and for all $\tilde{\Omega} \subset\subset \Omega$ it holds

$$\|\nabla u\|_{L^{2,\mu}(\tilde{\Omega})^{d^2}} \leq C\,(\|\nabla u\|_2 + \|F\|_{L^{2,\mu}(\Omega)} + \|g\|_{L^{2,\mu}(\Omega)})$$

with a constant $C = C(\lambda, \mu, d, \|A\|_{C(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \text{dist }(\tilde{\Omega}, \Omega), \text{diam }\Omega) > 0$. Thanks to the inequalities (1.68), (1.80), it follows that $p \in L^{2,\mu}_{loc}(\Omega)$ and we also obtain the inequality (1.75). The proof is complete. □

Theorem 1.6. *Suppose that the assumptions* (1.64), (1.65) *are satisfied,* $A \in C^{0,\delta}(\Omega)^{d^4}$, $F \in C^{0,\delta}(\Omega)^{d^2}$, $g \in C^{0,\delta}(\Omega)$ *with* $0 < \delta < 1$ *and* $(u, p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ *be a weak solution of system* (1.63). *Then* $\nabla u \in C^{0,\delta}_{loc}(\Omega)^{d^2}$, $p \in C^{0,\delta}_{loc}(\Omega)$, *and for all* $\tilde{\Omega} \subset\subset \Omega$ *we have the estimates*

$$\|\nabla u\|_{C^{0,\delta}(\tilde{\Omega})^{d^2}} + \|p\|_{C^{0,\delta}(\tilde{\Omega})} \leq C[\|\nabla u\|_{2,\Omega'} + \|F\|_{C^{0,\delta}(\overline{\Omega'})} + \|g\|_{C^{0,\delta}(\overline{\Omega'})}] \quad (1.81)$$

where the constant

$$C = C(\lambda, \delta, d, \|A\|_{C^{0,\delta}(\overline{\Omega'})}\|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \|B^{-1}\|_{1,\infty}, \text{dist }(\tilde{\Omega}, \Omega), \text{diam }\Omega) > 0.$$

Here $\Omega' = \{x \in \Omega; \text{dist}(x, \partial\Omega) > \frac{1}{2}\text{ dist}(\tilde{\Omega}, \partial\Omega)\}$.

Proof. Fix $x_0 \in \Omega, R > 0$ sufficiently small. Let v, q be the weak solution to

$$\begin{aligned} -\text{div }(\bar{A}(x_0)\nabla v) + \nabla q = 0 \ \text{ in } B_R(x_0), \\ \text{div } v = (g)_{x_0,R},\ u - v \in W^{1,2}_0(B_R(x_0)). \end{aligned}$$

The existence of such a solution is guaranteed by Lax-Milgram Theorem. Clearly, as v solves a system with constant coefficients and zero right hand side, div $v = (g)_{x_0,R}$, also ∇v is a solution of the same problem, div $\nabla v = 0$. Thus (1.74) is valid for ∇v i.e. if $0 < \rho \leq R < \frac{1}{2}\text{dist}(\tilde{\Omega}, \partial\Omega)$ then

$$\|\nabla v - (\nabla v)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \leq C(\frac{\rho}{R})^{d+2}\|\nabla v - (\nabla v)_{x_0,R}\|^2_{2,B_R(x_0)}.$$

Set $w = u - v$, then

$$\begin{aligned} \|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \leq \\ C[(\frac{\rho}{R})^{d+2}\|\nabla v - (\nabla v)_{x_0,R}\|^2_{2,B_R(x_0)} + \|\nabla w\|^2_{2,B_R(x_0)}]. \end{aligned} \quad (1.82)$$

As u also solves the system

$$\begin{aligned} \int_{B_R(x_0)} \bar{A}(x)\nabla u : \nabla\varphi\, dx \\ = -\int_{B_R(x_0)} [(H\nabla u).\varphi - (p - (p)_{x_0,R})\text{ div }\varphi + (F - (F)_{x_0,R})\ \nabla(B^{-1}\varphi)]\, dx, \end{aligned}$$

for all $\varphi \in W_0^{1,2}(B_R(x_0))^d$, then $w \in W_0^{1,2}(B_R(x_0))^d$ solves the system

$$\int_{B_R(x_0)} \bar{A}(x_0)\nabla w : \nabla\varphi \, dx$$

$$= \int_{B_R(x_0)} [(\bar{A}(x_0) - \bar{A}(x))\nabla u : \nabla\varphi - (H\nabla u).\varphi + (F - (F)_{x_0,R})\, \nabla(B^{-1}\varphi)]\, dx$$

$$+ \int_{B_R(x_0)} [(p - (p)_{x_0,R}) - (q - (q)_{x_0,R})]\text{div}\, \varphi \, dx, \quad \forall \varphi \in W_0^{1,2}(B_{R(x_0)})^d,$$

$$\text{div}\, w = g - (g)_{x_0,R}.$$

As in the proof of Theorem 1.5, we choose $\varphi = w$ and get

$$\|\nabla w\|^2_{2,B_\rho(x_0)} \le C[(\omega^2(R) + R^2 H_c^2)\|\nabla u\|^2_{2,B_R(x_0)}$$
$$+\|F - (F)_{x_0,R}\|^2_{2,B_R(x_0)} + \|g - (g)_{x_0,R}\|^2_{2,B_R(x_0)}].$$

Since $\bar{A} \in C^{0,\delta}(\overline{\Omega})^{d^4}$ we have $\omega(R) \le \|\bar{A}\|_{C^{0,\delta}(\overline{\Omega})} R^\delta$. Thanks to the assumptions on F, g it implies that

$$\begin{aligned}\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} &\le C[(\frac{\rho}{R})^{d+2}\|\nabla u - (\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)} \\ &+R^{2\delta}\|\nabla u\|^2_{2,B_R(x_0)} + ([F]^2_{2,d+2\delta;\Omega'} + [g]^2_{2,d+2\delta;\Omega'})R^{d+2\delta}].\end{aligned} \quad (1.83)$$

For any $\varepsilon > 0$, we have $F, g \in C^{0,\delta}(\Omega) \simeq \mathcal{L}^{2,d+2\delta}(\Omega) \subset \mathcal{L}^{2,d-\varepsilon}(\Omega) = L^{2,d-\varepsilon}(\Omega)$. According to Theorem 1.5, we have $\nabla u \in L^{2,d-\varepsilon}(\tilde{\Omega})^{d^2}$ and inequality

$$\|\nabla u\|^2_{2,B_R(x_0)} \le C[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')}]R^{d-\varepsilon}. \quad (1.84)$$

It implies

$$\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C[(\frac{\rho}{R})^{d+2}\|\nabla u - (\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)} + (\|\nabla u\|^2_{2,\Omega'}$$
$$+\|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')})R^{d+2\delta-\varepsilon} + ([F]^2_{2,d+2\delta;\Omega'} + [g]^2_{2,d+2\delta;\Omega'})R^{d+2\delta}].$$

Applying Lemma 0.3, we obtain

$$\|\nabla u - (\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{L^{2,d-\varepsilon}(\Omega')^{d^2}} + \|g\|^2_{L^{2,d-\varepsilon}(\Omega')}]\rho^{d+2\delta-\varepsilon}$$

which implies that $\nabla u \in C^{0,\delta-\varepsilon/2}_{loc}(\Omega)^{d^2}$ for $0 < \varepsilon < \delta$, and the inequality

$$\|\nabla u\|_{C^{0,\delta-\varepsilon/2}(\tilde{\Omega})^{d^2}} \le C[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{\mathcal{L}^{2,d+2\delta}(\Omega')^{d^2}} + \|g\|^2_{\mathcal{L}^{2,d+2\delta}(\Omega')}]$$

holds. In particular, ∇u is locally bounded and thus

$$\|\nabla u\|^2_{2,B_R(x_0)} \le C[\|\nabla u\|^2_{2,\Omega'} + \|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}} + \|g\|^2_{C^{0,\delta}(\overline{\Omega'})}]R^d. \quad (1.85)$$

Substituting this inequality into (1.83), we get

$$\|\nabla u-(\nabla u)_{x_0,\rho}\|^2_{2,B_\rho(x_0)} \le C(\frac{\rho}{R})^{d+2}\|\nabla u-(\nabla u)_{x_0,R}\|_{2,B_R(x_0)} \quad (1.86)$$
$$+C[\|\nabla u\|^2_{2,\Omega'}+\|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}}+\|g\|^2_{C^{0,\delta}(\overline{\Omega'})}]R^{d+2\delta}.$$

Applying again the Lemma 0.2, we conclude that $\nabla u\in C^{0,\delta}_{loc}(\Omega)^{d^2}$, and it holds

$$\|\nabla u\|_{C^{0,\delta}(\tilde{\Omega})^{d^2}} \le C[\|\nabla u\|^2_{2,\Omega'}+\|F\|^2_{C^{0,\delta}(\overline{\Omega'})^{d^2}}+\|g\|^2_{C^{0,\delta}(\overline{\Omega'})}]$$

with a constant $C=C(\lambda,\delta,d,\|A\|_{C^{0,\delta}(\overline{\Omega})},\|B^{-1}\|_{C(\Omega)},\text{dist }(\tilde{\Omega},\Omega),\text{diam }\Omega)>0$. We see easily that $\nabla u-(\nabla u)_{x_0,R}$ solves the system (1.66) with the right hand side $F-F_{x_0,R}$ and we obtain an inequality

$$\|(p-p_{x_0,R})\|^2_{2,B_R(x_0)} \le C[\|\nabla u-(\nabla u)_{x_0,R}\|^2_{2,B_R(x_0)}+R^2\|\nabla u\|^2_{2,B_R(x_0)}$$
$$+\|F-F_{x_0,R}\|^2_{2,B_R(x_0)}]. \quad (1.87)$$

Therefore, we conclude that $p\in C^{0,\delta}_{loc}(\Omega)^d$ and it satisfies the required estimate. □

3.2. Schauder regularity up to the boundary

Suppose that $\Omega=B_1^+(0):=B_1(0)\cap\{x\in\mathbb{R}^d;x_d>0\}$. We continue to study system (1.63) in $\Omega=B_1^+(0)$, namely we have system

$$\begin{aligned} -\text{div }(A\nabla u)+B\nabla p &= \text{div } F \ \text{ in } B_1^+(0),\\ \text{div } u &= g \ \text{ in } B_1^+(0),\\ u &= 0 \ \text{ on } \Gamma. \end{aligned} \quad (1.88)$$

As A,B satisfy the condition (1.64), we can denote by

$$\Lambda=\max(\|\bar{A}\|_{L^\infty(B_1^+(0))},\|H\|_{L^\infty(B_1^+(0))},\|B^{-1}\|_{W^{1,\infty}(B_1^+(0))}).$$

We start with an energy estimate

Theorem 1.7. *(Caccioppoli's inequality). Suppose that* (1.64), (1.8) *are satisfied in* $B_1^+(0)$ *and* $F\in L^2(B_1^+(0))^{d^2}$, $g\in L^2(B_1^+(0))$. *Then there is a positive constant* C^* *depending only on* λ,Λ *such that for any* (u,p) *which is a weak solution to system* (1.88), *for all* $z_0\in\Gamma$ *and all* ρ,R *with* $0<\rho<R<\text{dist}(z_0,\partial B_1^+(0)\backslash\Gamma)$, *we have*

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \le C^*[\frac{1}{(R-\rho)^2}\|u\|^2_{2,B_R^+(z_0)}+\|F\|^2_{2,B_R^+(z_0)}+\|g\|^2_{2,B_R^+(z_0)}], \quad (1.89)$$

$$\|p-p_{B_\rho^+(z_0)}\|^2_{2,B_\rho^+(z_0)} \le C^*[\frac{1}{(R-\rho)^2}\|u\|^2_{2,B_R^+(z_0)}+\|F\|^2_{2,B_R^+(z_0)}+\|g\|^2_{2,B_R^+(z_0)}]. \quad (1.90)$$

Proof. Let u, p be a weak solution of system (1.88). Choose a test function $\varphi = \eta^2 u$, where $\eta \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function such that

$$\text{supp}\,\eta \subset B_R(z_0), 0 \le \eta \le 1 \text{ on } \mathrm{B}_\mathrm{R}^+(\mathrm{z}_0),\ \eta \equiv 1 \text{ on } \mathrm{B}_\rho^+(\mathrm{z}_0);\ |\nabla \eta| \le \frac{\mathrm{C}}{\mathrm{R} - \rho}.$$

Then (1.66) implies

$$\int_{B_R^+(z_0)} \bar{A}\nabla(\eta u) : \nabla(\eta u)\, dx = \int_{B_R^+(z_0)} \bar{A}(\nabla\eta \otimes u) : (\nabla \eta \otimes u + \nabla(\eta u))\, dx$$
$$- \int_{B_R^+(z_0)} \bar{A}\nabla(\eta u) : (\nabla\eta \otimes u)\, dx$$
$$- \int_{B_R^+(z_0)} \Big((p - (p)_{B_R^+(z_0)})(\eta^2 g + 2\eta\nabla\eta u) + F\nabla(\eta^2 B^{-1} u) + \eta^2(H\nabla u)u\Big)\, dx.$$

As we assumed that $\bar{A}$ generates an elliptic bilinear form on $W_0^{1,2}(B_1^+(0))^{d^2}$, we have for $v = \eta u$

$$\lambda\|\nabla(\eta u)\|^2_{2,B_R^+(z_0)} \le \|\bar{A}\|_{\infty,B_R^+(z_0)}[2\|\nabla(\eta u)\|_{2,B_R^+(z_0)} + \|u \otimes \nabla\eta\|_{2,B_R^+(z_0)}]$$
$$\|u\otimes\nabla\eta\|_{2,B_R^+(z_0)} + \|\eta(p-(p)_{B_R^+(z_0)})\|_{2,B_R^+(z_0)}[2\|u\nabla\eta\|_{2,B_R^+(z_0)} + \|g\|_{2,B_R^+(z_0)}]$$
$$+ [\|\nabla(B^{-1})\|_{\infty,B_R^+(z_0)}\|\eta u\|^2_{2,B_R^+(z_0)} + \|B^{-1}\|_{\infty,B_R^+(z_0)}(\|\eta\nabla u\|_{2,B_R^+(z_0)}$$
$$+ 2\|u\nabla\eta\|_{2,B_R^+(z_0)}]\|F\|_{2,B_R^+(z_0)} + \|H\|_{\infty,B_R^+(z_0)}\|\eta\nabla u\|_{2,B_R^+(z_0)}\|\eta u\|_{2,B_R^+(z_0)}.$$

By Young's inequality we have estimate

$$\|\eta\nabla u\|^2_{2,B_R^+(z_0)} \le \varepsilon\|\eta(p-(p)_{B_R^+(z_0)})\|^2_{2,B_R^+(z_0)} + C(\varepsilon)[\frac{1}{(R-\rho)^2}\|u\|^2_{2,B_R^+(z_0)}$$
$$+\|F\|^2_{2,B_R^+(z_0)} + \|g\|^2_{2,B_R^+(z_0)}] \quad (1.91)$$

where $C(\varepsilon)$ depends on norms of λ and Λ. Properties of the cut off function η imply

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \le \varepsilon\|p-(p)_{B_R^+(z_0)})\|^2_{2,B_R^+(z_0)} + C(\varepsilon)[\frac{1}{(R-\rho)^2}\|u\|^2_{2,B_R^+(z_0)}$$
$$+\|F\|^2_{2,B_R^+(z_0)} + \|g\|^2_{2,B_R^+(z_0)}]. \quad (1.92)$$

For the estimate of pressure term we use Lemma 0.1 for $\Omega = \Omega_0 = B_R^+(z_0)$ and

$$[f,\varphi] = \int_{B_R^+(z_0)} \big(\bar{A}\nabla u : \nabla\varphi + H\nabla u\varphi + F\nabla(B^{-1}\varphi)\big)\, dx.$$

As u and $q = p-(p)_{B_R^+(z_0)}$ are weak solutions to system (2.44) we have $[f,\varphi] = 0$ for any $\varphi \in W_0^{1,2}(B_R^+(z_0))^d$ with div $\varphi = 0$. Moreover, $\int_{B_R^+(z_0)} q(x)dx = 0$.

Further

$$\|f\|_{W^{-1,2}(B_R^+(z_0))} \le C\Big(\|\nabla u\|_{2,B_R^+(z_0)}(1+R\|H\|_{\infty,B_R^+(z_0)})$$
$$+\|F\|_{2,B_R^+(z_0)}(\|\nabla B^{-1}\|_{\infty,B_R^+(z_0)}R+\|B^{-1}\|_{\infty,B_R^+(z_0)})\Big). \qquad (1.93)$$

Thus also $f=\nabla q=\nabla p$ and

$$\|p-(p)_{B_R^+(z_0)}\|_{2,B_\rho^+(z_0)} \le C(\|\nabla u\|_{2,B_R^+(z_0)}(1+R^2\|H\|_{\infty,B_R^+(z_0)}$$
$$+\|F\|_{2,B_R^+(z_0)}\|B\|_{1,\infty,B_R^+(z_0)}). \qquad (1.94)$$

From formula (1.92) (1.94) and by application of Lemma 0.2, we conclude the proof.

□

Now, we consider systems with constant coefficients

$$\begin{aligned} -\text{div}\,(A_0\nabla u)+\nabla p=0 &\quad \text{in } B_1^+(0),\\ \text{div}\, u=0 &\quad \text{in } B_1^+(0),\\ u=0 &\quad \text{on } \Gamma \end{aligned} \qquad (1.95)$$

where A_0 is a $d^2\times d^2$ constant matrix.
Assume A_0 is a constant matrix satisfying strong Legendre-Hadamard condition (1.65).

Let (u,p) be weak solutions to system (1.95), then in an analogous way to the proof of Lemma 1.2 we prove that u,p are the smooth functions near Γ. Therefore, using the standard proof of Campanato's inequality we can get first estimates for the Hölder continuity by a following proposition.

Proposition 1.4. *Suppose that A_0 is a constant matrix satisfying strong Legendre-Hadamard condition (see (1.65)). Then there is a positive constant C such that for any weak solution (u,p) of system (1.96), for all $z_0\in\Gamma$ and all ρ,R with $0<\rho<R<\text{dist}(z_0,\partial B_1^+(0)\backslash\Gamma)$, the following two estimates are valid:*

$$\|u\|^2_{2,B_\rho^+(z_0)} \le C(\frac{\rho}{R})^d\|u\|^2_{2,B_R^+(z_0)} \qquad (1.96)$$

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \le C(\frac{\rho}{R})^d\|\nabla u\|^2_{2,B_R^+(z_0)} \qquad (1.97)$$

with a constant C depending on $\frac{|A_0|}{\lambda}$, d and dist $(z_0,\partial B_1^+(0)\backslash\Gamma)$.

Proof. Proposition 1.4 can be proof as in Proposition 1.9, Part I in [13]. □

Next, we state the first regularity result for solutions to system (1.88) up to the boundary in Morrey spaces.

Theorem 1.8. *Suppose that A, B satisfy the conditions (1.64), (1.65) in $B_1^+(0)$, $A \in C(\overline{B_1^+(0)})^{d^4}$ and $F \in L^{2,\mu}(B_1^+(0))^{d^2}, g \in L^{2,\mu}(B_1^+(0))$ with $0 < \mu < d$. If $(u,p) \in W_0^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ is a weak solution of system (1.88) then for all $\bar{x} \in \Gamma$, there exists $\sigma > 0$ such that $\nabla u \in L^{2,\mu}(B_\sigma^+(\bar{x}))^{d^2}$, $p \in L^{2,\mu}(B_\sigma^+(\bar{x}))$, and we have the estimates*

$$\begin{aligned}\|\nabla u\|_{L^{2,\mu}(B_\sigma^+(\bar{x}))} + \|p\|_{L^{2,\mu}(B_\sigma^+(\bar{x}))} \leq C(&\|\nabla u\|_{2,B_1^+(0)} \\ &+\|F\|_{L^{2,\mu}(B_1^+(0))} + \|g\|_{L^{2,\mu}(B_1^+(0))})\end{aligned} \tag{1.98}$$

with a constant $C = C(\lambda, \Lambda, \mu, d, \|A\|_{C(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \text{dist } (\bar{x}, \partial B_1^+(0)\backslash\Gamma)) > 0$.

Proof. Fix $\bar{x} \in \Gamma$ and R with $0 < R < R_0$, where R_0 sufficiently small will be defined later. Set $\sigma < \frac{R}{4}$. We shall prove that there is a constant $C_H > 0$ such that for all $z_0 \in \overline{B}_\sigma^+(\bar{x})$, $\rho < \frac{R-\sigma}{2}$ an inequality

$$\int_{B_\rho^+(z_0)} |\nabla u(z)|^2 + |p - (p)_{B_\rho^+(z_0)}|^2 \, dz \leq C_H \rho^\mu \tag{1.99}$$

holds.

Case 1 : $z_0 \in \Gamma$.
Let v, q be the weak solution to system

$$\begin{aligned}-\text{div } (\bar{A}(z_0)\nabla v) + \nabla q = 0 \quad &\text{in } B_R^+(z_0), \\ \text{div } v = 0, \ u - v \in W_0^{1,2}&(B_R^+(z_0))^d.\end{aligned} \tag{1.100}$$

The solvability of this system is guaranteed by Lax-Milgram theorem. By Proposition 1.4 we get that if $0 < \rho < R$

$$\|\nabla v\|^2_{2,B_\rho^+(z_0)} \leq C(\frac{\rho}{R})^d \|\nabla v\|^2_{2,B_R^+(z_0)}.$$

Set $w = u - v$, then

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \leq C\left[(\frac{\rho}{R})^d \|\nabla u\|^2_{2,B_R^+(z_0)} + \|\nabla w\|^2_{2,B_R^+(z_0)}\right]. \tag{1.101}$$

It is easily seen that $w \in W_0^{1,2}(B_R^+(z_0))^d$ satisfies

$$\begin{aligned}\int_{B_R^+(z_0)} \bar{A}(z_0)\nabla w : \nabla\varphi \, dx &= \int_{B_R^+(z_0)} \{[\bar{A}(z_0) - \bar{A}(x)]\nabla u : \nabla\varphi - (H\nabla u).\varphi \\ &+F \, \nabla(B^{-1}\varphi)\} \, dx + \int_{B_R^+(z_0)} [(p - p_{B_R^+(z_0)}) - (q - q_{B_R^+(z_0)})]\text{div } \varphi \, dx, \\ &\text{for all } \varphi \in W_0^{1,2}(B_R^+(z_0))^d; \quad \text{div } w = g.\end{aligned} \tag{1.102}$$

By choosing $\varphi = w$, using the assumptions of Theorem 1.8 and Poincaré's inequality, we get

$$\|\nabla w\|^2_{2,B_R^+(z_0)} \leq C[(\omega^2(R) + R^2H_c^2)\|\nabla u\|^2_{2,B_R^+(z_0)} + \|F\|^2_{2,B_R^+(z_0)}$$
$$+ |\int_{B_R^+(z_0)} [(p - p_{B_R^+(z_0)}) - (q - q_{B_R^+(z_0)})]g\, dx|] \tag{1.103}$$

where

$$\omega(R) = \max_{i,j,\alpha,\beta}[\sup_{B_R^+(z_0)} |\bar{A}_{ij}^{\alpha\beta}(x) - \bar{A}_{ij}^{\alpha\beta}(z_0)|],\ H_c = \|H\|_\infty.$$

On the other hand, from system (1.102) applying Lemma 0.1 we have

$$\|(p-q) - (p-q)_{B_R^+(z_0)}\|_{2,B_R^+(z_0)}$$
$$\leq C[\|\nabla w\|_{2,B_R^+(z_0)} + (\omega(R) + RH_c)\|\nabla u\|_{2,B_R^+(z_0)} + \|F\|_{2,B_R^+(z_0)}].$$

Therefore

$$|\int_{B_R^+(z_0)} [(p - p_{B_R^+(z_0)}) - (q - q_{B_R^+(z_0)})]g\, dx|$$
$$\leq \|(p-q) - (p-q)_{B_R^+(z_0)}\|_{2,B_R^+(z_0)}\|g\|_{2,B_R^+(z_0)}$$
$$\leq C[\|\nabla w\|_{2,B_R^+(z_0)} + (\omega(R) + RH_c)\|\nabla u\|_{2,B_R^+(z_0)} + \|F\|_{2,B_R^+(z_0)}]\|g\|_{2,B_R^+(z_0)}$$
$$\leq \varepsilon\|\nabla w\|^2_{2,B_R^+(z_0)} + C(\varepsilon)[(\omega^2(R) + R^2H_c^2)\|\nabla u\|^2_{2,B_R^+(z_0)} + \|F\|^2_{2,B_R^+(z_0)}$$
$$+ \|g\|^2_{2,B_R^+(z_0)}] \tag{1.104}$$

with $R \leq R_1$ small enough and not dependent on z_0.
From the inequalities (1.93), (1.101), (1.103), (1.104), we obtain a following estimate

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \leq C[((\frac{\rho}{R})^d + \omega^2(R) + R^2)\|\nabla u\|^2_{2,B_R^+(z_0)} + \|F\|^2_{2,B_R^+(z_0)} + \|g\|^2_{2,B_R^+(z_0)}], \tag{1.105}$$

Thanks to the assumptions $F \in L^{2,\mu}(B_1^+(0))^{d^2}$, $g \in L^{2,\mu}(B_1^+(0))$, it follows

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \leq C[(\frac{\rho}{R})^d + (\omega^2(R) + R^2)\|\nabla u\|^2_{2,B_R^+(z_0)} + MR^\mu],$$

where $M = [\|F\|^2_{L^{2,\mu}(B_1^+(0))} + \|g\|^2_{L^{2,\mu}(B_1^+(0))}]$.
Thus, for all $\varepsilon > 0$ there exists $R_2 > 0$ such that if $R < R_2$ then $\omega^2(R) + R^2 < \varepsilon$.
Setting

$$\mathrm{R_0 = \min\ (dist\ (z_0, \partial B_1^+(0)\backslash\Gamma), R_1, R_2)}$$

and applying Lemma 0.3 for $\rho < R < R_0$ we have

$$\|\nabla u\|^2_{2,B_\rho^+(z_0)} \leq C[\frac{1}{R^d}\|\nabla u\|^2_{2,B_R^+(z_0)} + M]\rho^\mu. \tag{1.106}$$

Thanks to inequality (1.90), hence we have

$$\|p-(p)_{B_\rho^+(z_0)}\|^2_{2,B_\rho^+(z_0)} \le C[\frac{1}{R^d}\|\nabla u\|^2_{2,B_R^+(z_0)} + M]\rho^\mu. \tag{1.107}$$

Case 2 : $z_0 \in B_\sigma^+(\bar{x})$.
We denote x_0 the projection of z_0 into Γ, $d_{z_0} = \text{dist}\,(z_0, \Gamma)$.
(i) If $d_{z_0} \le \rho < \frac{R-\sigma}{2}$, then $B_\rho^+(z_0) := B_\rho(z_0) \cap B_1^+(0) \subset B_{2\rho}^+(x_0)$. Thus

$$\begin{aligned}\rho^{-\mu}\int_{B_\rho^+(z_0)} &\left(|\nabla u|^2 + |p-(p)_{B_\rho^+(z_0)}|^2\right) dx\\ &\le 2^\mu(2\rho)^{-\mu}\int_{B_{2\rho}^+(x_0)}\left(|\nabla u|^2 + |p-(p)_{B_{2\rho}^+(x_0)}|^2\right) dx. \end{aligned}\tag{1.108}$$

Employing Case 1 for $x_0 \in \Gamma$, $2\rho < R-\sigma < R < R_2$ we have

$$(2\rho)^{-\mu}\int_{B_{2\rho}^+(x_0)}\left(|\nabla u|^2 + |p-(p)_{B_{2\rho}^+(x_0)}|^2\right) dx \;\le\; C[\frac{1}{R^d}\|\nabla u\|^2_{2,B_R^+(x_0)} + M]. \tag{1.109}$$

Thus

$$\rho^{-\mu}\int_{B_\rho^+(z_0)}\left(|\nabla u|^2 + |p-(p)_{B_\rho^+(z_0)}|^2\right) dx \;\le\; 2^\mu C[\frac{1}{R^d}\|\nabla u\|^2_{2,B_R^+(x_0)} + M]. \tag{1.110}$$

(ii) If $\rho < d_{z_0}$, we apply inequality (1.75) to obtain

$$\begin{aligned}\int_{B_\rho(z_0)}\left(|\nabla u|^2 + |p-(p)_{B_\rho^+(z_0)}|^2\right) dx \le C(d_{z_0}^{-\mu}\|\nabla u\|^2_{2,B_{d_{z_0}}(z_0)} + \|F\|^2_{L^{2,\mu}(B_1^+(0))}\\ + \|g\|^2_{L^{2,\mu}(B_1^+(0))})\rho^\mu.\end{aligned} \tag{1.111}$$

Employing now inequality (1.106), we get

$$\|\nabla u\|^2_{2,B_{d_{z_0}}(z_0)} \le \|\nabla u\|^2_{2,B^+_{2d_{z_0}}(x_0)} \le C[\frac{1}{R^d}\|\nabla u\|^2_{2,B_R^+(z_0)} + M^2](2d_{z_0})^\mu. \tag{1.112}$$

Therefore,

$$\int_{B_\rho^+(z_0)}\left(|\nabla u|^2 + |p-(p)_{B_\rho^+(z_0)}|^2\right) dx \le C[\frac{2^\mu}{R^d}\|\nabla u\|^2_{2,B_R^+(z_0)} + M]\rho^\mu. \tag{1.113}$$

Finally, if $0 < R < R_0$, by the above way we show that there exists a constant $C_H > 0$ such that (1.99) holds for all $z_0 \in B_\delta^+(\bar{x})$. We have proved the statement of Theorem 1.8. □

From above results, by using a standard covering argument we have following results

Theorem 1.9. *Let the assumption* (1.64), (1.65) *be satisfied in* $B_1^+(0)$), $A \in C(\overline{B_1^+(0)})^{d^4}$, $F \in L^{2,\mu}(B_1^+(0))^{d^2}$, $g \in L^{2,\mu}(B_1^+(0))$ *with* $0 < \mu < d$ *and* $(u,p) \in W^{1,2}(B_1^+(0)))^d \times L^2(B_1^+(0)))$ *be a weak solution of system* (1.88) *such that* $\mathrm{supp}\,\mathrm{u}, \mathrm{supp}\,\mathrm{p} \subset \mathrm{B}_1^*(0)$. *Then* $\nabla u \in L^{2,\mu}(B_1^+(0))^{d^2}$, $p \in L^{2,\mu}(B_1^+(0))$, *and we have the estimates*

$$\|\nabla u\|_{L^{2,\mu}(B_1^+(0))} + \|p\|_{L^{2,\mu}(B_1^+(0))} \leq C[\|\nabla u\|_{2,B_1^+(0)} + \|F\|_{L^{2,\mu}(B_1^+(0))} + \|g\|_{L^{2,\mu}(B_1^+(0))}] \quad (1.114)$$

with a constant $C = C(\lambda, \Lambda, \mu, d, \|A\|_{C(\overline{\Omega})}, \|B^{-1}\|_{C^{0,1}(\overline{\Omega})}, \mathrm{dist}\,(\bar{x}, \partial B_1^+(0)\backslash\Gamma)) > 0$.

Chapter 2

Nonlinear problems

Let $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$, be a bounded domain with boundary $\partial\Omega$. We study a following problem : For given $f = (f_1, \cdots, f_d) : \Omega \longrightarrow \mathbb{R}^d$ and stress tensor $T(Du, p) : \mathbb{R}^{d\times d} \times \mathbb{R} \longrightarrow \mathbb{R}^{d\times d}$ we look for $v = (v_1, \cdots, v_d) : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ solving

$$\begin{aligned} \sum_{k=1}^{d} v_k \frac{\partial v}{\partial x_k} - \text{div } T(p, Dv) + \nabla p = f \ \text{ in } \Omega, \\ \text{div } v = 0 \ \text{ in } \Omega, \\ v = 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Dv denotes the symmetric part of the velocity gradient ∇v, i.e.

$$Dv = \frac{1}{2}(\nabla v + \nabla^T v) \quad \text{with} \quad D_{ij}v = \frac{1}{2}(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i}).$$

We assume throughout this section that

$$T(p, Dv) = \nu(p, |D|^2)\, Dv,$$

where a generalized viscosity ν is supposed to be continuously differentiable function of both variables. Moreover, there exist positive constants λ_0, λ_1 and ν_0 such that for arbitrary symmetric $d \times d$- matrices ξ, D and any $p \in \mathbb{R}$ the following estimates hold

$$\begin{aligned} \lambda_0|\xi|^2 \le \frac{\partial T}{\partial D}(p, D)\xi : \xi \le \lambda_1|\xi|^2, \\ \left|\frac{\partial \nu}{\partial p}(p, |D|)D\right| \le \nu_0. \end{aligned} \tag{2.2}$$

The existence of solutions to the system (2.1) is proved in [8] under assumptions on growth conditions of T. In [8], authors prove that there exists a solution to system

$$
\begin{aligned}
(\text{div } v^\varepsilon)\frac{v^\varepsilon}{2} + \sum_{k=1}^{d} v_k \frac{\partial v^\varepsilon}{\partial x_k} - \text{div } T(p^\varepsilon, Dv^\varepsilon) + \nabla p^\varepsilon = f \ \text{ in } \Omega, \\
-\varepsilon \triangle p^\varepsilon + \varepsilon(p^\varepsilon - p_0)\text{div } v^\varepsilon = 0 \ \text{ in } \Omega, \qquad (2.3) \\
\frac{\partial p^\varepsilon}{\partial n} = 0 \ \text{ and } \ v^\varepsilon = 0 \ \text{ on } \partial\Omega,
\end{aligned}
$$

with $p_0 := (p^\varepsilon)_\Omega$.
By allowing $\epsilon \to 0$, authors obtain solution to (2.1) from solutions to (2.3).
In a completely similar way, we get the following existence results to (2.1)under condition (2.2) and ν_0 small with respect to $\lambda_1, \ \lambda_2$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, be a bounded domain with Lipschitz boundary $\partial\Omega$. If the assumption* (2.2) *is satisfied, $\nu_0 < \frac{\lambda_0}{(\lambda_0+\lambda_1)C_{\text{div},\Omega}}$, $f \in W_0^{-1,2}(\Omega)$ then there exists a solution $v \in W_0^{1,2}(\Omega), p \in L^2(\Omega)$ to the system* (2.1).

In Theorem 2.1, $C_{\text{div},\Omega}$ is the norm of solution operator

$$
L \ : \ \{f \in L^2(\Omega); \int_\Omega f \, dx = 0\} \longrightarrow W_0^{1,2}(\Omega)
$$

defined by $Lf = u$ where u is a solution of the problem $\text{div u} = \text{f} \ \text{ in } \Omega, \ \text{u} = 0 \ \text{ on } \partial\Omega$.
The main aim of this chapter is to present the partial regularity of solutions to system (2.1) up to the boundary. It splits into two sections. In section 1, we study higher differentiability of solutions v, p in $W^{2,2}(\Omega)^d \times W^{1,2}(\Omega)$. In section 2, we state the partial regularity of solutions in the interior of domain Ω and near the boundary of Ω.

1. Higher differentiability of solutions

The higher differentiability of solutions to the system (2.1) in the interior is studied in [19] for T having subquadratic growth m. In [19] under the suitable conditions the authors show that $v \in W^{2,2}_{\text{loc}}(\Omega)^d, p \in W^{1,2}_{\text{loc}}(\Omega)$. By a similar way we get the following result

Theorem 2.2. *(Interior estimates) Let the assumption* (2.2) *be satisfied, $\nu_0 < \frac{\lambda_0}{(\lambda_0+C_{\text{div},\Omega}\lambda_1)C_{\text{div},\Omega}}$ and $f \in L^2(\Omega)$. Let $(v, p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ is a weak* *solution to the system* (2.1). *Then* $v \in W^{2,2}_{\mathrm{loc}}(\Omega), p \in W^{1,2}_{\mathrm{loc}}(\Omega)$. *Moreover, for any subdomain* $\Omega' \subset\subset \Omega$ *the inequality*

$$\|\nabla^2 v\|_{2;\Omega'} + \|\nabla p\|_{2;\Omega'} \leq C \tag{2.4}$$

holds with a positive constant C *which depends on* $\|v\|_2, \|p\|_2, \|f\|_2$ *and* dist (Ω', Ω).

In this section, we prove higher differentiability of solution up to the boundary. Suppose that $\Omega = B_1^+(0)$. Then we have the following theorem

Theorem 2.3. *Let the assumption* (2.2) *be satisfied,* $\nu_0 < \min\left(\frac{\lambda_0}{(\lambda_0 + C_{\mathrm{div}}\lambda_1)C_{\mathrm{div}}}, \frac{\lambda_0}{\lambda_0 + 3(d-1)(\lambda_1 - \lambda_0)}\right)$ *with* (C_{div} *defined in Remark after Lemma* 0.1*),* $f \in L^2(B_1^+(0))$. *Let* $(v, p) \in W_0^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ *be a weak solution to the system* (2.1) *and* supp v, supp $p \subset B_1^*(0)$. *Then* $v \in W^{2,2}(B_1^+(0)), p \in W^{1,2}(B_1^+(0))$. *Moreover, we have an estimate*

$$\|\nabla^2 v\|_{2;B_1^+(0)} + \|\nabla p\|_{2;B_1^+(0)} \leq C \tag{2.5}$$

with a positive constant C *which depends on* $\|v\|_2, \|p\|_2,$ *and* $\|f\|_2$.

Proof. By the same procedure as in the proof of higher differentiability of solution to (2.1) in the interior of domain Ω (see Theorem 5.1 in [8]) we get

$$D\frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2}, \ \frac{\partial p}{\partial x_s} \in L^2(B_1^+(0)) \text{ for all } s = 1, \cdots, d-1,$$

and we have estimate

$$\|D\nabla' v\|_2 + \|\nabla' p\|_2 \leq C \tag{2.6}$$

with a constant $C = C(\|v\|_2, \|p\|_2, \|f\|_2) > 0$.
From the assumption supp $v \subset B_1^*(0)$, $v = 0$ on Γ, it follows that $\frac{\partial v}{\partial x_s} \in W_0^{1,2}(B_1^+(0))^d$ for all $s = 1, \cdots, d-1$. We have from Korn's inequality that $\nabla \frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2}$ for all $s = 1, \cdots, d-1$.
Since div $v = 0$, $\nabla \frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2}$ for all $s = 1, \cdots, d-1$, we get

$$\frac{\partial^2 v_d}{\partial^2 x_d} \in L^2(B_1^+(0)). \tag{2.7}$$

Therefore, it is sufficient to prove that $\frac{\partial D_{id} v}{\partial x_d} \in L^2(B_1^+(0))$, $i = 1, \cdots, d-1$ and $\frac{\partial p}{\partial x_d} \in L^2(B_1^+(0))$.
Next, Theorem 2.2 guarantees the existence of second derivatives of v and first derivatives of p which are locally square integrable on $B_1^+(0)$. Thus a.e. on $B_1^+(0)$ it holds

$$\sum_{k=1}^{d} v_k \frac{\partial v_i}{\partial x_k} - \sum_{j=1}^{d-1} \frac{\partial T_{ij}(p, Dv)}{\partial D} \frac{\partial Dv}{\partial x_j} - \sum_{j=1}^{d-1} \frac{\partial T_{ij}(p, Dv)}{\partial p} \frac{\partial p}{\partial x_j}$$
$$- \frac{\partial T_{id}(p, Dv)}{\partial x_d} + \nabla p = f \ , \ i = 1, \cdots, d \tag{2.8}$$

or

$$-\frac{\partial T_{id}(p,Dv)}{\partial D}\frac{\partial Dv}{\partial x_d}-\frac{\partial T_{id}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_d}+\sum_{k=1}^{d}v_k\frac{\partial v_i}{\partial x_k}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial D}\frac{\partial Dv}{\partial x_j}$$
$$-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_j}+\nabla p=f \;\;,\;\; i=1,\cdots,d. \qquad (2.9)$$

Hence

$$\sum_{k=1}^{d-1}2\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}\frac{\partial D_{kd}v}{\partial x_d}+(\frac{\partial T_{id}(p,Dv)}{\partial p}-\delta_{id})\frac{\partial p}{\partial x_d}$$
$$=\sum_{k=1}^{d}v_k\frac{\partial v_i}{\partial x_k}+F_i \;\;,\;\; i=1,\cdots,d \qquad (2.10)$$

where F_i are given by

$$F_i:=-\sum_{k,l=1}^{d-1}2\frac{\partial T_{id}(p,Dv)}{\partial D_{kl}}\frac{\partial D_{kl}v}{\partial x_d}-\frac{\partial T_{id}(p,Dv)}{\partial D_{dd}}\frac{\partial D_{dd}v}{\partial x_d}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial D}$$
$$\frac{\partial Dv}{\partial x_j}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_j}+(1-\delta_{id})\frac{\partial p}{\partial x_i}-f_i \;\;,\;\; i=1,\cdots,d. \qquad (2.11)$$

It is easy to see that (2.6) and (2.7) implies

$$F_i\in L^2(B_1^+(0));\; i=1,\cdots,d. \qquad (2.12)$$

Now, we consider the system (2.10) at first as a linear system in the unknowns $\frac{\partial D_{kd}v}{\partial x_d},\;\; k=1,\cdots,d-1,\; \frac{\partial p}{\partial x_d}$.
Denote $(R_{ik})_{i,k=1}^d$ the matrix of system (2.10) i.e.

$$R_{ik}=2\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}} \text{ if k} < \text{d, i} < \text{d; } \mathrm{R_{id}}=\frac{\partial T_{id}(p,Dv)}{\partial p} \text{ if k} = \text{d, i} < \text{d;}$$
$$R_{dd}=\frac{\partial T_{dd}(p,Dv)}{\partial p}-1.$$

Multiply d^{th} row of R by $(1-\frac{\partial T_{dd}(p,Dv)}{\partial p})^{-1}\frac{\partial T_{id}(p,Dv)}{\partial p}$ and then subtract it from i^{th} row $(i=1,\cdots,d-1)$. We obtain a new matrix whose determinant det $(R)=(\frac{\partial T_{dd}(p,Dv)}{\partial p}-1)2^{d-1}$det S where S is a $(d-1)\times(d-1)$ matrix given by

$$S_{ik}=\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}+(1-\frac{\partial T_{dd}(p,Dv)}{\partial p})^{-1}\frac{\partial T_{id}(p,Dv)}{\partial p}\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}}.$$

Moreover, we have

$$\sum_{k=1}^{d-1} S_{ik}\frac{\partial D_{kd}v}{\partial x_d} = \sum_{k=1}^{d} v_k\frac{\partial v_i}{\partial x_k} + F_i + [\sum_{k=1}^{d} v_k\frac{\partial v_d}{\partial x_k} + F_d]$$
$$[1 - \frac{\partial T_{dd}(p, Dv)}{\partial p}]^{-1}\frac{\partial T_{id}(p, Dv)}{\partial p} =: H_i, \quad i = 1, \cdots, d-1. \quad (2.13)$$

Next, we show that S is positive definite matrix. In fact, we have from (2.2)

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_i\zeta_k = \sum_{i,k=1}^{d-1} \frac{\partial T_{id}(p, Dv)}{\partial D_{kd}}\zeta_i\zeta_k + \sum_{i,k=1}^{d-1}[(1 - \frac{\partial T_{dd}(p, Dv)}{\partial p})^{-1}$$
$$\frac{\partial T_{id}(p, Dv)}{\partial p}\frac{\partial T_{dd}(p, Dv)}{\partial D_{kd}}]\zeta_i\zeta_k \geq \frac{\lambda_0}{4}|\zeta|^2 - \sum_{i,k=1}^{d-1}\frac{\nu_0}{1-\nu_0}|\frac{\partial T_{dd}(p, Dv)}{\partial D_{kd}}||\zeta_i||\zeta_k|$$

for all $\zeta \in \mathbb{R}^{d-1}$.
On the other hand, for every $k \in \{1, \cdots, d-1\}$ by choosing $\xi_{dd} = \xi_{kd} = \xi_{dk} = 1, \xi_{ij} = 0$ if $(i, j) \neq (d, d), (d, k)$ and (k, d) the inequality (2.2) shows that

$$3\lambda_1 \geq 4\frac{\partial T_{kd}(p, Dv)}{\partial D_{kd}} + 4\frac{\partial T_{dd}(p, Dv)}{\partial D_{kd}} + \frac{\partial T_{dd}(p, Dv)}{\partial D_{dd}} \geq 3\lambda_0.$$

We also have

$$\lambda_1 \geq \frac{\partial T_{dd}(p, Dv)}{\partial D_{dd}} \geq \lambda_0; \; 2\lambda_1 \geq 4\frac{\partial T_{kd}(p, Dv)}{\partial D_{kd}} \geq 2\lambda_0.$$

Hence

$$|\frac{\partial T_{dd}(p, Dv)}{\partial D_{kd}}| \leq \frac{3}{4}(\lambda_1 - \lambda_0).$$

Therefore, we obtain an inequality

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_i\zeta_k \geq \left[\frac{\lambda_0}{4} - \frac{3}{4}(d-1)\frac{\nu_0}{1-\nu_0}(\lambda_1 - \lambda_0)\right]|\zeta|^2 = \lambda|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^{d-1}. \quad (2.14)$$

Thanks to $\nu_0 < \frac{\lambda_0}{\lambda_0 + 3(d-1)(\lambda_1 - \lambda_0)}$ we get $\lambda > 0$. From this we conclude that matrix S is positive definite. Thus there exists a positive constant C such that det $(N) \geq C$ (thanks Lemma 0.8). Therefore

$$\det(R) \geq 2^{d-1}C \quad (2.15)$$

and system (2.10) can be solved for the unknowns $\frac{\partial D_{kd}v}{\partial x_d}$, $k = 1, \cdots, d-1$, $\frac{\partial p}{\partial x_d}$, for almost all $x \in B_1^+(0)$.

Sobolev embedding theorem shows that $L^6(B_1^+(0))^d \subset W_0^{1,2}(B_1^+(0))^d$. Using Young's inequality we get

$$\int_{B_1^+(0)} (v.\nabla v)^{\frac{3}{2}} \le \frac{1}{4}\int_{B_1^+(0)} |v|^6\, dx + \frac{3}{4}\int_{B_1^+(0)} |\nabla v|^2\, dx \le C.$$

It implies $v.\nabla v \in L^{\frac{3}{2}}(B_1^+(0))^d$. From this, (2.15) and the calculation of $\frac{\partial D_{kd}v}{\partial x_d}$, $k = 1,\cdots,d-1$, $\frac{\partial p}{\partial x_d}$ from (2.9) where $F_i \in L^2(B_1^+(0))$, $i = 1,\cdots,d$; $v.\nabla v \in L^{\frac{3}{2}}(B_1^+(0))$, we deduce that

$$\frac{\partial D_{kd}v}{\partial x_d} \in L^{\frac{3}{2}}(B_1^+(0)),\ \ k = 1,\cdots,d-1,\ \frac{\partial p}{\partial x_d} \in L^{\frac{3}{2}}(B_1^+(0)).$$

It implies $v \in W_0^{2,\frac{3}{2}}(B_1^+(0))^d$.
(i) $d = 2$ Sobolev embedding theorem implies that $\nabla^2 v \in L^{\frac{3}{2}}(B_1^+(0))$, then $\nabla v \in L^6(B_1^+(0))$. Hence $v \in L^\infty(B_1^+(0))$ and $v.\nabla v \in L^2(B_1^+(0))^d$.
(ii) $d = 3$ Sobolev embedding theorem implies that $\nabla v \in L^3(B_1^+(0))$, $v \in L^6(B_1^+(0))$. Using Young's inequality we get

$$\int_{B_1^+(0)} (v.\nabla v)^2 \le \frac{1}{3}\int_{B_1^+(0)} |v|^6\, dx + \frac{2}{3}\int_{B_1^+(0)} |\nabla v|^3\, dx \le C.$$

It implies $v.\nabla v \in L^2(B_1^+(0))^d$ and $H_i \in L^2(B_1^+(0))$ for all $i = 1,\cdots,d-1$.
By a similar way, we obtain

$$\frac{\partial D_{kd}v}{\partial x_d} \in L^2(B_1^+(0)),\ \ k = 1,\cdots,d-1\ ,\ \frac{\partial p}{\partial x_d} \in L^2(B_1^+(0)). \tag{2.13}$$

Next, we estimate for $\frac{\partial D_{kd}v}{\partial x_d}$, $k = 1,\cdots,d-1$, $\frac{\partial p}{\partial x_d}$.
By setting $\zeta_k = \frac{\partial D_{id}v}{\partial x_d}$, $k = 1,\cdots,d-1$; , we get from (2.13)

$$\sum_{k=1}^{d-1} S_{ik}\zeta_k = H_i,\ i = 1,\cdots,d-1. \tag{2.16}$$

Thus

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_k\zeta_i = \sum_{i=1}^{d-1} H_i\zeta_i. \tag{2.17}$$

Consequently, $\lambda|\zeta|^2 \le |H||\zeta|$ or

$$\lambda\sum_{k=1}^{d-1} |\frac{\partial D_{kd}v}{\partial x_d}| \le |H| \ \text{ a.e. in } B_1^+(0).$$

Hence

$$\sum_{k=1}^{d-1} \|\frac{\partial D_{kd}v}{\partial x_d}\|_{2,B_1^+(0)} \le C(\|v\|_2, \|p\|_2, \|f\|_2).$$

Now the d^{th} equation of system (2.10) implies

$$\|\frac{\partial p}{\partial x_d}\|_{2,B_1^+(0)} \leq C(\|v\|_2, \|p\|_2, \|f\|_2),$$

and we obtain the inequality (2.5). Theorem is proved. $\square$

2. Partial regularity of solutions

In [19] the interior partial regularity of solutions to (2.1) is studied for stress tensor with subquadratic growth $m \in (\frac{3d}{d+2}, 2)$. By the same way we get an analogous result for $m = 2$.

Theorem 2.4. *(Interior case) Let the assumptions* (2.2) *are satisfied,* $\nu_0 < \frac{\lambda_0}{(\lambda_0 + C_{\text{div}}\lambda_1)C_{\text{div}}}$ *and* $f \in \text{BMO}(\Omega)$. *Let* $(w, q) \in W^{1,2}(\Omega)^d \times L^2(\Omega)$ *be a weak solution to the system* (2.1). *Then there exists a closed subset* $\Sigma \subset \Omega$ *so that* $|\Sigma| = 0$ *and* $\nabla v, p$ *are locally Hölder continuous in* $\Omega \setminus \Sigma$.

The aim of this section is to study Hölder continuity of solutions to system (2.1) near the boundary. Suppose that $\Omega = B_1^+(0)$. We recall that $\Gamma := B_1(0) \cap \{x \in \mathbb{R}^d;\ x_d = 0\}$, $B_1^+(0) := B_1(0) \cap \{x \in \mathbb{R}^d;\ x_d > 0\}$ and $B_1^\star(0) := B_1(0) \cap \{x \in \mathbb{R}^d;\ x_d \geq 0\}$. We start with several auxiliary results about Caccioppoli's and Campanato's type inequalities for solutions to a class of generalized Stokes systems with constant coefficients.

We consider a generalized Stokes system

$$\begin{aligned} -\text{div}\,(ADw) + (E - B)\nabla q = 0 \ \text{ in } B_1^+(0), \\ \text{div}\, w = 0 \ \text{ in } B_1^+(0), \end{aligned} \tag{2.18}$$

where A is a $d^2 \times d^2$ matrix ; B is a $d \times d$ matrix ; A, B are a constant symmetric matrices; and E is the identity $d \times d$ matrix.

We assume throughout this part that there exist positive constants λ_0, λ_1, $\nu_0 < 1$ such that

$$\begin{aligned} &\lambda_0|\xi|^2 \leq A\xi : \xi \ \text{ for all symmetric matries } \xi \in \mathbb{R}^{d\times d} \text{and} \\ &|A_{ij}^{kl}| \leq \lambda_1,\ i, j, k, l = 1, \cdots, d;\ |B| \leq \nu_0. \end{aligned} \tag{2.19}$$

Moreover, suppose that

$$\nu_0 < \min\,(\frac{\lambda_0}{\lambda_0 + \sqrt{2}\lambda_1}, \frac{\lambda_0}{\lambda_0 + 3(\text{d} - 1)(\lambda_1 - \lambda_0)}). \tag{2.20}$$

We first establish Caccioppoli's inequality for solutions to the system (2.18):

Lemma 2.1. *(Caccioppoli's inequality) Let* $(w,q) \in W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ *be a weak solution of system (2.18) and* $w = 0$ *on* Γ. *Then for all* $\tau \in (0,1)$, $R \leq 1$, *we have* $(w,q) \in W^{2,2}(B_{\tau R}^+(0))^d \times W^{1,2}(B_{\tau R}^+(0))$ *and the inequalities*

$$\begin{aligned} \|\nabla^2 w\|_{2,B_{\tau R}^+(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}, \\ \|\nabla q\|_{2,B_{\tau R}^+(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)} \end{aligned} \tag{2.21}$$

hold with constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.

Proof. The assumption (2.20) implies that there exists $(E-B)^{-1}$. Therefore, we can denote

$$H := (E-B)^{-1} - E = \sum_{n=1}^{\infty} B^n.$$

It is obvious that

$$|(E-B)^{-1}| \leq \frac{1}{1-\nu_0}, \ |H| \leq \frac{\nu_0}{1-\nu_0}. \tag{2.22}$$

Then the system (2.18) can be written as

$$\text{div } w = 0, \quad -\text{div }[(E+H)ADw] + \nabla q = 0 \ \text{ in } B_1^+(0), \tag{2.23}$$

where $[(E+H)A]_{ij}^{kl} = \sum_{n=1}^{d}(E+H)_{in}A_{nj}^{kl}$.
Let $\eta \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function :

$$\text{supp } \eta \subset B_{\frac{R+\tau R}{2}}(0), \ 0 \leq \eta \leq 1 \text{ on } B^+_{\frac{R+\tau R}{2}}(0), \ \eta \equiv 1 \text{ on } B_{\tau R}^+(0);$$
$$|\nabla \eta| \leq \frac{C}{R(1-\tau)}, |\nabla^2 \eta| \leq \frac{C}{R^2(1-\tau)^2}.$$

Then we get

$$\text{div } (\eta w) = g, \quad -\text{div }[(E+H)AD(\eta w)] + \nabla[\eta(q-(q)_{B_R^+(0)})] = f \ \text{ in } B_1^+(0), \tag{2.24}$$

where $g(x) := \nabla\eta.w$ and $f(x) := (f_1(x), \cdots, f_d(x))$ is given by

$$\begin{aligned} f_i = \frac{1}{2}[(E+H)A]_{ij}^{kl}[&\frac{\partial \eta}{\partial x_l}\frac{\partial w_k}{\partial x_j} + \frac{\partial \eta}{\partial x_k}\frac{\partial w_l}{\partial x_j} + \frac{\partial^2 \eta}{\partial x_j \partial x_l}w_k + \frac{\partial^2 \eta}{\partial x_j \partial x_k}w_l + 2\frac{\partial \eta}{\partial x_j}D_{kl}w] \\ &+ \frac{\partial \eta}{\partial x_i}(q-(q)_{B_R^+(0)}), \ \ i = 1, \cdots, d. \end{aligned}$$

The properties of η show that

$$\begin{aligned}
\|\nabla g\|_{2,B_R^+(0)} &\le \|\nabla\eta\|_{2,B_R^+(0)}\|\nabla w\|_{2,B_R^+(0)} + \|\nabla^2\eta\|_{2,B_R^+(0)}\|w\|_{2,B_R^+(0)} \\
&\le \frac{C}{R(1-\tau)}\|\nabla w\|_{2,B_R^+(0)} + \frac{C}{R^2(1-\tau)^2}\|w\|_{2,B_R^+(0)} \\
\|f\|_{2,B_R^+(0)} &\le C[\|\nabla\eta\|_{2,B_R^+(0)}\|\nabla w\|_{2,B_R^+(0)} + \|\nabla^2\eta\|_{2,B_R^+(0)}\|w\|_{2,B_R^+(0)}] \\
&+ \|\nabla\eta\|_{2,B_R^+(0)}\|q-(q)_{B_R^+(0)}\|_{2,B_R^+(0)}.
\end{aligned}$$

Applying poincaré's inequality and Lemma 0.1 we get

$$\begin{aligned}
\|\nabla w\|_{2,B_R^+(0)} &\le CR\|\nabla w\|_{2,B_R^+(0)} \\
\|q-(q)_{B_R^+(0)}\|_{2,B_R^+(0)} &\le C\|\nabla w\|_{2,B_R^+(0)}.
\end{aligned}$$

It implies the inequalities

$$\begin{aligned}
\|\nabla g\|_{2,B_R^+(0)} &\le \frac{C}{R(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)} \\
\|f\|_{2,B_R^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}
\end{aligned} \tag{2.25}$$

hold with a constant $C = C(\lambda_1, \nu_0, d) > 0$.

We consider $s = 1, \cdots, d-1$. Set the difference quotient in the x_s direction $\triangle_{\delta,s}(\eta w) = \frac{1}{\delta}[\eta w(x+\delta e_s) - \eta w(x)]$ with $0 < |\delta| < \frac{R(1-\tau)}{2}$.
Since (w, q) solve the system (2.23) we have

$$\text{div } u_\delta = \triangle_{\delta,s} g, \qquad -\text{div } [(E+H)ADu_\delta] + \nabla p_\delta = \triangle_{\delta,s} f \ \text{ in } B_R^+(0), \tag{2.26}$$

where $u_\delta := \triangle_{\delta,s}(\eta w)$, $p_\delta := \triangle_{\delta,s}[\eta(q-(q)_{B_R^+(0)})]$.
Thanks to Lemma 0.1 the quotient $\triangle_{\delta,s} g(x)$ can be written in the form $G_\delta \in W_0^{1,2}(B_R^+(0))^d$ such that $\|\nabla G_\delta\|_{2,B_R^+(0)} \le C\|\triangle_{\delta,s} g\|_{2,B_R^+(0)}$ with C not depending on δ and s.
On the other hand, a calculation yields

$$\triangle_{\delta,s} f = \frac{\partial}{\partial x_s}[\frac{1}{\delta}\int_0^\delta f(x+te_s)dt]. \tag{2.27}$$

This shows that $\triangle_{\delta,s} f$ has the form div $F_{\delta,s}$ with some $F_{\delta,s} \in L^2(B_R^+(0))^{d^2}$ defined by (2.27). Using Hölder's inequalities we have

$$\|\nabla G_\delta\|_{2,B_R^+(0)} \le C\|\triangle_{\delta,s} g\|_{2,B_R^+(0)}$$

with C not depending on δ and s.
Using a weak formulation of the system and setting test a function $\varphi = (w_\delta -$

$G_\delta) \in W^{1,2}_{0,\mathrm{div}}(B^+_R(0))$ we get

$$\int_{B^+_R(0)} AD(u_\delta - G_\delta) : D(u_\delta - G_\delta) + AD(G_\delta) : D(u_\delta - G_\delta) + (HA)D(u_\delta - G_\delta)$$
$$: \nabla(u_\delta - G_\delta) + (HA)DG_\delta : \nabla(u_\delta - G_\delta)\, dx = -\int_{B^+_R(0)} F_\delta.[\nabla(u_\delta - G_\delta)]\, dx.$$

Denote $I := \|D(u_\delta - G_\delta)\|_{2,B^+_R(0)}$. Then the assumption (2.19) and Young's inequality give

$$I^2 \le \frac{1}{\lambda_0}\int_{B^+_R(0)} AD(u_\delta - G_\delta) : D(u_\delta - G_\delta) \le$$
$$\frac{C}{\lambda_0}\left(\lambda_1\|\triangle_{\delta,s}g\|_{2,B^+_R(0)}I + \|f\|_{2,B^+_R(0)}\right) + \frac{\lambda_1}{\lambda_0}|H|\|\nabla(u_\delta - G_\delta)\|_{2,B^+_R(0)}I. \quad (2.28)$$

Korn's inequality implies $\|\nabla(u_\delta - G_\delta)\|_{2,B^+_R(0)} \le \sqrt{2}I$. Inserting this inequality into (2.28) we obtain

$$(1 - \sqrt{2}\frac{\lambda_1}{\lambda_0}|H|)I^2 \le \frac{C}{\lambda_0}\left(\lambda_1\|\triangle_{\delta,s}g\|_{2,B^+_R(0)} + \|f\|_{2,B^+_R(0)}\right)I.$$

The assumption (2.20) and (2.22) imply that $1 - \sqrt{2}\frac{\lambda_1}{\lambda_0}|H| > 0$. Henceforth,

$$\|D(u_\delta - G_\delta)\|_{2,B^+_R(0)} \le C\left(\|\triangle_{\delta,s}g\|_{2,B^+_R(0)} + \|f\|_{2,B^+_R(0)}\right) \quad (2.29)$$

and

$$\|D\triangle_{\delta,s}(\eta w)\|_{2,B^+_R(0)} = \|Du_\delta)\|_{2,B^+_R(0)} \le \|D(u_\delta - G_\delta)\|_{2,B^+_R(0)} + \|DG_\delta\|_{2,B^+_R(0)}$$
$$\le C\left(\|\triangle_{\delta,s}g\|_{2,B^+_R(0)} + \|f\|_{2,B^+_R(0)}\right) \quad (2.30)$$

with a constant $C = C(\lambda_0, \lambda_1, d) > 0$.
Due to $\int_{B^+_R(0)} p_\delta\, dx = 0$, we can apply Lemma 0.1 for the system (2.26) and using (2.30) we have estimate

$$\|\triangle_{\delta,s}(\eta(q - (q)_{B^+_R(0)}))\|_{2,B^+_R(0)} \le C\left(\|\triangle_{\delta,s}g\|_{2,B^+_R(0)} + \|f\|_{2,B^+_R(0)}\right). \quad (2.31)$$

Let $\delta \to 0$ in inequalities (2.30), (2.31) , we deduce $D\frac{\partial(\eta w)}{\partial x_s} \in L^2(B^+_R(0))^{d^2}$, $\frac{\partial(\eta(q-(q)_{B^+_R(0)}))}{\partial x_s} \in L^2(B^+_R(0))^d$, $s = 1, \cdots, d-1$. Thanks to (2.25) and Nirenberg's lemma -see [12], II.3, Proposition , we also have estimates

$$\|D\frac{\partial(\eta w)}{\partial x_s}\|_{2,B^+_R(0)} \le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B^+_R(0)},$$
$$\|\frac{\partial(\eta(q - (q)_{B^+_R(0)}))}{\partial x_s}\|_{2,B^+_R(0)} \le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B^+_R(0)},\ s = 1, \cdots, d-1. \quad (2.32)$$

Thus

$$\|D\eta\frac{\partial w}{\partial x_s}\|_{2,B_R^+(0)} \le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\ s=1,\cdots,d-1. \tag{2.33}$$

Using Korn's inequality we also have $D(\eta\frac{\partial w}{\partial x_s}) \in L^2(B_R^+(0))^{d^2}$. Since div $w=0$, $\nabla(\eta\frac{\partial w}{\partial x_s}) \in L^2(B^+(0,R))^{d^2}$ for all $s=1,\cdots,d-1$, it implies that $\frac{\partial\eta\frac{\partial w_d}{\partial x_d}}{\partial x_d} \in L^2(B_{\tau R}^+(0)))$.
Therefore, it is sufficient to prove that $\frac{\partial D_{id}w}{\partial x_d} \in L^2(B_{\tau R}^+(0))$ for all $i=1,\cdots,d-1$ and $\frac{\partial q}{\partial x_d} \in L^2(B_{\tau R}^+(0))$.
Combining these inequalities and assertions, we can conclude that

$$(\nabla'\nabla w,\ \nabla' q) \in L^2(B_{\tau R}^+(0))^{d(d-1)} \times L^2(B_{\tau R}^+(0))^{d-1},\ \frac{\partial\frac{\partial w_d}{\partial x_d}}{\partial x_d} \in L^2(B_{\tau R}^+(0))$$

and

$$\begin{aligned}\|\nabla'\nabla w\|_{2,B_{\tau R}^+(0)} + \|\frac{\partial^2 w_d}{\partial^2 x_d}\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}\\ \|\nabla' q\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}\end{aligned} \tag{2.34}$$

hold with a constant $C=C(\lambda_0,\lambda_1,\nu_0,d)>0$.
As in the proof of Theorem 2.3, we can show that $\frac{\partial D_{id}w}{\partial x_d} \in L^2(B_{\tau R}^+(0))$ for all $i=1,\cdots,d-1$; $\frac{\partial q}{\partial x_d} \in L^2(B_{\tau R}^+(1))$ and inequalities

$$\begin{aligned}\|\frac{\partial D_{id}w}{\partial x_d}\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\quad i=1,\cdots,d-1\\ \|\frac{\partial q}{\partial x_d}\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}\end{aligned} \tag{2.35}$$

hold with a constant $C=C(\lambda_0,\lambda_1,\nu_0,d)>0$.
Finally, all above estimates implies $(\nabla^2 w,\ \nabla q) \in L^2(B_{\tau R}^+(0))^{d^2} \times L^2(B_{\tau R}^+(0))^d$ and we obtain the inequalities (2.21). □

Remark. Let $(w,\ q) \in W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ be a weak solution of system (2.18) and $w=0$ on Γ, then by the procedure of Lemma 2.1 and an induction argument, we can show that $w \in W^{k+2,2}(B_r^+(0))^d$, $q \in W^{k+1,2}(B_r^+(0))(r<1,\ k\in\mathbb{N})$.

Let $x\in\Gamma, 0<R\le \text{dist}\ \partial(x, B_1^+(0)\setminus\Gamma)$, we set

$$\begin{aligned}E_0^{w,q}(x,R) &:= \frac{1}{R^{\frac{d-2}{2}}}\|\nabla^2 w\|^2_{2,B_R^+(x)} + \frac{1}{R^{\frac{d-2}{2}}}\|\nabla q\|^2_{2,B_R^+(x)} + R^\alpha,\\ E^{w,q}(x,R) &= E_0^{w,q}(x,R) + R^\alpha\end{aligned} \tag{2.36}$$

where $0 < \alpha < 1$.

We have a following lemma for decay for the system (2.18)

Lemma 2.2. *Let $(w,\ q) \in W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ be a weak solution of system (2.18) and $w = 0$ on Γ. Then for all $\tau, \alpha \in (0,1)$, $R \le 1$, there is a positive constant C^* such that*

$$E_0^{w,q}(0, \tau R) \le C^* \tau^\alpha E_0^{w,q}(0, R) \text{ and } E^{w,q}(0, \tau R) \le C^* \tau^\alpha E^{w,q}(0, R) \quad (2.37)$$

where C^* depend only on $\lambda_0, \lambda_1, \nu_0, d$.

Proof. It is clear that (2.37) holds if $\tau \ge \frac{1}{2}$.

Let $0 < \tau < \frac{1}{2}$. Fix $k > \frac{d}{2}$. As all the tangent derivatives of w, q solve (2.23) and higher derivatives in normal direction are then calculated from (2.23) we can use repeatedly Lemma 2.1 for $j = 1, \cdots, k$ and $\nabla^j w, \nabla^j q$ on half balls with radii $\frac{1}{2}(1 + \frac{j-1}{k})$ and $\frac{1}{2}(1 + \frac{j}{k})$. Sobolev embedding implies $\nabla^2 w$, ∇q are essentially bounded on $B_{\frac{1}{2}}^+(0)$ and we get

$$\begin{aligned}\|\nabla^2 w\|_{L^\infty(B_{\frac{1}{2}}^+(0))} + \|\nabla q\|_{L^\infty(B_{\frac{1}{2}}^+(0))} &\le C(\|\nabla^2 w\|_{k,2;B_{\frac{1}{2}}^+(0)} + \|\nabla q\|_{k,2;B_{\frac{1}{2}}^+(0)}) \\ &\le C\|\nabla w\|_{2,B_1^+(0)}. \end{aligned} \quad (2.38)$$

with a constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.
Thus inequalities

$$\begin{aligned} E_0^{w,q}(0,\tau) &= \frac{1}{\tau^{\frac{d-2}{2}}} \|\nabla^2 w\|_{2,B_\tau^+(0)} + \frac{1}{\tau^{\frac{d-2}{2}}} \|\nabla q\|_{2,B_\tau^+(0)} \\ &\le C\tau(\|\nabla^2 w\|_{L^\infty(B_{\frac{1}{2}}^+(0))} + \|\nabla q\|_{L^\infty(B_{\frac{1}{2}}^+(0))}) \le C^* \tau E_0^{w,q}(0,1), \end{aligned}$$

hold with a constant $C^* = C(\lambda_0, \lambda_1, \nu_0, d) > 1$.
By simple scaling argument, from (2.18) we deduce

$$E_0^{w,q}(0, \tau R) \le C^* \tau E_0^{w,q}(0, R).$$

Estimate for $E^{w,q}(0, \tau R)$ is obvious. □

Return to the nonlinear system (2.1).

Lemma 2.3. *(Decay for nonlinear system) Let the assumptions (2.2) are satisfied, $\nu_0 < \frac{\lambda_0}{(\lambda_0 + C_0\lambda_1)C_0}$) with $C_0 = \max\,(C_{\mathrm{div}}, \sqrt{3(d-1)})$ and $f \in L^{2,\mu}(B_1^+(0))$ for $\mu > d - 2\alpha$. For all $M \in (0,\infty)$ and for all $\tau \in (0,1)$, there exists an $\varepsilon > 0$ and a constant $C^* > 0$ having the property : Let $(v,p) \in W^{1,2}(B_1^+(0))^d) \times L^2(B_1^+(0))$ with $\operatorname{supp} v$, $\operatorname{supp} p \subset B_1^*(0)$ be a weak solution of system (2.1), $v = 0$ on Γ, let for $x \in \Gamma, R \le \operatorname{dist} \partial(\mathrm{x}, \mathrm{B}_1^+(0) \setminus \Gamma)$ the inequalities*

$$E^{v,p}(x,R) < \varepsilon,\ |(Dv)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| \le M \quad (2.39)$$

hold. Then

$$E^{v,p}(x, \tau R) \le 2C^* \tau^\alpha E^{v,p}(x, R). \quad (2.40)$$

Proof. We use the indirect method to prove this lemma. Assume that there exist τ, $M, \varepsilon_h \to 0$, x_h, $R_h \to 0$ and weak solutions v_h, p_h to (2.1) whose supports are contained in $B_1^*(0)$ for which

$$E^{v_h,p_h}(x_h, R_h) = \varepsilon_h, |(\nabla v_h)_{B^+_{R_h}(x_h)}| + |(p_h)_{B^+_{R_h}(x_h)}| \leq M \tag{2.41}$$

and, at the same time,

$$E^{v_h,p_h}(x_h, \tau R_h) > 2C^* \tau^\alpha E^{v_h,p_h}(x_h, R_h). \tag{2.42}$$

Thanks to (2.41) we can assume

$$(p_h)_{B^+_{R_h}(x_h)} \longrightarrow a \quad \text{in} \quad \mathbb{R},$$
$$(D^* v_h)_{B^+_{R_h}(x_h)} \longrightarrow e \quad \text{in} \quad \mathbb{R}^{d^2}$$

where $D^* v_h$ is a symmetric $d \times d$ matrix given by

$$D^*_{ij} v_h = 0 \text{ if } i, j = 1, \cdots, d-1, \; D^*_{dd} v_h = \frac{\partial (v_h)_d}{\partial x_d},$$
$$D^*_{id} v_h = D^*_{di} v_h = \frac{1}{2} \frac{\partial (v_h)_i}{\partial x_d} \; i, j = 1, \cdots, d-1. \tag{2.43}$$

It is clear that e is a symmetric matrix and $|a| + |e| \leq M$.
Denote

$$A := \frac{\partial T(a,e)}{\partial D}, \; B := \frac{\partial T(a,e)}{\partial p}. \tag{2.44}$$

Recall that for any a, e the ellipticity constant λ_0 and bounds λ_1, ν_0 for A, B remain the same as well as C^*.

Now we use a blow-up technique for p_h and v_h.

Step 1. Scaling and uniform estimates.
We consider functions w_h, q_h defined on $B_1^+(0)$, given by

$$w_h(y) := \frac{v_h(x_h + R_h y) - (\nabla v_h)_{B^+_{R_h}(x_h)} \cdot (0, \cdots, 0, y_d)^T R_h}{R_h \varepsilon_h},$$
$$q_h(y) := \frac{p(x_h + R_h y) - (p_h)_{B^+_{R_h}(x_h)}}{\varepsilon_h}, \tag{2.45}$$
$$f_h(y) = \frac{R_h}{\varepsilon_h} f(x_h + R_h y).$$

It follows that

$$\nabla_y w_h(y) = \frac{\nabla_x v_h(x_h + R_h y) - (\nabla v_h)_{B^+_{R_h}(x_h)} (0, \cdots, 0, 1)^T}{\varepsilon_h},$$
$$\nabla_y q_h(y) = \frac{R_h \nabla_x p_h(x_h + R_h y)}{\varepsilon_h},$$

$$\nabla_y^2 w_h(y) = \frac{R_h \nabla_x^2 v_h(x_h + R_h y)}{\varepsilon_h}, \tag{2.46}$$

$$D_y w_h(y) = \frac{D_x v_h(x_h + R_h y) - (D^* v_h)_{B^+_{R_h}(x_h)}}{\varepsilon_h},$$

where $D^* v_h$ is in (2.43).
It easy to verify that

$$w_h = \nabla' w_h = 0 \ \text{ on } \ \Gamma,$$

$$(q_h)_{B_1^+(0)} = (\frac{\partial (w_h)_i}{\partial x_d})_{B_1^+(0)} = 0 \ \ i = 1, \cdots, d. \tag{2.47}$$

The assumption (2.41) allows us to use Poincaré's inequality to have

$$\|\nabla^2 w_h(y)\|^2_{B_1^+(0)} = \frac{1}{\varepsilon_h^2 R_h^{d-2}} \|\nabla^2 v_h\|^2_{B^+_{R_h}(x_h)} \le \frac{E_h^v(x_h, \tau R_h)^2}{\varepsilon_h^2} \le 1,$$

$$\|\nabla q_h(y)\|^2_{B_1^+(0)} = \frac{1}{\varepsilon_h^2 R_h^{d-2}} \|\nabla p_h\|^2_{B^+_{R_h}(x_h)} \le \frac{E_h^v(x_h, \tau R_h)^2}{\varepsilon_h^2} \le 1,$$

$$\|\nabla w_h(y)\|^2_{B_1^+(0)} \le C_P \|\nabla^2 w_h(y)\|^2_{B_1^+(0)} \le C_P,$$

$$\|q_h(y)\|^2_{B_1^+(0)} \le C_P \|\nabla q_h(y)\|^2_{B_1^+(0)} \le C_P,$$

$$\|w_h(y)\|^2_{B_1^+(0)} \le C_P \|\nabla w_h(y)\|^2_{B_1^+(0)} \le C_P^2, \tag{2.48}$$

where C_P is the constant from Poincaré's inequality on the half ball $B_1^+(0)$. Hence w_h, q_h are bounded in $W^{2,2}(B_1^+(0))^d$, $W^{1,2}(B_1^+(0))$. Then there exists a subsequence, still denoted by w_h, q_h such that

$$(w_h, q_h) \longrightarrow (w, q), \quad \text{weakly in} \quad W^{2,2}(B_1^+)^d \times W^{1,2}(B_1^+(0)),$$

$$(w_h, q_h) \longrightarrow (w, q) \quad \text{strongly in} \quad W^{1,2}(B_1^+)^d \times L^2(B_1^+(0))^d,$$

$$w_h \longrightarrow w, \ \ \nabla w_h \longrightarrow \nabla w, \ \ q_h \longrightarrow q \ \text{ a.e. on } \ (B_1^+(0)). \tag{2.49}$$

Next, putting instead of p_h, v_h the rescaled qualities q_h, w_h and $y = \frac{x - x_h}{R_h}$ into system (2.1), we obtain

$$\int_{B_1^+(0)} \frac{1}{\varepsilon_h} T(q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)}) : D_y \psi \, dy$$

$$+ \int_{B_1^+(0)} \frac{1}{\varepsilon_h} [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)} . (0, \cdots, 0, y_d)^T R_h] \otimes [w_h R_h \varepsilon_h$$

$$+ (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)} . (0, \cdots, 0, y_d)^T R_h] . D_y \psi \, dy$$

$$- \int_{B_1^+(0)} \frac{1}{\varepsilon_h} [q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}] \text{div}_{\ y} \psi \, dy - \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h \psi \, dy = 0$$

$$\text{for all } \ \psi \in W_0^{1,2}(B_1^+(0))^d),$$

$$\text{div } w_h = \frac{1}{\varepsilon_h} (\frac{\partial (v_h)_d}{\partial x_d})_{B^+_{R_h}(x_h)} \ \text{ on } \ B_1^+(0). \tag{2.50}$$

We denote the four terms on the left hand side of (2.50) by D_1, D_2, D_3, D_4. Now we pass to the limit in (2.50) for $h \to 0$.

Step 2. Passage to the limit.
From the estimates (2.48), we deduce

$$\begin{aligned} w_h R_h \varepsilon_h &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0))^d, \\ \nabla w_h \varepsilon_h &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0))^{d^2}, \\ q_h \varepsilon_h &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0)), \\ w_h R_h &\longrightarrow 0, \ \nabla w_h \varepsilon_h \longrightarrow 0, \ q_h \varepsilon_h \longrightarrow 0 \text{ a.e. on } B_1^+(0). \end{aligned} \tag{2.51}$$

Moreover, we can assume that there exist functions $\tilde{w} \in L^2(B_1^+(0))^d$, $w^* \in L^2(B_1^+(0))^{d^2}$ such that $|w_h R_h \varepsilon_h| \le \tilde{w}, |\nabla w_h \varepsilon_h| \le w^*$ a.e. on $B_1^+(0)$. (Note that we pass to a not relabelled subsequence)
It implies that

$$\begin{aligned} Dw_h\varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)} &\longrightarrow a \text{ a.e. on } B_1^+(0), \\ q_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)} &\longrightarrow e \text{ a.e. on } B_1^+(0). \end{aligned} \tag{2.52}$$

We have

$$\begin{aligned} D_1 &= \int_{B_1^+(0)} \frac{1}{\varepsilon_h} T(q_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, Dw_h\varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)}).D_y\psi \, dy \\ &= \frac{1}{\varepsilon_h}\int_{B_1^+(0)} \left(\int_0^1 \frac{\partial}{\partial s} T(sq_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, sDw_h\varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)}) \, ds\right) \\ &\qquad .D\psi \, dy \\ &= \int_{B_1^+(0)} \left(\int_0^1 \frac{\partial T(sq_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, sDw_h\varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)})}{\partial D} \, ds\right) \\ &\qquad Dw_h : D\psi(y) \, dy \\ &+ \int_{B_1^+(0)} \left(\int_0^1 \frac{\partial T(sq_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, sDw_h\varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)})}{\partial p} \, ds\right) \\ &\qquad q_h.D\psi(y) \, dy. \end{aligned} \tag{2.53}$$

Because of the continuity of T and the convergences (2.49), (2.52), we can pass to the limit in (2.53) and deduce that

$$D_1 \longrightarrow \int_{B_1^+(0)} ADw_h : D\psi(y) \, dy + \int_{B_1^+(0)} Bq_h.D\psi(y) \, dy \text{ as } h \longrightarrow 0,$$

where A, B are given by (2.44).

We continue to pass to the limit in D_2 :

$$D_2 = \int_{B_1^+(0)} [w_h R_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)} .(0, \cdots, 0, y_d)^T \frac{R_h}{\varepsilon_h}] \otimes [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)} .(0, \cdots, 0, y_d)^T R_h] . D_y \psi \, dy. \quad (2.54)$$

Now, inequality (2.41) implies $\frac{R_h}{\varepsilon_h} = \frac{R_h^\alpha}{\varepsilon_h} R_h^{1-\alpha} \to 0$ as $h \to 0$. Combining this with (2.41), (2.49), (2.51), we can summarize $D_2 \to 0$ as h tend to zero.
Next

$$D_3 = \int_{B_1^+(0)} \frac{1}{\varepsilon_h}[q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}] \text{div}\,_y \psi \, dy = \int_{B_1^+(0)} q_h \text{div}\,_y \psi \, dy \longrightarrow \int_{B_1^+(0)} q \text{div}\,_y \psi \, dy \text{ as } h \longrightarrow 0.$$

Recall the assumption $f \in L^{2,\mu}(B_1^+(0))$ for $\mu > d - 2\alpha$, hence for all $\psi(y) \in W_0^{1,2}(B_1^+(0))^d$ and $\mu = d - 2\alpha + \varepsilon$ for ε small enough we obtain

$$|D_4| = |\frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \psi \, dy| \le \left(\int_{B_1^+(0)} |f_h|^2 \, dy \int_{B_1^+(0)} |\psi|^2 \, dy \right)^{\frac{1}{2}} = \left(\frac{R_h^{2\alpha}}{\varepsilon_h^2} R_h^{\varepsilon} \frac{1}{R_h^{d-2\alpha+\varepsilon}} \int_{B^+_{R_h}(x_h)} |f|^2 dx \int_{B_1^+(0)} |\psi|^2 \, dy \right)^{\frac{1}{2}} \longrightarrow 0 \text{ as } h \longrightarrow 0. \quad (2.55)$$

Finally, it is easily seen that div $w_h \to$ div $w = 0$ as $h \to 0$. It follows that w, q solve the system (2.18) with A, B given by (2.44). On the other hand, it is easy to check that A, B satisfy the assumptions of Lemma 2.2. Therefore, we have

$$E^{w,q}(x, \tau R) \le C^* \tau^\alpha E^{w,q}(x, R). \quad (2.56)$$

Step 3. Strong convergence on $B_\tau^+(0)$.
Thanks to Theorem 2.23 and the system (2.1), derivatives $\frac{\partial v_h(x)}{\partial x_s}$, $\frac{\partial p_h(x)}{\partial x_s}$ for $s = 1, \cdots, d$ satisfy a weak formulation

$$\int_{B^+_{R_h}(x_h)} [\frac{\partial T(p_h(x), Dv_h(x))}{\partial D} D \frac{\partial v_h(x)}{\partial x_s} : D\varphi(x) + \frac{\partial T(p_h(x), Dv_h(x))}{\partial p_h} \frac{\partial p_h(x)}{\partial x_s} : D\varphi(x) + \frac{\partial (v_h \otimes v_h)}{\partial x_s} . D_{jk}\varphi(x) - \frac{\partial p_h(x)}{\partial x_s} \text{div } \varphi(x)] \, dx = - \int_{B^+_{R_h}(x_h)} f(x) \frac{\partial \varphi(x)}{\partial x_s} \, dx \quad (2.57)$$

for all $\varphi(x) \in W_0^{1,2}(B_{R_h}^+(x_h))^d$.
Substituting $y = \frac{x-x_h}{R_h}$ and using (2.45), (2.46); (2.57), we have

$$\int_{B_1^+(0)} [\frac{\partial T(q_h\varepsilon_h + (p_h)_{B_{R_h}^+}, (D^*v_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial D} D\frac{\partial w_h(y)}{\partial y_s} : D\psi(y) + \frac{R_h}{\varepsilon_h}$$
$$\{[\frac{\partial w_h}{\partial y_s}\varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+}.(0,\cdots,0,1)^T] \otimes [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+}.(0,\cdots,0,y_d)^T R_h]$$
$$+ [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+} R_h(0,\cdots,0,y_d)^T] \otimes [\frac{\partial w_h}{\partial y_s}\varepsilon_h + (\frac{\partial v)}{\partial x_s})_{B_{R_h}^+}.(0,\cdots,0,1)^T]\}$$
$$D\psi(y) + \frac{\partial T(q_h(y)\varepsilon_h + (p_h)_{B_{R_h}^+}, (D^*v_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial p}\frac{\partial q_h(y)}{\partial y_s} : D\psi(x)$$
$$- \frac{\partial q_h}{\partial y_s}\text{div } \psi(y)]\, dy = -\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \psi(y)}{\partial y_s}\, dy, \quad s = 1,\cdots,d, \quad (2.58)$$

and div $w_h = \frac{R_h}{\varepsilon_h}(\frac{\partial (v_h)_d}{\partial x_d})_{B_{R_h}^+}$ on $B_1^+(0)$, where $\psi(y) = \varphi(x_h + R_h y) \in W_0^{1,2}(B_1^+(0))^d$, $B_{R_h}^+ := B_{R_h}^+(x_h)$.
The inequalities (2.42) and (2.41) imply

$$E^{w_h,q_h}(0,\tau) > 2C^*\tau^\alpha E^{w_h,q_h}(0,1),\ E^{w_h,q_h}(0,1) = 1. \qquad (2.59)$$

Denote

$$A_h := \frac{\partial T(q_h\varepsilon_h + (p_h)_{B_{R_h}^+}, (Dw_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial D},$$
$$B_h = \frac{\partial T(q_h(y)\varepsilon_h + (p_h)_{B_{R_h}^+}, (Dw_h)_{B_{R_h}^+} + Dw_h\varepsilon_h}{\partial p}. \qquad (2.60)$$

It is clear that A_h, B_h are uniformly bounded, $A_h \to A$, $B_h \to B$ a.e. on $B_1^+(0)$.
For simplicity, system (2.58) can rewritten in the following way

$$\int_{B_1^+(0)} \left(A_h D\frac{\partial w_h}{\partial y_s} : D\psi + (B_h - E)\frac{\partial q_h}{\partial y_s} : D\psi\right) dy + I_h$$
$$= -\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \psi(y)}{\partial y_s}\, dy \quad (2.61)$$

for all $\psi(y) \in W_0^{1,2}(B_1^+(0))^d$, where I_h is the second term on the left hand side of (2.58).

Set $u_h = \frac{\partial w_h}{\partial y_s}, \pi_h = \frac{\partial q_h}{\partial y_s}$; $u = \frac{\partial w}{\partial y_s}, \pi = \frac{\partial q}{\partial y_s}$, $s = 1,\cdots,d$. Then u_h, q_h

satisfy a weak formulation

$$\int_{B_1^+(0)} (A_h Du_h : D\psi + (B_h - E)\pi_h : D\psi)\, dy$$
$$= -I_h - \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \psi_h(y)}{\partial y_s}\, dy \quad (2.62)$$

for all $\psi \in W_0^{1,2}(B_1^+(0))^d$.
Thanks to (2.48) and (2.49), we have

$$\|\nabla u_h(y)\|^2_{2,B_1^+(0)} \leq 1 \quad \|\pi_h(y)\|^2_{2,B_1^+(0)} \leq 1 \quad \|u_h(y)\|^2_{2,B_1^+(0)} \leq C_P \quad (2.63)$$

and

$$\begin{aligned} (u_h, \pi_h) &\longrightarrow (u, \pi), \quad \text{weakly in} \quad W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0)), \\ u_h &\longrightarrow u \quad \text{strongly in} \quad L^2(B_1^+)^d \quad u_h \longrightarrow u \;\text{ a.e. on }\; (B_1^+(0)). \end{aligned} \quad (2.64)$$

Now consider $s = 1, \cdots, d-1$.
As $u_h = 0,\ u = 0$ on Γ, we can choose a test function $\psi_h = \eta^2(u_h - u)$, where $\eta \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function : supp $\eta \subset B_{\frac{1+\tau}{2}}(0)$, $0 \leq \eta \leq 1$ on $\mathrm{B}^+_{\frac{1+\tau}{2}}(0)$, $\eta \equiv 1$ on $\mathrm{B}^+_\tau(0)$; $|\nabla \eta| \leq \frac{C}{1-\tau}$. Functions u, π solve the system (2.18), hence (2.62) can be written as

$$\int_{B_1^+(0)} (A_h D(u_h - u) : D\psi_h + (\pi_h - \pi)(B_h - E).D\psi_h)\, dy = -I_h - G_h$$

where

$$G_h := \int_{B_1^+(0)} (A_h Du : D\psi_h + \pi(B_h - E).D\psi_h)\, dy + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h \frac{\partial \psi_h}{\partial y_s}\, dy.$$

Then

$$\int_{B_1^+(0)} \eta^2 A_h D(u_h - u) : D(u_h - u)\, dy = \int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\, dy$$
$$- I_h - G_h - H_h,$$

where

$$H_h = \int_{B_1^+(0)} 2\eta A_h D(u_h - u) : [(u_h - u) \otimes \nabla \eta]\, dy.$$

Using the assumption (2.2), we get

$$\int_{B_1^+(0)} |\eta D(u_h - u)|^2 dy \leq \frac{1}{\lambda_0} \int_{B_1^+(0)} \eta^2 A_h D(u_h - u) : D(u_h - u)\, dy$$
$$= \frac{1}{\lambda_0} [\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\, dy - I_h - G_h - H_h]. \quad (2.65)$$

Since $\psi_h = \eta^2(u_h - u)$, $s = 1, \cdots, d-1$, (2.64), we deduce

$$
\begin{aligned}
&(\psi_h) \longrightarrow 0, \quad \text{weakly in} \quad W^{1,2}(B_1^+(0)),\\
&(\psi_h) \longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^d \quad \text{and a.e. on} \quad B_1^+(0).
\end{aligned}
$$

Thanks to $\frac{R_h}{\varepsilon_h} \to 0$, $\varepsilon_h \to 0$, (2.41), (2.48), (2.51) we obtain $I_h \to 0$ as $h \to 0$.
As u, π are satisfy the identity (2.18), we have

$$
G_h = \int_{B_1^+(0)} ((A_h - A)Du : D\psi_h + \pi(B_h - B).D\psi_h)\, dy \\
+ \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h \frac{\partial \psi_h}{\partial y_s}\, dy.
$$

The coefficients $A_h - A$, $B_h - B$ are equibounded and tend to 0 a.e. on $B_1^+(0)$, $Du \in L^2(B_1^+)^{d^2}$, $\pi \in L^2(B_1^+)$ and $|\frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \psi_h}{\partial y_s}\, dy| \to 0$ as $h \to 0$, which implies that $G_h \to 0$ as $h \to 0$.
It is easily seen that $H_h \to 0$ as $h \to 0$.
Denote

$$
\begin{aligned}
W_h &= \|\eta D(u_h - u)\|_{2;B_1^+(0)}, \; c_h = (\eta(\pi_h - \pi))_{B_1^+(0)}\\
Q_h &= \|\eta(\pi_h - \pi) - c_h\|_{2;B_1^+(0)}.
\end{aligned} \tag{2.66}
$$

Next, we have

$$
\begin{aligned}
&|\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\, dy|\\
&= |\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D[\eta^2(u_h - u)]dy|\\
&\leq |2 \int_{B_1^+(0)} \eta(\pi_h - \pi)(E - B_h).[(u_h - u) \otimes \nabla\eta]\, dy|\\
&+| \int_{B_1^+(0)} c_h \eta(E - B_h).D(u_h - u)\, dy|\\
&+| \int_{B_1^+(0)} \eta[\eta(\pi_h - \pi) - c_h]\text{div}\,(u_h - u) dy|\\
&+| \int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h]B_h.D(u_h - u)\eta\, dy| = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Using Hölder's inequality, (2.48) and (2.49), we get

$$
\begin{aligned}
I_1 &\leq \frac{C}{1-\tau}\|(\pi_h - \pi)\|_{2;B_1^+(0)}\|(u_h - u)\|_{2;B_1^+(0)}\\
&\leq C\|(u_h - u)\|_{2;B_1^+(0)} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.
\end{aligned}
$$

As $E - B_h$ is uniformly bounded in $L^\infty(B_1^+(0)^{d^2}$, $D(u_h - u)$ is uniformly bounded in $L^2(B_1^+(0))^d$ and $c_h \to 0$ as $h \to 0$, it gives $I_2 \to 0$ as $h \to 0$.
Obviously, $I_3 = 0$ due to div $w =$ div $w_h = 0$, $w, w_h \in W^{2,2}(B_1^+(0))$.
For last term I_4, we have estimates

$$I_4 \leq \nu_0 W_h Q_h.$$

With all these above estimates, we can conclude that

$$W_h^2 \leq \frac{\nu_0}{\lambda_0} Q_h W_h + o(h), \text{ as } h \longrightarrow 0. \tag{2.67}$$

Our next aim is to estimate Q_h.
Let $\varphi_h \in W_0^{1,2}(B_1^+(0))^d$ be a solution to problem

$$\text{div } \varphi_h = \eta(\pi_h - \pi) - c_h \text{ in } B_1^+(0). \tag{2.68}$$

Recall that $\|\nabla\varphi_h\|_{2,B_1^+(0)} \leq C_{\text{div}} \|\eta(\pi_h - \pi) - c_h\|_{2,B_1^+(0)}$. Combining this inequality and (2.63), we can assert that φ_h are bounded uniformly with respect to h in $W^{1,2}(B_1^+(0))^d$. Moreover, $\eta(\pi_h - \pi) - c_h \longrightarrow 0$ weakly in $L^2(B_1^+(0))$ and the solution operator of problem (2.68) is linear and continuous from $L^2(B_1^+(0))$ to $W^{1,2}(B_1^+(0))^d$ so that

$$\begin{aligned} (\varphi_h) &\longrightarrow 0, \quad \text{weakly in } \ W^{1,2}(B_1^+(0))^d, \\ (\varphi_h) &\longrightarrow 0 \ \text{ strongly in } \ L^2(B_1^+(0))^d. \end{aligned} \tag{2.69}$$

Choosing test function $\phi_h = \eta\varphi_h$ in the formulation (2.62), we get

$$\begin{aligned} \int_{B_1^+(0)} \pi_h \text{ div } \phi_h \, dy = \int_{B_1^+(0)} & A_h D u_h : D\phi_h + \pi_h B_h . D\phi_h dy \\ & + I_h + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \phi_h(y)}{\partial y_s} \, dy, \end{aligned} \tag{2.70}$$

where I_h is the second term on the left hand side of (2.58) with test function φ_h replaced by ϕ_h.
From the definition of ϕ_h we have

$$\begin{aligned} \int_{B_1^+(0)} \pi_h \text{ div } \phi_h \, dy = \int_{B_1^+(0)} & [\eta(\pi_h - \pi) - c_h] \text{div } \varphi_h \, dx \\ & + \int_{B_1^+(0)} (\pi_h - \pi) \nabla\eta . \varphi_h \, dy + \int_{B_1^+(0)} \pi \text{ div } \phi_h \, dy. \end{aligned}$$

Hence

$$\begin{aligned} Q_h^2 = \int_{B_1^+(0)} & A_h D u_h : D\phi_h + \pi_h B_h . D\phi_h dy + I_h + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \phi_h(y)}{\partial y_s} \, dy \\ & - \int_{B_1^+(0)} (\pi_h - \pi) \nabla\eta . \varphi_h \, dy - \int_{B_1^+(0)} \pi \text{ div } \phi_h \, dy. \end{aligned} \tag{2.71}$$

By similar estimates as in the above proof, we can see that

$$I_h + \frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \phi_h(y)}{\partial y_s}\, dy \longrightarrow 0 \;\text{ as }\; h \longrightarrow 0.$$

As $(\varphi_h) \to 0$ strongly and div $\varphi_h \to 0$ weakly in $L^2(B_1^+(0))^d$, we have that div $\phi_h \to 0$ weakly in $L^2(B_1^+(0))^d$; moreover π_h, π are the uniformly bounded in $L^2(B_1^+(0))$ and $(\pi_h - \pi) \to 0$ weakly in $L^2(B_1^+(0)$. Thus

$$\int_{B_1^+(0)} (\pi_h - \pi)\nabla\eta.\varphi_h\, dy + \int_{B_1^+(0)} \pi \operatorname{div} \phi_h\, dy \longrightarrow 0 \;\text{ as }\; h \longrightarrow 0.$$

Next we have

$$\begin{aligned}
S_h = &\int_{B_1^+(0)} A_h Du_h : D\phi_h + \pi_h B_h.D\phi_h\, dy = \int_{B_1^+(0)} A_h\eta D(u_h - u) : D\varphi_h\, dy \\
&+ \int_{B_1^+(0)} \eta A_h Du : D\varphi_h\, dy + \int_{B_1^+(0)} A_h Du_h : (\nabla\eta \otimes \varphi_h)\, dy \\
&+ \int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h]B_h.D\varphi_h\, dy + \int_{B_1^+(0)} (\pi\eta + c_h)B_h.D\varphi_h\, dy \\
&+ \int_{B_1^+(0)} \pi_h B_h.(\nabla\eta \otimes \varphi_h) dy.
\end{aligned} \tag{2.72}$$

Employing Hölder's inequality, we get

$$\begin{aligned}
&|\int_{B_1^+(0)} A_h\eta D(u_h - u) : D\varphi_h\, dy| \leq \lambda_1 C_{\text{div}} W_h Q_h, \\
&|\int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h]B_h.D\varphi_h\, dy| \leq \nu_0 C_{\text{div}} Q_h^2.
\end{aligned} \tag{2.73}$$

Due to $\|Du_h\|_{B_1^+(0)} \leq 1$, (2.2), (2.69) and the lower semicontinuity of the L^2 norm with respect to weak convergence, we have $\|Du\|_{B_1^+(0)} \leq 1$. It implies

$$\int_{B_1^+(0)} \eta A_h Du : D\varphi_h\, dy + \int_{B_1^+(0)} A_h Du_h : (\nabla\eta \otimes \varphi_h)\, dy \longrightarrow 0 \;\text{ as }\; h \longrightarrow 0.$$

Combining the fact $\pi, \pi_h \in L^2_{B_1^+(0)}$, $c_h \to 0$ with (2.69) yields

$$\begin{aligned}
&\int_{B_1^+(0)} (\pi\eta + c_h)]B_h.D\varphi_h\, dy + \int_{B_1^+(0)} \pi_h B_h.(\nabla\eta \otimes \varphi_h) dy \\
&\qquad - \int_{B_1^+(0)} c_h B_h.D\varphi_h\, dy \longrightarrow 0 \;\text{ as }\; h \longrightarrow 0.
\end{aligned}$$

Using all the above estimates, we may conclude that

$$Q_h^2 \leq \lambda_1 C_{\text{div}} Q_h W_h + \nu_0 C_{\text{div}} Q_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Applying Young's inequality, we have

$$\lambda_1 C_{\text{div}} Q_h W_h \leq \frac{\lambda_1 C_{\text{div}}}{2(\lambda_1 C_{\text{div}} + \lambda_0)} Q_h^2 + \frac{\lambda_1 C_{\text{div}}(\lambda_1 C_{\text{div}} + \lambda_0)}{2} W_h^2,$$

and thus

$$(1 - \nu_0 C_{\text{div}} - \frac{\lambda_1 C_{\text{div}}}{2(\lambda_1 C_{\text{div}} + \lambda_0)}) Q_h^2 \leq \frac{\lambda_1 C_{\text{div}}(\lambda_1 C_{\text{div}} + \lambda_0)}{2} W_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Due to the assumption $\nu_0 < \frac{\lambda_0}{(\lambda_0 + C_0\lambda_1)C_0}$ and $C_0 \geq C_{\text{div}}$, we deduce

$$Q_h^2 \leq (\lambda_1 C_{\text{div}} + \lambda_0)^2 W_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Recall that (2.67) implies

$$W_h^2 \leq \frac{\nu_0}{\lambda_0} Q_h W_h + o(h) \leq \frac{\nu_0(\lambda_1 C_{\text{div}} + \lambda_0)}{\lambda_0} W_h^2 + o(h).$$

It is clear that $\frac{\nu_0(\lambda_1 C_{\text{div}} + \lambda_0)}{\lambda_0} \leq \frac{\nu_0(\lambda_1 C_0 + \lambda_0)}{\lambda_0} \leq \frac{1}{C_0} < 1$. Hence

$$W_h,\ Q_h \longrightarrow 0 \text{ as } h \longrightarrow 0 \text{ i.e.}$$

$$\|\eta D(u_h - u)\|_{2;B_1^+(0)} + \|\eta(\pi_h - \pi) - c_h\|_{2;B_1^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus $D(u_h - u) \to 0$ strongly in $L^2(B_\tau^+(0))^d$. Moreover, as $(u_h - u) \to 0$ strongly in $L^2(B_\tau^+(0))^d$, it implies

$$\|D[\eta(u_h - u)]\|_{2;B_1^+(0)}, \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Applying Korn's inequality, we have

$$\nabla[\eta(u_h - u)] \longrightarrow 0 \text{ strongly in } L^2(B_1^+(0))^d \text{ as } h \longrightarrow 0.$$

We also have $\|\eta(\pi_h - \pi)\|_{2;B_1^+(0)} \to 0$ as $h \to 0$.
Henceforth, for all $s = 1, \cdots, d-1$ it holds

$$\begin{aligned}
\eta\nabla(\frac{\partial(w_h - w)}{\partial y_s}) &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0))^{d^2} \\
\eta D(\frac{\partial(w_h - w)}{\partial y_s}) &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0))^{d^2} \qquad (2.74) \\
\eta\frac{\partial(p_h - p)}{\partial y_s} &\longrightarrow 0 \text{ strongly in } L^2(B_1^+(0)).
\end{aligned}$$

Thanks to the condition div $w = 0$, we also have

$$\frac{\partial[\eta \frac{\partial(w_h - w)_d}{\partial y_d}]}{\partial y_d} \to 0 \text{ strongly in } L^2(B_1^+(0)) \text{ as } h \longrightarrow 0. \tag{2.75}$$

Next we shall prove $D_{kd}(\frac{\partial(w_h - w)}{\partial y_d}) \to 0$ strongly in $L^2(B_\tau^+(0))$ for all $k = 1, \cdots, d-1$ and $\frac{\partial(p_h - p)}{\partial y_d} \to 0$ strongly in $L^2(B_\tau^+(0))$ (this is sufficient to make $w_h \to w$ strongly in $W^{2,2}(B_\tau^+(0))^d$, $p_h \to p$ strongly in $W^{1,2}(B_\tau^+(0))$.
In fact, because v_h, p_h are solutions to the system (2.1), in an analogous way as in the proof of Theorem 2.3, we have

$$\sum_{j=1}^{d} \frac{\partial T_{ij}(p_h, Dv_h)}{\partial D} \frac{\partial Dv_h}{\partial x_j} + \sum_{j=1}^{d} \frac{\partial T_{ij}(p_h, Dv_h)}{\partial p} \frac{\partial p_h}{\partial x_j} - \nabla p_h = -f + v_h \otimes \frac{\partial (v_h)_i}{\partial x} \tag{2.76}$$

for all $i = 1, \cdots, d$.
Then we put instead of p_h, v_h, Dv_h the rescaled quantities q_h, w_h, Dw_h and we obtain

$$\sum_{j=1}^{d} (A_h)_{ij} \frac{\partial Dw_h}{\partial y_j} + \sum_{j=1}^{d} (B_h)_{ij} \frac{\partial q_h}{\partial y_j} - \nabla q_h = -\frac{R_h}{\varepsilon_h}(f_h)_i + (J_h)_i \tag{2.77}$$

for all $i = 1, \cdots, d$, where A_h, B_h, f_h were already defined in the beginning of this proof and

$$\begin{aligned}(J_h)_i = \frac{R_h}{\varepsilon_h}[w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}} R_h(0, \cdots, 0, y_d)^T] \otimes [\frac{\partial (w_h)_i}{\partial y_s} \varepsilon_h \\ + (\frac{\partial (v_h)_i}{\partial x_s})_{B^+_{R_h}} .(0, \cdots, 0, 1)^T]; \text{ for all } i = 1, \cdots, d.\end{aligned}$$

Recall that w, q are solution to system (2.18), thus

$$\begin{aligned}\sum_{j=1}^{d} (A_h)_{ij} \frac{\partial D(w_h - w)}{\partial y_j} + \sum_{j=1}^{d} (B_h)_{ij} \frac{\partial (q_h - q)}{\partial y_j} - \nabla (q_h - q) \\ = -(\bar{G}_h)_i + (J_h)_i \text{ for all } i = 1, \cdots, d,\end{aligned} \tag{2.78}$$

with $(\bar{G}_h)_i := \sum_{j=1}^{d} (A_h - A)_{ij} \frac{\partial Dw}{\partial y_j} + \sum_{j=1}^{d} (B_h - B)_{ij} \frac{\partial q}{\partial y_j} + \frac{R_h}{\varepsilon_h}(f_h)_i$ for all $i = 1, \cdots, d$.
The system can be written as

$$\sum_{k=1}^{d-1} 2(A_h)_{id}^{kd} \frac{\partial D_{kd}(w_h - w)}{\partial y_d} + (B_{id} - \delta_{id}) \frac{\partial (q_h - q)}{\partial y_d} = (\bar{F}_h)_i - (\bar{G}_h)_i + (J_h)_i \tag{2.79}$$

for all $i = 1, \cdots, d$, where $\bar{F}_h$ are given by

$$(\bar{F}_h)_i := -\sum_{k,l=1}^{d-1} 2(A_h)_{id}^{kl} \frac{\partial D_{kl}(w_h - w)}{\partial y_d} - (A_h)_{dd}^{dd} \frac{\partial D_{dd}(w_h - w)}{\partial y_d}$$
$$-\sum_{j=1}^{d-1} (A_h)_{ij} \frac{\partial D(w_h - w)}{\partial y_j} - \sum_{j=1}^{d-1} B_{ij} \frac{\partial (q_h - q)}{\partial x_j} + (1 - \delta_{id}) \frac{\partial (q_h - q)}{\partial y_i} \quad (2.80)$$

for all $i = 1, \cdots, d$. Denote

$$K_h := \bar{F}_h - \bar{G}_h + J_h$$

Let us start with limit procedure for $h \to 0$. A_h, B_h are uniformly bounded, $A_h \to A$, $B_h \to B$ a.e. on $B_1^+(0)$. Next thanks to (2.48), (2.49), (2.51), (2.74), (2.75) and $|\frac{R_h}{\varepsilon_h} \|f_h(y)\|_{2;B_1^+(0)} \to 0$ we deduce

$$\|\bar{F}_h\|_{2;B_\tau^+(0)}, \ \|\bar{G}_h\|_{2;B_\tau^+(0)}, \ \|J_h\|_{2;B_\tau^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0. \quad (2.81)$$

Henceforth, $\|K_h\|_{2;B_\tau^+(0)} \to 0$ as $h \to 0$.
By multiplying d^{th} row of system (2.79) by $[1-(B_h)_{dd}]^{-1}(B_h)_{id}$ and then adding the product to i^{th} row ($i = 1, \cdots, d-1$), we obtain

$$\sum_{k=1}^{d-1} \bar{S}_{ik} \frac{\partial D_{kd} v}{\partial x_d} = \frac{1}{2}\{K_i + K_d[1 - (B_h)_{dd}]^{-1}(B_h)_{id}\}, \quad i = 1, \cdots, d-1 \quad (2.82)$$

where $\bar{S}_{ik} = (A_h)_{id}^{kd} + [1 - (B_h)_{dd}]^{-1}(B_h)_{id}(A_h)_{dd}^{kd}$.
By the same argument as in the proof of Theorem 2.3, we get that $\bar{S}$ is the positive definite matrix. Namely, we have

$$\sum_{i,k=1}^{d-1} \bar{S}_{ik}\zeta_i\zeta_k \geq \left[\frac{\lambda_0}{4} - \frac{3}{4}(d-1)\frac{\nu_0}{1-\nu_0}(\lambda_1 - \lambda_0)\right] |\zeta|^2 = \lambda|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^{d-1}, \quad (2.83)$$

where we denote $\lambda := [\frac{\lambda_0}{4} - \frac{3}{4}(d-1)\frac{\nu_0}{1-\nu_0}(\lambda_1 - \lambda_0)]$. Note that $\lambda > 0$ because of the assumptions of Lemma 2.3.
In a similar way as in the proof of Theorem 2.3 we can conclude

$$\lambda \sum_{k=1}^{d-1} \|\frac{\partial [D_{kd}(v_h - v)]}{\partial y_d}\|_{2;B_\tau^+(0)} \leq \sum_{i=1}^{d-1} \frac{1}{2}\|K_i + K_d[1 - (B_h)_{dd}]^{-1}(B_h)_{id}\|_{2;B_\tau^+(0)}.$$

It is easily seen that $\sum_{i=1}^{d-1} \frac{1}{2}\|K_i + K_d[1-(B_h)_{dd}]^{-1}(B_h)_{id}\|_{2;B_\tau^+(0)} \to 0$ as $h \to 0$, $\lambda > 0$. Hence

$$\|\frac{\partial [D_{kd}(v_h - v)]}{\partial y_d}\|_{2;B_\tau^+(0)} \longrightarrow 0 \ \text{ as } \ h \longrightarrow 0, \ \ k = 1, \cdots, d-1.$$

From the d^{th}-equation of system (2.79), it follows

$$\|\frac{\partial(p_h - p)}{\partial y_d}\|_{2;B_\tau^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus we proved that $w_h \to w$ strongly in $W^{2,2}(B_\tau^+(0))^d$, $q_h \to q$ strongly in $W^{1,2}(B_\tau^+(0)))$ as $h \to 0$. Hence

$$E^{w_h,q_h}(0,\tau) \longrightarrow E^{w,q}(0,\tau) \text{ as } h \longrightarrow 0.$$

Combining (2.59), (2.56) and using the lower semicontinuty of the L^2 norm with respect to weak convergence, we get

$$\begin{aligned}2C^*\tau^\alpha < E^{w_h,q_h}(0,\tau) \longrightarrow E^{w,q}(0,\tau) \le C^*\tau^\alpha E^{w,q}(0,1) \\ \le C^*\tau^\alpha \liminf_{h\longrightarrow 0} E^{w_h,q_h}(0,1) \le C^*\tau^\alpha,\end{aligned}$$

which is a contradiction. □

Next, we prove an iterated result

Lemma 2.4. *Let the assumptions of Lemma 2.3 hold. If $v \in W^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0)))$ be a weak solution of system (2.1) with* supp v, supp $p \subset B_1^*(0)$ *and $v = 0$ on Γ. Then for all $M > 0$, $\gamma \in (0,\alpha)$, there exists $\tau \in (0,1)$, $\varepsilon_1 > 0$ such that for any $x \in \Gamma$, $R \le dist\, \partial(x, \Omega\setminus\Gamma)$ satisfying*

$$E^{v,p}(x,R) < \varepsilon_1,\ |(\nabla v)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| \le \frac{M}{2}, \tag{2.84}$$

we have the inequalities

$$\begin{aligned}E^{v,p}(x,\tau^k R) \le& \tau^{k\gamma} E^v(x,R),\\ |(\nabla v)_{B^+_{\tau^{k-1}R}(x)}| +& |(p)_{B^+_{\tau^{k-1}R}(x)}| \le M.\end{aligned} \tag{2.85}$$

for all $k \in \mathbb{N}$.

Remark. Here we choose $\tau \in (0,1)$, $\varepsilon_1 > 0$ such that $2C^*\tau^{\alpha-\gamma} \le 1$, $\varepsilon_1 = \min\,(\varepsilon,\ \frac{M(\tau^d\kappa_d)^{\frac{1}{2}}(1-\tau^\gamma)}{C_P})$, C_P is the constant in Poincaré's inequality for both half ball and ball, κ_d is measure of unit half ball in $\mathbb{R}^d$.

Proof. We shall prove this assertion by an induction.

In fact, for $k = 1$ the second part of the assertion is exactly (2.84) and Lemma 2.3 implies that

$$E^{v,p}(x,\tau R) \le 2C^*[\tau^\alpha E^{v,p}(x,R)] \le \tau^\gamma E^{v,p}(x,R).$$

Therefore, we obtain the assertion (2.85) for $k = 1$.
Suppose that (2.85) holds for $i = 1,\cdots,k$ and we will prove that it holds also

for $i = k+1$.
Applying the Poincaré 's inequality, we get

$$\begin{aligned}
|(\nabla v)_{B^+_{\tau^i R}(x)} - (\nabla v)_{B^+_{\tau^{i-1} R}(x)}| &\leq \frac{1}{(\tau^i R)^d \kappa_d} \int_{B^+_{\tau^i R}(x)} |\nabla v - (\nabla v)_{B^+_{\tau^{i-1} R}(x)}| \\
&\leq \frac{1}{((\tau^i R)^d \kappa_d)^{\frac{1}{2}}} \|\nabla v - (\nabla v)_{B^+_{\tau^{i-1} R}(x)}\|_{2; B^+_{\tau^{i-1} R(x)}} \\
&\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1}) R^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2; B^+_{\tau^{i-1} R}(x)}, \ i = 1, \cdots, k.
\end{aligned} \tag{2.86}$$

Hence

$$\begin{aligned}
|(\nabla v)_{B^+_{\tau^k R}(x)}| &\leq \sum_{i=1}^{k} |(\nabla v)_{B^+_{\tau^i R}(x)} - (\nabla v)_{B^+_{\tau^{i-1} R}(x)}| + |(\nabla v)_{B^+_R}(x)| \\
&\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} \frac{1}{(\tau^{i-1} R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2; B^+_{\tau^{i-1} R}(x)} + |(\nabla v)_{B^+_R(x)}|.
\end{aligned} \tag{2.87}$$

By an analogous way, we have

$$\begin{aligned}
|(p)_{B^+_{\tau^k R}(x)}| &\leq \sum_{i=1}^{k} |(p)_{B^+_{\tau^i R}(x)} - (p)_{B^+_{\tau^{i-1} R}(x)}| + |(p)_{B^+_R(x)}| \\
&\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} \frac{1}{(\tau^{i-1} R)^{\frac{d-2}{2}}} \|\nabla p\|_{2; B^+_{\tau^{i-1} R}(x)} + |(p)_{B^+_R(x)}|.
\end{aligned} \tag{2.88}$$

Adding (2.87) to (2.88) and using the hypothesis of induction in (2.85) we obtain

$$\begin{aligned}
|(\nabla v)_{B^+_{\tau^k R}(x)}| + |(p)_{B^+_{\tau^k R}(x)}| &\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} E^v(x, \tau^{i-1} R) + \frac{M}{2} \\
&\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \sum_{i=1}^{k} (\tau^\gamma)^{i-1} + \frac{M}{2} \leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \frac{1}{1-\tau^\gamma} + \frac{M}{2}.
\end{aligned} \tag{2.89}$$

Thanks to the choice of ε_1 it holds $\frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \frac{1}{1-\tau^\gamma} \leq \frac{M}{2}$ so that

$$|(\nabla v)_{B^+_{\tau^k R}(x)}| + |(p)_{B^+_{\tau^k R}(x)}| \leq M.$$

Thus the assumption of Lemma 2.3 is satisfied and we deduce

$$E^{v,p}(x, \tau^{k+1} R) \leq \tau^\gamma E^{v,p}(x, \tau^k R) \leq \tau^{(k+1)\gamma} E^{v,p}(x, R).$$

□

The partial regularity of solutions to the system (2.1) with subquadratic growth in the interior of domain Ω was already stated in [19], together with the decay lemma for system (2.1). In our case, by an analogous way we also have the same conclusion of Lemma 2.3.

Let $x \in B_1^+(0), 0 < R \leq \text{dist}\, \partial(x, B_1^+(0))$ we set

$$E^{w,q}(x,R) = E_0^{w,q}(x,R) + R^\alpha := \frac{1}{R^{\frac{d-2}{2}}}\|\nabla^2 w\|^2_{2,B_R(x)} + \frac{1}{R^{\frac{d-2}{2}}}\|\nabla q\|^2_{2,B_R(x)} + R^\alpha, \tag{2.90}$$

where $0 < \alpha < 1$.

Lemma 2.5. *(Decay for nonlinear system - interior case) Let the assumptions of Lemma 2.3 hold, $(v,p) \in W^{1,2}(B_1^+(0))^d) \times L^2(B_1^+(0))$ be a weak solution of system (2.1) with* supp v, supp $p \subset B_1^*(0)$. *Then For all $M \in (0,\infty)$ and for all $\tau \in (0,1)$, there exists an $\varepsilon > 0$ so that for any $x \in B_1^+(0), R \leq$* dist $(\mathrm{x}, \partial \mathrm{B}_1^+(0))$ *satisfying*

$$E^{v,p}(x,R) < \varepsilon,\ |(v)_{B_R(x)}| + |(\nabla v)_{B_R(x)}| + |(p)_{B_R(x)}| \leq M, \tag{2.91}$$

there is a constant C^* such that

$$E^{v,p}(x,\tau R) \leq 2C^*[\tau^\alpha E^{v,p}(x,R)]. \tag{2.92}$$

Remark. We will write the same notation M, τ, ε, C^* in Lemma 2.3 and Lemma 2.5. Further, we also obtain the result in a ball with radius $\tau^k R$, where k is the arbitrary nonnegative integer number.

Lemma 2.6. *(Interior case) Let the assumptions of Lemma 2.3 hold, let $v \in W^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0)))$ with* supp v, supp $p \subset B_1^*(0)$ *is a weak solution of system (2.1). Then for all $M > 0$, $\gamma \in (0,\alpha)$, there exists $\tau \in (0,1)$, $\varepsilon_2 > 0$ so that for any $x \in B_1^+(0), R \leq$* min (dist $(x, \partial\Omega), \frac{1}{2})$ *satisfying*

$$E^{v,p}(x,R) < \varepsilon_2,\ |(v)_{B_R(x)}| + |(\nabla v)_{B_R(x)}| + |(p)_{B_R(x)}| \leq \frac{M}{4}, \tag{2.93}$$

we have for all $k \in \mathbb{N}$ the inequalities

$$\begin{aligned} &E^{v,p}(x,\tau^k R) \leq \tau^{k\gamma} E^{v,p}(x,R), \\ &|(p)_{B_{\tau^{k-1}R}(x)}| + |(\nabla v)_{B_{\tau^{k-1}R}(x)}| + |(p)_{B_{\tau^{k-1}R}(x)}| \leq M. \end{aligned} \tag{2.94}$$

Remark. Here we choose $\tau \in (0,1)$, $\varepsilon_2 > 0$ such that $2C^*\tau^{\alpha-\gamma} \leq 1$, $\varepsilon_2 =$ min $(\varepsilon, \frac{M(2\tau^d\kappa_d)^{\frac{1}{2}}(1-\tau^\gamma)}{4C_P'})$, where $C_P' = \max(\mathrm{C_P}, \mathrm{C_P^2})$.

Proof. We shall prove this lemma by an induction

In fact, for $k = 1$ the second assertion follows from (2.93). The Lemma 2.5 implies that

$$E^{v,p}(x,\tau R) \leq 2C^*[\tau^\alpha E^{v,p}(x,R)] \leq \tau^\gamma E^{v,p}(x,R),$$

where we defined γ as in the previews case .
Suppose that (2.94) holds for $i = 1, \cdots, k$ and we will prove that it holds also for $i = k+1$.
Applying Poincaré 's inequality, we get

$$\begin{aligned} |(\nabla v)_{x,\tau^i R} - (\nabla v)_{x,\tau^{i-1}R}| &\leq \frac{1}{2(\tau^i R)^d \kappa_d} \int_{B_{\tau^i R}(x)} |\nabla v - (\nabla v)_{x,\tau^{i-1}R}| \\ &\leq \frac{1}{(2(\tau^i R)^d \kappa_d)^{\frac{1}{2}}} \|\nabla v - (\nabla v)_{x,\tau^{i-1}R}\|_{2;B_{\tau^{i-1}R}(x)} \\ &\leq \frac{C_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)}, \; i = 1, \cdots, k. \end{aligned} \tag{2.95}$$

Using repeatedly (2.95) for i then

$$\begin{aligned} |(\nabla v)_{x,\tau^k R}| &\leq \sum_{i=1}^{k} |(\nabla v)_{x,\tau^i R} - (\nabla v)_{x,\tau^{i-1}R}| + |(\nabla v)_{x,R}| \\ &\leq \frac{C_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} + |(\nabla v)_{B_R(x)}|. \end{aligned} \tag{2.96}$$

By an analogous way, we have

$$\begin{aligned} |(p)_{x,\tau^k R}| &\leq \sum_{i=1}^{k} |(p)_{B_{\tau^i R}(x)} - (p)_{B_{\tau^{i-1}R}(x)}| + |(p)_{B_R(x)}| \\ &\leq \frac{C_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla p\|_{2;B_{\tau^{i-1}R}(x)} + |(p)_{B_R(x)}|. \end{aligned} \tag{2.97}$$

Next, let use the Poincaré 's inequality for estimates of $|(v)_{B_{\tau^k R}(x)}|$

$$\begin{aligned} |(v)_{x,\tau^i R} - (v)_{x,\tau^{i-1}R}| &\leq \frac{1}{2(\tau^i R)^d \kappa_d} \int_{B_{\tau^i R}(x)} |v(z) - (v)_{x,\tau^{i-1}R} \\ &\quad - (\nabla v)_{x,\tau^{i-1}R}(z-x)| \, dz + \tau^i R |(\nabla v)_{x,\tau^{i-1}R}| \\ &\leq \frac{1}{(2(\tau^i R)^d \kappa_d)^{\frac{1}{2}}} \|v(z) - (v)_{x,\tau^{i-1}R} - (\nabla v)_{x,\tau^{i-1}R}(z-x)\|_{2;B_{\tau^{i-1}R}(x)} \\ &\quad + \tau^i R |(\nabla v)_{x,\tau^{i-1}R}| \leq \frac{C_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla v - (\nabla v)_{x,\tau^{i-1}R}\|_{2;B_{\tau^{i-1}R}(x)} \\ &\quad + \tau^i R |(\nabla v)_{x,\tau^{i-1}R}| \leq \frac{C_P^2 \tau^{i-1} R}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} \\ &\quad + \tau^i R |(\nabla v)_{x,\tau^{i-1}R}|. \end{aligned} \tag{2.98}$$

Thus

$$|(v)_{x,\tau^k R}| \le \sum_{i=1}^{k} |(v)_{x,\tau^i R} - (v)_{x,\tau^{i-1}R}| + |(v)_{x,R}|$$

$$\le \sum_{i=1}^{k} \left(\frac{C_P^2 \tau^{i-1} R}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} + \tau^i R |(\nabla v)_{x,\tau^{i-1}R}| \right) + |(\nabla v)_{x,R}|. \quad (2.99)$$

Summing (2.96), (2.97), (2.99) and using the hypothesis we obtain

$$|(v)_{x,\tau^k R}| + |(\nabla v)_{x,\tau^k R}| + |(p)_{x,\tau^k R}| \le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} E^v(x, \tau^{i-1} R) + \sum_{i=1}^{k} (\tau^i R |(\nabla v)_{x,\tau^{i-1}R}|) + \frac{M}{4}$$

$$\le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \sum_{i=1}^{k} (\tau^\gamma)^{i-1} + \frac{MR}{1-\tau} + \frac{M}{4} \le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \frac{1}{1-\tau^\gamma} + \frac{MR}{1-\tau} + \frac{M}{4} \quad (2.100)$$

where $C'_P = \max\,(C_P, C_P^2)$.
Thanks to the choice of ε_2, R it holds $\frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \frac{1}{1-\tau^\gamma} \le \frac{M}{4}$, $\frac{R}{1-\tau} \le \frac{1}{2}$ so that

$$|(v)_{x,\tau^k R}| + |(\nabla v)_{x,\tau^k R}| + |(p)_{x,\tau^k R}| \le M.$$

Applying Lemma (2.5), we obtain

$$E^{v,p}(x, \tau^{k+1} R) \le \tau^\gamma E^{v,p}(x, \tau^k R) \le \tau^{(k+1)\gamma} E^{v,p}(x, R).$$

□

Next, we present the main result of this part to show that in regular points solutions belong to Hölder's space up to the boundary.

Theorem 2.5. *Let the assumptions of Lemma 2.3 hold,* $v \in W^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0)))$ *with* supp v, supp $p \subset B_1^*(0)$ *be a weak solution of system* (2.1) *and* $v = 0$ *on* Γ. *If* $\bar{x} \in \Gamma$ *is such that*

$$\liminf_{R \to 0^+} E^{v,p}(\bar{x}, R) = 0,$$
$$\limsup_{R \to 0^+} |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \infty, \quad (2.101)$$

then there exists $\delta > 0$ *such that* ∇v *and* p *are Hölder continuous on* $\overline{B_\delta^+(\bar{x})}$.

Proof. Let $\bar{x} \in \Gamma$ satisfies (2.101). Then there are $R_0 \in (0,1)$ and $M \in (0,\infty)$ such that

$$\sup_{R \le R_0} |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \frac{M}{32}.$$

Take $\gamma \in (0,\alpha)$, $\tau \in (0, \min\ (\frac{1}{2}(2C^*)^{\frac{1}{\gamma-\alpha}}, \frac{1}{2}))$ and find $\varepsilon, \varepsilon_1, \varepsilon_2$ from Lemma 2.3, Lemma 2.4, Lemma 2.5, Lemma 2.6. Choose $\varepsilon_3 > 0$ so that it satisfies inequalities

$$\begin{aligned} [(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^{\gamma} + 1]\varepsilon_3 \le \varepsilon_2, \\ \frac{C_P'}{(\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{1-\tau^\gamma}\varepsilon_3 \le \frac{M}{32}, \\ \frac{C_P'}{\sqrt{2\kappa_d}}(\frac{2}{\tau})^{\frac{d}{2}}\tau^{\frac{2-d}{2}}(\frac{1}{\tau R})^{\gamma}\varepsilon_3 \le \frac{M}{16}. \end{aligned} \tag{2.102}$$

Then (2.101) guarantees the existence of $R \in (0, \min\ (\frac{1}{4}, R_0, \text{dist}\ (\bar{x}, \partial B_1^+(0)\backslash\Gamma))$ such that

$$|E^{v,p}(\bar{x}, R)| < \varepsilon_3 \text{ and } |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \frac{M}{32}. \tag{2.103}$$

It implies that there is $\delta \in (0, \frac{R}{4})$ such that for all $x \in B_\delta^*(\bar{x}) := B_\delta(\bar{x}) \cap B_1^*(0)$ inequalities

$$|E^{v,p}(x, R)| < \varepsilon_3 \text{ and } |(v)_{B_R^+(x)}| + |(\nabla v)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| < \frac{M}{32} \tag{2.104}$$

where

$$\begin{aligned} B_R^+(x) := B_R(x) \cap B_1^+(0), \\ E^{v,p}(x,R) := \frac{1}{R^{\frac{d-2}{2}}}\|\nabla^2 w\|_{2,B_R^+(x)} + \frac{1}{R^{\frac{d-2}{2}}}\|\nabla p\|_{2,B_R^+(x)} + R^\alpha. \end{aligned} \tag{2.105}$$

Now we want to show that there exists a constant C_H such that

$$E_0^{v,p}(x,\rho) := \frac{1}{\rho^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2,B_\rho^+(x)} + \frac{1}{\rho^{\frac{d-2}{2}}}\|\nabla p\|_{2,B_\rho^+(x)} \le C_H\ \rho^\gamma \tag{2.106}$$

for all $x \in B_\delta^*(\bar{x})$ and $0 < \rho < \frac{R-\delta}{2}$.

Case 1 : $x \in B_\delta(\bar{x}) \cap \Gamma$.

We can find $k \in \mathbb{N}$ such that $\tau^{k+1}R < \rho \le \tau^k R$. Then

$$E_0^{v,p}(x,\rho) \le \tau^{\frac{2-d}{2}} E_0^{v,p}(x, \tau^k R). \tag{2.107}$$

Due to (2.104) and $\varepsilon_3 \leq \varepsilon_1$ so that we can apply Lemma 2.4 and have inequalities

$$E^{v,p}(x, \tau^k R) \leq \tau^{k\gamma} E^{v,p}(x, R) \leq \tau^{k\gamma}\varepsilon_3 \leq (\frac{1}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{2.108}$$

Thus

$$E_0^{v,p}(x, \rho) \leq \tau^{\frac{2-d}{2}} E^v(x, \tau^k R_x) \leq \tau^{\frac{2-d}{2}}(\frac{1}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{2.109}$$

Case 2 : $x \in B_\delta^+(\bar{x})$.

Denote x_0 be projection of x into Γ. It is clear that

$$x_0 \in B_\delta^+(\bar{x}),\ x_0 = (x', 0),\ x_d = |x_0 - x|,\ 2x_d < 2\delta < R - \delta.$$

(i) if $x_d \leq \rho \leq \frac{R-\delta}{2}$, it follows $2\rho \leq R - \delta$ and $B_\rho^+(x) \subset B_{2\rho}^+(x_0)$. Henceforth,

$$E_0^{v,p}(x, \rho) \leq 2^{\frac{d-2}{2}} E_0^{v,p}(x_0, 2\rho). \tag{2.110}$$

By the same way in case 1 we obtain

$$E_0^{v,p}(x_0, 2\rho) \leq \tau^{\frac{2-d}{2}}(\frac{2}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{2.111}$$

Hence

$$E_0^{v,p}(x, \rho) \leq\ [(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma \varepsilon_3]\rho^\gamma. \tag{2.112}$$

(ii) $0 < \rho < x_d$. We start with proving estimates

$$|E^{v,p}(x, x_d)| < \varepsilon_2, \tag{2.113}$$

$$|(v)_{B_{x_d}(x)}| + |(\nabla v)_{B_{x_d}(x)}| + |(p)_{B_{x_d}(x)}| < \frac{M}{4}. \tag{2.114}$$

In fact, applying the inequality (2.112) for $\rho = x_d$ we get

$$E_0^{v,p}(x, x_d) \leq\ [(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma \varepsilon_3] x_d^\gamma.$$

Thanks to choice ε_3 satisfies (2.102) we deduce the inequality (2.113)
Next we can find $l \in \mathbb{N}$ such that $\tau^{l+1}R < 2x_d \leq \tau^l R$ and denote $r_0 = \tau^l R$. Then $B_{x_d}(x) \subset B_{r_0}^+(x_0)$ and applying Poincaré 's inequality, we get

$$|(\nabla v)_{x,x_d} - (\nabla v)_{B_{r_0}^+(x_0)}| \leq \frac{1}{2x_d^d \kappa_d} \int_{B_{x_d}(x)} |\nabla v - (\nabla v)_{B_{r_0}^+(x_0)}|\, dz$$

$$\leq \frac{1}{(2x_d^d \kappa_d)^{\frac{1}{2}}} \|\nabla v - (\nabla v)_{B_{r_0}^+(x_0)}\|_{2;B_{r_0}^+(x_0)} \leq \frac{C_P r_0^{\frac{d}{2}}}{(2x_d^d \kappa_d)^{\frac{1}{2}}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_{r_0}^+(x_0)}. \tag{2.115}$$

Hence

$$
\begin{aligned}
|((\nabla v)_{x,x_d}| &\leq |(\nabla v)_{x,x_d} - (\nabla v)_{B^+_{r_0}(x_0)}| + |(\nabla v)_{B^+_{r_0}(x_0)}| \\
&\leq \frac{C_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac{d}{2}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B^+_{r_0}(x_0)} + |(\nabla v)_{B^+_{r_0}(x_0)}|.
\end{aligned} \tag{2.116}
$$

By an analogous way, we have

$$
\begin{aligned}
|((p)_{x,x_d}| &\leq |(p)_{x,x_d} - (p)_{B^+_{r_0}(x_0)}| + |(p)_{B^+_{r_0}(x_0)}| \\
&\leq \frac{C_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac{d}{2}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla p\|_{2;B^+_{r_0}(x_0)} + |(p)_{B^+_{r_0}(x_0)}|.
\end{aligned} \tag{2.117}
$$

Next we have following estimates for $|(v)_{x,x_d}|$

$$
\begin{aligned}
|(v)_{x,x_d} - (v)_{B^+_{r_0}(x_0)}| \leq \frac{1}{2x_d^d\kappa_d} \int_{B_{x_d}(x)} & |v(z) - (v)_{B^+_{r_0}(x_0)} \\
& - (\nabla v)_{B^+_{r_0}(x_0)}(z-x)| \, dz + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| \\
\leq \frac{1}{(2(x_d)^d\kappa_d)^{\frac{1}{2}}} & \|v(z) - (v)_{B^+_{r_0}(x_0)} - (\nabla v)_{B^+_{r_0}(x_0)}(z-x)\|_{2;B^+_{r_0}(x_0)} \\
+ x_d|(\nabla v)_{B^+_{r_0}(x_0)}| & \leq \frac{C_P r_0^{\frac{d}{2}}}{(2x_d^d\kappa_d)^{\frac{1}{2}}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla v - (\nabla v)_{B^+_{r_0}(x_0)}\|_{2;B^+_{r_0}(x_0)} \\
+ x_d|(\nabla v)_{B^+_{r_0}(x_0)}| \leq \frac{C_P^2 r_0^{\frac{d}{2}} r_0}{(2x_d^d\kappa_d)^{\frac{1}{2}}} & \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}|.
\end{aligned} \tag{2.118}
$$

Thus

$$
\begin{aligned}
|(v)_{x,x_d}| &\leq |(v)_{x,x_d} - (v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}| \\
&\leq C_P^2(\frac{r_0^{d+2}}{2x_d^d\kappa_d})^{\frac{1}{2}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}| \\
&\leq \frac{C_P^2}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac{d}{2}} \frac{1}{r_0^{\frac{d-2}{2}}} \|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}|.
\end{aligned} \tag{2.119}
$$

Summing (2.116), (2.117), (2.119) and using (2.104), we obtain

$$
\begin{aligned}
|(v)_{x,x_d}| + |(\nabla v)_{x,x_d}| + |(p)_{x,x_d}| &< \frac{C'_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac{d}{2}} E_0^{v,p}(x_0, r_0) \\
+x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + \Big(|(v)_{B^+_{r_0}(x_0)}| &+ |(\nabla v)_{B^+_{r_0}(x_0)}| + |(p)_{B^+_{r_0}(x_0)}|\Big).
\end{aligned} \tag{2.120}
$$

As $x = x_0, \rho = r_0$ the inequality (2.109) gives

$$
E_0^{v,p}(x_0, r_0) \leq \tau^{\frac{2-d}{2}}(\frac{1}{\tau R})^{\gamma}\varepsilon_3.
$$

Therefore, from (2.102) we deduce

$$\frac{C'_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac{d}{2}}E_0^{v,p}(x_0,r_0) \le \frac{C'_P}{\sqrt{2\kappa_d}}(\frac{2}{\tau})^{\frac{d}{2}}E_0^{v,p}(x_0,r_0) \le \frac{M}{16}.$$

Thanks to (2.104), (2.102) it holds

$$|E^{v,p}(x_0,R)| < \varepsilon_3 \ , \ |(v)_{B_R^+(x_0)}| + |(\nabla v)_{B_R^+(x_0)}| + |(p)_{B_R^+(x_0)}| < \frac{M}{32}$$

$$\text{and} \ \frac{C'_P}{(\tau^d\kappa_d)^{\frac{1}{2}}}\varepsilon_3\frac{1}{1-\tau^\gamma} \le \frac{M}{32}.$$

By the similar way in Lemma 2.6, we have for all $k \in \mathbb{N}$ the inequalities

$$|(p)_{B^+_{\tau^{k-1}R}(x_0)}| + |(\nabla v)_{B^+_{\tau^{k-1}R}(x_0)}| + |(p)_{B^+_{\tau^{k-1}R}(x_0)}| \le \frac{M}{8}.$$

Hence,

$$|(p)_{B^+_{r_0}(x_0)}| + |(\nabla v)_{B^+_{r_0}(x_0)}| + |(p)^+_{B_{r_0}(x_0)}| \le \frac{M}{8} \ \text{and} \ x_d|(\nabla v)_{B^+_{r_0}(x_0)}| \le \frac{M}{16}.$$

From the above estimates we can conclude the assertion (2.114) holds.

Return to prove the inequality (2.106). There is $k \in \mathbb{N}$ such that $\tau^{k+1}x_d < \rho \le \tau^k x_d$. The assertion (2.114) holds so that we can apply Lemma 2.6 and obtain

$$E_0^{v,p}(x,\rho) \le \tau^{\frac{d-2}{2}}E_0^{v,p}(x,\tau^k x_d) \le \tau^{\frac{2-d}{2}}E^{v,p}(x,\tau^k x_d) \le \tau^{\frac{2-d}{2}}\tau^{k\gamma}E^{v,p}(x,x_d). \tag{2.121}$$

On the other hand, applying (2.112) for $\rho := x_d$ we have

$$E^{v,p}(x,x_d) \le \ [(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma\varepsilon_3]x_d^\gamma + x_d^\gamma.$$

Thus

$$E_0^{v,p}(x,\rho) \le \tau^{\frac{2-d}{2}}[(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma\varepsilon_3 + 1]\tau^{k\gamma}x_d^\gamma$$
$$\le \tau^{\frac{2-d}{2}}[(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma\varepsilon_3 + 1]\tau^{-\gamma}\rho^\gamma. \tag{2.122}$$

Summarize that the inequality (2.106) holds for all $x \in B_\delta^+(\bar{x})$, where C_H given

$$C_H = \max\ (\tau^{\frac{2-d}{2}}(\frac{1}{\tau R})^\gamma\varepsilon_3, (\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma\varepsilon_3,$$
$$\tau^{\frac{2-d}{2}}[(\frac{2}{\tau})^{\frac{d-2}{2}}(\frac{2}{\tau R})^\gamma\varepsilon_3 + 1]\tau^{-\gamma}). \tag{2.123}$$

Therefore, $\nabla^2 v \in L^{2,d-2+\gamma}(B_\delta^+(x))^{d^3}$ and $\nabla p \in L^{2,d-2+\gamma}(B_\delta^+(x))^d$. It implies that $\nabla v \in \mathcal{L}^{2,d+\gamma}(B_\delta^+(x))^{d^2} \simeq C^{\frac{\gamma}{2}}\overline{(B_\delta^+(x)}^{d^2}$ and $p \in \mathcal{L}^{2,d+\gamma}(B_\delta^+(x)) \simeq C^{\frac{\gamma}{2}}\overline{(B_\delta^+(x)}$. Thus prove that ∇v, p are Hölder continuous on $\overline{B_\delta^+(x)}$. □

Denote

$$\begin{aligned}
\Sigma_1^v &= \{x \in \Gamma;\ \liminf_{R\to 0+} \frac{1}{R^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_R^+(x)} > 0\},\\
\Sigma_1^p &= \{x \in \Gamma;\ \liminf_{R\to 0+} \frac{1}{R^{\frac{d-2}{2}}} \|\nabla p\|_{2;B_R^+(x)} > 0\},\\
\Sigma_2^v &= \{x \in \Gamma;\ \limsup_{R\to 0+} |(v)_{2;B_R^+(x)}| + |(\nabla v)_{2;B_R^+(x)}| = \infty\},\\
\Sigma_2^p &= \{x \in \Gamma;\ \limsup_{R\to 0+} |(p)_{2;B_R^+(x)}| = \infty\}.
\end{aligned} \tag{2.124}$$

Applying Lemma 0.6 and Lemma 0.7, for any $\epsilon > 0$ it holds

$$\begin{aligned}
H^{d-2}(\Sigma_1^v) = H^{d-2}(\Sigma_1^p) = 0,\\
H^{d-2+\epsilon}(\Sigma_2^v) = H^{d-2+\epsilon}(\Sigma_2^p) = 0.
\end{aligned} \tag{2.125}$$

Then we have

Theorem 2.6. *Let the assumptions in Theorem 2.5 are satisfied. Then there exists* $\Gamma_0 \subset \Gamma$ *relatively open in* Γ *so that* $H_{d-2+\epsilon}(\Gamma - \Gamma_0) = 0$ *for any* $\epsilon > 0$*. Moreover, for any* $x \in \Gamma_0$ *there exists* $\delta > 0$ *so that* ∇v *and* p *are Hölder continuous on* $\overline{B_\delta^+(x)}$*.*

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