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Navier's Slip and Evolutionary Navier-Stokes-Fourier-Like Systems with Pressure, Shear-Rate and Temperature Dependent Viscosity

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Contents

Abstract

There are plenty of experimental studies suggesting to model behavior of viscous materials as incompressible fluids with the viscosity depending on the shear-rate, the temperature and the mean normal stress (the pressure). In this thesis we investigate mathematical properties of internal unsteady three-dimensional flows of such fluids subject to Navier's slip at the boundary. We establish the large-data and long-time existence of weak solution provided that the viscosity and heat conductivity depend on the shear rate, temperature and the pressure in a suitably specified manner. Note that specific relationship however includes the classical Navier-Stokes equations and power-law fluid (with power law index $r-2, r \leq 2$) as special cases.

The achieved results are based on two observation. First, although for smooth functions completely equivalent, in the context of weak solutions the formulation of the balance of total energy share better mathematical properties than the equation for the temperature, balancing the internal energy. Second, for evolutionary models, again in the context of weak solutions, Navier's slip boundary conditions are well suitable to defining the global pressure needed if the viscosity is pressure-dependent. Except for the special case, the Navier-Stokes equations, when one identifies the Navier-Stokes system with the evolutionary Stokes system, is open how to define the pressure globally for no-slip boundary conditions.

1 Introduction

1.1 A formulation of the problem and its importance in fluid mechanics

The Navier-Stokes-Fourier equations (NSEs in short) represent the key reference model in fluid mechanics, both from the point of view of modeling in continuum mechanics and from the point of view of theoretical analysis. In continuum mechanics, non-Newtonian fluids are those whose behavior cannot be captured by the NSEs. In mathematical analysis, the question of the well-posedness of the NSEs is set-up as the most fundamental problem of the theory of partial differential equations.

NSEs for a homogeneous incompressible fluid are usually written as

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\nu \mathbf{D}(\boldsymbol{v})) &= -\nabla p + \boldsymbol{f}, \\ \theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \operatorname{div}(k\nabla\theta) &= \nu|\mathbf{D}(\boldsymbol{v})|^2, \end{aligned} \tag{1.1}$$

where $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity, p is the pressure, θ is the temperature, $\boldsymbol{f}$ are given specific body forces; $\mathbf{D}(\boldsymbol{v})$ denotes the symmetric part of the velocity gradient $\nabla \boldsymbol{v}$, which means that $2\mathbf{D}(\boldsymbol{v}) = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T$. The material properties of the fluid are encoded into the viscosity ν and the heat conductivity k. For a Navier-Stokes fluid, the viscosity and the heat conductivity are equal to positive constants. Consequently, $-\operatorname{div}(\nu \mathbf{D}(\boldsymbol{v})) = -\frac{\nu}{2}\triangle \boldsymbol{v}$ and $\operatorname{div}(k\nabla\theta) = k\triangle\theta$.

Note that if this is the case one can start solving only first two equations in (1.1) and then find the temperature θ as a solution of $(1.1)_3$.

The most of the mathematical studies of NSEs, considering flows in a fixed bounded domain $\Omega \subset \mathbb{R}^3$, deal with internal flows where

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } (0,T) \times \partial\Omega \qquad (\partial\Omega \text{ denotes the boundary of } \Omega) \tag{1.2}$$

and these flows are subject to the no-slip boundary conditions

$$\boldsymbol{v}_\tau := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} = \mathbf{0} \quad \text{on } (0,T) \times \partial\Omega. \tag{1.3}$$

Note that $\boldsymbol{v}_\tau = \boldsymbol{v}$ if (1.2) holds.

The task to provide a suitable mathematical description for flows of (incompressible and compressible) fluids in terms of differential equations and to find the same for interactions of the fluid with the boundary was addressed and intensively studied by such scientists as Newton, Euler, Coulomb, Poisson, Navier, St. Venant and Stokes.

Stokes [30] while formulating (1.1)-(1.3) was very careful about the validity and applicability of the involved assumptions. Particularly, he thoroughly discusses the following assumptions

(**A1**) ν is independent of the pressure,

(**A2**) the velocity adheres to the boundary, after he included them into the Navier-Stokes model.

In this thesis we will relax both the assumptions **(A1)** and **(A2)**. Before doing so, we formulate the basic balance laws of continuum physics for incompressible homogeneous fluids in their differential forms. Denoting the Cauchy stress by $\mathbf{T}$, the specific internal energy by e, the heat flux by $\boldsymbol{q}$ and the constant density by $\rho^*(>0)$, the basic balance equations capturing flows of incompressible fluids take the form

$$\operatorname{div}\boldsymbol{v} = 0, \tag{1.4}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\frac{\mathbf{T}}{\rho^*} = \boldsymbol{f}, \tag{1.5}$$

$$(e + \frac{|\boldsymbol{v}|^2}{2})_{,t} + \operatorname{div}\left((e + \frac{|\boldsymbol{v}|^2}{2})\boldsymbol{v}\right) - \operatorname{div}\frac{\boldsymbol{q}}{\rho^*} = \operatorname{div}\left(\frac{\mathbf{T}}{\rho^*}\boldsymbol{v}\right) + \boldsymbol{f}\cdot\boldsymbol{v}. \tag{1.6}$$

In the classical Navier-Stokes theory, one assumes

$$\frac{\mathbf{T}}{\rho^*} = -p\mathbf{I} + \nu\mathbf{D}(\boldsymbol{v}) \qquad \text{with } \nu\in\mathbb{R}^+, \tag{1.7}$$

$$\frac{\boldsymbol{q}}{\rho^*} = k\nabla\theta \qquad \text{with } k\in\mathbb{R}^+ \tag{1.8}$$

and

$$e = c_V\theta \qquad \text{with } c_V \equiv 1 \text{ (for simpklicity)}. \tag{1.9}$$

Note that (1.7) implies that (1.4), (1.5) and (1.7) can be solved independently of (1.9), and after doing so, (1.6), (1.8) and (1.9) can be used to compute the temperature.

As said above, we assume the validity of (1.9) in this thesis, but we relax the assumption **(A1)**, i.e., (1.7), and rather consider an incompressible fluid with the viscosity depending on the pressure, the temperature and the shear rate, i.e., the Cauchy stress $\mathbf{T}$ takes the form

$$\mathbf{T} = -p\mathbf{I} + \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}). \tag{1.10}$$

We also relax the assumption on the heat conductivity . Instead of constant k, i.e., (1.8), we consider the model where the heat flux $\boldsymbol{q}$ takes the form

$$\boldsymbol{q} = -k(\theta,p,|\mathbf{D}(\boldsymbol{v})|^2)\nabla\theta. \tag{1.11}$$

Models of the type (1.9), (1.10) and (1.11) are used in various engineering areas; elastohydrodynamics or mechanics of granular or visco-elastic materials where their deformation are subject to high pressures can serve as appropriate examples. We refer the reader to [17], [21] and [13] for more details related to (1.10) and for a list of references confirming experimentally the pressure-viscosity, temperature-viscosity, temperature-pressure-shear-heat conductivity relationships and their relevance to the assumptions of incompressibility, as well.

The assumptions on the structure of $\mathbf{T}$ specified above include as special cases both the Navier-Stokes fluid (ν is constant), the fluid with shear-rate dependent viscosity, where

$$\mathbf{T} = -p\mathbf{I} + \nu(|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \tag{1.12}$$

and the fluid with temperature dependent viscosity, where

$$\mathbf{T} = -p\mathbf{I} + \nu(\theta)\mathbf{D}(\boldsymbol{v}). \tag{1.13}$$

Note that the popular power-law-like fluid $\nu(\mathbf{D}) = |\mathbf{D}|^{r-2}$ or its non-degenerate variant $\nu(\mathbf{D}) = (A + |\mathbf{D}|^2)^{\frac{r-2}{2}}$ where $A > 0$, $(r-2)$ is the power-law index, fall into (1.12) as particular cases. Also note that the popular model where $\nu(\theta) = \exp(\frac{1}{\theta})$ is included in (1.13).

Regarding the second Stokes assumption **(A2)**, we relax it as well assuming that the fluid-boundary interactions are well-described by the Navier's slip boundary conditions:

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ and } (\mathbf{T}\boldsymbol{n})_\tau + \alpha \boldsymbol{v}_\tau = \mathbf{0} \quad (\alpha \geq 0). \tag{1.14}$$

Note that letting $\alpha \to 0_+$ we obtain the so-called no-stick (slippery) boundary conditions (and this case is included into our analysis). Note also, that the limit $\alpha \to +\infty$ formally leads ($(1.14)_2$ is multiplied by $\frac{1}{\alpha}$ first) to the no-slip boundary conditions (1.3). This case is however omitted from our studies in what follows.

It remains to specify which types of boundary condition are relevant for temperature θ. The most physical situations are well described by the conditions

$$\begin{aligned} \theta &= \theta_1 \quad && \text{on } \Gamma_1, \\ \nabla \boldsymbol{q} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{1.15}$$

where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$. For simplicity we consider in this paper that $|\Gamma_1| = 0$. The condition (1.15) then reduces to

$$\boldsymbol{q} \cdot \boldsymbol{n} = k \nabla\theta \cdot \boldsymbol{n} = 0 \quad \text{ on } \partial\Omega. \tag{1.16}$$

There is one point concerning the general formulation of the balance laws (1.5) and (1.6) that deserves our attention. If $\boldsymbol{v}$ is sufficiently smooth, we can take scalar product of (1.5) with $\boldsymbol{v}$ and obtain

$$\left(\frac{|\boldsymbol{v}|^2}{2}\right)_{,t} + \operatorname{div}\left(\frac{|\boldsymbol{v}|^2}{2}\boldsymbol{v}\right) + \frac{\mathbf{T}\cdot\mathbf{D}(\boldsymbol{v})}{\rho^*} = \boldsymbol{f}\cdot\boldsymbol{v} + \operatorname{div}\left(\frac{\mathbf{T}}{\rho^*}\boldsymbol{v}\right). \tag{1.17}$$

Subtracting (1.17) from (1.6), and using (1.9) we obtain the equation for internal energy

$$\theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \operatorname{div}(k\nabla\theta) = \frac{\mathbf{T}\cdot\mathbf{D}(\boldsymbol{v})}{\rho^*}. \tag{1.18}$$

It seems that (1.18) is equivalent to (1.6). But as we assumed this is true only if $\boldsymbol{v}$ is *sufficiently* smooth. In general, because we use the concept of weak solutions we do not have *sufficiently* smooth $\boldsymbol{v}$ to take scalar product of (1.5) with $\boldsymbol{v}$ and we cannot conclude that (1.18) and (1.6) are equivalent. The best information that we are usually able to show is only the following so-called *kinetic energy* inequality

$$\left(\frac{|\boldsymbol{v}|^2}{2}\right)_{,t} + \operatorname{div}\left(\frac{|\boldsymbol{v}|^2}{2}\boldsymbol{v}\right) + \frac{\mathsf{T}\cdot\mathsf{D}(\boldsymbol{v})}{\rho^*} \leq \boldsymbol{f}\cdot\boldsymbol{v} + \operatorname{div}\left(\frac{\mathsf{T}}{\rho^*}\boldsymbol{v}\right). \tag{1.19}$$

And consequently, we have to change the equality sign in (1.18) to inequality sign to obtain

$$\theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \operatorname{div}(k\nabla\theta) \geq \frac{\mathsf{T}\cdot\mathsf{D}(\boldsymbol{v})}{\rho^*}. \tag{1.20}$$

From the physician point of view, it also seems to be reasonable to assume validity of (1.6) (the equation for global energy) instead validity of (1.18) (the equation for internal energy) because in general, we should always prefer balance of global energy to balance of one part of the energy. Also note that the same approach (taking into account the equation for global energy (1.6) and assuming only inequality (1.20)) was first used in [12] for incompressible Navier-Stokes equations with temperature dependent viscosity and it has very similar meaning to approach that was developed in [11] where the author used the concept of entropy inequality instead of energy equality (for compressible Navier-Stokes equations).

Our problem then takes the following form: We would like to find a triple $(\boldsymbol{v},\theta,p)$ solving the following problem $(\mathcal{P})$.

$$(\mathcal{P})\qquad \begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\nu(\cdot)\mathsf{D}(\boldsymbol{v}) + \nabla p &= \boldsymbol{f}\\
\operatorname{div}\boldsymbol{v} &= 0\\
\left.\begin{aligned}
\left(\theta + \frac{|\boldsymbol{v}^2|}{2}\right)_{,t} + \operatorname{div}\left(\boldsymbol{v}\left(\frac{|\boldsymbol{v}|^2}{2} + \theta + p\right)\right) - \boldsymbol{f}\cdot\boldsymbol{v}\\
- \operatorname{div}\left(k(\cdot)\nabla\theta\right) - \operatorname{div}\left(\nu(\cdot)\mathsf{D}(\boldsymbol{v})\boldsymbol{v}\right)
\end{aligned}\right\} &= 0\\
\theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \operatorname{div}\left(k(\cdot)\nabla\theta\right) - \nu(\cdot)|\mathsf{D}(\boldsymbol{v})|^2 &\geq 0
\end{aligned}\right\} \quad \text{in } \Omega\times(0,T),\\
&\left.\begin{aligned}
\left(\nu(\cdot)\mathsf{D}(\boldsymbol{v})\boldsymbol{n}\right)_{\tau} + \alpha\boldsymbol{v}_{\tau} &= 0\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0\\
k(\cdot)\nabla\theta\cdot\boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \partial\Omega\times(0,T),\\
&\int_\Omega p(x,t)\,dx = 0 \qquad \text{a. a. } t\in(0,T),\\
&\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0 \qquad \text{in } \Omega,\\
&\theta(\cdot,0) = \theta_0 \qquad \text{in } \Omega,
\end{aligned}$$

where we considered that

$$
\begin{aligned}
k(\cdot) &:= k(\theta, p|\mathbf{D}(\boldsymbol{v})|^2),\\
\nu(\cdot) &:= \nu(\theta, p|\mathbf{D}(\boldsymbol{v})|^2).
\end{aligned}
$$

1.2 Basic notations

We write that $\Omega \in \mathcal{C}^{0,1}$ if $\Omega \subseteq \mathbb{R}^d, d \geq 2$ is a bounded open connected set with Lipschitz boundary $\partial\Omega$. If in addition the boundary $\partial\Omega$ is locally $\mathcal{C}^{1,1}$ mapping then we write $\Omega \in \mathcal{C}^{1,1}$.

Let $r \in [1, \infty]$. The Lebesgue spaces $L^r(\Omega)$ equipped with the norm $\|\cdot\|_r$ and the Sobolev spaces $W^{1,r}(\Omega)$ with the norm $\|\cdot\|_{1,r}$ are defined in the standard way. If X is a Banach space then

$$
X^d = \underbrace{X \times X \times \ldots \times X}_{d-\text{times}}.
$$

The trace of Sobolev function u is denoted through $\operatorname{tr} u$, if $\boldsymbol{u} \in (W^{1,r}(\Omega))^d$ then $\operatorname{tr} \boldsymbol{u} := (\operatorname{tr} u_1, \ldots, \operatorname{tr} u_d)$. For our purposes we introduce the subspaces of vector-valued Sobolev functions which have zero normal part on the boundary. Let $1 \leq q \leq \infty$. We define

$$
\begin{aligned}
W^{1,q}_{\boldsymbol{n}} &:= \overline{\left\{\boldsymbol{v} \in (\mathcal{C}^\infty(\Omega))^d \cap (\mathcal{C}(\overline{\Omega}))^d; \operatorname{tr} \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\right\}}^{\|\cdot\|_{1,q}},\\
W^{1,q}_{\boldsymbol{n},\operatorname{div}} &:= \left\{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}}; \operatorname{div} \boldsymbol{v} = 0\right\},\\
L^q_{\boldsymbol{n}} &:= \overline{\left\{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n},\operatorname{div}}\right\}}^{\|\cdot\|_q}.
\end{aligned}
$$

We also introduce the notation for the dual spaces:

$$
W^{-1,q'}_{\boldsymbol{n}} := \left(W^{1,q}_{\boldsymbol{n}}\right)^* \text{ and } W^{-1,q'}_{\boldsymbol{n},\operatorname{div}} := \left(W^{1,q}_{\boldsymbol{n},\operatorname{div}}\right)^*.
$$

All above introduced spaces are Banach spaces. Moreover, if $1 < q < \infty$ then they are also reflexive and separable. For $r, q \in [1, +\infty]$, we also introduce relevant spaces of a Bochner-type, namely,

$$
\begin{aligned}
X^{r,q} &:= \{\boldsymbol{u} \in L^r(0,T; W^{1,r}_{\boldsymbol{n}}) \cap L^q(0,T; L^q(\Omega)^d), \operatorname{tr} \boldsymbol{u} \in L^2(0,T; (L^2(\partial\Omega))^d)\},\\
X^{r,q}_{\operatorname{div}} &:= \{\boldsymbol{u} \in X^{r,q}, \operatorname{div} \boldsymbol{u} = 0\},\\
Y^{r,q} &:= \{\boldsymbol{u} \in L^r(0,T; W^{1,r}_{\boldsymbol{n}}); \operatorname{div} \boldsymbol{v} \in L^q(0,T; L^q(\Omega)); \operatorname{tr} \boldsymbol{u} \in L^2(0,T; (L^2(\partial\Omega))^d\}.
\end{aligned}
$$

For simplicity we will often write (a, b) instead $\int_\Omega ab\; dx$ whenever the integral makes a good sense. We often also do not explicitly write the subscript symbol in the duality, i.e., for $a \in X$ and $b \in X^*$ we use the symbol

$$
\langle a, b\rangle := \langle a, b\rangle_{X,X^*}.
$$

We also introduce the so-called Helmholtz decomposition. For $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}}$, let $g^{\boldsymbol{v}}$ be the solution of the problem

$$\begin{aligned} \triangle g^{\boldsymbol{v}} &= \operatorname{div} \boldsymbol{v} && \text{in } \Omega, \\ \nabla g^{\boldsymbol{v}} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\ \int_\Omega g^{\boldsymbol{v}} \, dx &= 0. \end{aligned} \tag{1.21}$$

Then we set

$$\boldsymbol{v}_{\mathrm{div}} := \boldsymbol{v} - \nabla g^{\boldsymbol{v}},$$

that is of course equivalent to

$$\boldsymbol{v} := \boldsymbol{v}_{\mathrm{div}} + \nabla g^{\boldsymbol{v}} \quad \text{(Helmholtz decomposition)}.$$

Note that from the definition of $\boldsymbol{v}_{\mathrm{div}}$ it is clear that $\operatorname{div} \boldsymbol{v}_{\mathrm{div}} = 0$ a.e. Moreover, Lemma B.1 on the existence and regularity of the solution of (1.21) implies that

$$\|g^{\boldsymbol{v}}\|_{2,q} \le C_{reg}(\Omega, q)\| \operatorname{div} \boldsymbol{v}\|_q \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_{1,q} \le (C_{reg}(\Omega, q) + 1)\|\boldsymbol{v}\|_{1,q}, \tag{1.22}$$

$$\|g^{\boldsymbol{v}}\|_{1,s} \le C(\Omega, s)\|\boldsymbol{v}\|_s \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_s \le (C(\Omega, s) + 1)\|\boldsymbol{v}\|_s. \tag{1.23}$$

1.3 Structure of the thesis, main results and bibliography

This thesis is organized as follows. In Section 2 we consider a generalization of the NSEs where the viscosity and the heat conductivity depend only on the temperature. Section 3 is devoted to the case when viscosity depends on the pressure and the shear rate. Finally, in Section 4 we solve full system, i.e., the viscosity and the heat conductivity can depend on the pressure, the temperature and the shear rate. Some properties of the viscosity that follow from structural assumptions are discussed in Appendix A while in Appendix B we describe several theorems from functional analysis used in the text.

In this thesis we restrict ourselves three-dimensional flows only. In each section we first describe assumptions on viscosity and heat conductivity and we give several examples of them that fulfil these assumptions. Then we precisely formulate the notation of that what we mean by weak solution and then formulate the main theorem of that section. The rest of sections are devoted to the proof of these theorems.

We wish to note that all results presented in this thesis seem to be new. In Section 2 we give a detailed proof of the following result:

Theorem 2.1: Let $\Omega \in \mathcal{C}^{1,1}$. Let $\boldsymbol{f} \in L^2(0, T; W^{-1,2}_{\boldsymbol{n}})$. Let ν, k be continuous, strictly positive bounded functions. Then there exists weak solution to the problem $(\mathcal{P})$.

Next, in Section 3 we extend our theory for the models where the viscosity depends on the pressure and the shear-rate. We will consider the models of the type[1]

$$\nu(p, |\mathbf{D}|^2) \sim \nu_0(1 + \gamma(p) + |\mathbf{D}|^2)^{\frac{r-2}{2}}, \tag{1.24}$$

where the function γ is smooth, non-negative, bounded by suitable constant and satisfies $|\gamma'(p)| \leq \gamma_0 \ll 1$ with sufficiently small constant γ_0. For these types of models there is proved in Section 3 the following theorem:

Theorem 3.1: Let $\Omega \in \mathcal{C}^{1,1}$. Let ν satisfy (1.24) with

$$2 > r > \frac{8}{5},$$

and γ_0 being sufficiently small. Then there exists weak solution to the problem $(\mathcal{P})$.

Finally, Section 4 is devoted to study the class of fluids that can be describe by the following relations

$$\begin{aligned} \nu(p, \theta, |\mathbf{D}|^2) &\sim \nu_1(\theta)(1 + \gamma(p, \theta) + |\mathbf{D}|^2)^{\frac{r-2}{2}}, \\ k_1\theta^\beta \leq k(p, &\theta, |\mathbf{D}|^2) \leq k_2\theta^\beta, \end{aligned} \tag{1.25}$$

where again the function γ is supposed to be nonnegative, smooth, bounded and there holds for all θ, p that $\left|\frac{\partial\gamma(\theta,p)}{\partial p}\right| \leq \gamma_0 \ll 1$. Under this hypothesis on the structure of the viscosity and the heat conductivity[2] we will prove in Section 4 the following result:

Theorem 4.1: Let $\Omega \in \mathcal{C}^{1,1}$. Let $\boldsymbol{f} \in L^{r'}(0, T; W_{\boldsymbol{n}}^{-1,r'})$. Let ν, k satisfy (1.25) with

$$\frac{9}{5} < r < 2,$$

$$\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$$

and γ_0 being sufficiently small. Then there exists weak solution to the problem $(\mathcal{P})$.

Note that for simplicity we give complete (rigorous) proof of the existence theorem in Section 2 and then in Sections 3 and 4 we omit the proof of those things that will be clear from the previous Section 2 (for example we will not prove the attainment of initial condition).

We also wish to make several bibliographical remarks. Mathematical analysis of the incompressible Navier-Stokes equations is one of the most celebrated problem in the theory of partial differential equations. The first attempt is due to Leray [19] who established the existence of weak solution to Navier-Stokes

[1]For precise formulation of assumptions on the viscosity see Section 3, where they are detailed described.

[2]The exact assumptions on the viscosity and the heat conductivity are given in Section 4.

equation with constant viscosity for Cauchy problem. The extension also for no-slip boundary conditions was done by Hopf [16]. The full model, i.e., the model with constant viscosity completed by the equation for temperature is for example discussed in [20].

The model with temperature dependent viscosity were usually supposed with the equations for internal energy (1.18). Except it is not physically to consider this equation as it was already explain in preceding subsection it also makes the problems from the mathematical point of view because the *bad* term $\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2$ that appears on the right-hand side of (1.18) belongs only to the Lebesgue space L^1 that is not proper space for weak convergence that is usually used for the proof of the existence of the global solution. First method that does not use the equation (1.18) and changes it to the inequality (1.20) (for incompressible fluids) is described in [25]. The author completed the system of equations by so-called equation for total energy of the fluid motion that has the form

$$\frac{d}{dt}\int_\Omega \frac{|\boldsymbol{v}(x,t)|^2}{2} + \theta(x,t)\,dx = \int_\Omega \boldsymbol{f}(x,t)\cdot\boldsymbol{v}(x,t)\,dx \tag{1.26}$$

and was able to establish the existence of weak solution that satisfies (1.4)-(1.5), (1.20) and (1.26) with no-slip boundary condition for velociy. Note that very similar procedure was first described by Feireisl [11] for compressible flows. Also note that for no-slip boundary conditions the equation (1.6) (after integration over Ω) directly implies (1.26) and if $\boldsymbol{v}$ is sufficiently smooth then the equations (1.26) and (1.6) are equivalent. The first work that establishes the existence of weak solution to our problem $(\mathcal{P})$ subject to space-periodic boundary conditions is by Feireisl and Málek [12] and the existence theorem in Section 2 is generalization of the case that has been studied in [12] and we get existence of weak solution also for bounded domain. Note that the key-role in this generalization plays the fact that we are able to construct the pressure that is possible if one works with Navier's slip boundary conditions. Also note that for no-slip boundary conditions the existence of weak solution is still open and the reason is that we do not know (up to now) how to construct the pressure because it is needed in the equation (1.6) and cannot be omitted by using divergence-free test functions as it is usual for incompressible fluids.

The fluids of the type (1.24) with $\gamma(p) \equiv 0$ or its degenerate variant (the so-called power law fluid)

$$\nu(|\mathbf{D}|^2) := \nu_0|\mathbf{D}|^{r-2}$$

are other important examples of real fluids. The existence of weak solution for such fluid was first studied by Ladyzhenskaya and she established the existence of weak solution for parameters $r \geq \frac{11}{5}$ for no-slip boundary conditions by using monotone operator theory (see [18]). Next improvement for the range of parameters r was established by Málek, Nečas and Růžička [23] and they were able to prove the existence of weak solution to the problem $(\mathcal{P})$ (with hypothesis (1.24), $\gamma_0 \equiv 0$) subject to space periodic boundary conditions for all $r > \frac{9}{5}$ by using regularity technique. For no-slip boundary conditions the existence of weak solution is established in [24]. For $r > \frac{8}{5}$ and no-slip boundary conditions the existence of weak solution was established by Wolf [33] and the proof is based on the using of L^∞ truncation test functions. Very recently there were established the existence of weak solution for all $r > \frac{6}{5}$ by using the method of Lipsichtz truncation function for no-slip boundary conditions (see [9]).

The first results on the solution for fluids with viscosity depending on the pressure were proved by Renardy [26] and Gazzola [15] but under very restrictive condition on viscosity and the authors established only local-in-time and small-data existence of solution. The first global-in-time and large-data existence result for fluids with viscosity depending on the pressure and the shear rate was established by Málek, Nečas and Rajagopal [21] and they established the existence of weak solution for fluids described by (1.24) subjected to the space periodic boundary conditions for parameter $r > \frac{9}{5}$. The existence of (global) weak solution for fluid of the type (1.24) in bounded domain was first established in [6]. Theorem 3.1 generalizes the results presented in [6] such that not so restrictive assumptions on the viscosity is needed. Note that the key-role in this generalization plays the decomposition of the pressure that is possible if one works with Navier's slip boundary conditions and that will be clearly described in Section 3.

Finally, in Section 4, the full system is studied. It seems that Theorem 4.1 is the first result that establishes the existence of global weak solution to the problem $(\mathcal{P})$ with viscosity and heat conductivity depending on the temperature, the pressure and the shear rate. Moreover, we do not need to restrict ourselves onto space periodic problem and we are able to prove our theorem for bounded domain assuming Navier's boundary conditions. There are several works that deal with shear rate and temperature dependent viscosity (see for example [4] and [7] where the fluids of the type (1.25) is studied with $\gamma_0 \equiv 0$) and establish the existence of local-in-time and small-data solution but up to know there are not any texts that established global weak solution for full system i.e., viscosity and heat conductivity depending on the temperature, the pressure and the shear rate.

2 Fluids with temperature dependent viscosity

2.1 Definition of weak solution and main existence theorem

In this section we consider the case

$$\nu(p,\theta,|\mathbf{D}|^2) \equiv \nu(\theta) \text{ with } \qquad 0 < C_1 \le \nu(\theta) \le C_2, \tag{2.1}$$
$$k(p,\theta,|\mathbf{D}|^2) \equiv k(\theta) \text{ with } \qquad 0 < C_1 \le k(\theta) \le C_2, \tag{2.2}$$

for all $\theta \in [0,\infty)$ and ν, k are continuous functions of the temperature θ. The problem $(\mathcal{P})$ then reduces to the following problem $(\mathcal{P}_{ns})$:

$$(\mathcal{P}_{ns}) \qquad \begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}(\nu(\theta)\mathbf{D}(\boldsymbol{v})) + \nabla p &= \boldsymbol{f}\\
\operatorname{div}\boldsymbol{v} &= 0\\
\left.\begin{aligned}(\theta + \frac{|\boldsymbol{v}|^2}{2})_{,t} + \operatorname{div}(\boldsymbol{v}(\frac{1}{2}|\boldsymbol{v}|^2 + \theta + p)) - \boldsymbol{f}\cdot\boldsymbol{v}\\
- \operatorname{div}(k(\theta)\nabla\theta) - \operatorname{div}(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{v})\end{aligned}\right\} &= 0\\
\theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \operatorname{div}(k(\theta)\nabla\theta) - \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 &\ge 0
\end{aligned}\right\} && \text{in } \Omega\times(0,T),\\
&\left.\begin{aligned}
(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n})_\tau + \alpha\boldsymbol{v}_\tau &= 0\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0\\
\nabla\theta\cdot\boldsymbol{n} &= 0
\end{aligned}\right\} && \text{on } \partial\Omega\times(0,T),\\
&\int_\Omega p(x,t)\,dx = 0 && \text{a. a. } t\in(0,T),\\
&\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0 && \text{in } \Omega,\\
&\theta(\cdot,0) = \theta_0 && \text{in } \Omega.
\end{aligned}$$

Note that in this system the classical incompressible (NSEs) are included.

Next, we precisely define what we mean by weak solution to the problem $(\mathcal{P}_{ns})$. For this purpose we define for all $\psi \in \mathcal{C}(\Omega)$, the *energetic* functional

$$E(t,\psi) := \int_\Omega (\theta(t,x) + \frac{|\boldsymbol{v}(t,x)|^2}{2})\psi(x)\,dx$$

and we also define the *initial* energetic functional $E_0(\psi)$ as

$$E_0(\psi) := \int_\Omega (\theta_0 + \frac{|\boldsymbol{v}_0|^2}{2})\psi\,dx.$$

The definition of that what we mean by weak solution to the problem $(\mathcal{P}_{ns})$ is the following.

Definition 2.1. *Let* $\Omega \in \mathcal{C}^{0,1}$, $\boldsymbol{f} \in L^2(0,T;W^{-1,2}_{\boldsymbol{n}})$. *Let* $\nu, k : \mathbb{R} \to [0,\infty)$ *be continuous functions satisfying* (2.1)-(2.2).
Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\operatorname{div}}$ *and* $\theta_0 \in L^1(\Omega)$, $\theta_0 \ge C_3 > 0$ *for a.a.* $x \in \Omega$.

We say that a triple $(\boldsymbol{v},\theta,p)$ is weak solution to the problem $(\mathcal{P}_{ns})$ if for all $m\in(1,\frac{5}{3})$, $n\in(1,\frac{5}{4})$ and all $\psi\in\mathcal{C}(\Omega)$

$$\boldsymbol{v}\in\mathcal{C}(0,T;L^2_{weak}(\Omega)^3)\cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{2.3}$$

$$\boldsymbol{v}_{,t}\in L^{\frac{5}{3}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5}{3}}), \tag{2.4}$$

$$p\in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega)), \tag{2.5}$$

$$\theta\in L^m(0,T;L^m(\Omega))\cap L^n(0,T;W^{1,n}(\Omega)), \tag{2.6}$$

$$E(t,\psi)\in\mathcal{C}(0,T) \tag{2.7}$$

$$\lim_{t\to0+}\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2=0, \tag{2.8}$$

$$\lim_{t\to0+}E(t,\varphi)=E_0(\varphi) \tag{2.9}$$

and if

$$\begin{aligned}\int_0^T\Big(\langle\boldsymbol{v}_{,t},\boldsymbol{\varphi}\rangle-(\boldsymbol{v}\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi})+(\nu(\theta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\varphi}))+\alpha\int_{\partial\Omega}\boldsymbol{v}\boldsymbol{\varphi}\,dS\Big)\,dt\\=\int_0^T\left((p,\operatorname{div}\boldsymbol{\varphi})+\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\right)\,dt,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\int_0^T\Big(-\Big(\varphi_{,t},\frac{|\boldsymbol{v}|^2}{2}+\theta\Big)-\Big(\boldsymbol{v}(\frac{|\boldsymbol{v}|^2}{2}+\theta+p),\nabla\varphi\Big)+(k(\theta)\nabla\theta,\nabla\varphi)\\+\alpha\int_{\partial\Omega}|\boldsymbol{v}|^2\varphi\,dS+(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{v},\nabla\varphi)-\langle\boldsymbol{f},\boldsymbol{v}\varphi\rangle\Big)\,dt=\frac{1}{2}|\boldsymbol{v}_0|^2+\theta_0,\end{aligned} \tag{2.11}$$

are valid for all $\varphi\in\mathcal{D}(-\infty,T;\mathcal{C}^\infty(\Omega))$ and all $\boldsymbol{\varphi}\in L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}})$. Moreover, for all $\psi\geq0$, $\psi\in\mathcal{D}(-\infty,T;\mathcal{C}^\infty(\Omega))$ there holds

$$\begin{aligned}\int_0^T\big(-(\theta,\psi_{,t})-(\boldsymbol{v}\theta,\nabla\psi)+(k(\theta)\nabla\theta,\nabla\psi)-(\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2,\psi)\big)\,dt\\\geq(\theta_0,\psi(0)).\end{aligned} \tag{2.12}$$

The meaning of the equations (2.10)-(2.12)is the following. We formally multiply the first equation in the definition of the problem $(\mathcal{P}_{ns})$ by $\boldsymbol{\varphi}$, the third one by φ and the fourth one by ψ, integrate over Ω and time $(0,T)$, use integration per parts to get the resulting equations (2.10)-(2.11) and inequality (2.12). There is only one difference between the weak formulation for no-slip boundary conditions and for Navier's boundary conditions and it is the boundary integral. We try to explain why the boundary integral appears in the weak formulation. Multiplying the dissipative term in the first equation by of the problem $(\mathcal{P}_{ns})$ by $\boldsymbol{\varphi}$, we have (after integration over time and space)

$$\begin{aligned}-\int_0^T\int_\Omega\operatorname{div}\left(\nu(\theta)\mathbf{D}(\boldsymbol{v})\right)\boldsymbol{\varphi}\,dx\,dt=\int_0^T\int_\Omega\nu(\theta)\mathbf{D}(\boldsymbol{v})\cdot\mathbf{D}(\boldsymbol{\varphi})\,dx\,dt\\-\int_0^T\int_{\partial\Omega}(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n})\cdot\boldsymbol{\varphi}\,dS\,dt.\end{aligned} \tag{2.13}$$

Next, for the boundary integral we can compute

$$\int_0^T \int_{\partial\Omega} (\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}) \cdot \boldsymbol{\varphi} \, dS \, dt = \int_0^T \int_{\partial\Omega} ((\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}) \cdot \boldsymbol{n}) \, \boldsymbol{n} \cdot \boldsymbol{\varphi} \, dS \, dt$$
$$\int_0^T \int_{\partial\Omega} (\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi} \, dS \, dt := I_1 + I_2.$$

Because the normal part of the function $\boldsymbol{\varphi}$ is equal to zero on the boundary, we have $I_1 = 0$. For second integral we use the Navier's slip boundary conditions to get

$$I_2 = -\alpha \int_0^T \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{\varphi} \, dS \, dt$$

that is exactly the same integral as in the equation (2.10). Similar computation shows why the boundary integral is presented also in the equation (2.11).

Important questions is if all integrals in (2.10)-(2.12) are finite. First note that Corollary B.3 together with (2.3) imply that

$$\operatorname{tr} \boldsymbol{v} \in L^2(0, T; L^2(\partial\Omega)^3). \tag{2.14}$$

Therefore, all boundary integrals are meaningful. We see that then the most critical integral is the second one in (2.11). But having estimates (2.3) we can easily show that $\boldsymbol{v} \in L^{\frac{10}{3}}(\Omega \times (0,T))^3$. Because $p \in L^{\frac{5}{3}}(\Omega \times (0,T))$ Hölder inequality leads to the the fact that

$$\boldsymbol{v}(\frac{1}{2}|\boldsymbol{v}|^2 + p) \in L^{\frac{10}{9}}(\Omega \times (0,T))^3.$$

Moreover, using (2.6) and Hölder inequality again we obtain that

$$\boldsymbol{v}\theta \in L^q(\Omega \times (0,T))^3$$

for all $q \in [1, \frac{10}{9})$. Hence also the second integral in (2.11) is finite.

Other integrals in the definition of weak solution can be bound directly from the estimates (2.3)-(2.6) and the assumptions on ν, k (3.1)-(2.2). Thus, we conclude that all terms in weak formulation, i.e., in (2.10)-(2.12) are meaningful.

Now, we formulate the main theorem of this section.

Theorem 2.1. *Let $\Omega \in \mathcal{C}^{1,1}$. Let $\boldsymbol{f} \in L^2(0,T; W^{-1,2}_{\boldsymbol{n}})$. Let ν, k satisfy (2.1)-(2.2). Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ and $\theta_0 \in L^1(\Omega)$, $\theta_0 \geq C_3 > 0$ for a.a. $x \in \Omega$. Then there exists weak solution to the problem $(\mathcal{P}_{ns})$.*

Theorem 2.1 is only an easy generalization of the results presented in [12]. Under the same assumptions on ν, k the authors established the existence of weak solution for spatially periodic problem. The rest of the Section 2 is devoted to the proof of Theorem 2.1.

Note also that in the definition of weak solution we assume only $\Omega \in \mathcal{C}^{0,1}$ but in Theorem 2.1 the stronger assumption is considered, namely $\Omega \in \mathcal{C}^{1,1}$.

The reason is the necessity of having apriori estimates of the pressure p that appears in (2.11) and cannot be omitted by using divergenceless functions as test functions as it is usual in studying NSEs without temperature. Note that the better assumption on the boundary is also adopted for theory of (NSEs) with no-slip boundary condition if one wants to reconstruct the pressure globally, see for example [29].

2.2 Proof of the theorem

The proof is split into several steps, all of them form particular subsubsections of the Subsection 2.2 of this paper.

In Subsection 2.2.1 we introduce the so-called quasi-compressible approximative problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$ that consists of two levels of approximations and where the equation for the global energy (1.6) is replaced by the equation for internal energy $(1.1)_3$. In order to have some pressures in hands from the beginning, we first perturb for all $\varepsilon > 0$, the incompressible constraint, i.e., $\operatorname{div} \boldsymbol{v} = 0$, by the Neumann problem for the pressure of the form[3]

$$
\begin{aligned}
-\varepsilon \triangle p + \operatorname{div} \boldsymbol{v} &= 0 \text{ in } \Omega \times (0,T), \\
\frac{\partial p}{\partial \boldsymbol{n}} &= 0 \text{ on } \partial\Omega \times (0,T).
\end{aligned}
\tag{2.15}
$$

In order to preserve apriori estimates we also modify the convective term by a suitable $\eta-$approximation.

The proof of the existence of a solution to $(\mathcal{P}_{ns})^{\varepsilon,\eta}$-approximations for all $\varepsilon > 0$, $\eta > 0$ fixed will be done via Galerkin approximations incorporating the compactness of velocities and temperatures.

In Subsection 2.2.3 we will pass to the limit $\varepsilon \to 0+$ (i.e. we will obtain the "incompressible" limit). Here the fact that we deal with Navier's boundary condition together with the assumption $\Omega \in \mathcal{C}^{1,1}$ will play important role. Note that the similar procedure can be used also for spatially-periodic problem, but it seems (up to now) that there is no chance how to pass to the limit in the case of no-slip boundary conditions. In order to let $\varepsilon \to 0+$ we need to obtain estimates for the pressure that are uniform w.r.t. $\varepsilon, \eta > 0$.

This is performed by taking a test function $\boldsymbol{\varphi}$ in the weak formulation of the

[3]Note that this perturbation is in fact not needed in this section. The pressure can be simply reconstructed from the equation as it is usual in the theory of NSEs and because we consider Navier's slip boundary conditions, we can then easily show that the pressure belongs to the expected space $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega))$. The reason why to construct the pressure from the beginning also in this section is that we want to prepare the theory for this approximation in details in section for relatively easy model and then we want to use it in the following sections where the pressure appears also in the viscosity and the heat conductivity and where this approximation plays important (key) role because for those models it is needed to have the pressure from the beginning.

problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$ of the form $\boldsymbol{\varphi} = -\nabla g^{\varepsilon,\eta}$ where $g^{\varepsilon,\eta}$ solves

$$\begin{aligned}
-\triangle g^{\varepsilon,\eta} &= |p^{\varepsilon,\eta}|^{\alpha-2}p^{\varepsilon,\eta} - \frac{1}{|\Omega|}\int_\Omega |p^{\varepsilon,\eta}|^{\alpha-2}p^{\varepsilon,\eta} && \text{in } \Omega \quad (\alpha>1),\\
\frac{\partial g^{\varepsilon,\eta}}{\partial \boldsymbol{n}} &= 0 && \text{on } \partial\Omega,\\
\int_\Omega g^{\varepsilon,\eta}\,dx &= 0.
\end{aligned} \tag{2.16}$$

Clearly, the term involving the pressure leads to

$$(p^{\varepsilon,\eta}, \operatorname{div}\boldsymbol{\varphi}) = (p^{\varepsilon,\eta}, -\triangle g^{\varepsilon,\eta}) = \|p^{\varepsilon,\eta}\|_\alpha^\alpha. \tag{2.17}$$

The task is to control the remaining terms, where the time derivative is the most critical. Using the Helmholtz decomposition

$$\boldsymbol{v}^{\varepsilon,\eta} = \boldsymbol{v}^{\varepsilon,\eta}_{\mathrm{div}} + \nabla g^{\boldsymbol{v}^{\varepsilon,\eta}},$$

where $g^{\boldsymbol{v}^{\varepsilon,\eta}}$ is a solution of the auxiliary problem (1.21), we observe by comparing (2.15) with (1.21) that $g^{\boldsymbol{v}^{\varepsilon,\eta}} = \varepsilon p^{\varepsilon,\eta}$. Consequently, we have

$$\begin{aligned}
\langle \boldsymbol{v}^{\varepsilon,\eta}_{,t}, \boldsymbol{\varphi}\rangle &= \langle \boldsymbol{v}^{\varepsilon,\eta}_{\mathrm{div},t} + (\nabla g^{\boldsymbol{v}^{\varepsilon,\eta}})_{,t}, -\nabla g^{\varepsilon,\eta}\rangle\\
&= -\langle \nabla g^{\boldsymbol{v}^{\varepsilon,\eta}}_{,t}, \nabla g^{\varepsilon,\eta}\rangle\\
&= \langle -g^{\boldsymbol{v}^{\varepsilon,\eta}}_{,t}, -\triangle g^{\varepsilon,\eta}\rangle\\
&= -\varepsilon\langle p^{\varepsilon,\eta}_{,t}, |p^{\varepsilon,\eta}|^{\alpha-2}p^{\varepsilon,\eta}\rangle = -\varepsilon\frac{d}{dt}\|p^{\varepsilon,\eta}\|_\alpha^\alpha \leq^{4} 0.
\end{aligned}$$

Thus, the time derivative acting on the test function $\nabla g^{\varepsilon,\eta}$ has a correct sign and we separate the pressure from time derivative. In analysis of the remaining terms we will apply, in Subsection 2.2.3 below, the following standard result on the solvability of (1.21). If $\Omega \in \mathcal{C}^{1,1}$ then

$$\begin{aligned}
\|g^{\boldsymbol{v}}\|_{2,q} &\leq C_{reg}(\Omega,q)\|\operatorname{div}\boldsymbol{v}\|_q & \|\boldsymbol{v}_{\mathrm{div}}\|_{1,q} &\leq (C_{reg}(\Omega,q)+1)\|\boldsymbol{v}\|_{1,q},\\
\|g^{\boldsymbol{v}}\|_{1,s} &\leq C(\Omega,s)\|\boldsymbol{v}\|_s & \|\boldsymbol{v}_{\mathrm{div}}\|_s &\leq (C(\Omega,s)+1)\|\boldsymbol{v}\|_s,
\end{aligned}$$

whenever the right hand sides make a good sense (see Lemma B.1 for details).

Finally, in Subsection 2.2.4, we let $\eta \to 0+$. Here we replace the equation for internal energy by that for global energy. Note that this change is possible at this order of approximations because we have the validity of the balance of kinetic energy (1.19) (we have equality sign in (1.19)). Finally, by using apriori estimates and Aubin-Lions lemma we will able to pass to the limit and get (2.10)-(2.11). To obtain (2.12) it is enough to use only weak lower semicontinuity of the term $\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2$.

[4] After integration over time $t \in (0,T)$ and using the fact that $p^{\varepsilon,\eta}(0,x) = 0$ as $\operatorname{div}\boldsymbol{v}^{\varepsilon,\eta}(0,x) = 0$.

2.2.1 ε, η - approximation

We define an approximative problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$ as (for simplicity we write $(\boldsymbol{v}, p, \theta)$ instead of $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta}, \theta^{\varepsilon,\eta})$)

$$
(\mathcal{P}_{ns})^{\varepsilon,\eta} \qquad
\begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div}\left(\nu(\theta)\mathbf{D}(\boldsymbol{v})\right) + \nabla p &= \boldsymbol{f} \\
-\varepsilon \triangle p + \operatorname{div} \boldsymbol{v} &= 0 \\
\theta_{,t} + \operatorname{div}(\theta \boldsymbol{v}_\eta) - \operatorname{div}\left(k(\theta)\nabla\theta\right) - \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 &= 0
\end{aligned}\right\} \quad \text{in } \Omega \times (0,T), \\
&\left.\begin{aligned}
\left(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right)_\tau + \alpha \boldsymbol{v}_\tau &= 0 \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\
\nabla\theta \cdot \boldsymbol{n} &= 0 \\
\nabla p \cdot \boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \partial\Omega \times (0,T), \\
&\int_\Omega p(x,t)\, dx = 0 \qquad \text{a.a. } t \in (0,T), \\
&\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0 \qquad \text{in } \Omega, \\
&\theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega.
\end{aligned}
$$

The definition of $\boldsymbol{v}_\eta$ is following. Let $\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}}$ and $\eta > 0$ be fixed. We define the function φ_η

$$
\varphi_\eta(x) := \begin{cases} 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \le 2\eta, \\ 1 & \text{elsewhere.} \end{cases}
$$

We define $\boldsymbol{v}_\eta := ((\varphi_\eta \boldsymbol{v}) * \omega_\eta)_{\mathrm{div}}$ where the symbol $u * \omega_\eta$ is the standard regularization of an integrable function u with kernel ω_η having the support in a ball of radii η. The symbol $(.)_{\mathrm{div}}$ then comes from the Helmholtz decomposition (Note that we can use Helmholtz decomposition (1.21) because $(\varphi_\eta \boldsymbol{v}) * \omega^\eta \in (\mathcal{C}_0^\infty(\Omega))^3$.). Note also that this definition leads to the identity

$$
\int_\Omega \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) \cdot \boldsymbol{v} \; = \int_\Omega \boldsymbol{v}_\eta \cdot \nabla \frac{|\boldsymbol{v}|^2}{2}\, dx = -\frac{1}{2} \int_\Omega \operatorname{div} \boldsymbol{v}_\eta |\boldsymbol{v}|^2 \, dx = 0. \tag{2.18}
$$

Moreover, if $\operatorname{div} \boldsymbol{v} = 0$ then $\boldsymbol{v}_\eta \to \boldsymbol{v}$ in $L^q(0,T; L^q_{\boldsymbol{n}})$ provided $\boldsymbol{v} \in L^q(0,T; L^q_{\boldsymbol{n}})$. To show it, we define the function g_η such that $\triangle g_\eta = \operatorname{div}((\varphi_\eta \boldsymbol{v}) * \omega_\eta)$ in Ω, $\frac{\partial g_\eta}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$ and mean value of g_η is equal to zero. Then it follows from the definition of $\boldsymbol{v}_\eta$ that

$$
\boldsymbol{v}_\eta = (\varphi_\eta \boldsymbol{v}) * \omega_\eta - \nabla g_\eta.
$$

But $(\varphi_\eta \boldsymbol{v}) * \omega_\eta \to \boldsymbol{v}$ as $\eta \to 0$ in $L^q(0,T; L^q_{\boldsymbol{n}})$ and it remains to show that $\nabla g_\eta \to 0$ in $L^q(0,T; L^q_{\boldsymbol{n}})$. From theory for Laplace equation it follows that $g_\eta \to g$ in $L^q(0,T; W^{1,q}(\Omega))$ and g solves for all smooth φ and for almost all $t \in (0,T)$

$$
\int_\Omega \nabla g \cdot \nabla \varphi \, dx = -\int_\Omega \boldsymbol{v} \cdot \nabla \varphi \, dx \overset{\operatorname{div} \boldsymbol{v} = 0}{=} 0
$$

and from uniqueness of solution for Laplace equation we get that $g = 0$.

The proof of the existence of solutions to problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$ will be done by using Galerkin approximations.

First, we define the continuous mapping $\mathcal{F} : W^{1,2}_{\boldsymbol{n}} \to W^{2,2}(\Omega)$: to some $\boldsymbol{v} \in W^{1,2}_{\boldsymbol{n}}$ we assign $p \in W^{2,2}(\Omega)$ solving the equation

$$\begin{aligned} &\varepsilon \triangle p = \operatorname{div} \boldsymbol{v} && \text{in } \Omega \\ &\int_\Omega p \, dx = 0, && \\ &\nabla p \cdot \boldsymbol{n} = 0 && \text{on } \partial\Omega. \end{aligned}$$

The existence of such p is a consequence of Lemma B.1. This lemma also implies that $\mathcal{F} : W^{1,2}_{\boldsymbol{n}} \to W^{2,2}(\Omega)$ is continuous.

Next, let $\{\boldsymbol{w}_j\}_{j=1}^\infty$ be a basis of $W^{1,2}_{\boldsymbol{n}}$ such that $\boldsymbol{w}_j \in W^{1,4}_{\boldsymbol{n}}$ for all j and $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \; dx = \delta_{ij}$. It is the standard result that such basis exists. Let $\{w_j\}_{j=1}^\infty$ be a basis of $W^{1,2}(\Omega)$ which is again orthonormal in the space $L^2(\Omega)$. We construct Galerkin approximations $\{\boldsymbol{v}^{N,M}, \theta^{N,M}, p^{N,M}\}_{N,M=1}^\infty$ being of the form

$$\begin{aligned} \boldsymbol{v}^{N,M} &:= \sum_{i=1}^N c_i^{N,M}(t) \boldsymbol{w}_i, \\ \theta^{N,M} &:= \sum_{i=1}^M d_i^{N,M}(t) w_i, \\ p^{N,M} &:= \mathcal{F}(\boldsymbol{v}^{N,M}), \end{aligned}$$

where $\boldsymbol{c}^{N,M} := (c_1^{N,M}, \ldots, c_N^{N,M})$, $\boldsymbol{d}^{N,M} := (d_1^{N,M}, \ldots, d_M^{N,M})$ solve the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{v}^{N,M}, \boldsymbol{w}_j) - (\boldsymbol{v}_\eta^{N,M} \otimes \boldsymbol{v}^{N,M}, \nabla \boldsymbol{w}_j) + (\nu(\theta^{N,M}) \mathbf{D}(\boldsymbol{v}^{N,M}), \nabla \boldsymbol{w}_j) \\ + \alpha \int_{\partial\Omega} \boldsymbol{v}^{N,M} \cdot \boldsymbol{w}_j \; dS - (\mathcal{F}(\boldsymbol{v}^{N,M}), \operatorname{div} \boldsymbol{w}_j) = \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{d}{dt}(\theta^{N,M}, w_k) - (\boldsymbol{v}_\eta^{N,M} \theta^{N,M}, \nabla w_k) + (k(\theta^{N,M}) \nabla \theta^{N,M}, \nabla w_k) \\ = \left(\nu(\theta^{N,M}) |\mathbf{D}(\boldsymbol{v}^{N,M})|^2, w_k \right), \end{aligned} \tag{2.20}$$

for all $j = 1, 2, \ldots, N$ and for all $k = 1, 2, \ldots, M$. We assume that $\boldsymbol{v}^{N,M}$ and $\theta^{N,M}$ satisfy the following initial conditions

$$\begin{aligned} \boldsymbol{v}^{N,M}(\cdot, 0) &= \boldsymbol{v}_0^{N,M}, \\ \theta^{N,M}(\cdot, 0) &= \theta_0^{N,M}, \end{aligned}$$

where $\boldsymbol{v}_0^{N,M} := \sum_{j=1}^N c_{0,j}^N \boldsymbol{w}_j$ are the projections of $\boldsymbol{v}_0$ onto linear hulls of $\{\boldsymbol{w}_j\}_{j=1}^N$ and $\theta_0^{N,M}$ has the following meaning. We first regularize θ_0 with regularization kernel $\omega_{\frac{1}{N}}$ of radii $\frac{1}{N}$. It means we define $\theta_0^N := (\omega_{1/N} * \theta_0)$ (we use the convection that $\theta_0(x) := C_3$ for $x \in \mathbb{R}^3 \setminus \Omega$). Then we apply the projection onto the linear hull of $\{w_j\}_{j=1}^M$. Thus, $\theta_0^{N,M}$ has the form $\theta_0^{N,M} := \sum_{j=1}^M d_{0,j}^M w_j$. Note that

$$
\begin{aligned}
\theta_0^{M,N} &\overset{M\to\infty}{\to} \theta_0^N && \text{strongly in } L^2(\Omega),\\
\boldsymbol{v}_0^N &\overset{N\to\infty}{\to} \boldsymbol{v}_0 && \text{strongly in } L^2(\Omega)^3,\\
\theta_0^N &\overset{N\to\infty}{\to} \theta_0 && \text{strongly in } L^1(\Omega).
\end{aligned}
$$

Let us define $\mathsf{C}(t) := (c_1^{N,M}(t), \dots, c_N^{N,M}(t), d_1^{N,M}(t), \dots, d_M^{N,M}(t))$ and $\mathsf{C}_0 := (\boldsymbol{c}_0^N, \boldsymbol{d}_0^M) := (c_{0,1}^N, \dots, c_{0,N}^N, d_{0,1}^M, \dots, d_{0,M}^M)$. The system (2.19)-(2.20) can be rewritten as

$$
\begin{aligned}
\frac{d}{dt}\mathsf{C} &= \mathcal{G}(t, \mathsf{C}),\\
\mathsf{C}(0) &= \mathsf{C}_0.
\end{aligned}
\tag{2.21}
$$

Because the operator $\mathcal{F}$ is continuous and the viscosity ν and the heat conductivity k have the same property, we see that $\mathcal{G}$ is continuous with respect to C. Because $\boldsymbol{f} \in L^2(0, T; W_{\boldsymbol{n}}^{-1,2})$, the term $\langle \boldsymbol{f}(t), \boldsymbol{w}_j\rangle$ is at least measurable (after defining $\boldsymbol{f}(t, x) \equiv \mathbf{0}$ for $t < 0$). Hence $\mathcal{G}$ is measurable in t. Moreover, it is easy to show that there exists an integrable function G such that

$$
|\mathcal{G}(t, \mathsf{C})| \le G(t)
$$

at least for all $(t, \mathsf{C}) \in (-T, T) \times D$ where D is $(M + N)$-dimensional cube with radii $R := 2 \max_i |C_0^i|$.

These three simple observations are in fact Caratheodory's conditions and using Theorem B.1, we observe that there exist some $\delta > 0$ and a continuous function C such that $\frac{d}{dt}\mathsf{C}$ exists for almost all $t \in (0, \delta)$ and C solves (2.21).

In the next subsection we show that owing to the bounds uniform w.r.t. N, M solution exists for all $t \in (0, T)$.

2.2.2 Apriori estimates and limit $N, M \to \infty$

In this subsection we derive apriori estimates and we pass to the limit in Galerkin approximation. First, we set $M \to \infty$ and then $N \to \infty$. Note that some of estimates will be independent of the order of approximation and will be frequently used later (after using weak lower semicontinuity of norm in a reflexive space). We also have to note that in what follows the constant C denotes some universal constant depending only on the data of the problem $(\mathcal{P}_{ns})$. If there is some dependence on the order of approximation it will be clearly denoted.

Estimates independent of M: Multiplying j-th equation in (2.19) by $c_j^{N,M}$, summing over $j = 1, ..., M$, integrating over $(0, T)$, using the assumption on the viscosity ν and the definition of the operator $\mathcal{F}$, we observe

$$\begin{aligned}\sup_{t\in(0,T)} \|\boldsymbol{v}^{N,M}(t)\|_2 + \int_0^T \Big(\|\mathbf{D}(\boldsymbol{v}^{N,M})\|_2^2 + \alpha \int_{\partial\Omega} |\boldsymbol{v}^{N,M}|^2 \, dS \\ + \varepsilon \|\nabla p^{N,M}\|_2^2\Big) \, d\tau \le C \left(1 + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}^{N,M}\rangle \, d\tau\right).\end{aligned} \tag{2.22}$$

Multiplying of the j-th equation in (2.20) by $d_j^{N,M}$, summing over $k = 1, \dots, M$, integrating over time $t \in (0,T)$, using the previous estimate (2.22), assumption on k, ν and the fact that $\boldsymbol{w}_j \in W^{1,4}_{\boldsymbol{n}}$ lead to

$$\|\theta^{N,M}(t)\|_2^2 + \int_0^t \|\nabla\theta^{N,M}\|_2^2 \, d\tau \le C(N) \left(1 + \int_0^t \|\theta^{N,M}\|_2^2 \, d\tau\right). \tag{2.23}$$

Finally, we apply Korn's inequality (Lemma B.2) and Young's inequality to (2.22) to get

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{N,M}(t)\|_2 + \int_0^T \|\boldsymbol{v}^{N,M}\|^2_{W^{1,2}_{\boldsymbol{n}}} + \varepsilon \|\nabla p^{N,M}\|_2^2 \, dt \le C. \tag{2.24}$$

Applying Gronwall's lemma to (2.23), we are led to

$$\sup_{t\in(0,T)} \|\theta^{N,M}(t)\|_2^2 + \int_0^T \|\nabla\theta^{N,M}\|_2^2 \, d\tau \le C(N). \tag{2.25}$$

To get some compactness of the velocity and temperature, we also estimate the norms of their time derivative. Multiplying the j-equation in (2.19) by $\frac{d}{dt} c_j^{N,M}$, summing over $j = 1, \dots, N$ and integrating it over time, we get (after using the estimate (2.24)) that

$$\int_0^T \left(\frac{d}{dt}\boldsymbol{c}^{N,M}\right)^2 \, dt \le C(N). \tag{2.26}$$

Let $\varphi \in L^2(0,T;W^{1,2}(\Omega))$ be arbitrary. We denote by $L^2(0,T;V^M)$ the subspace of $L^2(0,T;W^{1,2}(\Omega))$ where $V^M := \text{Lin hull}\{w_j\}_{j=1}^M$, and P^M denotes $P^M : L^2(0,T;W^{1,2}(\Omega)) \to L^2(0,T;V^M)$, the projections. The norm of $\theta^{N,M}_{,t}$ in the space $L^2(0,T;W^{-1,2}(\Omega))$ then be computed as (after using orthonormality of w_j) the supremum over all $\varphi \in L^2(0,T;W^{1,2}(\Omega))$, $\|\varphi\| \le 1$. Hence,

$$\begin{aligned}\|\theta^{N,M}_{,t}\| &= \sup_{\varphi} \int_0^T \langle \theta^{N,M}_{,t}, \varphi\rangle \\ &= \sup_{\varphi} \int_0^T (\theta^{N,M}_{,t}, P(\varphi)) \\ &\overset{(2.20)}{=} \sup_{\varphi} \Big(\int_0^T (-\boldsymbol{v}^{N,M}_{\eta} \theta^{N,M}, \nabla P\varphi) + (k(\theta^{N,M} \nabla\theta^{N,M}, \nabla P\varphi) \\ &\quad - (\nu(\theta^{N,M})\mathbf{D}(\boldsymbol{v}^{N,M}), P\varphi)\Big) \le C(N),\end{aligned} \tag{2.27}$$

where we used the continuity of the projection P, the fact that $\boldsymbol{w}_i \in W^{1,4}_{\boldsymbol{n}}$ and the estimates (2.24) and (2.25).

Limit $M \to \infty$ (N fixed): Having apriori estimates (2.24)-(2.27), we can let $M \to \infty$ and find subsequences $\{\boldsymbol{c}^{N,M}, \theta^{N,M}\}_{M=1}^{\infty}$ (that is not relabeled) such that

$$\boldsymbol{c}^{N,M}_{,t} \rightharpoonup \boldsymbol{c}^{N}_{,t} \qquad \text{weakly in } L^2(0,T), \tag{2.28}$$

$$\boldsymbol{c}^{N,M} \rightharpoonup^* \boldsymbol{c}^{N} \qquad \text{weakly}^* \text{ in } L^\infty(0,T), \tag{2.29}$$

$$\theta^{N,M} \rightharpoonup^* \theta^{N} \qquad \text{weakly }^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{2.30}$$

$$\theta^{N,M} \rightharpoonup \theta^{N} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{2.31}$$

$$\theta^{N,M}_{,t} \rightharpoonup \theta^{N}_{,t} \qquad \text{weakly in } L^2(0,T;W^{-1,2}(\Omega)). \tag{2.32}$$

Morover, using generalized version of Aubin-Lions compactness lemma (Theorem B.3), we have after using standard interpolation inequalities that (we again do not relabeled the sequences)

$$\theta^{N,M} \to \theta^{N} \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \text{ for all } m \in \left\langle 1, \frac{10}{3} \right). \tag{2.33}$$

Finally, we will deduce that (after taking not relabeled subsequence)

$$\boldsymbol{c}^{N,M} \to \boldsymbol{c}^{N} \qquad \text{strongly in } \mathcal{C}(0,T). \tag{2.34}$$

We use Arsela-Ascoli theorem to prove this. Indeed, the inequality

$$\sup_t |\boldsymbol{c}^{N,M}(t)| \overset{(2.24)}{\leq} C,$$

implies that the sequence $\{\boldsymbol{c}^{N,M}\}_{M=1}^{\infty}$ is uniformly bounded. To get uniform continuity, one can compute

$$\begin{aligned} |\boldsymbol{c}^{N,M}(t_1) - \boldsymbol{c}^{N,M}(t_2)| = \left| \int_{t_1}^{t_2} \frac{d}{dt}\boldsymbol{c}^{N,M}(t)\, dt \right| &\overset{\text{Hölder}}{\leq} |t_1 - t_2|^{1/2} \left(\int_{t_1}^{t_2} |\frac{d}{dt}\boldsymbol{c}(t)|^2\, dt \right)^{1/2} \\ &\overset{(2.26)}{\leq} C(N)|t_1 - t_2|^{1/2}. \end{aligned}$$

Let $\varepsilon > 0$. We define $\delta := \varepsilon / C(N)$ and it leads to

$$\sup_N |\boldsymbol{c}^{N,M}(t_2) - \boldsymbol{c}^{N,M}(t_1)| \leq \varepsilon \quad \text{ for all } t_1, t_2 \in (0,T); |t_1 - t_2| \leq \delta$$

which is exactly the definition of uniform continuity. Arsela-Ascoli theorem then implies (2.34). Moreover, it is a simple consequence of our choice of basis and (2.34) that

$$\boldsymbol{v}^{N,M} \to \boldsymbol{v}^{N} \qquad \text{strongly in } L^4(0,T;W^{1,4}_{\boldsymbol{n}}), \tag{2.35}$$

and consequently, because $\mathcal{F}$ is continuous, we have

$$p^{N,M} \to p^N \qquad\qquad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{2.36}$$

Convergence (2.28)-(2.36) proved above allow us to pass to the limit in (2.19) and in (2.20). Indeed, we take an arbitrary $\varphi \in \mathcal{D}(0,T)$ and multiply j-th equation in (2.19) and (2.20) by it and then integrate over time $t \in (0,T)$. Then it is easy to pass to the limit in all terms to get the following systems

$$\begin{aligned}\int_0^T \Big(\frac{d}{dt}(\boldsymbol{v}^N, \boldsymbol{w}_j) - (\boldsymbol{v}^N_\eta \otimes \boldsymbol{v}^N, \nabla \boldsymbol{w}_j) + (\nu(\theta^N)\mathbf{D}(\boldsymbol{v}^N), \nabla \boldsymbol{w}_j)\\ +\alpha \int_{\partial\Omega} \boldsymbol{v}^N \cdot \boldsymbol{w}_j \, dS - (p^N, \operatorname{div} \boldsymbol{w}_j) - \langle \boldsymbol{f}, \boldsymbol{w}_j\rangle\Big)\varphi(t)\, dt = 0\end{aligned} \tag{2.37}$$

for all $j = 1, 2, \ldots, N$ and

$$\begin{aligned}\int_0^T \big((\theta^N_{,t}, w_j) - (\boldsymbol{v}^N_\eta \theta^N, \nabla w_j) + (k(\theta^N)\nabla\theta^N, \nabla w_j)\big)\varphi(t)\, dt\\ = \int_0^T \big(\big(\nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2, w_j\big)\, \varphi(t)\big)\, dt\end{aligned} \tag{2.38}$$

for all $j = 1, 2, \ldots, \infty$. Because φ can be chosen arbitrarily we can conclude that

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{v}^N, \boldsymbol{w}_j) - (\boldsymbol{v}^N_\eta \otimes \boldsymbol{v}^N, \nabla \boldsymbol{w}_j) + (\nu(\theta^N)\mathbf{D}(\boldsymbol{v}^N), \nabla \boldsymbol{w}_j)\\ +\alpha \int_{\partial\Omega} \boldsymbol{v}^N \cdot \boldsymbol{w}_j \, dS - (p^N, \operatorname{div} \boldsymbol{w}_j) - \langle \boldsymbol{f}, \boldsymbol{w}_j\rangle = 0\end{aligned} \tag{2.39}$$

for $j = 1, 2, \ldots, N$ and a.a. time $t \in (0,T)$. From the same reason and from the fact that $\{w_j\}_{j=1}^\infty$ is a basis of $W^{1,2}(\Omega)$ we conclude that

$$\langle \theta^N_{,t}, \varphi\rangle - (\boldsymbol{v}^N_\eta \theta^N, \nabla\varphi) + (k(\theta^N)\nabla\theta^N, \nabla\varphi) = \big(\nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2, \varphi\big)\,, \tag{2.40}$$

is valid for all $\varphi \in W^{1,2}(\Omega)$ and for a.a. $t \in (0,T)$.

Because (2.34) and the fact that $\boldsymbol{c}^{N,M}(0) = \boldsymbol{c}^N_0$ for all M we see that $\boldsymbol{c}^N(0) = \boldsymbol{c}^N_0$ and consequently from the definition of $\boldsymbol{v}^N$ and $\boldsymbol{v}^N_0$ it is clear that

$$\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}^N_0.$$

It remains to show that $\theta^N(0,\cdot) = \theta^N_0$. First note that apriori estimates after using Lemma B.4 imply that $\theta^N \in \mathcal{C}(0,T;L^2(\Omega))$. Thus, it makes a good sense to define an initial condition. To prove our goal, we integrate the equation (2.20) over time $t \in (0,t_1)$. Then we pass to limit with M to get

$$\begin{aligned}(\theta^N(t_1), w_j) - \int_0^{t_1} (\boldsymbol{v}^N\theta^N, \nabla w_j) + (k(\theta^N)\nabla\theta^N, \nabla w_j) - (\nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2, w_j)\, dt\\ = (\theta^N_0, w_j).\end{aligned}$$

Hence, setting $t_1 \to 0$ we obtain

$$(\theta^N(t), w_j) \to (\theta_0^N, w_j).$$

But as θ^N is continuous into the space $L^2(\Omega)$ and weak limit as time tends to zero is θ_0^N, we have that

$$\lim_{t\to 0} \|\theta^N(t) - \theta_0^N\|_2^2 = 0.$$

Minimum principle: Next, applying another standard tool for parabolic problems we show that

$$\theta^N(x,t) \geq \operatorname*{essinf}_{x\in\Omega} \theta_0^N \geq C_3 > 0 \text{ for a.a. } (x,t) \in \Omega \times (0,T). \tag{2.41}$$

We use the weak formulation (2.40) with the function $\varphi := \chi_{[0,\tau]}(t) \min(0, \theta^N - C_3) \leq 0$. Integrating it over time $t \in (0,T)$ we have (after using $\nu(\theta^N) \geq 0$) that

$$I_1 + I_2 + I_3 := \int_0^T \langle \theta^N_{,t}, \varphi\rangle - (\boldsymbol{v}_\eta^N \theta^N, \nabla\varphi) + (k(\theta^N)\nabla\theta^N, \nabla\varphi)\, dt \leq 0. \tag{2.42}$$

Next, we can compute

$$\begin{aligned}
I_2 &= -\int_0^T \int_\Omega \theta^N \boldsymbol{v}_\eta^N \cdot \nabla\varphi \, dx\, dt = \int_0^T \int_\Omega v_{i,\eta}^N \frac{\partial\theta^N}{\partial x_i} \varphi \, dx\, dt \\
&= \int_0^T \int_\Omega v_{i,\eta}^N \frac{\partial\varphi}{\partial x_i} \varphi \, dx\, dt = \frac{1}{2}\int_0^T \int_\Omega v_{i,\eta}^N \frac{\partial\varphi^2}{\partial x_i} \, dx\, dt \overset{\operatorname{div} \boldsymbol{v}_\eta^N = 0}{=} 0, \\
I_3 &= \int_0^T \int_\Omega k(\theta)\nabla\theta^N \cdot \nabla\varphi \, dx\, dt = \int_0^T \int_\Omega k(\theta)\nabla\varphi \cdot \nabla\varphi \, dx\, dt \geq 0, \\
I_1 &= \int_0^T \langle \theta^N_{,t}, \varphi\rangle \, dt = \int_0^\tau \langle \varphi_{,t}, \varphi\rangle \, dt = \frac{1}{2}\|\varphi(\tau)\|_2^2 - \frac{1}{2}\|\varphi(0)\|_2^2.
\end{aligned} \tag{2.43}$$

Hence, insertion of (2.43) into (2.42) and using the fact that $\varphi(0,x) = 0$ a. e., implies that

$$\|\varphi(\tau)\|_2 = 0.$$

This procedure can be used for all $\tau \in (0,T)$. Then one can easily obtain the desired conclusion (2.41). Note that we always deal with at least weakly converging sequences of the temperatures θ^N (or other their approximations). The convexity of the set $\{y \in \mathbb{R}; y \geq C_3\}$ then will imply that all limit functions will also satisfy the minimum principle (2.41).

Estimates independent of N: Next, we observe apriori estimates being independent of N. Using weak lower semicontinuity of norms we find that (2.24) holds (without superscript M). Testing (2.40) by $\varphi \equiv 1$ and using (2.24) leads to

$$\sup_t \|\theta^N(t)\|_1 \leq C. \tag{2.44}$$

Next, we want to show some estimates on the gradient of the temperature that are independent of the order of approximation. To do it we set $\varphi := (\theta^N)^\lambda$ with $-1 < \lambda < 0$. Note that (2.41) implies that $0 \le \varphi \le C$ for almost all (t,x). We use φ as a test function in (2.40) and we integrate it over $t \in (0,T)$ to get

$$\begin{aligned}
&\int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2(\theta^N)^\lambda - \lambda k(\theta^N)(\theta^N)^{\lambda-1}|\nabla\theta^N|^2 \, dx \, dt \\
&\quad = \int_0^T \int_\Omega -(\boldsymbol{v}_\eta^N \theta^N \cdot \nabla\theta^N)^\lambda \, dx \, dt + \int_0^T \langle \theta^N_{,t}, (\theta^N)^\lambda \rangle \, dt =: I_1 + I_2.
\end{aligned} \tag{2.45}$$

Now, we estimate I_1, I_2. For I_1 we have

$$\begin{aligned}
-I_1 &= \int_0^T \int_\Omega \lambda v^N_{\eta,i}(\theta^N)^\lambda \frac{\partial\theta^N}{\partial x_i} \, dx \, dt \\
&= \int_0^T \int_\Omega \frac{\lambda}{\lambda+1} v^N_{\eta,i} \frac{\partial(\theta^N)^{\lambda+1}}{\partial x_i} \, dx \, dt \overset{\operatorname{div} \boldsymbol{v}_\eta = 0}{=} 0.
\end{aligned} \tag{2.46}$$

I_2 can be estimated[5]

$$\begin{aligned}
I_2 &= \int_0^T (\theta^N_{,t}, (\theta^N)^\lambda) \, dt = \frac{1}{1+\lambda} \int_0^T \int_\Omega ((\theta^N)^{\lambda+1})_{,t} \, dx \, dt \\
&= \int_\Omega (\theta^N(T))^{\lambda+1} - (\theta^N_0)^{\lambda+1} \, dx.
\end{aligned} \tag{2.47}$$

Inserting (2.46)-(2.47) into (2.45), using $k \ge C_1 > 0$ and (2.44), we conclude that

$$\int_{\Omega\times(0,T)} |\nabla(\theta^N)^{\frac{\lambda+1}{2}}|^2 \, dx \, dt \le C(\lambda). \tag{2.48}$$

Embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ then implies that $(\theta^N)^{\frac{\lambda+1}{2}}$ is bounded in the space $L^2(0,T;L^6(\Omega))$ for all $\lambda < 0$. Combination of (2.44) and (2.48) then leads to the following conclusion (after using standard interpolation inequality)

$$\|\theta^N\|_{L^n(0,T;L^n(\Omega))} \le C(n) \text{ for all } n \in \left\langle 1, \frac{5}{3} \right). \tag{2.49}$$

Finally, to get an estimate of the gradient of temperature we can compute for all $s \in \langle 1, 2)$ (For simplicity, we write $Q := \Omega \times (0,T)$.)

$$\begin{aligned}
\int_Q |\nabla\theta^N|^s \, dx \, dt &= \int_Q |\nabla\theta^N|^s (\theta^N)^{(\lambda-1)\frac{s}{2}} (\theta^N)^{(1-\lambda)\frac{s}{2}} \, dx \, dt \\
&\le \left(\int_Q |\nabla\theta^N|^2 (\theta^N)^{(\lambda-1)} \, dx \, dt \right)^{\frac{s}{2}} \left(\int_Q (\theta^N)^{(1-\lambda)\frac{s}{2-s}} \, dx \, dt \right)^{\frac{2-s}{2}}.
\end{aligned} \tag{2.50}$$

[5] This computation is formal. But because the first integral is meaningful one can argue by density of smooth functions.

Combining (2.50), (2.49) and (2.48), we conclude

$$\int_0^T \|\theta^N\|^s_{W^{1,s}(\Omega)}\, dt \le C \quad \text{for all } s \in \left(1, \frac{5}{4}\right). \tag{2.51}$$

Thus, similarly as in the preceding paragraph, we can estimate

$$\|\theta^N_{,t}\|_{L^1(0,T;W^{-1,q'}(\Omega))} \le C \text{ for } q \text{ being sufficiently large.} \tag{2.52}$$

Moreover, estimate (2.24) gives us the following information

$$\|\boldsymbol{v}^N_{,t}\|L^2(0,T;W^{-1,2}_{\boldsymbol{n}}) \le C(\varepsilon,\eta). \tag{2.53}$$

Limit $N \to \infty$: Using generalized version of Aubin-Lions lemma (Theorem B.3) and (2.49)- (2.52) we have (after taking subsequence that is not relabeled)

$$\theta^N \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \qquad \text{for } s \in \left(1, \frac{5}{4}\right), \tag{2.54}$$

$$\theta^N \to \theta \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \qquad \text{for } m \in \left(1, \frac{5}{3}\right), \tag{2.55}$$

$$(\theta^N)^{\frac{\lambda+1}{2}} \rightharpoonup \theta^{\frac{\lambda+1}{2}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \qquad \text{for } \lambda < 0, \tag{2.56}$$

using (2.24), (2.53) then implies

$$\boldsymbol{v}^N_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}), \tag{2.57}$$

$$\boldsymbol{v}^N \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{2.58}$$

$$\boldsymbol{v}^N \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}), \tag{2.59}$$

$$\boldsymbol{v}^N \to \boldsymbol{v} \qquad \text{strongly in } L^n(0,T;L^n(\Omega)^3) \qquad \text{for } n \in \left(1, \frac{10}{3}\right), \tag{2.60}$$

$$p^N \rightharpoonup p \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{2.61}$$

Finally, Corollary B.1 together with (2.24) and (2.53) imply that

$$\operatorname{tr} \boldsymbol{v}^N \to \operatorname{tr} \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)). \tag{2.62}$$

These convergences are enough to pass to the limit in (2.39) to get for all $\boldsymbol{\varphi} \in L^2(0,T;W^{1,2}_{\boldsymbol{n}})$

$$\begin{aligned} &\int_0^T \big(\langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla\boldsymbol{\varphi}) + (\nu(\theta)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{\varphi})) \\ &+\alpha \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{\varphi}\, dS\big)\, dt = \int_0^T \left((p, \operatorname{div}\boldsymbol{\varphi}) + \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle\right)\, dt \end{aligned} \tag{2.63}$$

and we are also able to get

$$-\varepsilon(\nabla p(t), \nabla\varphi) = (\varphi, \operatorname{div} \boldsymbol{v}(t)) \tag{2.64}$$

for all $\varphi \in W^{1,2}(\Omega)$ and a.a. $t \in (0,T)$.

Passing to the limit in all terms on the left hand side of (2.38) is standard. To get the limit also for the term on the right hand side we need to establish the convergence of the most critical term $\nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2$. To do it, we test (2.39) by $\boldsymbol{v}^N$ (i.e., we multiply the i-th equation by $c_i^N(t)$) and limit equation (2.63) by $\boldsymbol{v}$ and we set $N \to \infty$ to get

$$\lim_{N\to\infty} \int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2 \, dx \, dt = \int_0^T \int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt. \tag{2.65}$$

Then we can compute

$$\begin{aligned} C\int_0^T \|\mathbf{D}(\boldsymbol{v}^N - \boldsymbol{v})\|_2^2 \, dt &\le \int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N - \boldsymbol{v})|^2 \, dx \, dt \\ = \int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2 &+ \nu(\theta^N)|\mathbf{D}(\boldsymbol{v})|^2 - 2\nu(\theta^N)\mathbf{D}(\boldsymbol{v}^N)\cdot\mathbf{D}(\boldsymbol{v}) \, dx \, dt. \end{aligned}$$

Using Lebesgue dominated convergence theorem and weak convergence (2.59) we get that

$$\begin{aligned} -2\int_0^T \int_\Omega \nu(\theta^N)\mathbf{D}(\boldsymbol{v}^N)\cdot\mathbf{D}(\boldsymbol{v}) \, dx \, dt &\to -2\int_0^T \int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt, \\ \int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt &\to \int_0^T \int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt, \end{aligned}$$

and (2.65) implies that

$$\int_0^T \int_\Omega \nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2 \, dx \, dt \to \int_0^T \int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \, dx \, dt.$$

Thus we conclude that

$$\mathbf{D}(\boldsymbol{v}^N) \to \mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^2(0,T;L^2(\Omega)^{3\times 3}). \tag{2.66}$$

The relation (2.66) and (2.55), assumptions on ν and Lebesgue dominated convergence theorem then imply that

$$\nu(\theta^N)|\mathbf{D}(\boldsymbol{v}^N)|^2 \to \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

Hence, we can pass to the limit also in (2.38) to get for all $\varphi \in \mathcal{D}^\infty(-\infty,T;W^{1,q}(\Omega))$ (q being sufficiently large)

$$\begin{aligned} \int_0^T -(\theta,\varphi_{,t}) - (\theta\boldsymbol{v}_\eta,\nabla\varphi) + (k(\theta)\nabla\theta,\nabla\varphi) \, dt \\ = \int_0^T (\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2,\varphi) \, dt + (\theta_0,\varphi(0)). \end{aligned} \tag{2.67}$$

Note that the point-wise convergence of θ^N and Fatou's lemma also imply that

$$\|\theta(t)\|_1 \leq \liminf_{N\to\infty} \|\theta^N(t)\|_1 \leq C \tag{2.68}$$

for almost all $t \in (0,T)$. Next, we can define a set $S := \{t \in (0,T); \int_\Omega \theta(t,x)\,dx < \infty\}$ and χ_τ being characteristic function of the time interval $(0,\tau)$. Let $\psi \in \mathcal{C}^1(\Omega)$ be arbitrary. Then for an arbitrary but fixed $\tau \in S$ we set in (2.40) $\varphi := \chi_\tau \psi$. After integration per partes w.r.t. time and passing to the limit with N we obtain that

$$(\theta(\tau),\psi) - (\theta_0,\psi) = \int_0^\tau -(k(\theta),\nabla\theta,\nabla\psi) + (\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2,\psi) + (\theta\boldsymbol{v}_\eta,\nabla\psi)\;dt. \tag{2.69}$$

Subtracting this formula for $\tau_1 \in S$ from that one for $\tau_2 \in S$ and passing with $\tau_1 \to \tau_2$, we see that the functional of *internal* energy

$$E_{int}(t,\psi) := \int_\Omega \theta(t,x)\psi(x)\;dx$$

is continuous function of time on S. Moreover, it can be defined also for $t \in (0,T)\setminus S$ by the formula (2.69). Because of density of $\mathcal{C}^1$ functions in $\mathcal{C}$ and estimate (2.68), we can also define $E_{int}(t,\psi)$ for all $\psi \in \mathcal{C}(\Omega)$. Formula (2.69) also leads to the conclusion that

$$\lim_{t\to 0+} E_{int}(t,\psi) = \int_\Omega \theta_0(x)\psi(x)\;dx.$$

It remains to prove attainment of initial condition $\boldsymbol{v}_0$. Using the same tools as above (and density of smooth functions in the space $L^2(\Omega)^3$) we can simply observe that

$$\boldsymbol{v} \in \mathcal{C}(0,T;L^2_{weak}(\Omega)^3).$$

and that

$$\boldsymbol{v}(t) \rightharpoonup \boldsymbol{v}_0 \text{ weakly in } L^2(\Omega)^3 \text{ as time } t \to 0_+.$$

For the strong convergence of $\boldsymbol{v}(t)$ to $\boldsymbol{v}_0$ it is enough to show that $\|\boldsymbol{v}(t)\|_2^2 \to \|\boldsymbol{v}_0\|_2^2$ as time $t \to 0_+$. To do it we multiply the j-th equation in (2.39) by c_j^N sum over $j = 1,\dots N$ and integrate over time $t \in (0,\tau)$. Passing to the limit with N and using weak lower semicontinuity of norm in reflexive space, we get for a.a. $\tau \in (0,T)$

$$\|\boldsymbol{v}(\tau)\|_2^2 + \int_0^\tau \left(\varepsilon\|\nabla p\|_2^2 + \int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2\;dx + \alpha\int_{\partial\Omega}|\boldsymbol{v}|^2\;dS - \langle \boldsymbol{f},\boldsymbol{v}\rangle\right)dt \leq \|\boldsymbol{v}_0\|_2^2.$$

Therefore we have that

$$\limsup_{t\to 0+}\|\boldsymbol{v}(t)\|_2^2 \leq \|\boldsymbol{v}_0\|_2^2 \leq \liminf_{t\to 0+}\|\boldsymbol{v}(t)\|_2^2$$

and we conclude.

2.2.3 Limit $\varepsilon \to 0$

Here we will pass to the limit with $\varepsilon \to 0$ to get the weak solution of the following problem $(\mathcal{P}_{ns})^\eta$ (for simplicity we denote $(\boldsymbol{v},\theta,p) := (\boldsymbol{v}^\eta,\theta^\eta,p^\eta)$)

$$
(\mathcal{P}_{ns})^\eta \qquad
\begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div}\left(\nu(\theta)\mathbf{D}(\boldsymbol{v})\right) + \nabla p &= \boldsymbol{f}\\
\operatorname{div}\boldsymbol{v} &= 0\\
\theta_{,t} + \operatorname{div}(\theta \boldsymbol{v}_\eta) - \operatorname{div}\left(k(\theta)\nabla\theta\right) - \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 &= 0
\end{aligned}\right\} && \text{in } \Omega\times(0,T),\\
&\left.\begin{aligned}
\left(\nu(\theta)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right)_\tau + \alpha \boldsymbol{v}_\tau &= 0\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0\\
\nabla\theta\cdot\boldsymbol{n} &= 0
\end{aligned}\right\} && \text{on } \partial\Omega\times(0,T),\\
&\int_\Omega p(x,t)\, dx = 0 && \text{a. a. } t\in(0,T),\\
&\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0 && \text{in } \Omega,\\
&\theta(\cdot,0) = \theta_0 && \text{in } \Omega.
\end{aligned}
$$

At this step we denote by $(\boldsymbol{v}^\varepsilon,\theta^\varepsilon,p^\varepsilon) := (\boldsymbol{v}^{\varepsilon,\eta},\theta^{\varepsilon,\eta},p^{\varepsilon,\eta})$ weak solution of the problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$.

Using the weak lower semicontinuity of norm, Fatou's lemma and the relations (2.24), (2.49) and (2.51), we get

$$
\begin{aligned}
\sup_{t\in(0,T)} \left(\|\boldsymbol{v}^\varepsilon(t)\|_2^2 + \|\theta^\varepsilon(t)\|_1\right) + \int_0^T \|\mathbf{D}(\boldsymbol{v}^\varepsilon)\|_2^2 &+ \|\nabla(\theta^\varepsilon)^{\frac{\lambda+1}{2}}\|_2^2\\
&+\varepsilon\|\nabla p^\varepsilon\|_2^2\, dt \le C
\end{aligned}
\tag{2.70}
$$

for all $\lambda < 0$. Consequently, standard interpolation inequalities then lead to the estimate

$$
\int_0^T \|\boldsymbol{v}^\varepsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + \|\nabla\theta^\varepsilon\|_n^n + \|\theta^\varepsilon\|_m^m \le C \tag{2.71}
$$

for all $n \in \left\langle 1, \frac{5}{4}\right)$ and all $m \in \left\langle 1, \frac{5}{3}\right)$.

To get apriori estimates on the pressures p^ε (independently of ε), we set φ^ε such that is solves the following problem $\mathcal{Q}$

$$
\triangle\varphi^\varepsilon = |p^\varepsilon|^{\beta-2}p^\varepsilon - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon|^{\beta-2}p^\varepsilon \qquad \text{in } \Omega, \tag{2.72}
$$

$$
\nabla\varphi^\varepsilon\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{2.73}
$$

$$
\int_\Omega \varphi^\varepsilon = 0, \tag{2.74}
$$

for some $\beta \le 2$. Having the theory for Laplace equation (Lemma B.1), we get

$$
\|\nabla\varphi^\varepsilon\|_{W^{1,\beta'}(\Omega)}^{\beta'} \le C\|p^\varepsilon\|_\beta^\beta. \tag{2.75}
$$

We use $\nabla\varphi^\varepsilon$ as a test function in the problem $(\mathcal{P}_{ns})^{\varepsilon,\eta}$, i.e., we set in (2.63) $\boldsymbol{\varphi} := \nabla\varphi^\varepsilon$ and it leads to

$$\int_0^T \|p^\varepsilon\|_\beta^\beta \, dt := I_1 + I_2 + I_3 + I_4 + I_5, \tag{2.76}$$

where (for simplicity we use the notation $Q := \Omega \times (0,T)$)

$$\begin{aligned}
I_1 &= \int_Q \nu(\theta^\varepsilon)\mathbf{D}(\boldsymbol{v}^\varepsilon)\cdot\mathbf{D}(\nabla\varphi^\varepsilon)\,dx\,dt \overset{(2.75)}{\leq} C\int_Q |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^\beta\,dx\,dt + \frac{1}{8}\int_0^T \|p^\varepsilon\|_\beta^\beta\,dt,\\
I_2 &= -\int_0^T \langle \boldsymbol{f}, \nabla\varphi^\varepsilon\rangle\,dt \leq C\int_0^T \|\boldsymbol{f}\|^2_{W_{\boldsymbol{n}}^{-1,2}}\,dt + C + \frac{1}{8}\int_0^T \|p^\varepsilon\|_\beta^\beta\,dt,\\
I_3 &= \alpha\int_0^T\int_{\partial\Omega} \boldsymbol{v}^\varepsilon\cdot\nabla\varphi^\varepsilon\,dS\,dt \leq \alpha\int_0^T \|\operatorname{tr}\boldsymbol{v}^\varepsilon\|_2\|\operatorname{tr}\nabla\varphi^\varepsilon\|_2\,dt\\
&\overset{(B.5)}{\leq} C\int_0^T \|\operatorname{tr}\boldsymbol{v}^\varepsilon\|_2^2 + \|\nabla\varphi^\varepsilon\|^2_{W^{1,2}(\Omega)}\,dt\\
&\overset{\beta\leq 2}{\leq} C\int_0^T (1+\|\operatorname{tr}\boldsymbol{v}^\varepsilon\|_2^2)\,dt + \frac{1}{8}\int_0^T \|p^\varepsilon\|_\beta^\beta\,dt,\\
I_4 &= -\int_Q (\boldsymbol{v}^\varepsilon_\eta\otimes\boldsymbol{v}^\varepsilon)\cdot\nabla^2\varphi^\varepsilon\,dx\,dt \leq C\int_Q |\boldsymbol{v}^\varepsilon_\eta\otimes\boldsymbol{v}^\varepsilon|^\beta\,dx\,dt + \frac{1}{8}\int_0^T \|p^\varepsilon\|_\beta^\beta\,dt,\\
I_5 &= \int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \nabla\varphi^\varepsilon\rangle\,dt.
\end{aligned}$$

First, we will show that $I_5 \leq 0$. We find a sequence of smooth function $\boldsymbol{v}^{\delta,\varepsilon}$ such that $\boldsymbol{v}^{\delta,\varepsilon}_{,t} \to \boldsymbol{v}^\varepsilon_{,t}$ strongly in $L^2(0,T;W^{-1,2}_{\boldsymbol{n}})$ and $\boldsymbol{v}^{\delta,\varepsilon} \to \boldsymbol{v}^\varepsilon$ strongly in $L^2(0,T;W^{1,2}_{\boldsymbol{n}})$. Then we find a sequence $p^{\delta,\varepsilon}$ solving $\varepsilon\triangle p^{\delta,\varepsilon} = \operatorname{div}\boldsymbol{v}^{\delta,\varepsilon}$ with homogeneous Neuman boundary condition and mean value equal to zero. Then we define functions $\varphi^{\delta,\varepsilon}$ as the solution of the problem $\mathcal{Q}$ ((2.72)-(2.74)) where we replace p^ε by $p^{\delta,\varepsilon}$. Then it holds that

$$I_5 = \lim_\delta I_5^\delta := \lim_\delta \int_0^T \langle \boldsymbol{v}^{\delta,\varepsilon}_{,t}, \nabla\varphi^{\delta,\varepsilon}\rangle\,dt.$$

Next, we apply to the velocity $\boldsymbol{v}^{\delta,\varepsilon}$ the Helmholtz decomposition (1.21) to obtain

$$\begin{aligned}
I_5^\delta &= \int_0^T \langle \boldsymbol{v}^{\delta,\varepsilon}_{\mathrm{div},t} + \nabla g^{\boldsymbol{v}^{\delta,\varepsilon}}_{,t}, \nabla\varphi^{\delta,\varepsilon}\rangle\,dt \overset{\operatorname{div}\boldsymbol{v}^{\delta,\varepsilon}_{\mathrm{div}}=0}{=} \int_0^T \langle \nabla g^{\boldsymbol{v}^{\delta,\varepsilon}}_{,t}, \nabla\varphi^{\delta,\varepsilon}\rangle\,dt\\
&\overset{\nabla\varphi^{\delta,\varepsilon}\cdot\boldsymbol{n}=0|_{\partial\Omega}}{=} -\int_0^t (g^{\boldsymbol{v}^{\delta,\varepsilon}}_{,t}, \triangle\varphi^{\delta,\varepsilon})\,dt.
\end{aligned}$$

By using the definition of $g^{\boldsymbol{v}^{\delta,\varepsilon}}$, $\varphi^{\delta,\varepsilon}$ and the equation (2.64) we have

$$\frac{1}{\varepsilon}\triangle g^{\boldsymbol{v}^{\delta,\varepsilon}} = \frac{1}{\varepsilon}\operatorname{div}\boldsymbol{v}^{\delta,\varepsilon} = \triangle p^{\delta,\varepsilon}.$$

The uniqueness of the solution to the Laplace equation then gives

$$\frac{1}{\varepsilon}g^{\boldsymbol{v}^{\delta,\varepsilon}} = p^{\delta,\varepsilon}.$$

Hence (after using $\int_\Omega p^{\delta,\varepsilon}(x,t)\,dx = 0$),

$$\begin{aligned}I_5^\delta &= -\varepsilon\int_0^T (p^{\delta,\varepsilon}_{,t}, |p^{\delta,\varepsilon}|^{\beta-2}p^{\delta,\varepsilon})\,dt = -\frac{\varepsilon}{\beta}\int_0^T \frac{d}{dt}\|p^{\delta,\varepsilon}\|_\beta^\beta \\ &= -\frac{\varepsilon}{\beta}\|p^{\delta,\varepsilon}(t)\|_\beta^\beta + \frac{\varepsilon}{\beta}\|p^{\delta,\varepsilon}(0)\|_\beta^\beta \le 0,\end{aligned}$$

where we used the fact that $p^{\delta,\varepsilon}(0) = 0$ ($\operatorname{div}\boldsymbol{v}^{\delta,\varepsilon}(0) = 0$).
By setting $\beta = 2$ we obtain (by using (2.71)) that

$$I_4 \le C(\eta) + \frac{1}{8}\int_0^T \|p^\varepsilon\|_2^2\,dt.$$

Consequently, inserting the estimates for $I_1 - I_5$ into (2.76) (with $\beta = 2$), we observe

$$\int_0^T \|p^\varepsilon\|_2^2\,dt \le C(\eta). \tag{2.77}$$

If we set $\beta := \frac{5}{3}$ we get that (after using (2.71))

$$I_4 \le C + \frac{1}{8}\int_0^T \|p^\varepsilon\|_\beta^\beta\,dt,$$

and conclude that

$$\int_0^T \|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}}\,dt \le C \tag{2.78}$$

independently of the order of approximation. Finally, (2.63), (2.67), (2.70) and (2.77) imply that

$$\|\boldsymbol{v}^\varepsilon_{,t}\|_{L^2(0,T;W^{-1,2}_{\boldsymbol{n}})} + \|\theta^\varepsilon_{,t}\|_{L^1(0,T;W^{-1,q'}(\Omega))} \le C(\eta) \quad \text{for } q \text{ sufficiently large} \tag{2.79}$$

Having (2.70), (2.71), (2.79), (2.77) and Aubin-Lions lemma, we can take a subsequence that is not relabeled such that

$$
\begin{aligned}
\boldsymbol{v}^\varepsilon_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}),\\
\boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}),\\
\boldsymbol{v}^\varepsilon &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3),\\
(\theta^\varepsilon)^{\frac{\lambda+1}{2}} &\rightharpoonup (\theta)^{\frac{\lambda+1}{2}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) && \text{for } \lambda<0,\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^q(0,T;L^q(\Omega)^3) && \text{for } q\in\left\langle 1,\frac{10}{3}\right),\\
\theta^\varepsilon &\rightharpoonup \theta && \text{weakly in } L^n(0,T;W^{1,n}(\Omega)) && \text{for } n\in\left\langle 1,\frac{5}{4}\right),\\
\theta^\varepsilon &\to \theta && \text{strongly in } L^m(0,T;L^m(\Omega)) && \text{for } m\in\left\langle 1,\frac{5}{3}\right),\\
\nu(\theta^\varepsilon)\mathbf{D}(\boldsymbol{v}^\varepsilon) &\rightharpoonup \nu(\theta)\mathbf{D}(\boldsymbol{v}) && \text{weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}),\\
p^\varepsilon &\rightharpoonup p && \text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned}
$$

Moreover, using Corollary B.1, we have

$$
\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^3).
$$

First, we pass to the limit in (2.64). Let $\varphi \in L^2(0,T;W^{1,2}(\Omega))$ be arbitrary, then

$$
\begin{aligned}
\left|\int_0^T (\operatorname{div}\boldsymbol{v},\varphi)\,dt\right| &= \lim_{\varepsilon\to 0}\left|\int_0^T (\operatorname{div}\boldsymbol{v}^\varepsilon,\varphi)\,dt\right| \overset{(2.64)}{=} \lim_{\varepsilon\to 0}\left|\varepsilon\int_0^T\int_\Omega \nabla\varphi\cdot\nabla p^\varepsilon\,dt\right|\\
&\le \lim_{\varepsilon\to 0}\underbrace{\left(\int_Q \varepsilon|\nabla\varphi|^2\,dx\,dt\right)^{\frac12}}_{\to 0}\underbrace{\left(\int_Q \varepsilon|\nabla p^\varepsilon|^2\right)^{\frac12}}_{\overset{(2.70)}{\le} C} = 0.
\end{aligned}
\tag{2.80}
$$

Hence, (2.80) implies that

$$
\operatorname{div}\boldsymbol{v} = 0 \qquad \text{a. e. in } \Omega\times(0,T). \tag{2.81}
$$

Using again weak convergence, that has already been shown above, we can pass to the limit in (2.63). It means that we replace $(\boldsymbol{v},\theta,p)$ in (2.63) by $(\boldsymbol{v}^\varepsilon,\theta^\varepsilon,p^\varepsilon)$ and set $\varepsilon\to 0$ to get

$$
\begin{aligned}
&\int_0^T \big(\langle \boldsymbol{v}_{,t},\boldsymbol{\varphi}\rangle - (\boldsymbol{v}_\eta\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi}) + (\nu(\theta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\varphi}))\\
&+\alpha\int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{\varphi}\,dS\big)\,dt = \int_0^T (p,\operatorname{div}\boldsymbol{\varphi}) + \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle\,dt.
\end{aligned}
\tag{2.82}
$$

Moreover, we will show that

$$\int_Q \nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 \overset{\varepsilon\to 0}{\to} \int_Q \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2. \tag{2.83}$$

As in preceding subsection, (2.83) then will imply that

$$\mathbf{D}(\boldsymbol{v}^\varepsilon) \to \mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^2(0,T;L^2(\Omega)^{3\times 3}).$$

Thus, we will able to pass to the limit in (2.67) (where we will replace $(\boldsymbol{v},\theta)$ by $(\boldsymbol{v}^\varepsilon,\theta^\varepsilon)$) to get

$$\begin{aligned}\int_0^T \big(-(\theta,\varphi_{,t}) - (\theta\boldsymbol{v}_\eta,\nabla\varphi) + (k(\theta)\nabla\theta,\nabla\varphi)\big)\,dt \\ = \int_0^T (\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2,\varphi)\,dt + (\theta_0,\varphi(0))\end{aligned} \tag{2.84}$$

for all $\varphi \in \mathcal{D}(-\infty,T;W^{1,q}(\Omega))$, q being sufficiently large.

It remains to show that (2.83) is valid. First we show that

$$\int_Q \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2 \le \liminf_{\varepsilon\to 0} \int_Q \nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2. \tag{2.85}$$

To observe it we can split

$$\nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 = \nu(\theta^\varepsilon)(|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 - |\mathbf{D}(\boldsymbol{v})|^2) + \nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v})|^2 =: I_1^\varepsilon + I_2^\varepsilon.$$

Lebesgue convergence theorem implies that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega I_2^\varepsilon\,dx\,dt = \int_0^T\int_\Omega \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2\,dx\,dt.$$

Hence, it is enough to show that $\liminf \int_0^T\int_\Omega I_1^\varepsilon \ge 0$ to prove (2.85). But using the assumption on the viscosity ν, we have for all ε that

$$\int_0^T\int_\Omega I_1^\varepsilon\,dx\,dt \ge \begin{cases} \int_0^T\int_\Omega C_1(|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 - |\mathbf{D}(\boldsymbol{v})|^2)\,dx\,dt \text{ if } \int_0^T\int_\Omega I_1^\varepsilon \ge 0, \\ \int_0^T\int_\Omega C_2(|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 - |\mathbf{D}(\boldsymbol{v})|^2)\,dx\,dt \text{ if } \int_0^T\int_\Omega I_1^\varepsilon < 0. \end{cases}$$

Therefore, weak lower semicontinuity of norm then implies that

$$\liminf_{\varepsilon\to 0} I_1^\varepsilon \ge 0.$$

To show (2.85) with opposite inequality sign, one can compute

$$
\begin{aligned}
\int_Q \nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 \,dx\,dt &\leq \int_Q \nu(\theta^\varepsilon)|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2 + \varepsilon|\nabla p^\varepsilon|^2 \,dx\,dt \\
&\overset{(2.63)}{=} -\frac{1}{2}\|\boldsymbol{v}^\varepsilon(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon\rangle \,dt - \alpha\int_{\partial\Omega} |\boldsymbol{v}^\varepsilon|^2 \,dS\,dt \\
&\overset{\varepsilon\to 0}{\leq} -\frac{1}{2}\|\boldsymbol{v}(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}\rangle \,dt - \alpha\int_{\partial\Omega} |\boldsymbol{v}|^2 \,dS\,dt \\
&\overset{(2.82)}{=} \int_Q \nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2,
\end{aligned}
\tag{2.86}
$$

and we see that (2.83) is satisfied.

2.2.4 Limit $\eta \to 0$

Here, we prove the existence of weak solution to our original problem $(\mathcal{P}_{ns})$. We use approximative problems $(\mathcal{P}_{ns})^\eta$ and we denote by $(\boldsymbol{v}^\eta, \theta^\eta, p^\eta)$ their solutions. Using weak lower semicontinuity of norm, we have the estimates (2.70) and (2.78). As in the preceding section, the relations (2.70) and (2.78) allow us to get

$$
\|\boldsymbol{v}^\eta_{,t}\|_{L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}})} + \|\theta^\eta_{,t}\|_{L^1(0,T;W^{1,s'}(\Omega))} \leq C \qquad \text{for } s \text{ sufficiently large.}
\tag{2.87}
$$

Hence, Aubin-Lions lemma and (2.70), (2.78), (2.87) imply that we can take a subsequence such that

$$
\begin{aligned}
\boldsymbol{v}^\eta_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^2(0,T;W_{\boldsymbol{n}}^{-1,2}), \\
\boldsymbol{v}^\eta &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), \\
\boldsymbol{v}^\eta &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;W_{\boldsymbol{n}}^{1,2}), \\
\theta^\eta &\to \theta && \text{strongly in } L^m(0,T;L^m(\Omega)) \qquad \text{for } m \in \left\langle 1, \frac{5}{3}\right), \\
(\theta^\eta)^{\frac{\lambda+1}{2}} &\rightharpoonup (\theta)^{\frac{\lambda+1}{2}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \qquad \text{for } \lambda < 0, \\
\boldsymbol{v}^\eta &\to \boldsymbol{v} && \text{strongly in } L^n(0,T;L^n(\Omega)^3) \qquad \text{for } n \in \left\langle 1, \frac{10}{3}\right), \\
\theta^\eta &\rightharpoonup \theta && \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \qquad \text{for } s \in \left\langle 1, \frac{5}{4}\right), \\
\nu(\theta^\eta)\mathbf{D}(\boldsymbol{v}^\eta) &\rightharpoonup \nu(\theta)\mathbf{D}(\boldsymbol{v}) && \text{weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}), \\
p^\eta &\rightharpoonup p && \text{weakly in } L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega)).
\end{aligned}
$$

Next, it is easy to pass to the limit in (2.82) to get (2.10). Finally, let $\varphi \in \mathcal{D}(-\infty,T;\mathcal{C}^1(\Omega))$. We set $\boldsymbol{\varphi}^\eta := \boldsymbol{v}^\eta\varphi$ in (2.82) and we test (2.84) by φ. We add the result equations to obtain

$$\int_0^T -\left(\frac{|\boldsymbol{v}^\eta|^2}{2}+\theta^\eta,\varphi_{,t}\right)-\left(\boldsymbol{v}^\eta_\eta(\frac{|\boldsymbol{v}^\eta|^2}{2}+\theta^\eta)+\boldsymbol{v}^\eta p^\eta,\nabla\varphi\right)+(k(\theta^\eta)\nabla\theta^\eta,\nabla\varphi)$$
$$+\alpha\int_{\partial\Omega}|\boldsymbol{v}^\eta|^2\varphi\,dS+(\nu(\theta^\eta)\mathbf{D}(\boldsymbol{v}^\eta)\boldsymbol{v}^\eta,\nabla\varphi)-\langle\boldsymbol{f},\boldsymbol{v}^\eta\varphi\rangle\,dt=\frac{1}{2}|\boldsymbol{v}_0|^2+\theta_0. \tag{2.88}$$

But it is easy to pass to the limit in (2.88) to get (2.11).

It remains to prove the inequality (2.12). To do it, we set an arbitrary ψ, $\psi\geq 0$ a.e. in $\Omega\times(0,T)$ and $\psi\in\mathcal{D}(-\infty,T;\mathcal{C}^\infty(\Omega))$. Then we test (2.84) by ψ. It is easy to pass to the limit in all terms on the left hand side. To pass in the remaining term we use weak-lower-semicontinuity, i.e.,

$$\int_0^T\int_\Omega\nu(\theta)|\mathbf{D}(\boldsymbol{v})|^2\psi\,dx\,dt\leq\liminf_{\eta\to 0+}\int_0^T\int_\Omega\nu(\theta^\eta)|\mathbf{D}(\boldsymbol{v}^\eta)|^2\psi\,dx\,dt$$

provided ψ is non-negative, θ^η converges point-wisely and $\mathbf{D}(\boldsymbol{v}^\eta)$ weakly in $L^2(0,T;L^2(\Omega)^{3\times 3})$. This completes the proof of (2.12).

The rest of the proof, i.e., attainment of initial condition and continuity of the functional of global energy $E(t,\varphi)$ can be done by using the same tools as in the preceding subsection.

3 Non-Newtonian fluids with pressure and shear rate dependent viscosity ($r < 2$)

3.1 The assumptions on the structure of the viscosity

Here, we consider the case when the viscosity depends only on the pressure and the shear rate in suitable form. It means

$$\nu(\theta, p, |\mathsf{D}|^2) := \nu(p, |\mathsf{D}|^2).$$

We assume that the viscosity ν is a $\mathcal{C}^1$-mapping of $\mathbb{R} \times \mathbb{R}_0^+$ into $\mathbb{R}^+$ satisfying for some fixed (but arbitrary) $r \in [1, 2)$ and all $\mathsf{D} \in \mathbb{R}^{3\times 3}_{\text{sym}}$, $\mathsf{B} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ and $p \in \mathbb{R}$ the following inequalities

$$C_1(1 + |\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 \le \frac{\partial \nu(p, |\mathsf{D}|^2)\mathsf{D}_{ij}}{\partial \mathsf{D}_{kl}}\mathsf{B}_{ij}\mathsf{B}_{kl} \le C_2(1 + |\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2, \tag{3.1}$$

$$\left|\frac{\partial \nu(p, |\mathsf{D}|^2)}{\partial p}\right| |\mathsf{D}| \le \gamma_0(1 + |\mathsf{D}|^2)^{\frac{r-2}{4}}, \tag{3.2}$$

where $\gamma_0 > 0$ is a constant whose value will be restricted in the formulation of the main theorem.

If ν is independent of p we see that (3.2) holds trivially and (3.1) is fulfilled by the so-called generalized power-law-like fluids. On the other hand, our assumptions do not permit to consider any model where the viscosity depends on the pressure only.

Before giving the definition of the problem, we list some examples of viscosities fulfilling (3.1) and (3.2).

Example 3.1. *Consider*

$$\nu_i(p, |\mathsf{D}|^2) = \left(1 + \gamma_i(p) + |\mathsf{D}|^2\right)^{\frac{r-2}{2}}, \quad i = 1, 2, \tag{3.3}$$

where $\gamma_i(p)$ have the form ($s \ge 0$)

$$\left.\begin{aligned} &\gamma_1(p) = (1 + \alpha^2 p^2)^{-s/2}, \\ &\gamma_2(p) = \begin{cases} (1 + \exp(\alpha p))^{-s} & \text{if } p > 0 \\ 1 & \text{if } p \le 0 \end{cases} \end{aligned}\right\} \Rightarrow 0 \le \gamma_i(p) \le 1, \ (i = 1, 2). \tag{3.4}$$

We show, that the viscosities given by (3.3)-(3.4) satisfy (3.1)-(3.2) with any parameter $r \in (1, 2)$. In fact, the relation (3.1) is proved in [21] and we skip the proof of it here. To prove (3.2) we observe that

$$\begin{aligned}\left|\frac{\partial \nu(p,|\mathbf{D}|^2)}{\partial p}\right||\mathbf{D}| &= \left|\frac{r-2}{2}\right|\left(1+\gamma_i(p)+|\mathbf{D}|^2\right)^{\frac{r-4}{2}}|\gamma_i'(p)||\mathbf{D}| \\ &= \frac{2-r}{2}(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}(1+|\mathbf{D}|^2)^{\frac{2-r}{4}}|\gamma_i'(p)||\mathbf{D}|\left(1+\gamma_i(p)+|\mathbf{D}|^2\right)^{\frac{r-4}{2}} \\ &\le \frac{2-r}{2}|\gamma_i'(p)|(1+|\mathbf{D}|^2)^{\frac{4-r}{4}}(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}\left(1+\gamma_i(p)+|\mathbf{D}|^2\right)^{\frac{r-4}{2}} \\ &\le \frac{2-r}{2}\frac{|\gamma_i'(p)|}{(1+\gamma_i(p))^{\frac{4-r}{4}}}(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}.\end{aligned}$$

For γ_1 we have

$$\begin{aligned}\frac{|\gamma_1'(p)|}{(1+\gamma_1(p))^{\frac{4-r}{4}}} &\le \alpha^2 s|p|\frac{(1+\alpha^2p^2)^{\frac{-s-2}{2}}}{(1+\alpha^2p^2)^{-\frac{s}{2}\frac{4-r}{4}}} \le \alpha s\frac{(1+\alpha^2p^2)^{\frac{-s-1}{2}}}{(1+\alpha^2p^2)^{-\frac{s}{2}\frac{4-r}{4}}} \\ &\le \alpha s(1+\alpha^2p^2)^{-\frac{rs}{8}-\frac{1}{2}} \le \sqrt{2}\alpha s(1+\alpha|p|)^{-(1+\frac{rs}{4})}.\end{aligned}$$

Thus, (3.2) is satisfied with $\gamma_0 := \sqrt{2}\alpha s\frac{2-r}{2}$.
For γ_2 we first notice that $\gamma_2'(p)=0$ if $p\le 0$. For $p>0$ we have

$$\begin{aligned}\frac{|\gamma_2'(p)|}{(1+\gamma_2(p))^{\frac{4-r}{4}}} &\le \alpha s\frac{(1+\exp(\alpha p))^{-s}}{(1+\exp(\alpha p))^{-s\frac{4-r}{4}}} \le \alpha s(1+\exp(\alpha p))^{-\frac{rs}{4}} \\ &\le \alpha s(1+\alpha|p|)^{-1},\end{aligned}$$

and we can set $\gamma_0 := \frac{2-r}{2}\alpha s$. □

3.2 Definition of the problem and main existence theorem

Because the temperature does not appear in the viscosity we can solve only the first two equations in $(\mathcal{P})$ and then we can reconstruct the temperature as is shown for example in [20]. For simplicity we use the same method and do not reconstruct the temperature in this section. Our problem $(\mathcal{P})$ then reduces to the following problem $(\mathcal{P}_1)$

$$(\mathcal{P}_1)\qquad \begin{aligned} \left.\begin{aligned}\boldsymbol{v}_{,t}+\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})-\operatorname{div}\left(\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\right)+\nabla p=\boldsymbol{f} \\ \operatorname{div}\boldsymbol{v}=0\end{aligned}\right\} &\quad \text{in } \Omega\times(0,T), \\ \left.\begin{aligned}\left(\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right)_\tau+\alpha\boldsymbol{v}_\tau=0 \\ \boldsymbol{v}\cdot\boldsymbol{n}=0\end{aligned}\right\} &\quad \text{on } \partial\Omega\times(0,T), \\ \int_\Omega p(x,t)\,dx=0 &\quad \text{a.a. } t\in(0,T), \\ \boldsymbol{v}(\cdot,0)=\boldsymbol{v}_0 &\quad \text{in } \Omega. \end{aligned}$$

The requirement on zero mean value of the pressure p over domains Ω needs some comments. It is obvious that if ν is independent of p then the pressure is determined up to a function of time, and usually in analysis in bounded domain it is fixed so that $\int_\Omega p(x,t)\,dx = 0$ a.e. in $(0,T)$. Since only the gradient of the pressure occurs in such systems, this choice has no influence on the solution $\boldsymbol{v}$ of the problem. This is however not true if ν depends on the pressure. First of all, there is a question whether the pressure should be fixed at all by a condition like $\int p(t,x)\,dx = g(t)$, or whether it is uniquely determined just by equations and boundary conditions. The analysis of spatially-periodic problem suggests that at least in some cases there is an undetermined value of p that has to be fixed (and in spatially-periodic case the choice $\int_\Omega p(t,x)\,dx = 0$ seems to be reasonable). Here, we also use the same condition (even if it does not seem to be very natural).

Next, we define the notion of a solution to the problem $(\mathcal{P}_1)$ if the viscosity satisfies assumptions (3.1)-(3.2)

Definition 3.1. *Let $\Omega \in \mathcal{C}^{0,1}$. Let ν satisfy the assumptions* (3.1) *and* (3.2) *with parameter $r \in (\frac{6}{5}, 2)$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}})$ and $0 < T < \infty$. We say that a couple $(\boldsymbol{v},p)$ is weak solution to the problem $(\mathcal{P}_1)$ if there hold*

$$\boldsymbol{v} \in \mathcal{C}(0,T;L^2_{weak}) \cap L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}), \tag{3.5}$$

$$\boldsymbol{v}_{,t} \in L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{\boldsymbol{n}}), \tag{3.6}$$

$$p \in L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)) \text{ and } \int_\Omega p(x,t) = 0 \text{ for a.a. } t \in (0,T), \tag{3.7}$$

$$\lim_{t\to 0+} \int_\Omega \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0, \tag{3.8}$$

and the following weak formulation is valid

$$\begin{aligned} \int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{\varphi}) + \big(\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{\varphi})\big)\,dt \\ +\alpha\int_0^T \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{\varphi}\,dS\,dt = \int_0^T (p, \operatorname{div}\boldsymbol{\varphi}) + \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle\,dt \end{aligned} \tag{3.9}$$

for all $\boldsymbol{\varphi} \in L^{\frac{5r}{5r-6}}(0,T;W^{1,\frac{5r}{5r-6}}_{\boldsymbol{n}})$.

There is a remarkable difference between the introducing the pressure in time independent models and evolutionary ones for no-slip boundary condition. While for the stationary problems, the pressure can be easily identified using for example de Rham's theorem, the same tool cannot be in general used to evolutionary models since time derivative of the velocity is not a distribution (it usually belongs to a dual space of divergenceless functions only).

There is also remarkable difference in introducing the pressure between the evolutionary NSEs and time-dependent models with non-constant viscosity. For the NSEs, we can identify the model with evolutionary Stokes system, where the convective term is included into the right-hand side, and apply the results on L^p estimates available for such systems (see [29]). For the models where ν is not constant (and may depend on p or $|\mathbf{D}(\boldsymbol{v})|^2$) an analogous theory for generalized Stokes system is up to now not available. This section (or all thesis) says that the Navier's slip boundary conditions, on contrary to no-slip boundary conditions, do not suffer such deficiency and it is possible to introduce the pressure globally.

Now, we formulate the main theorem of this section.

Theorem 3.1. *Let $\Omega \in \mathcal{C}^{1,1}$. Let ν satisfy* (3.1)-(3.2) *with*

$$2 > r > \frac{8}{5},$$

and with γ_0 fulfilling

$$\gamma_0 < \frac{C_1}{C_{reg}(\Omega,2)(C_1 + C_2)}, \tag{3.10}$$

where $C_{reg}(\Omega,2)$ appears in (1.22). *Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ and $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}})$. Then there exists weak solution to the problem $(\mathcal{P}_1)$.*

3.3 Proof of the theorem

3.3.1 The main ingredience of the proof

The proof is split into several steps, all of them forming particular subsections of the Section 3.3 of this thesis.

We use the same approximation as in the Section 2 and we omit the equation for temperature. The proof of the existence of a solution to $(\mathcal{P}_1)^{\varepsilon,\eta}$-approximations for all $\varepsilon > 0$, $\eta > 0$ fixed will be done via Galerkin approximations incorporating the compactness of velocities and compactness of the pressures based on the observation that

$$\begin{aligned}\varepsilon\|\nabla p^N - \nabla p\|_2^2 &= (\operatorname{div}\boldsymbol{v}^N - \operatorname{div}\boldsymbol{v}, p^N - p)\\ &= -(\boldsymbol{v}^N - \boldsymbol{v}, \nabla p^N - p) \to 0 \text{ as } \boldsymbol{v}^N \to \boldsymbol{v} \text{ strongly in } (L^2(\Omega))^3.\end{aligned}$$

Using the methods of monotone operators we can easily pass to the limit with ε. To pass to the limit with η we use the so-called method of L^∞ truncation functions. Note that this method can be used for model where the "worst" nonlinear term (in the problem $(\mathcal{P}_1)$ it is the convective term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$) is at least integrable. This is the reason why we have the restriction $r > \frac{8}{5}$.

Because of the pressure in the viscosity we need to have some information about behavior of the pressure. We simplify the method introduced in [6]. The authors assumed more restrictive condition on ν (instead of (3.2)) and they were able to pass to the limit with η. Here, we use the similar procedure but before we do so, we split the pressure into two parts. The first one will converge strongly in some suitable space and the second one will converge only weakly but in a better space, say $L^{r'}(0,T;L^{r'}(\Omega))$. Finally, in Subsection 3.3.4, we set $\eta \to 0_+$ and apply the method of $L^\infty(0,T;L^\infty)$ truncation function. This method for the time dependent models describing flows of fluids where viscosity ν depends only on $|\mathbf{D}|^2$ was first used in [14] for space-periodic setting or for no-stick boundary conditions. For parabolic system, the method was developed in studies by Boccardo and Murat [5]. We will slightly modify this method and the key role will play (3.2) and splitting of the pressure.

3.3.2 ε, η - approximation

We start again with the so-called quasi-compressible approximation $(\mathcal{P}_1)^{\varepsilon,\eta}$ (for simplicity, we write $(\boldsymbol{v}, p)$ instead of $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$)

$$
(\mathcal{P}_1)^{\varepsilon,\eta} \qquad
\begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div}\big(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\big) + \nabla p &= \boldsymbol{f} \\
-\varepsilon \triangle p + \operatorname{div} \boldsymbol{v} &= 0
\end{aligned}\right\} \quad \text{in } \Omega \times (0,T), \\
&\left.\begin{aligned}
\big(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big)_\tau + \alpha \boldsymbol{v}_\tau &= 0 \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\
\nabla p \cdot \boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \partial\Omega \times (0,T), \\
&\int_\Omega p(x,t)\, dx = 0, \\
&\boldsymbol{v}(x,0) = \boldsymbol{v}_0.
\end{aligned}
$$

Galerkin approximation: For a fixed $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}}$ there exists unique p solving $\varepsilon \triangle p = \operatorname{div} \boldsymbol{v}$, $\int_\Omega p\, dx = 0$ with homogeneous Neumann boundary condition. From the regularity theory for the Laplace equation it follows that we can define mapping $\mathcal{F} : W^{1,r}_{\boldsymbol{n}} \to W^{2,r}(\Omega) (\hookrightarrow W^{1,2}(\Omega) (r \geq \frac{6}{5}))$ such that $\mathcal{F}(\boldsymbol{v}) := p$. Moreover, this mapping is continuous.

Thus, the system $(\mathcal{P}_1)^{\varepsilon,\eta}$ can be rewritten to the form

$$
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div} \nu(\mathcal{F}(\boldsymbol{v}), |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) + \nabla \mathcal{F}(\boldsymbol{v}) = \boldsymbol{f}
$$

with the same boundary conditions on $\boldsymbol{v}$ as in the problem $(\mathcal{P}_1)^{\varepsilon,\eta}$.

Let $\{\boldsymbol{w}_j\}_{j=1}^\infty$ be a basis of $W^{1,r}_{\boldsymbol{n}}$. We construct Galerkin approximations $\{\boldsymbol{v}^N\}_{N=1}^\infty$ being of the form

$$
\boldsymbol{v}^N(x,t) := \sum_{i=1}^N c_j^N(t) \boldsymbol{w}_j(x),
$$

where $\boldsymbol{c}^N(t)$ solve the system of ordinary differential equations:

$$
\begin{aligned}
&\frac{d}{dt}(\boldsymbol{v}^N, \boldsymbol{w}_j) - (\boldsymbol{v}^N_\eta \otimes \boldsymbol{v}^N, \nabla \boldsymbol{w}_j) + (\nu(\mathcal{F}(\boldsymbol{v}^N), |\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N), \nabla \boldsymbol{w}_j) \\
&+\alpha \int_{\partial\Omega} \boldsymbol{v}^N \cdot \boldsymbol{w}_j\, dS - (\mathcal{F}(\boldsymbol{v}^N), \operatorname{div} \boldsymbol{w}_j) = \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle \quad \text{for } j = 1, 2, \ldots, N.
\end{aligned} \tag{3.11}
$$

Note that the term $(\mathcal{F}(\boldsymbol{v}^N), \operatorname{div} \boldsymbol{w}_j)$ makes a good sense whenever $r \geq \frac{6}{5}$. Due to continuity of ν, $\mathcal{F}$ and definition of $\boldsymbol{v}^N_\eta$ the local-in-time existence follows from Caratheodory theory (for details see Section 2.2 and Theorem B.1). The global-in-time existence will be established by means of apriori estimates proved below.

Apriori estimates: Testing the second equation in $(\mathcal{P}_1)^{\varepsilon,\eta}$ by p^N and the first one by $\boldsymbol{v}^N$, integrating it over time and using (B.2), we are led to the following estimate:

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^N(t)\|_2^2 + \int_0^T \left(\|\boldsymbol{v}^N\|_{1,r}^r + \varepsilon\|\nabla p^N\|_2^2 + \int_{\partial\Omega} |\boldsymbol{v}^N|^2 \, dS \right) dt \le C. \quad (3.12)$$

From (3.11) and (3.12) it then follows that ($X^{rs'}_{\mathrm{div}}$ is defined in Section 1.2)

$$\begin{aligned} \|\boldsymbol{v}^N_{,t}\|_{(X^{r,2})^*} &\le C(\varepsilon,\eta), \\ \|\boldsymbol{v}^N_{,t}\|_{(X^{r,\sigma'}_{\mathrm{div}})^*} &\le C \text{ uniformly w.r.t. } \varepsilon,\eta, \end{aligned} \quad (3.13)$$

where $\sigma := \frac{5r}{8} > 1$ (as $r > \frac{8}{5}$).

Using (3.1), (3.12), (3.13), Corollary B.1, Korn's inequality and Aubin-Lions compactness lemma we have

$$\begin{aligned}
\boldsymbol{v}^N_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } (X^{r,2})^*, && (3.14)\\
\boldsymbol{v}^N &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), && (3.15)\\
\boldsymbol{v}^N &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), && (3.16)\\
\operatorname{tr}\boldsymbol{v}^N &\to \operatorname{tr}\boldsymbol{v} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)), && (3.17)\\
\boldsymbol{v}^N &\to \boldsymbol{v} && \text{strongly in } L^z(0,T;L^z(\Omega)^3) \text{ for } z\in\langle 1,\frac{5r}{3}), && (3.18)\\
p^N &\rightharpoonup p && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), && (3.19)
\end{aligned}$$

and

$$\nu(p^N,|\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N) \rightharpoonup \overline{\nu\mathbf{D}} \text{ weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times3}). \quad (3.20)$$

Passing to the limit in (3.11), we get for all $\psi \in L^2(0,T;W^{1,2}(\Omega))$, $\boldsymbol{\varphi} \in X^{r,2}$ that

$$\int_0^T \varepsilon(\nabla p, \nabla\psi)\, d\tau = \int_0^T (\boldsymbol{v},\nabla\psi)\, d\tau, \quad (3.21)$$

$$\begin{aligned} \int_0^T \left\{ \langle \boldsymbol{v}_{,t},\boldsymbol{\varphi}\rangle - (\boldsymbol{v}^\eta\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi}) + (\overline{\nu\mathbf{D}},\mathbf{D}(\boldsymbol{\varphi})) + \alpha\int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{\varphi} dS \right\} d\tau \\ = \int_0^T \{(p,\operatorname{div}\boldsymbol{\varphi}) + \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle\}\, d\tau. \end{aligned} \quad (3.22)$$

To get the strong convergence of the pressure we can compute

$$\begin{aligned} \int_0^T \|\nabla(p^N-p)\|_2^2\, dt &= \int_0^T (\nabla p^N,\nabla p^N) - (\nabla p^N,\nabla p) - (\nabla p,\nabla(p^N-p))\, dt \\ &= \int_0^T -(\nabla p,\nabla(p^n-p)) - (\nabla p^N,\nabla p) + \frac{1}{\varepsilon}(\boldsymbol{v}^N,\nabla p^N)\, dt \\ &\overset{(3.18)}{\to} \int_0^T -(\nabla p,\nabla p) + \frac{1}{\varepsilon}(\boldsymbol{v},\nabla p)\, dt \overset{(3.21)}{=} 0, \end{aligned}$$

where we used the strong convergence of $\boldsymbol{v}^N$ stated in (3.18). Hence

$$p^N \to p \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)).$$

By using (A.7) we have for all $\boldsymbol{\varphi} \in X^{r,2}$

$$0 \le \int_0^T \Big((\nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N) - \nu(p, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}^N - \boldsymbol{\varphi})) \\ + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 \Big)\, dt.$$

We use the Galerkin approximation (3.11) to replace the term

$$(\nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N), \mathbf{D}(\boldsymbol{v}^N))$$

and pass to the limit as $N \to \infty$. Using the strong convergence of pressure, (3.14)-(3.18) and (3.22) we conclude that

$$0 \le \int_0^T (\overline{\nu\mathbf{D}} - \nu(p, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v} - \boldsymbol{\varphi}))\, d\tau.$$

A possible choice $\boldsymbol{\varphi} := \boldsymbol{v} \pm \lambda \boldsymbol{u}$ then implies (after using standard Minty's trick) that

$$\overline{\nu\mathbf{D}} = \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \text{ a.e. in } \Omega \times (0,T).$$

The solvability of the problem $(\mathcal{P}_1)^{\varepsilon,\eta}$ is finished.

3.3.3 Limit $\varepsilon \to 0$

In this subsection we establish the existence of weak solution to the following problem $(\mathcal{P}_1)^{\eta}$ (we write for simplicity $(\boldsymbol{v}, p)$ instead $(\boldsymbol{v}^{\eta}, p^{\eta})$) :

$$(\mathcal{P}_1)^{\eta} \qquad \begin{aligned} \left.\begin{aligned} \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_{\eta} \otimes \boldsymbol{v}) - \operatorname{div}\big(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\big) + \nabla p = \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\} &\quad \text{in } \Omega \times (0,T), \\ \left.\begin{aligned} \big(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big)_{\tau} + \alpha \boldsymbol{v}_{\tau} = 0 \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \end{aligned}\right\} &\quad \text{on } \partial\Omega \times (0,T), \\ \int_{\Omega} p(x,t)\, dx = 0, & \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0. & \end{aligned}$$

To prove the existence of the solution to $(\mathcal{P}_1)^{\eta}$ we use the problem $(\mathcal{P}_1)^{\varepsilon,\eta}$ and we pass to the limit in ε.

Here, we abbreviate by $(\boldsymbol{v}^{\varepsilon}, p^{\varepsilon})$ the solutions $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ of $(\mathcal{P}_1)^{\varepsilon,\eta}$. After using weak lower semicontinuity of all terms (independent of ε) in (3.12) and in (3.13), we have

$$\sup_{t \in (0,T)} \|\boldsymbol{v}^{\varepsilon}(t)\|_2^2 + \|\boldsymbol{v}^{\varepsilon}_{,t}\|_{(X^{r,\sigma'}_{\mathrm{div}})^*} + \int_0^T \left(\|\boldsymbol{v}^{\varepsilon}\|_{1,r}^r + \int_{\partial\Omega} |\boldsymbol{v}^{\varepsilon}|^2\, dS \right) dt \le C. \quad (3.23)$$

with $\sigma := \frac{5r}{8} > 1$.

We also need an estimate on the pressure p^ε that is uniform w.r.t. $\varepsilon > 0$. To do it, we use the same procedure as in Section 2, but we get uniform estimates that depend on the parameter r. We consider g^ε solving the following homogeneous Neumann problem for the Laplace equation:

$$\begin{aligned} \triangle g^\varepsilon(t) &= |p^\varepsilon(t)|^{r'-2}p^\varepsilon(t) - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon(t)|^{r'-2}p^\varepsilon(t) \quad \text{ in } \Omega, \\ \int_\Omega g^\varepsilon(t)\, dx &= 0, \\ \frac{\partial g^\varepsilon}{\partial \boldsymbol{n}} &= 0 \quad \text{ on } \partial\Omega. \end{aligned} \tag{3.24}$$

Note that $\|g^\varepsilon(t)\|_{2,r}^r \le 2C_{reg}(\Omega, r)\|p^\varepsilon(t)\|_{r'}^{r'}$ for a.a. $t \in (0,T)$. Taking $\boldsymbol{\varphi} := \nabla g^\varepsilon$ in (3.22), we obtain

$$\int_0^t \|p^\varepsilon\|_{r'}^{r'}\, dt = \sum_{i=1}^5 I_j,$$

where (we use just standard Hölder's, Young's and embedding inequalities to estimate $I_2, \dots, I_4$)

$$\begin{aligned} I_1 &:= \alpha \int_0^t \int_{\partial\Omega} \boldsymbol{v}^\varepsilon \cdot \nabla g^\varepsilon \, dS\, d\tau, \\ I_2 &:= -\int_0^t (\boldsymbol{v}_\eta^\varepsilon \otimes \boldsymbol{v}^\varepsilon, \nabla^2 g^\varepsilon) d\tau \le C(\eta) + \frac{1}{8}\int_0^t \|p^\varepsilon\|_{r'}^{r'}, \\ I_3 &:= \int_0^t \langle \boldsymbol{f}, \nabla g^\varepsilon\rangle \le C + \frac{1}{8}\int_0^t \|p^\varepsilon\|_{r'}^{r'}, \\ I_4 &:= \int_0^t (\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon), \nabla^2 g^\varepsilon)\, d\tau \le C + \frac{1}{8}\int_0^t \|p^\varepsilon\|_{r'}^{r'}\, d\tau, \\ I_5 &:= \int_0^t \langle \boldsymbol{v}_{,t}^\varepsilon, \nabla g^\varepsilon\rangle\, d\tau. \end{aligned}$$

Next, we will bound I_1 and I_5. To do it for I_1, we use Corollary B.1 and we have

$$\begin{aligned} I_1 &\le \int_0^t \|\operatorname{tr} \boldsymbol{v}\|_{\frac{2r}{3(r-1)}} \|\operatorname{tr} \nabla g^\varepsilon\|_{\frac{2r}{3-r}}\, d\tau \le \int_0^t \|\operatorname{tr} \boldsymbol{v}\|_{\frac{2r}{3(r-1)}} \|g^\varepsilon\|_{2,r}\, d\tau \\ &\overset{(B.1)}{\le} C + \frac{1}{8}\int_0^t \|p^\varepsilon\|_{r'}^{r'}\, d\tau. \end{aligned}$$

Finally, we will show that $I_5 \le 0$. Note that I_5 is well defined. Assume for a moment that p^ε and $\boldsymbol{v}^\varepsilon$ are smooth. Then the function g^ε introduced in (3.24) is smooth as well. To the velocity $\boldsymbol{v}^\varepsilon$ we apply the Helmholtz decomposition to obtain

$$I_5 = \int_0^t \langle \boldsymbol{v}^\varepsilon_{\mathrm{div},t} + \nabla g^{\boldsymbol{v}^\varepsilon}_{,t}, \nabla g^\varepsilon \rangle d\tau \overset{\operatorname{div} \boldsymbol{v}^\varepsilon_{\mathrm{div}}=0}{=} \int_0^t \langle \nabla g^{\boldsymbol{v}^\varepsilon}_{,t}, \nabla g^\varepsilon \rangle d\tau$$
$$\overset{\nabla \varphi^\varepsilon \cdot \boldsymbol{n}=0|_{\partial\Omega}}{=} -\int_0^t (g^{\boldsymbol{v}^\varepsilon}_{,t}, \triangle g^\varepsilon) d\tau.$$

By using the definition of $g^{\boldsymbol{v}^\varepsilon}$, g^ε and the equation (3.21) we have

$$\frac{1}{\varepsilon}\triangle g^{\boldsymbol{v}^\varepsilon} = \frac{1}{\varepsilon}\operatorname{div} \boldsymbol{v}^\varepsilon = \triangle p^\varepsilon.$$

The uniqueness of the solution to the Laplace equation then gives

$$\frac{1}{\varepsilon} g^{\boldsymbol{v}^\varepsilon} = p^\varepsilon.$$

Hence (after using $\int_\Omega p^\varepsilon \, dx = 0$),

$$I_5 = -\varepsilon \int_0^t (p^\varepsilon_{,t}, |p^\varepsilon|^{r'-2} p^\varepsilon) \, d\tau = \frac{\varepsilon}{r'}\left(-\|p^\varepsilon(t)\|^{r'}_{r'} + \|p^\varepsilon(0)\|^{r'}_{r'}\right) = -\frac{\varepsilon}{r'}\|p^\varepsilon(t)\|^{r'}_{r'}$$
$$\leq 0,$$

where we used the fact that $p^\varepsilon(0) = 0$ ($\operatorname{div} \boldsymbol{v}^\varepsilon(0) = 0$). Thus, for smooth functions is I_5 non-positive. By means of the density of smooth functions we conclude that the same holds also for the original couple $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$. To summarize, the above computation implies that

$$\|p^\varepsilon\|_{L^{r'}(0,T;L^{r'}(\Omega))} \leq C(\eta). \tag{3.25}$$

As consequences of (3.23) and (3.25), Aubin-Lions lemma, Korn's inequality and Corollary B.1 we then conclude that

$$\boldsymbol{v}^\varepsilon_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}), \tag{3.26}$$
$$\boldsymbol{v}^\varepsilon \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{3.27}$$
$$\boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{3.28}$$
$$\operatorname{tr} \boldsymbol{v}^\varepsilon \to \operatorname{tr} \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)), \tag{3.29}$$
$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } L^z(0,T;L^z(\Omega)^3) \text{ for } z \in \langle 1, \frac{5r}{3}), \tag{3.30}$$
$$p^\varepsilon \rightharpoonup p \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)), \tag{3.31}$$

and

$$\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) \rightharpoonup \overline{\nu\mathbf{D}} \text{ weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}). \tag{3.32}$$

Moreover, after passing to the limit in the first equation of $(\mathcal{P}_1)^{\varepsilon,\eta}$, we have

$$(\operatorname{div} \boldsymbol{v}, \varphi) = 0 \quad \forall \varphi \in \mathcal{C}^\infty(\Omega),$$

which implies that $\operatorname{div} \boldsymbol{v} = 0$ a.e. The convergence (3.26)-(3.32) is sufficient to pass to the limit in the convective term and in the boundary term. The same is true also for linear terms.

To take the limit in the viscosity term we need almost everywhere point wise convergence for the velocity gradients $\nabla \boldsymbol{v}^\varepsilon$ and pressures p^ε. One can then use the Vitali's theorem (Theorem B.2) to pass to the limit also in the viscosity.

The rest of this subsection is devoted to the proof of pointwise convergence of the velocity gradient and pressure. We use Lemma A.2 for functions $(\boldsymbol{v}, p)$ and $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ and we get

$$\begin{aligned} \frac{C_1}{2} \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx &\le \frac{\gamma_0^2}{2C_1} \|p^\varepsilon - p\|_2^2 \\ &+ \int_\Omega \big(\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) - \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\big) \cdot \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\, dx. \end{aligned} \tag{3.33}$$

Using the weak formulation (3.22) of $(\mathcal{P}_1)^{\varepsilon,\eta}$ with $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon - \boldsymbol{v}$ to replace the term

$$\big(\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)$$

in (3.33), we conclude that

$$\frac{C_1}{2} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, d\tau \le \frac{\gamma_0^2}{2C_1} \|p^\varepsilon - p\|_2^2\, d\tau + f(\varepsilon) \tag{3.34}$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next, we consider g^ε solving

$$\triangle g^\varepsilon(t) = p^\varepsilon(t) - p(t) \text{ in } \Omega, \frac{\partial g^\varepsilon(t)}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, \int_\Omega g^\varepsilon(t)\, dx = 0 \text{ for a.a. } t \in (0,T)$$

Note that (3.31) implies that

$$g^\varepsilon \rightharpoonup 0 \qquad \text{weakly in } L^{r'}(0,T; W^{2,r'}(\Omega)). \tag{3.35}$$

Inserting $\boldsymbol{\varphi} = \nabla g^\varepsilon$ into (3.22) leads to

$$\begin{aligned} \int_0^T \|p^\varepsilon - p\|_2^2\, dt = \int_0^T (p^\varepsilon, p^\varepsilon - p) - (p, p^\varepsilon - p)\, dt &= \sum_{i=1}^5 J_i \\ &- \underbrace{\int_0^T (p, p^\varepsilon - p)\, dt}_{\to 0}, \end{aligned} \tag{3.36}$$

where

$$J_1 := \int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \nabla g^\varepsilon \rangle \; dt, \tag{3.37}$$

$$J_2 := -\int_0^T (\boldsymbol{v}^\varepsilon_\eta \otimes \boldsymbol{v}^\varepsilon, \nabla^2 g^\varepsilon) \; dt, \tag{3.38}$$

$$J_3 := \alpha \int_0^T \int_{\partial\Omega} \boldsymbol{v}^\varepsilon \cdot \nabla g^\varepsilon \; dS \; dt, \tag{3.39}$$

$$J_4 := -\int_0^T \langle \boldsymbol{f}, \nabla g^\varepsilon \rangle \; dt, \tag{3.40}$$

and

$$J_5 := \int_0^T \Big(\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2) \mathbf{D}(\boldsymbol{v}^\varepsilon), \nabla^2 g^\varepsilon \Big) \; dt. \tag{3.41}$$

To estimate J_1 we split $g^\varepsilon = g_1^\varepsilon - g_2$ where

$$\begin{aligned}
&\triangle g_1^\varepsilon(t) = p^\varepsilon(t) \text{ in } \Omega, && \frac{\partial g_1^\varepsilon(t)}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, && \int_\Omega g_1^\varepsilon \; dx = 0, \\
&\triangle g_2(t) = p(t) \text{ in } \Omega, && \frac{\partial g_2(t)}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, && \int_\Omega g_2 \; dx = 0.
\end{aligned}$$

Then

$$\begin{aligned}
\limsup_{\varepsilon\to 0} J_1 &= \limsup_{\varepsilon\to 0} \left(\int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \nabla g_1^\varepsilon - \nabla g_2 \rangle \; dt \right) \\
&= \underbrace{\limsup_{\varepsilon\to 0} \left(\int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \nabla g_1^\varepsilon \rangle \; dt \right)}_{\leq 0} - \underbrace{\lim_{\varepsilon\to 0} \left(\int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \nabla g_2 \rangle \; dt \right)}_{\overset{\operatorname{div} \boldsymbol{v}=0}{\to} 0} \leq 0.
\end{aligned} \tag{3.42}$$

Regarding J_2, J_3 and J_4 we easily conclude that

$$\lim_{\varepsilon\to 0} J_2 \overset{(3.30),(3.35)}{=} 0, \tag{3.43}$$

$$\lim_{\varepsilon\to 0} J_3 \overset{(3.29),(3.35)}{=} 0, \tag{3.44}$$

$$\lim_{\varepsilon\to 0} J_4 \overset{(3.35)}{=} 0. \tag{3.45}$$

Finally, we apply Lemma A.2 to the term J_5 to obtain

$$\begin{aligned}
J_5 &= \int_0^T \Big(\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) - \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon\Big)\, dt \\
&\quad + \underbrace{\int_0^T \Big(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon\Big)\, dt}_{\to 0} \\
&\overset{\text{(A.8)+Hölder}}{\leq} f(\varepsilon) + \int_0^T \Big(C_2 C_{reg}(\Omega,2)\Big(\int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\Big)^{\frac{1}{2}} \|p^\varepsilon - p\|_2 \qquad (3.46) \\
&\quad + \gamma_0 C_{reg}(\Omega,2)\|p^\varepsilon - p\|_2^2\Big)\, dt \\
&\overset{\text{Young}}{\leq} f(\varepsilon) + \Big(\gamma_0 C_{reg}(\Omega,2) + \frac{1 - \gamma_0 C_{reg}(\Omega,2)}{2}\Big)\int_0^T \|p^\varepsilon - p\|_2^2\, dt \\
&\quad + \frac{C_2^2 C_{reg}^2(\Omega,2)}{2(1 - \gamma_0 C_{reg}(\Omega,2))}\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt.
\end{aligned}$$

Combining (3.36)-(3.46), we get

$$\int_0^T \|p^\varepsilon - p\|_2^2\, dt \leq C f(\varepsilon) + \frac{C_2^2 C_{reg}^2(\Omega,2)}{(1 - \gamma_0 C_{reg}(\Omega,2))^2}\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt. \quad (3.47)$$

Substituting (3.47) into (3.34), and using Young's inequality we obtain

$$\begin{aligned}
\frac{C_1}{2}\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt &\leq C f(\varepsilon) \\
&\quad + \frac{\gamma_0^2}{2C_1}\frac{C_2^2 C_{reg}^2(\Omega,2)}{(1 - \gamma_0 C_{reg}(\Omega,2))^2}\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt.
\end{aligned}$$

Using the assumption on γ_0, we know that $\frac{C_1}{2} > \frac{\gamma_0^2}{2C_1}\frac{C_2^2 C_{reg}^2(\Omega,2)}{(1-\gamma_0 C_{reg}(\Omega,2))^2}$. Therefore,

$$\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt \to 0 \qquad (3.48)$$

and consequently using (3.36)-(3.48) we come to the conclusion that

$$\int_0^T \|p^\varepsilon - p\|_2^2\, dt \to 0, \qquad \text{as } \varepsilon \to 0. \qquad (3.49)$$

Finally, we will show that (3.48) implies a.e. convergence of $\mathbf{D}(\boldsymbol{v}^\varepsilon)$. Since

$\int_0^T \|\nabla \boldsymbol{v}^\varepsilon\|_r^r \le C$ we have

$$\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})| \, dx d\tau \\
&\quad \le \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{\frac{1}{2}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \left(\int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}^\varepsilon) - s(\mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}))|^2)^{\frac{r-2}{2}} \right)^{-\frac{1}{2}} \, dx \, dt \\
&\le C \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{\frac{1}{2}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) (1 + |\mathbf{D}(\boldsymbol{v}^\varepsilon)| + |\mathbf{D}(\boldsymbol{v})|)^{\frac{2-r}{2}} \, dx \, dt \\
&\overset{\text{Hölder}}{\le} C \left(\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T (1 + \|\nabla \boldsymbol{v}^\varepsilon\|_{2-r}^{2-r} + \|\nabla \boldsymbol{v}\|_{2-r}^{2-r} \, dt) \right)^{\frac{1}{2}} \\
&\le C \left(\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt \right)^{\frac{1}{2}} \overset{(3.48)}{\to} 0,
\end{aligned}$$

and we conclude at least for a subsequence that

$$\mathbf{D}(\boldsymbol{v}^\varepsilon) \to \mathbf{D}(\boldsymbol{v}) \text{ a.e. in } \Omega \times (0,T). \tag{3.50}$$

Hence, we can use Vitali's convergence theorem (Theorem B.2), (3.50) and (3.49) to conclude that

$$\overline{\nu \mathbf{D}} = \nu(p, |\mathbf{D}(\boldsymbol{v})|^2) \mathbf{D}(\boldsymbol{v}), \text{ a.e. in } \Omega \times (0,T).$$

This completes the solvability of the problem $(\mathcal{P}_1)^\eta$.

3.3.4 Limit $\eta \to 0$

Considering the constructed solutions $(\boldsymbol{v}^\eta, p^\eta)$ of the problem $(\mathcal{P}_1)^\eta$, we summarize first the estimates that are uniform w.r.t. $\eta > 0$. Then letting $\eta \to 0_+$ we aim to show that the limit functions solve the problem $(\mathcal{P})$.

Note that the estimates (3.23) are uniform w.r.t. $\eta > 0$ and consequently hold also for $\boldsymbol{v}^\eta$. This is not however true for the estimate (3.25) and we have to modify the argument slightly. We decompose the pressure p^η into two particular pressures $p^\eta := p_1^\eta + p_2^\eta$ such that p_2^η will satisfy (3.25) and p_1^η will converge strongly in some suitable space to p_1.

Decomposition of the pressure: We will find p_1^η such that it solves the following problem (at each time level t)

$$-(p_1^\eta, \triangle \varphi) = \left(\boldsymbol{v}_\eta^\eta \otimes \boldsymbol{v}^\eta, \nabla^2 \varphi \right) \tag{3.51}$$

for all $\varphi \in W^{2,r}$, $\nabla \varphi \in W^{1,r}_{\boldsymbol{n}}$ and $\int_\Omega p_1^\eta \, dx = 0$. It is easy to find such uniquely defined pressure. To do it we can consider an approximative problem $(\mathcal{M})$

$$(\mathcal{M}) \qquad \begin{aligned}
-\triangle p_1^N &= \operatorname{div} \operatorname{div} \mathbf{F}^N && \text{in } \Omega, \\
\nabla p_1^N \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\
\int_\Omega p_1^N \, dx &= 0,
\end{aligned}$$

where $\mathsf{F}^N := (\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{v}^\eta \chi_N) * \omega_{\frac{1}{N}}$. Here χ_N denotes the characteristic function of the set $\{x \in \Omega;\ \text{dist}\,(x, \partial\Omega) > \frac{1}{N}\}$ and $\omega_{\frac{1}{N}}$ is regularization kernel with radii $\frac{1}{N}$. The solvability of the problem $(\mathcal{M})$ easily follows from standard theory for Laplace equation (see Lemma B.1). Next, let $g^{N,M}$ solves the following problem:

$$\triangle g^{N,M} = |p_1^N - p_1^M|^{r'-2}(p_1^N - p_1^M) - \frac{1}{|\Omega|}\int_\Omega |p_1^N - p_1^M|^{r'-2}(p_1^N - p_1^M)\, dx$$

with homogeneous Neuman boundary condition and such that $\int_\Omega g^{N,M}\, dx = 0$. Next, subtracting problem $(\mathcal{M})$ for N from that one for M and multiplying the resulting system by $g^{N,M}$, we observe the inequality (after integration per partes and using Young's inequality)

$$\|p_1^N - p_1^M\|_{r'}^{r'} \le C\|\mathsf{F}^N - \mathsf{F}^M\|_{r'}^{r'}.$$

But the sequence $\{\mathsf{F}^N\}_{N=1}^\infty$ is Cauchy and therefore the existence of the solution to (3.51) easily follows. Finally, we define $p_2^\eta := p^\eta - p_1^\eta$. It is a consequence of (3.51) that p_2^η solves at each time level

$$(p_2^\eta, \triangle\varphi) = -\langle \boldsymbol{f}, \nabla\varphi\rangle + \alpha\int_{\partial\Omega} \boldsymbol{v}^\eta \cdot \nabla\varphi\, dS + (\nu(p^\eta, |\mathbf{D}(\boldsymbol{v}^\eta)|^2)\mathbf{D}(\boldsymbol{v}^\eta), \nabla^2\varphi) \tag{3.52}$$

for all $\varphi \in W^{2,r}(\Omega)$ with $\nabla\varphi \in W^{1,r}_{\boldsymbol{n}}$.

A priori estimates and their consequences The same procedure as in the preceding subsection implies that (we use uniform estimates (3.23) and the formulations (3.51), (3.52))

$$\int_0^T \|p^\eta\|_{\frac{5r}{6}}^{\frac{5r}{6}}\, dt \le C, \tag{3.53}$$

and

$$\int_0^T \|p_2^\eta\|_{r'}^{r'}\, dt \le C. \tag{3.54}$$

The uniform estimates (3.23), (3.53) and (3.54) again imply that (after taking a subsequence)

$$\begin{aligned}
\boldsymbol{v}^\eta &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), && (3.55)\\
\boldsymbol{v}^\eta &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), && (3.56)\\
\operatorname{tr}\boldsymbol{v}^\eta &\to \operatorname{tr}\boldsymbol{v} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)), && (3.57)\\
\boldsymbol{v}^\eta &\to \boldsymbol{v} && \text{strongly in } L^z(0,T;L^z(\Omega)^3) \text{ for } z \in \langle 1, \frac{5r}{3}), && (3.58)\\
p^\eta &\rightharpoonup p && \text{weakly in } L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)), && (3.59)\\
p_2^\eta &\rightharpoonup p_2 && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)), && (3.60)
\end{aligned}$$

and

$$\nu(p^\eta,|\mathbf{D}(\boldsymbol{v}^\eta)|^2)\mathbf{D}(\boldsymbol{v}^\eta) \rightharpoonup \overline{\nu\mathbf{D}} \text{ weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times3}). \tag{3.61}$$

Consequently (as in the previous subsection)

$$\boldsymbol{v}^\eta_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } (X^{r,\frac{5r}{5r-8}}_{\mathrm{div}})^*. \tag{3.62}$$

$$\boldsymbol{v}^\eta_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{\boldsymbol{n}}), \tag{3.63}$$

Note that here we need $r > \frac{8}{5}$ to have the relation (3.62) meaningful. To end the proof it remains to show the a.e. convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$ and p^η. Then using all of the weak convergence shown above and Vitali's theorem completes the proof of our theorem. First note that (3.58) together with (3.51) imply that

$$p_1^\eta \rightharpoonup p_1 \quad \text{strongly in } L^y(0,T;L^y(\Omega)) \text{ for } y \in \langle 1, \frac{5r}{6}), \tag{3.64}$$

We will follow the approach described in [14]. We relabel the sequence $\{\boldsymbol{v}^\eta\}$ to $\{\boldsymbol{v}^j\}_{j=1}^\infty$. We split the rest of the proof into two steps.

Step 1: Let $k \in \mathbb{N}$ be arbitrary and fixed. We set

$$\begin{aligned} g^k := |\nabla \boldsymbol{v}^k|^r + |\nabla \boldsymbol{v}|^r + \Big(&|\nu(p^k,|\mathbf{D}(\boldsymbol{v}^k)|^2)\mathbf{D}(\boldsymbol{v}^k)| + \\ &+ |\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})|\Big)\left(|\mathbf{D}(\boldsymbol{v}^k)| + |\mathbf{D}(\boldsymbol{v})|\right). \end{aligned} \tag{3.65}$$

It follows from apriori estimates that

$$0 \le \int_0^T \int_\Omega g^k \, dx \, dt \le K$$

with some constant $K, 1 \le K < \infty$. We prove the following property:

$$\begin{aligned} &\text{for every } \varepsilon^* > 0 \text{ there is } L \le \frac{\varepsilon^*}{K} \text{ and there is } \{\boldsymbol{v}^l\}_{l=1}^\infty \subset \{\boldsymbol{v}^j\}_{j=1}^\infty \\ &\text{and sets } E^l := \left\{(x,t) \in \Omega \times (0,T); L^2 \le |\boldsymbol{v}^l(x,t) - \boldsymbol{v}(x,t)| < L\right\} \\ &\text{such that } \int_{E^l} g^l \, dx \, dt \le \varepsilon^*. \end{aligned} \tag{3.66}$$

To see it, we fix $\varepsilon^* \in (0,1)$, set $L_1 := \frac{\varepsilon^*}{K}$ and take $N \in \mathbb{N}$ such that $N > \frac{K}{\varepsilon^*}$. We define iteratively $L_i := L_{i-1}^2$ for $i = 2,3,\ldots,N$ and we set

$$E_i^j := \left\{(x,t) \in \Omega \times (0,T); L_i^2 \le |\boldsymbol{v}^j(x,t) - \boldsymbol{v}(x,t)| < L_i\right\} (i = 1,\ldots,N).$$

For j fixed, E_i^j are mutually disjoint. Consequently

$$\sum_{i=1}^N \int_{E_i^j} g^j \, dx \, dt \le K.$$

As $N\varepsilon^* > K$, for each $j \in \mathbb{N}$ there is $i_0(j) \in \{1, \ldots, N\}$ such that

$$\int_{E^j_{i_0(j)}} g^j \, dx \, dt \leq \varepsilon^*.$$

As, $i_0(j)$ are taken from the finite set of indices we can take a subsequence $\{\boldsymbol{v}^l\}_{l=1}^\infty \subset \{\boldsymbol{v}^j\}_{j=1}^\infty$ such that $i_0(l) = i_0^*$ for each l. The property (3.66) is then proved by setting $L := L_{i_0^*}$ and $E^l := E^l_{i_0^*}$ $\square$.

Let $\varepsilon^* > 0$ be arbitrary but fixed. We find the sequence $\{\boldsymbol{v}^l\}_{l=1}^\infty$ and L satisfying (3.66). Then we define $\boldsymbol{u}^l$ as

$$\boldsymbol{u}^l := (\boldsymbol{v}^l - \boldsymbol{v})(1 - \min\left(\frac{|\boldsymbol{v} - \boldsymbol{v}^l|}{L}, 1)\right). \tag{3.67}$$

and define the sets Q^l, Q as

$$Q^l := \left\{(x,t); |\boldsymbol{v} - \boldsymbol{v}^l| < L\right\}, \quad Q := (0,T) \times \Omega.$$

By using (3.55)-(3.58) and the fact that $\|\boldsymbol{u}^l(t)\|_\infty \leq C$ we have

$$\boldsymbol{u}^l \stackrel{l\to\infty}{\rightharpoonup} \mathbf{0} \qquad \text{weakly in } L^r(0,T; W^{1,r}_{\boldsymbol{n}}), \tag{3.68}$$

$$\boldsymbol{u}^l \stackrel{l\to\infty}{\to} \mathbf{0} \qquad \text{strongly in } L^s(0,T; L^s(\Omega)^3) \quad \forall s < \infty. \tag{3.69}$$

$$\operatorname{tr} \boldsymbol{u}^l \stackrel{l\to\infty}{\to} \mathbf{0} \qquad \text{strongly in } L^2(0,T; L^2(\partial\Omega)^3). \tag{3.70}$$

Let us compute the divergence of $\boldsymbol{u}^l$:

$$\operatorname{div} \boldsymbol{u}^l = -\frac{1}{L}(\boldsymbol{v} - \boldsymbol{v}^l) \cdot (\nabla|\boldsymbol{v}^l - \boldsymbol{v}|)\chi_{Q^l}$$

where χ_U denotes the characteristic function of the set U. Hence, the L^r-norm of $\operatorname{div} \boldsymbol{u}^l$ can be estimated by using (3.66) as

$$\begin{aligned} \int_0^T \|\operatorname{div} \boldsymbol{u}^l\|_r^r &\leq \int_Q \frac{|\boldsymbol{v} - \boldsymbol{v}^l|^r}{L^r} |\nabla(\boldsymbol{v} - \boldsymbol{v}^l)|^r \chi_{Q^l} \, dx \, dt \\ &= \int_{Q^l \setminus E^l} \ldots dx \, dt + \int_{E^l} \ldots dx \, dt \leq C(L + \varepsilon^*) \leq C\varepsilon^*. \end{aligned} \tag{3.71}$$

The Helmholtz decomposition then gives $\boldsymbol{u}^l = \boldsymbol{u}^l_{\mathrm{div}} + \nabla g^{\boldsymbol{u}^l}$ where

$$\int_0^T \|g^{\boldsymbol{u}^l}\|_{2,r}^r \, dt \leq C\varepsilon^*. \tag{3.72}$$

Moreover, (3.69) and the L^p-theory for the Laplace equation imply

$$\boldsymbol{u}^l_{\mathrm{div}} \stackrel{l\to\infty}{\to} \mathbf{0} \quad \text{strongly in } L^s(0,T; L^s(\Omega)^3) \quad \forall s < \infty. \tag{3.73}$$

For simplicity, we denote for $m \in \mathbb{N}$

$$U^m := \nu(p^m, |\mathbf{D}(\boldsymbol{v}^m)|^2)\mathbf{D}(\boldsymbol{v}^m) \in L^{r'}(\Omega \times (0,T))$$
$$W^m := \nu(p_1^m + p_2, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \in L^{r'}(\Omega \times (0,T)).$$

The integration of (A.7) over Q^l (with setting: $\boldsymbol{u} := \boldsymbol{v}$, $\boldsymbol{v} := \boldsymbol{v}^l$, $p := p_1^l + p_2$, $q := p^l$) leads to

$$\begin{aligned} \frac{C_1}{2}\int_{Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}, \boldsymbol{v}^l)\, dx\, dt &\leq \frac{\gamma_0^2}{2C_1}\int_{Q^l} |p_2^l - p_2|^2\, dx\, dt \\ &+ \int_{Q^l} W^l \cdot \mathbf{D}(\boldsymbol{v} - \boldsymbol{v}^l)\, dx\, dt - \int_{Q^l} U^l \cdot \mathbf{D}(\boldsymbol{v} - \boldsymbol{v}^l)\, dx\, dt \\ &=: Y_1 + Y_2 + Y_3. \end{aligned} \tag{3.74}$$

First, we observe that as p_1^l converges point-wise Lebesgue's dominated convergence theorem then implies that

$$W^m \to \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^{r'}(0,T; L^{r'}(\Omega)^{3\times 3}). \tag{3.75}$$

Therefore

$$Y_2 := \int_{Q^l} W^l \mathbf{D}(\boldsymbol{v} - \boldsymbol{v}^l)\, dx\, dt \to 0.$$

Next, taking $\boldsymbol{\varphi} = \boldsymbol{u}_{\mathrm{div}}^l$ as the test function in the weak formulation of the problem $(\mathcal{P}_1)^{\eta(l)}$, we obtain

$$\begin{aligned} Y_3 &:= \int_{Q^l} U^l \mathbf{D}(\boldsymbol{v}^l - \boldsymbol{v}) \\ &= \int_Q U^l \mathbf{D}(\boldsymbol{u}^l)\, dx\, dt + \int_{Q^l} U^l \mathbf{D}((\boldsymbol{v} - \boldsymbol{v}^l)\frac{|\boldsymbol{v} - \boldsymbol{v}^l|}{L})\, dx\, dt \\ &= \int_Q U^l \mathbf{D}(\boldsymbol{u}_{\mathrm{div}}^l)\, dx\, dt + \int_Q U^l \mathbf{D}(\nabla g^{\boldsymbol{u}^l})\, dx\, dt \\ &\quad + \int_{Q^l} U^l \mathbf{D}((\boldsymbol{v} - \boldsymbol{v}^l)\frac{|\boldsymbol{v} - \boldsymbol{v}^l|}{L}) \\ &\overset{(3.66),(3.72)}{\leq} \int_Q U^l \mathbf{D}(\boldsymbol{u}_{\mathrm{div}}^l)\, dx\, dt + C\varepsilon^* := \sum_{i=1}^{4} I_i + C\varepsilon^* \end{aligned} \tag{3.76}$$

where

$$I_1 = -\int_0^T \langle \boldsymbol{v}^l_{,t}, \boldsymbol{u}^l_{\mathrm{div}} \rangle = \int_0^T \langle \boldsymbol{v}_{,t} - \boldsymbol{v}^l_{,t}, \boldsymbol{u}^l_{\mathrm{div}} \rangle - \underbrace{\int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{u}^l_{\mathrm{div}} \rangle}_{\to 0}$$

$$\leq f(l) + \int_0^T \langle \boldsymbol{v}_{,t} - \boldsymbol{v}^l_{,t}, \boldsymbol{u}^l_{\mathrm{div}} \rangle \overset{\mathrm{div}\, \boldsymbol{v} - \boldsymbol{v}^l = 0}{=} f(l) + \int_0^T \langle \boldsymbol{v}_{,t} - \boldsymbol{v}^l_{,t}, \boldsymbol{u}^l \rangle \leq {}^6 f(l),$$

$$I_2 = -\int_Q \left(\boldsymbol{v}^l_{\eta(l)} \nabla \boldsymbol{v}^l \right) (\boldsymbol{u}^l_{\mathrm{div}}) \leq C \|\boldsymbol{u}^l\|_{L^s(0,T;L^s)} \quad s \text{ sufficiently large},$$

$$I_3 = -\alpha \int_{(0,T)\times\partial\Omega} \boldsymbol{v}^l \boldsymbol{u}^l_{\mathrm{div}} \, dS \, dt \leq C \|\boldsymbol{u}^l\|_{L^2(0,T;L^2(\partial\Omega))},$$

$$I_4 = \int_0^T \langle \boldsymbol{f}, \boldsymbol{u}^l_{\mathrm{div}} \rangle \, dt \leq f(l),$$

where $f(l) \overset{l\to\infty}{\to} 0$.

We also estimate Y_1 in (3.74). For this purpose we consider g^l as a solution of Neumann problem

$$\begin{aligned} &\triangle g^l = p_2^l - p_2 \quad \text{ in } \Omega \times (0,T), \\ &\frac{\partial g^l}{\partial \boldsymbol{n}} = 0 \text{ on } \Omega \times (0,T), \int_\Omega g^l(t) \, dx = 0 \text{ for a.a. } t \in (0,T). \end{aligned}$$

The weak convergence (3.60) implies that

$$g^l \rightharpoonup 0 \quad \text{ weakly in } L^{r'}(0,T;L^{r'}(\Omega)). \tag{3.77}$$

[6]Here we have to note that the second equality in estimate for I_1 is only formal (the duality does not make a good sense). To prove the estimate for I_1 rigorously we set $\boldsymbol{w} := \boldsymbol{v}^l - \boldsymbol{v}$. From the density of smooth functions we can find a sequence $\boldsymbol{w}^n \in \mathcal{C}^\infty(0,T;\mathcal{C}^\infty_{\boldsymbol{n},\mathrm{div}}(\Omega))$, $\boldsymbol{w}^n(0) = 0$ such that

$$-\int_0^T \langle \boldsymbol{v}_{,t} - \boldsymbol{v}^l_{,t}, \boldsymbol{u}^l_{\mathrm{div}} \rangle \, dt = \lim_{n\to\infty} \int_0^T \langle \boldsymbol{w}^n_{,t}, (\boldsymbol{w}^n(1 - \min(\frac{|\boldsymbol{w}^n|}{L}, 1)))_{\mathrm{div}} \rangle \, dt$$

By using $\mathrm{div}\, \boldsymbol{w}^n = 0$ we have

$$\begin{aligned} &= \lim_{n\to\infty} \int_0^T \langle \boldsymbol{w}^n_{,t}, \boldsymbol{w}^n(1 - \min(\frac{|\boldsymbol{w}^n|}{L}, 1)) \rangle \, dt \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega \frac{1}{2} |\boldsymbol{w}^n|^2_{,t} (1 - \min(\frac{|\boldsymbol{w}^n|}{L}, 1)) \, dx \, dt \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega F^n(x,t)_{,t} \, dx \, dt \\ &= \lim_{n\to\infty} \int_\Omega F^n(x,T) - F^n(x,0) \, dx \overset{F^n(.,0)=0}{=} \lim_{n\to\infty} \int_\Omega F^n(x,T) \, dx \geq 0, \end{aligned}$$

where F^n is defined as

$$F^n(x,t) := \begin{cases} |\boldsymbol{w}^n|^2 (1 - \frac{2}{3} \frac{|\boldsymbol{w}^n|}{L}) & \text{if } |\boldsymbol{w}^n| < L \\ \frac{1}{3} L^2 & \text{if } |\boldsymbol{w}^n| \geq L. \end{cases}$$

Moreover, we have

$$\operatorname{tr} \nabla g^l \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;L^2(\partial\Omega)). \tag{3.78}$$

Next, we can compute

$$\begin{aligned}\int_0^T \|p_2^l - p_2\|_2^2 \, dt &= \int_0^T (p_2^l, p_2^l - p_2) \; dt + \underbrace{\int_0^T (p_2, p_2 - p_2^l) \, dt}_{\to 0} \\ &\le f(l) + \int_0^T (p_2^l, p_2^l - p_2) \, dt.\end{aligned} \tag{3.79}$$

Now testing the weak formulation of $(\mathcal{P}_1)^\eta$ by ∇g^l and using the definition of p_2^l, p_1^l, we get that

$$\int_Q \|p_2^l - p_2\|_2^2 \, dx \, dt \le f(l) + \sum_{a=1}^4 I_a, \tag{3.80}$$

where

$$\begin{aligned}
I_1 &= \int_0^T \langle \boldsymbol{v}^l_{,t}, \nabla g^l\rangle \, dt \overset{\operatorname{div} \boldsymbol{v}^l = 0}{=} 0,\\
I_2 &= \alpha \int_{(0,T)\times\partial\Omega} (\boldsymbol{v}^l)(\nabla g^l) dS \, dt \overset{(3.78),(3.57)}{\le} k(l),\\
I_3 &= \int_Q U^l \nabla^2 g^l \, dx \, dt = \int_Q (U^l - W^l)\nabla^2 g^l \, dx \, dt + \underbrace{\int_Q W^l \nabla^2 g^l \, dx \, dt}_{\overset{(3.75),(3.77)}{\to} 0}\\
&\overset{(A.8)}{\le} \gamma_0 C_{reg}(2,\Omega) \int_Q |p_2^l - p_2|^2 \, dx \, dt + C_2 \int_Q J \, dx \, dt + f(l),
\end{aligned}$$

where the symbol J is defined as

$$J := \left| \mathbf{D}(\boldsymbol{v}^l - \boldsymbol{v}) \int_0^1 \left(1 + |\mathbf{D}(\boldsymbol{v}^l + s(\boldsymbol{v} - \boldsymbol{v}^l))|^2\right)^{\frac{r-2}{2}} \; ds \right| |\nabla^2 g^l|.$$

Therefore, we can estimate

$$\begin{aligned}
\int_{Q\setminus Q^l} J \, dx \, dt &\le \left(\int_{Q\setminus Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^l, \boldsymbol{v})\right)^{\frac{1}{2}} \left(\int_{Q\setminus Q^l} |g^l|^{r'}\right)^{\frac{1}{r'}} |Q \setminus Q^l|^{\frac{r'-2}{2r'}}\\
&\le C|Q\setminus Q^l|^{\frac{r'-2}{2r'}} \overset{(r<2)}{\le} f(l),\\
\int_{Q^l} J \, dx \, dt &\le C_{reg}(2,\Omega) \left(\int_{Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^l, \boldsymbol{v}) \right)^{\frac{1}{2}} \left(\int_{Q^l} |p_2^l - p_2|^2)\right)^{\frac{1}{2}}\\
&\le \frac{C_{reg}^2(\Omega,2) C_2}{2(1-\gamma_0 C_{reg}(\Omega,2))} \int_{Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}, \boldsymbol{v}^l) \, dx \, dt\\
&\quad + \frac{1-\gamma_0 C_{reg}(\Omega,2)}{2C_2} \int_{Q^l} |p_2^l - p_2|^2 \, dx \, dt.
\end{aligned} \tag{3.81}$$

Combination of (3.80) and (3.81) gives

$$\int_Q |p_2^l - p_2|^2 \; dx \; dt \leq \frac{C_{reg}^2(\Omega,2) C_2^2}{(1-\gamma_0 C_{reg}(\Omega,2))^2} \int_{Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v},\boldsymbol{v}^l) \; dx \; dt + f(l). \qquad (3.82)$$

Step 2: Using the same procedure as in previous section ($\varepsilon \to 0$), combination of the estimates (3.74) with (3.76), (3.82)and using the assumption on γ_0, we obtain the following inequality

$$\int_{Q^l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v},\boldsymbol{v}^l) \; dx \; dt + \int_Q |p_2^l - p_2|^2 \; dx \; dt \leq f(l) + \varepsilon^* \qquad (3.83)$$

with some function f such that $f(l) \overset{l\to\infty}{\to} 0$. To get point-wise convergence of the velocity gradient we can compute

$$\begin{aligned} \int_Q |\mathbf{D}(\boldsymbol{v}^l - \boldsymbol{v})| dx \; dt &\leq \left(\int_{Q^l} \ldots \; dx \; dt + \int_{Q\setminus Q^l} \ldots \; dx \; dt \right) \\ &\overset{(3.83)+\text{Hölder}}{\leq} C \left(\varepsilon^* + \int_{Q_l} \mathcal{I}_{\mathbf{D}}(\boldsymbol{v},\boldsymbol{v}^l) \; dx \; dt + |Q \setminus Q^l| \right)^\alpha \\ &\leq C(\varepsilon^* + f(l))^\alpha \end{aligned} \qquad (3.84)$$

for some $\alpha > 0$. As $f(l)$ tends to zero, (3.84) implies that

$$\limsup_{l\to\infty} \int_Q |\mathbf{D}(\boldsymbol{v}^l - \boldsymbol{v})| \; dx \; dt \leq C\varepsilon^*.$$

Because ε^* can be chosen arbitrarily small, we can use diagonal procedure to conclude that there exists subsequence that is again not relabeled such that

$$\lim_{l\to\infty} \int_Q |\mathbf{D}(\boldsymbol{v}^l - \boldsymbol{v})| \; dx \; dt = 0.$$

It implies point-wise convergence of the velocity gradient. The same conclusion is true also for point-wise convergence of the pressures p_2^l (after using (3.83)). Hence, Vitali's lemma then completes the proof of the existence of weak solution to the problem ($\mathcal{P}_1$).

4 Non-Newtonian fluids with pressure, temperature and shear rate dependent viscosity

This section is devoted to the analysis of flows of an incompressible fluid where viscosity can depend on the temperature, on the pressure and also on the shear rate, i.e., the Cauchy stress T has the form (1.10). Because of the presence of the temperature in the viscosity we also have to assume some structural condition on the heat flux, i.e., $\boldsymbol{q}$ takes the form (1.11).

Next we give assumptions on the viscosity ν and on the heat conductivity k.

Assumption on ν: We assume that the viscosity ν is a $\mathcal{C}^1$-mapping of $\mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ into $\mathbb{R}^+$ satisfying for some fixed (but arbitrary) $r \in \langle 1, 2)$ and all $\mathsf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, $\mathsf{B} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $p \in \mathbb{R}$,$\theta \in \mathbb{R}_0^+$ the following inequalities

$$\begin{aligned} C_1 \gamma_1(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{2}} |\mathsf{B}|^2 &\leq \frac{\partial \nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}_{ij}}{\partial \mathsf{D}_{kl}} \mathsf{B}_{ij}\mathsf{B}_{kl}, \\ C_2 \gamma_1(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{2}} |\mathsf{B}|^2 &\geq \frac{\partial \nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}_{ij}}{\partial \mathsf{D}_{kl}} \mathsf{B}_{ij}\mathsf{B}_{kl}, \end{aligned} \tag{4.1}$$

$$\left| \frac{\partial \nu(p,\theta,|\mathsf{D}|^2)}{\partial p} \right| |\mathsf{D}| \leq \gamma_0 \gamma_2(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{4}}, \tag{4.2}$$

where $\gamma_0 \geq 0$ is a constant whose value will be restricted in the next text and $\gamma_1(\theta) \geq 1$ is continuous non-increasing function and γ_2 is non-negative function of the temperature θ. We also define constants B_1, B_2, B_3 such that

$$\begin{aligned} B_1 &:= B_1(C_3) = \sup_{\theta \geq C_3} \gamma_1(\theta), \\ B_2 &:= B_2(C_3) = \sup_{\theta \geq C_3} \frac{\gamma_2^2(\theta)}{\gamma_1(\theta)}, \\ B_3 &:= B_3(C_3) = \sup_{\theta \geq C_3} \gamma_2(\theta). \end{aligned}$$

Note that this setting is reasonable. There are many experiences that the incompressible fluid should satisfy:

• . . . ν is increasing with respect to the pressure,
• . . . ν is decreasing with respect to the temperature.
• . . . ν is decreasing with respect to shear rate.

More sophistic discussion the reader can find for example in [2] where the dependence of the viscosity on the shear rate and the temperature is considered.

Assumption on k: We assume that k is continuous mapping of $\mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ into $\mathbb{R}^+$ such that for some arbitrary but fixed $\beta \in \mathbb{R}$ there exist positive constants C_4, C_5 such that

$$C_4 \theta^\beta \leq k(p,\theta,|\mathsf{D}|^2) \leq C_5 \theta^\beta. \tag{4.3}$$

Note that if ν is independent of θ then (4.1)-(4.2) reduce to the system (3.1)-(3.2) that has been studied in Section 3.

Let us list several examples that fulfill the assumptions (4.1)-(4.2). Note that all assumptions (4.1)-(4.3) are satisfied by the models given in the preceding section.

Example 4.1. *Consider*

$$\nu(p,\theta,|\mathbf{D}|^2) = \left(1+\gamma_i(p,\theta)+|\mathbf{D}|^2\right)^{\frac{r-2}{2}}, \tag{4.4}$$

where $\gamma_i(p,\theta)$ have the form ($s \geq 0$)

$$\gamma_1 = (1+a\frac{p^2}{1+|\theta|})^{-s}, \tag{4.5}$$

$$\gamma_2 = \begin{cases} (1+\exp(\dfrac{\alpha p}{|\theta|}))^{-s} & \text{if } p>0, \\ 1 & \text{if } p \leq 0. \end{cases} \tag{4.6}$$

It is easy to observe that the viscosity constructed in Example 4.1 satisfies (4.1)-(4.2). The proof of this observation is the same as in preceding section. Next, we give another example that is for some parameters exactly equal to the so-called *Arrhenius law*.

Example 4.2. *Consider*

$$\nu(p,\theta,|\mathbf{D}|^2) = \nu_0\exp(\frac{1}{\theta}-\frac{1}{\theta_0})(1+\alpha\gamma_i+|\mathbf{D}|^2)^{\frac{r-2}{2}} \tag{4.7}$$

where the functions γ_i can be defined as for example in previous Example 4.1 and $\alpha \geq 0$.

4.1 Definition of the weak solution and the existence theorem

In this section, we finally consider the case introduced in Section 1, i.e., we would like to find a triple $(\boldsymbol{v}, \theta, p)$ solving the following problem $(\mathcal{P})$:

$$
(\mathcal{P}) \qquad
\begin{aligned}
&\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{T} &= \boldsymbol{f} \\
\operatorname{div} \boldsymbol{v} &= 0 \\
\left.\begin{aligned}
(\theta + \frac{1}{2}|\boldsymbol{v}^2|)_{,t} + \operatorname{div}(\boldsymbol{v}(\frac{1}{2}|\boldsymbol{v}|^2 + \theta)) \\
-\boldsymbol{f} \cdot \boldsymbol{v} + \operatorname{div} \boldsymbol{q} - \operatorname{div} \mathbf{T} \cdot \mathbf{D}(\boldsymbol{v})
\end{aligned}\right\} &= 0 \\
\theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) + \operatorname{div} \boldsymbol{q} - \mathbf{T} \cdot \mathbf{D}(\boldsymbol{v}) &\geq 0
\end{aligned}\right\} \quad \text{in } \Omega \times (0,T), \\
&\left.\begin{aligned}
(\mathbf{T}\boldsymbol{n})_\tau + \alpha \boldsymbol{v}_\tau &= 0 \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\
\nabla\theta \cdot \boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \partial\Omega \times (0,T), \\
&\int_\Omega p(x,t)\, dx = 0 \qquad \text{a.a. } t \in (0,T), \\
&\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0 \qquad \text{in } \Omega, \\
&\theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega,
\end{aligned}
$$

where we considered the structural assumptions

$$
\begin{aligned}
\mathbf{T} &:= -p\mathbf{I} + \nu(p, \theta, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\
\boldsymbol{q} &:= -k(p, \theta, |\mathbf{D}(\boldsymbol{v})|^2)\nabla\theta.
\end{aligned}
$$

We also define as in Section 2 functional $E(t, \psi)$, such that for all all $t \in (0,T)$ and all $\psi \in \mathcal{C}(\Omega)$ it has the form

$$
E(t, \psi) := \int_\Omega (\theta(t,x) + \frac{|\boldsymbol{v}(t,x)|^2}{2})\psi(x)\, dx,
$$

and also define $E_0(\psi)$ as

$$
E_0(\psi) := \int_\Omega (\theta_0(x) + \frac{|\boldsymbol{v}_0(x)|^2}{2})\psi(x)\, dx.
$$

Next, we define what we mean by weak solution to the problem $(\mathcal{P})$.

Definition 4.1. *Let $\Omega \in \mathcal{C}^{0,1}$, $\boldsymbol{f} \in L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'})$ and $0 < T < \infty$. Let ν satisfy the assumptions (4.1)-(4.2) with parameter $r \in \left(\frac{9}{5}, 2\right)$. Let k satisfy (4.3) with parameter $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\theta_0 \in L^1(\Omega)$, $\theta_0 \geq C_3 > 0$ a.e. in Ω. We say that a triple $(\boldsymbol{v}, p, \theta)$ is weak solution to the problem $(\mathcal{P})$ if*

$$\boldsymbol{v} \in \mathcal{C}(0,T;L^2_{weak}(\Omega)^3) \cap L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}), \tag{4.8}$$

$$\boldsymbol{v}_{,t} \in L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}}), \tag{4.9}$$

$$p \in L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)) \text{ and } \int_\Omega p(x,t)\,dx = 0 \text{ for a.a. } t \in (0,T), \tag{4.10}$$

$$\theta \in L^\infty(0,T;L^1(\Omega)), \tag{4.11}$$

$$\theta^{\frac{\beta+\lambda+1}{2}} \in L^2(0,T;W^{1,2}(\Omega)) \text{ for all } \lambda < 0, \tag{4.12}$$

$$E(t,\psi) \in \mathcal{C}(0,T) \qquad \text{for all } \psi \in \mathcal{C}(\Omega), \tag{4.13}$$

$$\lim_{t\to 0+} \int_\Omega \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0, \tag{4.14}$$

$$\lim_{t\to 0+} E(t,\psi) = E_0(\psi), \tag{4.15}$$

and the following weak formulations hold

$$\int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \boldsymbol{\varphi}) + \alpha\int_{\partial\Omega} \boldsymbol{v}\boldsymbol{\varphi}\,dS + (\mathbf{T}, \mathbf{D}(\boldsymbol{\varphi})) - \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle\,dt = 0, \tag{4.16}$$

$$\left.\begin{aligned} &\int_0^T -(\varphi_{,t}, \theta + \frac{1}{2}|\boldsymbol{v}|^2) - (\boldsymbol{v}(\frac{1}{2}|\boldsymbol{v}|^2 + \theta), \nabla\varphi) + \alpha\int_{\partial\Omega} |\boldsymbol{v}|^2\varphi\,dS \\ &\quad + (\mathbf{T}\boldsymbol{v} + \boldsymbol{q}, \nabla\varphi) - \langle \boldsymbol{f}, \boldsymbol{v}\varphi\rangle\,dt - (\frac{1}{2}|\boldsymbol{v}_0(x)|^2 + \theta_0(x), \varphi(0,x)) \end{aligned}\right\} = 0, \tag{4.17}$$

$$\int_0^T -(\theta, \psi_{,t}) - (\boldsymbol{v}\theta, \nabla\psi) + (\boldsymbol{q}, \nabla\psi) - (\mathbf{T}\cdot\mathbf{D}(\boldsymbol{v}), \psi) - (\theta_0, \psi(0)) \geq 0, \tag{4.18}$$

for all $\boldsymbol{\varphi} \in L^{\frac{5r}{5r-6}}(0,T;W_{\boldsymbol{n}}^{1,\frac{5r}{5r-6}})$, all $\varphi \in \mathcal{D}(-\infty,T;\mathcal{C}^1(\Omega))$ and for all $\psi \geq 0$, $\psi \in \mathcal{D}(-\infty,T;\mathcal{C}^1(\Omega))$.

One can see that there is a remarkable difference in the lower bound on r in Section 3. Namely, in Section 3 there was assumed that $r > \frac{6}{5}$. On the other hand, we prescribe the condition $r > \frac{9}{5}$ in the Definition 4.1. But this is the price that one has to pay for equation (4.17). Indeed, one needs integrability of the term $\boldsymbol{v}|\boldsymbol{v}|^2$ that is presented in (4.17). As we have the estimate (4.8), we can obtain (after using standard interpolation) that

$$\boldsymbol{v} \in L^{\frac{5r}{3}}(0,T;L^{\frac{5r}{3}}(\Omega)^3). \tag{4.19}$$

Because we cannot expect more information about the velocity $\boldsymbol{v}$ (for example some regularity), (4.19) is the best information that we have. The bound $r > \frac{9}{5}$

then follows from the requirement of the integrability of $\boldsymbol{v}|\boldsymbol{v}|^2$ (and also the integrability of $\boldsymbol{v}p$).

Second bound in Definition 4.1 is $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$. But this is also *natural* requirement if we want to have weak formulation of the global energy equation (4.17) because we need that $\boldsymbol{v}\theta \in L^1(0,T;L^1(\Omega)^3)$. To show it we use only standard interpolation inequalities together with relations (4.8), (4.11) and (4.12). First we have for all $2 \le q \le \frac{3r}{3-r}$ and for all $1 \le s \le 3(\beta+\lambda+1)$ that

$$\|\cdot\|_q \le \|\cdot\|_2^{\frac{6r-6q+2qr}{q(5r-6)}} \|\cdot\|_{1,r}^{\frac{3r(q-2)}{q(5r-6)}}, \tag{4.20}$$

$$\|\cdot\|_s \le \|\cdot\|_1^{\frac{3(\beta+\lambda+1)-s}{s(3\beta+3\lambda+2)}} \|\cdot\|_{3(\beta+\lambda+1)}^{\frac{3(\beta+\lambda+1)(s-1)}{s(3\beta+3\lambda+2)}}. \tag{4.21}$$

Then we can estimate

$$\begin{aligned}
\int_0^T \|\boldsymbol{v}\theta\|_1 &\le \int_0^T \|\boldsymbol{v}\|_q \|\theta\|_{q'} \overset{(4.8),(4.11),(4.20),(4.21)}{\le} C\int_0^T \|\boldsymbol{v}\|_{1,r}^{\frac{3r(q-2)}{q(5r-6)}} \|\theta\|_{3(\beta+\lambda+1)}^{\frac{3(\beta+\lambda+1)(q'-1)}{q'(3\beta+3\lambda+2)}} \\
&\le C\Big(1+\int_0^T \|\boldsymbol{v}\|_{1,r}^{\frac{3r(q-2)}{q(5r-6)}} \|\theta^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^{\frac{6(q'-1)}{q'(3\beta+3\lambda+2)}}\Big) \\
&\overset{(4.8)}{\le} C + \int_0^T \|\theta^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^{\frac{6}{q(3\beta+3\lambda+2)}\frac{q(5r-6)}{5rq-9q+6}}
\end{aligned}$$

provided $2 \le q \le \frac{3r}{3-r}$ and $q' \le 3(\beta+\lambda+1)$. Using (4.12) we see that we need at least

$$\frac{6}{q(3\beta+3\lambda+2)}\frac{q(5r-6)}{5rq-9q+6} \le 2 \Leftrightarrow \frac{5r-6}{5rq-9q+6} - \frac{2}{3} \le \beta+\lambda$$

to bound the term on the right hand side. Because $r > \frac{9}{5}$ we see that the first term on the right hand side of the second inequality is decreasing with respect to q and we are led to chose maximal q. Hence, if we set $q := \frac{3r}{3-r}$ then we get

$$\beta+\lambda \ge \frac{3-r}{3(r-1)} - \frac{2}{3}. \tag{4.22}$$

It remains to show that

$$q' \le 3(\beta+\lambda+1) \Leftrightarrow \frac{r}{4r-3} - 1 \le \beta+\lambda. \tag{4.23}$$

But for $r \in (1,3)$ we have that

$$\frac{3-r}{3(r-1)} - \frac{2}{3} \ge \frac{r}{4r-3} - 1$$

and then (4.22) implies (4.23). Hence, because λ can be chosen arbitrarily small we see that that (4.22) leads to the assumption $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$ that is exactly the assumption considered in Definition 4.1.

Next, we give the main theorem of this section.

Theorem 4.1. *Let $\Omega \in \mathcal{C}^{1,1}$. Let $\boldsymbol{f} \in L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'})$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\theta_0 \in L^1(\Omega)$, $\theta_0 \geq C_3 > 0$ a.e. in Ω. Let ν satisfy* (4.1)-(4.2) *with $\frac{9}{5} < r < 2$ and γ_0 fulfilling*

$$\gamma_0 < \frac{C_1}{C_{reg}(2,\Omega)(C_1 B_3 + C_2\sqrt{B_1 B_2})}. \tag{4.24}$$

Let k satisfy (4.3) *with $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$. Then there exists weak solution to the problem $(\mathcal{P})$.*

Note that there is only one difference between assumptions of Definition 4.1 and assumptions of Theorem 4.1. This different assumption is $\Omega \in \mathcal{C}^{0,1}$ vs. $\Omega \in \mathcal{C}^{1,1}$. However, this is needed to have pressure globally, because L^p theory for the Laplace equation will be used in the proof.

This theorem combines the results proved in Section 2.2 and 3 and it is the first result about global-in-time and large-data existence for model where viscosity ν and heat conductivity k can depend on $\theta, p, |\mathbf{D}|^2$. Note that similar systems were studied in [4] where the authors assumed the case $k \equiv$ *const.*, $\nu = \nu(\theta, |\mathbf{D}|^2)$ and they studied stationary non-linear Stokes system (without convective term). Non-stationary models were for example studied in [7]. The authors considered again the case $k \equiv$ *const.*, $\nu = \nu(\theta)|\mathbf{D}|^r$. They established local-in-time and small-data existence result for nonlinear Stokes system and they used non-stationary equation for temperature. They did not need to assume that $r > \frac{9}{5}$ because they used equation for internal energy $(1.1)_3$ where this bound is not needed.

We have to note that the classical NSEs are not included in Theorem 4.1 because we have $r < 2$. However, it will be clear from the proof that if the viscosity ν and the heat conductivity k do not depend on the pressure then we will not need to have point-wise convergence of the pressures and the bound $r < 2$ will be irrelevant. Hence, we can formulate the following theorem:

Theorem 4.2. *Let $\Omega \in \mathcal{C}^{1,1}$. Let $\boldsymbol{f} \in L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'})$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\theta_0 \in L^1(\Omega)$, $\theta_0 \geq C_3$ a.e. in Ω. Let ν satisfy* (4.1) *with $r > 2$ Let k satisfy* (4.3) *with $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$. Then there exists weak solution to the problem $(\mathcal{P})$.*

Remark 4.1. *We have to modify the definition of weak solution for $r > \frac{11}{5}$. The solution constructed in Theorem 4.2 for such r does not satisfy* (4.10) *but it satisfies*

$$p \in L^{r'}(0,T;L^{r'}(\Omega)),$$

and (4.16) *holds for all $\boldsymbol{\varphi} \in L^r(0,T;W_{\boldsymbol{n}}^{1,r})$.*

4.2 Proof of the theorem

4.2.1 The structure of the proof

We use the methods that were described in preceding two sections. We again use the same quasi-compressible approximation for the pressure, i.e.,

$$-\varepsilon \triangle p + \operatorname{div} \boldsymbol{v} = 0,$$

that helps us to have some pressure from the beginning. We also mollify the convective term to have validity of equation for kinetic energy. Because we will want to use the standard theory for parabolic systems we also modify the function k in proper way such that k will be a bounded function. Moreover we add a perturbation $\eta(1+|\theta|)^\beta$ into the equation of internal energy $(1.1)_3$ that helps us to have expected apriori estimates of the temperature. Then we will follow almost step by step the method (or procedure) developed in Sections 2-3 with only small modification in the limit passage for the internal energy equation.

4.2.2 ε,η - approximation

We start with the following approximation. We define the function k_ε such that

$$k_\varepsilon(p,\theta,|\mathbf{D}|^2) := \begin{cases} k(\cdot,\cdot,\cdot) & \text{if } \varepsilon \le k \le \dfrac{1}{\varepsilon} \\ \dfrac{1}{\varepsilon} & \text{if } k > \dfrac{1}{\varepsilon} \\ \varepsilon & \text{if } k < \varepsilon. \end{cases}$$

Finally, we define the problem $(\mathcal{P})^{\varepsilon,\eta}$ such that (we use the notation $\nu := \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)$, $k_\varepsilon := k_\varepsilon(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)$, and $Q := \Omega\times(0,T)$, $\partial Q := \partial\Omega\times(0,T)$)

$$(\mathcal{P})^{\varepsilon,\eta} \qquad \begin{aligned} &\left.\begin{aligned} \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta\otimes\boldsymbol{v}) - \operatorname{div}(\nu\mathbf{D}(\boldsymbol{v})) + \nabla p &= \boldsymbol{f} \\ -\varepsilon\triangle p + \operatorname{div}\boldsymbol{v} &= 0 \\ \theta_{,t} + \operatorname{div}(\boldsymbol{v}_\eta\theta) - \operatorname{div}\big((\eta(1+|\theta|)^\beta + k_\varepsilon)\nabla\theta\big) - \nu|\mathbf{D}(\boldsymbol{v})|^2 &= 0 \end{aligned}\right\} \quad \text{in } Q, \\ &\left.\begin{aligned} \big(\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big)_\tau + \alpha\boldsymbol{v}_\tau &= 0 \\ \boldsymbol{v}\cdot\boldsymbol{n} &= 0 \\ \nabla p\cdot\boldsymbol{n} &= 0 \\ \nabla\theta\cdot\boldsymbol{n} &= 0 \end{aligned}\right\} \quad \text{on } \partial Q, \\ &\int_\Omega p(x,t)\,dx = 0, \\ &\boldsymbol{v}(x,0) = \boldsymbol{v}_0, \\ &\theta(x,0) = \theta_0, \end{aligned}$$

where we wrote $(\boldsymbol{v},\theta,p)$ instead of $(\boldsymbol{v}^{\varepsilon,\eta},\theta^{\varepsilon,\eta},p^{\varepsilon,\eta})$. The definition of the mollified function $\boldsymbol{v}_\eta$ is given in Subsection 2.2.1.

To solve the approximative system we use Galerkin approximations and by very similar methods as those described in Subsection 2.2.1 we will get the solution of the system $(\mathcal{P})^{\varepsilon,\eta}$.

Galerkin approximation First, we define the continuous mapping $\mathcal{F} : W^{1,r}_{\boldsymbol{n}} \to W^{2,r}(\Omega)$ as follows. For some $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}}$ we find $p \in W^{2,r}(\Omega)$ solving the equation

$$
\begin{aligned}
\varepsilon \triangle p &= \operatorname{div} \boldsymbol{v}, && \text{in } \Omega,\\
\int_\Omega p \, dx &= 0, &&\\
\nabla p \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

and we define $\mathcal{F}(\boldsymbol{v}) := p$. It follows from the theory for Laplace equation that this operator is continuous.

Next, let $\{\boldsymbol{w}_j\}_{j=1}^\infty$ be a basis of $W^{1,r}_{\boldsymbol{n}}$ such that $\boldsymbol{w}_j \in W^{1,2r}_{\boldsymbol{n}}$ for all j and $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, dx = \delta_{ij}$. Let $\{w_j\}_{j=1}^\infty$ be a basis of $W^{1,2}(\Omega)$ which is again orhtonormal in the space $L^2(\Omega)$.

We construct Galerkin approximations $\{\boldsymbol{v}^{N,M}, \theta^{N,M}, p^{N,M}\}_{N,M=1}^\infty$ in the form

$$
\begin{aligned}
\boldsymbol{v}^{N,M} &:= \sum_{i=1}^N c_i^{N,M}(t) \boldsymbol{w}_i,\\
\theta^{N,M} &:= \sum_{i=1}^M d_i^{N,M}(t) w_i,\\
p^{N,M} &:= \mathcal{F}(\boldsymbol{v}^{N,M}),
\end{aligned}
$$

where $\boldsymbol{c}^{N,M}$ solve the system of ordinary differential equations

$$
\begin{aligned}
\frac{d}{dt}(\boldsymbol{v}^{N,M}, \boldsymbol{w}_j) - (\boldsymbol{v}_\eta^{N,M} \otimes \boldsymbol{v}^{N,M}, \nabla \boldsymbol{w}_j) + (\nu_N^{N,M}) \mathbf{D}(\boldsymbol{v}^{N,M}), \nabla \boldsymbol{w}_j)&\\
+\alpha \int_{\partial\Omega} \boldsymbol{v}^{N,M} \cdot \boldsymbol{w}_j \, dS - (\mathcal{F}(\boldsymbol{v}^{N,M}), \operatorname{div} \boldsymbol{w}_j) = \langle \boldsymbol{f}, \boldsymbol{w}_j, \rangle&
\end{aligned} \tag{4.25}
$$

for all $j = 1, 2, \dots, N$ and $\boldsymbol{d}^{N,M}$ solve

$$
\begin{aligned}
\frac{d}{dt}(\theta^{N,M}, w_j) + ((\eta(1 + |\theta^{N,M}|)^\beta + k_\varepsilon^{N,M}) \nabla \theta^{N,M}, \nabla w_j)&\\
-(\boldsymbol{v}_\eta^{N,M} \theta^{N,M}, \nabla w_j) = \big(\nu^{N,M} |\mathbf{D}(\boldsymbol{v}^{N,M})|^2, w_j\big)&
\end{aligned} \tag{4.26}
$$

for all $j = 1, 2, \dots, M$. We have to explain the meaning of $\nu_N^{N,M}$. It is defined as

$$
\nu_N^{N,M} := \begin{cases} \nu^{N,M} := \nu(\theta^{N,M}, p^{N,M}, |\mathbf{D}(\boldsymbol{v}^{N,M}|^2) & \text{if } \theta^{N,M} \geq C_3,\\ \nu^{N,M} := \nu(C_3, p^{N,M}, |\mathbf{D}(\boldsymbol{v}^{N,M}|^2)) & \text{if } \theta^{N,M} < C_3. \end{cases}
$$

Note that in the next step we will show that the temperature satisfies minimum principle and therefore the definition of our approximative viscosity will coincide with the non-approximative one.

We also assume that $\boldsymbol{v}^{N,M}$ and $\theta^{N,M}$ satisfy the following initial conditions

$$
\begin{aligned}
\boldsymbol{v}^{N,M}(\cdot, 0) &= \boldsymbol{v}_0^{N,M},\\
\theta^{N,M}(\cdot, 0) &= \theta_0^{N,M},
\end{aligned}
$$

where the definition of $\boldsymbol{v}_0^{N,M}$ and $\theta_0^{N,M}$ is the same as that introduced in Section 2.2.1 i.e., we define $\boldsymbol{v}_0^{N,M} := \sum_{j=1}^N c_0^N \boldsymbol{w}_j$ being the projections of $\boldsymbol{v}_0$ onto linear hulls of $\{\boldsymbol{w}_j\}_{j=1}^N$ and $\theta_0^{N,M}$ has the following meaning. We first regularize θ_0 with regularization kernel $\omega_{\frac{1}{N}}$ of radii $\frac{1}{N}$ (after defining $\theta_0 := C_3$ outside Ω). It means we define $\theta_0^N := (\omega_{1/N} * \theta_0)$. Then we apply the projection onto the linear hull of $\{w_j\}_{j=1}^M$. Thus, $\theta_0^{N,M}$ has the form $\theta_0^{N,M} := \sum_{j=1}^M d_0^M w_j$.

Let us define $\mathsf{C}(t) := (c_1^{N,M}(t), \dots c_N^{N,M}(t), d_1^{N,M}(t), \dots d_M^{N,M}(t))$ and $\mathsf{C}_0 := (\boldsymbol{c}_0^N, \boldsymbol{d}_0^M)$. The system (2.19)-(2.20) can be rewritten as

$$\begin{aligned} \frac{d}{dt}\mathsf{C} &= \mathcal{G}(t, \mathsf{C}), \\ \mathsf{C}(0) &= \mathsf{C}_0. \end{aligned} \tag{4.27}$$

By using the same methods as those described in Subsection 2.2.1 (Caratheodory's theory) we can get the local existence of the solution to the system (4.25)-(4.26).

In the next subsection we will show the uniform boundedness of C and by means of Theorem B.1 we will able to extend the solution to the whole time interval $(0, T)$.

Next, we derive apriori estimates and we pass to the limit in Galerkin approximation. First, we set $M \to \infty$ and then $N \to \infty$.

Estimates independent of M: Multiplying j-th equation in (4.25) by $c_j^{N,M}$, summing over $j = 1, ..., M$, integrating over $(0, T)$ and using the assumption on the viscosity ν and Lemma A.1, we observe that

$$\begin{aligned} \sup_{t\in(0,T)} \|\boldsymbol{v}^{N,M}(t)\|_2 + \int_0^T \|\mathbf{D}(\boldsymbol{v}^{N,M})\|_r^r + \int_{\partial\Omega} |\boldsymbol{v}^{N,M}|^2 \, dS \\ + \varepsilon \|\nabla p^{N,M}\|_2^2 \, d\tau \le C \left(1 + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}^{N,M} \rangle \, d\tau \right) \end{aligned} \tag{4.28}$$

Multiplying the j-th equation in (4.26) by $d_j^{M,N}$, integrating over time $t \in (0, T)$, using the previous estimate (2.22), assumption on k_ε, ν and the fact that $\boldsymbol{w}_j \in W_{\boldsymbol{n}}^{1,2r}$ we get

$$\begin{aligned} \|\theta^{N,M}(t)\|_2^2 + \int_0^t \int_\Omega (\eta(1 + |\theta^{N,M}|)^\beta + 1) |\nabla\theta^{N,M}|_2^2 \, dx \, d\tau \\ \le C(N)\Big(1 + \int_0^t \|\theta^{N,M}\|_2^2 \, d\tau\Big). \end{aligned} \tag{4.29}$$

Finally, we apply Korn's inequality (Lemma B.2) and Young's inequality to (4.28) to get

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{N,M}(t)\|_2 + \int_0^T \|\boldsymbol{v}^{N,M}\|_{1,r}^r + \varepsilon \|\nabla p^{N,M}\|_2^2 \, dt \le C. \tag{4.30}$$

An application of Gronwall's lemma to (4.29) then gives

$$\sup_{t\in(0,T)} \|\theta^{N,M}(t)\|_2^2 + \int_0^T \int_\Omega (k_\varepsilon^{N,M} + \eta(1+|\theta^{N,M}|)^\beta)|\nabla\theta^{N,M}|^2 \, dx \, d\tau \le C(N). \tag{4.31}$$

The inequality (4.31) also implies that (after using the definition of k_ε)

$$\int_0^T \int_\Omega |\nabla\theta^{N,M}|^2 + |\nabla(1+|\theta^{N,M}|)^{\frac{\beta+2}{2}}|^2 \, dx \, dt \le C(\eta,\varepsilon,N).$$

Consequently, for $\beta \le 0$ we have

$$\int_0^T \|\theta^{N,M}\|_{W^{1,2}(\Omega)}^2 \, dt \le C(\varepsilon, N).$$

For $\beta > 0$ we can compute (we omit the superscripts N, M)

$$\begin{aligned}
\int_0^T \|(1+|\theta|)^{\frac{\beta+2}{2}}\|_{W^{1,2}(\Omega)}^2 \, dt &= \int_0^T \|(1+|\theta|)^{\frac{\beta+2}{2}}\|_2^2 + \|\nabla(1+|\theta|)^{\frac{\beta+2}{2}}\|_2^2 \\
&\overset{(4.31)}{\le} C(\varepsilon,N,\eta) + \int_0^T \|(1+|\theta|)\|_{\beta+2}^{\beta+2} \, dt \\
&\le C(\varepsilon,N,\eta) + \int_0^T \|(1+|\theta|)\|_1^{\frac{2}{3\beta+5}} \|(1+|\theta|)\|_{3(\beta+2)}^{3\frac{\beta+1}{3\beta+5}} \, dt \\
&\le C(1 + \int_0^T \|(1+|\theta|)^{\frac{\beta+2}{2}}\|_6^{3\frac{\beta+1}{3\beta+5}\frac{6}{3(\beta+2)}} \, dt) \\
&\le C(1 + \int_0^T \|(1+|\theta|)^{\frac{\beta+2}{2}}\|_{W^{1,2}(\Omega)}^{3\frac{\beta+1}{3\beta+5}\frac{6}{3(\beta+2)}} \, dt).
\end{aligned}$$

Because $3\frac{\beta+1}{3\beta+5}\frac{6}{3(\beta+2)} < 2$ we can apply Young's inequality to get

$$\int_0^T \|(1+|\theta|)^{\frac{\beta+2}{2}}\|_{W^{1,2}(\Omega)}^2 \, dt \le C(\varepsilon,\eta,N). \tag{4.32}$$

Next, for $\beta \le 0$ we know that $\int_0^T \|\theta^{N,M}\|_{W^{1,2}(\Omega)}^2 \le C(\varepsilon, N)$. Thus, it is a simple consequence of standard interpolation inequality and (4.31) that

$$\int_0^T \|\theta^{N,M}\|_{\frac{10}{3}}^{\frac{10}{3}} \le C(N,\varepsilon).$$

If $\beta \ge 0$, we can use the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ to obtain

$$\int_0^T \|(1+|\theta^{N,M}|)\|_{3(\beta+2)}^{\beta+2} = \int_0^T \|(1+|\theta^{N,M}|)^{\frac{\beta+2}{2}}\|_6^2 \overset{(4.32)}{\le} C(N,\eta).$$

Finally, using

$$\|\cdot\|_p \le \|\cdot\|_2^{\frac{2(3(\beta+2)-p)}{p(3\beta+4)}} \, \|\cdot\|_{3(\beta+2)}^{\frac{3p\beta-6(\beta+2)+6p}{p(3\beta+4)}}$$

leads to

$$\int_0^T \|\theta^{N,M}\|_p^p \le \int_0^T \|\theta^{N,M}\|_2^{\frac{2(3(\beta+2)-p)}{3\beta+4}} \, \|\theta^{N,M}\|_{3(\beta+2)}^{\frac{3p\beta-6(\beta+2)+6p}{3\beta+4}}. \tag{4.33}$$

The right hand side of (4.33) is bounded if $\frac{3p\beta-6(\beta+2)+6p}{3\beta+4} \le \beta+2$ which is equivalent to $p \le \beta + \frac{10}{3}$. Thus, we have

$$\int_0^T \|\theta^{N,M}\|_{\beta+\frac{10}{3}}^{\beta+\frac{10}{3}} \le C(\eta, N). \tag{4.34}$$

To get some compactness of the velocity and temperature, we also estimate the norms of their time derivatives. Multiplying the j-equation in (4.25) by $\frac{d}{dt}c_j^{N,M}$ and integrating it over time, we obtain (after using the estimate (4.30))

$$\int_0^T \left(\frac{d}{dt}\boldsymbol{c}^{N,M}\right)^2 \, dt \le C(N). \tag{4.35}$$

From the estimates proved above we see that we can estimate for sufficiently large σ

$$\|\theta_{,t}^{N,M}\|_{L^{\sigma'}(0,T;W^{-1,\sigma'}(\Omega))} \le C(N). \tag{4.36}$$

Limit $M \to \infty$**:** Having apriori estimates (2.24)-(2.27), we can set $M \to \infty$ and find subsequences $\{\boldsymbol{c}^{N,M}, \theta^{N,M}\}_{M=1}^{\infty}$ (that are not relabeled) such that

$$\begin{aligned}
\boldsymbol{c}_{,t}^{N,M} &\rightharpoonup \boldsymbol{c}_{,t}^{N} && \text{weakly in } L^2(0,T), && (4.37)\\
\boldsymbol{c}^{N,M} &\rightharpoonup^* \boldsymbol{c}^{N} && \text{weakly}^* \text{ in } L^\infty(0,T), && (4.38)\\
\theta^{N,M} &\rightharpoonup \theta^{N} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), && (4.39)\\
(1+|\theta^{N,M}|)^{\frac{\beta+2}{2}} &\rightharpoonup \overline{(1+|\theta^{N}|)^{\frac{\beta+2}{2}}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) && (4.40)\\
\theta_{,t}^{N,M} &\rightharpoonup \theta_{,t}^{N} && \text{weakly in } L^{\sigma'}(0,T;W^{-1,\sigma'}(\Omega)). && (4.41)
\end{aligned}$$

Morover, after using Aubin-Lions compactness lemma (Lemma B.3) we have

$$\theta^{N,M} \to \theta^{N} \qquad \text{strongly in } L^m(0,T;L^m(\Omega)). \tag{4.42}$$

for all $m \in \left\langle 1, \max(\beta + \frac{10}{3}, \frac{10}{3})\right)$ and consequently we get that

$$\overline{(1+|\theta^N|)^{\frac{\beta+2}{2}}} = (1+|\theta^N|)^{\frac{\beta+2}{2}}.$$

Finnaly, we deduce by using Arzela-Ascoli theorem (see Section 2.2.1) that

$$\boldsymbol{c}^{N,M} \to \boldsymbol{c}^{N} \qquad \text{strongly in } \mathcal{C}(0,T). \tag{4.43}$$

Moreover, it is a simple consequence of our choice of basis and (4.43) that

$$\boldsymbol{v}^{N,M} \to \boldsymbol{v}^N \qquad \text{strongly in } L^{2r}(0,T;W^{1,2r}_{\boldsymbol{n}}), \tag{4.44}$$

and consequently as $\mathcal{F}$ is continuous

$$p^{N,M} \to p^N \qquad \text{strongly in } L^{2r}(0,T;W^{1,2r}(\Omega)). \tag{4.45}$$

The convergences (4.37)-(4.45) allow us to pass to the limit in (4.25) to get that the following system of equations

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{v}^N,\boldsymbol{w}_j) - (\boldsymbol{v}^N_\eta \otimes \boldsymbol{v}^N, \nabla \boldsymbol{w}_j) + (\nu^N_N \mathbf{D}(\boldsymbol{v}^N), \nabla \boldsymbol{w}_j)& \\ +\alpha \int_{\partial\Omega} \boldsymbol{v}^N \cdot \boldsymbol{w}_j \, dS - (p^N, \operatorname{div} \boldsymbol{w}_j) = \langle \boldsymbol{f}, \boldsymbol{w}_j\rangle&\end{aligned} \tag{4.46}$$

holds for $j = 1, 2, \ldots, N$ and almost all times $t \in (0,T)$. It is also easy to pass to the limit in (4.26) in time derivatives, the convective term and in the term that appears on the right hand side. It remains to prove that for all $\varphi \in L^\infty(0,T;\mathcal{C}^1(\Omega))$ the following holds

$$\begin{aligned}&\int_0^T \left((k_\varepsilon^{N,M} + \eta(1+|\theta|^{N,M})^\beta)\nabla\theta^{N,M}, \nabla\varphi\right)\, dt \\ &\overset{M\to\infty}{\to} \int_0^T \left((k_\varepsilon^{N} + \eta(1+|\theta|^{N})^\beta)\nabla\theta^{N}, \nabla\varphi\right)\, dt.\end{aligned}$$

As $k_\varepsilon^{N,M}$ are bounded function and $k_\varepsilon^{N,M} \to k_\varepsilon^N$ a.e. in $(0,T)\times\Omega$ it is easy to get that

$$\int_0^T \left(k_\varepsilon^{N,M}\nabla\theta^{N,M}, \nabla\varphi\right)\, dt \overset{M\to\infty}{\to} \int_0^T \left(k_\varepsilon^{N}\nabla\theta^{N}, \nabla\varphi\right)\, dt.$$

Next, if $\beta \le 0$ then the same procedure can be used on the remaining term. If $\beta > 0$ then we define the function

$$\Theta^{N,M}(x,t) := \int_0^{\theta^{N,M}(x,t)} (1+s^2)^{\frac{\beta}{4}}\, ds.$$

The estimate (4.31) then implies that

$$\int_0^T \|\nabla\Theta^{N,M}\|_2^2\, dt = \int_0^T \int_\Omega (1+(\theta^{N,M})^2)^{\frac{\beta}{2}} |\nabla\theta^{N,M}|^2\, dx\, dt \le \frac{C(N)}{\eta}.$$

Therefore we have

$$\nabla\Theta^{N,M} \rightharpoonup \nabla\Theta^N \text{ weakly in } L^2(0,T;L^2(\Omega)^3), \tag{4.47}$$

where $\Theta^N(x,t) = \int_0^{\theta^N(x,t)} (1+s^2)^{\frac{\beta}{4}}\, ds$. Thus, we can compute

$$
\begin{aligned}
&\int_0^T \left(\eta(1+|\theta|^{N,M})^\beta \nabla\theta^{N,M}, \nabla\varphi\right)\, dt \\
&\quad = \int_0^T \left(\eta \underbrace{\frac{(1+|\theta|^{N,M})^\beta}{(1+(\theta^{N,M})^2)^{\frac{\beta}{4}}}}_{\text{converges strongly in } L^2(0,T;L^2)} \underbrace{\nabla\Theta}_{\text{weakly in} L^2(0,T;L^2)}, \nabla\varphi \right) dt \\
&\quad \overset{M\to\infty}{\to} \eta \int_0^T \left(\frac{(1+|\theta|^N)^\beta}{(1+(\theta^N)^2)^{\frac{\beta}{4}}} \nabla\Theta, \nabla\varphi \right) dt \\
&\quad = \int_0^T \left(\eta(1+|\theta|^N)^\beta)\nabla\theta^N, \nabla\varphi\right)\, dt.
\end{aligned}
$$

Thus, we are able to get that

$$
\begin{aligned}
\langle \theta^N_{,t}, \varphi\rangle - (\boldsymbol{v}^N_\eta \theta^N, \nabla\varphi) + ((k^N_\varepsilon + \eta(1+|\theta^N|)^\beta)\nabla\theta^N, \nabla\varphi) \\
= \left(\nu^N |\mathbf{D}(\boldsymbol{v}^N)|^2, \varphi\right)
\end{aligned}
\tag{4.48}
$$

is valid for all $\varphi \in W^{1,\sigma}(\Omega)$ and for a.a. $t \in (0,T)$. The attainment of initial condition can be proved by using in the same procedure as in Section 2.2.1.

Minimum principle: Next we prove another standard result for parabolic problems. We will show that

$$
\theta^N(x,t) \geq \operatorname*{essinf}_{x\in\Omega} \theta^N_0 \geq C_3 > 0 \text{ for a.a. } (x,t) \in \Omega \times (0,T). \tag{4.49}
$$

We use the weak formulation (4.48) with the function $\varphi := \min(0, \theta^N - C_3) \leq 0$. If φ is an admissible test function then we can easily prove (4.49). Thus, for $\beta \leq 0$ it is clear that φ is possible test function and (4.49) follows. For $\beta > 0$ we see that the worst term is

$$
\int_0^T \left((1+|\theta^N|)^\beta \nabla\theta^N, \nabla\varphi\right).
$$

But it can be rewritten as

$$
\int_0^T \left(\operatorname{sign}(\theta^N)(1+|\theta^N|)^{\frac{\beta}{2}} \nabla(1+|\theta^N|)^{\frac{\beta+2}{2}}, \nabla\varphi\right)\, dt.
$$

But if $\nabla\varphi \in L^2(0,T;L^2(\Omega,\mu))$ where the measure μ is defined as $d\mu := \operatorname{sign}(\theta^N)(1+|\theta^N|)^{\frac{\beta}{2}} dx$ then the integral makes a good sense. Consequently, we can estimate the norm of time derivatives in corresponding space. Hence, we can test the equation (4.48) by $\varphi := \min(0, \theta^N - C_3)$ to get (4.49).

Note that (4.49) implies that $\nu^N_N = \nu^N$ a.e. in $\Omega \times (0,T)$.

Estimates independent of N: Next, we obtain apriori estimates being independent of N. Using weak lower semicontinuity of norms we find that (4.30) holds (without superscript M). Testing (4.48) by $\varphi \equiv 1$ and using (4.28) leads to

$$\sup_t \|\theta^N(t)\|_1 \leq C. \tag{4.50}$$

Next, we set $\varphi := (\theta^N)^\lambda$ with $\lambda < 0$. Note that (4.49) implies that $0 \leq \varphi \leq C$ for almost all (t, x). We use φ as a test function in (4.48) and we integrate it over time $t \in (0, T)$ to get (for details see Section 2):

$$\int_0^T \int_\Omega (k_\varepsilon^N + \eta(1+\theta^N)^\beta)(\theta^N)^{\lambda-1} |\nabla \theta^N|^2 \, dx \, dt \leq C(\lambda). \tag{4.51}$$

It leads to the following inequality (after using $k_\varepsilon^N \geq \varepsilon$)

$$\int_{\Omega \times (0,T)} |\nabla (\theta^N)^{\frac{\lambda+1}{2}}|^2 + |\nabla (A + \theta^N)^{\frac{\beta+\lambda+1}{2}}|^2 \, dx \, dt \leq C(\lambda, \varepsilon, \eta), \tag{4.52}$$

where

$$A := \begin{cases} 1 \text{ if } \beta \leq 0 \\ 0 \text{ if } \beta > 0. \end{cases}$$

Embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, (4.31) and the first part of the estimate (4.52) then imply that $(\theta^N)^{\frac{\lambda+1}{2}}$ is bounded in $L^2(0, T; L^6(\Omega))$ for all $\lambda < 0$. This fact together with estimate (4.50) then lead to the following conclusion (after using standard interpolation inequality)

$$\|\theta^N\|_{L^n(0,T;L^n(\Omega))} \leq C \text{ for all } n \in \left\langle 1, \frac{5}{3} \right). \tag{4.53}$$

Moreover, if $\beta > 0$ we can get a better result. Let us recall that (4.52) implies

$$\int_{\Omega \times (0,T)} (A + \theta^N)^{\frac{\beta+\lambda+1}{2}}|^2 \, dx \, dt \leq C(\lambda, \varepsilon, \eta).$$

We can compute similarly as in the preceding paragraph (we omit for simplicity the superscript N)

$$\begin{aligned}
\int_0^T \|(A+\theta)^{\frac{\beta+\lambda+1}{2}}\|^2_{W^{1,2}(\Omega)} &\overset{(4.52)}{\leq} C(\eta, \lambda, \varepsilon) + \int_0^T \|(A+\theta)^{\frac{\beta+\lambda+1}{2}}\|_2^2 \\
&= C(\eta, \lambda, \varepsilon) + \int_0^T \|(A+\theta)\|_{\beta+\lambda+1}^{\beta+\lambda+1} \\
&\leq C(\eta, \lambda, \varepsilon) + \int_0^T \|(A+\theta)\|_1^{\frac{2(\beta+\lambda+1)}{3\beta+3\lambda+2}} \|(A+\theta)\|_{3(\beta+\lambda+1)}^{\frac{3(\beta+\lambda)(\beta+\lambda+1)}{3\beta+3\lambda+2}} \\
&\leq C(\eta, \lambda, \varepsilon) \left(1 + \int_0^T \|(A+\theta)^{\frac{\beta+\lambda+1}{2}}\|_6^{\frac{6(\beta+\lambda)}{3\beta+3\lambda+2}}\right) \\
&\leq C(\eta, \lambda, \varepsilon) \left(1 + \int_0^T \|(A+\theta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,2}(\Omega)}^{\frac{6(\beta+\lambda)}{3\beta+3\lambda+2}}\right).
\end{aligned}$$

Because $\frac{6(\beta+\lambda)}{3\beta+3\lambda+2} < 2$ we can use Young's inequality to obtain

$$\int_0^T \|(A+\theta)^{\frac{\beta+\lambda+1}{2}}\|^2_{W^{1,2}(\Omega)} \leq C(\eta,\lambda,\varepsilon).$$

Embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ then implies that

$$\int_0^T \|(A+\theta^N)\|^{\beta+\lambda+1}_{3(\beta+\lambda+1)} = \int_0^T \|(A+\theta^N)^{\frac{\beta+\lambda+1}{2}}\|^2_6 \leq C(\eta,\lambda,\varepsilon). \tag{4.54}$$

Using interpolation inequality

$$\|\cdot\|_p \leq \|\cdot\|_1^{\frac{3(\beta+\lambda+1)-p}{p(3\beta+3\lambda+2)}} \|\cdot\|_{3(\beta+\lambda+1)}^{\frac{3(p-1)(\beta+\lambda+1)}{p(3\beta+3\lambda+2)}},$$

we obtain

$$\int_0^T \|\theta^N\|_p^p \leq \int_0^T \|\theta^N\|_1^{\frac{3(\beta+\lambda+1)-p}{3\beta+3\lambda+2}} \|\theta\|_{3(\beta+\lambda+1)}^{\frac{3(p-1)(\beta+\lambda+1)}{3\beta+3\lambda+2}} \overset{(4.31),(4.54)}{\leq} C(\eta,\lambda,\varepsilon) \tag{4.55}$$

provided $\frac{3(p-1)(\beta+\lambda+1)}{3\beta+3\lambda+2} \leq \beta+\lambda+1$. Hence setting $p := \beta+\lambda+\frac{5}{3}$ leads to the conclusion

$$\int_0^T \|\theta^N\|_s^s \, dt \leq C(\eta,\lambda,\varepsilon) \quad \text{ for all } s \in \left\langle 1, \beta+\frac{5}{3}\right). \tag{4.56}$$

Note that due to the assumption on β we have $\beta+\frac{5}{3} > 1$. Finally, to get an estimate on the gradient of the temperature we can compute for all $s \in \langle 1,2)$ $(Q := (\Omega \times (0,T))$

$$\begin{aligned}\int_Q |\nabla\theta^N|^s \, dx\, dt &= \int_Q |\nabla\theta^N|^s (\theta^N)^{(\lambda-1)\frac{s}{2}} (\theta^N)^{(1-\lambda)\frac{s}{2}} \, dx\, dt \\ &\leq \left(\int_Q |\nabla\theta^N|^2 (\theta^N)^{(\lambda-1)} \, dx\, dt\right)^{\frac{s}{2}} \left(\int_Q (\theta^N)^{(1-\lambda)\frac{s}{2-s}} \, dx\, dt\right)^{\frac{2-s}{s}}.\end{aligned} \tag{4.57}$$

Combining (4.57), (4.53) and (4.52) we conclude

$$\int_0^T \|\theta^N\|^s_{W^{1,s}(\Omega)} \, dt \leq C(\varepsilon) \quad \text{for all } s \in \left\langle 1, \frac{5}{4}\right). \tag{4.58}$$

For $\beta > 1$ we directly obtain from the equation (4.51) that

$$\int_0^T \|\theta^N\|^2_{W^{1,2}(\Omega)} \, dt \leq C(\eta). \tag{4.59}$$

If $\beta \in (0,1\rangle$ then

$$\begin{aligned}\int_0^T \|\nabla\theta^N\|_s^s \, dt &= \int_0^T \int_\Omega (A+\theta^N)^{(\beta+\lambda-1)\frac{s}{2}} |\nabla\theta^N|^s (A+\theta^N)^{-(\beta+\lambda-1)\frac{s}{2}} \, dx\, dt \\ &\leq C \int_0^T \int_\Omega (A+\theta^N)^{\beta+\lambda-1} |\nabla\theta^N|^2 + (A+\theta^N)^{-(\beta+\lambda-1)\frac{s}{2-s}} \, dx\, dt.\end{aligned}$$

We see that the right hand side is bounded if $-(\beta+\lambda-1)\frac{s}{2-s} < \beta + \frac{5}{3}$. An easy computation then leads to

$$\int_0^T \|\nabla\theta^N\|_s^s \le C(\eta, s) \quad \text{for all } s \in \langle 1, \frac{3\beta+5}{4}). \tag{4.60}$$

Thus, similarly as in the preceding paragraph, we can estimate

$$\|\theta^N_{,t}\|_{L^1(0,T;W^{-1,q'}(\Omega))} \le C, \quad q \text{ being sufficiently large.} \tag{4.61}$$

Moreover, estimate (4.30) gives us the following information

$$\|\boldsymbol{v}^N_{,t}\|_{(X^{r,2})^*} \le C. \tag{4.62}$$

Limit $N \to \infty$**:** Using generalized version of Aubin-Lions lemma (Theorem B.3) and (4.53)- (4.61), we have (after taking subsequence that is not relabeled)

$$\theta^N \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \tag{4.63}$$

$$\text{for all } s \in \begin{cases} \left\langle 1, \dfrac{5}{4}\right) & \text{if } \beta < 0, \\ \left\langle 1, \dfrac{3\beta+5}{4}\right) & \text{if } 0 \le \beta < 1, \\ \langle 1, 2\rangle\,, & \text{if } \beta > 1, \end{cases}$$

$$\theta^N \to \theta \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \tag{4.64}$$

$$\text{for all } m \in \begin{cases} \left\langle 1, \dfrac{5}{3}\right) & \text{if } \beta \le 0, \\ \left\langle 1, \dfrac{5}{3}+\beta\right), & \text{if } \beta > 0, \end{cases}$$

$$(\theta^N)^{\frac{\lambda+1}{2}} \rightharpoonup \theta^{\frac{\lambda+1}{2}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.65}$$

$$(1+\theta^N)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup (1+\theta)^{\frac{\beta+\lambda+1}{2}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.66}$$

using (4.28), (4.62) then implies

$$\boldsymbol{v}^N_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } (X^{r,2})^*, \tag{4.67}$$

$$\boldsymbol{v}^N \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{4.68}$$

$$\boldsymbol{v}^N \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{4.69}$$

$$\boldsymbol{v}^N \to \boldsymbol{v} \qquad \text{strongly in } L^n(0,T;L^n(\Omega)^3), \tag{4.70}$$

for all $n \in \langle 1, \frac{5r}{3}\big)$,

$$
\begin{aligned}
p^N &\rightharpoonup p && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). && (4.71)\\
\nu^N \mathbf{D}(\boldsymbol{v}^N) &\to \overline{\nu \mathbf{D}} && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}) && (4.72)
\end{aligned}
$$

Finally, Corollary B.1 together with (4.28) and (4.62) imply that

$$\operatorname{tr} \boldsymbol{v}^N \to \operatorname{tr} \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)). \tag{4.73}$$

These convergence are enough to pass to the limit in the time derivative, convective terms, in boundary integrals and also in the term involving external body forces. For passing to the limit we also need to know that

$$\overline{\nu \mathbf{D}} = \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \quad \text{ a.e. in } \Omega \times (0,T). \tag{4.74}$$

First, we can use the same procedure as in the previous section to obtain

$$p^N \to p \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.75}$$

Using Lemma A.2, we obtain for all $\boldsymbol{\varphi} \in X^{r,2}$ that

$$0 \le \int_0^T \big(\nu(p^N,\theta^N,|\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N) - \nu(p^N,\theta^N,|\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}^N - \boldsymbol{\varphi})\big)\ dt.$$

Using strong convergence (4.64) and (4.75) and Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
\lim_{N\to\infty} \int_0^T &\big(\nu(p^N,\theta^N,|\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}^N - \boldsymbol{\varphi})\big)\ dt\\
&= \int_0^T \big(\nu(p,\theta,|\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v} - \boldsymbol{\varphi})\big)\ dt.
\end{aligned}
$$

We also use (4.46) to replace the term

$$\int_0^T \big(\nu(p^N,\theta^N,|\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N), \mathbf{D}(\boldsymbol{v}^N)\big)\ dt.$$

Using a standard parabolic trick, we are finally led to the following inequality

$$0 \le \int_0^T \Big(\overline{\nu \mathbf{D}} - \nu(p,\theta,|\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v} - \boldsymbol{\varphi})\Big)\ dt.$$

Choosing $\boldsymbol{\varphi} := \boldsymbol{v} \pm \lambda \boldsymbol{u}$ then completes the proof of (4.74). Hence, we can pass to the limit in (4.46) to get for all $\boldsymbol{w} \in X^{r,2}$, $\varphi \in L^2(0,T;W^{1,2}(\Omega))$ and a.a. $t \in (0,T)$ that

$$
\begin{aligned}
&\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w}) + (\nu \mathbf{D}(\boldsymbol{v}), \nabla \boldsymbol{w})\\
&+\alpha \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{w}\ dS - (p, \operatorname{div} \boldsymbol{w}) = \langle \boldsymbol{f}, \boldsymbol{w}\rangle,
\end{aligned} \tag{4.76}
$$

and

$$-\varepsilon(\nabla p, \nabla\varphi) = (\operatorname{div}\boldsymbol{v}, \varphi). \tag{4.77}$$

Moreover, using Lemma A.2 leads to

$$C_1 \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^N, \boldsymbol{v})\, dx\, dt \le \int_0^T \Big(\nu_N^N \mathbf{D}(\boldsymbol{v}^N) - \nu(p^N, \theta^N, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\ \mathbf{D}(\boldsymbol{v}^N - \boldsymbol{v})\Big)\, dt.$$

Following again the same procedure as above, we conclude

$$\lim_{N\to\infty} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^N, \boldsymbol{v})\, dx\, dt = 0. \tag{4.78}$$

Finally, we prove the strong convergence of the symmetric part of the velocity gradient in the space $L^r(0,T;L^r(\Omega)^{3\times 3})$. To do it we use (4.78) as follows.

$$\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{D}(\boldsymbol{v}^N - \boldsymbol{v})|^r\, dx\, dt = \\
&\quad = \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{\frac{r}{2}}(\boldsymbol{v}^N, \boldsymbol{v}) \left(\int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}^N) - s\mathbf{D}(\boldsymbol{v}^N - \boldsymbol{v})|^2)^{\frac{r-2}{2}}\, ds\right)^{-\frac{r}{2}} dx\, dt \\
&\quad \le C \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{\frac{r}{2}}(\boldsymbol{v}^N, \boldsymbol{v}) \left(1 + |\mathbf{D}(\boldsymbol{v}^N)| + |\mathbf{D}(\boldsymbol{v})|\right)^{r\frac{2-r}{2}}\Big)^{-\frac{r}{2}} dx\, dt \\
&\quad \le C \left(\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^N, \boldsymbol{v})\, dx\, dt\right)^{\frac{r}{2}} \left(\int_0^T (1 + \|\mathbf{D}(\boldsymbol{v}^N)\|_r^r + \|\mathbf{D}(\boldsymbol{v})\|_r^r)\, dx\, dt\right)^{\frac{2-r}{2}}
\end{aligned}$$

Hence, we have that (after using Korn's inequality (B.2))

$$\boldsymbol{v}^N \to \boldsymbol{v} \quad \text{strongly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}). \tag{4.79}$$

It is a simple consequence of (4.79) and Lemma A.1 that also

$$\nu_N^N |\mathbf{D}(\boldsymbol{v}^N)|^2 \to \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)|\mathbf{D}(\boldsymbol{v})|^2 \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

Thus, we are able to pass to the limit on the right hand side of (4.48). To get the limit of $k_\varepsilon^N \nabla\theta^N$ it is enough use the fact that k_ε^N converges point-wise and (4.63). To pass to the limit in the remaining term of (4.48) for $\beta \le 0$ we can use the same procedure. For $\beta > 0$ we can compute for all $\varphi \in L^\infty(0,T;C^1(\Omega))$

and all $-1 < \lambda < 0$.

$$
\begin{aligned}
&\int_0^T ((1+\theta^N)^\beta \nabla\theta^N, \nabla\varphi)\, dt \\
&\qquad = \frac{2}{\beta+\lambda+1}\int_0^T \left(\nabla(1+\theta^N)^{\frac{\beta+\lambda+1}{2}}(1+\theta^N)^{\frac{\beta-\lambda+1}{2}}, \nabla\varphi\right)\, dt \\
&\qquad \overset{(4.64),(4.66)}{\longrightarrow} \frac{2}{\beta+\lambda+1}\int_0^T \left(\nabla(1+\theta)^{\frac{\beta+\lambda+1}{2}}(1+\theta)^{\frac{\beta-\lambda+1}{2}}, \nabla\varphi\right)\, dt \\
&\qquad = \int_0^T \left((1+\theta)^\beta \nabla\theta, \nabla\varphi\right)\, dt.
\end{aligned}
\tag{4.80}
$$

Thus, (4.63)-(4.65) and (4.78)-(4.80) then give that

$$
\begin{aligned}
&\int_0^T -(\theta, \varphi_{,t}) - (\boldsymbol{v}_\eta\theta, \nabla\varphi) + ((k_\varepsilon + \eta(1+\theta)^\beta)\nabla\theta, \nabla\varphi)\, dt \\
&\qquad = \int_0^T \left(\nu|\mathbf{D}(\boldsymbol{v})|^2, \varphi\right)\, dt + (\theta_0, \varphi(0))
\end{aligned}
\tag{4.81}
$$

is valid for all $\varphi \in \mathcal{D}(-\infty, T; W^{1,\sigma}(\Omega))$ with $\sigma > 3$.

The attainment of initial conditions $\boldsymbol{v}_0$ can be established by means of the same methods as those introduced in preceding subsection.

4.2.3 Limit $\varepsilon \to 0$.

Here, we will establish the existence of weak solution to the following problem $(\mathcal{P})^\eta$ (we write $(\boldsymbol{v}, \theta, p)$ instead $(\boldsymbol{v}^\eta, \theta^\eta, p^\eta)$)

$$
(\mathcal{P})^\eta \qquad
\begin{aligned}
\left.\begin{aligned}
\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div}(\nu\mathbf{D}(\boldsymbol{v})) + \nabla p = \boldsymbol{f} \\
\operatorname{div}\boldsymbol{v} = 0 \\
\theta_{,t} + \operatorname{div}(\boldsymbol{v}_\eta\theta) - \operatorname{div}\left((\eta(1+|\theta|)^\beta + k)\nabla\theta\right) - \nu|\mathbf{D}(\boldsymbol{v})|^2 = 0
\end{aligned}\right\} &\quad \text{in } Q, \\
\left.\begin{aligned}
\left(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right)_\tau + \alpha\boldsymbol{v}_\tau = 0 \\
\boldsymbol{v}\cdot\boldsymbol{n} = 0 \\
\nabla\theta\cdot\boldsymbol{n} = 0
\end{aligned}\right\} &\quad \text{on } \partial Q, \\
\int_\Omega p(x,t)\, dx = 0, & \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0. & \\
\theta(x,0) = \theta_0. &
\end{aligned}
$$

where for simplicity we defined

$$
\begin{aligned}
\nu &:= \nu(p, \theta, |\mathbf{D}(\boldsymbol{v})|^2), \\
k &:= k(p, \theta, |\mathbf{D}(\boldsymbol{v})|^2).
\end{aligned}
$$

We use the problem $(\mathcal{P})^{\varepsilon,\eta}$ and pass to the limit in ε to get the solution of the problem $(\mathcal{P})^{\eta}$. For this purpose, we denote by $(\theta^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon)$ the weak solution of $(\mathcal{P})^{\varepsilon,\eta}$.

After using weak lower semicontinuity in (4.30), we find that

$$\sup_t \|\boldsymbol{v}^\varepsilon(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^\varepsilon\|_{1,r}^r \leq C, \tag{4.82}$$

and consequently

$$\|\boldsymbol{v}^\varepsilon\|_{(X^{r,\sigma'}_{\mathrm{div}})^*} \leq C \tag{4.83}$$

for $\sigma = \frac{5r}{8}$ independently of η, ε. We also need to estimate the norm of the pressure p^ε. To do it, one can follow the procedure presented in Section 3.3.3 to obtain that

$$\int_0^T \|p^\varepsilon\|_{r'}^{r'} \leq C(\eta). \tag{4.84}$$

Moreover, using Fatou's lemma and (4.50), we get

$$\sup_t \|\theta^\varepsilon\|_1 \leq C. \tag{4.85}$$

Next, we derive apriori estimates on the gradient of the temperatures θ^ε. We use (4.51) and weak lower semicontinuity to get

$$\eta \int_0^T \|\nabla(1+\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}}\|_2^2 \, dt \leq C. \tag{4.86}$$

But this estimate still depends on η. For deriving estimates independent of η we study two cases:

$\beta \leq 0$: After using the fact that $k^N := k(p^N, \theta^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \geq C_3(\theta^N)^\beta$ and minimum principle (4.49), we know that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$k_\varepsilon^N \geq k^N \geq C(\theta^N)^\beta.$$

Therefore,

$$\int_0^T \|\nabla(\theta^N)^{\frac{\beta+\lambda+1}{2}}\|_2^2 \, dt \leq C \int_0^T k_\varepsilon^N |\nabla\theta^N|^2 \, dt \overset{(4.51)}{\leq} C.$$

Lower semicontinuity then implies that

$$\int_0^T \|\nabla(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}}\|_2^2 \, dt \leq C,$$

and consequently, applying the same tools as in the preceding subsection, we obtain

$$\int_0^T \|(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,2}(\Omega)}^2 \, dt \leq C. \tag{4.87}$$

$\beta > 0$**:** Here, we choose arbitrary fixed $s \in (1,2)$. Hölder's inequality and (4.51) imply that

$$\int_0^T \int_\Omega (k_\varepsilon^N)^{\frac{s}{2}} |\nabla\theta^N|^s (\theta^N)^{(\lambda-1)\frac{s}{2}} \, dx \, dt \le C. \tag{4.88}$$

Moreover, the definition of k_ε^N and the assumption on k imply that (we use the notation $Q_\varepsilon^N := \{(x,t); \theta^N \ge \frac{1}{\varepsilon}\}$ and $k^N := k(p^N, \theta^N, |\mathbf{D}(\boldsymbol{v}^N)|^2)$)

$$\begin{aligned}
&\int_0^T \int_\Omega (k^N)^{\frac{s}{2}} (\theta^N)^{(\lambda-1)\frac{s}{2}} |\nabla\theta^N|^{\frac{s}{2}} \, dx \, dt \\
&\quad \le C\Big(\int_0^T \int_\Omega (k_\varepsilon^N)^{\frac{s}{2}} (\theta^N)^{(\lambda-1)\frac{s}{2}} |\nabla\theta^N|^{\frac{s}{2}} \, dx \, dt + \int_{Q_\varepsilon^N} (k^N)^{\frac{s}{2}} (\theta^N)^{(\lambda-1)\frac{s}{2}} |\nabla\theta^N|^{\frac{s}{2}} \, dx \, dt \Big) \\
&\quad \overset{(4.88),(4.3)}{\le} C\big(1 + \int_{Q_\varepsilon^N} (\theta^N)^{\frac{s}{2}(\beta+\lambda-1)} |\nabla\theta^N|^{\frac{s}{2}} \, dx \, dt\big) \\
&\quad \overset{(4.51),\text{Hölder}}{\le} C + C(\eta) |Q_\varepsilon^N|^{\frac{2-s}{2}}.
\end{aligned}$$

Finally, using the assumption on k and weak lower semicontinuity, we conclude that

$$\int_0^T \|(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,s}(\Omega)}^s \, dt \le C + C(\eta) |Q_\varepsilon|^{\frac{2-s}{2}}, \tag{4.89}$$

where $Q_\varepsilon := \{(x,t); \theta^\varepsilon(x,t) \ge \frac{1}{\varepsilon}\}$.

Inequalities (4.82), (4.86) also imply that

$$\int_0^T \|\theta_{,t}^\varepsilon\|_{W^{-1,q'}(\Omega)} \, dt \le C \quad \text{ for } q > 3. \tag{4.90}$$

Using Aubin-Lions lemma, (4.82), (4.83), (4.84), (4.86), we can find a subsequence such that (note that $\frac{3\beta+5}{4} > 1$ as $r < 2$ and $\beta > \frac{3-r}{3(r-1)} - \frac{2}{3}$)

$$\theta^\varepsilon \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \tag{4.91}$$

$$\text{for all } s \in \begin{cases} \left\langle 1, \dfrac{3\beta+5}{4} \right) & \text{if } \beta \le 1, \\ \langle 1, 2 \rangle, & \text{if } \beta > 1, \end{cases}$$

$$\theta^\varepsilon \to \theta \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \tag{4.92}$$

for all $m \in \big\langle 1, \frac{5}{3} + \beta \big)$,

$$\begin{aligned}
(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}} &\rightharpoonup (\theta)^{\frac{\beta+\lambda+1}{2}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), && (4.93)\\
\boldsymbol{v}_{,t}^\varepsilon &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'}), && (4.94)\\
\boldsymbol{v}^\varepsilon &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), && (4.95)\\
\boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W_{\boldsymbol{n}}^{1,r}), && (4.96)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^n(0,T;L^n(\Omega)^3) && (4.97)
\end{aligned}$$

for all $n \in \langle 1, \frac{5r}{3}\big)$,

$$p^\varepsilon \rightharpoonup p \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)). \tag{4.98}$$

$$\nu^\varepsilon \mathbf{D}(\boldsymbol{v}^\varepsilon) \to \overline{\nu \mathbf{D}} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}) \tag{4.99}$$

Finally, Corollary B.1 together with (4.28) and (4.62) imply that

$$\operatorname{tr} \boldsymbol{v}^\varepsilon \to \operatorname{tr} \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)). \tag{4.100}$$

Next, it is easy to pass to the limit in (4.77) to observe that

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{ in } \Omega \times (0,T).$$

As in the preceding section we need at least point-wise convergence of the pressures p^ε and velocity gradients $\nabla \boldsymbol{v}^\varepsilon$. To show this, we use Lemma A.2 and we obtain

$$\begin{aligned}\frac{C_1}{2}\int_0^T\int_\Omega \gamma_1 \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dx\, dt \le \frac{B_2\gamma_0^2}{2C_1}\int_0^T \|p^\varepsilon - p\|_2^2\, dt \\ + \int_0^T \big(\nu(p^\varepsilon,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) - \nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)\, dt.\end{aligned} \tag{4.101}$$

First, (4.99), (4.92) and Lebesgue theorem imply that

$$\int_0^T\int_\Omega \big(\nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)\, dt \overset{\varepsilon\to 0}{\to} 0.$$

For the remaining term in (4.101), i.e., for

$$\int_0^T \big(\nu(p^\varepsilon,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)\ dt,$$

we use weak formulation of $(\mathcal{P})^{\varepsilon,\eta}$ to observe that

$$\limsup_{\varepsilon\to 0}\int_0^T \big(\nu(p^\varepsilon,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)\ dt \le 0.$$

Finally, inserting these estimates into (4.101), we have that

$$\frac{C_1}{2}\int_0^T\int_\Omega \gamma_1(\theta^\varepsilon)\mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})\, dt \le f(\varepsilon) + \frac{\gamma_0 B_2(C_3)}{2C_1}\int_0^T \|p^\varepsilon - p\|_2^2\, dt, \tag{4.102}$$

where $f(\varepsilon) \overset{\varepsilon\to 0}{\to} 0$. To estimate the right hand side of (4.102), we need information about the behavior of the pressures p^ε. To get such information, we can compute

$$\begin{aligned}\int_0^T \|p^\varepsilon - p\|_2^2 \, dt &= \int_0^T (p^\varepsilon, p^\varepsilon - p) \, dt - \underbrace{\int_0^T (p, p^\varepsilon - p) \, dt}_{\to 0} \\ &\leq f(\varepsilon) + \int_0^T (p^\varepsilon, p^\varepsilon - p) \, dt.\end{aligned} \tag{4.103}$$

Next, we find g^ε solving

$$\begin{aligned} \triangle g^\varepsilon &= p^\varepsilon - p && \text{in } \Omega, \\ \frac{\partial g^\varepsilon}{\partial \boldsymbol{n}} &= 0 && \text{on } \partial\Omega, \\ \int_\Omega g^\varepsilon \, dx &= 0. \end{aligned}$$

It is a consequence of (4.98) that

$$g^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^{r'}(0,T; L^{r'}(\Omega)). \tag{4.104}$$

Next we test the problem $(\mathcal{P})^{\varepsilon,\eta}$ by ∇g^ε, i.e., we set in (4.76) $\boldsymbol{w} := \nabla g^\varepsilon$. The same procedure as in Section 3.3.3 then gives that

$$\begin{aligned}\int_0^T (p^\varepsilon, p^\varepsilon - p) \, dt &\leq f(\varepsilon) + \int_0^T (\nu(p^\varepsilon, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon), \nabla^2 g^\varepsilon) \, dt \\ &= f(\varepsilon) + \int_0^T (\nu(p^\varepsilon, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) - \nu(p, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon) \, dt \\ &\quad + \int_0^T (\nu(p, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon) \, dt =: f(\varepsilon) + I_1 + I_2.\end{aligned}$$

To estimate I_2, we can use Lebesgue theorem and (4.64), (4.104) and obtain

$$I_2 \overset{\varepsilon \to 0}{\to} 0.$$

To the term I_1, we apply Lemma A.2 to get

$$\begin{aligned} I_1 &\leq \int_0^T \int_\Omega C_2 \gamma_1(\theta^\varepsilon) |\mathcal{I}_{\mathbf{D}}^{\frac{1}{2}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v})| |\nabla^2 g^\varepsilon| + \gamma_0 \gamma_2(\theta^\varepsilon) |p^\varepsilon - p| |\nabla^2 g^\varepsilon| \, dx \, dt \\ &\leq C_2 \sqrt{B_1} C_{reg}(\Omega, 2) \int_0^T \left(\int_\Omega \gamma_1(\theta^\varepsilon) \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \right)^{\frac{1}{2}} \|p^\varepsilon - p\|_2 \, dt \\ &\quad + \gamma_0 B_3 C_{reg}(\Omega, 2) \int_0^T \|p^\varepsilon - p\|_2^2 \, dt \\ &\leq \int_0^T \frac{1 + \gamma_0 B_3 C_{reg}(\Omega, 2)}{2} \|p^\varepsilon - p\|_2^2 \, dt \\ &\quad + \frac{C_2^2 B_1 C_{reg}^2(\Omega, 2)}{2(1 - \gamma_0 B_3 C_{reg}(\Omega, 2))} \int_0^T \int_\Omega \gamma_1(\theta^\varepsilon) \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt. \end{aligned}$$

Inserting estimates for I_1, I_2 into (4.103), we find that

$$\begin{aligned}\int_0^T \|p^\varepsilon - p\|_2^2 \, dt \le f(\varepsilon) \\ + \frac{C_2^2 B_1 C_{reg}^2(\Omega,2)}{(1-\gamma_0 B_3 C_{reg}(\Omega,2))^2} \int_0^T \int_\Omega \gamma_1(\theta^\varepsilon)\mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt.\end{aligned} \tag{4.105}$$

Finally, (4.102) and (4.105) imply that

$$\begin{aligned}\frac{C_1}{2}\int_0^T \gamma_1(\theta^\varepsilon)\mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt \le f(\varepsilon) \\ + \frac{\gamma_0^2 B_2}{2C_1}\frac{C_2^2 B_1 C_{reg}^2(\Omega,2)}{(1-\gamma_0 B_3 C_{reg}(\Omega,2))^2} \int_0^T \int_\Omega \gamma_1(\theta^\varepsilon)\mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt.\end{aligned}$$

Using assumption on γ_0, we have the inequality

$$\frac{C_1}{2} > \frac{\gamma_0^2 B_2}{2C_1}\frac{C_2^2 B_1 C_{reg}^2(\Omega,2)}{(1-\gamma_0 B_3 C_{reg}(\Omega,2))^2}$$

and therefore (because $\gamma_1 \ge 1$),

$$\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\boldsymbol{v}^\varepsilon, \boldsymbol{v}) \, dx \, dt \overset{\varepsilon \to 0}{\to} 0,$$

$$\int_0^T \|p^\varepsilon - p\|_2^2 \, dt \overset{\varepsilon \to 0}{\to} 0.$$

The same procedure as (4.78)-(4.79) then implies that

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{4.106}$$

$$p^\varepsilon \to p \qquad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{4.107}$$

Hence, using Vitali's theorem then complete the limit process in (4.76).

To pass to the limit also in (4.81) with the term on the right hand side it is enough to take into account (4.106), (4.107), (4.92) and the assumption (4.2). We can also use the same procedure as in previous section to get the limit of time derivative, convective term and $\eta(1+\theta^\varepsilon)^\beta \nabla\theta^\varepsilon$ in (4.81). For the remaining term we can compute

$$\int_0^T (k_\varepsilon \nabla\theta^\varepsilon, \nabla\varphi) \, dt = \int_0^T (\underbrace{\nabla((\theta^\varepsilon)^{\frac{\beta+1+\lambda}{2}})}_{\rightharpoonup \text{ weakly in } L^2(0,T;L^2(\Omega))}, (\theta^\varepsilon)^{\frac{1-\beta-\lambda}{2}} k_\varepsilon \nabla\varphi) \, dt$$

If we show that $G^\varepsilon := (\theta^\varepsilon)^{\frac{1-\beta-\lambda}{2}} k_\varepsilon^\varepsilon \nabla\varphi$ converges strongly in $L^2(0,T;L^2(\Omega))$ to $G := \theta^{\frac{1-\beta-\lambda}{2}} k\nabla\varphi$ then the proof will be completed. First, it is consequence of (4.106), (4.107) and (4.92) that G^ε converges point-wise. Next, we have that

$$\int_Q |k_\varepsilon^\varepsilon| (\theta^\varepsilon)^{\frac{-\beta+1-\lambda}{2}} |\nabla\varphi| \, dxdt \le C \int_Q (\theta^\varepsilon)^{\frac{1+\beta-\lambda}{2}} \, dx \, dt \overset{(4.92)}{\le} C|Q|^\alpha,$$

with some $\alpha > 0$. Vitali's theorem then completes the proof of the existence of the problem $(\mathcal{P})^\eta$.

4.2.4 Limit $\eta \to 0$

Here, we will pass to the limit with η in the problem $(\mathcal{P})^\eta$ to get weak solution to the problem $(\mathcal{P})$. We denote by $(\boldsymbol{v}^\eta, \theta^\eta, p^\eta)$ the solution of $(\mathcal{P})^\eta$. Using weak lower semicontinuity and Fatou's lemma and (4.82),(4.50), we find that

$$\sup_t(\|\boldsymbol{v}(t)\|_2^2 + \|\theta^\eta(t)\|_1) + \int_0^T \|\boldsymbol{v}^\eta\|_{1,r}^r \leq C. \tag{4.108}$$

Using the same procedure as in 3.3.4 we also observe that

$$\int_0^T \|p^\eta\|_{\frac{5r}{6}}^{\frac{5r}{6}} \, dt \leq C(\lambda). \tag{4.109}$$

Next, if $\beta \leq 0$ then (4.87) implies that for all $-1 < \lambda < 0$

$$\int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,2}(\Omega)}^2 \leq C(\lambda). \tag{4.110}$$

For $\beta > 0$ we get from (4.86) that

$$\eta \int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,2}(\Omega)}^2 \, dt \leq C \tag{4.111}$$

and from (4.89) we derive

$$\int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,s}(\Omega)}^s \, dt \leq \limsup_{\varepsilon \to 0}(C + C(\eta)|Q_\varepsilon|^{\frac{2-s}{2}}) = C(\lambda). \tag{4.112}$$

Finally, Lesbegue theorem and (4.111) imply that

$$\int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,2}(\Omega)}^2 = \lim_{s \to 2} \int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{W^{1,s}(\Omega)}^s \, dt \overset{(4.112)}{\leq} C(\lambda). \tag{4.113}$$

We also easily obtain that time derivatives can be estimated as

$$\|\boldsymbol{v}^\eta_{,t}\|_{(X^{r,\sigma'}_{\mathrm{div}})^*} + \|\boldsymbol{v}^\eta_{,t}\|_{L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{\boldsymbol{n}})} + \|\theta^\eta_{,t}\|_{L^1(0,T;W^{-1,q'}(\Omega))} \leq C \tag{4.114}$$

for $\sigma := \frac{5r}{8}$ and q being sufficiently large. Hence, we can find a subsequence such that

$$\theta^\eta \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \tag{4.115}$$

$$\text{for all } s \in \begin{cases} \left\langle 1, \dfrac{3\beta+5}{4} \right) & \text{if } \beta \leq 1, \\ \langle 1, 2 \rangle, & \text{if } \beta > 1, \end{cases}$$

$$\theta^\eta \to \theta \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \tag{4.116}$$

for all $m \in \langle 1, \frac{5}{3} + \beta)$,

$$
\begin{aligned}
(\theta^\eta)^{\frac{\beta+\lambda+1}{2}} &\rightharpoonup (\theta)^{\frac{\beta+\lambda+1}{2}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), && (4.117)\\
\boldsymbol{v}^\eta_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}}), && (4.118)\\
\boldsymbol{v}^\eta_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } (X^{r,\sigma'})^*, && (4.119)\\
\boldsymbol{v}^\eta &\rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^3), && (4.120)\\
\boldsymbol{v}^\eta &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), && (4.121)\\
\boldsymbol{v}^\eta &\to \boldsymbol{v} && \text{strongly in } L^n(0,T;L^n(\Omega)^3) && (4.122)
\end{aligned}
$$

for all $n \in \langle 1, \frac{5r}{3})$,

$$
\begin{aligned}
p^\eta &\rightharpoonup p && \text{weakly in } L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)), && (4.123)\\
\nu^\eta \mathbf{D}(\boldsymbol{v}^\eta) &\to \overline{\nu \mathbf{D}} && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}). && (4.124)
\end{aligned}
$$

Finally, Corollary B.1 together with (4.28) and (4.62) imply that

$$
\operatorname{tr} \boldsymbol{v}^\eta \to \operatorname{tr} \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)). \qquad (4.125)
$$

Moreover, if we use step by step the procedure that was described in Section 3.3.4 ((3.51)-(3.54)) we find that $p^\eta := p_1^\eta + p_2^\eta$ such that

$$
\begin{aligned}
p_1^\eta &\to p_1 && \text{strongly in } L^s(0,T;L^s(\Omega)) \text{ for all } s < \frac{5r}{6},\\
p_2^\eta &\rightharpoonup p_2 && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)).
\end{aligned}
$$

The proof of the point-wise convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$ and p_2^η can be done by using the methods developed in Section 3.3.4 and in Section 4.2.3. Consequently, we are able to pass to the limit to get (4.16) and using Fatou's lemma, we can get (4.18). To get (4.17) we test (2.67) by φ and we set $\boldsymbol{\varphi} := \boldsymbol{v}^\eta \varphi$ in (2.82). Adding the resulting equations, integrating per partes and passing to the limit, we easily obtain (4.17). The inequality (4.18) can be proved by using Fatou's lemma. $\square$

A Some consequences of the assumptions on the viscosity

In this Appendix, we summarize important properties of the viscosity, that were frequently used in the preceding text. We assume that the viscosity ν is a $\mathcal{C}^1$-mapping of $\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}_0^+$ into $\mathbb{R}^+$ satisfying for some fixed (but arbitrary) $r\geq 1$ and all $\mathsf{D}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\mathsf{B}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$ and $p\in\mathbb{R}$, $\theta\in\mathbb{R}_0^+$ the following inequalities

$$\begin{aligned} C_1\gamma_1(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 &\leq \frac{\partial\nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}_{ij}}{\partial\mathsf{D}_{kl}}\mathsf{B}_{ij}\mathsf{B}_{kl},\\ C_2\gamma_1(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 &\geq \frac{\partial\nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}_{ij}}{\partial\mathsf{D}_{kl}}\mathsf{B}_{ij}\mathsf{B}_{kl}, \end{aligned} \tag{A.1}$$

$$\left|\frac{\partial\nu(p,\theta,|\mathsf{D}|^2)}{\partial p}\right||\mathsf{D}| \leq \gamma_0\gamma_2(\theta)(1+|\mathsf{D}|^2)^{\frac{r-2}{4}}, \tag{A.2}$$

with continuous functions $\gamma_1,\gamma_2\geq 1$.

The first standard property of the viscosity that follows from (A.1) is:

Lemma A.1. *Let ν satisfy* (A.1). *Then there exist positive constants $\hat{C},\overline{C}$ such that*

$$\hat{C}\gamma_1(\theta)(|\mathsf{D}|^r-1) \leq \nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}\cdot\mathsf{D}, \tag{A.3}$$

$$\left|\nu(p,\theta,|\mathsf{D}|^2)\mathsf{D}\right| \leq \overline{C}\gamma_1(\theta)(|\mathsf{D}|^{r-1}+1). \tag{A.4}$$

Proof. See Lemma 1.19, page 198 in [22]. □

The next lemma gives important information on the monotonicity of the term $\nu(p,|\mathsf{D}|^2)\mathsf{D}$. For simplicity, we set

$$\mathcal{I}_{\mathsf{D}}(\boldsymbol{u},\boldsymbol{v}) := |\mathsf{D}(\boldsymbol{u}-\boldsymbol{v})|^2\int_0^1(1+|\mathsf{D}(\boldsymbol{v})+s(\mathsf{D}(\boldsymbol{u}-\boldsymbol{v}))|^2)^{\frac{r-2}{2}}\,ds, \tag{A.5}$$

$$\mathcal{I}_p(p,q) := |p-q|^2, \tag{A.6}$$

Lemma A.2. *Let ν satisfy* (A.1)-(A.2) *with $r\in(1,2)$. Then*

$$\begin{aligned} \frac{C_1}{2}\gamma_1(\theta)\mathcal{I}_{\mathsf{D}}(\boldsymbol{u},\boldsymbol{v}) &\leq \frac{\gamma_0^2\gamma_2^2(\theta)}{2C_1\gamma_1(\theta)}\mathcal{I}_p(p,q)\\ &\quad+\big(\nu(p,\theta,|\mathsf{D}(\boldsymbol{u})|^2)\mathsf{D}(\boldsymbol{u})-\nu(q,\theta,|\mathsf{D}(\boldsymbol{v})|^2)\mathsf{D}(\boldsymbol{v})\big)\cdot\mathsf{D}(\boldsymbol{u}-\boldsymbol{v}) \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} &|\nu(p,\theta,|\mathsf{D}(\boldsymbol{u})|^2)\mathsf{D}(\boldsymbol{u})-\nu(q,\theta,|\mathsf{D}(\boldsymbol{v})|^2)\mathsf{D}(\boldsymbol{v})|\\ &\leq \gamma_0\gamma_2(\theta)\mathcal{I}_p^{\frac12}(p,q)+C_2\gamma_1(\theta)|\mathsf{D}(\boldsymbol{u}-\boldsymbol{v})|\int_0^1(1+|\mathsf{D}(\boldsymbol{v})+s(\mathsf{D}(\boldsymbol{u}-\boldsymbol{v}))|^2)^{\frac{r-2}{2}}\,ds\\ &\leq \gamma_0\gamma_2(\theta)\mathcal{I}_p^{\frac12}(p,q)+C_2\gamma_1(\theta)\mathcal{I}_{\mathsf{D}}^{\frac12}(\boldsymbol{u},\boldsymbol{v}). \end{aligned} \tag{A.8}$$

Proof. We follow the idea presented in [21] where the same inequalities are shown for $\gamma_1, \gamma_2, \equiv 1$; Lemma A.2 is thus an easy generalization that we prove here for the sake of completeness.
We set $p_s := q - s(q-p)$, $\boldsymbol{w}_s = \boldsymbol{v} - s(\boldsymbol{v} - \boldsymbol{u})$. Then

$$\begin{aligned} M(\boldsymbol{u},\boldsymbol{v}) &:= (\nu(p,\theta,|\mathsf{D}(\boldsymbol{u})|^2)\mathsf{D}(\boldsymbol{u}) - \nu(q,\theta,|\mathsf{D}(\boldsymbol{v})|^2)\mathsf{D}(\boldsymbol{v}))\cdot\mathsf{D}(\boldsymbol{u}-\boldsymbol{v}) \\ &= \int_0^1 \frac{d}{ds}\nu(p_s,\theta,|\mathsf{D}(\boldsymbol{w}_s)|^2)\mathsf{D}(\boldsymbol{w}_s)\cdot\mathsf{D}(\boldsymbol{u}-\boldsymbol{v})\,ds \\ &= \int_0^1 \frac{\partial\nu(p_s,\theta,|\mathsf{D}(\boldsymbol{w}_s)|^2)\mathsf{D}_{kl}(\boldsymbol{w}_s)}{\partial\mathsf{D}_{ij}}\mathsf{D}_{ij}(\boldsymbol{u}-\boldsymbol{v})\mathsf{D}_{kl}(\boldsymbol{u}-\boldsymbol{v})\,ds \\ &\quad + \int_0^1 \frac{\partial\nu(p_s,\theta,|\mathsf{D}(\boldsymbol{w}_s)|^2)}{\partial p_s}(p-q)\mathsf{D}(\boldsymbol{w}_s)\cdot\mathsf{D}(\boldsymbol{u}-\boldsymbol{v})\,ds. \end{aligned}$$

Using the assumptions (A.1)-(A.2), the fact that $r \le 2$ and Hölder's inequality to the second term, one concludes

$$M(\boldsymbol{u},\boldsymbol{v}) \ge C_1\gamma_1(\theta)\mathcal{I}_{\mathsf{D}}(\boldsymbol{u},\boldsymbol{v}) - \gamma_0\gamma_2(\theta)\mathcal{I}_{\mathsf{D}}^{\frac{1}{2}}(\boldsymbol{u},\boldsymbol{v})\mathcal{I}_p^{\frac{1}{2}}(p,q).$$

Young's inequality then completes the proof of (A.7).

The inequality (A.8) can be verified by similar arguments. □

B Theorems on properties of Sobolev functions

Lemma B.1. *Let $\Omega \in \mathcal{C}^{1,1}$. Then there exist $C_{reg}(\Omega, r)$ such that for all $f \in L^r(\Omega)$, $\int_\Omega f\,dx = 0$, there exists unique φ solving*

$$\begin{aligned} \triangle\varphi &= f \quad \text{in } \Omega, \\ \nabla\varphi\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega \\ \int_\Omega \varphi\,dx &= 0, \end{aligned}$$

satisfying

$$\|\varphi\|_{2,r} \le C_{reg}(\Omega,r)\|f\|_r.$$

Moreover, there exists $C(\Omega, s)$ such that if $f = \operatorname{div}\boldsymbol{v}$ and $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}} \cap L^s(\Omega)^d$ then φ satisfies

$$\|\varphi\|_{1,s} \le C(\Omega,s)\|\boldsymbol{v}\|_s.$$

.

By using previous lemma, we define the so-called Helmholtz decomposition. Let $\boldsymbol{v} \in W^{1,r}_{\boldsymbol{n}} \cap L^s(\Omega)^d$, then we denote by $g^{\boldsymbol{v}}$ the solution of (B.1) with $f = \operatorname{div}\boldsymbol{v}$. Moreover, we define

$$\boldsymbol{v}_{\text{div}} := \boldsymbol{v} - \nabla g^{\boldsymbol{v}}. \tag{B.1}$$

It follows from Lemma B.1 that

$$\|\boldsymbol{v}_{\mathrm{div}}\|_{1,r} \leq (C_{reg}(\Omega, r) + 1\|\boldsymbol{v}\|_{1,r}$$
$$\|\boldsymbol{v}_{\mathrm{div}}\|_{s} \leq C(\Omega, s) + 1)\|\boldsymbol{v}\|_{s}$$

Here we give some important inequalities which will be frequently used in what follows.

Lemma B.2. (Korn's inequality) *Let $q \in (1, \infty)$. Then there exists positive constant C depending only on Ω and q such that for all $\boldsymbol{v} \in W^{1,q}(\Omega)^d$ which has the trace $\operatorname{tr} \boldsymbol{v} \in L^2(\partial\Omega)^d$ there holds*

$$C\|\boldsymbol{v}\|_{1,q} \leq \|\mathbf{D}(\boldsymbol{v})\|_q + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}. \tag{B.2}$$

Proof. First, we know that there exist $C'(\Omega, q)$ such that for all $\boldsymbol{v} \in W^{1,q}(\Omega)^d$

$$\|\boldsymbol{v}\|_{1,q} \leq C' \left(\|\mathbf{D}(\boldsymbol{v})\|_q + \|\boldsymbol{v}\|_q\right). \tag{B.3}$$

For proof see for example first part of Theorem 1.10 p. 196 in [22]. Thus, it is enough to show that for all $\boldsymbol{v} \in W^{1,q}(\Omega)^d$ with $\operatorname{tr} \boldsymbol{v} \in L^2(\partial\Omega)^d$

$$\|\boldsymbol{v}\|_q \leq C'' \left(\|\mathbf{D}(\boldsymbol{v})\|_q + \|\operatorname{tr} \boldsymbol{v}\|_2\right)$$

with some positive constant C''. To prove it, we assume contrary. We take sequence $\{\boldsymbol{v}^n\}_{i=1}^{\infty}$ such that $\|\boldsymbol{v}^n\|_q = 1$ and

$$1 > n \left(\|\mathbf{D}(\boldsymbol{v}^n)\|_q + \|\operatorname{tr} \boldsymbol{v}^n\|_2\right).$$

It implies

$$\|\mathbf{D}(\boldsymbol{v}^n)\|_q \to 0,$$
$$\|\operatorname{tr} \boldsymbol{v}^n\|_2 \to 0.$$

With help of (B.3) we have $\|\boldsymbol{v}^n\|_{1,q} \leq C' < \infty$. As the space $W^{1,q}(\Omega)^d$ is reflexive ($1 < q < \infty$), we can take a subsequence which is not relabelled such that

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \quad \text{weakly in } W^{1,q}(\Omega)^d.$$

From the compact embedding we also have

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{strongly in } L^q(\Omega)^d.$$

This convergence then leads to conclusion that $\|\boldsymbol{v}\|_q = 1$. On the other hand $\boldsymbol{v} \in W_0^{1,q}(\Omega)^d$ and $\mathbf{D}(\boldsymbol{v}) \equiv \mathbf{0}$. Then using of the Korn's inequality for functions vanishing on the boundary implies that $\boldsymbol{v} = \mathbf{0}$, which is a contradiction. □

The next lemma gives an important information about the behavior of some functions on the boundary.

Lemma B.3. *Let $d = 2, 3$ and $\{\boldsymbol{v}^i\}_{i=1}^{\infty}$ be bounded in $\mathcal{S}$ defined for some $1 < q_1, q_2 < \infty$ through*

$$\mathcal{S} := \{\boldsymbol{v}; \boldsymbol{v} \in L^{\infty}(0,T; L^2(\Omega)^d) \cap L^r(0,T; W^{1,r}_{\boldsymbol{n}}), \boldsymbol{v}_{,t} \in L^{q_1}(0,T; W^{-1,q_2}_{\boldsymbol{n},\mathrm{div}})\}.$$

Let $2 \geq r > \frac{2d}{d+2}$. Then $\{\operatorname{tr} \boldsymbol{v}^i\}_{i=1}^{\infty}$ is precompact in $L^p(0,T; L^s(\partial\Omega)^d)$ for all $p, s \in \langle 1, \infty)$ such that

$$p < s\frac{dr + 2r - 2d}{sd - 2d + 2}; \quad s \in (2\frac{d-1}{d}, \frac{r(d-1)}{d-r}). \tag{B.4}$$

Proof. According to [31][7] there is a continuous *trace* operator tr such that for $m \in \mathbb{R}_+$, $n \geq 1$ and $m > \frac{1}{n}$

$$\operatorname{tr} : W^{m,n}(\Omega)^d \to W^{m-\frac{1}{n},n}(\partial\Omega)^d, \tag{B.5}$$

Next, using the Aubin-Lions compactness lemma we observe that

$$\mathcal{S} \hookrightarrow\hookrightarrow L^r(0,T; W^{1-\varepsilon_1,r}(\Omega)^d).$$

Then (B.5) implies that we can take a subsequence (not relabelled) $\{\boldsymbol{v}^i\}_{i=1}^{\infty}$ such that

$$\operatorname{tr} \boldsymbol{v}^i \to \operatorname{tr} \boldsymbol{v} \quad \text{strongly in } L^1(0,T; L^1(\partial\Omega)^d). \tag{B.6}$$

Consider fixed and arbitrary s, p satisfying (B.4). We show that

$$\{\operatorname{tr} \boldsymbol{v}^i\}_{L^{p+\varepsilon_2}(0,T;L^{s+\varepsilon_2}(\partial\Omega)^d)} \leq K < \infty \tag{B.7}$$

holds for some positive constant K and sufficiently small $\varepsilon_2 > 0$. The combination of (B.6) and (B.7) completes the proof of Lemma B.3.

To prove (B.7) we take $\delta \in (0,1)$ small enough and for $r \in (\frac{2d}{d+2}, 2)$ we observe that

$$W^{1,r}(\Omega) \hookrightarrow W^{k,2}(\Omega) \qquad k = \frac{dr + 2r - 2d}{2r} > 0, \tag{B.8}$$

$$W^{\ell,2}(\Omega) \hookrightarrow W^{\frac{1}{s}+\delta,s}(\Omega) \qquad \ell = \frac{sd - 2d + 2}{2s} + \delta \leq k, \tag{B.9}$$

$$\|\cdot\|_{\ell,2} \leq \|\cdot\|_{k,2}^{\frac{\ell}{k}} \|\cdot\|_2^{1-\frac{\ell}{k}}. \tag{B.10}$$

[7] In fact in [31] there is not proved exactly the relation (B.5) but we can get it as a simple consequence of several theorems that are also proved there. First in Subsection 2.2.2 (Remark 3) there is shown that $W^{s,p}(\Omega) = \Lambda^s_{p,p}(\Omega)$ for noninteger $s > 0$ and $1 \leq p < \infty$ (the first spaces denotes the Sobolev-Slobodetski space and the second one is the Besov space). These spaces are introduced in the same Section 2.2.2. Then in Subsection 2.3.5 there is proved that $\Lambda^s_{p,q}(\Omega) = B^s_{p,q}(\Omega)$ for $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ (the first is again the Besov space and the second one is the Triebel space, introduced in Subsection 2.3.1). Finally, in Subsection 3.3.3 (Trace theorem) the following trace theorem is established tr : $B^s_{p,q}(\Omega) \to B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ for $s - \frac{1}{p} > 0$, $p > 1$, $q > 0$. As a simple consequence of these three facts we finally get (B.5).

Thus, we have

$$\int_0^T \|\operatorname{tr} \boldsymbol{v}^i\|_{s+\varepsilon_2}^{p+\varepsilon_2}\, dt \overset{(B.5)}{\leq} C \int_0^T \|\boldsymbol{v}^i\|_{\frac{1}{s}+\delta,s}^{p+\varepsilon_2}\, dt \overset{(B.9)}{\leq} C \int_0^T \|\boldsymbol{v}^i\|_{\ell,2}^{p+\varepsilon_2}\, dt$$
$$\overset{(B.8),(B.10)}{\leq} C\|\boldsymbol{v}^i\|_{L^\infty(0,T;L^2(\Omega)^d)}^{(1-\frac{\ell}{k})(p+\varepsilon_2)} \int_0^T \|\boldsymbol{v}^i\|_{1,r}^{(p+\varepsilon_2)\frac{\ell}{k}}\, dt.$$

We want to choose ε_2 and δ so small that $(p+\varepsilon_2)\frac{\ell}{k} \leq r$. It is possible if and only if $p < s\frac{dr+2r-2d}{sd-2d+2}$, which is exactly (B.4). ☐

By using Lemma B.3 we can prove the following

Corollary B.1. *Let $d = 2,3$ and $r > \frac{2(d+1)}{d+2}$. Let $\{\boldsymbol{v}^i\}_{i=1}^\infty$ be bounded in $\mathcal{S}$. Then $\{\operatorname{tr} \boldsymbol{v}^i\}_{i=1}^\infty$ is precompact in $L^2(0,T;L^2(\partial\Omega)^d)$ and also in $L^{r'}(0,T;L^{\frac{(d-1)r}{d(r-1)}}(\partial\Omega)^d)$.*

Before we give next lemma, we introduce notation for ordinary differential equations. We want to find a function $\boldsymbol{c} : (t_0-\delta, t_0+\delta) \to \mathbb{R}^N$ such that it solves the following ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\boldsymbol{c}(t) &= \mathcal{G}(t,\boldsymbol{c}(t)), \qquad \text{for all } t \in (t_0-\delta, t_0+\delta), \\ \boldsymbol{c}(t_0) &= \boldsymbol{c}_0 \end{aligned} \tag{B.11}$$

where $\boldsymbol{c}_0 \in \mathbb{R}^N$. Consider $\mathcal{G} : (t_0-\delta, t_0+\delta) \times B_\varepsilon(\boldsymbol{c}_0) \to \mathbb{R}^N$ for some $\varepsilon > 0$, where $B_\varepsilon(\boldsymbol{c}_0)$ is the ball with center $\boldsymbol{c}_0$ and radii ε.

Theorem B.1. (Carathéodory) *Let $\mathcal{G} : (t_0-\delta, t_0+\delta) \times B_\varepsilon(\boldsymbol{c}_0) \to \mathbb{R}^N$ satisfy*

$$\begin{aligned} &\mathcal{G}_i(\cdot,\boldsymbol{c}) \textit{ is measurable for all } i = 1,\dots,N \textit{ and for all } \boldsymbol{c} \in B_\varepsilon(\boldsymbol{c}_0) \\ &\mathcal{G}_i(t,\cdot) \textit{ is continuous for almost all } t \in (t_0-\delta, t_0+\delta) \\ &|\mathcal{G}(t,\boldsymbol{c})| \leq G(t) \textit{ for all } (t,\boldsymbol{c}) \in (t_0-\delta, t_0+\delta) \times B_\varepsilon(\boldsymbol{c}_0), \end{aligned} \tag{B.12}$$

where $G(t) \in L^1(t_0-\delta, t_0+\delta)$. Then there exist $\delta_1 \in (0,\delta)$ and absolutely continuous function $\boldsymbol{c} : (t_0-\delta_1, t_0+\delta_1) \to \mathbb{R}^N$ such that $\boldsymbol{c}$ solves (B.11) *for almost all $t \in (t_0-\delta_1, t_0+\delta_1)$.*

Moreover, there exists $\delta_2 \in \langle \delta_1, \delta \rangle$ such that $\boldsymbol{c}$ solves (B.11) *for almost all $t \in (0,\delta_2)$ and either $|\boldsymbol{c}(t)| \to \infty$ as $t \to \delta_2$ or $\delta_2 = \delta$.*

Proof. For proof of the first parts see for example [8], Chapter 2 or [32], Chapter 1. For the second part see for example [34], Chapter 30. ☐

Next lemma gives an important information when the limiting process in integral is possible.

Theorem B.2. (Vitali) *Let Ω be a bounded measurable domain in $\mathbb{R}^d$ and $f^n : \Omega \to \mathbb{R}$ be integrable for every $n \in \mathbb{N}$. Let*

$$\lim_{n\to\infty} f^n(x) \textit{ exists and is finite for almost all } x \in \Omega,$$

and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_B |f^n(x)|\, dx \le \varepsilon \quad \text{for all } B \subset \Omega, |B| < \delta.$$

Then

$$\lim_{n\to\infty} \int_\Omega f^n(x)\, dx = \int_\Omega \lim_{n\to\infty} f^n(x)\, dx.$$

Proof. See for example [1], page 63 or [10]. □

The next Lemma shows how one can integrate per partes in some Bochner spaces.

Lemma B.4. *Let $V \subset H \cong H^* \subset V^*$. Let $\mathcal{V} := \{v; v \in L^p(0,T;V), v_{,t} \in L^{p'}(0,T;V^*)\}$. Then $\mathcal{V} \hookrightarrow \mathcal{C}(0,T;H)$ and there holds for all $0 \le t_1 \le t_2 \le T$ and all $u, v \in \mathcal{V}$*

$$(u(t_2), v(t_2)) - (u(t_1), v(t_1)) = \int_{t_1}^{t_2} \langle u_{,t}, v\rangle + \langle u, v_{,t}\rangle\, dt.$$

Proof. See for example Lemma 7.3, page 191 in [28]. □

The last theorem of this section is the so-called Aubin-Lions lemma.

Theorem B.3. *Let V_1, V_2 be Banach spaces, and V_3 be a metrizable Hausdorf locally convex space, V_1 be separable and reflexive,*

$$V_1 \hookrightarrow\hookrightarrow V_2 \text{ and } V_2 \hookrightarrow V_3.$$

Let $1 < p < \infty$, $1 \le q \le +\infty$ and $0 < T < \infty$. Then

$$\mathcal{M} := \{v; v \in L^p(0,T;V_1), v_{,t} \in L^q(0,T;V_3)\} \hookrightarrow\hookrightarrow L^p(0,T;V_2).$$

Proof. For original version with V_3 Banach space and $1 < q < \infty$ see [3]. For more general case see [27]. □

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