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BACHELOR THESIS

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Generalized Metric and Gravity

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Názov práce: Zovšeobecnená Metrika a Gravitácia

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Abstrakt: Na základe znalostí z diferenciálnej geometrie je predstavená zovšeobecnená geometria. V dôsledku symetrií tejto novej geometrie sa prirodzene vynára B-pole známe z teórie strún. Taktiež bola zkonštruovaná zovšeobecnená metrika pozostávajúca z klasickej metriky a už spomínaného B-poľa. Hore uvedené štruktúry umožňujú zaviesť konexiu na zovšeobecnenej geometrii a rozvinúť Riemannovskú zovšeobecnenú geometriu. Nahradením obyčajnej krivosti za zovšeobecnenú v Einstein-Hilbertovej akcii dostávame akciu nápadne podobnú bozónovej časti akcie supergravitácie.

Kľúčové slová: zovšeobecnená metrika, Courantová zátvorka, B-pole, Einstein-Hilbertová akcia

Title: Generalized Metric and Gravity

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Abstract: Based on the knowledge from differential geometry, the generalized geometry is introduced. As a consequence of the symmetries in this new geometry, a B-field, known from the string theory, inherently emerges. Generalized metric based on ordinary metric tensor and the B-field will be established as well. This allows to construct connection in the framework of generalized geometry and develop a Riemannian generalized geometry. From this point, it is a straightforward way to the replacement of an ordinary scalar curvature by the generalized one in Einstein-Hilbert action. Obtained action closely resembles the supergravity action, especially the bosonic part.

Keywords: generalized metric, Courant bracket, B-field, Einstein-Hilbert action

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Prague, May 20, 2014 author signature

To those who have ignited the passion...

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Preface

Generalized geometry has come to light thanks to Nigel Hitchin's work [7, 8] and was further elaborated by his former students: Marco Gualtieri (in his DPhil thesis [6]) and Gil Cavalcanti (in his DPhil thesis [3]). Nowadays, it plays an important role in string theory, particularly in supersymmetric flux compactifications. These compactifications establish a link between 10-dimensional string theory and 4-dimensional worlds. The concept of generalized geometry become a powerful tool describing numerous features of field theories as well.

One may think of generalized geometry as a sort of geometrical structure based on the direct sum of tangent and cotangent bundle $T\mathcal{M} \oplus T^*\mathcal{M}$, alias a generalized tangent bundle $\mathcal{D}T$, treating the tangent and cotangent bundle on the equal footing, hence providing a natural framework for implementing T-duality. As we will see, this structure posses two natural pairing operations (one symmetric, the other skew-symmetric). It is equipped with a bracket operation, called the Courant bracket, as well. In 1997 Zhang-Ju Liu, Alan Weinstein and Ping Xu introduced, see [11], a concept of a Courant algebroid which is perfectly suited for[1] generalized geometry. The classification of exact Courant algebroid leads naturally to a skew-symmetric field, called B-field,[2] with remarkable properties, which we will elaborate.

Surprisingly, the ordinary Riemannian geometry also hints towards introducing the notion of generalized metric. One can explicitly write this metric with the aid of ordinary metric tensor, B-field and natural symmetric pairing [8].

The aim is to describe a construction, starting from generalized geometry, of the gravity action corresponding to the bosonic part of the supergravity action. In order to do so, a connection on generalized tangent bundle has to be introduced first. This can be achieved thanks to the Courant bracket [5]. Hence, the connection introduced in such manner guides us to build up a Riemannian generalized geometry. To be specific, we will describe the corresponding Riemann and Ricci tensors and scalar curvature. Replacing an ordinary scalar curvature R^{LC} in the Einstein-Hilbert action with scalar curvature R extracted from generalized geometry yields to the desired action. Additionally, we work out the equations of motion for metric tensor and B-field.

Organization of the thesis

In the first chapter, we start recalling the most essential structures, identities and objects from differential geometry and we build up a bridge between index and index free notation as well. The chapter continues with various objects of Riemannian geometry as connection and its torsion, Christoffel symbols of the second kind, curvature tensor and its contractions. To the end of the chapter the reader will be acquainted with the Lie algebroid.

In the subsequent chapter, devoted to the generalized geometry, we describe the concept of a natural pairing and its symmetries; introduce the Courant and Dorfman brackets and point out some of their properties. We will also provide the basics of the Courant algebroid. The next object of interest here will be the B-field and its action on sections of the generalized tangent bundle and on the Courant bracket, which will lead us smoothly to the twisting.

The penultimate chapter is dedicated to the Riemannian generalized geometry. It starts with the notion of the generalized metric and continues with the connection in generalized geometry.

[1] Or maybe Hitchin made generalized geometry fit to Courant algebroid.

[2] Also well known as Kalb–Ramond field in string theory. However, we will refer to it as B-field.

The last objective of this chapter is the computation of generalized curvature tensor and other tensors associated with it.

The fourth shortest and very last chapter concerns gravity, Einstein field equations and equations of motion for B-field. These are obtained from the new generalized geometry based action. At the end of this chapter we will uncover an interesting fact that the curvature of an empty space depends on the dimension of this space.

Chapter 1

Introduction

Let us briefly summarize some of the basic notions from differential geometry which are essential for further reading. Reader interested in this topic can find more information in countless books (e.g. [4, 9]) that cover the subject in more detail, thus, proofs are easy to find.

This chapter will begin with differential geometry and fundamental statements necessary for moving towards generalized geometry and its further development.

1.1 Differential Geometry preliminaries

First, we introduce notation. Let $\mathcal{M}$ be a smooth n-dimensional manifold, $T\mathcal{M}$ and $T^*\mathcal{M}$ its tangent and cotangent bundle, respectively. Then let $\Gamma(T\mathcal{M})$ denote the space of smooth sections of $T\mathcal{M}$ and $\Gamma(T^*\mathcal{M})$ the space of smooth sections of $T^*\mathcal{M}$. It is clear that $\Gamma(T\mathcal{M})$ and $\Gamma(T^*\mathcal{M})$ are vector spaces; elements of the first one are vector fields and elements of the second one are 1-forms.

Let $\mathcal{F}(\mathcal{M})$ be the space of smooth (i.e. $C^\infty(\mathcal{M})$) functions on $\mathcal{M}$. Then we can look at $\Gamma(T\mathcal{M})$ as a set of first order differential operators with an action on $\mathcal{F}(\mathcal{M})$. These operators satisfy the Leibniz rule

$$X(fg) = X(f)g + fX(g)\,,$$

where $X \in \Gamma(T\mathcal{M})$ and $f, g \in \mathcal{F}(\mathcal{M})$.

Definition 1.1.1 (Lie Algebra)**.** Lie algebra is a vector space V together with a binary operation $[\cdot,\cdot] : V \times V \to V$, called Lie bracket, satisfying skew-symmetry, bi-linearity and Jacobi identity.

Definition 1.1.2 (Lie Bracket)**.** The Lie bracket is a binary operation on vector fields

$$[\cdot,\cdot] : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})\,,$$

defined by

$$[X, Y] := \mathcal{L}_X Y\,,$$

where $X, Y \in \Gamma(T\mathcal{M})$ and $\mathcal{L}_X$ is the Lie derivative along a vector field X.

The operation from the preceding definition is skew-symmetric, bi-linear[1] and satisfies Jacobi identity. Hence, we have obtained a Lie Algebra.

Definition 1.1.3. One can introduce following operations on k-forms $\Omega^k(\mathcal{M}) = \Gamma(\wedge^k T^*\mathcal{M})$.

[1] On functions it behaves according to the Leibniz rule (i.e. $[X, fY] = X(f)Y + f[X, Y]$, where $f \in \mathcal{F}(\mathcal{M})$.)

i) An exterior differential $\mathrm{d} : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$

$$(\mathrm{d}\omega)(X_0, \ldots, X_k) := \sum_{i=0}^{k} (-1)^i X_i \omega\left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right) + \sum_{0 \leq i \leq j \leq k} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k\right),$$

where $X_i \in \Gamma(T\mathcal{M})$ and $\hat{X}_i$ is omitted.

ii) An interior product $\iota_X : \Omega^{k+1}(\mathcal{M}) \to \Omega^k(\mathcal{M})$

$$\iota_X \omega(X_1, \ldots, X_k) := \omega(X, X_1, \ldots, X_k),$$

where $X, X_i \in \Gamma(T\mathcal{M})$.

iii) A Lie derivative $\mathcal{L}_X : \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$

$$\mathcal{L}_X := \mathrm{d}\iota_X + \iota_X \mathrm{d},$$

where $X \in \Gamma(T\mathcal{M})$.

The aim of a subsequent proposition is to exhibit existing identities among the operations listed above. These will accompany us through large part of the text.

Proposition 1.1.1 (Cartan formulas). *The following identities hold*

$$\begin{aligned} \{\iota_X, \iota_Y\} &= 0, \\ \{\mathrm{d}, \iota_X\} &= \mathcal{L}_X, \\ [\mathrm{d}, \mathcal{L}_X] &= 0, \\ [\mathcal{L}_X, \iota_Y] &= \iota_{[X,Y]}, \\ [\mathcal{L}_X, \mathcal{L}_Y] &= \mathcal{L}_{[X,Y]}, \end{aligned} \tag{1.1}$$

where $X, Y \in \Gamma(T\mathcal{M})$ and $\iota_X, \mathcal{L}_X, \mathrm{d}, [X, Y]$ are as above. Additionally, $\{\cdot, \cdot\}$ denotes the anti-commutator.

The transition between index and index-free notation may be confusing at the beginning, mainly in various operations such as Lie derivative or exterior differential. We will examine these two, particularly, in the next definition and proposition. They later turn out to be of considerable importance.

Definition 1.1.4 (Lie derivative). The Lie derivative of tensor-field A along W in index notation is

$$\mathcal{L}_W A \to W^m A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q, m} - \sum_{i=1}^{p} W^{a_i}{}_{,m} A^{a_1 \ldots m \ldots a_p}{}_{b_1 \ldots b_q} + \sum_{i=1}^{q} W^m{}_{,b_i} A^{a_1 \ldots a_p}{}_{b_1 \ldots m \ldots b_q},$$

where $A \in \mathcal{T}^p_q(\mathcal{M})$, for $\mathcal{T}^p_q(\mathcal{M})$ to be a space of tensor-fields of the type (p,q) on $\mathcal{M}$ and $W \in \Gamma(T\mathcal{M})$.

Proposition 1.1.2. *Let $\alpha \in \Omega^p(\mathcal{M})$, then its components are*

$$(\mathrm{d}\alpha)_{i \ldots jk} := (-1)^p (p+1) \alpha_{[i \ldots j,k]}.$$

Proof. From the definition of forms we get

$$\mathrm{d}\alpha = \frac{1}{(p+1)!}(\mathrm{d}\alpha)_{i\dots jk}\mathrm{d}x^i \wedge \dots \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \,.$$

And the definition of the exterior differential reads

$$\begin{aligned}\mathrm{d}\alpha &= \mathrm{d}\Big(\frac{1}{p!}\alpha_{i\dots j}\mathrm{d}x^i \wedge \dots \wedge \mathrm{d}x^j\Big)\\ &= \frac{1}{p!}\alpha_{i\dots j,k}\mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \dots \wedge \mathrm{d}x^j\\ &= \frac{(-1)^p}{p!}\alpha_{[i\dots j,k]}\mathrm{d}x^i \wedge \dots \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \,.\end{aligned}$$

Equating these two yields the conclusion. □

Choosing an orientation of the basis, we can introduce the Hodge dual also known as the Hodge star operator.

Definition 1.1.5 (Hodge dual)**.** The Hodge dual $*_g : \Omega^P(\mathcal{M}) \to \Omega^{n-p}(\mathcal{M})$ is defined by

$$(*_g\alpha)_{a\dots b} := \frac{1}{p!}\alpha^{c\dots d}\omega_{c\dots da\dots b} \,,$$

where ω_g is a metric volume form, $\alpha \in \Omega^p(\mathcal{M})$ and we rise indices on the p-form with assistance of the inverse metric tensor.

This specific operator provides a way to propose an adjoint operator of the exterior differential d. The new operator is called the codifferential. An index g here represents a conformity with the metric.

Definition 1.1.6 (Codifferential)**.** The codifferential $\delta_g : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ is defined by

$$\delta_g := (-1)^p *_g^{-1} \mathrm{d} *_g \,.$$

The Hodge dual provide a way to establish inner product on p-forms as follows.

Definition 1.1.7 (Inner product on p-forms)**.** We define an inner product on p-forms by

$$(\alpha, \beta)_g\omega_g := \alpha \wedge *_g\beta \,,$$

where $\alpha, \beta \in \Omega^p(\mathcal{M})$ and ω_g is a metric volume form. In indices this reads[2]

$$(\alpha, \beta)_g := \frac{1}{p!}\alpha_{a\dots b}\beta^{a\dots b} \,,$$

where we used the metric tensor to rise the indices on β.

1.2 Connection, Riemann and Ricci

Let us continue with the notion of linear connection. We will do it in a manner of [4]. Properties of this structure sketched here will be helpful later in definition of connection on generalized tangent bundle.

Definition 1.2.1 (Linear connection)**.** Linear connection assigns every vector field $X \in \Gamma(T\mathcal{M})$ an operator ∇_X, called the covariant derivative along the vector field X. This operator has following properties.

[2]For a norm of α we will use $\|\alpha\|^2 := \sqrt{(\alpha, \alpha)_g}$.

i) It is a linear operator on tensor algebra preserving the degree of this algebra

$$\begin{aligned}\nabla_X &: \mathcal{T}^p_q(\mathcal{M}) \to \mathcal{T}^p_q(\mathcal{M})\,,\\ \nabla_X(A+\lambda B) &= \nabla_X A + \lambda \nabla_X B\,,\end{aligned}$$

where $A,\ B \in \mathcal{T}^p_q(\mathcal{M})$ and $\lambda \in \mathbb{R}$.

ii) It behaves according to Leibniz rule on tensor product

$$\nabla_X(A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B)\,,$$

where where $A \in \mathcal{T}^p_q(\mathcal{M})$ and $B \in \mathcal{T}^{p'}_{q'}(\mathcal{M})$.

iii) On function it yields

$$\nabla_X f = X(f) = \mathcal{L}_X f\,,$$

where $f \in \mathcal{F}(\mathcal{M}) \equiv \mathcal{T}^0_0(\mathcal{M})$.

iv) It commutes with contractions

$$\nabla_X \circ C = C \circ \nabla_X\,,$$

where C is an arbitrary contraction.

v) It is $\mathcal{F}$-linear in the direction argument[3]

$$\nabla_{X+fY} = \nabla_X + f\nabla_Y\,,$$

where $f \in \mathcal{F}(\mathcal{M})$ and $Y \in \Gamma(T\mathcal{M})$.

The tensor, we acquaint next, plays a fundamental role in describing how a moving frame twists around a curve.

Definition 1.2.2 (Torsion tensor)**.** The torsion tensor $T : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ is defined by

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]\,,$$

where ∇ is a linear connection on a manifold $\mathcal{M}$ and $X,\ Y \in \Gamma(T\mathcal{M})$. In a coordinate representation (i.e. in index notation) the tensor reads

$$\begin{aligned}\langle \mathrm{d}x^i, T(\partial_j,\partial_k)\rangle &\equiv T^i{}_{jk}\\ &= \Gamma^i_{kj} - \Gamma^i_{jk}\,,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is an ordinary inner product.

The covariant derivative is fully determined by the connection coefficients also known as Christoffel symbols of the second kind. In a frame field[4] e_a of $\mathcal{M}$ it reads

$$\Gamma^c_{ba} e_c := \nabla_a e_b\,, \tag{1.2}$$

where $\nabla_a := \nabla_{e_a}$. The connection is said to be metric when

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,,$$

where $X, Y, Z \in \Gamma(T\mathcal{M})$.

Proposition 1.2.1 (Levi-Civita connection)**.** *The connection that is both metric and torsion-free is unique and called the Levi-Civita connection. We will denote it by* ∇^{LC}.

[3] This is the only property which makes difference between covariant derivative and Lie derivative. (Second one is just $\mathbb{R}$-linear in the direction argument.)

[4] The frame field is a set of orthonormal fields forming a basis for the manifold $\mathcal{M}$.

One can easily see that the Christoffel symbols of the second kind can be decomposed into its symmetric part[5] and skew-symmetric part[6] as follows

$$\Gamma^i_{jk} := \Gamma^i_{(jk)} - \frac{1}{2} T^i{}_{jk} \,. \tag{1.3}$$

It is possible to express exterior differential and codifferential in terms of the covariant derivative. This comes out from subsequent proposition.

Proposition 1.2.2. *Following expressions for the exterior differential* d *and the codifferential* δ *are equivalent to the original definitions.*

$$\begin{aligned} \mathrm{d} &= j^a \nabla^{LC}_a \equiv e^a \wedge \nabla^{LC}_a \,, \\ \delta &= -\iota^a \nabla_a \,. \end{aligned}$$

We will continue with definition of three objects linked to the curvature of a manifold. The first one is a tensor measuring failure of the Riemannian manifold, determined by a given metric tensor, to be locally isometric to the Euclidean space. The next one, again a tensor, presents difference in volume of the geodesic ball in Riemannian manifold and standard ball in Euclidean space. The last one is a number (scalar) assigned to each point on the manifold depending on the intrinsic geometry around that point.

Definition 1.2.3 (Riemann curvature tensor)**.** The Riemann curvature tensor is defined by

$$R(X,Y)Z := \big([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\big) Z \,,$$

where $X, Y, Z \in \Gamma(T\mathcal{M})$. In the index notation

$$\langle \mathrm{d}x^i, R(\partial_k, \partial_l)\partial_j \rangle =: R^i{}_{jkl} \,.$$

Definition 1.2.4 (Ricci curvature tensor)**.** The Ricci curvature tensor is defined by

$$Ric(Z,Y) := \mathrm{tr}\big(X \to R(X,Y)Z\big) \,,$$

where $X \to R(X,Y)Z$ is a map and tr is a trace of this map. In index notation

$$R_{ij} := R^l{}_{ilj} \,,$$

here it was possible to use a shortcut $Ric \to R$ since there is no risk of confusion.

Definition 1.2.5 (Ricci scalar)**.** The scalar curvature or the Ricci scalar is defined by

$$R := \mathrm{tr}_g\big(Ric\big) \,,$$

where tr_g means trace[7] with respect to the metric g. In indices

$$R := R_{ij} g^{ji} \,.$$

1.3 Lie Algebroid

In this section we introduce Lie algebroid. For a more comprehensive treatment see [12].

One can say that Lie algebroids play the same role in the theory of Lie groupoids as Lie algebras play in the theory of Lie groups. It means that Lie groups have their infinitesimal objects called Lie algebras and so have Lie groupoids their infinitesimal objects called Lie algebroids. One can think of Lie algebroids as a generalization of both Lie algebras and tangent vector bundles.

[5] These are the connection coefficients (in coordinate representation) of Levi-Civita connection.
[6] This is the torsion tensor from the Definition 1.2.2.
[7] This can be also written as $\mathrm{tr}\big(X \to (g^{-1}Ric)(X)\big)$.

Definition 1.3.1 (Lie Algebroid)**.** Lie algebroid $(E, [\cdot,\cdot], \rho)$ consists of a vector bundle E over $\mathcal{M}$, Lie bracket $[\cdot,\cdot]$ on smooth sections $\Gamma(E)$ and smooth bundle map $\rho : E \to T\mathcal{M}$ called the anchor, satisfying

$$[X, fY] = f[X, Y] + (\rho(X)f)Y\,, \tag{1.4}$$

where $X, Y \in \Gamma(E)$ and $f \in \mathcal{F}(\mathcal{M})$.

Remark. Note that Jacobi identity and the Equation (1.4) yield

$$\rho([X, Y]) = [\rho(X), \rho(Y)]\,,$$

where $X, Y \in \Gamma(E)$. For proof see [2] or expand $[X, [Y, fZ]]$ using Jacobi and the Equation (1.4); then equate the two expressions.

On Lie algebroids it is possible to introduce operations like exterior differential, inner product or Lie derivative as we did in previous paragraph.

Chapter 2

Generalized Geometry

Generalized geometry naturally emerges from differential geometry by replacing two structures; namely the tangent bundle $T\mathcal{M}$ with the double space structure $T\mathcal{M} \oplus T^*\mathcal{M}$ and the Lie bracket with Courant bracket. For typographical clarity we denote $T\mathcal{M} \oplus T^*\mathcal{M}$ by $\mathcal{D}T$. This new structure was formalized by Hitchin [8, 7] and further developed by his student Gualtieri [6].

We will start with preliminaries resulting from linear algebra and will continue by building structure of generalized geometry. Next we will introduce Courant bracket and Courant algebroid. To the end we will describe a notion of twisted Courant bracket and B-field transformation.

2.1 Linear Algebra preliminaries

In this section we will introduce generalized tangent bundle which is nothing but $\mathcal{D}T$. If we have a vector field $X \in \Gamma(T\mathcal{M})$ and a 1-form $\xi \in \Gamma(T^*\mathcal{M})$ it is straightforward to create element $X + \xi$ from sections of generalized tangent bundle $\Gamma(\mathcal{D}T)$. This bundle is equipped with two natural bi-linear non-degenerate forms. One symmetric and one skew-symmetric

$$\begin{aligned} \langle X+\xi, Y+\eta\rangle_+ &:= \frac{1}{2}(\iota_Y\xi + \iota_X\eta) = \frac{1}{2}\begin{pmatrix} X & \xi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Y \\ \eta \end{pmatrix}, \\ \langle X+\xi, Y+\eta\rangle_- &:= \frac{1}{2}(\iota_Y\xi - \iota_X\eta) = \frac{1}{2}\begin{pmatrix} X & \xi \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Y \\ \eta \end{pmatrix}, \end{aligned}$$

where $X + \xi,\ Y + \eta \in \Gamma(\mathcal{D}T)$. In this thesis $\langle \cdot, \cdot \rangle$ refers to the symmetric one and is called inner product since it is more significant. The signature[1] of the inner product is (n, n), where n is a dimension of the manifold $\mathcal{M}$.

The orthogonal group and its Lie algebra preserving the symmetric form are

$$\begin{aligned} O(\mathcal{D}T) &:= \{O \in GL(\mathcal{D}T) \mid \langle O\cdot, O\cdot\rangle = \langle \cdot, \cdot\rangle\}, \\ o(\mathcal{D}T) &:= \{Q \in gl(\mathcal{D}T) \mid \langle Q\cdot, \cdot\rangle + \langle \cdot, Q\cdot\rangle = 0\}. \end{aligned}$$

One can easily see that $O(\mathcal{D}T) \cong O(n, n)$ because of the signature of inner product. Additionally wanting to preserve the canonical orientation of $\mathcal{D}T$, the group will be $SO(\mathcal{D}T) \cong SO(n, n)$ and its Lie algebra $so(\mathcal{D}T)$; see [6].

From what was stated so far one can see by eyeball the structure[2] of Q.

$$Q = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}, \tag{2.1}$$

[1] The sketch of proof will be presented later, where we will talk about generalized metric.
[2] By decomposition into block matrix.

where

$$A : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}) , \qquad A^T : \Gamma(T^*\mathcal{M}) \to \Gamma(T^*\mathcal{M}) ,$$
$$B : \Gamma(T\mathcal{M}) \to \Gamma(T^*\mathcal{M}) , \qquad \beta : \Gamma(T^*\mathcal{M}) \to \Gamma(T\mathcal{M}) .$$

Since Q is in Lie algebra of $O(\mathcal{D}T)$, B and β have to be skew-symmetric[3]. Thus, we can treat B as 2-form ($B \in \Omega^2(\mathcal{M})$) and β as bi-vector ($\beta \in \Gamma(\wedge^2 T\mathcal{M})$).

Exponentiation of the matrix $Q \in o(\mathcal{D}T)$ yields an element of the orthogonal group preserving inner product. In following examples we show the two most essential symmetries of $\mathcal{D}T$ which preserve the inner product.

Example 2.1.1 (B-transformation)**.** Seeing the matrix

$$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

is nilpotent of second degree. We can write

$$\mathrm{e}^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} . \tag{2.2}$$

Therefore we have $\mathrm{e}^B(X + \xi) = X + \xi + \iota_X B$.

Example 2.1.2 (β-transformation)**.** For the same reason as in the Example 2.1.1 we are allowed to set

$$\mathrm{e}^\beta := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} . \tag{2.3}$$

The transformation gives $\mathrm{e}^\beta(X + \xi) = X + \xi + \iota_\xi \beta$.

β-transformation is not a counterpart of B-transformation. As we will see later the second one plays a more fundamental role and in some sense provides additional structure-preserving transformation of Courant bracket.

2.2 Courant Bracket

Here we introduce the the Courant bracket which replaces within the generalized geometry the usual Lie bracket of vector fields. In fact, there are two structures which can replace the Lie bracket. The alternative is called Dorfman bracket. We will see that these two are equivalent (i.e. one is nothing but skew-symmetrization of another).

First, we introduce the skew-symmetric one since it shows up more frequently in this thesis.

Definition 2.2.1 (Courant bracket)**.** The Courant bracket is defined by

$$[X + \xi, Y + \eta] := [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}\mathrm{d}(\iota_X \eta - \iota_Y \xi) ,$$

where $X + \xi, Y + \eta \in \Gamma(\mathcal{D}T)$.

Note. Restriction[4] on vector fields gives reduction of the Courant bracket to the ordinary Lie bracket from the Definition 1.1.2. That means that if we have a natural projection to the first factor $\pi : \mathcal{D}T \to T\mathcal{M}$ and $A, B \in \Gamma(\mathcal{D}T)$ then

$$\pi([A, B] = [\pi(A), \pi(B)] .$$

However restriction on 1-forms yields vanishing Courant bracket.

[3]It is also clear from the fact that $\langle Q\cdot, \cdot\rangle + \langle \cdot, Q\cdot\rangle = 0$ since $Q \in o(\mathcal{D}T)$.

[4]If $X + \xi \in \Gamma(\mathcal{D}T)$ then the restriction means that $\xi = 0$.

Before presenting some features of this bracket let us continue with the definition of the second one.

Definition 2.2.2 (Dorfman bracket)**.** The Dorfman bracket is defined by

$$[X+\xi, Y+\eta]_{\mathrm{D}} := [X,Y] + \mathcal{L}_X\eta - \iota_Y \mathrm{d}\xi\,,$$

where $X+\xi, Y+\eta \in \Gamma(\mathcal{D}T)$.

In the following proposition we show how Courant and Dorfman bracket are related to each other. Since the proof is elementary, it is omitted.

Proposition 2.2.1 (Courant-Dorfman relations)**.**

$$[A,B] = [A,B]_{\mathrm{D}} - \mathrm{d}\langle A,B\rangle\,,$$

and as stated above, the Courant is a skew-symmetrization of the Dorfman. Therefore

$$[A,B] = \frac{1}{2}\left([A,B]_{\mathrm{D}} - [B,A]_{\mathrm{D}}\right)\,,$$

where $A,B \in \Gamma(\mathcal{D}T)$*.*

The next goal is to demonstrate that the Dorfman bracket satisfies the Leibniz rule. We will use this result later.

Proposition 2.2.2 (Leibniz rule)**.** *Let* $A,B,C \in \Gamma(\mathcal{D}T)$ *then*

$$[A,[B,C]_{\mathrm{D}}]_{\mathrm{D}} = [[A,B]_{\mathrm{D}},C]_{\mathrm{D}} + [B,[A,C]_{\mathrm{D}}]_{\mathrm{D}}\,. \tag{2.4}$$

Proof. Substituting $A = X+\xi,\ B = Y+\eta,\ C = Z+\zeta$ and expanding the Dorfman brackets on the right hand side we get

$$\begin{aligned}
&[[X,Y],Z] + [Y,[X,Z]] + \mathcal{L}_{[X,Y]}\zeta - \iota_Z \mathrm{d}(\mathcal{L}_X\eta - \iota_Y\mathrm{d}\xi) + \mathcal{L}_Y(\mathcal{L}_X\zeta - \iota_Z\mathrm{d}\xi) - \iota_{[X.Z]}\mathrm{d}\eta \\
=&[X,[Y,Z]] + \mathcal{L}_X(\mathcal{L}_Y\zeta - \iota_Z\mathrm{d}\eta) - \iota_{[Y,Z]}\mathrm{d}\xi \\
=&[A,[B,C]_{\mathrm{D}}]_{\mathrm{D}}\,.
\end{aligned}$$

In the first line, Leibniz rule for an ordinary Lie bracket and the Cartan formulas (1.1) were applied. Rearrangement and back-substitution lead us to the desired result. □

As it will be revealed, the Courant bracket does not satisfy the Jacobi identity. Thus, we define an operator measuring failure of the Courant bracket to meet Jacobi identity.

Definition 2.2.3 (Jacobiator)**.** Define a tri-linear operator called Jacobiator as follows

$$\mathrm{Jac}(A,B,C) := [[A,B],C] + [[B,C],A] + [[C,A],B]\,,$$

where $A,B,C \in \Gamma(\mathcal{D}T)$.

We will use all machinery worked out previously to show[5] that the Courant bracket satisfies the Jacobi identity up to an exact term.

Theorem 2.2.3. *Let* $A,B,C \in \Gamma(\mathcal{D}T)$ *and let*

$$\mathrm{Nij}(A,B,C) := \frac{1}{3}\big(\langle [A,B],C\rangle + \langle [B,C],A\rangle + \langle [C,A],B\rangle\big)$$

be a so-called Nijenhuis operator. Then

$$\mathrm{Jac}(A,B,C) = \mathrm{d}(\mathrm{Nij}(A,B,C))\,. \tag{2.5}$$

[5] Also see [6].

Proof. For the main part of the proof we will need the following equation

$$
\begin{aligned}
[[A,B],C] &= [[A,B]_{\mathrm{D}} - \mathrm{d}\langle A,B\rangle, C]_{\mathrm{D}} - \mathrm{d}\langle [A,B],C\rangle \\
&= [[A,B]_{\mathrm{D}}, C]_{\mathrm{D}} - \mathrm{d}\langle [A,B],C\rangle \,,
\end{aligned}
$$

in the first equality the Proposition 2.2.1 was used twice, and in the second one we adopt the fact that $[\alpha, C]_{\mathrm{D}}$ vanishes if α is a closed 1-form. Now we are able to perform the main part of the proof.

$$
\begin{aligned}
\mathrm{Jac}(A,B,C) &= [[A,B],C] + \text{c.p.} \\
&= \frac{1}{4}\big([[A,B]_{\mathrm{D}},C]_{\mathrm{D}} - [C,[A,B]_{\mathrm{D}}]_{\mathrm{D}} - [[B,A]_{\mathrm{D}},C]_{\mathrm{D}} + [C,[B,A]_{\mathrm{D}}]_{\mathrm{D}} + \text{c.p.}\big) \\
&= \frac{1}{4}\big([A,[B,C]_{\mathrm{D}}]_{\mathrm{D}} - [B,[A,C]_{\mathrm{D}}]_{\mathrm{D}} - [C,[A,B]_{\mathrm{D}}]_{\mathrm{D}} \\
&\quad - [B,[A,C]_{\mathrm{D}}]_{\mathrm{D}} + [A,[B,C]_{\mathrm{D}}]_{\mathrm{D}} + [C,[B,A]_{\mathrm{D}}]_{\mathrm{D}} + \text{c.p.}\big) \\
&= \frac{1}{4}\big([A,[B,C]_{\mathrm{D}}]_{\mathrm{D}} - [B,[A,C]_{\mathrm{D}}]_{\mathrm{D}} + \text{c.p.}\big) \\
&= \frac{1}{4}\big([[A,B]_{\mathrm{D}},C]_{\mathrm{D}} + \text{c.p.}\big) \\
&= \frac{1}{4}\big([[A,B],C] + \mathrm{d}\langle [A,B],C\rangle + \text{c.p.}\big) \\
&= \frac{1}{4}\big(\mathrm{Jac}(A,B,C) + 3\mathrm{d}(\mathrm{Nij}(A,B,C))\big) \,,
\end{aligned}
$$

where c.p. denotes cyclic permutation. Considering this line by line, the Proposition 2.2.1 was applied in the second line twice; the third equality comes from the Leibniz rule (2.4); in the next one we notice that all explicit terms except first two vanished thanks to c.p.; in the fifth equality we employed the Equation (2.4) again; the next one follows our remark from the beginning of this proof. Finally, the last one is nothing but the definition of the Jacobiator and the Nijenhuis operator. □

Two upcoming properties of the Courant bracket are of considerable importance. Both will occur later in the introduction of a connection on generalized geometry.

Proposition 2.2.4. *Let $\pi : \mathcal{D}T \to T\mathcal{M}$ be the natural projection to the first factor and let $A, B, C \in \Gamma(\mathcal{D}T)$ then*

$$
\pi(A)\big(\langle B,C\rangle\big) = \langle [A,B] + \mathrm{d}\langle A,B\rangle, C\rangle + \langle B, [A,C] + \mathrm{d}\langle A,C\rangle\rangle \,.
$$

Proof. Astute reader can see that there are hidden Dorfman brackets on the right hand side. Substituting $A = X + \xi,\ B = Y + \eta,\ C = Z + \zeta$ we get

$$
\begin{aligned}
&\langle [A,B] + \mathrm{d}\langle A,B\rangle, C\rangle + \langle B, [A,C] + \mathrm{d}\langle A,C\rangle\rangle \\
=&\langle [A,B]_{\mathrm{D}}, C\rangle + \langle B, [A,C]_{\mathrm{D}}\rangle \\
=&\frac{1}{2}\left(\iota_{[X,Y]}\zeta + \iota_Z\left(\mathcal{L}_X\eta - \iota_Y\xi\right) + \iota_{[X,Z]}\eta + \iota_Y\left(\mathcal{L}_X\zeta - \iota_Z\xi\right)\right) \\
=&\frac{1}{2}\mathcal{L}_X\left(\iota_Y\zeta + \iota_Z\eta\right) \\
=&\pi(A)\big(\langle B,C\rangle\big) \,.
\end{aligned}
$$

As was remarked above, in the first equality are hidden Dorfman brackets; in the second one we expand both the Bracket and the inner product; the next line follows from the Cartan formulas (1.1); in the last one we used that $\mathcal{L}_X f = X(f)$, where $f \in \mathcal{F}(\mathcal{M})$ and performed back-substitution. □

Proposition 2.2.5. *Let $\pi : \mathcal{D}T \to T\mathcal{M}$ be the natural projection to the first factor. Let $A, B \in \Gamma(\mathcal{D}T)$ and let $f \in \mathcal{F}(\mathcal{M})$ then*

$$[A, fB] = f[A, B] + (\pi(A)f)B - \langle A, B\rangle \mathrm{d}f\,.$$

Proof. Substitutions $A = X + \xi,\ Y + \eta$ give

$$\begin{aligned}[X+\xi, f(Y+\eta)] &= [X, fY] + \mathcal{L}_X f\eta - \mathcal{L}_{fY}\xi - \frac{1}{2}\mathrm{d}(\iota_X f\eta - \iota_{fY}\xi)\\ &= f[X,Y] + X(f)Y + f\mathcal{L}_X\eta + X(f)\eta - f\mathcal{L}_Y\xi - \iota_Y\xi\mathrm{d}f\\ &\quad - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\xi) - \frac{1}{2}(\iota_X\eta - \iota_Y\xi)\mathrm{d}f\\ &= f[A,B] + (\pi(A)f)B - \langle A, B\rangle\mathrm{d}f\,,\end{aligned}$$

to go from the first line to second we have applied properties of the Lie bracket, the Lie derivative[6] and of the exterior differential. With an help of the back-substitution we acquire the conclusion. □

It has been proven that the Courant bracket fails to satisfy the Jacobi identity and therefore the triplet $(\mathcal{D}T, [\cdot,\cdot], \pi)$ cannot be a Lie algebroid. However, this leads us to define a new structure called the Courant algebroid which was originally introduced by Zhang-Ju Liu, Alan Weinstein and Ping Xu in their paper [11][7]

Definition 2.2.4 (Courant algebroid)**.** The Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ consists of vector bundle E over $\mathcal{M}$, non-degenerate symmetric fiber-wise bi-linear form $\langle\cdot,\cdot\rangle$, skew-symmetric bracket $[\cdot,\cdot]$ and a smooth bundle map $\pi : E \to T\mathcal{M}$ called anchor. These structures induce a natural differential operator $\mathcal{D} : \mathcal{F}(\mathcal{M}) \to \Gamma(E)$ through the definition $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\pi(A)f$ for $f \in \mathcal{F}(\mathcal{M}), A \in \Gamma(E)$. All above listed structures are subjected to hold the following axioms:

$$\begin{aligned}\pi([A,B]) &= [\pi(A), \pi(B)]\,,\\ \mathrm{Jac}(A,B,C) &= \mathcal{D}(\mathrm{Nij}(A,B,C))\,,\\ [A, fB] &= f[A,B] + (\pi(A)f)B - \langle A,B\rangle\mathcal{D}f\,,\\ \pi\circ\mathcal{D} &= 0\,,\\ \pi(A)\langle B, C\rangle &= \langle [A,B] + \mathcal{D}\langle A,B\rangle, C\rangle + \langle B, [A,C] + \mathcal{D}\langle A, C\rangle\rangle\,,\end{aligned}$$

where $A, B, C \in \Gamma(E),\ f \in \mathcal{F}(\mathcal{M})$.

Example 2.2.1 (Courant algebroid)**.**

$$(\mathcal{D}T, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$$

is a Courant algebroid.

2.3 B-field and twisted Courant Bracket

In the current section we will show, as we promised earlier, the difference between β-transformation (Example 2.1.2) and B-transformation (Example 2.1.1). The first one preserves the Courant bracket in some sense but the second one does not. We also introduce the concept of twisted Courant bracket and twisted Courant algebroid. At the end of the section we will present an interesting relation between generalized geometry and Levi-Civita connection from the Proposition 1.2.1.

Proposition 2.3.1. *Let $B \in \Omega^2(\mathcal{M})$. Then B-transformation (i.e. e^B) is an automorphism of the Courant bracket if and only if 2-form B is closed (i.e. $\mathrm{d}B = 0$).*

[6] In second term with Lie derivative (i.e. $\mathcal{L}_{fY}\xi$) it is helpful to use the Cartan formulas (1.1).

[7] As Definition 2.1.

Proof. For B-transformation to be an automorphism of Courant bracket it means that

$$e^B[X+\xi, Y+\eta] = [e^B(X+\xi), e^B(Y+\eta)]\,,$$

where $X+\xi,\ Y+\eta \in \Gamma(\mathcal{D}T)$. Taking a closer look

$$\begin{aligned}[e^B(X+\xi), e^B(Y+\eta)] &= [X+\xi+\iota_X B, Y+\eta+\iota_Y B]\\ &= [X+\xi, Y+\eta] + \mathcal{L}_X\iota_Y B - \frac{1}{2}d\iota_X\iota_Y B - \mathcal{L}_Y\iota_X B + \frac{1}{2}d\iota_Y\iota_X\\ &= [X+\xi, Y+\eta] + \iota_{[X,Y]}B + \iota_Y\iota_X dB\\ &= e^B[X+\xi, Y+\eta] + \iota_Y\iota_X dB\,,\end{aligned}$$

in the second equality we expand the bracket in third terms; the next line emerges directly from the Cartan formulas (1.1) and the last one follows from the definition of the B-transformation again. Therefore, $\iota_X\iota_Y dB = 0$ for $X, Y \in \Gamma(T\mathcal{M})$, is equivalent to $dB = 0$. □

This B-transformation with closed B is often called B-field transformation. Now let's consider what β-transformation does.

$$\begin{aligned}[e^\beta(X+\xi), e^\beta(Y+\eta)] &= [X+\xi+\iota_\xi\beta, Y+\eta+\iota_\eta\beta]\\ &= [X+\xi, Y+\eta] + [\iota_\xi\beta+\xi, \iota_\eta\beta+\eta] + [X, \iota_\eta\beta] + [\iota_\xi\beta, Y]\,,\end{aligned}$$

from what is obvious that there is no chance to change this messy equation to the desired form

$$e^\beta[X+\xi, Y+\eta]$$

necessary for β-transformation to be an automorphism of the Courant bracket.

Moreover, there is an interesting result[8] which says that there are only two orthogonal automorphisms of $\mathcal{D}T$ preserving Courant bracket, B-field transformations and diffeomorphisms of $\mathcal{M}$.

Now we can define another bracket on $\mathcal{D}T$ closely related to the original one by using a 3-form H.

Definition 2.3.1 (Twisted Courant bracket)**.** The twisted Courant bracket is defined by

$$[X+\xi, Y+\eta]_H := [X+\xi, Y+\eta] + \iota_Y\iota_X H\,,$$

where $X+\xi,\ Y+\eta \in \Gamma(\mathcal{D}T)$ and $H \in \Omega^3(\mathcal{M})$.

Remark. If H is exact, one can denote $dB = H$ and obtain $[\cdot,\cdot]_H$ form $e^{-B}[e^B(\cdot), e^B(\cdot)]$

Notice, there is no relation between D and H in the Dorfman bracket $[\cdot,\cdot]_D$ and the twisted Courant bracket $[\cdot,\cdot]_H$. The D in the first one is just a labeling index but the H in the second is 3-form.

We can similarly define twisted Nijenhuis operator

$$\mathrm{Nij}(A,B,C)_H := \mathrm{Nij}(A,B,C) + H(X,Y,Z)$$

and twisted Jacobiator

$$\mathrm{Jac}(A,B,C)_H := d(\mathrm{Nij}(A,B,C)_H) + \iota_Z\iota_Y\iota_X dH\,,$$

where $A,B,C \in \Gamma(\mathcal{D}T)$; $X = \pi(A),\ Y = \pi(B),\ Z = \pi(C)$ and $\pi : \mathcal{D}T \to T\mathcal{M}$ is a natural projection to the first factor.

[8] See proposition 3.24 in [6].

Example 2.3.1 ("Twisted" Courant algebroid)**.**

$$(\mathcal{D}T, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$$

is a Courant algebroid if and only if $\mathrm{d}H = 0$.

The following proposition presents B-field transformation as a symmetry preserving the twisted Courant bracket.

Proposition 2.3.2. *Let $B \in \Omega^2(\mathcal{M})$. Then B-transformation (i.e. e^B) is an automorphism of the twisted Courant bracket if and only if 2-form B is closed (i.e. $\mathrm{d}B = 0$).*

Proof. We want to show that

$$[\mathrm{e}^B C, \mathrm{e}^B D]_H = \mathrm{e}^B [C, D]_H \,,$$

for some $H \in \Omega^3(\mathcal{M})$. But from the Proposition 2.3.1 we know that

$$\begin{aligned}[\mathrm{e}^B C, \mathrm{e}^B D] &= \mathrm{e}^B [C, D] + \iota_Y \iota_X \mathrm{d}B \\ &= \mathrm{e}^B [C, D]_{\mathrm{d}B} \,,\end{aligned}$$

where $C, D \in \Gamma(\mathcal{D}T)$; $X = \pi(C)$, $Y = \pi(D)$ and $\pi : \mathcal{D}T \to T\mathcal{M}$ is a natural projection to the first factor. Putting all together we acquire

$$[\mathrm{e}^B C, \mathrm{e}^B D]_H = \mathrm{e}^B [C, D]_{H+\mathrm{d}B} \,.$$

□

Thus, we now know that this transformation is a symmetry of both ordinary and twisted Courant bracket.

Let's consider what would happen if we performed B-transformation on the Courant bracket, but instead of 2-form B we would choose the metric tensor g on $\mathcal{M}$. Thus, the transformation alters to

$$\mathrm{e}^g := \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} ,$$

mapping $X + \xi$ to $X + \xi + \iota_X g$. Later proposition displays the remarkable fact that the unique, metric torsion-free connection pops out from this transform.

Proposition 2.3.3. *Let g be a metric tensor on a manifold $\mathcal{M}$. $X + \xi$, $Y + \eta \in \Gamma(\mathcal{D}T)$ and let ∇^{LC} be a Levi-Civita connection on $\mathcal{M}$. Then*

$$[\mathrm{e}^g (X + \xi), \mathrm{e}^g (Y + \eta)] = \mathrm{e}^g [X + \xi, Y + \eta] + \big(g(\nabla^{LC}_{\bullet} X, Y) - g(X, \nabla^{LC}_{\bullet} Y)\big) \,.$$

Proof. To verify this proposition we will use both index and index-free notation. We also have to be careful, because the exterior derivative is not defined on general (in our case symmetric) covariant tensor-fields as the metric tensor $g \in \mathcal{T}^0_2(\mathcal{M})$ is. Therefore, some of the Cartan formulas (1.1) do not hold here anymore.

We examine, on the first sight, a interesting equation in index notation before the very proof, by using the Definition 1.1.4 from the first chapter. Later we will consider it as really useful.

$$\begin{aligned}\iota_Y \mathcal{L}_X g - \mathcal{L}_Y \iota_X g \to &+ Y^i X^k g_{ij,k} + \cancel{Y^i X^k{}_{,i} g_{kj}} + Y^i X^k{}_{,j} g_{ik} \\ &- \cancel{Y^k X^i{}_{,k} g_{ij}} - Y^k X^i g_{ij,k} - Y^k{}_{,j} X^i g_{ik} \\ = &+ g_{ki} Y^i \Big(X^k{}_{,j} + X^m \frac{1}{2} g^{nk} (g_{nj,m} + g_{nm,j} - g_{mj,n}) \Big) \\ &- g_{ik} X^i \Big(Y^k{}_{,j} + Y^m \frac{1}{2} g^{nk} (g_{nj,m} + g_{nm,j} - g_{mj,n}) \Big) \\ = &+ g_{ki} Y^i \Big(X^k{}_{,j} + X^m \Gamma^k_{(mj)} \Big) - g_{ik} X^i \Big(Y^k{}_{,j} + Y^m \Gamma^k_{(mj)} \Big) \\ \to &\, g(\nabla^{LC}_{\bullet} X, Y) - g(X, \nabla^{LC}_{\bullet} Y) \,,\end{aligned}$$

here the first two lines follow from the above mentioned definition and from the fact that the two opposite terms cancel out; reorganization terms, addition and subtraction of $\frac{1}{2}X^nY^mg_{nm,j}$ have given subsequent two lines; afterwards we employed the definition of the torsion-free Christoffel symbols. In the last step we recognized the Levi-Civita connection and switched back to index-free notation. What remains is to show that

$$
\begin{aligned}
[\mathrm{e}^g(X+\xi), \mathrm{e}^g(Y+\eta)] &= [X+\xi+\iota_X g, Y+\eta+\iota_Y g] \\
&= [X+\xi, Y+\eta] + \mathcal{L}_X\iota_Y g - \mathcal{L}_Y\iota_X g - \frac{1}{2}\mathrm{d}\big(\iota_X\iota_Y g - \iota_Y\iota_X g\big) \\
&= [X+\xi, Y+\eta] + \iota_{[X,Y]}g + \iota_Y\mathcal{L}_X g - \mathcal{L}_Y\iota_X g \\
&= \mathrm{e}^g[X+\xi, Y+\eta] + \big(g(\nabla_{\cdot} X, Y) - g(X, \nabla_{\cdot} Y)\big)\,,
\end{aligned}
$$

where in the second equality we adopt the fact that the metric is symmetric[9]; then we applied $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$ and in the last step we implemented our previous strange but useful calculation. □

[9]Cartan formula $\{\iota_X, \iota_Y\} = 0$ does not hold here but we have $[\iota_X, \iota_Y] = 0$ instead.

Chapter 3

Riemannian Generalized Geometry

One can think of the generalized geometry as something where it is impossible to institute an Riemannian geometry but it is not. To show it we will employ our knowledge and built up structures from the preceding chapter to exhibit that actually the inverse is true. Along our path we encounter objects such as a generalized metric, generalized connections and tensors associated with these connections. These objects are counterparts of those from the ordinary Riemannian geometry we are familiar with.

3.1 Generalized Metric

The definition of generalized metric differs in various papers in generalized geometry. There are at least two equivalent approaches to this notion[1]. To describe them we will follow in the first case [8] and the second case is according to [10]. In this thesis we will adopt the second one which seems easier to grasp. We begin with its definition, next we show the equivalent approach and to the end we will see the block matrix form of generalized metric.

Definition 3.1.1 (Generalized metric)**.** Generalized metric is defined by a positive-definite symmetric form

$$\langle A, B\rangle_\tau := \langle A, \tau(B)\rangle\,,$$

where $A, B \in \Gamma(\mathcal{D}T)$; $\tau : \Gamma(\mathcal{D}T) \to \Gamma(\mathcal{D}T)$ is $\mathcal{F}(\mathcal{M})$-linear map satisfying $\tau^2 = \mathrm{id}$.

Proposition 3.1.1. *The symmetry of* $\langle\cdot,\cdot\rangle_\tau$ *implies the symmetry of* τ

$$\langle A, \tau(B)\rangle = \langle \tau(A), B\rangle\,,$$

where $A, B \in \Gamma(\mathcal{D}T)$.

The map τ has two eigenvalues $+1$ and -1 since $\tau^2 = 1$. Therefore, we have two eigenbundles V_+ and V_-, each of dimension n identical with a manifold $\mathcal{M}$, corresponding to these eigenvalues, respectively.

Proposition 3.1.2. *Let* V_+ *and* V_- *be as above. Then the generalized tangent bundle can be decomposed into orthogonal sum*

$$\mathcal{D}T = V_+ \oplus V_-\,.$$

Proof. To prove this, we adopt the fact that the inner product $\langle\cdot,\cdot\rangle$ on sections of V_+ is positive-definite, and on sections of V_- it is negative-definite, since $\langle\cdot,\cdot\rangle_\tau$ is positive-definite and τ has the eigenvalues ± 1. From what we get the orthogonality of V_+ and V_-. If we consider the signature (n, n) of inner product, the dimension of each V_+ and V_- and the fact $V_+^\perp = V_-$ we get the conclusion. □

[1] See [8]

As we stated above there is an equivalent[2] way to define generalized metric.

Proposition 3.1.3. *Let g and B be a Riemannian metric and an arbitrary 2-form on $\mathcal{M}$, respectively. Then we can think of V_+ and V_- as follows*

$$\Gamma(V_+) = \{X + \iota_X(g+B) \mid X \in \Gamma(T\mathcal{M})\} = \{\xi + \iota_\xi(g+B)^{-1} \mid \xi \in \Gamma(T^*\mathcal{M})\},$$
$$\Gamma(V_-) = \{X - \iota_X(g-B) \mid X \in \Gamma(T\mathcal{M})\} = \{\xi - \iota_\xi(g-B)^{-1} \mid \xi \in \Gamma(T^*\mathcal{M})\}.$$

Proof. We have $V_+ \cap T\mathcal{M} = 0$ and $V_+ \cap T^*\mathcal{M} = 0$ reflecting[3] the positive-definiteness of the inner product on V_+, and similarly for V_- and negative-definiteness. From this it is easy to observe that $\Gamma(V_+) = \{X + \iota_X T \mid X \in \Gamma(T\mathcal{M})\}$, where $T \in \mathcal{T}^0_2(\mathcal{M})$ which can be decomposed into the symmetric part g and the skew-symmetric part B. Thus, $T = g + B$. One can argue g to be a Riemannian metric from the positive-definiteness of V_+. In the situation with V_- instead of V_+ we use the fact that V_+ and V_- are orthogonal. What is left undetermined is an arbitrary 2-form B. □

From this analogy, by using g and B, it is possible to uniquely reconstruct our generalized metric from the Definition 3.1.1.

Proposition 3.1.4. *Let g be a Riemannian metric and let B be a 2-form. Sections $X+\xi$, $Y+\eta \in \Gamma(\mathcal{D}T)$ are as always. Then*

$$\langle X+\xi, Y+\eta\rangle_\tau := \frac{1}{2}\begin{pmatrix} X \\ \xi \end{pmatrix}^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix}$$

is generalized metric in the block matrix form.

Proof. First, notice that the action of τ in block matrix form yields

$$\tau \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix}. \tag{3.1}$$

From what one can easily see that

$$\begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}^2 = \mathrm{id}.$$

Next we take a closer look on the matrix form. It is possible to rewrite it in a clearer way

$$\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}.$$

Thus, we get

$$\begin{aligned} \langle X+\xi, Y+\eta\rangle_\tau &= \frac{1}{2}\left(\mathrm{e}^{-B}\begin{pmatrix} X \\ \xi \end{pmatrix}\right)^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \mathrm{e}^{-B}\begin{pmatrix} Y \\ \eta \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} X \\ \xi - \iota_X B \end{pmatrix}^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} Y \\ \eta - \iota_Y B \end{pmatrix}, \end{aligned}$$

and if we let $X+\xi$, $Y+\eta \in \Gamma(V_+)$ then thanks to the Proposition 3.1.3 we can write ξ and η as $\iota_X(g+B)$ and $\iota_Y(g+B)$, respectively. Hence

$$\begin{aligned} \langle X+\xi, Y+\eta\rangle_\tau &= \frac{1}{2}\begin{pmatrix} X \\ \iota_X g \end{pmatrix}^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} Y \\ \iota_Y g \end{pmatrix} \\ &= g(X,Y) \\ &= \langle X+\xi, Y+\eta\rangle, \end{aligned}$$

[2]According to [8] generalized metric is sub-bundle $V_+ \subset \mathcal{D}T$ with a positive-definite inner product.

[3]One can imagine this as plane with x-axis corresponding to $T\mathcal{M}$ and y-axis to $T^*\mathcal{M}$. Then V_+ represents an arbitrary line in the first quadrant going through origin. (Thus, the line have to continue also into the third quadrant.) Therefore the intersection of each two is zero.

what reflects the fact that V_+ is the eigenbundle of τ with eigenvalue $+1$ (i.e. $\tau(V_+) = V_+$) and in like manner for V_-. $\square$

Remark. There is a way to define a new generalized metric $\langle \cdot, \cdot \rangle_{\tilde{\tau}}$ using the symmetry group[4]

$$SO(\mathcal{D}T) \cong SO(n,n)\,.$$

Consequently, $\tilde{\tau} = O^{-1}\tau O$ and $\tilde{\tau}^2 = 1$, where $O \in SO(n,n)$. Since the group is $SO(\mathcal{D}T)$, the new metric preserves distances.

3.2 Connection in Generalized Geometry

Here we get acquainted with lifts of $\Gamma(T\mathcal{M})$ and $\Gamma(T^*\mathcal{M})$ and introduce to the crucial notion of a generalized connection. It will turn out that, in fact, there are two connections one differing from the other by the sign in its torsion tensor. But stop making spoilers and let's get the show on the road!

The Proposition 3.1.3 and the fact that $g \pm B$ is always invertible since g is Riemannian metric (i.e. it is positive-definite) allow us to define Lifts. Hence, we have in hand two alternatives how to lift a vector field X. This is shown in following definition.

Definition 3.2.1 (The Lifts)**.** The lifts to sections of $V_\pm$ are defined by

$$\begin{aligned} X^\pm &:= X \pm \iota_X(g \pm B)\,, \\ \xi^\pm &:= \xi \pm \iota_\xi(g \pm B)^{-1}\,, \end{aligned}$$

where $X \in \Gamma(T\mathcal{M})$, $\xi \in \Gamma(T^*\mathcal{M})$ and $X^\pm,\ \xi^\pm \in \Gamma(V_\pm)$.

The vector bundle $\mathcal{D}T$ is also endowed with orthogonal projections $\pi_{V_\pm} : \mathcal{D}T \to V_\pm$. In subsequent examples we examine this projections in more (practical for computation) detail where we put to use the map τ from the Equation (3.1) and its properties from the Proposition 3.1.2.

Example 3.2.1. The orthogonal projection of $A \in \Gamma(\mathcal{D}T)$ to sections of V_+ is as follows

$$\begin{aligned} \pi_{V_+}(A) &:= \frac{1}{2}\big(A + \tau(A)\big) \\ &= \frac{1}{2}\big(A^+ + A^- + \tau(A^+ + A^-)\big) \\ &= A^+\,, \end{aligned}$$

where the fact that $\tau(V_\pm) = \pm V_\pm$ and the linearity of τ were applied.

Example 3.2.2. The orthogonal projection of $A \in \Gamma(\mathcal{D}T)$ into sections of V_- is as follows

$$\begin{aligned} \pi_{V_-}(A) &:= \frac{1}{2}\big(A - \tau(A)\big) \\ &= A^-\,. \end{aligned}$$

Note. In the Example 3.2.1 and the Example 3.2.2 the notation A^+ and A^- does not mean lifts of vector fields or 1-forms. It is just a convenient way to label the parts of the A which are in $\Gamma(V_+)$ and $\Gamma(V_-)$, respectively.

This provides us a way to define connections[5] on $V_\pm$ (as it is in [8, 5]). We establish such connections in following theorem.

[4] It works also for $O(\mathcal{D}T)$ but we want to preserve length and orientation.

[5] Strictly speaking this is not a connection according to the Definition 1.2.1. In the proof we will see the differences. However, this is not a big deal since we will extend this definition (using Christoffel symbols) later to fit the requirements on Linear connection.

Theorem 3.2.1 (Generalized Connections)**.** *Let A and B be sections of V_+ and V_-, respectively. Let $X \in \Gamma(T\mathcal{M})$. Then*

$$\nabla^+_X A := \pi_{V_+}[X^-, A]\,,$$
$$\nabla^-_X B := \pi_{V_-}[X^+, B]\,,$$

(where $[\cdot,\cdot]$ is the Courant bracket) define connections on V_+ and V_-. Moreover, this connections preserve the inner product.

Proof. To verify correctness of a definition of the connections we have to satisfy five properties in the Definition 1.2.1.

Let us kick off with $\mathcal{F}$-linearity.

$$\begin{aligned}\nabla^+_{fX} A &= \pi_{V_+}[fX^-, A]\\ &= \pi_{V_+}\big(f[X^-, A] - (\pi(A)f)X^- + \langle X^-, A\rangle \mathrm{d}f\big)\\ &= \pi_{V_+}\big(f[X^-, A]\big)\\ &= f\nabla^+_X A\,,\end{aligned}$$

where $\pi : \mathcal{D}T \to T\mathcal{M}$ is the natural projection into the first factor. To go from the first to the second line we have applied the Proposition 2.2.5; next we noticed the fact that $\pi_{V_+}X^- = 0$ and $\langle X^-, A\rangle = 0$ since $A \in \Gamma(V_+)$.

The linearity naturally emerges from the Definition 2.2.1 of the Courant bracket and from the Theorem 3.2.1 itself. There is no need for caring about the preservation of a degree, the property on functions and the commutation with contractions since this connections are defined[6] only on $\Gamma(V_+)$ and $\Gamma(V_-)$. The very same reason is telling us that we just have to check up the Leibniz rule on a product of a function and a section of $V_\pm$.

$$\begin{aligned}\nabla^+_X(fA) &= \pi_{V_+}[X^-, fA]\\ &= \pi_{V_+}\big(f[X^-, A] + X(f)A + \langle X^-, A\rangle \mathrm{d}f\big)\\ &= X(f)A + f\pi_{V_+}[X^-, A]\,,\end{aligned}$$

where was used the Proposition 2.2.5 and $\langle X^-, A\rangle = 0$ as above.

Finally, we can take a closer look on the compatibility with the inner product. Let $C \in \Gamma(V_+)$ then we can write

$$\begin{aligned}\langle \nabla^+_X A, C\rangle + \langle A, \nabla^+_X C\rangle &= \langle [X^-, A], C\rangle + \langle A, [X^-, C]\rangle\\ &= \langle [X^-, A] + \mathrm{d}\langle X^-, A\rangle, C\rangle + \langle A, [X^-, C] + \mathrm{d}\langle X^-, C\rangle\rangle\\ &= X\big(\langle A, C\rangle\big)\\ &= \nabla^+_X\langle A, C\rangle\,.\end{aligned}$$

The first equality is justified thanks to the definition of the connection, and the fact that the inner product with an element from $\Gamma(V_+)$ take care of the projecting into V_+; hence, we can get rid of the orthogonal projection π_{V_+}. The next line yields from the addition of the two zero terms, namely $\langle X^-, A\rangle$ and $\langle X^-, C\rangle$; in the before last one we took advantage of the Proposition 2.2.4. ☐

As was proclaimed in the first chapter the sufficient items for full determination of a connection are the Christoffel symbols. Let's consider what comes by applying this on our new connections from the Theorem 3.2.1. The following Lemma is proved in [8] with vanishing B-field. With non-trivial B-field it is stated in [5] without proof.

[6]This is a consequence of the definition of Courant bracket which is defined as an bi-linear operator on $\Gamma(\mathcal{D}T)$.

Lemma 3.2.2 (Christoffel symbols)**.** *Let ∂_j be a coordinate-induced basis on $\Gamma(T\mathcal{M})$. Then Γ^i_{kj} and $\widetilde{\Gamma}^i_{kj}$ are Christoffel symbols of the connections ∇^+ and ∇^-, respectively. Moreover, the torsion tensor of both connections reads*

$$-\widetilde{T} = T = -g^{-1}(\mathrm{d}B)\,. \tag{3.2}$$

Proof. What we need to show is

$$\begin{aligned}
\nabla^+_j \partial^+_k &= \pi_{V_+}[\partial_j - (g-B)_{mj}\mathrm{d}x^m, \partial_k + (g+B)_{nk}\mathrm{d}x^n] \\
&= \pi_{V_+}\big((g+B)_{nk,j}\mathrm{d}x^n + (g-B)_{mj,k}\mathrm{d}x^m - (g-B)_{kj,m}\mathrm{d}x^m\big) \\
&= \frac{1}{2}\big((g+B)_{mk,j} + (g-B)_{mj,k} - (g-B)_{kj,m}\big)\big(\mathrm{d}x^m + \tau(\mathrm{d}x^m)\big) \\
&= \frac{1}{2}\big((g+B)_{mk,j} + (g-B)_{mj,k} - (g-B)_{kj,m}\big)\big(g_{li}g^{im}\mathrm{d}x^l + g^{im}\partial_i + B_{li}g^{im}\mathrm{d}x^l)\big) \\
&= \frac{1}{2}g^{im}\big(g_{mk,j} + g_{mj,k} - g_{kj,m} + B_{mk,j} - B_{mj,k} + B_{kj,m}\big)\big(\partial_i + (g+B)_{il}\mathrm{d}x^l\big) \\
&= \Gamma^i_{kj}\partial^+_i\,.
\end{aligned}$$

Let's think about this line by line. We start with the definition following with the fact from the differential geometry that $[\partial_j, \partial_k] = 0$. Next comes from the Definition 1.1.4; in the third line there was applied the Example 3.2.1 shadowed by usage of the Equation (3.1). A little rearrangement yields the conclusion. Treating $\nabla^-_j\partial^-_k$ with the very same procedure we obtain

$$\begin{aligned}
\nabla^-_j \partial^-_k &= \widetilde{\Gamma}^i_{kj}\partial^-_i \\
&= \Gamma^i_{jk}\partial^-_i\,.
\end{aligned}$$

Finally, the torsion tensor reads

$$\begin{aligned}
T^i{}_{jk} &= \Gamma^i_{kj} - \Gamma^i_{jk} \\
&= -\frac{3!}{2}g^{im}B_{[mj,k]} \\
&= -g^{im}(\mathrm{d}B)_{mjk} \\
&\to -g^{-1}(\mathrm{d}B)\,.
\end{aligned}$$

where we have omitted the derivations of metric tensor since they cancel out thanks to symmetry. Everything else comes from the definition of skew-symmetrization and form the Proposition 1.1.2. □

Remark. Since $T\mathcal{M}$ and $V_\pm$ are isomorphic, we can think of the Christoffel symbols from the previous lemma as an ordinary Christoffel symbols on $T\mathcal{M}$.

One can argue that there is also a unified covariant derivation on $\mathcal{D}T$ defined as

$$\nabla_X(A) := \nabla^+_X(A^+) + \nabla^-_X(A^-)\,, \tag{3.3}$$

where $A \in \Gamma(\mathcal{D}T)$ and $X \in \Gamma(T\mathcal{M})$. Again the superscript[7] on A indicates the corresponding part of A either in $\Gamma(V_+)$ or $\Gamma(V_-)$. Moreover, previous lemma provides a tremendous possibility to extend the definition of the covariant derivative from just sections of $V_\pm$ to arbitrary tensor products of these sections (e.g. $\Gamma(V_+)\otimes\cdots\otimes\Gamma(V_+)$). First, we define these tensor fields.

Definition 3.2.2. A spaces of tensor fields of the type p on V_+ and V_- are defined by

$$\begin{aligned}
\mathcal{T}^p(V_+) &\in \underbrace{V_+\otimes\cdots\otimes V_+}_{p\,\text{copies}}\,, \\
\mathcal{T}^p(V_-) &\in \underbrace{V_-\otimes\cdots\otimes V_-}_{p\,\text{copies}}\,.
\end{aligned}$$

[7]It can also be denoted as $A^+ = \pi_{V_P}(A)$. Thus, $\nabla_X(A) := \nabla^+_X\big(\pi_{V_P}(A)\big) + \nabla^-_X\big(\pi_{V_P}(A^-)\big)$.

Theorem 3.2.3. *Let $A \in \mathcal{T}^p(V_+)$, $B \in \mathcal{T}^p(V_-)$ and let $X, \in \Gamma(T\mathcal{M})$. Then*

$$\nabla^+_X A := X^j \Big(A^{i_1 \dots i_p}{}_{,j} + \Gamma^{i_1}_{kj} A^{k i_2 \dots i_p} + \dots + \Gamma^{i_p}_{kj} A^{i_1 \dots i_{p-1} k} \Big) \partial^+_{i_1} \otimes \dots \otimes \partial^+_{i_p} ,$$
$$\nabla^-_X B := X^j \Big(B^{i_1 \dots i_p}{}_{,j} + \widetilde{\Gamma}^{i_1}_{kj} B^{k i_2 \dots i_p} + \dots + \widetilde{\Gamma}^{i_p}_{kj} B^{i_1 \dots i_{p-1} k} \Big) \partial^-_{i_1} \otimes \dots \otimes \partial^-_{i_p} .$$

Proof. From the requirement that the connection have to satisfy the Leibniz rule on the tensor product we obtain

$$\begin{aligned} \nabla^+_X A &= X^j \nabla^+_j (A^{i_1 \dots i_p} \partial^+_{i_1} \otimes \dots \otimes \partial^+_{i_p}) \\ &= X^j \Big(A^{i_1 \dots i_p}{}_{,j} \partial^+_{i_1} \otimes \dots \otimes \partial^+_{i_p} \\ &\quad + A^{i_1 \dots i_p} \big(\nabla^+_j (\partial^+_{i_1}) \otimes \dots \otimes \partial^+_{i_p} + \dots + \partial^+_{i_1} \otimes \dots \otimes \nabla^+_j (\partial^+_{i_p}) \big) \Big) . \end{aligned}$$

Applying the Lemma 3.2.2 and relabeling some of dummy indices give rise to the conclusion. □

This can also be extended on the unified covariant derivative. The general expression would be quite messy; hence we will not provide it. But simply expanding the index-free notation (e.g. as we did in proof of the Theorem 3.2.3) and obeying the Leibniz rule render desired unified covariant derivative applicable on an ordinary tensor field in $\mathcal{D}T$.

3.3 Riemann and Ricci again

In the first chapter we presented, following [5], the definition of the Riemann curvature tensor corresponding to a connection ∇. What we did not give is the piece of information that this tensor can also be expressed in terms of the Cristoffel symbols

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk} . \tag{3.4}$$

Here and now we have two connections differing only in a sign in torsion tensor otherwise the very same. Consequently, there are two curvature tensors matching two connections ∇^+ and ∇^-. They will be denoted by R^+ and R^-, in the order already mentioned. We will manifest in an upcoming definition only the first one since the other arrive when swapping either $\nabla^- \to \nabla^+$ or $T(X,Y) \to -T(X,Y)$.

Definition 3.3.1 (generalized Riemann curvature tensor)**.** The generalized Riemann curvature tensor is defined by

$$R^+(X,Y,Z) := \big([\nabla^+_X, \nabla^+_Y] - \nabla^+_{[X,Y]} \big) Z^+ ,$$

where $X, Y, Z \in \Gamma(T\mathcal{M})$.

Theorem 3.3.1 (Decomposition of Riemann curvature tensor)**.** *Let R^{LC} be the Riemann curvature tensor of the Levi-Civita connection*[8] *∇^{LC}. Then we can decompose the Riemann curvature tensor of the generalized connection to the Levi-Civita part and the torsion part*

$$R^+(X,Y,Z) = R^{LC}(X,Y,Z) + R^T(X,Y,Z) ,$$

where $X, Y, Z \in \Gamma(T\mathcal{M})$ and

$$\begin{aligned} R^T(X,Y,Z) &= \frac{1}{2} \big(\nabla^{LC}_X T(Y,Z) - \nabla^{LC}_Y T(X,Z) \big) \\ &\quad + \frac{1}{4} \big((TT)(X,Y,Z) - (TT)(Y,X,Z) \big) . \end{aligned}$$

[8]We sometimes refer to the Levi-Civita connection but we actually mean symmetric part of the generalized connection. (It is the same for both ∇^+ and ∇^- since their Christoffel symbols equal.)

Proof. We are going to take the advantage of the index notation and the alternative definition of the curvature tensor from the Equation (3.4).

$$
\begin{aligned}
(R^+)^i{}_{jkl} &= \Gamma^i_{(jl),k} - \Gamma^i_{(jk),l} + \Gamma^m_{(jl)}\Gamma^i_{(mk)} - \Gamma^m_{(jk)}\Gamma^i_{(ml)} \\
&\quad + \frac{1}{4}\big(T^i{}_{mk}T^m{}_{jl} - T^i{}_{ml}T^m{}_{jk}\big) \\
&\quad - \frac{1}{2}\big(T^i{}_{jl,k} + \Gamma^i_{(mk)}T^m{}_{jl} - \Gamma^m_{(jk)}T^i{}_{ml}\big) \\
&\quad + \frac{1}{2}\big(T^i{}_{jk,l} + \Gamma^i_{(ml)}T^m{}_{jk} - \Gamma^m_{(jl)}T^i{}_{mk}\big) \\
&= (R^{LC})^i{}_{jkl} + \frac{1}{2}\big(\nabla^{LC}_k T^i{}_{lj} - \nabla^{LC}_l T^i{}_{kj}\big) \qquad (3.5) \\
&\quad + \frac{1}{4}\big(T^i{}_{km}T^m{}_{lj} - T^i{}_{lm}T^m{}_{kj}\big)\,.
\end{aligned}
$$

To go through the first equality we have recalled the Equation (1.3). This was followed by addition and subtraction of the term $\Gamma^m_{(lk)}T^i{}_{jm}$. We have also implemented the skew-symmetry of the torsion. Dressing this in the index-free notation we obtain

$$
\begin{aligned}
R^+(X,Y,Z) = R^{LC}(X,Y,Z) \\
&+ \frac{1}{2}\big(\nabla^{LC}_X T(Y,Z) - \nabla^{LC}_Y T(X,Z)\big) \\
&+ \frac{1}{4}\big((TT)(X,Y,Z) - (TT)(Y,X,Z)\big)\,.
\end{aligned}
$$

□

Using the other connection ∇^- we end up with the same expression as in the Theorem 3.3.1 unless the term with the coefficient $\frac{1}{2}$ has a minus sign.

Since our generalized connections and an ordinary connection with skew-torsion have both the same Christoffel symbols, they also have the same curvature tensors and therefore the same symmetries of this tensors. We refer a reader interested in the symmetry of the Riemann tensor to [5].

Contracting first and third index in the Equation 3.5 we come by the Ricci tensor from the Definition 1.2.4.

Proposition 3.3.2 (generalized Ricci tensor)**.** *Let R^+ be the generalized Riemann curvature tensor and let δ_g be the codifferential from the Definition 1.1.6. Then the generalized Ricci tensor gained from this curvature tensor is*

$$
Ric^+(Z,Y) := Ric^{LC}(Z,Y) - \frac{1}{4}\mathrm{tr}\big(X \to (TT)(Y,X,Z)\big) - \frac{1}{2}\delta_g(gT)(Y,Z)\,.
$$

Proof. If we recall how was the torsion tensor derived in the Lemma 3.2.2, and if we take into account the fact that it is totally skew-symmetric[9] then it is clear that an contraction inside vanishes. Hence, the Equation (3.5) yields

$$
\begin{aligned}
(R^+)_{jl} &= (R^{LC})_{jl} + \frac{1}{2}\big(\nabla^{LC}_i T^i{}_{lj} - \nabla^{LC}_l T^i{}_{ij}\big) \\
&\quad + \frac{1}{4}\big(T^i{}_{im}T^m{}_{lj} - T^i{}_{lm}T^m{}_{ij}\big) \\
&= (R^{LC})_{jl} - \frac{1}{4}T^i{}_{lm}T^m{}_{ij} + \frac{1}{2}\nabla^{LC}_i T^i{}_{lj}\,. \qquad (3.6)
\end{aligned}
$$

Switching to the another notation and the Proposition 1.2.2 provide the result. □

[9]Here we are talking about the torsion tensor with lowered first index T_{ijk}.

It is obvious form the Equation (3.6) that the first two terms are symmetric and the last one is skew-symmetric. The other connection, namely ∇^-, leads to the expression with opposite sign in the last term.

The Ricci scalar is the last object remaining to be computed in this chapter. This is easy to do since the last term in the Equation (3.6) will vanish if we perform the contraction with the metric tensor. The first two result in the Levi-Civita scalar curvature and an additional term from the torsion. Explicitly

$$R := R^{LC} - \frac{3}{2}\|T\|^2 \,, \tag{3.7}$$

where $\|T\|^2 = \frac{1}{6}T^{ijk}T_{ijk}$ is from the Definition 1.1.7. This scalar curvature is the same for both ∇^+ and ∇^- since the only difference between Ric^+ and Ric^- is in the skew-symmetric term which vanishes anyway.

Chapter 4

Gravity

This chapter is custom-built for variation principle, or principle of least action, in gravity. It consists from relative short sections in which we will conquer modified Einstein field equations (EFE). The alteration of EFE compared to the classic ones is in additional terms emerging from the torsion part of the connection. We adapt the general discussion, see for example [1], to the case of the torsion originating form the generalized geometry and defined by B-field.

4.1 Einstein-Hilbert action

Firstly, we formally introduce the Einstein-Hilbert action as it is in many books devoted to Einstein gravity. Afterwards, we amend the action to suit our purposes and perform the variation from what we capture the left-hand side of EFE.

But not so quickly! The very start is about two identities that will tackle with variation of the Einstein-Hilbert (EH) action much easier. The first one reads

$$\delta M^{-1} = -M^{-1}\delta(M)M^{-1} .$$

We obtain it from differentiation of $MM^{-1} = \mathrm{id}$, which gives $(\partial M)M^{-1} + M\partial M^{-1} = 0$. The second one arise from a cool matrix identity $\log \det M = \operatorname{tr} \log M$. Differentiate it to get $(\det M)^{-1}\partial M = \operatorname{tr}(M^{-1}\partial M)$ what results in

$$\delta \det(M) = \det(M)\mathrm{Tr}(M^{-1}\delta M) .$$

Application of these two on the metric tensor yields

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{mn}\delta g_{mn} , \tag{4.1}$$

and

$$\delta g^{jl} = -g^{jm}g^{nl}\delta g_{mn} . \tag{4.2}$$

A reader may have noticed an absolute value in $\sqrt{|g|}$. It is because we silently switch from the Riemannian manifolds to the pseudo-Riemannian manifold. Therefore, a determinant of the metric is either positive or negative depending on the metric signature. Note also that we denote the determinant of the metric $\det(g)$ simply as g since there is no danger of confusion.

Definition 4.1.1 (Einstein-Hilbert action)**.** The Einstein-Hilbert action is defined by

$$S_{EH} := \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{|g|}R^{LC} . \tag{4.3}$$

We replace the Ricci scalar of the Levi-Civita connection in EH action with our generalized Ricci scalar from the Equation (3.7). Varying the new action according to the metric tensor we get modified equations of motion. But before doing so, we have a look on something else we will need.

Proposition 4.1.1 (Palatini identity). *A variation of the Levi-Civita curvature tensor reads*

$$\delta(R^{LC})^i{}_{jkl} = \nabla^{LC}_k \delta\Gamma^i_{(jl)} - \nabla^{LC}_l \delta\Gamma^i_{(jk)}\,.$$

Theorem 4.1.2 (Equations of motion for metric). *Let R be the Ricci scalar acquired from an connection with a totally*[1] *skew-symmetric torsion. Than the action*

$$\tilde{S}_{EH} := \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{|g|}R$$

gives equations of motion determining the metric tensor

$$G^{mn} + \frac{3}{4}\big(\|T\|^2 g^{mn} - T^{mij}T^n{}_{ij}\big) = 0\,,$$

where $G^{mn} = (R^{LC})^{mn} - \frac{1}{2}R^{LC}g^{mn}$ is the usual Einstein tensor.

Proof. We have done the first part of the variation in the Equation 4.1. The remaining part is

$$\begin{aligned}\delta\big(g^{jl}(R^+)_{jl}\big) &= \delta\big(g^{jl}(R^{LC})^i{}_{jil}\big)\\ &\quad - \frac{1}{4}T_{ijk}T_{lmn}\delta(g^{il}g^{jm}g^{kn}) + \frac{1}{2}\delta(g^{jl}\nabla^{LC}_i T^i{}_{lj})\\ &= -(\delta g_{mn})(R^{LC})^{mn} + g^{jl}\big(\nabla^{LC}_i\delta\Gamma^i_{(jl)} - \nabla^{LC}_l\delta\Gamma^i_{(ji)}\big)\\ &\quad + \frac{3}{4}T^{mij}T^n{}_{ij}\delta g^{mn}\,.\end{aligned}$$

Firstly, the Equation 3.6 was applied. Going through next equality one has to implement the result of the Equation 4.2 and the Palatini identity from the Proposition 4.1.1. Notice also that the term with $\frac{1}{2}$ coefficient vanished thanks to symmetry of the metric. Putting this together we arrive with

$$\begin{aligned}\delta\tilde{S}_{EH} &= \frac{1}{16\pi}\delta\int \mathrm{d}^4x\sqrt{-g}R\\ &= \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\delta g_{mn}\Big(\frac{1}{2}R^{LC}g^{mn} - \frac{3}{4}\|T\|^2 g^{mn} - (R^{LC})^{mn} + \frac{3}{4}T^{mij}T^n{}_{ij}\Big)\,,\end{aligned}$$

where an integration by parts was performed in order to get rid of the surface term $\nabla^{LC}_i\delta\Gamma^i_{(jl)} - \nabla^{LC}_l\delta\Gamma^i_{(ji)}$. □

4.2 Constraints on B-field

This section enlightens what B has to satisfy to fit into the Einstein field equations. Its properties spontaneously emerge also from the variation principle what we will see to the end of this section.

Starting with an observation that the divergence, in terms of an arbitrary connection with totally skew-symmetric torsion, of an arbitrary symmetric tensor, reduces to the divergence in terms of the Levi-Civita connection. In the subsequent proof we will utilize the Equation (1.3) from the First chapter.

Proposition 4.2.1. *Let $\widehat{\nabla}$ and ∇^{LC} be an arbitrary connection with totally skew-symmetric torsion T and the Levi-Civita connection, respectively. Let S be a totally symmetric, but otherwise arbitrary, tensor field. Then*

$$\widehat{\nabla}_i S^{ij\ldots}{}_{kl\ldots} = \nabla^{LC}_i S^{ij\ldots}{}_{kl\ldots}\,.$$

[1] This means that the 3-covariant tensor gT is a 3-form.

Proof. This is an elementary proof since every term containing the torsion tensor vanishes thanks either to its total skew-symmetry or to symmetry of S.

$$\begin{aligned}\widehat{\nabla}_i S^{ij\cdots}{}_{kl\ldots} = \nabla_i^{LC} S^{ij\cdots}{}_{kl\ldots} \\ -\frac{1}{2}\left(T^i{}_{mi}S^{mj\cdots}{}_{kl\ldots} + T^j{}_{mi}S^{im\cdots}{}_{kl\ldots} + \cdots\right) \\ +\frac{1}{2}\left(T^m{}_{ki}S^{ij\cdots}{}_{ml\ldots} + T^m{}_{li}S^{ij\cdots}{}_{km\ldots} + \cdots\right).\end{aligned}$$

□

Preceding proposition gives a permission to calculate the divergence, regardless of the chosen connection[2], of the equations of motion gained from the variation principle above. We want the mentioned divergence to vanish, not just because on the right side sits zero. Calculation yields

$$\nabla_n^{LC} G^{mn} + \frac{3}{4} g^{mn} \nabla_n^{LC} \|T\|^2 - \frac{3}{4} \nabla_n^{LC} (T^{mij} T^n{}_{ij}) = 0\,,$$

We can get rid of the first term owing to the Bianchi identity[3]

$$\nabla_{[m|}^{LC} (R^{LC})^i{}_{j|kl]} = (R^{LC})^i{}_{j[kl;m]} = 0\,. \tag{4.4}$$

Thus, we end up with[4]

$$T_{ijk} T^{ijk;m} = 3T^{mij;k} T_{ijk} + 3T^{mij} T_{ijk}{}^{;k}\,,$$

where we have done little work with dummy indices. Notice that the first term on the right-hand side can be rewritten as $3T^{m[ij;k]}T_{ijk} = (T^{mij;k} + T^{mjk;i} + T^{mki;j})T_{ijk}$ since T_{ijk} is totally skew-symmetric. Throwing it on the other side and rearranging the indices yields

$$(T^{ijk;m} - T^{jkm;i} + T^{kmi;j} - T^{mij;k})T_{ijk} = 3T^{mij} T_{ijk}{}^{;k}\,.$$

The astute reader undoubtedly see that the part in parenthesis is nothing but $4T^{[ijk;m]}$. (It comes directly from the skew-symmetry of the first three indices in each term.) If we recall the Proposition 1.1.2 from the first chapter, we are allowed to write

$$(\mathrm{d}T)^{mijk} T_{ijk} = 3T^{mij} T_{ijk}{}^{;k}\,.$$

However, one can argue that $T = \mathrm{d}B$ therefore the left-hand side disappears by default. Hence, we finish with

$$T^{mij} T_{ijk}{}^{;k} = 0\,. \tag{4.5}$$

In particular $T_{ijk}{}^{;k}$ is the same as $-\delta T(\partial_i, \partial_j)$. (Here δ is the codifferential from the Definition 1.1.6 or the Proposition 1.2.2 and has nothing to do with variation.) We will see that the condition $\delta \mathrm{d}B = 0$ arises naturally as equations of motion obtained from variation of EH action, but this time according to B.

Theorem 4.2.2 (Equations of motion for 2-form B)**.** *Let R be as above. Than the action*

$$\tilde{S}_{EH} = \frac{1}{16\pi} \int \mathrm{d}^4 x \sqrt{|g|} R$$

gives equations of motion determining the 2-form B

$$(\mathrm{d}B)_{ijk}{}^{;k} = 0 \quad \Leftrightarrow \quad \delta \mathrm{d}B = 0\,.$$

[2] We have two such connections, namely ∇^+ and ∇^-.

[3] Contract i with k and l with m to obtain the wished result.

[4] For typographical clarity we will here denote ∇^{LC} by semicolon notation.

Proof. There is just a single term we have to carry out an variation on. So let's do it.

$$\begin{aligned}\delta\|T\|^2 &= \frac{2}{3!}T^{ijk}\delta(-3)(B)_{[ij;k]}\\ &= -T^{ijk}(\delta B)_{ij;k}\,,\end{aligned}$$

where we are allowed to interchange the Levi-Civita covariant derivative and the variation since they do not depend on each other. We have also employed the total skew-symmetry of T^{ijk}. Hence, the the variation takes the form

$$\begin{aligned}\delta\tilde{S}_{EH} &= \frac{3}{32\pi}\int \mathrm{d}^4x\sqrt{|g|}T^{ijk}(\delta B)_{ij;k}\\ &\overset{pp}{=} -\frac{3}{32\pi}\int \mathrm{d}^4x\sqrt{|g|}(\delta B)^{ij}T_{ijk}{}^{;k}\,.\end{aligned}$$

To go from the first line to the second we have integrated by parts and swapped the indices. □

4.3 Einstein field equation

Naturally, the right-hand side of EFE has to contain the stress-energy tensor. This can also be obtained from the variation principle. One can vary an action of matter or particles or electromagnetic field according to metric and arrive with the corresponding stress-energy tensor $\delta S_M = \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g|}T^{mn}\delta g_{mn}$.

Therefore, the field equations acquire the form

$$G^{mn} + \frac{3}{4}\big(\|T\|^2 g^{mn} - T^{mij}T^n{}_{ij}\big) = 8\pi T^{mn}\,. \tag{4.6}$$

Contraction of the previous equation with g_{mn} gives

$$\frac{2-n}{2}R^{LC} + \frac{3(n-6)}{4}\|T\|^2 = 8\pi T\,, \tag{4.7}$$

where n denotes a dimension of the manifold $\mathcal{M}$ and $T = T^{mn}g_{mn}$. Elimination of R^{LC} in the Equation (4.6) yields

$$(R^{LC})^{mn} + \frac{3}{n-2}\|T\|^2 g^{mn} - \frac{3}{4}T^{mij}T^n{}_{ij} = 8\pi\big(T^{mn} - \frac{1}{n-2}Tg^{mn}\big)\,. \tag{4.8}$$

An obvious observation, that the Equations (4.6) and (4.8) reduce to the classic ones,[5] will take place, if T^{ijk} vanish. One can also notice that the curvature of an empty[6] space depends on the dimension. If we set the right-hand side in the Equation (4.7) equal to zero then we are allowed to write

$$R^{LC} = \frac{3}{2}\frac{n-6}{n-2}\|T\|^2\,.$$

Hence, for the dimension n satisfying $2 < n < 6$ we inevitably obtain a space with the negative curvature.

[5] $G^{mn} = 8\pi T^{mn}$ and $(R^{LC})^{mn} = 8\pi\big(T^{mn} - \frac{1}{n-2}Tg^{mn}\big)$, respectively.

[6] This means that there isn't any matter nor electromagnetic field. In other words $T^{mn} = 0$.

Epilogue

The object of this thesis was to introduce and outline the basic properties of generalized geometry. The knowledge of differential geometry had naturally lead us towards the replacement of the Lie bracket with the Courant one. A remarkable abundant mathematical structure known as the Courant algebroid appeared. We explored various operation of generalized geometry such as the action of the B-field and the β-transformation, twisting of the Courant bracket. We have even investigated the action of a metric tensor on the Courant bracket.

The Riemannian geometry is naturally included within the generalized geometry. We introduced generalized metric together with the connection in the framework of this new geometry. Additionally, we worked out some properties of the two mentioned structures. The main aim of the Riemannian generalized geometry was to introduce the generalized scalar curvature gained from the most prominent tensor of the Riemannian geometry.

All this work had to be done in order to acquire the modified Einstein-Hilbert action similar to the one from the supergravity. We have also derived the equations of motion for both the metric tensor and the B-field. We showed that the curvature of a emerging from such field equations depends on its dimension and is negative in 4-dimensional space-time. This observation is associated with a question, which comes to mind, whether the cosmological constant can somehow be interpreted by additional terms in Einstein field equations.

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List of Abbreviations

$\mathcal{D}T \equiv T\mathcal{M} \oplus T^*\mathcal{M}$	generalized tangent bundle
$\mathcal{F}(\mathcal{M})$	algebra of (smooth) function on $\mathcal{M}$
Γ^i_{jk}	Christoffel symbols of the second kind
$\Gamma(\mathcal{D}T)$	space of sections of $\mathcal{D}T$
$\Gamma(T\mathcal{M})$	space of sections of $T\mathcal{M}$
$\Gamma(T^*\mathcal{M})$	space of sections of $T^*\mathcal{M}$
$\Gamma(V_-)$	space of sections of V_-
$\Gamma(V_+)$	space of sections of V_+
$\langle \cdot, \cdot \rangle_\tau$	generalised metric
$\mathcal{M}$	differentiable manifold
$\Omega(\mathcal{M})$	Cartan algebra of differentiable forms on $\mathcal{M}$
π	natural projection
$\pi_{V_\pm}$	orthogonal projections
$T\mathcal{M}$	tangent bundle of a differentiable manifold $\mathcal{M}$
$\mathcal{T}^p(V_-)$	space of tensor fields of the type p on V_-
$\mathcal{T}^p(V_+)$	space of tensor fields of the type p on V_+
$T^*\mathcal{M}$	cotangent bundle of a differentiable manifold $\mathcal{M}$
$\mathcal{T}^p_q(\mathcal{M})$	space of tensor-fields of the type (p,q) on $\mathcal{M}$
V_-	negative eigenbundle of a generalized metric map τ
V_+	positive eigenbundle of a generalized metric map τ
EFE	Einstein field equations
EH	Einstein-Hilbert action
GL	general linear group
gl	general linear Lie algebra
O	orthogonal group
s	orthogonal Lie algebra
SO	special orthogonal group
so	special orthogonal Lie algebra