Charles University in Prague

Faculty of Social Sciences
Institute of Economic Studies

MASTER THESIS

Realized Jump GARCH model:
Can decomposition of volatility improve its forecasting?

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature, and the thesis has not been used to obtain a different or the same degree.

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Prague, June 14, 2014

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Signature

Acknowledgments

Abstract

The present thesis focuses on exploration of the applicability of realized measures in volatility modeling and forecasting. We provide a first comprehensive study of jump variation impact on future volatility of Central and Eastern European stock markets. As a main workhorse, the recently proposed Realized Jump GARCH model, which enables a study of the impact of jump variation on future volatility forecasts, is used. In addition, we estimate Realized GARCH and heterogeneous autoregressive (HAR) models using one-minute and five-minute high frequency data. We find that jumps are important for future volatility, but only to a limited extent due to the high level of information aggregation within the stock market index. Moreover, Realized (Jump) GARCH models outperform the standard GARCH model in terms of data fit and forecasting performance. Comparison of forecasts with HAR models reveals that Realized (Jump) GARCH models capture higher portion of volatility variation. Eventually, Realized Jump GARCH compared to other Realized GARCH models provides comparable or even better forecasting performance.

Abstrakt

Tato diplomová práce se zabývá aplikací neparametrických odhadů realizované volatility při modelování a predikcích volatility. Analýza dopadu skoků v cenovém procesu na budoucí volatilitu na akciových trzích střední a východní Evropy je provedena pomocí Realized Jump GARCH modelu, který umožňuje analyzovat vliv skokové variace na budoucí volatilitu. Portfolio odhadovaných modelů dále zahrnuje Realized GARCH a HAR modely pro porovnání predikčních a odhadních vlastností. Výsledky analýzy naznačují, že skoková složka volatility není zanedbatelná. Zároveň je ale její vliv značně omezen. To může být způsobeno vysokou mírou informační agregace v rámci akciového indexu. Porovnání Realized (Jump) GARCH modelů se standardním GARCH modelem naznačuje, že zahrnutí odhadů realizované volatility implikuje lepší odhadní a predikční vlastnosti. Srovnání predikcí získaných použitím HAR modelů a Realized (Jump) GARCH modelů naznačuje, že Realized (Jump) GARCH modely mají lepší predikční vlastnosti především ve vyšší míře zachycené variability volatility. Porovnání predikcí Realized Jump GARCH modelu s ostatními Realized GARCH modely naznačuje, že jeho predikce jsou srovnatelné nebo mírně lepší.

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Acronyms

ARCH	Autoregressive Conditional Heteroskedasticity (model)
ARMA	Autoregressive–moving-average model
BPV	Bipower variation
BUX	BUX Index
C	Continuous sample path of volatility
GARCH	Generalized Autoregressive Conditional Heteroskedasticity (model)
HAR	Heterogenous Autoregressive (model)
HAR-RV	Heterogenous Autoregressive model using Realized variance
HAR-CJ	Heterogenous Autoregressive model using decomposition of volatility into its continuous sample path and jump component
J	Jump variation
MSE	Mean Square Error
OLS	Ordinary Least Squares
PX	PX Index
QV	Quadratic variation
RV	Realized variance
SDE	Stochastic differential equation
TSRV	Two-scale realized variance
TQ	Tripower quadricity
WIG	WIG 20 Index

Master Thesis Proposal

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Proposed Topic:

Realized Jump GARCH model: Can decomposition of volatility improve its forecasting?

Topic Characteristics:

Volatility of financial time-series forms a significant part of current econometric research as volatility is a key component of modern finance. Understanding of volatility is crucial to large span of fields such as risk management, hedging or asset pricing which closely rely on the volatility estimates. In other words, estimation of volatility and its accuracy is very important because this measure is relevant for all market participants and for their decision-making. The base stone of the volatility estimation forms the GARCH methodology of Engle (1982), which opened new avenues to the financial engineering. In recent years, the availability of high frequency data brought new measures of volatility and initiated the discussion about the accuracy and goodness of forecasts given by GARCH models.

Realized measures of volatility offered a possibility to gain a consistent estimate of quadratic returns variation out of high frequency data. This measure was used in a variety of models, for example Heterogeneous Auto-Regressive model. HAR models showed promising results that exceeded those gained by the GARCH model.

Promising area of research is to combine parametric models with high frequency measures. One of the most current GARCH extensions, Realized GARCH (Hansen, 2011), offers the inclusion of realized measures of volatility into variance equation and yet shows promising results. Realized measures of volatility also enable us to decompose quadratic variation into the integrated and jump variation. Current research shows that the role of jumps in the analysis of financial series is very important. The goal of this thesis is to incorporate the jumps component into the Realized GARCH model, find if there is a significant effect of jumps on volatility forecast and provide comparison with HAR and ordinary GARCH model.

Hypotheses:

1. Incorporating realized measure of volatility into GARCH framework improves accuracy of predictions relative to standard GARCH model.
2. Jumps have significant effect on volatility forecast in the Realized GARCH framework.
3. Realized GARCH framework is a suitable alternative to HAR models in terms of efficiency of forecasts.

Methodology:

The goal of the thesis is to enrich the Realized GARCH frameworks by including jumps as in Baruník, Vácha (2013) and provide benchmark based on comparison of predicting efficiency between the following models: GARCH, Realized GARCH and HAR.

The key point of the analysis is the choice of the estimator of quadratic variation from the high frequency data of three CEE stock indices (PX, WIG, BUX). For this reason, various types of measures are employed such as Realized variance (Andersen and Bollerslev, 1998), Two-scale Realized volatility (Zhang, 2005) and Bi-power variation (Barndorff-Nielsen

and Shephard, 2004). In order to separate the effect of jumps, the estimate of price jumps is obtained as the difference between the Realized variance (contains both integrated and jump variation) and bi-power variation (estimator is robust to jumps).

For each measure of quadratic variation and each time-series the models included in the benchmark portfolio are estimated and forecasted. Realized GARCH model will be estimated first, using the estimates of quadratic variation and later, with the inclusion of jump variation into the variance equation. In case, that the impact of jumps on the volatility is important, we should obtain significant coefficient of this variable. Results and forecasts of these two models will be compared to ordinary GARCH model and HAR model. The test of the efficiency will be provided by tested using Mincer and Zarnowitz test (1969).

Outline:

1. Introduction
2. Literature Review
3. Theoretical framework of realized measures of volatility
4. GARCH models
5. HAR models
6. Data description
7. Forecasting
8. Conclusions

Core Bibliography:

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1 Introduction

Understanding volatility can be considered as one of the highest priorities of all financial practioners. As volatility measures riskiness of an asset over certain period of time, the ability to model it as well as to accurately forecast it has strinking implications for all market participants. Volatility as a measure of risk is simply needed for every day decision making, as well as an input for many asset-pricing models. Last but not least, volatility has strong implications for risk management.

Recently, many different methodological approaches to volatility estimation were proposed thanks to increased availability of high-frequency data, which opened new perspectives for research. Andersen et al. (2001) proposed construction of an ex-post non-parametric measure of volatility - realized variance. Following works such as Barndorff-Nielsen et al. (2002), Andersen et al. (2003) investigated properties of realized variance and further enriched the theoretical backgrounds behind realized measures. Consequently, other realized measures were proposed to capture different components of volatility. Barndorff-Nielsen (2004, 2006) proposed construction of bipower variation - a realized measure of volatility robust to jumps in return process. As a consequence, the possibility to decompose volatility into continuous sample path and jump variation brought new perspectives to volatility research. The present thesis aims to contribute to this branch of research as we focus on inclusion of realized measures into the traditional Generalized Autoregressive Conditionally Heteroscedastic model (henceforth GARCH) and Heterogenous autoregressive model (henceforth HAR). The recently proposed Realized GARCH model by Hansen (2011) extends the GARCH framework and proposes inclusion of realized measures into standard GARCH operating on daily data. Baruník and Vácha (2012) enrich and further develop the former concept and embed decomposition of volatility into the Realized GARCH framework. Their Realized Jump GARCH model therefore incorporates measures of continuous sample path of volatility and jump variation into the Realized GARCH framework. HAR models, proposed by Corsi (2004), directly build on the usage of realized measures and provide a parsimonious and simple tool for volatility estimation. Extension of the HAR model proposed by Andersen et al. (2007) is based on inclusion of realized measures of continuous sample path of volatility and jump variation. Further and detailed literature review will be subject to the first two chapters, which aim to define and described realized measures and volatility models used in this thesis.

The major attention of this thesis is dedicated to the impact of jumps in price process to the future volatility of the stock market index. Recently, many studies have suggested strong implications of jumps in a single asset price process to its volatility. Naturally, the importance of jumps within a single asset price process is higher compared to the stock market index because indices are constructed as a weighted average of the constituent stocks. Therefore, the impact of information about individual jumps is very likely to be diminished on an aggregated level. On the other hand, arrival of information with macroeconomic nature can induce jumps in a larger portion of constituent stocks' prices and be apparent even on the aggregate level. In other words, the role of jumps in price process of the stock market index remains unclear. All in all, this thesis aims to explore the role of jumps in aggregated information provided by the stock market index and determine whether accounting for the jump component of volatility improves its forecasting.

To do so, the empirical part of this thesis offers insight into the applicability of realized measures and reveals the impact of volatility decomposition on its forecasting. To do so, we estimate following volatility models: GARCH(1,1), Realized GARCH(1,1) with various realized measures, Realized Jump GARCH(1,1), HAR-RV and HAR-CJ over a triplet of stock market indices of the Prague Stock Exchange, Warsaw Stock Exchange and Budapest Stock Exchange. We aim to find out if the inclusion of realized measures into the GARCH framework brings additional explanatory power to GARCH models. Consequently, comparison of out-of-sample forecasts on a rolling basis of all estimated volatility models shows whether inclusion of realized measures into GARCH framework improves forecasting performance relative to standard GARCH(1,1) and whether forecasts by Realized GARCH models outperform forecasts provided by HAR models. Estimation of Realized Jump GARCH models then shows the impact of jumps in price process to future volatility of a stock market index and whether the decomposition of volatility brings improvement to the model fit and to its forecasting.

Estimation results indicate that inclusion of realized measures into the GARCH framework improves the model fit and forecasting performance relative to standard GARCH models. Such results also suggest that jumps bring additional significant information about future volatility, but their impact is rather limited. Comparison of forecasting performance shows that even though the role of jumps in a price process is not so important, the forecast provided by the Realized Jump GARCH model is comparable or slightly better than forecasts of other Realized GARCH models. In comparison with the forecasting performance of HAR models, we can conclude that

Realized GARCH and Realized Jump GARCH models outperform forecasts provided by HAR models, especially in the volume of variation of volatility captured.

The present thesis is organized in following matter. Chapter 2 is dedicated to introduction of realized measures and to the theory behind them. It aims to introduce all realized measures used for the estimation of volatility models. Chapter 3 is dedicated to the description of volatility models. In the first part of this chapter, attention is paid to the group of GARCH models (GARCH, Realized GARCH and Realized Jump GARCH). Its second part focuses on HAR models (HAR-RV and HAR-CJ). Datasets used for the estimation of volatility models are described in chapter 4, and estimation results are presented and discussed within chapter 5. Finally, chapter 6 concludes and summarizes the results.

2 Realized measures of volatility

The aim of this chapter is to define and introduce the concept of quadratic variation as a representation of realized sample path variation of return process. After the initial price process setup, the major attention will be dedicated to the decomposition of return process which will allow a proper definition of the quadratic variation, the derivation of the realized variance measure and reveal rationale behind other realized measures. The guideline of this chapter is based on Andersen et al. (2010) with respect to work of Barndorff-Nielsen et al. (2002), Tankov (2004), Andersen et al. (2003, 2011), Protter (1992) and Back (1991).

Asset returns are a key interest of every investor and many financial models have been developed in order to provide relevant information that would enable an investor to make qualified decisions. Many of these models are based on the discrete time setup. This framework enables an investor to calculate an expected return of a certain asset based on a suitable model. Very often, the investor's expectations differ from the reality. Intuitively, expectations about the price of an asset will always include an error which occurs from improper specification of model or simply from unexpected events that appeared in the market. Consequently, return process can be decomposed into two basic components: predictable part (for simplicity, let's assume the expected return) and innovations that oppose the predictable part (an error made by investor). Assuming continuous-time return process, frictionless and no-arbitrage opportunity condition, the relationship between expected return and innovations differs. Under the continuous setup, we can expect that the role of innovations increases significantly. This requires different methodological approaches to the modeling of return processes.

2.1 Definition of price process

The price process in present thesis is based on the univariate continuous logarithmic price process $p(t)$ which evolves over time interval $[0, T]$, where T is a finite integer. The price process is defined on probabilistic space $(\Omega, \mathcal{F}, P)$. An important part of its definition is the filtration $\mathcal{F}$. It enables us to take the dynamic aspect of price process into account. The filtration $\mathcal{F}$ can be viewed as an information filter which ensures that pieces of information are revealed dynamically in the time. At time t, $t \in [0, T]$

market participants don't know what the price is at $t + 1$ and only possess information about the value of the price process at time t revealed by information $\mathcal{F}_t$. Such process is called non-anticipating price process with respect to the information structure. Moreover, if the available information about the price process consists only of its past values, filtration of this price process is referred as natural. (Tankov, 2004, p.40).

Eventually, the set of initial requirements for the price process should be completed with assumption of frictionless markets and no-arbitrage opportunities. The first named ensures that selling and buying can proceed without any costs: absence of transaction costs, no margin requirements or bid/ask spreads. The former implies that any zero investment trading strategy with a positive probability of gains must also have a positive probability of losses.

Following definition of Andersen et al. (2010, p.70), we assume finite time period $[0, T]$ and continuously compounded return over the time interval $[t - h, t]$ as:

$$r(t, h) = p(t) - p(t - h), \qquad 0 \le h \le t \le T \qquad (2.1)$$

Usage of continuously compounded returns allows us to extend former definition and define cumulative return up the time t (return over the time period $[0, t]$):

$$r(t) \equiv p(t) - p(0), \qquad 0 \le t \le T \qquad (2.2)$$

Former definitions imply the following relation between cumulative returns and period-by-period:

$$r(t, h) = r(t) - r(t - h), \qquad 0 \le h \le t \le T \qquad (2.3)$$

Due to the setting, which has been defined so far, price process needs to be restricted to remain strictly positive because the price process was logarithmically transformed. Thus, any violation of this restriction would imply that $r(t)$ and $p(t)$ wouldn't be well defined over the time interval $[0, T]$. Moreover, return process $r(t)$ has a finite and countable number of jumps and we assume price and return process to be square integrable.

The inclusion of jumps into return process requires further definition of the return process because, due to their existence, process discontinuities are brought to play. Convenient tool for dealing with discontinuities is class of processes called *càdlàg (resp. càglàd).*[1] Defining $r(t-) = \lim_{s \to t, s < t} r(s)$, $r(t+) = \lim_{s \to t, s > t} r(s)$ we are

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[1] Word *càdlàg* is an acronym for French "continu à droite, limite à gauche" meaning continuous from the right with limit from the left. Respectively, word *càglàd* is an acronym for French "continu à gauche, limite à droite" meaning continuous from the left with limit from the right.

able to uniquely determine process for which $r(t+) = r(t)$ as càdlàg (right continuous with left limit) and càglàd (left continuous with right limit) for $r(t-) = r(t)$ for all $t \in [0, T]$. The definition of these two processes has also an economic rationale behind. The continuity of the process implies whether the jump was predictable or not. Assuming càdlàg process, the jump at time t wasn't forseeable from the previous sample path of the process. From that point of the view, the jump at time t was unpredictable. Conversely, in case of càglàd process, value of the process at time t could have been predicted based on observations made approaching time t. In the context of financial modeling, jumps are usually considered to be sources of uncertainty that unpredictably appear in the process. (Tankov, 2004, p. 38) Right from the definition of càdlàg, we can define jump in cumulative price and return process as:

$$\Delta r(t) = r(t) - r(-), \qquad 0 \leq t \leq T \qquad (2.4)$$

Consequently, we can conclude that in the continuity points of the return process we have $\Delta r(t) = 0$. As previously stated, there is a countable number of jumps in the process. This means that continuity points must exist and that for arbitrarily chosen times t matching the continuity points, we have $P(\Delta r(t) \neq 0) = 0$. The importance of condition of countability of jumps is obvious. In case that there would be an infinite number of jumps, the price process would "explode". For this reason, regular processes are defined on existence of instantaneous jump intensity. (Andersen, 2010, p.70)

2.2   Return decomposition

Having price and return process defined, attention can be finally dedicated to the decomposition of returns. The basic theoretical framework, discussed in Back (1991), uses a concept of (special) semimartingale of log-price process (under assumption of no-arbitrage, frictionless setting, finite-expected returns) and is further enriched by Protter (1992) who shows possibility of unique canonical return decomposition.

Before we come up with a proposition that allows canonical return decomposition, let's explain the role of martingales and semimartingales in this setting. Their definition and further usage are very beneficial, mainly for their suitability for stochastic integration. First of all, let us remind the definition of martingale following Tankov (2004, p.40).

Definition 1: Martingale
A process $(X_t)_{t\in[0,T]}$ is called a martingale if for any $t \in [0,T]$ of the realized sequence, the process is adapted to the filtration $\mathcal{F}_t$ and has finite mean, and (for $\forall s > t$), we have $E[X_s|\mathcal{F}_t] = X_t$.

In other words, the best prediction of martingales future value is given by the present values of the process.

Definition of semimartingale allows decomposition of the process into the sum of local martingales and adapted (non-anticipating) finite-variation process. Assuming that the finite variation process is predictable[2], we talk about special semimartingale process. (Protter, 1992, p. 107) Semimartingales have beneficial properties for the purpose of stochastic integration because they remain stable under stochastic integration and smooth nonlinear transformation. For example, its associativity property ensures that stochastic integral with respect to semimartingale is still a semimartingale. (Tankov, 2004, p. 245). Definition of local martingale arises from Sampling theorem which suggests that a martingale stopped at a non-anticipating random time is still a martingale. Following Tankov (2004, p. 42), definition of local martingale is provided.

Definition 2: Local martingale
A process $(X_t)_{t\in[0,T]}$ is a called a local martingale if there exists a sequence of stopping times (T_n) with $T_n \to \infty$ such that $\left(X_{t \wedge T_n}\right)_{t\in[0,T]}$ is a martingale.

Having all important features of theory of martingale and semimartingale introduced, we can finally come to the key return decomposition proposed by Protter (1992).

Proposition 1: Return decomposition
Any arbitrage-free logarithmic price process subject to the regularity conditions outlined above may be uniquely represented as

$$r(t) \equiv p(t) - p(0) = \mu(t) + M(t) = \mu(t) + M^C(t) + M^J(t), \qquad (2.5)$$

where $\mu(t)$ is a predictable and finite-variation process, $M(t)$ is a local martingale that may be further decomposed into $M^C(t)$, a continuous sample path, infinite-variation local martingale component and $M^J(t)$, a compensated jump martingale. We may normalize that initial conditions such that all components may be assumed to

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[2] For example, *càglàd* process which is left continuous process whose value at time t is also known in a fraction of a moment before t.

have initial conditions normalized such that $\mu(0) \equiv M^C(0) \equiv M^J(0) \equiv 0$, which implies that $r(t) \equiv p(t)$. (Andersen, 2010, p. 71)

Such decomposition of returns into predictable finite variation process and local martingale can create an illusion that jumps arise in the process only through the compensated jump martingale M^J. This setting also enables to introduce jumps in the predictable mean component.[3] For the purpose of following discussion, let's allow decomposition of mean predictable process $\mu(t)$ into continuous process $\mu^C(t)$ and pure jump process $\mu^J(t)$.

Following discussion should clarify why the martingale components contribute to the innovations of the return process more than the predictable mean process. First of all, let's compare the influence of continuous terms $\mu^C(t)$ and $M^C(t)$. In case that the variation of $\mu^C(t)$ would be higher than the variation of $M^C(t)$ over small time intervals, investor could easily perfectly diversify his portfolio, thanks to law of large numbers, simply by creating many long positions in risky asset over any time intervals. The perfect diversification would be ensured because martingale component is from the definition uncorrelated. Preserving risk-return relationship and no-arbitrage condition, it is implied that the impact of $\mu^C(t)$ is exceeded by $M^C(t)$. No-arbitrage condition also explains why the role predictable jump $\mu^J(t)$ is always accompanied by existence of jump in the martingale $M^J(t)$. The predictable jump can be explained as arrival of expected information in the market. If there weren't any concurrent jump in the $M^J(t)$, investors could simply expoit this jump by creating an offsetting position without any risk. Moreover, if we assume continuous framework and uncertainty about the exact timing of the information arrival into the market, the role of this predictable jump diminishes even more. (Andersen, 2010, p.72) Based on the intuitions above we can come with consequential proposition to Proposition 1 based on Andersen (2003, p.583):

Proposition 2
The predictable jumps are associated with genuine jump risk in the sense that if $\mu^J(t) \neq 0$, then

$$P\left[sgn\left(\mu^J(t)\right) = -sgn\left(\mu^J(t) + M^J(t)\right)\right] > 0, \qquad (2.6)$$

Where $sgn(x) = 1$ for $x \geq 0$ and $sgn(x) = -1$ for $x < 0$.

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[3] For further details see Andersen (2003, 2010).

2.3 Quadratic variation and notional volatility

One of the base stones of theory of volatility is the concept of the quadratic variation. In this section, quadratic variation process will be defined and its main properties will be discussed. Then the concept of quadratic variation will be introduced.

Definition 3: Quadratic variation process

Let $X(t)$ denote any (special) semimartingale. The unique quadratic variation process, $[X, X]_t, t \in [0, T]$, associated with $X(t)$ is formally defined as

$$[X, X]_t \equiv X^2(t) - 2 \int_0^t X(s-)dX(s), \ \ 0 < t \leq T, \tag{2.7}$$

where the stochastic integral of the adapted càglàd process, $X(s-)$, with respect to the càdlàg semimartingale, $X(s)$, is well defined. (Andersen, 2010, p.75)

Based on this definition we can conclude main properties of this process. We get that $[X, X]_t$ is an increasing process. The jumps in the process are concurrently caused by jumps in the X_t process ($\Delta[X, X] = (\Delta X)^2$). Moreover, if X is continuous and has paths of finite variation then $[X, X] = 0$. (Tankov, 2004, p.264)

Barndorff-Nielsen (2002, p.463) show important simplification implied by assumption that predictable finite variation process $\mu(t)$ is continuous. Based on Jacod and Shiryaev (1987) and decomposition of semimartingale, it is shown that quadratic variation process is robust to "smooth" $\mu(t)$ processes.

Eventually, it shows out that the martingale component is the key interest from the perspective of the quadratic variation process. Unfortunately, martingale requires us to have excess to continuous data. In reality, these data aren't available. For this reason, we will focus on a discrete setting and define quadratic variation of return process $QV(t, h)$; also average or notional volatility; over certain period of time. If we choose this time interval small enough we can get a close approximation of continuous setting. Based on Andersen (2010, 2003) we define quadratic variation of return process as follows:

Definition 4: Quadratic variation

Let r_t be a semi-martingale return process. Assuming that there are no predictable jumps in the return process, the quadratic variation $QV(t, h)$ of over time interval $[t - h, t], 0 < h \leq t \leq T$, is:

$$QV_{t,h} = [r, r]_t - [r, r]_{t-h} = [M^C, M^C]_t - [M^C, M^C]_{t-h} + \sum_{t-h<s\leq t} \Delta M^2(s \tag{2.8}$$

$$QV_{t,h} = [r, r]_t - [r, r]_{t-h} = [M^C, M^C]_t - [M^C, M^C]_{t-h} + \sum_{t-h<s\leq t} \Delta r^2(s \tag{2.9}$$

The quadratic variation over time interval $[t - h, t]$ can be understood as an increment to the quadratic variation process. Andersen (2010) uses the term notional volatility to express the variation of the process over a small time interval. The equation (2.9) in Definition 4 suggests that the quadratic variation measures the variability of squared returns process. This has wide application in finance because it enables us to construct a model-free nonparametric ex-post estimate of the return variation - Realized variance. The large popularity of this concept is mainly caused by its simplicity and also by increased availability of high frequency data. The concept of realized variance will be subject of next section. Last remark will be dedicated to the Martingale representation theorem (Protter, 1992) which creates an important notional interconnection between conventional stochastic differential equations and abstract integral representations for continuous sample path semimartingales. Before the quadratic variation was defined, it had been suggested that the continuous setting has certain limitation due to the continuous data unavailability. The usage of high frequency data brings us close to the continuous setting. For this reason, literature assumes the continuous setting, thus stochastic differential equations are preferred way of describing the corresponding diffusion process. The Proposition 3 than creates an equivalency of the integral representation to the standard stochastic differential equation representation of the price process given by:

$$\mathrm{d}p(t) = \mu(t)\mathrm{d}t + \sigma(t)dW(t), \ 0 \leq t \leq T, \qquad (2.10)$$

where $\mu(s)$ is an integrable, predictable, and finite-variation stochastic process, spot volatility process $\sigma(s)$ is strictly positive càdlàg and square integrable and $W(t)$ denotes a standard Brownian motion.

Proposition 3: Martingale representation theorem

For any univariate, square-integrable, continuous sample path, logarithmic price process, which is not locally riskless, there exists a representation such that for all $0 \leq t \leq T$,

$$r(t, h) = \mu(t, h) + M(t, h) = \int_{t-h}^{t} \mu(s)\mathrm{d}s + \int_{t-h}^{t} \sigma(s)dW(s), \qquad (2.11)$$

Where $\mu(s)$ is an integrable, predictable, and finite-variation stochastic process, $\sigma(s)$ is strictly positive càdlàg stochastic process satisfying

$$P\left[\int_{t-h}^{t} \sigma^2(s)\mathrm{d}s < \infty \right] = 1, \qquad (2.12)$$

and $W(s)$ is a standard Brownian motion. (Andersen, 2010, p.79)

Therefore, if we assume that the price process is within the class of continuous sample path semimartingale, Proposition 3 allows us to conclude that the quadratic variation $QV_{t,h}$ (increment of quadratic variation process) equals to

$$QV_{t,h} = \int_{t-h}^{t} \sigma^2(s)\mathrm{d}s. \tag{2.13}$$

Under the condition of no jumps in price process, the quadratic variation coincides with the definition of integrated variance defined in the equation (2.14).

$$IV_{t,h} = \int_{t-h}^{t} \sigma^2(s)\mathrm{d}s \tag{2.14}$$

The equivalence of quadratic variation and of integrated variance no longer holds after inclusion of jumps into the price process. This topic will be subject of our interest in chapter 2.5.

2.4   Realized variance

Definition 4 through equation (2.9) shows that the realized sample-path variation process of the squared return process is measured by quadratic variation. Assuming that the mean process is smooth (doesn't include any predictable jumps), Definition 4 reveals the leading role of innovations to the return process for quadratic variation and outlines a possibility to construct a realized measure of the quadratic variation: summation of squared returns over a time interval. (Andersen, 2003, p.585) The construction of these ex-post measures of volatility has long tradition in finance, for example Poterba and Summers (1986) or French et al. (1987). Throughout the time, the construction of realized measures has changed significantly. Availability of high-frequency data allowed the shift from the daily returns to intra-day returns and opened new perspectives for the research. Interconnection between quadratic variation and realized sample-path variability is given by Proposition 4.

Proposition 4
Let a sequence of partitions of $[0, T]$, (τ_m), *be given by* $0 = \tau_{m,0} \leq \tau_{m,1} \leq \cdots \leq \tau_{m,m} = T$ *such that* $\sup_{j \geq 0}(\tau_{m,j+1} - \tau_{m,j}) \to 0$ *for* $m \to \infty$. *Then, for* $t \in [0, T]$,

$$\lim_{m \to \infty} \left\{ \sum_{j \geq 1} \left(X(t \wedge \tau_{m,j}) - X(t \wedge \tau_{m,j-1}) \right)^2 \right\} \to [X, X]_t$$

where $t \wedge \tau \equiv \min(t, \tau)$ *and the convergence is uniform in probability.*
(Andersen, 2010, p. 75)

Proposition 4 in other words suggests that the quadratic variation can be approximated by summation of squared high frequency returns. Such measure is in literature referred as the realized variance.

Definition 5: Realized variance

The realized variance over $[t-h, t]$, *for* $0 < h \leq t \leq T$, *is defined by*

$$\widehat{RV}_{t,h,n} \equiv \sum_{i=1,\ldots,n} r(t-h+(i/n) \cdot h, h/n)^2. \tag{2.15}$$

Andersen (2010, p. 109)

In other words, realized variance over time interval $[t-h, t]$ equals to the sum of n squared returns with scaling frequency h/n. In fact, it is second, uncentered sample moment of the return process. Sometimes, realized variance is in the literature referred as the realized volatility. In present thesis, the realized volatility denotes the square root of the realized variance. Therefore, we will refer to realized measures of volatility as to square-root of realized measures of variance.

Following paragraphs will be dedicated to important properties of the realized variance. Many studies have been dedicated to the issues of distribution of realized variance (for example Barndorff-Nielson (2002), Andersen at al. (2001)). They prove that although the concept of the realized variance is simple and easily interpretable, it has certain limitations arising from the usage of high-frequency data. The key feature of realized variance is its convergence in probability to quadratic variation as n → ∞. This property holds for all semimartingale processes. (Barndorff-Nielsen, 2002, p. 463) This enables us to assume that $\widehat{RV}_{t,h,n}$ is a suitable estimator of $QV_{t,h}$. Following two propositions will be dedicated to the important properties of unbiasness and consistency.

Proposition 5

If the return process is square-integrable and $\mu(t) \equiv 0$, *then for any value of* $n \geq 1$ *and* $h > 0$,

$$\xi^2(t, h) = E[QV_{t,h}|\mathcal{F}_{t-h}] = E[M^2(t, h)|\mathcal{F}_{t-h}] = E[\widehat{RV}_{t,h,n}|\mathcal{F}_{t-h}], \tag{2.16}$$

where $\xi^2(t, h)$ *denotes the expected ex-ante volatility.* (Andersen, 2010, p.110)

Proposition 6

The realized variance provides a consistent nonparametric measure of the quadratic variation (notional volatility),

$$\text{plim}_{n \to \infty} \widehat{RV}_{t,h,n} = QV_{t,h}, \quad 0 < h \leq t \leq T \tag{2.17}$$

where the convergence is uniform in probability. (Andersen, 2010, p. 111)

Proposition 5 states that ex-post realized variance is an unbiased estimator of expected ex-ante volatility (ex-ante variance of the return process). The Proposition 6 shows that in the limit for increasingly finely sampled returns, the realized variance is a consistent estimator of the quadratic variation. This convergence is also ensured by the semimartingale theory.

Moreover, combination of Proposition 5 and Proposition 6 allow us to construct the conditional expectations of the quadratic variation (conditional return variance) based on ex-post realized volatility measure which, consequently, helps us to specify the distribution of return process. Adopting the SDE setting given by equation (2.10) we have:

$$\mathrm{d}p(t) = \mu(t)\mathrm{d}t + \sigma(t)dW(t),\ 0 \le t \le T, \tag{2.18}$$

where processes $\mu(t)$ and $\sigma(t)$ denote the conditional mean and the conditional volatility of the return, respectively. If we further assume that the price process is square-integrable arbitrage-free process with continuous sample path and that $\mu(t), \sigma(t)$ and $dW(t)$ are mutually independent, then $r(t,h)$ is normally distributed conditional on the cumulative drift $\mu(t,h) = \int_{t-h}^{t} \mu(s)\mathrm{d}s$ and the quadratic variation $QV_{t,h} = \int_{t-h}^{t} \sigma(s)dW(s)$:

$$\left(r(t,h)\middle|\ \mu(t,h), QV_{t,h}\right) \sim N\left(\mu(t,h), QV_{t,h}\right) \tag{2.19}$$

Such mixed Gaussian distribution enables the return process to have features that are observed in the real-world data. For example, extreme return observations or volatility clustering caused by the persistence in the quadratic variation. On the other hand, in the real world, the quadratic variation is unobservable which raises the question of applicability of such mixed distribution. Fortunately, the realized variance provides a consistent estimate of the quadratic variation. Andersen (2008, p.8)

Furthermore, literature suggests that the usage of mixed Gaussian distribution is more problematic due to the violation of its assumptions. First, between some classes of assets the correlation between concurrent return and volatility innovations was observed. Second, the assumption of continuity of the price process is far too restrictive. (Andersen, 2003, p.592) Although we have so far neglected the jumps in the price process, it must be noted that they have economic rationale. They can signify the sudden unexpected arrival of the information to the market. Also, the occurrence of jumps in the price process has been reported by several studies. For example Andersen, Benzoni, Lund (2002, p.1278) find that models that didn't incorporate failed to accommodate the prominent features of the daily S&P 500.

Moreover, they find that inclusion of these features into stochastic volatility jump diffusion model brought significant improvement of results.

All in all, it seems that the setting which assumed the price process to have continuous sample path is too restrictive and ignoring the jumps in the price process could lead to important biases of analysis. Therefore, following section will bring insight into challenges implied by jumps in the price process.

2.5 Bipower variation

The aim of this chapter is to introduce the concept of the bipower variation proposed by Barndorff-Nielsen (2004) and define a consistent measure of the integrated volatility that will further allow us to determine the jump component in the quadratic variation.

As we have discussed, the omission of jump in the price process is too restrictive as some studies have reported their existence. For this reason, we suggest to redefine the return process in (2.10) and include jumps in the mean process.

$$\mathrm{d}p(t) = \mu(t)\mathrm{d}t + \sigma(t)dW(t) + \xi(t)dq_t, \ 0 \leq t \leq T, \qquad (2.20)$$

where q is Poisson process which is uncorrelated with W . The Poisson process has positive and finite jump intensity λ_t: $P(dq_t = 1) = \lambda_t dt$. This definition ensures that there is only finite number of jumps in the return process. Scaling factor $\xi(t)$ then determines the magnitude of jump in the time of occurrence t. (Andersen, 2008, p.9)

Under the setting given by equation (2.20), we come to the conclusion that the integrated variance no longer equals to the quadratic variation. Consequently, the quadratic variation needs to be further decomposed into the integrated variance component and into the jump variation. Then we can rewrite the quadratic variation over time interval $[t - h, t]$ as the sum of diffusive integrated variance and a sum of squared jumps.

$$\mathrm{QV}(t,h) = \int_{t-h}^{t} \sigma^2(s)\mathrm{d}s + \sum_{t-h \leq s \leq t} J^2(s) \equiv IV(t,h) + \sum_{t-h \leq s \leq t} J^2(s) \qquad (2.21)$$

Based on Proposition 6, the realized variance is a consistent estimator of the quadratic variation even after inclusion of jumps. In order to subtract the jump variation, we need a consistent measure of integrated variance. The concept of

bipower variation was proposed by Barndorff-Nielsen (2004) and allows construction of consistent estimator of integrated variance.

Definition 6: Realized bipower variation

For $0 \leq h \leq t \leq T$, the realized bipower variation over time interval $[t - h, t]$ is

$$\widehat{BV}(t,h,n) = \mu_r^{-2} \sum_{i=2}^{n} \left| r\left(t - h + \frac{ih}{n}, \frac{1}{n}\right) \right| \cdot \left| r\left(t - h + \frac{(i-1)h}{n}, \frac{1}{n}\right) \right|, \quad (2.22)$$

where $\mu_r = E|u|^r = 2^{r/2} \dfrac{\Gamma\left(\frac{1}{2}(r+1)\right)}{\Gamma\left(\frac{1}{2}\right)}$ with $u \sim N(0,1)$ and Γ denotes gamma function.

Term $\frac{1}{n}$ denotes scaling frequency. For $r = 1$, we have $\mu_r = \sqrt{\frac{2}{\pi}}$. (Andersen, 2008, p.9)

Barndorff-Nielsen (2004) shows that the realized bipower variation provides a consistent measure of the integrated variance. In case of zero jumps in the return process, he shows that the realized bipower variation provides the same results as the realized variance. When jumps in the mean process are present, they prove that the realized bipower variation is robust to jumps and consistently measures the integrated variance. Moreover they show that the consistency of both realized measures enables to estimate the contribution of jumps variation to the quadratic variation. Formally, we have:

$$\widehat{RV}_{t,h,n} - \widehat{BV}_{t,h,n} \xrightarrow[n \to \infty]{} QV_{t,h} - IV_{t,h} = \sum_{t-h \leq s \leq t} J^2(s) \quad (2.23)$$

Andersen (2011, p.178) provides another version of the bipower variation robust to microstructure noise. Therefore we decided to use following version instead.

Definition 7: Realized bipower variation (by Andersen (2011)

For $0 \leq h \leq t \leq T$, the realized bipower variation over time interval $[t - h, t]$ is

$$\widehat{BV}(t,h,n) = \mu_r^{-2} \frac{n}{n-2} \sum_{i=3}^{n} \left| r\left(t - h + \frac{ih}{n}, \frac{1}{n}\right) \right| \cdot \left| r\left(t - h + \frac{(i-2)h}{n}, \frac{1}{n}\right) \right|, \quad (2.24)$$

where $\mu_r = E|u|^r = 2^{r/2} \dfrac{\Gamma\left(\frac{1}{2}(r+1)\right)}{\Gamma\left(\frac{1}{2}\right)}$ with $u \sim N(0,1)$ and Γ denotes gamma function.

Term $\frac{1}{n}$ denotes scaling frequency. For $r = 1$, we have $\mu_r = \sqrt{\frac{2}{\pi}}$. (Andersen, 2008, p.9)

The availability of a consistent estimator of the integrated variance can also enable us to test whether there are jumps in the price process. Based on Barndorff-Nielsen (2006), many jump detection test has been published. Andersen (2011, p. 178) proposes indicator test based on the joint asymptotic distribution of the jump variation. Assuming the regulatory conditions and null hypothesis of no within-day

jumps, the test statistic Z_t defined by equation (2.25) is asymptotically standard normally distributed. Therefore, the test statistic Z_t can be used to test the null hypothesis of no within-day jumps.

$$Z_t = \frac{\frac{\widehat{RV}_{t,h,n} - \widehat{BV}_{t,h,n}}{\widehat{RV}_{t,h,n}}}{\sqrt{\left(\left(\frac{\pi}{2}\right)^2 + \pi - 5\right)\frac{1}{n}\max\left(1, \frac{RTQ_t}{BV^2_{t,h,n}}\right)}}, \tag{2.25}$$

where $RTQ_t = n\mu_{\frac{4}{3}}^{-3}\left(\frac{n}{n-4}\right)\sum_{j=5}^{n}\left|r\left(t-h+\frac{(i-4)h}{n}, \frac{1}{n}\right)\right|^{\frac{4}{3}}\left|r\left(t-h+\frac{(i-2)h}{n}, \frac{1}{n}\right)\right|^{\frac{4}{3}}\left|r\left(t-h+\frac{ih}{n}, \frac{1}{n}\right)\right|^{\frac{4}{3}}$.

The Z_t statistic can be further used to measure the contribution of jump variation to the overall quadratic variation of the return process. The measure is given by equation (2.26).

$$J_{t,h} = I(Z_t > \Phi_\alpha) \cdot \left(\widehat{RV}_{t,h,n} - \widehat{BV}_{t,h,n}\right), \tag{2.26}$$

where $I(\cdot)$ denotes the indicator function and Φ_α reffers to the critical value of the normal distribution. Similarly, the measure of the integrated variance is defined as

$$C_{t,h} = I(Z_t \leq \Phi_\alpha) \cdot \widehat{RV}_{t,h,n} + I(Z_t > \Phi_\alpha) \cdot \widehat{BV}_{t,h,n}, \tag{2.27}$$

(Andersen, 2011, p. 178)

2.6 Effect of microstructure noise

In previous chapters, we came to conclusion that the realized variance estimator is a consistent measure of the quadratic variation. Unfortunately, the usage of high-frequency data brings other troublesome complications which need to be taken into account. Although high-frequency data enable us to approach close to the continuity setting, the data are still discrete and thus, discretization error is inevitable. Moreover, high-frequency sampling, when constructing realized measures, can lead to misleading and highly biased results due to the effect of microstructure noise. Extremely short time intervals can induce spurious autocorrelation. (Andersen, 2008, p.10) Especially from the intra-day data perspective, there is large span of microstructure effects that have important implications for the construction of realized measures. First, the volatility clusters are observed during the intraday trading. Second, intraday volatility is determined by behavioral aspects of trading. For example, in the global markets it can be observed that volatility varies depending on the activity of different regions of the world. Third, high-frequency returns are usually negatively correlated and the magnitude of this correlations increases with the

sampling frequency. Moreover, when sampling frequency is high, spurious relationships can occur between returns of two assets that are independent (this is referred as non-synchronous trading effect). Last but not least, when the sampling frequency is high, the bid-ask spread effect comes into play. (Bai, Russell and Tiao, 2000, p.6) To sum up, literature suggests that the realized variance estimator is not robust to the sampling frequency. In other words, the construction of the realized measures and the choice of the sampling interval play a crucial role especially from the perspective of the sampling frequency and microstructure noise. The shorter interval we choose, the issue of microstructure noise becomes more pronounced. (Zhang, 2005. p. 1395)

Andersen (2000, p. 106) developed a tool which helps to determine the effect of the microstructure noise called Volatility signature plot. The idea is based on the plot of the estimates of realized variance as the function of the sampling frequency. In case, that the microstructure noise effect is absent, the plot should look like a horizontal line which doesn't distort from the average of the realized volatility estimator. Authors find that in the case of a liquid asset, the high sampling frequencies imply that the values of realized variances exceed the average. With the growing width of the sampling interval, the values diminish and stabilize around the values of K=20 (sampling interval is 20 minutes). Authors note that explanation can be found in negative serial correlation in returns (the presence of bid-ask spread). In case of less liquid assets, the values of the realized variance for high-frequencies are below the average. With the growing sampling interval, the values of the realized variance estimators increase and stabilize around the value of K=15 (15 minutes sampling interval). Literature suggests that the optimal sampling frequency lies in the interval from 5 minutes to 40 minutes. (Andersen, 2008, p.10)

Zhang (2005) proposes an alternative measure which enables the usage of all available data called the Two-scale realized variance. The estimator is based on the quantification and further correction of the microstructure bias. The estimator is given by following definition:

Definition 8: Two-scale realized variance
For $0 \leq h \leq t \leq T$, the Two-scale realized variance over time interval $[t-h, t]$ is

$$\widehat{TSRV}(t, h) = \widehat{RV}_{t,h}^{avarage} - \frac{\bar{n}}{n} \widehat{RV}_{t,h}^{all}, \tag{2.28}$$

where $\widehat{RV}_{t,h}^{all}$ is realized variance estimated over all available data in interval$[t-h, t]$ and $\widehat{RV}_{t,h}^{avarage}$ is an average of realized variances estimated over K subgrids of average size $\bar{n} = \frac{n}{K}$. Zhang (2005, p.1395)

Subgrids K are the subsamples of the available data in interval $[t - h, t]$ and they are constructed on the following logic. Let's assume 5 minutes sampling frequency. The first subsample contains first observation and then, every five minutes, an observation is added. The second subsample contains the second observation and then, following the previous logic, every five minutes, an additional observation is taken, etc.

The two-scale realized variance estimator got its name because of the two time scales which are used to construct the estimator. The first term in equation (2.28) covers the slow scale meaning that the dataset is divided into subgrids as described above. The second term covers the fast time (high-frequency) scale which includes all the available data. The two-scale realized variance is a consistent and asymptotic estimator of the quadratic variation of return process with rate of convergence $n^{-1/6}$. Moreover, authors provide the theory to an optimal choice of number of subgrids K so that the total asymptotic variance is minimized. (Zhang, 2005, p.1397)

3 Volatility models

Modeling and forecasting of asset volatility is certainly crucial for all agents in financial markets. Volatility can be viewed as a measure of risk and its understanding is naturally important for all market participants. The investigation of relation between risk and return has a very long tradition in economic literature and its implications are tremendous. Markowitz (1952) and Tobin (1958), for example, associated the risk with the variance in the value of a portfolio. Another example is the CAPM theory which explains the direct trade-off between the expected return and risk. Ability to estimate and forecast volatility has wide implications to risk-management and option pricing models. These estimates are required as an input for many pricing formulas, such as Black-Scholes formula for derivation of the prices of traded options, or for value-at-risk model. In other words, ability to predict the future volatility precisely is indeed important.

The simplest definition of volatility would explain it as the standard deviation of returns over some historical period. Usage of this 'historical' volatility is indeed problematic because it gives only a constant value. Observing many historical financial series of returns, one can directly see that volatility is not constant over the time and that more dynamic approach is needed. Moreover, the usage of historical volatility brings other issues which need to be resolved such as what is the optimal length of the historical sample.

In order to avoid these problematic features, various approaches to volatility modeling and estimation were developed. In this thesis, two basic groups of models are estimated and compared. One group of models perceives the volatility as an unobserved variable and uses the concept of conditional variance in order to analyze the latent volatility. This group is represented by large (G)ARCH family of models which in recent years became popular for its wide applicability and especially for the ability to accommodate important features that are common to many financial series. The other group of models is based on realized measures (HAR models).

3.1 ARCH models

The Autoregressive Conditionally Heteroscedastic model (ARCH) was first proposed by Engel (1982). Its derivation was motivated by Milton Friedman´s conjecture (1972) that the unpredictability of inflation was a primary cause of business cycle. He argued that the high uncertainty about future price levels restricted entrepreneurs from investing and that this would lead the economy into recession. Perception of dynamic development of uncertainty led directly to the problem of heteroskedasticity and introduction of ARCH model. Unfortunately, the macroeconomic application didn't confirm this effect on the U.S. and U.K. data but in both cases, the presence of ARCH residuals was reported. This opened new possibilities for modeling in finance, where the relationship between the risk and return was of primary importance. (Engel, 2003, p.327)

It has been noted that ARCH models enable us to model volatility. It is indeed an important feature because stylized facts about financial time series aren't in accordance with classical assumptions of econometric estimation. First, the observed volatility isn't constant but changes over time. Moreover, it can be observed that the level of volatility changes dynamically and periods of low volatility as well as the periods of high volatility can be observed. This phenomenon is frequently referred as volatility clustering (volatility pooling). It is also apparent that the volatility performs certain persistence. The large returns are often followed by other large returns which sustain the increased level of volatility for some time. These volatility clusters can be explained as arrivals of new information to the market. Each newly arrived piece of information requires immediate reaction of investors who revalue their assets and adjust their positions to new changes. As information very often come in bunches, the volatility clustering occurs. Another explanation arises from the similar background: newly arrived information requires immediate reaction but various market players may evaluate its value differently. Following their investment strategy, investors may eventually become unsure about their own estimates and revise them assuming that their models are outperformed. Herd behavior can result into over or undervaluation of assets which will later result into consequent adjustment which result in periods of high volatility, volatility clusters.

Second, volatility tends to asymmetrically react to different news. This leverage effect frequently refers to the fact that volatility reacts much stronger to bad news. (Engel, 2003, p.330) Third, financial return series exhibit leptokurtic distributions with fat tails and excess peakedness at mean. (Brooks 2008, p.380)

To sum up, financial return series show that the volatility isn't constant over time and needs to be further modeled and ARCH models offered a possible solution through the concept of conditional variance which allows the conditional variance of error term depend on its previous values of squared errors (ARCH residuals). Following Brooks (2008), the conditional variance of error term u_t is denoted as h_t and can be formally expressed as:

$$h_t = \text{var}(u_t|\mathcal{F}_{t-1}) = \text{var}(u_t|u_{t-1}, u_{t-2}, \dots)$$
$$h_t = E\left[\left(u_t - E(u_t)\right)^2 \middle| u_{t-1}, u_{t-2}, \dots\right]$$

(3.29)

Assuming that $E(u_t) = 0$, we get

$$h_t = \text{var}(u_t|\mathcal{F}_{t-1}) = E[u_t^2|u_{t-1}, u_{t-2}, \dots].$$

(3.30)

The ARCH setting enables us to also specify the conditional mean which can be represented by ARMA models or simply by constant. Accounting for these two features, the ARCH(q) model of return process consists of mean and variance equation and can be represented by following specification:

Model 1: ARCH(q)

$$r_t = \mu + \varepsilon_t$$
$$\varepsilon_t = z_t\sqrt{h_t}$$
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-1}^2$$

(3.31)

where r_t stands for return , $z_t \sim \text{i.i.d}(0,1)$ and coefficients in variance equation are restricted to non-negative values; that is $\alpha_i \geq 0, \forall i = 0,1,2, \dots, q$.

Such specification ensures that the conditional variance remains positive. Non-negativity of ARCH coefficients is indeed an important feature as any variance estimate providing negative value would be misleading. If this condition didn't hold and some of the ARCH coefficients were negative, a very large shock could eventually translate into negative conditional variance.

It is also obvious that the model deals with the volatility clustering. Simply, large past shocks translate into larger volatility now. Although ARCH model ment a breakthrough in financial series modeling it should be noted that it has certain weaknesses. It remains unclear what is the optimal number of lags included into model. Often, the model requires inclusion of many lags in order to capture the dependence in the conditional variance. Such model is no longer parsimonious and with no clear economic interpretation. Moreover, inclusion of too many parameters can eventually break the non-negativity constraint of ARCH coefficients.

Generalization of ARCH model by Tim Bollerslev (1986) brought solution to some of these problems. (Brooks, 2008, p.392)

3.2 GARCH models

As well as the ARCH model, the GARCH uses the concept of the conditional variance. Its own estimate from previous period is added to the variance equation. This means that the estimate of conditional variance is a weighted average of three components. First, it depends on the constant which stands for the long-run average variance; second, on the estimate of the conditional variance from the previous period; third, on the error from the previous period. The former can be understood as an information correction: the piece of information that was missing when the previous forecast was made. (Engel, 2003, p.328) Assuming previous setting with zero conditional mean ($\mu = 0$), the GARCH(p,q) is specified as Model 2.

Model 2: GARCH(p,q)

$$r_t = z_t\sqrt{h_t}$$
$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-1}^2 + \sum_{j=1}^{q} \beta_j h_{t-1} \tag{3.32}$$

where r_t stands for return , $z_t \sim \text{i.i.d}(0,1)$, $\alpha_0 > 0; \alpha_i \geq 0$ for $\forall i = 0,1,2,\dots,p$; $\beta_j \geq 0$ for $\forall j = 0,1,2,\dots,q$ and $\sum_{i=1}^{\max(p,q)}(\alpha_k + \beta_k) < 1$ assumimng $\alpha_i = 0$ for $i > p$ and $\beta_j = 0$ for $j > q$.

The popular version of GARCH(p,q) model is restricted to $p = q = 1$ and is known as GARCH(1,1). The model is specified as Model 3.

Model 3: GARCH(1,1)

$$r_t = z_t\sqrt{h_t}$$
$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1} \tag{3.33}$$

where r_t stands for return , $z_t \sim i.i.d(0,1)$, $\alpha_0 > 0; \alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$.

Non-negativity of coefficients and the restriction that their sum must be lower than one ensure that the model is stationary and that volatility tends to reverse back to unconditional variance which in case of GARCH(1,1) equals to $\text{var}(r_t) = \frac{\alpha_0}{1-(\alpha_1+\beta_1)}$. (Bollerslev 1986, p. 311) This feature helps to explain the volatility clusters and volatility persistence and enables the passage from periods of high volatility to the periods with relatively low volatility through the mechanism of mean reversion.

GARCH models gained wide popularity for their simplicity and for their easy interpretation. It can be shown, that GARCH model can represent a restricted infinite order ARCH model.[4] An important property of the GARCH model, as well as of the ARCH model, is that it implies features of leptokurtic distribution through its higher moments (excess kurtosis). (Milhoj (1984), Bollerslev (1986, p. 313)) It is also argued that the GARCH model doesn't capture all of the leptokurtosis in the unconditional distribution of returns. Thanks to this problem, the residuals of the GARCH model don't necessarily have to be normally distributed. This has certain implications for the estimation of models as the normality is assumed. As a consequence, the standard error estimates are inappropriate and different variance-covariance matrix (robust to non-normality) must be used. For this purpose Quasi-maximum likelihood method is used. (Brooks, 2008, p. 399)

As mentioned before, the estimation of coefficients is done through the maximalization of log-likelihood function given by following equation:

$$logL(\{r_t\}_{t=}^n; \theta) = \sum_{t=}^n \log f(r_t|\mathcal{F}_{t-1}) \tag{3.34}$$

Assuming the Gaussian specification of z_t of GARCH(1,1) specified as Model 3, we get following form of log-likelihood function.

$$l(r) = -0{,}5 \sum_{t=1}^n \left(\log(2\pi) + \log(h_t) + \frac{r_t^2}{h_t} \right) \tag{3.35}$$

GARCH models earned high popularity for their simplicity and ability to accommodate many stylized fact of financial time-series. Last but not least, they offer a very flexible framework which enables to adapt the original setting and capture various features of time-series. As result of this, family of GARCH models has grown significantly in last years. For example, the TGARCH deals with the impact of the asymmetric information on volatility; the IGARCH deals with the unit-root variance (coefficient $\alpha_1 + \beta_1 = 1$). One of the new applications of GARCH models is Realized GARCH model proposed by Hansen (2011) which is one of the key interests of this thesis.

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[4] For further details see Brooks (2008, p. 393)

3.3 Realized GARCH

Realized GARCH models are a natural extension of GARCH models. The embedment of realized measures into the GARCH framework offers a great possibility of model improvement as the realized measures contain much more information about volatility than the squared daily returns traditionally used in the variance equation of the (G)ARCH models. As a consequence of low information content derived only from the squared daily returns, GARCH models suffer from inability to accommodate rapid changes of volatility. In other words, the GARCH models, due to the slow adjustment of conditional variance, require a long period to adjust to the new level of volatility. As the availability of high frequency data increased in past years, the inclusion of realized measures of volatility was a natural step. Before formulation of the Realized GARCH model, several models based on realized measures were designed. For example, Engle (2002) proposed the GARCH-X model which treated realized measures as exogenous variable. Following Hansen (2011), the Realized GARCH(1,1) with linear specification is specified as Model 4.

Model 4: Realized GARCH (1,1) with linear specification

$$
\begin{aligned}
r_t &= z_t\sqrt{h_t} \\
h_t &= \alpha_0 + \beta h_{t-1} + \gamma x_{t-1} \\
x_t &= \xi + \varphi h_t + \tau(z_t) + u_t
\end{aligned}
\tag{3.36}
$$

where r_t is the return, x_t is realized measure of volatility, $z_t \sim i.i.d(0,1)$, $u_t \sim i.i.d(0, \sigma_u^2)$ with u_t and z_t mutually independent, $h_t = var(r_t|\mathcal{F}_{t-1})$ with $\mathcal{F}_t = \sigma(r_t, x_t, r_{t-1}, x_{t-1}, \dots)$ and $\tau(z_t)$ is leverage function.

The Realized GARCH(1,1) consists of three equations. First two equations very closely remind the GARCH framework. The first one stands for the mean equation. As the conditional mean is assumed to be zero, returns are determined directly by the conditional volatility and z_t. The second GARCH equation includes the realized measure such as realized variance or bipower variation. As discussed before, this feature is very important as the realized measures include more information about daily volatility than the daily squared returns. Of course, this framework also enables to transform realized measures. For the Realized GARCH(1,1) and the Realized Jump GARCH(1,1) we will restrict ourselves to usage of realized measures of volatility estimators. For example, to estimate the Realized GARCH(1,1) we will use the realized volatility, the square-root of bipower variation, ect.

The third equation is called measurement equation and allows the realized measure to be interpreted as the measure of conditional variance. If x_t is a consistent measure of integrated variance, the measurement equation then illustrates close relationship

between the realized measure and the conditional variance: simply because the integrated variance can be perceived as the conditional variance plus random innovation. The measurement equation also enables us to overcome a possible measurement bias caused by different time spans of the conditional variance and the realized measure. The conditional variance (in case of close-to-close returns) covers period of 24 hours meanwhile the realized measure covers only the daily trading hours. Assuming that the time spans mismatch, the coefficient φ is expected to be smaller than one. Another important feature of the measurement equation is the leverage function because it captures the dependence between returns and future volatility. Its inclusion enables asymmetric reaction of volatility to negative shocks in returns which followingly, through the realized measure, translates into higher levels of the conditional variance. For the purpose of this thesis, we restrict ourselves to the simple log-linear specification of the Realized GARCH(1,1) with Gaussian innovations which is given by following specification.

Model 5: Realized GARCH (1,1) with log-linear specification

$$r_t = z_t\sqrt{h_t}$$
$$\log(h_t) = \alpha_0 + \beta\log(h_{t-1}) + \gamma\log(x_{t-1}) \tag{3.37}$$
$$\log(x_t) = \xi + \varphi\log(h_t) + \tau_1 z_t + \tau_2 z_t^2 + u_t$$

where r_t is the return, x_t is realized measure of volatility, $z_t \sim i.i.d(0,1)$, $u_t \sim i.i.d(0,\sigma_u^2)$ with u_t and z_t mutually independent, $h_t = var(r_t|\mathcal{F}_{t-1})$ with $\mathcal{F}_t = \sigma(r_t, x_t, r_{t-1}, x_{t-1}, \dots)$ and $\tau_1 z_t + \tau_2 z_t^2$ is leverage function.

One of the attractive features of log-linear specification is (as Hansen (2011) shows) that it preserves the ARMA structure (each variable from Model 5 can be expressed as the ARMA model). This property is shared with G(ARCH) models which implies that the GARCH nomenclature is in case of Realized GARCH appropriate. Moreover, logarithmic specification has an obvious advantage that it ensures the positive levels of conditional variance. Also, existence of zero returns doesn't cause any specification problems as they don't enter the GARCH equation. Last but not least, The Realized GARCH specification induces flexible stochastic volatility structure. This is obvious if the measurement equation is plugged into the GARCH equation. We get:

$$\log(h_t) = \mu + \pi\log(h_{t-1}) + \gamma\tau(Z_{t-1}) + \gamma u_{t-1} \tag{3.38}$$

where $\mu = \alpha_0 + \xi\gamma; \pi = \beta + \gamma\varphi$.

This result implies that the future level of conditional variance is determined by its previous value, the leverage effect and additional stochastic component given by u_{t-1}.

In Hansen (2011), the Quasi-maximum likelihood analysis of the Realized GARCH specification is provided including asymptotic properties of the quasi-maximum likelihood estimator (QMLE) which is used for the parametric estimation. The structure of QMLE is similar to the GARCH framework but requires additional inclusion of realized measures through the factorization of joint conditional density. The log-likelihood function is given by

$$logL(\{r_t, x_t\}_{t=1}^n; \theta) = \sum_{t=1}^n logf(r_t, x_t | \mathcal{F}_{t-1}) \tag{3.39}$$

Because the standard GARCH model doesn't include the realized measures, the factorization of the joint conditional density is needed in order to be able to compare the Realized GARCH model to the standard GARCH. We get

$$f(r_t, x_t | \mathcal{F}_{t-1}) = f(r_t | \mathcal{F}_{t-1}) f(x_t | r_t, \mathcal{F}_{t-1}) \tag{3.40}$$

Then the partial log-likelihood $l(r) = \sum_{t=1}^n logf(r_t | \mathcal{F}_{t-1})$ can be used for the comparison with the standard GARCH models. Supposing Gaussian specification for z_t and u_t, the joint likelihood is split into sum $l(r_t, x_t) = l(r) + l(x|r)$ where:

$$\begin{aligned} l(r) &= -\frac{1}{2} \sum_{t=1}^n \left[\log(2\pi) + \log(h_t) + \frac{r_t^2}{h_t} \right] \\ l(x|r) &= -\frac{1}{2} \sum_{t=1}^n \left[\log(2\pi) + \log(\sigma_u^2) + \frac{u_t^2}{\sigma_u^2} \right] \end{aligned} \tag{3.41}$$

(For further details see Hansen 2011).

To conclude, the Realized GARCH offers a very flexible and general framework that brings significant improvements to the GARCH models. It enables the usage of different realized measures and can be further generalized to the Realized GARCH(p,q) specification. Moreover, this framework offers a possibility of multi-period forecasts thanks to the full description of dynamic properties of realized measures. Furthermore, the analysis of skewness and kurtosis of cumulative returns produced by the Realized GARCH provided by Hansen (2011) suggests that Realized GARCH can produce very strong and persistent skewness and kurtosis.

3.4 Realized Jump GARCH

The Realized GARCH framework enables to incorporate realized measures into the GARCH model. In the first chapter which was dedicated to the realized measures, we introduced a methodological approach which enabled us to decompose the volatility into its integrated and jump part. Referring to equation (2.23), The realized variance

provides us with a consistent measure of the quadratic variation. The bipower variation, on the other hand, provides a measure of the integrated variance robust to jumps. Following the methodology proposed by Barndorff-Nielsen (2006), we are able to decompose the volatility process into integrated and jump variation through the test statistic Z_t (equation (2.25)). Consequently, being equipped with measures of integrated and jump variation, their inclusion into the Realized Jump GARCH seems as a natural step. The Realized Jump GARCH model was proposed in Baruník and Vácha (2012). The basic idea is to incorporate the measures of the integrated variation and the jump variation into the variance equation. Although authors used this framework within wavelet decomposition analysis, their approach is suitable for the purpose of this thesis as well. The Realized Jump GARCH model can be formalized as an extended version of Model 5.

Model 6: Realized Jump GARCH(1,1) with log-linear specification

$$r_t = z_t\sqrt{h_t}$$
$$\log(h_t) = \alpha_0 + \beta\log(h_{t-1}) + \gamma\log(x_{t-1}) + \gamma_J\log(1 + x_{t-1}^J) \qquad (3.42)$$
$$\log(x_t) = \xi + \varphi\log(h_t) + \tau_1 z_t + \tau_2 z_t^2 + u_t$$

where r_t is the return, x_t is a realized measure of integrated variation given by (2.27), x_t^J is a realized measure of jump variation given by equation (2.26)[5], z_t and u_t are ussumed to be mutually independent and $z_t \sim i.i.d(0,1)$, $u_t \sim i.i.d(0, \sigma_u^2)$.

If the estimation of the Realized Jump GARCH model results with significant coefficients then the jumps have a significant impact on future volatility. The estimation of this model is done via QMLE.

Taking the existence of jumps in return process into account, adjustment to the original return series has to be performed. Fleming and Paye (2010) focus on the problem of distribution of returns. If the returns follow pure diffusion process given by equation (2.10) (there are no jumps present), then $\frac{r_t}{\sqrt{IV_t}} \sim N(0,1)$. Essencially, in this setting, the integrated variance equals the realized variance and provides a measure of the quadratic variation process. If we on the other hand assume the existence of jumps in the returns, the integrated variance provides only "a part of the story" as the jump variation isn't captured. This is indeed an important finding for the estimation of the Realized Jump GARCH model because we assume $z_t \sim i.i.d(0,1)$.

---

[5] Both measures are constructed from series of bipower variation and realized variance. As we noted before, we restricted ourselves to usage of realized measures of volatility (squared-root of realized measure of variance). Therefore in case of Realized Jump GARCH (1,1), both series x_{t-1}(integrated variance) and x_{t-1}^J(jump variation) need to be transformed by square-root.

Therefore, the presence of jumps in the return process could significantly violate this assumption.

Andersen (2010b) provides methodology to subtract the jumps from the return process. The framework is based on the asymptotic distribution theory in Barndorff-Nielsen (2006) presented in part 2.5. The test statistic Z_t enables us to find the jump contribution to the overall variation of a certain trading day. If the contribution is non-zero, we are sure that there are jumps present in the return process of that day. As jumps are usually associated to an extreme development in the return process, the maximum return can be considered as a jump. If we subtract this return from the daily returns, the test statistic Z_t over remaining subset from the previous step should suggest zero contribution of the jump variation. If we still reject the null hypothesis, we must repeat this procedure sequentially until the testing suggests that there are no more jumps in the daily returns series. Eventually, we arrive with the sum of daily jumps and we are able to create series of adjusted returns. Following Andersen (2010b, p.224), we can formalize previous section followingly. Assuming that there is J jumps during day t with T observations, jumps are defined as:

$$\hat{\kappa}_{t,i} \equiv r_{t,j_i}, \quad for \ i = 1, \dots, J, t = 1, \dots, T \tag{3.43}$$

where j_i denotes the exact interval of the intra-day return associated with the i-th jump $\hat{\kappa}_{t,i}$. Jump-adjusted daily retun $\widehat{R}_t$ is therefore given by:

$$\hat{R}_t = R_t - \sum_{i=1}^{J} \hat{\kappa}_{t,i}, \quad for \ t = 1, \dots, T \tag{3.44}$$

3.5 HAR models

In previous sections, we have described the family of GARCH models based on the concept of the conditional variance. These models have been often criticized for their inability to replicate the main features of financial data or for their difficult estimation. As an opposition to these models, Corsi (2004) suggests a different approach: the construction of proxy of latent volatility (realized measure of volatility) which enables a creation of simple conditional volatility model – The Heterogenous Autoregressive model of the Realized Volatility (HAR-RV).

The basic notion of this model was inspired by the Heterogenous market Hypothesis Müller et al. (1993) arising from an empirical observation of positive correlation between the volatility and the market presence: the more traders active within the market, the larger volatility. That directly contradicts to the assumption of homogenous market framework because identical traders would be more likely agree on the true price of an asset. The more traders would be active, the lower the

volatility would be: simply because of the fact that the consensus about the true price of an asset would be reached faster. Therefore the homogenous market hypothesis implies that negative correlation between the volatility and market presence. Consequently, rejecting the homogenous market hypothesis, the heterogeneity of market is proposed. This also suggests that different market participants have different investment horizons. In other words, different groups of agents have different trading strategies, different risk profiles and therefore different investment horizons. Market players, in other words, operate under different dealing frequencies. For example, market makers and intraday speculators operate on higher frequencies than pension fund. Naturally, their reaction to news and price developments differs. All in all, volatility can be decomposed into different components: short-term covering the daily frequencies, medium-term covering weekly frequencies and long-term with characteristic time of one or more months. (Corsi, 2004, p.8)

Moreover, the heterogeneous market framework enables to study interactions between volatility within different time horizons. It is empirically observed that volatility over longer time-span influences short-term volatility and that there is a certain hierarchic relationship of volatility which has a clear economic interpretation. Short-time traders are directly influenced by the long-time volatility because it reveals long-run trends which help them to form expectations about future. Any change in the long run volatility therefore means revision of these expectations and adjustments to trader's positions. In other words, the current volatility is closely determined by the changes in volatility over longer time horizons. Converse relationship is negligible. Long-term strategies are not likely to be changed by the short-term turbulences. (Corsi, 2004, p.9) Accepting the notion of the heterogonous market framework and the hierarchical structure of volatility, proposition of the HAR models seems to be natural. Following the definition of models is based on Corsi (2004).

3.6 HAR-RV model

Let's define partial volatility $\tilde{\sigma}_t^{(\cdot)}$ as volatility generated by a certain market component. The model is structured as a hierarchical process where at each level of the cascade the future partial volatility depends on historical observed volatility within the same time scale as well as on the partial volatility at the higher scale of the cascade (longer horizon). Moreover, let's assume three components of the model: the partial volatilities respective to time horizons of one day $\tilde{\sigma}_t^{(d)}$, one week $\tilde{\sigma}_t^{(w)}$ and one

month $\tilde{\sigma}_t^{(m)}$. Furthemore, let's assume that the return process is determined directly by the highest frequency component in the cascade with $\tilde{\sigma}_t^{(d)} = \sigma_t^{(d)}$ the daily integrated volatility. Then we have

$$r_t = \sigma_t^{(d)} \varepsilon_t \tag{3.45}$$

where ε_t is assumed to be NID(0,1).

Incorporating ideas from previous paragraphs, we can formalize the model as "AR(1)" where partial volatility process (unobserved) depends on previous experienced values of realized volatility at the same time scale. Moreover it also includes expectation about the future values of volatility at higher level of the cascade (longer term partial volatilities). Therefore we get a system of three equations:

$$\tilde{\sigma}_{t+1m}^{(m)} = c^{(m)} + \phi^{(m)} RV_t^{(m)} + \tilde{\omega}_{t+1m}^{(m)} \tag{3.46}$$

$$\tilde{\sigma}_{t+1w}^{(w)} = c^{(w)} + \phi^{(w)} RV_t^{(w)} + \gamma^{(w)} E_t\left[\tilde{\sigma}_{t+1m}^{(m)}\right] + \tilde{\omega}_{t+1w}^{(w)} \tag{3.47}$$

$$\tilde{\sigma}_{t+1d}^{(d)} = c^{(d)} + \phi^{(d)} RV_t^{(d)} + \gamma^{(d)} E_t\left[\tilde{\sigma}_{t+1w}^{(w)}\right] + \tilde{\omega}_{t+1d}^{(d)} \tag{3.48}$$

where $RV_t^{(m)}, RV_t^{(w)}$ and $RV_t^{(m)}$ are respectively monthly, weekly and daily ex-post observed realized volatility measures defined as an average of daily realized volatilities constructed in following matter:

$$RV_t^{(w)} = \frac{1}{5}\left(RV_t^{(d)} + RV_{t-1}^{(d)} + \cdots + RV_{t-4}^{(d)}\right)$$
$$RV_t^{(m)} = \frac{1}{22}\left(RV_t^{(d)} + RV_{t-1}^{(d)} + \cdots + RV_{t-21}^{(d)}\right) \tag{3.49}$$

and $\omega_{t+1m}^{(m)}$, $\omega_{t+1w}^{(w)}$ and $\omega_{t+1d}^{(d)}$ are contemporaneously and serially independent zero mean nuisance with appropriately truncated left tail so the positivity of partial volatility is ensured. Substituting into equation (3.48), we get following equation which can be viewed as three factor stochastic volatility model where each factor stands for the past realized volatility viewed from different frequencies.

$$\sigma_{t+1}^{(d)} = c + \beta^{(d)} RV_t^{(d)} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \tilde{\omega}_{t+1d}^{(d)} \tag{3.50}$$

Moreover, realized volatility can be treated as a measure of latent volatility process. Therefore we can specify it by following equation:

$$\sigma_{t+1}^{(d)} = RV_{t+1}^{(d)} + \omega_{t+1d}^{(d)} \tag{3.51}$$

where $\omega_{t+1d}^{(d)}$ stands for latent daily volatility measurement error.

Equation (3.51) enables us to eventually obtain functional form for time series model in terms of realized volatilities by substituting into equation (3.50). Eventually, we define the HAR-RV model.

Model 6: HAR-RV

$$RV_{t+1}^{(d)} = c + \beta^{(d)} RV_t^{(d)} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \omega_{t+1} \qquad (3.52)$$

where $\omega_{t+1} = \widetilde{\omega}_{t+1d}^{(d)} - \omega_{t+1d}^{(d)}$.

Eventually, we obtained a parsimonious model which is easily estimated by OLS method and which has a clear economic interpretation. Andersen (2007) addresses the issue of adjustment of standard errors due to the serial correlation of error term. They propose usage of Bartlett/Newey-West heteroskedasticity consistent covariance matrix estimator with 5 lags. This approach is also applied in the present thesis.

3.7 HAR-RV-CJ model

Natural extension of the HAR-RV model is to account for existence of jumps. Here we can rely again on the methodology proposed by Barndorff-Nielsen (2006) presented in section 2.5. Usage of the realized variance and the bipower variation enables us to decompose the quadratic variation into its jump and the integrated component through the construction of test statistic Z_t. Following the notation of section 2.5, we construct following series.

$$
\begin{aligned}
J_t^{(w)} &= \frac{1}{5}\left(J_t^{(d)} + J_{t-1}^{(d)} + \cdots + J_{t-4}^{(d)} \right) \\
C_t^{(w)} &= \frac{1}{5}\left(C_t^{(d)} + C_{t-1}^{(d)} + \cdots + C_{t-4}^{(d)} \right) \\
J_t^{(m)} &= \frac{1}{22}\left(J_t^{(d)} + J_{t-1}^{(d)} + \cdots + J_{t-21}^{(d)} \right) \\
C_t^{(m)} &= \frac{1}{22}\left(C_t^{(d)} + C_{t-1}^{(d)} + \cdots + C_{t-21}^{(d)} \right)
\end{aligned}
\qquad (3.53)
$$

Andersen (2007) suggested usage of these series as explanatory variables and proposed the HAR-CJ model defined as Model 7.

Model 7: HAR-RV-CJ

$$
\begin{aligned}
RV_{t+1}^{(d)} = c &+ \beta_C^{(d)} C_t^{(d)} + \beta_C^{(w)} C_t^{(w)} + \beta_C^{(m)} C_t^{(m)} + \beta_J^{(d)} J_t^{(d)} + \beta_J^{(w)} J_t^{(w)} \\
&+ \beta_J^{(m)} J_t^{(m)} + \omega_{t+1}
\end{aligned}
\qquad (3.54)
$$

Framework of HAR models enables us to use various measures of volatility and their transformation. Original version by Corsi (2004) suggests usage of realized measures of variance. Andersen (2007) proposes HAR models which use realized measures of volatility (square-root of the realized measure of variance) as well as logarithmic transformation of realized measures of variance. In this thesis, we restrict ourselves to the former logarithmic specification. Although this specification has less clear economic interpretation, it ensures that the estimated volatility remains strictly non-

negative. This becomes an important issue especially in Model 7. Sufficiently large jump variation could cause problems with the non-negativity constraint of estimated values. Following two models represent logarithmic specification used in this thesis.

Model 8: Logarithmic HAR-RV

$$log\left(RV_{t+1}^{(d)}\right) = c + \beta^{(d)} log\left(RV_t^{(d)}\right) + \beta^{(w)} log\left(RV_t^{(w)}\right)$$
$$+ \beta^{(m)} log\left(RV_t^{(m)}\right) + \omega_{t+1} \tag{3.55}$$

Model 9: Logarithmic HAR-RV-CJ

$$log\left(RV_{t+1}^{(d)}\right) = c + \beta_C^{(d)} log\left(C_t^{(d)}\right) + \beta_C^{(w)} log\left(C_t^{(w)}\right) + \beta_C^{(m)} log\left(C_t^{(m)}\right)$$
$$+ \beta_J^{(d)} log\left(1 + J_t^{(d)}\right) + \beta_J^{(w)} log\left(1 + J_t^{(w)}\right) \tag{3.56}$$
$$+ \beta_J^{(m)} log\left(1 + J_t^{(m)}\right) + \omega_{t+1}$$

3.8 Evaluation of forecasts

Being equipped with a set of different volatility models, a measure to evaluate their performance is needed. In this thesis, we rely on usage of two main approaches, Mincer-Zarnowitz regressions (1969) and Diebold and Mariano test (1995).

First, let's briefly describe the Mincer-Zarnowitz regression which takes following form:

$$V_{t+1}^{(m)} = \alpha + \beta V_t^{(k,m)} + \varepsilon_t \tag{3.57}$$

where $V_{t+1}^{(m)}$ refers to the integrated volatility provided by m-th measure volatility (square root of volatility estimator: Realized variance, Bipower variation, Two-scale Realized variance, C) and $V_t^{(k,m)}$ stands for 1–day ahead forecast of $V_{t+1}^{(m)}$ provided by k-th model based on m-th realized measure.

The Mincer-Zarnowitz regression is based on a simple idea that if a model provides unbiased and efficient forecast, the intercept is zero and insignificant and the coefficient β equals to one. R^2 of regression refers to the amount of volatility variation captured forecast.

Diebold and Mariano test (1995) evaluates two forecasts comparing respective expected losses associated with forecasts and assessing if their difference significantly differs. To further describe this idea; let's assume two forecasts $\{\hat{y}_t^A\}$, $\{\hat{y}_t^B\}$ of time-series $\{y_t\}$ with associated errors $e_t^A = \hat{y}_t^A - y_t$ and $e_t^B = \hat{y}_t^B - y_t$.

Moreover, let function $L(x)$ be a loss function specified as $L(x) = x^2$.[6] Consequently, to test if the expected losses significantly differ, we test under null hypothesis that the expected losses associated with each forecast are equal. Formally written, we test following null hypothesis against an alternative:

$$\begin{aligned} H_0: E\left(L(e_t^A)\right) &= E\left(L(e_t^B)\right) \\ A: E\left(L(e_t^A)\right) &\neq E\left(L(e_t^B)\right) \end{aligned} \qquad (3.58)$$

An asymptotic test is constructed using loss differential defined as the difference between two loss functions. Test statistic S is then defined as

$$S = \frac{\bar{d}}{\sqrt{\frac{\hat{\sigma}_d^2}{T}}} \qquad (3.59)$$

where $\bar{d}$ is mean sample loss differential, $\hat{\sigma}_d^2$ is estimate long-run variance given by $\hat{\sigma}_d^2 = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j$ with $\gamma_j = cov(d_t, d_{t-j})$. Test statistic S is asymptotically distributed with N(0,1).

In this thesis, we used quadratic specification of the loss function. From this perspective, Diebold and Mariano test actually compares Mean Square Errors (MSE) of forecasts and assesses if their difference is statistically significant. Moreover, the sign of the test statistic suggests which forecast provides higher MSE.

To conclude this chapter, Table 3.1 provides overview of models and respective volatility measures. RV refers to the Realized Variance, BP to the Bipower variation, TSRV to the Two-scale realized variance, C to the integrated component of volatility and J to its Jump part. We recall the notation presented in first chapter: the realized measures of variance are expressed in capital letters and realized measures of volatility (squared-root of realized measures of variance) in lower-case.

---

[6] Usage of Diebold and Mariano test enables different specifications of loss function: for example absolute value. In this thesis we restrict ourselves to usage of $L(x) = x^2$.

Table 3.1: Volatility models and respective volatility measures

Model	Equations	Volatility measures
GARCH(1,1)	$r_t = z_t\sqrt{h_t}$ $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1}$	-
Realized GARCH(1,1)	$r_t = z_t\sqrt{h_t}$ $log\,(h_t) = \alpha_0 + \beta log\,(h_{t-1}) + \gamma log\,(x_{t-1})$ $log\,(x_t) = \xi + \varphi log\,(h_t) + \tau_1 z_t + \tau_2 z_t^2 + u_t$	rv, bp, tsrv
Realized Jump GARCH (1,1)	$r_t = z_t\sqrt{h_t}$ $log\,(h_t) = \alpha_0 + \beta log\,(h_{t-1}) + \gamma log\,(x_{t-1}) + \gamma_J log\,(1 + x_{t-1}^J)$ $log\,(x_t) = \xi + \varphi log\,(h_t) + \tau_1 z_t + \tau_2 z_t^2 + u_t$	c, j
HAR-RV	$log\big(RV_{t+1}^{(d)}\big) = c + \beta^{(d)} log\big(RV_t^{(d)}\big) + \beta^{(w)} log\big(RV_t^{(w)}\big)$ $\qquad\qquad + \beta^{(m)} log\big(RV_t^{(m)}\big) + \omega_{t+1}$	RV
HAR-RV-CJ	$log\big(RV_{t+1}^{(d)}\big) = c + \beta_C^{(d)} log\big(C_t^{(d)}\big) + \beta_C^{(w)} log\big(C_t^{(w)}\big)$ $\qquad\qquad + \beta_C^{(m)} log\big(C_t^{(m)}\big) + \beta_J^{(d)} log\big(1 + J_t^{(d)}\big)$ $\qquad\qquad + \beta_J^{(w)} log\big(1 + J_t^{(w)}\big) + \beta_J^{(m)} log\big(1 + J_t^{(m)}\big)$ $\qquad\qquad + \omega_{t+1}$	C, J

4 Data description

Having fully described the theoretical background of realized measures and having introduced the set of models used in this thesis; our attention shall be dedicated to the description of our dataset. This thesis focuses on the estimation of volatility over the triplet of Central and Eastern Europe stock market indices of the Prague Stock Exchange, the Warsaw Stock Exchange and the Budapest Stock Exchange: PX index, WIG 20 Index and BUX Index respectively. Our dataset contains one-minute and five-minute high frequency data.

The title of this thesis suggests our interest in finding whether the decomposition of volatility improves its forecasting. Baruník and Vácha (2012) extended the Realized GARCH framework with the decomposition of volatility into the integrated variance and the jump variation. Using the dataset consisting of currency futures (GBP, CHF, EUR), they show that accounting for jumps existence is very important and that the decomposition of volatility improves its forecasting. Our dataset on the other hand contains three stock market indices; therefore, we are dealing with a high level of information aggregation. The prices changes of constituent stocks are likely to contain jumps. As stock market indices are constructed over the portfolio of constituent shares, it is possible that these individual jumps will not be apparent on an aggregate level. Therefore, significance of jumps in aggregated information (market stock index) would be indeed an interesting finding. Following sections of this chapter will be dedicated to description of datasets respective to indices used in this thesis. Before we do so, let's pay shortly our attention to description of main features of high frequency data and to challenges that their usage provides.

4.1 High frequency data

Thanks to better performance of computers and advances in data storing, the high frequency data became a key interest of economic community. They provide an excellent opportunity to gain more information from the daily time-series. On the other hand, their usage offers challenges to the econometricians as their application requires special treatment. In this thesis, we don't need to deal with these issues as your dataset has been filtered and "adjusted" by the data provider. (www.tickdata.com) Therefore, in this section reasons for this special treatment will

be discussed briefly. Following paragraphs are prevailingly based on the paper High Frequency Data Filtering by Thomas N. Falkenberry (2002).[7]

High frequency data contain information about every trade made on the stock market. The usage of every tick (data point) is problematic for many reasons. First, the data contain errors caused by the data processing such as decimal errors, loses of portion of number, etc. Many of these errors are caused by human factor. Falkenberry (2002) suggests that the more liquid stock (in terms of market capitalization) the higher presence of errors is. Simply larger volume of transaction translates into higher occurrence of these technical errors. Second, different securities have different frequencies of ticks. Naturally, the frequency of the most highly traded assets is higher. Moreover, it appears that the tick frequency varies within the day. Third, bid-ask bounce leaves us with noisy data which cause large systematic bias which can lead to misleading results of analysis. All these problems imply the need for the filtration of the data and their adjustment.

Filtration has many different aspects that need to be taken into account. First, filtration can be done through the identification of "bad" ticks. This task can often be very easy as bad ticks appear as outliers within the plot of the series. On the other hand, treating certain ticks as "bad" can be rather subjective with respect to the trading frequency of the traders. Simply, the main objective of such a filter is to create a clean series which would preserve the statistical features of real-time data and would enable the analysis of series. The challenge of this task can be found in the degree of filtration. If the data are filtered too loosely, the data might still not be usable. On the other hand, over scrubbing data would remove important information and leave us with rather useless dataset. All in all, it is clear that existence of optimal filter is negligible as identification of the bad ticks is rather subjective. Therefore the design of the filter must match the trading profile and trading frequency of the trader. Also, the data filtration needs to account for seasonal differences in tick frequency which varies during the day and also across assets. Fortunately, our dataset contains "clean" series of one-minute and five-minute data and doesn't imply the need to resolve issues sketched above.

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[7] This paper is freely accessible on the webpage of the provider.

4.2 PX Index

The PX index (henceforth PX) is a free-float capitalization-weighted index of the Prague Stock Exchange currently constituted of 14 most liquid stocks traded. The history of the Prague Stock Exchange indices starts in 1994 when PX 50 was founded. This index was calculated based on 50 most important stocks traded. Later in 2006, this index was renamed and transformed into PX index. Currently, constituents of PX are represented by stocks of banking institutions (40%) and companies operating within sector of electric utilities (20%) and insurance corporations (19%).[8]

Our dataset contains one-minute and five-minute high-frequency data covering period from 2.1.2008 to 25.2.2014 which enables us to report about the current development of volatility on the Prague Stock Exchange. Throughout the data set, the number of the daily observations differs. The period from 2.1.2008 to 1.2.2011 provides us with usual 79 daily five-minute observations and 393 one-minute observations as the daily trading hours start at 9:30 and finish at 16:00[9]. On 1.2.2011 the trading hours were extended to 9:15 – 16:20. Therefore, the usual number of daily observations climbed up to 86 and 429 respectively. In our dataset, the number of the daily observations changes due to unusual events (for example due to 31st December). For the sake of conservativeness, we decided to drop these observations as we argue that calculation of realized measures within these days could bring misleading outcomes. Therefore every day which contains less than 90% of usual daily observations is dropped. In case of PX, threshold value for minimal number daily five-minute observations is set to 71 which decrease the number of days in our dataset from 1537 to 1532. As noted before, our dataset contains "clean" data which don't require any special treatment. Therefore, we are able to construct the series of daily realized measures.

First, the dataset is split into cells with daily price observations. From every set of daily five-minute and one-minute prices, logarithmic returns are computed. Let's denote $r_{t,h}$ the intraday logarithmic return over time interval h. Calculating $r_{t,h} = \ln\left(\frac{P_t}{P_{t-h}}\right)$ we get series of intraday five and one minutes logarithmic returns (h=5min and h=1 respectively) from daily prices P_t. The daily return r_t is constructed as an

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[8] http://www.bcpp.cz/

[9] The covered trading period of one-minute data covers approximately the same time span as five-minute data. Usually, 1 minute data are provided with additional 2-3 observations prior the starting time of five-minute data.

open-to-close logarithmic return calculated as the sum of intraday logarithmic returns.[10] Calculation of logarithmic returns finally enables us to construct realized measures described in Chapter 2: series of the daily realized variance (RV_t), bipower variation (BP_t) and two-scale realized variance ($TSRV_t$) and tripower quadricity. Following the methodology presented in section 2.5, we are able to construct series of the continuous sample path of volatility C_t (integrated variance) and the jump variation J_t. To calculate these series, the level of confidence (alfa) was set to 95%. For the purpose of the estimation of the Realized Jump GARCH model, the sum of intraday jumps needs to be calculated. The procedure is done sequentially through the calculation of test statistic Z_t.[11] Series of intraday jumps enables us to construct adjusted daily returns which are used as in input to the Realized Jump GARCH estimation. Figure 4.1 presents QQ plot of daily returns and adjusted returns. It proves that the procedure to obtain the sum of intraday jumps is important. It shows that standardization after accounting for intraday jumps matches the standard normal quantiles more closely. Moreover, it also shows that returns standardized by realized volatility are approximately normal.

Figure 4.1 QQ plot: Comparison of daily returns and adjusted returns PX
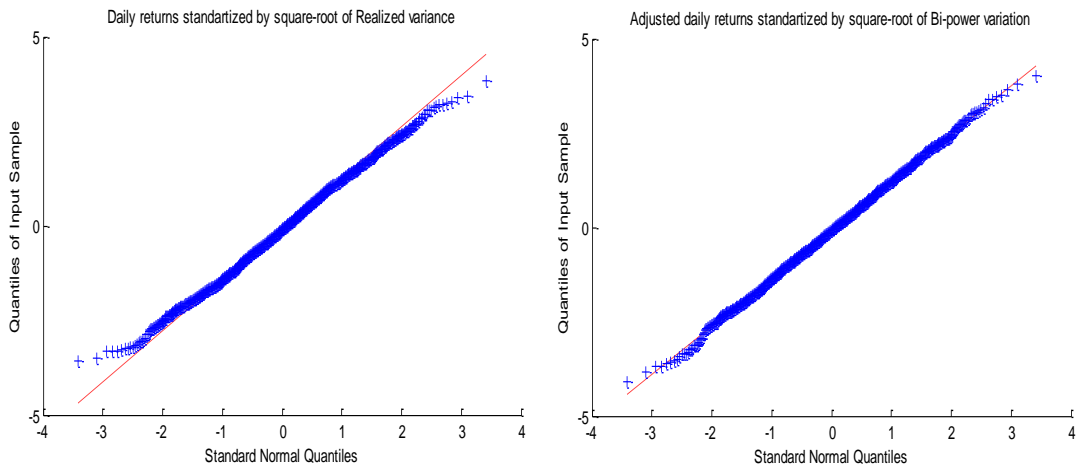
Source: Authors computations.

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[10] The filtration procedure which eliminates days with insufficient number of intraday observation doesn't cause any difficulties for construction of daily returns series. As open-to-close daily returns are computed as a sum in intraday log-return, for every daily realized measure we are left with respective daily open-to-close return.

[11] Procedure is in detail described in section 3.4.

To complete the data description, Table 4.1 provides reader with descriptive statistics of variables used for estimation of models.

Table 4.1: Descriptive statistics of PX

Notation of variables: $r_{t,h}$: five-minute logarithmic returns; r_t: daily logarithmic returns, r_t^{adj}: daily adjusted logarithmic returns; RV_t: Realized variance; BPV_t: Bipower variation; $TSRV_t$: Two-scale realized variance; C_t: Continuous sample path of volatility (integrated variance); J_t: Jump variation.

	$r_{t,h}$	r_t	r_t^{adj}	RV_t	BPV_t	$TSRV_t$	C_t	J_t
Mean	-4,43E-06	-1,05E-03	-9,53E-04	7,48E-05	5,72E-05	3,40E-05	6,21E-05	1,26E-05
Std. dev	0,002	0,012	0,011	1,27E-04	1,05E-04	5,53E-05	1,11E-04	3,44E-05
Skewness	-5,061	-1,281	-1,166	7,107	7,138	7,027	6,559	11,212
Kurtosis	970,00	14,94	12,84	72,81	70,04	71,61	58,46	195,79
Min	-0,149	-0,123	-0,084	4,40E-06	2,44E-06	1,85E-06	2,44E-06	0
Max	0,090	0,060	0,050	1,83E-03	1,41E-03	7,71E-04	1,41E-03	7,72E-04
Obs.	127747	1532	1532	1532	1532	1532	1532	1532

Source: Author's computations.

Figure 4.2 depicts the plot of the realized variance and its further decomposition into the integrated variance and the jump variation. The plot of realized variance shows high levels of volatility in year 2008. Turbulences caused by financial crises are clearly visible. Another period of higher volatility can be seen in year 2011 which reflects the consequences of the European debt crisis. It must be noted that in terms of magnitude of the volatility increase, the crisis of 2011 didn't live up to the large volatility levels of year 2008.

Plot of the integrated variance (continuous part of RV) shows the close relationship between the integrated variance and the realized variance. It is apparent that the integrated variance dominantly contributes to the total variation. As this feature is similar across all three indices, we decided to publish only plots of the integrated variance and the jump component in case of WIG and BUX.

The plot of jump contribution to the total variation suggests that the jumps are more likely to appear during the periods of higher volatility. Jumps occurred on 773 trading days which accounts for 50,46%. In addition for the purpose of estimation, the last step is to standardize every series by its standard deviation.

Figure 4.2: Plot of Realized variance and its components for PX
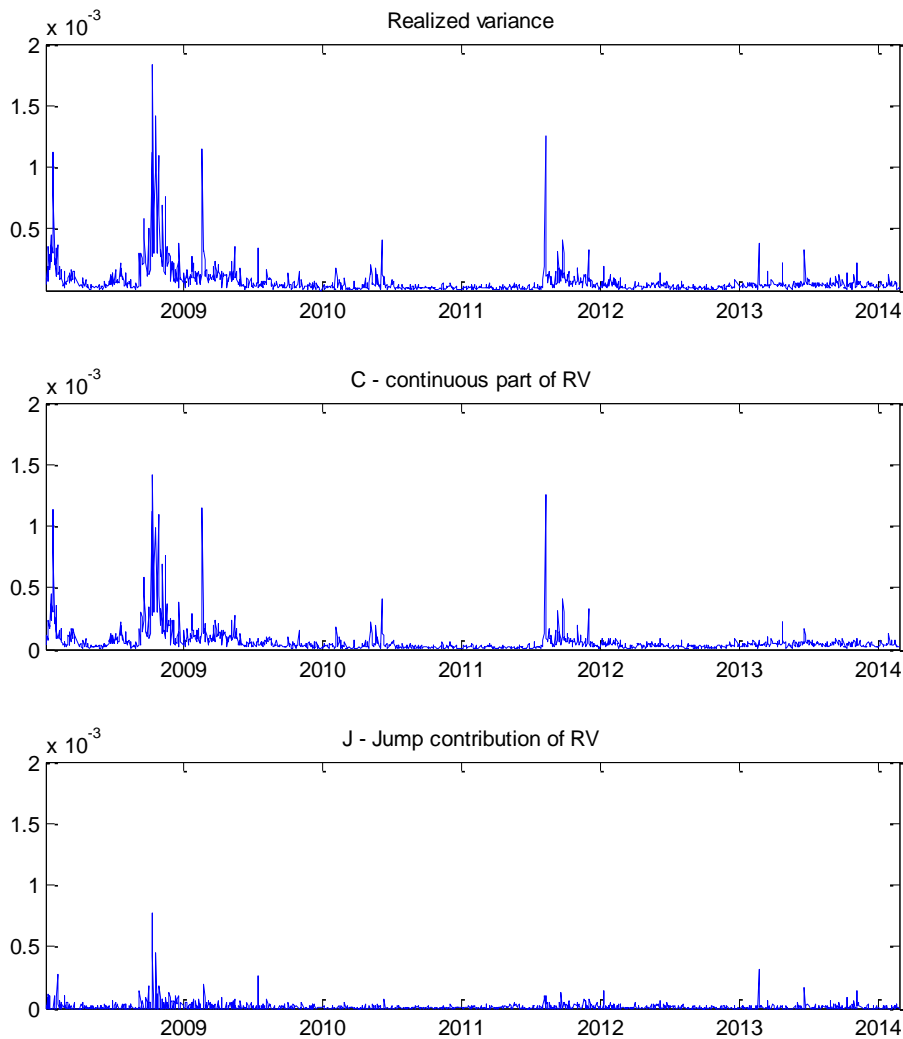
Source: Author's computations.

4.3 WIG 20 Index

The WIG 20 Index (henceforth WIG) is a capitalization-weighted index which is composed by 20 the most liquid stocks traded on the Warsaw Stock Exchange. WIG has been calculated since year 1994 and is based on prices of constituent shares. Their inclusion is restricted by the rule that no more than 5 companies from the single exchange sector can be represented.[12]

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[12] http://www.gpw.pl/

Our dataset includes one-minute and five-minute high frequency data covering the period from 2.1.2008 to 30.6.2011. Unfortunately, in case of WIG, dataset covering the same time span as PX and BUX isn't provided by the data provider. Therefore, we decided to estimate the model on the shorter time span. The dataset includes records of trading prices between 9:35 and 16:10[13] which provides us with usual 80 observations (five-minute data) and 399 observations (one-minute data) a day. As in case of PX, the number of daily observations differs through the dataset which requires us to remove days with insufficient number of observations. To exclude these days the threshold minimum number of observations was set to minimum of 72 daily five-minute observations. This elimination procedure decreased the number of days in our dataset from 878 days to 874.

Figure 4.3 QQ plot: Comparison of daily returns and adjusted returns WIG
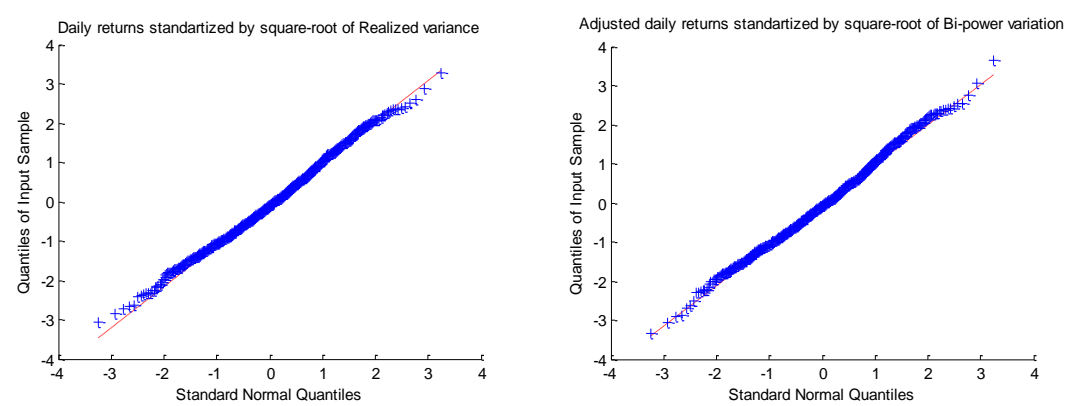
Source: Authors computations.

The calculation of time-series of realized measures is done in the same way as in case of PX. We will rely on the description provided in previous chapter and restrict ourselves to only comment on the main features of the data. Figure 4.3 shows comparison of QQ plots showing the effect of jumps extraction from the return process. As well as in case of PX, we can see that the adjusted daily returns provide better fit to the quantiles of normal distribution. The improvement can be seen especially in the left tail of the distribution.

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[13] The covered trading period of one-minute data covers approximately the same time span as five-minute data. Usually, the 1 minute data are provided with additional 3 observations prior the starting time of five-minute data.

Table 4.2: Descriptive statistics of WIG

Notation of variables: $r_{t,h}$: five-minute logarithmic returns; r_t: daily logarithmic returns, r_t^{adj}: daily adjusted logarithmic returns; RV_t: Realized variance; BPV_t: Bipower variation; $TSRV_t$: Two-scale realized variance; C_t: Continuous sample path of volatility (integrated variance); J_t: Jump variation.

	$r_{t,h}$	r_t	r_t^{adj}	RV_t	BPV_t	$TSRV_t$	C_t	J_t
Mean	-2,99E-06	-7,90E-04	-5,37E-04	1,84E-04	1,57E-04	9,19E-05	1,70E-04	1,38E-05
Std. dev	0,002	0,015	0,014	2,47E-04	2,15E-04	1,15E-04	2,43E-04	3,60E-05
Skewness	-0,428	-0,122	-0,083	5,044	5,072	5,199	5,391	3,899
Kurtosis	92,670	6,507	7,123	39,086	38,562	44,638	43,006	21,821
Min	-0,055	-0,082	-0,082	1,18E-05	1,08E-05	5,43E-06	1,18E-05	0
Max	0,052	0,064	0,064	2,69E-03	2,32E-03	1,42E-03	2,69E-03	3,18E-04
Obs.	7,01E+04	874	874	874	874	874	874	874

Source: Author's computations.

Figure 4.4: Plot of components of total variation for WIG
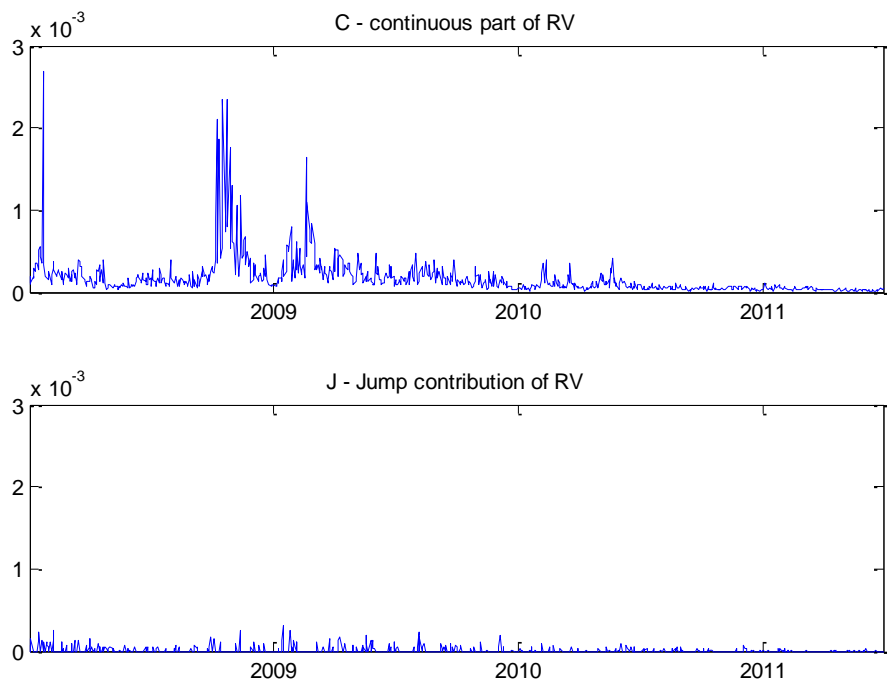
Source: Author's computations.

To complete the data description, Table 4.2 provides reader with descriptive statistics of variables used for estimation of models. Figure 4.4 shows the decomposition of total variation given by the realized variance into the integrated variance and the jump variation. The volatility of WIG develops in similar ways as PX. The period of the highest volatility is located around the end of 2008 and the beginning of 2009. As in the case of PX, high volatility can be dedicated to the implications of global financial crisis. The difference can be found in the jump variation process. Its magnitude doesn't seem to increase during periods of high volatility levels as in case

of PX. Jumps don't show any clear pattern and don't seem to follow the same development as integrated variance: its development seems to be rather stochastic. Jump variation can be observed on 211 days which contributes to 24,14% of the total number of days in the dataset. Based on these findings, we can expect that the models accounting for existence of jumps could bring less clear results in terms of contribution of jumps then in case of PX.

4.4 BUX Index

Last index in our dataset is the Hungarian BUX index (henceforth BUX) calculated at The Budapest Stock Exchange since 1991. It is constructed as a capitalization-weighted index adjusted for free float. It records the performance of largely traded stocks which means that underlying constituent stocks vary. The maximum number of included stock is 25 and 58% of shares account the domestic equity.[14]

The dataset includes price records from 2.1.2008 to 25.2.2014 which accounts for 1527 trading days. Between 2.1.2008 and 2.12.2010 the dataset includes daily price records covering trading period 9:05-16:30 for 5 minute data.[15] This leaves us with usual 90 and 449 daily five-minute and one-minute daily price records respectively. On 2.12.2010 the trading hours were extended by 30 minutes which extends the trading period covered in our dataset to 9:05-17:00. This increases the usual number of daily records to 96 (five-minute) and 479 (one-minute) observations. As in previous case, the number of daily observation varies which requires us to enforce the same filtration procedure as described before. We decided to set 81 daily five-minute observations as our threshold value to eliminate the days with insufficient number daily observations. This leaves us with 1523 trading days in our sample.

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[14] http://bse.hu

[15] The covered trading period of one-minute data covers approximately the same time span as five-minute data. Usually, the 1 minute data are provided with additional 3 observations prior the starting time of 5 minute data.

Figure 4.5 Realized measures over whole dataset BUX
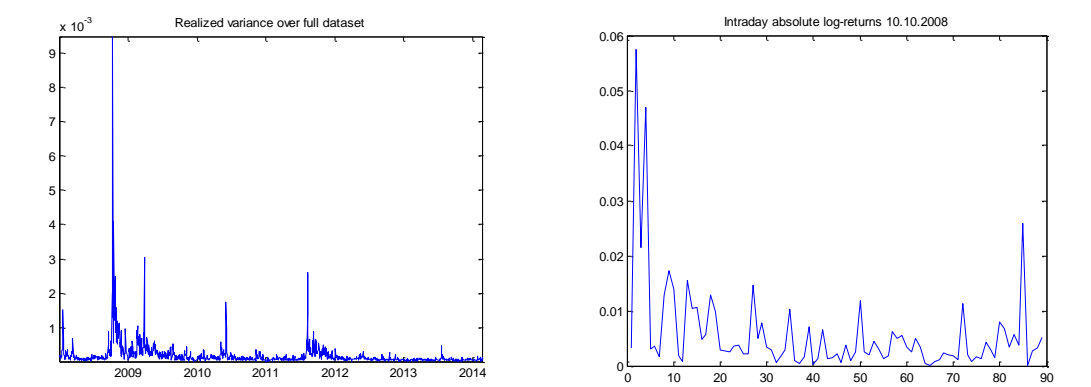
Source: Author's computations.

Having prepared the data, estimation of realized measures and daily open-to-close returns follows as in previous cases. The construction of realized measures of the BUX index revealed that the series of realized measures includes an outlier observation: 10.10.2008. Figure 4.5 offers detailed analysis of this problem. Its left plot shows the plot of the realized variance calculated over the whole unfiltered sample. The maximum value of the realized variance in our sample took place on 10.10.2008. Its value clearly and remarkably exceeds all other values of the realized variance. It equals to 0.0095. To offer suitable comparison, if we remove this daily observation, the maximum value of realized variance equals 0.004. This would suggest that this trading day experienced very high levels of volatility. The right part of Figure 4.5 reveals opposite. The average absolute logarithmic returns resulting from intraday trading were quite low with exception of two exceptionally large returns which took place right after the opening of trading. Existence of these two observations then causes that the daily realized variance provides misleading information about the daily volatility level. Therefore, we decided to remove this daily observation form our dataset as an outlier which leaves us with the final number of daily observations equal to 1522.

Figure 4.6 QQ plot: Comparison of daily returns and adjusted returns BUX
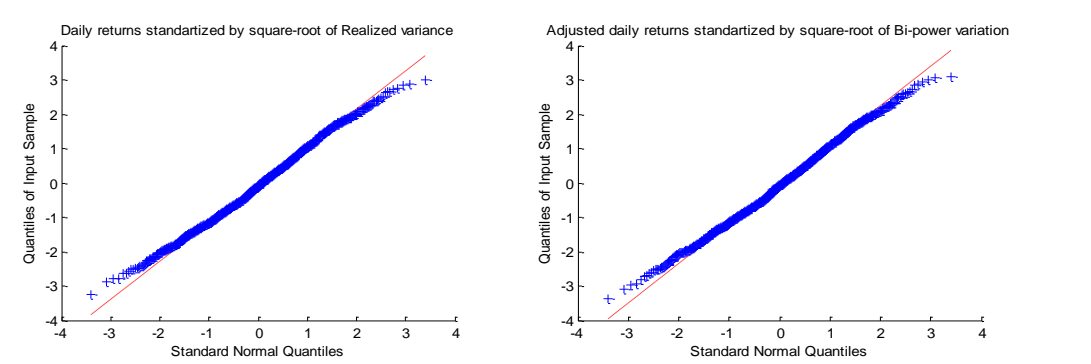
Source: Authors computations.

Figure 4.6 shows consequences of the construction of adjusted daily returns. In contrast to QQ plots of previous indices, we cannot graphically distinguish significant improvement. This can be caused by relatively low numbers of jumps within the series. The estimation of test statistic Z_t leaves us with 279 days which have significant jump contribution (18,33% of all days in our datasample).

Table 4.3: Descriptive statistics of BUX

Notation of variables: $r_{t,h}$: five-minute logarithmic returns; r_t: daily logarithmic returns, r_t^{adj}: daily adjusted logarithmic returns; RV_t: Realized variance; BPV_t: Bipower variation; $TSRV_t$: Two-scale realized variance; C_t: Continuous sample path of volatility (integrated variance); J_t: Jump variation.

	$r_{t,h}$	r_t	r_t^{adj}	RV_t	BPV_t	$TSRV_t$	C_t	J_t
Mean	-2,69E-06	-9,61E-04	-8,28E-04	1,75E-04	1,54E-04	8,85E-05	1,65E-04	1,04E-05
Std. dev	0,002	0,015	0,014	2,57E-04	2,20E-04	1,23E-04	2,32E-04	6,51E-05
Skewness	-1,069	-0,450	-0,429	6,832	6,312	7,174	6,156	22,399
Kurtosis	219,176	7,570	7,672	72,250	61,950	78,804	58,548	643,016
Min	-0,096	-0,103	-0,103	1,90E-05	1,65E-05	1,25E-05	1,65E-05	0
Max	0,069	0,067	0,067	0,004	0,003	0,002	0,003	0,002
Obs.	141981	1522	1522	1522	1522	1522	1522	1522

Source: Author's computations.

Figure 4.7: Plot of components of total variation for BUX
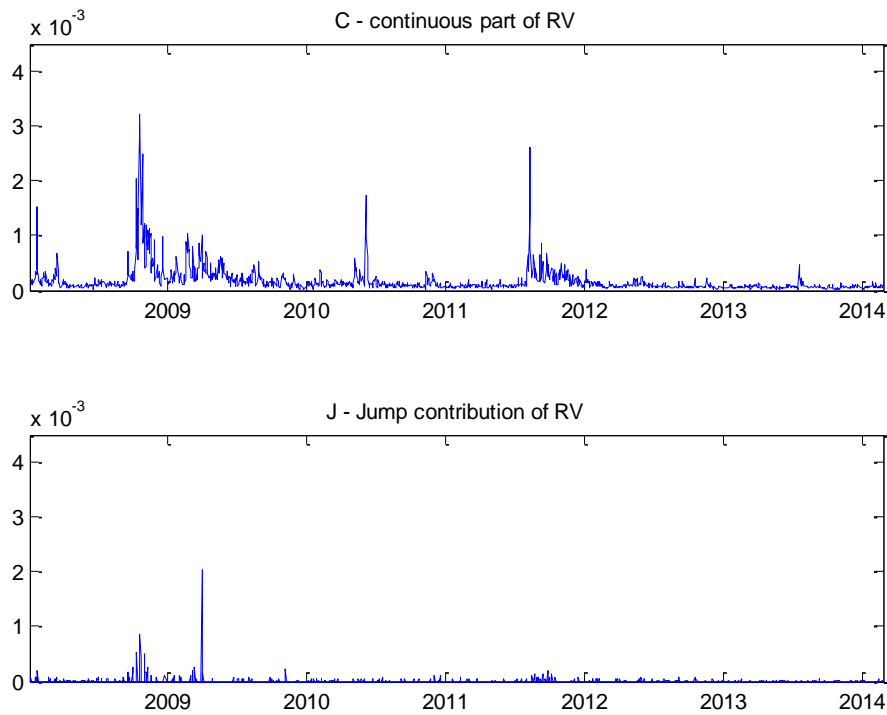
Source: Authors computations.

For the completeness of the data description, Table 4.3 provides descriptive statistics of variables used for estimation of models. Figure 4.7 shows the decomposition of total variation given by the realized variance into the integrated variance and the jump variation. It suggests similar findings as in previous cases. We can see large levels of volatility in late 2008 and beginning of 2009. In 2011, consequences of the European debt crises are observable in volatility cluster covering the second half of year. The jump variation component seems to have significant impact on the volatility, especially in periods of high volatility. As a consequence, we can expect the significant role of jumps resulting in significant coefficients of estimated volatility models.

5 Discussion of estimation results

Eventually, attention of last chapter is dedicated to the description of estimation results. But before we do so, let's briefly remind what the main motivations of this thesis are. First, the main focus of this thesis is nested in volatility estimation and exploration of applicability of two main groups of volatility models: GARCH and HAR models and their forecasting performance.

Second, we claim that realized measures are very important for understanding volatility, its estimation and its forecasting. Therefore, this thesis aims to test, if inclusion of realized measures of volatility into the GARCH framework brings additional significant information and if this information contributes to the improvement of forecasting.

Third, theory behind realized measures allows us to decompose volatility into its continuous sample path component (integrated variance) and the jump variation. Following the motivation behind the title of this thesis, we ask whether the volatility decomposition brings improvements to fits of volatility models and whether it improves their forecasting performance. In other words, we want to explore if the coefficient of jump component is significant. Furthermore, methodology of the volatility decomposition proposed by Barndorff-Nielsen (2006) is applicable within both, the Realized GARCH and the HAR, frameworks. Therefore, we aim to find out if and to which extent inclusion of measures of the jump variation into volatility models brings improvement to their forecasting.

Fourth, we would like to determine the impact of jumps in a price process of a stock market index on its future volatility. The role of jumps in a single asset price process has been investigated in recent years, and its existence and importance has been confirmed by various studies. Naturally, jumps in a return process of a single asset are more apparent compared to the jumps in a return series of an index which is constructed as a weighted average of constituent stock. Simply jumps in individual shares can be too small to be apparent on an aggregate level. On the other hand, large portion of constituent shares can commonly reflect arrival of new information relevant for their prices in similar way. Therefore jumps can occur and they are likely to be apparent even on the aggregate level. For this reasons, jumps are likely to have significant but limited influence of on the volatility of stock market index.

To conclude, in following sections of this chapter, we present results of the parametric estimation of volatility models presented in Table 3.1: GARCH, the Realized GARCH(1,1), the Realized Jump GARCH, the HAR-RV and the HAR-RV-CJ.[16] GARCH(1,1) model is estimated as a benchmark model. The Realized GARCH models then enable us to find out if inclusion of realized measures into the GARCH framework improves the model fit and its forecasting performance. Eventually, having estimated the Realized Jump GARCH model we find an answer to question whether jumps matter. Last but not least, HAR-RV and HAR-RV-CJ models are estimated in order to provide a benchmark to forecasting performance of the GARCH models.

Following text is divided into three sections covering the estimation over three stock market indices: PX, WIG and BUX. For each dataset, we provide a reader with a complex table which enables direct comparison of estimated models and their forecasting performance. In upper panel, estimated values of coefficients are presented with standard errors and respective values of t-statistics. Notation of coefficients follows the convention used in Table 3.1. For each model nested in the GARCH framework, the value of log-likelihood and partial log-likelihood is provided. HAR models are provided with the value of R^2. The lower panel contains the comparison of out-of-sample forecast - results of Mincer and Zamarowitz test and values of Mean Square Errors of forecast.

The computations were done using software MATLAB and MFE Toolbox by Kevin Sheppard. The estimation of Realized (Jump) GARCH model was performed using code provided by J. Baruník which was originally used in Baruník and Vácha (2012).

5.1 PX Index

5.1.1 GARCH models

Results of estimation are presented in Table 5.2. First, our attention shall be dedicated to group of GARCH models. The GARCH(1,1) is estimated as a benchmark model. We can see that the coefficients of the model are highly significant and we can observe high dependence of the current conditional volatility on its past estimates.

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[16] Table 3.1 also includes detailed information about specification of model and about the transformation of realized measures.

The Realized GARCH(1,1) models' coefficients are significant in all cases. It suggests that inclusion of realized measures into the model captures additional information about the conditional volatility. As mentioned before, this finding arises from the fact that the GARCH(1,1) captures volatility only from the daily data. Clearly, the information about intraday volatility is not in this case negligible. Therefore, inclusion of realized measures brings additional information about intraday volatility and can be considered as beneficial. To provide benchmark between different realized measures, the Realized GARCH models were estimated using rv (realized volatility), bpv (square-root of bipower variation) and $tsrv$ (two scale realized volatility). Suitable comparison to the GARCH(1,1) is provided by the values of partial log-likelihood function denoted as $l(r)$. Compared to the GARCH(1,1), the Realized GARCH models provide comparably better fit. The best obtained result is given by the Realized GARCH using the square-root of bipower variation as realized measure. Also, significance of coefficients τ_1 and τ_2 shows that inclusion of the leverage function into the measurement equation provides additional explanatory power to the model.

Summary of the Realized Jump GARCH estimation is presented in column (5). As in previous case, we can see that the coefficients of realized measures are significant. Interestingly, the coefficient of the jump variation is significant as well. This is a very interesting result as we are estimating model on the stock market index where jumps are likely to be less apparent compared to a single asset. In other words, it seems that jumps aren't negligible in volatility modeling. Compared to the benchmark GARCH(1,1), the improvement of the model fit is clearly obvious as the value of partial log-likelihood function is the highest among all GARCH models.

Despite the fact that coefficients of realized measures are similar across all models, values of log-likelihoods $l(r, x)$ vary among specifications with different realized measures. Amongst all GARCH models, the values are approximately around similar levels which would suggest that Realized Jump GARCH, doesn't bring significant improvement to the data fit. On the other hand, this can be caused by the fact, that the role of jumps in PX index volatility matter but isn't crucial due to the aggregation of information within stock market index. To confirm the usefulness of the Realized Jump GARCH specification we need to look at forecasting performance.

Out-of-sample forecasting was done on the basis of rolling sample starting on 24.10. 2012. To evaluate these forecasts, Mincer-Zarnowitz test, Mean Square Errors (MSE) and the Diebold and Mariano test were used. The results suggest that the Realized GARCH (using bpv) and the Realized Jump GARCH provide best forecasting in

terms of coefficients and R^2 which means that they can capture the largest portion of volatility variation. MSE are significantly smaller compared to the standard GARCH(1,1). Comparison between the Realized GARCH and the Realized Jump GARCH models in terms of MSE shows that their differences are statistically insignificant. (Table 5.1)

5.1.2 HAR models

The summary of estimation of the HAR-RV and the HAR-RV-CJ with logarithmic specification is presented in column 6 and 7 of Table 5.2. In case of the HAR-RV model, we can see that coefficients of all investing horizons (daily, weekly and monthly) are significant and that the largest weight which determines the current volatility is attributed to the coefficient $\beta^{(w)}$ (weekly average realized variance). HAR-RV-CJ model shows similar pattern. The measures representing the continuous sample path of volatility are significant for all time horizons. The coefficients of measures representing the contribution of the jump variation are insignificant with exception of weekly jump variation average $(\beta_J^{(w)})$. These findings suggest the importance of weekly investment horizon. In terms of increase in R^2, estimation of the HAR-RV-CJ provides only 1,12% increase. The estimation of HAR models supports finding from the previous section that jumps matter but only to limited extent.

Table 5.1 Diebold and Mariano test statistics PX

Diebold - Mariano test statistics: MSE	GARCH	Realized GARCH rv	Realized GARCH bpv	Realized GARCH tsrv	Realized Jump GARCH	HAR-RV	HAR-RV-CJ
GARCH – rv		46,89	52,86	30,13	44,58	45,10	45,69
Realized GARCH – bpv	-46,89		0,00	-0,98	-0,59	5,96	6,22
Realized GARCH – tsrv	-52,86	0,00		-1,00	-0,73	6,61	6,95
Realized GARCH	-30,13	0,98	1,00		0,80	4,43	4,51
Realized Jump GARCH	-44,58	0,59	0,73	-0,80		6,74	7,05
HAR-RV	-45,10	-5,96	-6,61	-4,43	-6,74		1,24
HAR-RV-CJ	-45,69	-6,22	-6,95	-4,51	-7,05	-1,24	

Source: Authors computations.

The forecasting performance of HAR models provides worse outcomes compared to GARCH forecasts. Although they provide lower values of MSE, the variation

captured by the forecast (in terms of R^2 of Mincer-Zarnowitz regression) is significantly lower.

Figure 5.1 presents the comparison of out-of-sample forecasting performance of selected models: the GARCH(1,1), the Realized Jump GARCH and the HAR-RV-CJ compared to observed measure of volatility – the realized volatility. We can see that the least accurate fit is provided by the GARCH(1,1). As the model was estimated over the period of structural shifts, such as recent financial crisis, the estimated parameters show high levels of volatility persistence which therefore results into the volatility over prediction in after crisis period. Although the HAR-RV-CJ forecast captures the main trend; it fails to accommodate steep changes in volatility and doesn't capture the variation of volatility to a larger extent. This finding is quite natural considering the high values of coefficients of weekly and monthly averages of realized variance. Realized Jump GARCH on the other hand captures the variation significantly better although its forecasts are slightly upwards shifted.

Figure 5.1 Comparison of out-of-sample forecasting performances PX
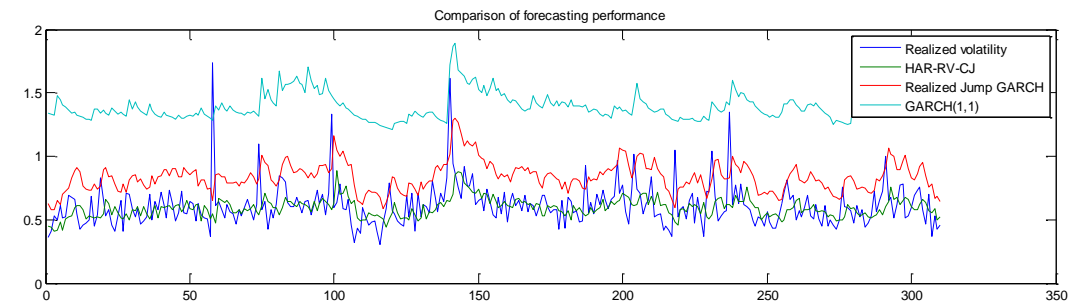
Source: Authors computations.

Table 5.2 Summary of model estimations PX

The upper part of the table includes results of in-sample estimation. Each model is presented with realized measures used. The summary of estimation presents coefficient values, robust standard errors and values of t-statistic. The lower part of the table contains evaluation of out-of-sample forecasts (Mincer-Zarnowitz test and Mean Square Errors).

	GARCH			Realized GARCH			Realized GARCH			Realized GARCH			Realized Jump GARCH			HAR-RV			HAR-RV-CJ		
				rv			bpv			tsrv			c,j			RV			C,J		
	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat
α_0	**0,019**	0,004	4,86	**0,191**	0,020	9,79	**0,200**	0,024	8,51	**0,175**	0,019	9,01	**0,200**	0,024	8,50						
α_1	**0,119**	0,013	9,08																		
β	**0,863**	0,012	73,95	**0,675**	0,024	28,57	**0,658**	0,049	13,53	**0,673**	0,027	24,66	**0,619**	0,027	22,82						
γ				**0,626**	0,047	13,42	**0,613**	0,071	8,60	**0,643**	0,056	11,45	**0,695**	0,048	14,43						
γ_J													**0,114**	0,033	3,42						
ξ				**-0,327**	0,015	-21,45	**-0,355**	0,017	-21,33	**-0,293**	0,015	-19,48	**-0,363**	0,017	-21,43						
φ				**0,470**	0,017	27,22	**0,503**	0,017	30,09	**0,460**	0,016	28,68	**0,482**	0,017	28,90						
τ_1				**-0,015**	0,007	-2,27	**-0,020**	0,007	-2,71	**-0,015**	0,007	-2,19	**-0,022**	0,007	-3,01						
τ_2				**0,062**	0,004	16,26	**0,060**	0,006	10,74	**0,055**	0,004	14,17	**0,054**	0,004	15,26						
c																**-0,185**	0,031	-5,91	**-0,305**	0,070	-4,34
$\beta^{(d)}$																**0,280**	0,036	7,79			
$\beta^{(w)}$																**0,391**	0,050	7,76			
$\beta^{(m)}$																**0,227**	0,050	4,54			
$\beta_C^{(d)}$																			**0,296**	0,033	8,87
$\beta_C^{(w)}$																			**0,313**	0,049	6,39
$\beta_C^{(m)}$																			**0,176**	0,055	3,22
$\beta_J^{(d)}$																			**-0,065**	0,068	-0,95
$\beta_J^{(w)}$																			**0,246**	0,097	2,54
$\beta_J^{(m)}$																			**0,155**	0,123	1,26
l(r)	-1885,4			-1873,3			-1870,5			-1877,0			-1857,1								
l(r,x)				-1900,9			-1952,6			-1889,1			-1990,7								
R^2																59,40%			60,68%		
Mincer and Zarnowitz test																					
α	-0,119			0,210			0,119			0,243			0,156			0,362			0,340		
β	0,519			0,475			0,560			0,474			0,518			0,423			0,458		
R^2	9,53%			12,35%			21,92%			9,90%			19,93%			4,13%			4,11%		
MSE	0,643			0,077			0,078			0,089			0,079			0,028			0,028		

*Source***:** Authors computations.

5.2 WIG 20 Index

5.2.1 GARCH models

The summary of estimation results of the Warsaw stock market index WIG is presented in Table 5.4. Discussion of results will be done in similar way as in case of PX; therefore, we will first focus on the results of GARCH models. First, the GARCH(1,1) model was estimated in order to provide benchmark for other types of GARCH models. As in previous case, GARCH and ARCH coefficients are significant revealing strong dependence in conditional variance. The estimation of the Realized GARCH models shows that the inclusion of realized measures into the GARCH framework provides additional significant information. For all three realized measures, we obtained significant μ coefficients and their value was approximately the same. The values of partial log-likelihoods show that the Realized GARCH models fit the data better than the standard GARCH(1,1). In all three specifications, significant coefficients of leverage function show the asymmetric reaction of volatility to negative shocks.

The Realized Jump GARCH provides additional information about the contribution of the jump variation to the conditional variance. As in case of PX, value of the coefficient of the realized measure of jump variation is significant. This again suggests the importance of jumps in volatility modeling. As in previous case, values of log-likelihood function don't show improvement of fit by the Realized Jump GARCH. Possible explanation can be found in percentage of days with occurring jumps which is significantly lower than in the case of the PX Index. This finding even reinforces conclusions resulting from the case of PX Index. Jumps provide significant information which has due to the high level of information aggregation limited implications.

Lower panel of Table 5.4 provides the evaluation of out-of-sample forecasts calculated on the basis of rolling samples starting on 14.2.2011. Forecasting performances of the Realized GARCH models and of the Realized Jump GARCH model follow similar pattern as in case of the PX Index. Again, R^2 of Mincer-Zarnowitz test is the highest for the Realized Jump GARCH. Similarly, values of coefficients suggest that the Realized Jump GARCH provides the best forecast. Evaluation in terms of MSE suggests that forecasts of the Realized GARCH models and the Realized Jump GARCH outperform those of the standard GARCH(1,1). MSE of the Realized GARCH and the Realized Jump GARCH forecasts are similar although MSE of the Realized GARCH specification with $tsrv$ are signicantly lower (in terms of Diebold-Mariano test statistic – presented in Table 5.3). But on the other

hand, the volume of variation captured by the Realized Jump GARCH is significantly higher compared to the specification with $tsrv$, therefore we suggest that the Realized Jump GARCH provides the best forecasts compared to other GARCH based models.

Table 5.3 Diebold and Mariano test statistics WIG

| Diebold - Mariano test statistics: MSE	GARCH	Realized GARCH RV	Realized GARCH BPV	Realized GARCH TSRV	Realized Jump GARCH	HAR-RV	HAR-RV-CJ
GARCH		62,66	61,28	58,32	57,03	59,35	59,31
Realized GARCH	-62,66		-0,91	2,77	0,08	4,91	4,89
Realized GARCH	-61,28	0,91		3,13	0,43	5,31	5,30
Realized GARCH	-58,32	-2,77	-3,13		-3,73	3,27	3,24
Realized Jump GARCH	-57,03	-0,08	-0,43	3,73		5,39	5,49
HAR-RV	-59,35	-4,91	-5,31	-3,27	-5,39		-0,17
HAR-RV-CJ	-59,31	-4,89	-5,30	-3,24	-5,49	0,17	

Source: Authors computations.

5.2.2 HAR models

The summary of estimation of HAR-RV and HAR-RV-CJ models with logarithmic specification is presented in column 6 and 7 of Table 5.4. In case of the HAR-RV model, we can see similar results as in case of the PX Index: all coefficients are significant. The largest weight is attributed to the weekly average realized variance. The estimation of the HAR-RV-CJ model provides significant coefficients for the continuous sample path of volatility in all investment horizons. Interestingly, the significant coefficients are obtained also for daily and weekly jump variation component. The decomposition of volatility increases R^2 only by 0,13%. This suggests similar results as in case of the GARCH models estimation. Jumps provide significant information about future volatility but its implications are rather limited.

The out-of-sample forecast was done in the same way as in case of PX starting on 8.2.2011. The comparison favors the performance of Realized GARCH models to HAR models. In terms of R^2 of Mincer-Zarnowitz test, Realized GARCH models and the Realized Jump GARCH model outperform both specifications of HAR models. Figure 5.2 provides comparison of forecasting performances of the GARCH(1,1), the Realized Jump GARCH and the HAR-RV-CJ with respect to the observed measure of volatility –the realized volatility. As the forecast period is shorter due to the shorter available dataset, the superiority of the Realized Jump GARCH is graphically less obvious. Generally, we can say the Realized Jump GARCH model captures more of

variation of volatility compared to HAR models and significantly better than the standard GARCH(1,1) model.

Figure 5.2 Comparison of out-of-sample forecasting performances WIG
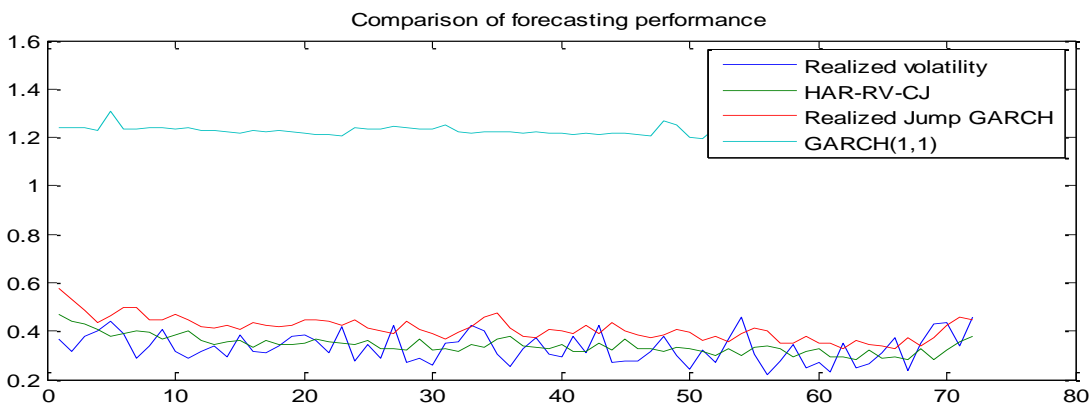
*Source***:** Authors computations.

Table 5.4 Summary of model estimations WIG

The upper part of the table includes results of in-sample estimation. Each model is presented with realized measures used. The summary of estimation presents coefficient values, robust standard errors and values of t-statistic. The lower part of the table contains evaluation of out-of-sample forecasts (Mincer-Zarnowitz test and Mean Square Errors).

	GARCH			Realized GARCH rv			Realized GARCH bpv			Realized GARCH tsrv			Realized Jump GARCH c,j			HAR-RV RV			HAR-RV-CJ C,J		
	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat
α_0	0,000	0,001	0,00	0,122	0,021	5,88	0,143	0,024	5,86	0,091	0,021	4,38	0,132	0,025	5,30						
α_1	0,043	0,006	6,89																		
B	0,957	0,005	177,14	0,684	0,028	24,07	0,671	0,030	22,45	0,691	0,023	30,60	0,644	0,033	19,61						
Γ				0,632	0,061	10,32	0,672	0,065	10,34	0,654	0,061	10,78	0,698	0,067	10,43						
γ_J													0,119	0,038	3,12						
Ξ				-0,207	0,021	-9,73	-0,226	0,023	-9,93	-0,153	0,021	-7,21	-0,238	0,023	-10,45						
φ				0,476	0,022	21,54	0,465	0,022	21,23	0,449	0,023	19,36	0,476	0,022	21,97						
τ_1				-0,019	0,008	-2,48	-0,021	0,008	-2,75	-0,028	0,008	-3,31	-0,022	0,008	-2,78						
τ_2				0,060	0,005	11,59	0,055	0,005	10,19	0,048	0,005	9,83	0,059	0,005	11,39						
C																-0,110	0,023	-4,81	-0,152	0,045	-3,39
$\beta^{(d)}$																0,218	0,048	4,52			
$\beta^{(w)}$																0,577	0,071	8,13			
$\beta^{(m)}$																0,156	0,046	3,43			
$\beta_C^{(d)}$																			0,206	0,046	4,45
$\beta_C^{(w)}$																			0,549	0,071	7,79
$\beta_C^{(m)}$																			0,153	0,051	3,02
$\beta_J^{(d)}$																			0,099	0,042	2,38
$\beta_J^{(w)}$																			0,177	0,075	2,37
$\beta_J^{(m)}$																			0,041	0,092	0,44
l(r)	-1076,8			-1068,1			-1067,6			-1068,0			-1058,6								
l(r,x)				-982,1			-995,0			-944,5			-986,1								
R^2																75,75%			75,88%		
Mincer and Zarnowitz test																					
α	-0,578			0,121			0,131			0,188			0,115			0,246			0,237		
B	0,761			0,530			0,511			0,460			0,528			0,255			0,276		
R^2	10,56%			19,40%			17,78%			10,39%			20,14%			2,44%			2,93%		
MSE	0,762			0,012			0,012			0,008			0,013			0,004			0,004		

Source: Authors computations.

5.3 BUX Index

5.3.1 GARCH models

Last index that was included in our dataset is the Hungarian BUX. Upper panel of Table 5.6 contains the summary of estimation results. The estimates of coefficients of the benchmark model GARCH(1,1) presented in first column are highly significant. As in previous cases, value of beta coefficient shows strong influence of past estimate of the conditional variance in estimating its current levels. The estimation of the Realized GARCH models results in significant coefficients for all realized measures. Values of coefficients are approximately the same. We can also see that compared to the GARCH(1,1), coefficient of lagged conditional variance β decreases. This change is caused by the fact that realized measure of volatility provides additional explanatory power relevant for derivation of current level of the conditional variance. The Realized GARCH models also account for higher values of partial log-likelihood function for all three specifications.

Estimation of coefficients of the Realized Jump GARCH suggests significant role of jumps, as the coefficient γ_J is significant. As in case of WIG, the number of the days with occurring jumps is quite low. This could be one of the reasons why, also in case of BUX, log-likelihood function doesn't bring in case of the Realized Jump GARCH significant improvement of the model fit.

Table 5.5 Diebold and Mariano test statistics BUX

Diebold-Mariano test statistics: MSE	GARCH	Realized GARCH RV	Realized GARCH BPV	Realized GARCH TSRV	Realized Jump GARCH	HAR-RV	HAR-RV-CJ
GARCH		76,40	73,57	71,09	69,03	72,28	72,20
Realized GARCH - RV	-76,40		0,46	2,13	-0,41	5,87	5,80
Realized GARCH - BPV	-73,57	-0,46		1,80	-0,50	5,28	5,21
Realized GARCH – TSRV	-71,09	-2,13	-1,80		-4,93	5,29	5,23
Realized Jump GARCH	-69,03	0,41	0,50	4,93		9,68	9,63
HAR-RV	-72,28	-5,87	-5,28	-5,29	-9,68		-0,83
HAR-RV-CJ	-72,20	-5,80	-5,21	-5,23	-9,63	0,83	

Source: Authors computations.

Lower panel of Table 5.6 contains evaluation of out-of-sample forecasts constructed on rolling sample basis starting on 4.12.2012. In terms of Mincer-Zarnowitz test, results suggest that the Realized Jump GARCH outperforms other GARCH models in terms of R^2 but also in terms of other coefficients. MSE of forecasts suggest that the GARCH(1,1) is outperformed by both Realized GARCH and Realized Jump GARCH

models. Differences between MSE of Realized GARCH and Realized Jump GARCH models are similar although MSE of specification with $tsrv$ is significantly lower. (Table 5.5) Although forecasting performance of Realized GARCH models and the Realized Jump GARCH model is comparable, it seems that accounting for jumps has some beneficial effect on forecasting.

5.3.2 HAR models

The estimation of the HAR-RV model provides us with significant coefficients for all time horizons. Compared to previous indices, the largest contribution of current volatility is given by the short-term horizon – the daily lagged realized variance. Coefficients of the HAR-RV-CJ model suggest interesting result. Meanwhile the coefficient of the continuous sample path of volatility are highly significant, coefficients of the jump variation are insignificant. As mentioned previously, the number of days in our dataset, when jumps occurred, is compared to other indices quite small. Also, the decomposition of volatility doesn't increase R^2 of regression which suggests that volatility decomposition for purpose of HAR modeling doesn't provide improvement of the model.

Lower panel of Table 5.6 shows evaluation of forecasts and suggests similar findings as in case of previous indices. The Realized Jump GARCH model outperforms the GARCH(1,1) and the HAR-RV-CJ especially in terms of variation of volatility it captures. Although the forecast is slightly upwards shifted, it accommodates changes in volatility quite well. The HAR-RV-CJ on the other hand rather "smoothes" the time-series and fails to accommodate the fast changes in volatility levels.

Figure 5.3 Comparison of out-of-sample forecasting performances BUX
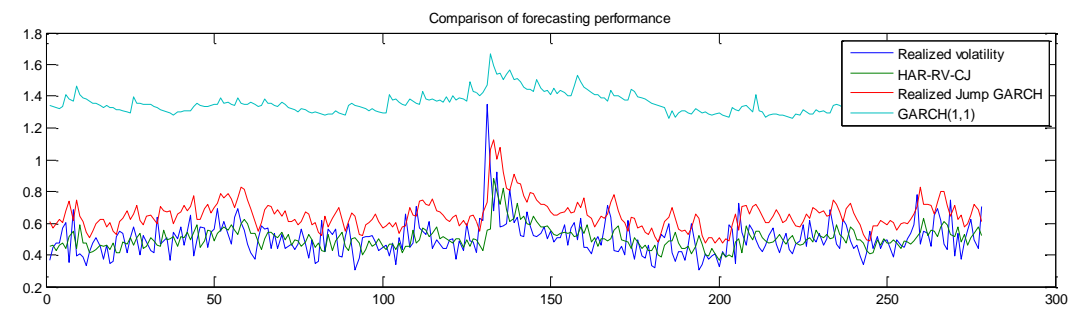
Source: Authors computations.

Table 5.6 Summary of model estimations BUX

The upper part of the table includes results of in-sample estimation. Each model is presented with realized measures used. The summary of estimation presents coefficient values, robust standard errors and values of t-statistic. The lower part of the table contains evaluation of out-of-sample forecasts (Mincer-Zarnowitz test and Mean Square Errors).

	GARCH			Realized GARCH rv			Realized GARCH bpv			Realized GARCH tsrv			Realized Jump GARCH c,j			HAR-RV RV			HAR-RV-CJ C,J		
	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat	coeff.	s.e.	t-stat
α_0	**0,011**	0,003	3,79	**0,194**	0,023	8,49	**0,170**	0,020	8,54	**0,177**	0,084	2,11	**0,160**	0,023	7,08						
α_1	**0,076**	0,008	9,83																		
β	**0,913**	0,009	98,49	**0,549**	0,031	17,97	**0,574**	0,026	22,37	**0,537**	0,108	4,95	**0,552**	0,030	18,21						
γ				**0,868**	0,066	13,12	**0,815**	0,037	21,93	**0,955**	0,315	3,03	**0,840**	0,060	14,08						
γ_J													**0,126**	0,032	3,99						
ξ				**-0,243**	0,016	-15,35	**-0,227**	0,017	-13,20	**-0,203**	0,027	-7,43	**-0,226**	0,017	-13,39						
φ				**0,468**	0,021	22,68	**0,471**	0,021	22,85	**0,437**	0,052	8,49	**0,476**	0,019	25,35						
τ_1				**-0,029**	0,006	-5,28	**-0,028**	0,006	-4,65	**-0,028**	0,005	-5,57	**-0,027**	0,005	-5,18						
τ_2				**0,053**	0,004	14,58	**0,052**	0,004	13,61	**0,044**	0,004	12,25	**0,052**	0,004	14,31						
c																**-0,108**	0,024	-4,46	**-0,190**	0,032	-5,91
$\beta^{(d)}$																**0,389**	0,040	9,62			
$\beta^{(w)}$																**0,295**	0,051	5,78			
$\beta^{(m)}$																**0,238**	0,039	6,10			
$\beta_C^{(d)}$																			**0,364**	0,041	8,97
$\beta_C^{(w)}$																			**0,310**	0,054	5,73
$\beta_C^{(m)}$																			**0,219**	0,042	5,22
$\beta_J^{(d)}$																			**0,039**	0,052	0,75
$\beta_J^{(w)}$																			**0,069**	0,100	0,69
$\beta_J^{(m)}$																			**0,081**	0,094	0,85
l(r)	-1940,6			-1916,3			-1916,6			-1922,9			-1925,8								
l(r,x)				-1632,1			-1686,8			-1547,7			-1688,3								
R^2																68,22%			68,18%		
Mincer and Zarnowitz test																					
α	-0,336			0,190			0,197			0,246			0,175			0,243			0,251		
β	0,615			0,484			0,484			0,455			0,515			0,516			0,500		
R^2	12,38%			15,52%			14,02%			14,38%			15,75%			9,38%			9,12%		
MSE	0,750			0,032			0,031			0,025			0,033			0,012			0,012		

Source: Authors computations.

6 Conclusion

The present thesis focuses on application of realized measures of volatility in volatility model estimation and volatility forecasting in case of three Central and Eastern Europe stock market indices: PX Index (Prague Stock Exchange), WIG 20 Index (Warsaw Stock Exchange) and BUX (Budapest Stock Exchange). Our dataset includes high-frequency one-minute and five-minute data, which allows investigation of the applicability of various realized measures in different groups of volatility models: GARCH models and HAR models.

The main motivation of this thesis was to explore the impact of jumps in price process to the future volatility of the stock market index. Despite the fact that many studies have confirmed the importance of jumps in a price process to the future volatility of a single asset, in the case of the stock market index, the role of jumps remains unclear. The natural expectation would suggest that the role of jumps in aggregated information (stock market index) would be lower relative to the single asset, as the individual jumps may not be apparent on the aggregate level. On the other hand, arrivals of new information with macroeconomic nature could induce jumps in large portion of constituent stocks and be apparent even on an aggregate level. To sum up, the aim of this thesis is to find out whether jumps in a price process of the stock market index have an impact on its future volatility and whether accounting for the jump component of volatility improves its forecasting.

In previous chapters, theoretical backgrounds of realized measures were presented. After the initial price process setup, the primary attention was dedicated to decomposition of return process, which allowed proper definition of quadratic variation and derivation of realized variance - realized measure, which enabled us to construct ex-post non-parametric estimates of volatility. Initial framework which assumed continuous sample path of price process was further extended and discontinuities caused by jumps were taken into account. Such approach enabled us to extend the portfolio of realized measures used in this thesis with bipower variation - realized measure robust to jumps. The former concept together with methodology proposed by Barndorff-Nielsen and Sheppard (2006) then enabled us to statistically distinguish the continuous part and jump component of volatility.

The primary attention was dedicated to the applicability of realized measures within GARCH and HAR frameworks. GARCH models offer a very popular and flexible

framework that has been recently extended by Hansen (2011), who proposed the Realized GARCH framework. This modification enables inclusion of different realized measures into the standard GARCH framework. Baruník and Vácha (2012) further develop the former concept and introduce the Realized Jump GARCH model, which enables incorporation of both measures of integrated variance and jump variation. HAR models proposed by Corsi (2004) directly build on the usage of realized measures and explain the current level of volatility by volatilities over different time horizons (daily, weekly and monthly). This basic HAR framework was extended by Andersen (2007), who incorporated decomposition of volatility into its continuous sample path of volatility and jump component.

Eventually, estimation of the portfolio of volatility models enabled us to answer the question proposed in the title of this thesis: Can decomposition of volatility improve its forecasting? Before we give a clear answer, we will briefly discuss estimation results and comment on them. First, estimation of Realized GARCH models shows significant improvement in all aspects of estimation compared to standard GARCH(1,1). Our expectation that inclusion of realized measures into the GARCH framework extends explanatory power and goodness of fit was confirmed across all data samples. Moreover, estimation of Realized Jump GARCH revealed, also across all datasets, significant coefficients of realized measures of jump variation. This finding is indeed interesting, as the price process of index suffers from a higher level of information aggregation that would suggest rather low importance of jumps. Although the significance of coefficients suggests that jumps significantly contribute to the volatility of stock indices, in terms of goodness of fit the message remains unclear.

Estimation of HAR-RV and HAR-RV-CJ provided similar findings. In the case of PX Index and WIG 20 Index, jump components were significant. In case of BUX, these coefficients were insignificant. This suggests similar findings to those sketched above. Jumps matter, but only to a certain extent.

Comparison of an out-of-sample volatility forecast revealed that forecasting performance of Realized GARCH and Realized Jump GARCH models is superior to forecasts provided by HAR-RV and HAR-RV-CJ models. Generally, Realized (Jump) GARCH models are able to explain more of volatility variation and accommodate moves in volatility levels. HAR models, on the other hand, fail to do so. Although their forecasts capture the main volatility trends, they fail to accommodate steep changes in volatility and do not capture the variation of volatility to a larger extent. We claim that this finding is quite natural considering the high

values of coefficients of weekly and monthly averages of realized variance, which imply the impossibility to accommodate fast changes in volatility levels.

To provide a reader with an answer from the title of this thesis: Can decomposition of volatility improve its forecasting? Based on evidence from three Central and Eastern Europe stock market indices, we can conclude that decomposition of volatility brings certain improvement to its forecasting using Realized Jump GARCH models. On the other hand, these improvements are rather small, and Realized Jump GARCH provides comparable or only slightly better forecasts than Realized GARCH models.

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