

Charles University in Prague

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BACHELOR THESIS

**The Nelson-Siegel Model: Present
Application and Alternative Lambda
Determination**

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Declaration of Authorship

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Prague, 13. května 2015

Signature

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Abstract

This thesis contributes to the topic of yield curve modelling by revaluing the famous Nelson-Siegel model in the relatively outdated but very parsimonious version. In order to make this framework applicable to present yield curves of government bonds, we introduce an alternative model dealing with an appropriateness of the possibly overlooked model parameter λ . By incorporating the sound methodology, we model the yield curves of the three currency regions - EUR, USD and GBP - and assess both in-sample fit and forecasting performance. Whereas the in-sample predicting generally achieves the best results with the alternative model predicting model coefficients, especially for longer maturities, the out-of-sample forecasting seems more complicated. Actually, the detail analysis show an interesting connection between efficiencies of the models and bond market volatilities. On the base of our research, the model directly extrapolating yields appears to be more suitable for more volatile markets.

JEL Classification C51, C53, C61, G17

Keywords Yield Curve, Nelson-Siegel, Newton Optimization Method

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Abstrakt

Tato bakalářská práce přispívá k tématu modelování výnosových křivek především novým náhledem na známý Nelson-Siegelův model v jeho prvotní verzi, která je velmi prostá, avšak ne zcela neaktuální. Za účelem současné aplikace na nynější data, při zachování jednoduché struktury modelu, uvádíme alternativní způsob pro výpočet optimálních hodnot koeficientů, především parametru λ . Tento přístup stavíme na ověřené metodologii a zkoumáme jeho účinnost při odhadování stávajících i budoucích hodnot výnosů státních dluhopisů pro tři měnové regiony - EUR, USD a GBP. Zatímco při odhadování stávajících hodnot náš alternativní přístup predikující modelové parametry dosahuje komplexně nejlepších výsledků, v případě předpovídání budoucích hodnot nejsou výsledky zcela jednoznačné. Detailní analýza v důsledku ukazuje zajímavou spojitost mezi efektivitou jednotlivých modelů a volatilitami daného trhu. Pro více volatilní trhy se jeví jako vhodnější model pracující přímo s výnosy jednotlivých dluhopisů namísto s koeficienty modelu.

Klasifikace JEL

C51, C53, C61, G17

Klíčová slova

Výnosová křivka, Nelson-Siegelův model,
Newtonova optimalizační metoda

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Bachelor Thesis Proposal

Author	Jan Marek
Supervisor	PhDr. Boril Šopov, MSc., LL.M.
Proposed topic	The Nelson-Siegel Model: Present Application and Alternative Lambda Determination

Topic characteristics When raising a question of modelling bond yield curves, the Nelson-Siegel framework provides truly efficient answer. Nelson and Siegel (1987) proposed the popular three-factor parsimonious model for fitting and forecasting bond yield curves in 1987. Later, Diebold and Li (2006) reinterpreted its coefficients as level, slope and curvature. Even being very simple, this framework achieves substantially good results in both disciplines - fitting and forecasting term structures of government bonds.

Hypotheses In this thesis we investigate, whether the Diebol-Li interpretation of Nelson-Siegel model is applicable on actual data and different currencies. Moreover, in order to improve overall predicative performance, we appeal to propose an alternative model especially for computing proper λ -factor for each time series.

Methodology

- Time-series econometrics
- Ordinary Least Squares
- Quasi-Newton optimization method
- Programming language Ox by Doornik (1996)

Outline

1. Introduction
2. Related Literature
3. Data and Methodology
4. The Mode
5. Empirical Results
6. Conclusion

Core bibliography

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Author

Supervisor

Chapter 1

Introduction

To introduce the topic of this thesis, let us start a little bit broadly and describe important and relevant terms connected with the Nelson-Siegel model used for fitting and forecasting yield curves of government bonds.

As we mention, we are in particular interested in debt securities known as bonds. Corporate or government bond, depending on an issuer, obliges an issuer to repay a holder borrowed money at given point in time. In general, at the time of maturity a face value of bond can significantly depart from an original issue price. Therefore, the holder is usually compensated via interest payments of given interest rate and given periodicity. These payments, or coupons, can be distributed repeatedly on the annual or semi-annual basis or at the expiration date as a single transfer. The related interest rate depends mainly on the time span between the issue date and the maturity date, which can differ from a few days up to 40 years. The obvious connection between maturities and rates is used as the core assumption when building yield curves, the subject of our primary interest.

A yield curve is a graphical representation of a relationship between two variables - the time to maturity and the interest rate. As a rule, in such plot the horizontal axis represents maturities while corresponding interest rates are depicted on the vertical axis. Besides a straightforward informative function about rates, yield curves reflect likewise, firstly, government decisions at the short end and, secondly, the expectations in the economy at the long end. These important information contained in yield curves are used in many ways from simple investment decisions to exact valuation of financial derivatives. All things considered, the precise modelling of yield curves becomes truly crucial, even more in the post-crisis times when we can observe for example nega-

tive interest rates. A very fitting and forecasting process is a demanding task, especially since yields curves take various shapes. Nevertheless, thanks to the econometricians Nelson and Siegel, the related literature provides an efficient solution to solving this problem via three-factor latent model. On one side, it can be characterized as very parsimonious and simple. On the other hand, if want to follow the mentioned simplicity, this model from 1987, as well as its Diebold-Li reinterpretation from 2006, are quite outdated.

In this thesis, we investigate where the Diebold-Li framework is sufficiently applicable on a present data for the three currency regions, the EUR, GBP and USD, in both disciplines - fitting and forecasting. In case of the former discipline, we appeal to improve the performance of the model by introducing an alternative model for more suitable determination of the model parameters. In case of the latter discipline, we evaluate the original model applied to our own estimates and compare it with a more trivial approach.

Including the introduction, this thesis is structured in six chapters. The second chapter aims on the literature related to fitting and forecasting yield curves, in particular on the Nelson-Siegel model and subsequent approaches. Besides the brief description of models in the 50-year long time-line, we provide a short reasoning why we choose exactly the prior version of the Nelson-Siegel model for our research.

In Chapter 3 we firstly summarize the data used for our research. Secondly, we provide an insight into the theory of mathematical optimization and describe the Quasi-Newton Broyden-Fletcher-Goldfarb-Shanno method. Thirdly, we introduce the core of our hypothesis that we prove in the following chapters.

The next chapter, Chapter 4, starts with a detail description of the Nelson-Siegel model and its Diebold-Li interpretation. In the second part, we introduce our alternative model for determination of λ -parameter and summarize forecasting techniques.

Empirical results are then provided in Chapter 5. For both, fitting and forecasting of yield curves, we compare all the models among themselves and express our findings via framework of tables and figures with short comments on each.

We end this thesis with Chapter 6 labelled as Conclusion where we finally provide a short discussion on obtained results and put them into wider economical context taking into account market volatilities.

Chapter 2

Literature Review

This chapter briefly summarizes the evolution of the literature related to fitting and forecasting yield curves, or term structures, of government bonds. Throughout the thesis we use, as a keystone, the popular Nelson-Siegel model, thus we provide an overview of development of Nelson-Siegel model extended by many authors in their seminal papers and books.

There has been paid a distinct attention to examining and forecasting bond yields dynamics since both sovereign and corporate bonds became a common investment instrument. As Nelson and Siegel (1987) mention, the prime insight into the issue of econometric modelling of yield curves was made in the early second half of the 20th century. Then, the majority of initial models was based on the methodology of multi-linear regression, eventually of exponential splines. Without being entirely complex and efficient, these models provided credible approach to capturing the basic set of empirically observed shapes associated with yield curves: monotonic, humped and S-shape. (Nelson and Siegel 1987)

Concerning the posterior models, difference and differential functions were considered as the ones that produce exactly correct estimation of forward rates of bonds. Nelson and Siegel (1987) enhanced this idea and introduced the parsimonious three-factor model. Their complex approach, and probably a certain milestone in the very evolution of yield curve modelling, examines estimated latent factors as short-, medium- and long-term components of yield curves. Despite of being that simple, it provides fairly good in-sample fit. Thanks to these properties, Nelson-Siegel model is widely used, directly or as a cornerstone for other models, in the global financial sector.

Later, Svensson (1995) modified the original Nelson-Siegel by including one

additional component in order to improve the overall model flexibility, especially in case of longer terms and extraordinary yield curve shapes.

Even later, a further research brought more comprehensive and modern models. Unfortunately, the vast of them is rather empirical without any solid theoretical background. (Christensen et al. 2009) Among the rest of approaches, Diebold and Li (2006) mention the arbitrage-free model and the equilibrium model as the two most outstanding. The former model is rather static in time attempting to fit yield curve as accurate as possible to preclude any possible arbitrage. Contrary, the later model is dynamic, capturing yields through the affine structure. Nevertheless, both models pay only a little attention to the question of forecasting. The non-arbitrage model provides only a static fit and the equilibrium model, although being dynamic and having good predicative potential, provides poor out-of-sample forecasts. (Diebold and Li 2006)

Therefore, despite the presence of modern competing models, Diebold and Li (2006) revise the very Nelson-Siegel. Firstly, the model becomes suitably dynamized and modern. Secondly, the economic interpretation of coefficients is greatly improved. Consequently, short-, medium- and long-term coefficients represent level, slope and curvature of a yield curve, respectively. In addition, in contrast to the original paper, Diebold and Li (2006) go further in terms of forecasting by incorporating a simple first-order autoregressive model. The AR(1) process provides surprisingly good, both in-sample and out-of-sample, forecasting results, especially at the horizon of six to twelve months ahead.

Subsequently, Diebold et al. (2006a) bring more to the previous Diebold-Li approach. It generally defines the model as non-affine and assess it as successfully capturing systematic risk through its parameters. Moreover, Diebold et al. (2008) extend the dynamic re-interpretation of the Nelson-Siegel model to a global context. Beside country-specific factors, they assume existence of global ones as well. To empirically support this hypothesis, they analyse yield curves for the exact currency regions, the U.S., Germany, Japan and the U.K..

Later, Christensen et al. (2009) evaluate appropriateness of the Nelson-Siegel and Svensson models for incorporating the non-arbitrage framework. Whereas the classic Nelson-Siegel is more efficient as arbitrage-free model, the Svensson's modification fits longer maturities more precisely. As a reaction Christensen et al. (2009) introduce the five-factor generalized model based on Svensson (1995), but adding one more slope coefficient.

Finally, Christensen et al. (2011) define a new class of affine arbitrage-free dynamic models that estimates the Nelson-Siegel interpretation. Empirically,

this approach even more increases fitting and forecasting performance.

In this thesis we build our hypothesis and all consequent models and computations exactly on the original Nelson-Siegel model and on its Diebold-Li dynamic interpretation, as proposed in Nelson and Siegel (1987) and Diebold and Li (2006), respectively. From the theoretical point of view, to clarify the reason why we do not use any later model, we argue, that the Nelson-Siegel model ensures very good estimation and forecasting performance itself, even in its prior version. Moreover, the Diebold-Li interpretation is still linear in parameters and therefore quite parsimonious. In contrast, the subsequent models, starting with Diebold et al. (2008), are substantially more complicated in terms of estimation and incorporate especially the state space framework which is beyond the scope of this thesis.

Chapter 3

Data and Theory

The following chapter introduces a methodology used for the fitting and forecasting yield curves. Firstly, we provide the data summary along with enclosed three-dimensional plot of the yield curves and the brief description of economic interactions. Secondly, we shortly describe the Broyden-Fletcher-Goldfarb-Shanno optimization method from class of the Quasi-Newton iterative algorithms. Finally, we introduce the main ideas and intentions of our conducted research in the overall core hypothesis.

3.1 Data

Concerning a data, we use interest rates of government bonds of selected currencies - the US Dollar (USD), the Euro (EUR) and the British Pound (GBP). In contrast with the related literature, we set two restrictions on the data selection. Firstly, instead of recalculated forward rates we use spot rates of AAA government bonds without any additional transformation. Secondly, only bonds without repetitive coupons, in other words, zero-coupon bonds, are included. Due to the diverse maturity structure, we use only tenors up to 10 years, exactly the tenors of 3, 6 and 9 (except for USD) months and 1, 2, 3, 5, 7 and 10 years. The data sets were downloaded from the official public databases of statistical units of the European Central Bank¹, the National Bank of England² and the Federal Reserve System³ for the EUR, GBP and USD yield curves, respectively. The time series are on the monthly basis with the time span between 10 and 15 years. Exact dates and numbers of observations are shown in Table (3.1).

¹<https://www.ecb.europa.eu/stats/money/yc/html/index.en.html>

²<http://www.bankofengland.co.uk/statistics/Pages/yieldcurve/archive.aspx>

³<http://www.federalreserve.gov/datadownload/default.htm>

Currency	Start Date	End Date	# observations	# tenors
EUR	30.9.2004	19.3.2015	127	9
GBP	31.1.2000	31.12.2012	156	9
USD	31.1.2000	28.2.2015	182	8

Table 3.1: Data summary
Source: Author's calculations.

More to the data structure is provided in Table (3.2) - mean and standard deviation written in brackets for each maturity and currency region. There can be seen that the standard deviation rises with increasing time to maturity. This fact is verified in the data representation provided in Figure (3.1). Among years, the long end of yield curves is less volatile than the short end.

Tenor	EUR	USD	GBP
3M	1.3991 (1.4580)	1.8441 (2.0085)	3.3625 (2.0292)
6M	1.4410 (1.4982)	1.9547 (2.0372)	3.3483 (2.0294)
9M	1.4840 (1.5125)	– –	3.3496 (2.0185)
1Y	1.5283 (1.5135)	2.0473 (1.9740)	3.3614 (1.9972)
2Y	1.7170 (1.4716)	2.3097 (1.8825)	3.4834 (1.8643)
3Y	1.9173 (1.4095)	2.5590 (1.7695)	3.6312 (1.7148)
5Y	2.3180 (1.2854)	3.0581 (1.5495)	3.8739 (1.4327)
7Y	2.6727 (1.1794)	3.4498 (1.3855)	4.0502 (1.1913)
10Y	3.0748 (1.0611)	3.7976 (1.1845)	4.2259 (0.9168)

Table 3.2: Data - descriptive statistics: mean (std. deviation)
Source: Author's calculations.

As to be seen in Figure (3.1), all the interest rates followed the down trend after the recession in early 2000, in the United States known as the "Dot Com bubble". In 2005 yields started to grow towards the peak in 2008, exactly before the subprime mortgage crisis leading to the complex global financial crisis. After

the consecutive dramatic drop in all rates, short-end yields have persevered on the low level near 0% return, while long-end rates humped as they recovered in 2012, but then reversed direction and followed the down trend of short-term bonds.

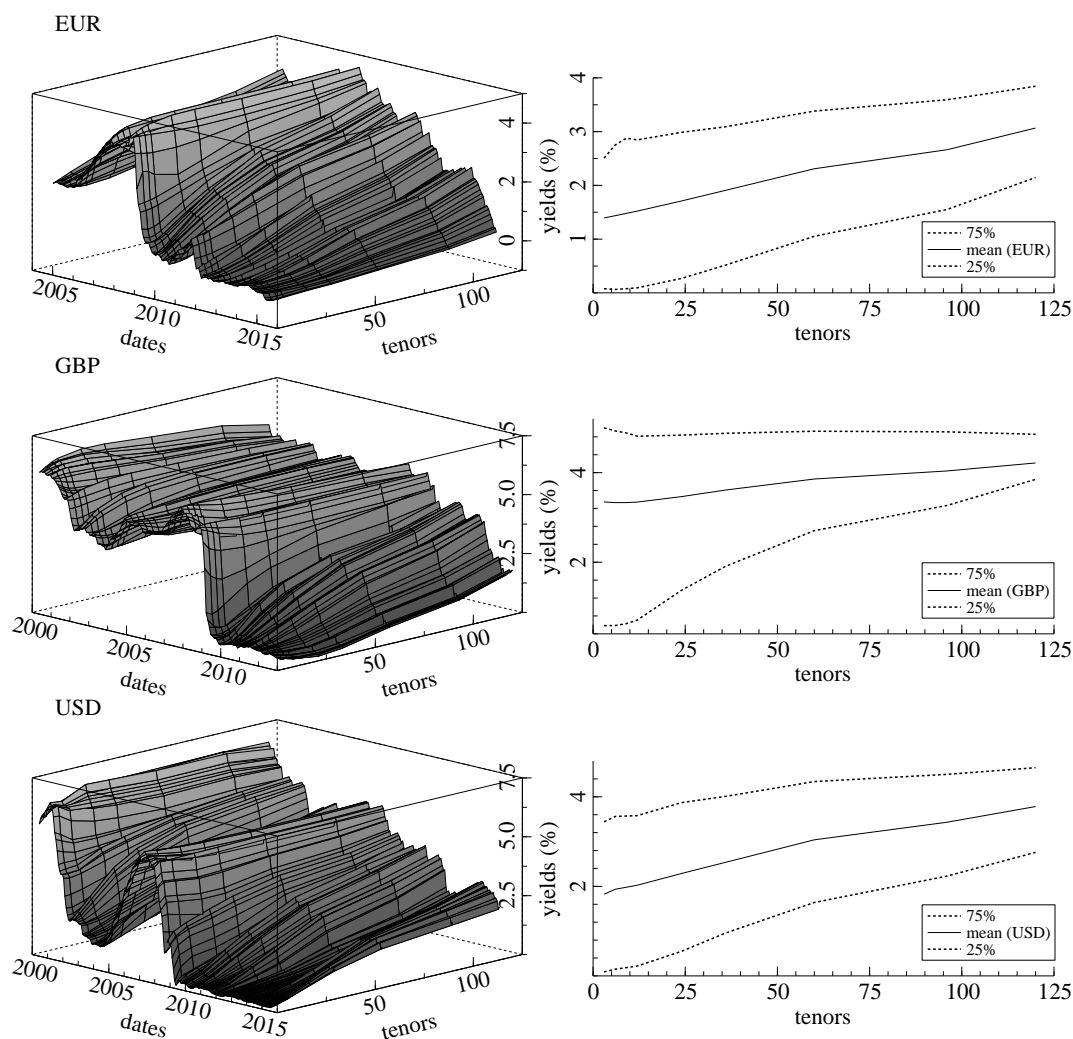


Figure 3.1: Left: Multidimensional projection of yield curves.

Right: Means by tenors against 25% and 75% quantiles.

Source: Author's calculations.

3.2 Optimization

Throughout the theoretical and empirical part of this thesis we use the advanced Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, an iterative algorithm

from the class of Quasi-Newton methods for optimizing non-linear problems. This class builds on the original Newton method evaluating second derivatives at points. Through iterations it seeks a stationary point, adopting the quadratic convergence principle. (Gill and Leonard 2001)

3.2.1 Newton Method

At the beginning, we make a point on the original Newton method. Let us consider a n -dimensional space. Then, let f denote a twice differentiable function $\mathbb{R}^n \rightarrow \mathbb{R}$, such that $f \in C^3$. In the running process, the quadratic approximation

$$f(x_i) + \nabla f(x_i)(x - x_i) + \frac{1}{2}\nabla^2 f(x_i)(x - x_i)^2, \quad (3.1)$$

as given in equation (3.1), of f at point x_i determines the direction of next search, p_i , mathematically

$$p_i = -\nabla^2 f(x_i)^{-1}\nabla f(x_i), \quad (3.2)$$

$$x_{i+1} = x_i + p_i \quad (3.3)$$

for $i = 0, 1, 2, \dots, K$ where K denotes number of iterations and i represents the actual step. Moreover, α_i denotes the step length obtained by inexact line search method with respect to Wolfe conditions, such that

$$f(x_i + \alpha_i p_i) \leq f(x_i) + c_1 \alpha_i \nabla f(x_i)^T p_i, \quad (3.4)$$

$$\nabla f(x_i + \alpha_i p_i)^T p_i \geq c_2 \nabla f(x_i)^T p_i. \quad (3.5)$$

These, the first and the second, provide a sufficient decrease and curvature of the function between iterations, respectively. Violation may lead to inefficient numerical progress in the optimization process and to finding some point being not even a local maximum or minimum. In case of the Quasi-Newton method, the usual value of length lies around 0.9 (Nocedal and Wright 1999, p. 35 - 40)

With respect to the Newton method, every repetition lowers the distance to the true optimum, until it is found. In fact, while being close enough to this point, at each step the difference, by which x_{i+1} is closer than x_i , is squared.

Subsequent adjusted modified Newton method works on similar principle

as the predecessor, except it replaces a second derivative with its constant estimation. Although being slower, this approach is more stable and parsimonious. (Brinkhuis and Tikhomirov 2005, p. 294 - 301)

3.2.2 Broyden-Fletcher-Goldfarb-Shanno Method

Following class of Quasi-Newton methods likewise approximates a second derivative, generally a Hessian matrix.

$$p_i = -H_i^{-1}\nabla f(x_i), \quad (3.6)$$

$$x_{i+1} = x_i + \alpha_i p_i \quad (3.7)$$

At each step, a symmetric and positive definite matrix H_{i+1} is obtained from a H_i via a low-rank update matrix. The most popular among these is the two-rank BFGS update formula (Gill and Leonard 2001), mathematically

$$H_{i+1} = H_i - \frac{1}{\delta_i^T H_i \delta_i} H_i \delta_i \delta_i^T H_i + \frac{1}{\gamma_i^T \delta_i} \gamma_i \gamma_i^T, \quad (3.8)$$

where $\delta_i = x_{i+1} - x_i$ and $\gamma_i = \nabla f(x_{i+1}) - \nabla f(x_i)$. The initial estimate of H_0 is usually set to be a multiple of identity matrix. Without direct computing of second derivatives, BFGS provides quick and efficient approach for solving unconstrained optimization problems. With the general claim of $\lim_{i \rightarrow \infty} \|\nabla f(x_i)\| = 0$, the process runs through iterations until the specific convergence criteria $\|\nabla f(x_i)\| > \epsilon > 0$ are met. (Nocedal and Wright 1999)

3.3 Hypothesis

As it is said at the very end of Chapter 2, we consider the Diebold-Li dynamic three-factor interpretation of the original Nelson-Siegel model as a cornerstone for building our hypothesis. In this thesis, we take the Diebold-Li approach and investigate whether it is still applicable on the actual data and the different currency regions, exactly the EUR, GBP and USD. Moreover, by proposing an alternative model for computing λ , we examine if any other value of this parameter is more suitable than the original one equal to 0.0609.

To be precise, we firstly assess the appropriateness of the Diebold-Li approach as a tool for fitting term structures of government bonds. Diebold and Li

(2006) label the Nelson-Siegel framework as very successful in capturing different shapes of yield curves. Therefore, the question of overcoming this original model is indeed challenging, particularly in case of the US Dollar curves. A possible way of improving the model consists in the parameter λ . Both, Nelson and Siegel (1987) and Diebold and Li (2006), drop this factor from the core estimation process and predetermine it on the base of the medium-term component *ex ante*.

In the second part, we are interested in an out-of-sample forecasting of bond yields. Diebold and Li (2006) highlight a surprising performance especially in the horizon between 6 to 12 months provided by applying the simple first-order autoregressive model to estimated regression coefficients and subsequent extrapolating of proper yields. Therefore, even in terms of forecasting, the Diebold-Li method ensures statistically strong results. Again, this raises a question whether this approach sufficiently well predicts values of yields when exercised on different sets of data. The strong persistence in factors, as shown by Nelson and Siegel (1987), can substantially affects results, for example, if we take into account present negative interest rates.

Chapter 4

The Model

This chapter provides a review of three models for the fitting and forecasting yield curves, two of them are current and one is newly proposed. At the beginning, we describe the original Nelson-Siegel model as a basis for further models. Then, we continue with the Diebold-Li re-interpretation. Finally, hand in hand with the idea of fixed parameter lambda, we introduce an alternative way of estimating λ in order to improve overall fitting and forecasting performance of the Nelson-Siegel model.

4.1 Nelson & Siegel Model

Nelson and Siegel (1987) give an ultimate solution to fitting and forecasting yield curves. The prior idea builds on the class of differential equations, owing to shapes produced by solutions of these differential equations. Consequently, the model provided by Nelson and Siegel (1987) explaining a forward rate as a function of maturity (τ) is given as a solution to a second-order differential equation, such that

$$r(\tau) = \beta_1 + \beta_2 \exp(-\tau\lambda) + \beta_3 [(\tau\lambda)\exp(-\tau\lambda)], \quad (4.1)$$

where λ is the time constant related to the equation and β_1 , β_2 and β_3 are parameters of the model. The model in equation (4.1) can be treated as a constant plus Laguerre function consisting of a polynomial times an exponential decay term. Generated forward rate curves are of shapes typical for yield curves - monotonic, humped and S-shaped. (Nelson and Siegel 1987)

To obtain a yield curve from a forward curve we need to integrate the expression in equation (4.1) with respect to x from 0 to τ

$$y(\tau) = \frac{1}{\tau} \int_0^\tau r(x) dx. \quad (4.2)$$

Solution to this exactly generates the popular figure of parsimonious Nelson & Siegel second-order model of three latent factors, presented as an equation where the dependent variable $y(\tau)$ denotes yield at maturity as the function of maturity τ ,

$$y_t(\tau) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} \right) + \beta_{3t} \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau} \right), \quad (4.3)$$

with λ as given parameter. Similar to the rest of coefficients, the coefficient λ , as provided by Nelson and Siegel (1987), does not have any exact economic meaning. In fact, from the econometric point of view, it is a constant assuring a specific slope of yield curve. In practice, higher values produce a faster decay of yield curve and vice versa. In conclusion, the factor λ plays a key role especially in fitting the longer end of yield curves.

As written in the previous paragraph, Nelson and Siegel (1987) interpret the coefficients β_1 , β_2 and β_3 in (4.3) in a specific way, exactly as the long-, short- and medium-term components, respectively. The factor loadings then represent such weights for beta coefficients in terms of yield curve composition.

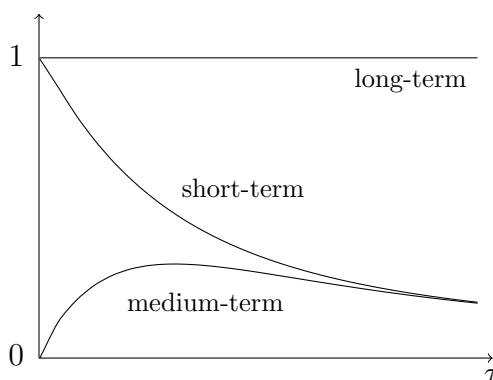


Figure 4.1: Factor loadings, tenors on x-axis

Source: Author's calculations.

As it can be seen in Figure 4.1, the long-term component remains 1 for all tenors. Thus it participates on the compositions of whole yield curve, hence the property of long-termness is quite obvious. The remaining two factors, the short- and medium-term, contrary to the long-term one, goes to zero in the limit. The short-term one starts at 1 and sharply falls to 0, thus the marginal

effect of the short-term factor continuously decreases with time to maturity. The medium-term factor starts at 0 and then it turns down and decay to 0. Hence, it affects rather longer maturities, but not as much as the long-term component. To conclude, this argumentation corresponds to the interpretation provided in detail by Nelson and Siegel (1987).

The general structure of model (4.3) is further developed by Nelson and Siegel (1987) and Siegel and Nelson (1988). Based on empirical results, they drop a part of third factor, $\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right)$, such that the model becomes even more parsimonious,

$$y_t(\tau) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t} (-e^{-\lambda_t\tau}). \quad (4.4)$$

Once again, given parameter λ , the function in (4.4) is linear in parameters β_1 , β_2 and β_3 , thus Nelson and Siegel (1987) estimate it by a simple linear method of least squares, in contrast with other contemporary models.

4.2 Diebold & Li Interpretation

Despite the wide variety of competing models, Diebold and Li (2006) revise the classic Nelson-Siegel three-factor approach. Besides a modernization, they improve the model to the extent of reinterpretation of the three coefficients β_1 , β_2 and β_3 . Moreover, they provide a satisfactory method of forecasting yield curves.

To begin with, Diebold and Li (2006) use the version as in equation (4.3) which actually Nelson and Siegel (1987) evaluate as worse for the fitting and forecasting yield curves. Nevertheless, Diebold and Li (2006) argue that the second and the third factor loadings in (4.4) are very similar. This consequently causes multicollinearity and prevents from an intuitive econometric interpretation of coefficients.

Next, as it is said in the previous paragraph, a huge significance is given to the interpretation of three latent factors. Diebold and Li (2006) keep the prior classification as long-, short- and medium-term components. On top of that, they re-value these parameters β_1 , β_2 and β_3 as level (L), slope (S) and curvature (C), respectively, as printed in the equation (4.5). Reasonable explanation for this comes from the maturity structure of the factor loadings.

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} \right) + C_t \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau} \right). \quad (4.5)$$

Firstly, following Diebold and Li (2006), the factor loading on the long-term coefficient, β_1 , do not change with maturities and multiplies L_t by 1, consequently any change in L_t vertically shifts the whole curve. Hence it might be said that L_t represents level.

Secondly, the short-term coefficient, β_2 or now S_t , can be evaluated as a slope which Diebold and Li (2006) define as the difference $y(\tau = 12) - y(\tau = 3)$ for tenors in months. Empirically, the factor loading of S_t quickly decays from 1 to 0 affecting rather short-end of yield curve, hence it is at most related to its slope.

Lastly, the curvature is connected with the medium-term factor, β_3 or now C_t , since its factor loading of quasi-concave shape influences at most the middle of yield curve. Diebold and Li (2006) again theoretically define curvature such as $2y(\tau = 24) - y(\tau = 120) - y(\tau = 3)$ for tenors in months.

Besides the beta coefficients, we still have one, but for the purpose of this thesis crucial, parameter remaining and unexplained in detail - the λ factor. Nelson and Siegel (1987) fix λ since this distinctly simplifies the computation allowing usage of the simple and linear ordinary least squares. Diebold and Li (2006) follow the same logic of exogenously predetermined λ . Emphasizing that λ determines the maturity at which the factor loading of the curvature component achieves maximum, Diebold and Li (2006) globally fix the value of λ at exactly 0.0609. Alternatively, in the related literature, Diebold et al. (2006b) instead use the value of 0.077. As they mention, these values every time maximize the medium-term factor for an average month in the range of maturities.

Finally, concerning forecasting, Diebold and Li (2006) give an efficient approach, although it follows a simple first-order autoregressive model, AR(1). Empirically, since the β -estimates appear to be strongly serially correlated, the model-based forecasting becomes truly feasible. (Diebold and Li 2006) All the β -coefficients are forecasted with regards to previously predicted values, such that

$$\hat{\beta}_{i,t+h} = \hat{\phi}_i + \hat{\psi}_i \hat{\beta}_{i,t}, \quad (4.6)$$

for $i = 1, 2, 3$ where ϕ and ψ are estimated AR(1) coefficients. Together this

gives expected yield at maturity in period $t + h$ in the consecutive form:

$$\hat{y}_{t+h}(\tau) = \hat{\beta}_{1,t+h} + \hat{\beta}_{2,t+h} \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} \right) + \hat{\beta}_{3,t+h} \left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau} \right). \quad (4.7)$$

As noted by Diebold and Li (2006), this provides surprisingly good results in out-of-sample forecasting especially one year ahead.

4.3 New Insight

Recall that the crucial seminal papers in the related literature, bringing innovative ideas of fitting and forecasting yield curves, predetermine the λ -factor exogenously. Diebold and Li (2006) derive λ as a value maximizing the medium-term factor for an average τ , in their case exactly 30 months. Given mathematically,

$$\hat{\lambda} = \max \left(\frac{1 - e^{-30\lambda}}{30\lambda} - e^{-30\lambda} \right). \quad (4.8)$$

In this thesis, we adopt a similar principle of constant λ and introduce an alternative way of its estimating. In order to capture time-development of yield curves, we particularly update the original Diebold-Li approach.

4.3.1 Alternative Two-Step Approach

To begin with, as a cornerstone let us once more consider the latent-variable model in the version of Nelson and Siegel (1987) as extended by Diebold and Li (2006),

$$y_t(\tau) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} \right) + \beta_{3t} \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right). \quad (4.9)$$

Naturally, to provide the highest estimation precision, consequently capable of efficient forecasting, remains our main motivation. The λ is now seen as full-fledged so we need to optimize the model (4.9) for the full set of parameters $\theta_t = (\beta_{1t}, \beta_{2t}, \beta_{3t}, \lambda)$. To deal with non-linearity we introduce the two-step approach estimating an overall λ and time-varying parameters β . Briefly described, an estimate of λ , denoted $\hat{\lambda}$, obtained in the first step is then used to get the rest of estimates $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$.

For better understanding, why we should compute λ in a separate process, let us now reverse the chronological order of explaining the above outlined steps. In other words, only for the explanatory purpose, we pretend that λ is already

somehow estimated by the method given as the first step. Therefore, having the exact λ computed, in the second step we confront the need of estimating the rest of parameters in equation (4.9). Since the lambda is again endogenously determined and fixed, we can implement the basic estimation method of ordinary least squares, henceforth OLS, used as well among the related literature, i.e. Nelson and Siegel (1987) or Diebold and Li (2006). In the matrix notation we can rewrite equation (4.9) as a latent-variable model for n tenors and given $\hat{\lambda}$ as follows

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_n) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\hat{\lambda}\tau_1}}{\hat{\lambda}\tau_1} & \frac{1-e^{-\hat{\lambda}\tau_1}}{\hat{\lambda}\tau_1} - e^{-\hat{\lambda}\tau_1} \\ 1 & \frac{1-e^{-\hat{\lambda}\tau_2}}{\hat{\lambda}\tau_2} & \frac{1-e^{-\hat{\lambda}\tau_2}}{\hat{\lambda}\tau_2} - e^{-\hat{\lambda}\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\hat{\lambda}\tau_n}}{\hat{\lambda}\tau_n} & \frac{1-e^{-\hat{\lambda}\tau_n}}{\hat{\lambda}\tau_n} - e^{-\hat{\lambda}\tau_n} \end{pmatrix} \begin{pmatrix} \beta_{1t} \\ \beta_{2t} \\ \beta_{3t} \end{pmatrix} + \begin{pmatrix} u_t(\tau_1) \\ u_t(\tau_2) \\ \vdots \\ u_t(\tau_n) \end{pmatrix}, \quad (4.10)$$

alternatively with bold latin letters representing corresponding matrices,

$$\mathbf{Y}_t = \mathbf{F} \cdot \boldsymbol{\beta}_t + \mathbf{u}_t, \quad (4.11)$$

where $u_t \sim N(0, \sigma^2)$ denotes disturbances for time periods $t = 1, \dots, T$. For every t in time range the matrix $\boldsymbol{\beta}_t$ is then estimated via OLS formula. In this way we follow the standard methodology provided, for example, in Wooldridge (2009), such that

$$\hat{\boldsymbol{\beta}}_t = (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{Y}_t. \quad (4.12)$$

Having a matrix of estimated values β_{1t}, β_{2t} and β_{3t} , we can follow Diebold and Li (2006) and interpret these coefficients as level, slope and curvature.

Let us now go back to the first step. As outlined above, the first step is related to deriving a proper value of λ . Such λ should provide reasonable basis for beta estimation (4.12) in order to ensure efficiency of the entire model when fitting and forecasting yield curves. Requirements of suitability and efficiency requires measurement if certain λ fulfils these demands or not. Therefore, we start building our sub-model for obtaining $\hat{\lambda}$ by measuring precision of estimated yields. As a quantitative measuring tool we use mean squared error (MSE), in literature, e.g. Wooldridge (2009), given as

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2, \quad (4.13)$$

where n usually represents number of observations. For our purpose, we compute MSE for each time-period t as a function of λ .

As we know, the lambda coefficient is a relevant part of equation (4.9) with certain econometric interpretation, see sections 4.1 and 4.2. Hence, in particular we try to keep the way how it contributes to the beta estimation by including the whole OLS from (4.12) into our efficiency measurement. Furthermore, no pre-estimated starting values of β -coefficients are required. Mathematically, we implement (4.11) and (4.12) into the MSE equation (4.13). With this adjustment, the complete MSE equation for observed yields in time t looks as follows

$$MSE_t(\lambda) = \frac{1}{n} \sum_{i=1}^n \{(\mathbf{Y}_t)_i - [\mathbf{F}((\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{Y}_t)]_i\}^2, \quad (4.14)$$

where bold letters again denote the matrices from (4.10), n denotes number of tensors and i refers to a relative position, exactly a row, of actual tensor in the tensor matrix

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix}. \quad (4.15)$$

Since the mean squared error function (4.14) compares true and fitted values of yields, we should minimize MSEs with respect to λ among all observations in time, in order to obtain the most efficient $\hat{\lambda}$. Broadly speaking, a smaller MSE provides stronger evidence of better fit. The presence of coefficient λ in exponential function causes that optimizing (4.14) requires some more sophisticated optimization method than only OLS. Based on the "differential background" of the Nelson-Siegel model, we use an approach examining second derivatives - the Newton method. As noted by Gill and Leonard (2001), among the class of newly modified Newton methods, exactly the Quasi-Newton class of iterative algorithms, the Broyden-Fletcher-Goldfarb-Shanno (BFGS) is the most popular one. For methodology details see section 3.2. To implement the BFGS method we use the object-oriented programming language `0x` proposed by Doornik (1996), see www.doornik.com, and as a starting value for optimization we reasonably take $\lambda_0 = 0.0609$.

Before completing the function for BFGS purpose, we adjust it, moreover, by differentiating between the older and newer observations. Our data-sets cover relatively long term structure of yields containing, inter alia, some non-

systematic radical shocks. In case of forecasting, without paying any attention to this, estimated values are affected by the present yields to the same extent as by the yields 10 years old. Thus, in order to improve adaptability to the development of interest rates, we weight our observations, consequently series of MSE_t , by making difference between the older and newer ones. The first and oldest MSE_1 is rated as the least significant, whereas the newest MSE_t is rated as the most significant. These weights are given by a formula dependent on t , exactly

$$w_2(t) = \frac{e^{\frac{t}{T}} - 1}{e - 1}, \quad (4.16)$$

where T denotes a number of observations. With time, computed mean square errors are given greater weights. Of course, we provide comparison with a non-weighted method, fixing w equal to 1 and giving all observations the same weight, hence

$$w_1(t) = 1. \quad (4.17)$$

Finally, to provide one overall value of λ , we put together weighted mean squared errors (WMSE) over time from $t = 1$ to T ,

$$f(\lambda) = \sum_{t=1}^T WMSE_t = \sum_{t=1}^T w_k(t) \cdot MSE_t, \quad (4.18)$$

for specified weight $w_k(t)$. This sum is then used as the minimized function with respect to λ . Because of the software limitations, we add the negative sign before the whole sum in order to turn the minimization into maximization. Therefore, our model for determining λ has the following form

$$\hat{\lambda} = \max_{\lambda} \left(- \sum_{t=1}^T w_k(t) \left(\frac{\sum_{i=1}^n [\{\mathbf{Y}_t\}_i - \{\mathbf{F}((\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{Y}_t)\}_i]^2}{n} \right) \right). \quad (4.19)$$

Once again, to summarize our two-step approach, we compute a data-specific value of λ in order to provide the most suitable value for the consequent fitting adopting the classic Nelson-Siegel approach.

4.3.2 Forecasting

In terms of predicting future values of bond yields, we take an advantage of two methods. Particularly, we follow the Diebold-Li framework as published in

Diebold and Li (2006). They compare variety of models, starting with a simple random walk and ending with more demanding VAR structures. Among these methods, Diebold and Li (2006) highlight the AR(1) model applied to term structures of estimated β -coefficients as the most efficient one. Since our intention is even to enhance the overall performance of the Nelson-Siegel model, we do not hesitate to directly adopt the Diebold-Li framework, as given in equations (4.6) and (4.7), and apply it on β -coefficients estimated via our alternative model introduced in section 4.3.

As a competing way of forecasting, we consider likewise the AR(1) model. However, in this case we directly predict bond yields which actually do not depend on any model for fitting yield curves. Although the application of this model seems to be too simple, our econometric explanation results from the empirical analysis. Basically, we argue that a development of term structures of government bonds changes over time and suffers from many deflections. Thus, by application of our first model (4.7), resulting estimated values of bond yields can be biased as a result of previous evolution in time series and suffer from a possibly very significant autocorrelation simultaneous for all parameters. If we go back to the very structure of this second forecasting approach, we can express it mathematically, such that

$$\hat{y}_{t+h}(\tau) = \hat{\gamma} + \hat{\delta}y_t(\tau), \quad (4.20)$$

where $\hat{\gamma}$ and $\hat{\delta}$ represent estimated autoregressive coefficients.

Chapter 5

Empirical Results

The fifth chapter applies the theoretical framework given in Chapter 4 to the exact data described in Chapter 3. We take three currency regions, EUR, GBP and USD, one after another, and evaluate both fitting and forecasting performance of the innovative two-step model. As a measure of success we use different statistical indicators and also compare our alternative approach with the original proven dynamic Nelson-Siegel model.

5.1 Fitting Yield Curves

In this thesis, estimating the term structure of sovereign bonds, as our prior intention, is ensured generally by two models. The former is the Diebold-Li dynamic interpretation of the Nelson-Siegel model and the latter is our new alternative insight.

For both models we use exactly the full data sets for all considered currency regions as described in section 3.1. Following the original idea of Nelson and Siegel (1987), we fit a yield curve for every single observation in time by incorporating the OLS. In case of Diebold-Li interpretation we use fixed value of $\lambda = 0.0609$ as proposed by Diebold and Li (2006). In case of our alternative approach we rather estimate overall λ for each currency based on given version of data weighting. This is done by the first step in two-step approach likewise introduced in section 4.3.

For each currency we mainly investigate a term structure of estimated level, slope and curvature parameters. Since we face multivariate econometric problems, we describe estimated parameters in terms of their means and standard deviations. For λ coefficients we only list exact values since they are concretely

determined through mathematical optimization and not essentially estimated. Next, we examine a persistency in these parameters incorporating the autocorrelation function (ACF) together with an unit root test, namely the augmented Dickey-Fuller test, henceforth ADF, performed in econometric software Gretl. An estimation accuracy is then measured on the base of estimated residuals, consequently by computing root mean squared errors (RMSE) across maturities. By residual we mean a difference between an observed and a predicted value of a yield at given maturity. RMSE is obtained by taking a square root of mean squared error (MSE), such that

$$RMSE(\tau) = \sqrt{\frac{1}{T} \sum_{i=1}^T (y_i(\tau) - \hat{y}_i(\tau))^2}. \quad (5.1)$$

To find out whether our approach outclasses the prior Diebold-Li version, we likewise provide RMSE comparison between those two models.

All the results are generally reported in the form of tables and figures. Technically, since we have more models, to shorten the notation we use rather specific abbreviations. Hence, let DL denote the Diebold-Li approach given in Diebold and Li (2006) as explained in detail in section 4.2. Next, in case of our alternative model (4.19), we need to distinguish between the version with the exponential weighting and the version with the unit weighting of mean squared errors in the process of obtaining λ . Therefore, let EXP denote the former model

$$\hat{\lambda} = \max_{\lambda} \left(- \sum_{t=1}^T \left(\frac{e^{\frac{t}{T}} - 1}{e - 1} \right) (\dots) \right) \quad (5.2)$$

and let UNIT denote the latter model

$$\hat{\lambda} = \max_{\lambda} \left(- \sum_{t=1}^T (\dots) \right). \quad (5.3)$$

5.1.1 EUR

To begin with, we consider data for the Euro currency bonds and estimate relevant parameters for both models. Their exact means and standard deviations are written in Table 5.1 along with proper λ parameters.

As we can see, our model produces level, slope and curvature more volatile among time compared to the Diebold-Li approach. Interestingly, the level factor acquires higher mean value in case of the new model, exactly it increases

	DL	EXP	UNIT
L	3.5982 (1.0242)	5.3380 (1.4112)	4.7103 (1.0384)
S	-2.1065 (1.2733)	-3.9892 (2.4358)	-3.3492 (1.9336)
C	-2.7345 (1.8950)	-2.3124 (3.2893)	-2.3155 (2.4285)
λ	0.0609	0.0181	0.0247

Table 5.1: EUR - summary statistics of regression results, standard deviation in brackets.

Source: Author's calculations.

up to 5.3 and 4.7 for the EXP and UNIT versions, respectively. Consequently, in the same version order the slope factor mean falls from -2.3 to 4 and -3.3 whereas the curvature mean remains nearly on the the same level. The λ for EXP and UNIT models is computed at a significantly lower value of about 0.02 compared to the initial value of 0.0609 . This is likewise obvious from Figure 5.1 where we plot the time-series of estimated coefficients.

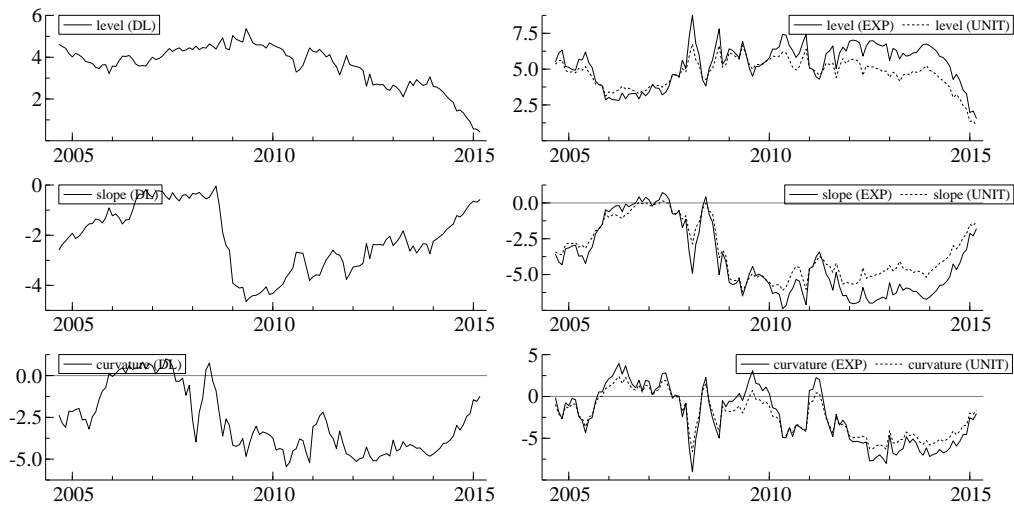


Figure 5.1: EUR - estimated regression coefficients

Source: Author's calculations.

Equation 5.4 relates to the ADF test containing test statistics, for the Students t distribution, computed with regard to the null hypothesis of unit root for level, slope and curvature. As it can be seen, for none of the coefficients, the hypothesis H_0 can be rejected since all values are generally below the 5% critical

value approximately equal 2. These results indicate possible non-stationarity of autoregressive process applied to time series of the level, slope and curvature coefficients.

$$t_{DL} = \begin{pmatrix} -1.4197 \\ -1.2706 \\ -1.3844 \end{pmatrix}, t_{EXP} = \begin{pmatrix} -1.1736 \\ -1.2730 \\ -1.7983 \end{pmatrix}, t_{UNIT} = \begin{pmatrix} 0.0845 \\ -1.3050 \\ -1.6954 \end{pmatrix} \quad (5.4)$$

In combination with the serial correlation plotted in Figure 5.2, we can conclude that all models show signs of significant persistence over time. As to be seen in the enclosed correlogram, both, level and slope, factors are serially correlated at least up to the lag of 40 months, followed by the less correlated level.

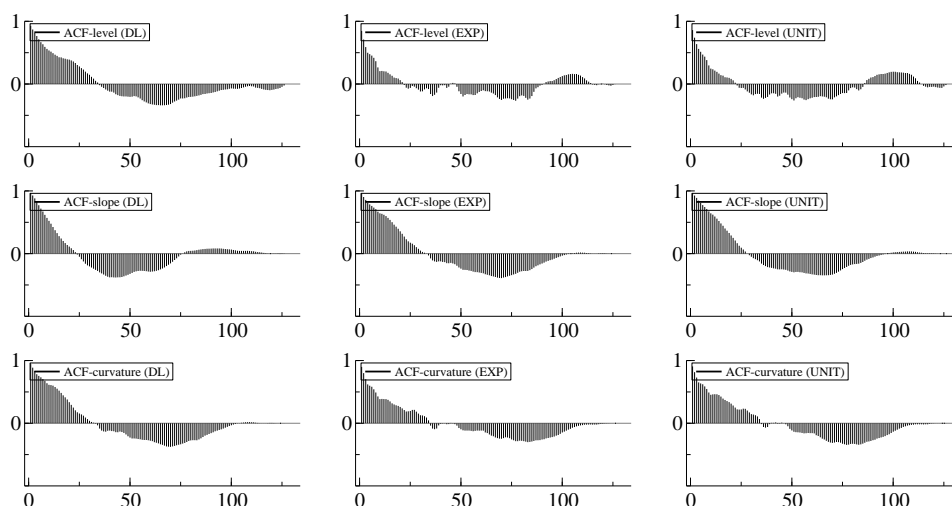


Figure 5.2: EUR - ACF, time lags on horizontal axis
Source: Author's calculations.

In order to assess the estimation efficiency, we compute regression residuals and RMSE for each model and maturity. The calculated residuals are listed in Table A.1, containing means of residuals for each maturity along with their standard deviation given below in brackets. Overall, based on results, except for one maturity our model predicts with substantially lower amount of inaccuracies. Mean squared errors are then plotted in Figure 5.3. All the models start at the RMSE value about 0.07 followed by stagnation around 0.05 and consecutive increase up 0.10. Compared to the Diebold-Li approach, both versions of our model then show significant decline in RMSE beginning as of 5 year maturity which consequently indicates a better in-sample prediction performance.

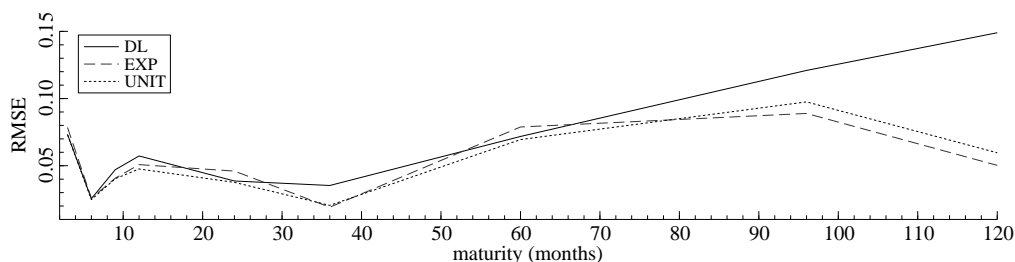


Figure 5.3: EUR - RMSE comparison

Source: Author's calculations.

5.1.2 GBP

Next, reporting results in a similar framework, we approximate the yield curve models for the British Pound (GBP). The computed estimates for each model along with their standard deviation are given in Table 5.2.

	DL	EXP	UNIT
L	4.5659 (0.7077)	4.8621 (0.6769)	4.7629 (0.6341)
S	-1.1806 (1.8285)	-1.5500 (2.3256)	-1.4361 (2.1595)
C	-1.5855 (2.6283)	-1.2773 (2.7954)	-1.3528 (2.6619)
λ	0.0609	0.0347	0.0409

Table 5.2: GBP - summary statistics of regression results, standard deviation in brackets.

Source: Author's calculations.

Interestingly, the estimated β -coefficients for the exponential and the unit weighting do depart from the Diebold-Li coefficients in absolute value approximately not more than by 0.3. Volatilities of level, slope and curvature among time expressed as values of standard deviations across models are likewise of very much alike values. Last but not least, compared to the DL model, λ -values 0.035 and 0.04 for the EXP and the UNIT models are respectively only by one third lower. As well as for the first considered currency we show time series of estimated coefficients in plot, exactly in Figure 5.4.

The results of the augmented Dickey-Fuller test for the GBP shows that we are not able to statistically eliminate a presence of unit-root for all the considered coefficients. All the values of t statistics appear to be below the two-sided 5%

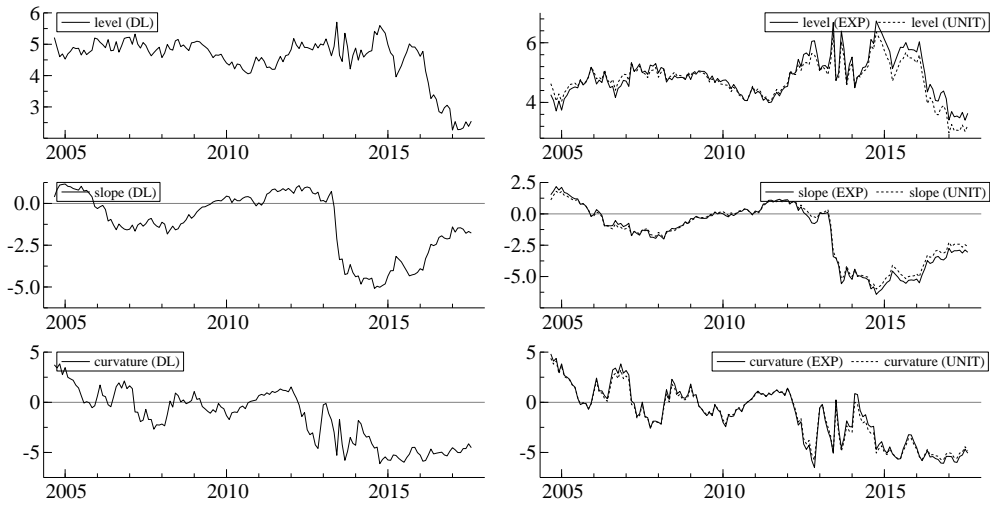


Figure 5.4: GBP - estimated regression coefficients

Source: Author's calculations.

critical boundary. If we enlarge the rejection interval up to 10%, the appropriate critical value equal to 1.645 is then exceeded at least in case of the EXP curvature and the UNIT slope, other coefficients except for the DL level and the UNIT level are only relatively close to 1.6. Nevertheless, we still assess all coefficients as considerably persistent. This viewpoint is supported also by observing serial correlation plotted in Figure 5.5. It can be seen that again at least for the 25 time lags, the slope and the curvature time series are positively correlated. The level factor indicates shorter correlation reaching zero at about 18 lags.

$$t_{DL} = \begin{pmatrix} 0.103 \\ -1.615 \\ -1.481 \end{pmatrix}, t_{EXP} = \begin{pmatrix} -1.765 \\ -1.596 \\ -1.730 \end{pmatrix}, t_{UNIT} = \begin{pmatrix} -1.005 \\ -1.708 \\ -1.615 \end{pmatrix} \quad (5.5)$$

The means of the OLS residuals are specified in Table A.2 along with their squared variances. The Diebold-Li model provides lower values up to the maturity of 5 years. For the long ends of the GBP yield curves the EXP model followed by the UNIT model predicts on average significantly better. We verify these findings by analysing the set of appropriate RMSEs. All three models go approximately hand in hand up to mentioned 5 years. After that, both versions of our model strongly decline from the value of 0.1 to 0.6 compared to 0.12 in case of the Diebold-Li approach. Concerning the EXP and UNIT models, for

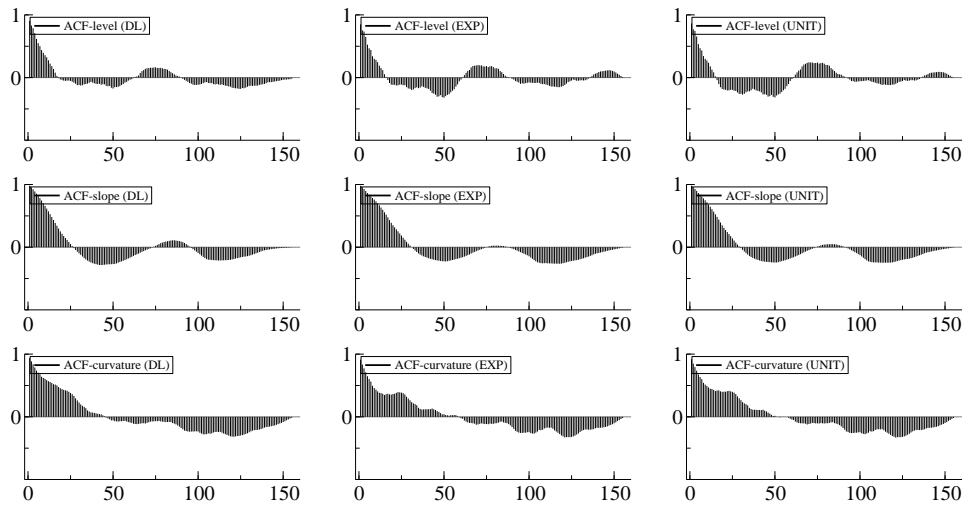


Figure 5.5: GBP - ACF, time lags on horizontal axis
Source: Author's calculations.

shorter maturities the UNIT model is more precise in predicting. On the other hand, in efficiency for longer maturities the EXP model slightly prevails.

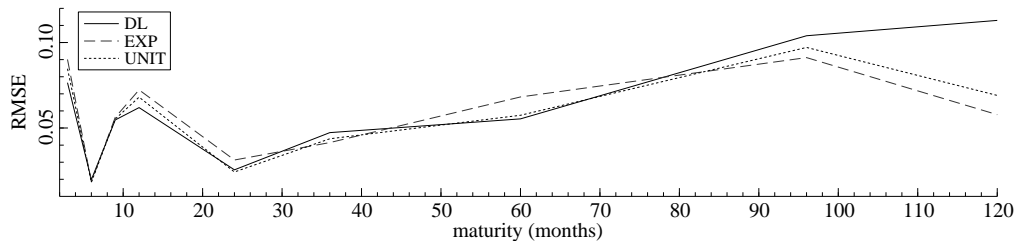


Figure 5.6: GBP - RMSE comparison
Source: Author's calculations.

5.1.3 USD

Finally, let us investigate the currency the very Nelson-Siegel model was designed for - the US Dollar. Contrary, we take into account the actual set of data from 31 January 2000 to 28 February 2015 as described in section 3.1. Estimated values, exactly their means and deviations, we again provide in the same look in Table 5.3.

The level means as well as the slope means are slightly higher in absolute values in case of the EXP and the UNIT models than in case of the DL appro-

	DL	EXP	UNIT
L	4.4229 (1.0423)	5.2454 (0.8850)	4.9906 (0.8807)
S	-2.5161 (1.6542)	-3.4556 (2.1992)	-3.1854 (2.0202)
C	-2.7674 (2.1305)	-1.8081 (2.5778)	-2.0517 (2.2598)
λ	0.0609	0.0272	0.0337

Table 5.3: USD - summary statistics of regression results, standard deviation in brackets.

Source: Author's calculations.

ach. On the other hand, the curvature means are lower. In terms of volatility, the level factor is then less volatile among time contrary to the more volatile slope and curvature parameters.

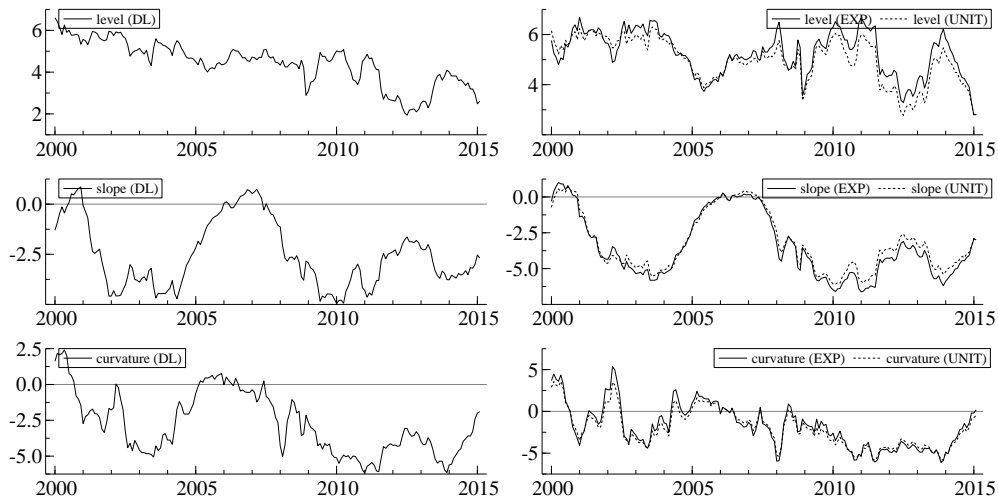


Figure 5.7: USD - estimated regression coefficients.

Source: Author's calculations.

The unit-root test statistics are listed in equation 5.6 again for level, slope and curvature from top of matrices respectively. Since the t -statistics for slope as well as for curvature are greater than 5% critical value 1.960, we can reject the null hypothesis of the unit-root for both slope and curvature for all considered models.

$$t_{DL} = \begin{pmatrix} -1.352 \\ -2.059 \\ -2.645 \end{pmatrix}, t_{EXP} = \begin{pmatrix} -2.056 \\ -2.273 \\ -3.001 \end{pmatrix}, t_{UNIT} = \begin{pmatrix} -1.659 \\ -2.229 \\ -2.920 \end{pmatrix} \quad (5.6)$$

Similar results are obtained by seeing serial correlation for all the models and all coefficients in the enclosed ACF plot in Figure 5.8. For all parameters we can still observe certain level of autocorrelation. Over the long run, the lowest one is shown by the curvature coefficient which is not even significantly correlated in the very short run. Similarly, the ACF for the level indicates quite low correlation compared to the original Diebold-Li model. These findings partly correspond with results provided by Nelson and Siegel (1987) who argue that AR(1) processes on the level and the slope might be non-stationary whereas the AR(1) on the curvature factor might be stationary.

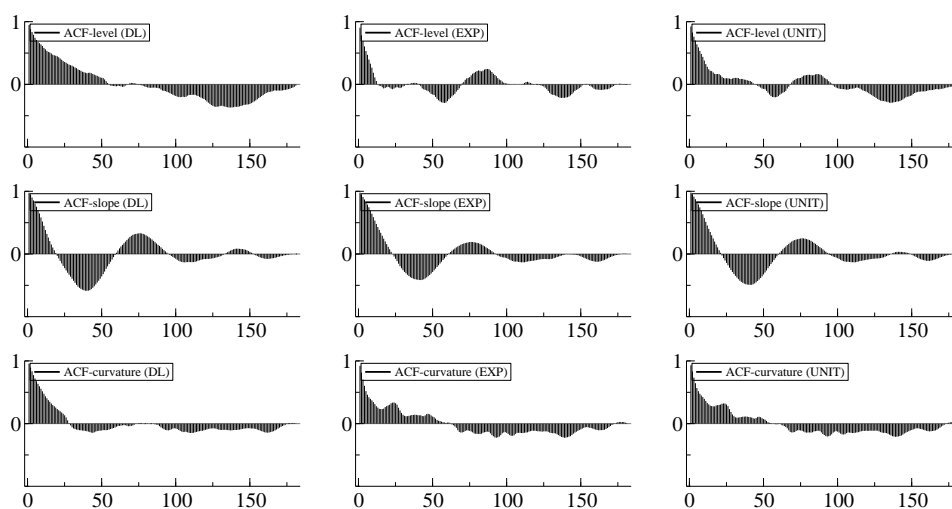


Figure 5.8: USD - ACF, time lags on horizontal axis.

Source: Author's calculations.

Average residuals and their volatilities are computed in Table A.3. Despite the relatively small values of residuals, both versions of our model generally seem to have better predictive performance. The root mean squared errors plotted in Figure 5.9 shows that the EXP and the UNIT models follow the DL model at the short end while at the long end they are strongly better.

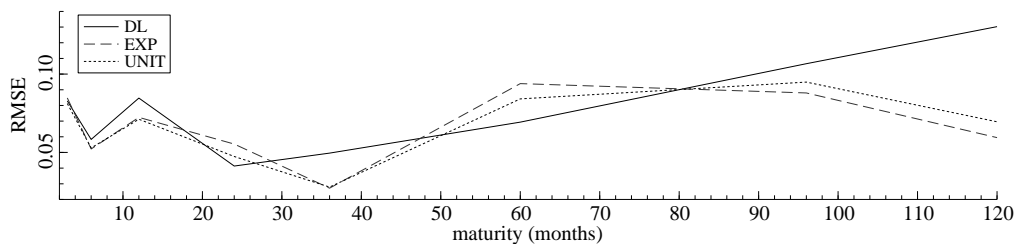


Figure 5.9: USD - RMSE comparison.

Source: Author's calculations.

5.2 Out-of-Sample Forecasting

Unlike the previous section where we apply given models on all observed data and examine whether consequential results adequately fit true yield values, in the following section we consider only a considerably large proportion of observations and using our models we try to forecast remaining values in a set. This is what we call out-of-sample forecasting.

For purposes of this part, we again use the data described in section 3.1. Since we do not have data of exactly same starting or ending dates, we can not set any specific date from which to start forecasting. Instead, we set an overall horizon of forecasting equal to 24 months. In practise, we drop the last 24 observations from the processes of level, slope, curvature and λ factor estimation. For forecasting these 24 yields, we then use the basic autoregressive model of first order, denoted AR(1), applied separately to both yields and factors. Further details on these approaches we state in subsection 4.3.2. In addition, as important assumption we take the proven persistency of beta coefficients over time. As Diebold and Li (2006) mention, this ensures solid meaningfulness of application of the AR(1) exactly to these coefficients, although a direct non-stationarity of the AR(1) model causes serious problems.

Similar to the previous section, we present our results of out-of-sample forecasting as a comparison between the Diebold-Li model and our models, with the exponential and the unit weighting of mean squared errors, and the model using the AR(1) applied directly to yields. For each currency, we list average values of predicted AR(1) coefficients, γ and δ , which are related to the general first-order autoregressive model

$$e_{t+h} = \gamma + \delta e_t + \epsilon_t. \quad (5.7)$$

Then we measure a predicative efficiency by incorporating the root mean squared error formula and by computing RMSE, not for every maturity but for every month forecasted. Mathematically,

$$RMSE(t) = \sqrt{\frac{1}{T} \sum_{i=1}^n (y_t(\tau_i) - \hat{y}_t(\tau_i))^2}. \quad (5.8)$$

where i refers to the actual number of row in the matrix 4.15. This gives us a brief ranking of models according to their forecasting accuracy. To illustrate the very process of yield curve prediction, we plot yield curves based on two best models against observed yield curves for selected points in time.

Last but not least, we likewise use the shorter notations DL, EXP and UNIT for the Diebold-Li model and the models with exponential and unit weighting, respectively. The added AR(1), taking into account only bond yields, we denote simply AR(1).

5.2.1 EUR

For the first currency, we start the forecasting on 31 December 2012. We estimate the AR(1) coefficients for each model and list averages of the resulting values in Table B.1. In the left part of the table, we can see average values for models extrapolating estimates of level, slope and curvature, while in the right part, there are average values for the simple AR(1) model which is applied directly to yields divided by tenors.

Next, the computed values of RMSE are plotted in Figure 5.10. We can see that all curves start together around 0.2 but then from the 8th forecasted month they start to diverge. Both AR(1) processes based on the EXP and the UNIT estimated regression coefficients climb nearly up to 2, while the simple AR(1) applied to observed yields indicates truly great predicting efficiency, especially from 9 to 18 months ahead. In Figure 5.11 we provide overview of the yield curves forecasted 1, 5, 10 and 18 months ahead. These are obtained by incorporating two best models with regards to our findings given in Figure 5.10, thus we take into account the Diebold-Li approach and the AR(1) applied to yields. As we can see, the plotted curves perfectly illustrate decreasing accuracy of forecasting with increasing time horizon. The Diebold-Li approach is more accurate especially in the shorter horizon. On the other hand, the parsimonious

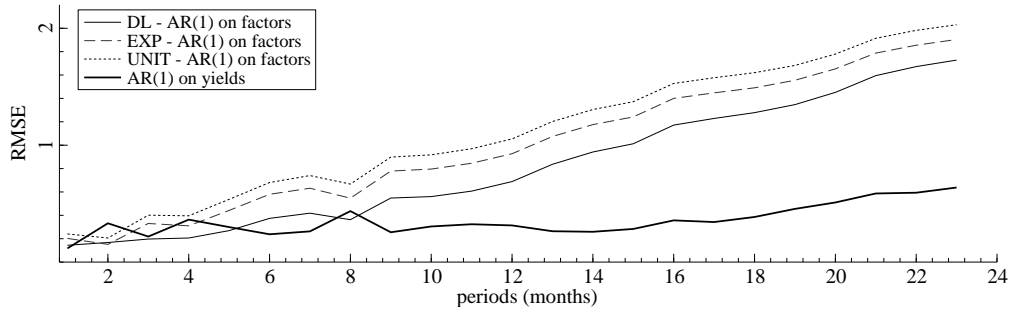


Figure 5.10: EUR - comparison of RMSEs by forecasting methods.
Source: Author's calculations.

AR(1) model constantly under-estimate the true yield curves, but in the longer horizon it starts getting closer and provides reasonably good results.

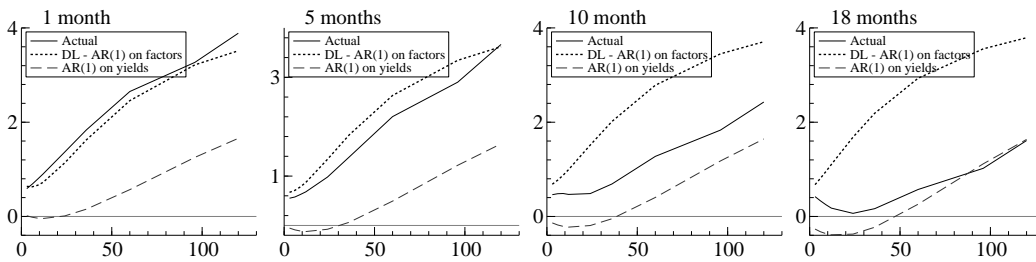


Figure 5.11: EUR - forecasted yield curves. Tenors on horizontal axis.
Source: Author's calculations.

5.2.2 GBP

In case of the British Pound, the forecasting process starts on 30 November 2010. Again, in Table B.2, we provide a complex overview of estimated coefficients of the AR(1) models. If we consider only models based on predicting level, slope and curvature, the obtained average values for these factors do not significantly differ from each other. This, in particular, coincides with our findings made while investigating only fitting of yield curves in subsection 5.1.2.

As to be seen in Figure 5.12, none of the RMSE curves indicates any significant difference from other curves, which verifies the information given in Table B.2. Up to 3 months, the best approximation is provided by the autoregressive process for the Diebold-Li level, slope and curvature factors. Nevertheless, in the horizon from 3 to 24 months, again the trivial AR(1) applied to yield ra-

tes gives the best results. Anyway, the differences between the RMSEs of this approach and the rest of models are not so great.

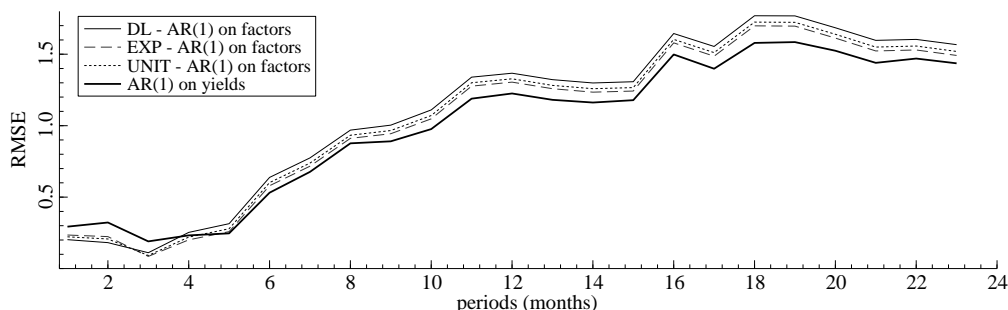


Figure 5.12: GBP - comparison of RMSEs by forecasting methods.

Source: Author's calculations.

What effect this similarity in coefficients actually has on the very yield curves we depict in Figure 5.13. In the long run, the yield curves, forecasted via two best-predicting models, our alternative with exponential weighting of MSEs and the simple AR(1), go hand in hand significantly over observed yield curves. On the other hand, up to 5 months, the forecasted and the observed curves are nearly equivalent.

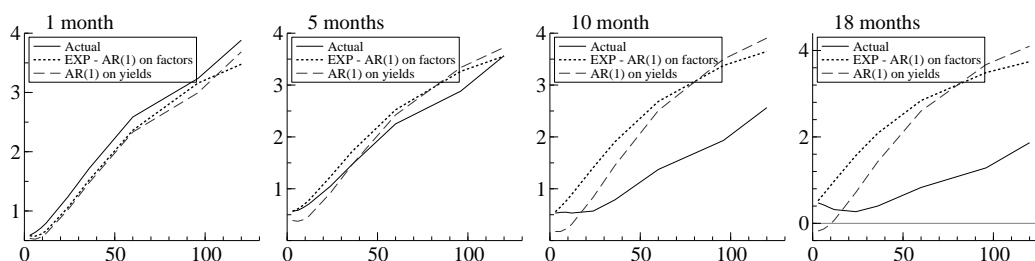


Figure 5.13: GBP - forecasted yield curves. Tenors on horizontal axis.

Source: Author's calculations.

5.2.3 USD

Forecasting the bond rates for the US Dollar begins as of 31 January 2013. As with the preceding currencies, we provide the average estimated coefficients in Table B.3. Again, on the left side we cover models forecasting β -coefficients and on the right side we cover AR(1) model directly predicting yields for each maturity.

Measuring the performance likewise brings us series of root means squared errors displayed in Figure 5.14. In the race for the best forecasting model we can directly eliminate AR(1) base on UNIT model which is worse among the entire horizon. Next, overall worse results are also provided by the original Diebold-Li approach. On the other hand, from 3 months up to 18 months the EXP model ensures solid forecast with such low RMSE approximately equal to 0.3 at maximum. The simple AR(1) to yields is then better in the very short and the very long horizon.

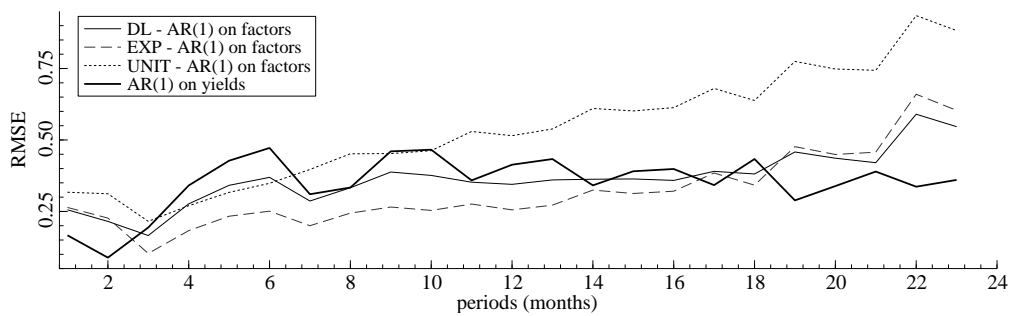


Figure 5.14: USD - comparison of RMSEs by forecasting methods.

Source: Author's calculations.

In spite of the results given in Figure 5.14, in Figure 5.15 we compare the Diebold-Li approach and our alternative model containing the exponential weighting of mean squared errors. The US Dollar is the currency for which Diebold and Li (2006) designed their AR(1) model, thus we emphasize this comparison as possibly more meaningful. Similar to the previous paragraph,

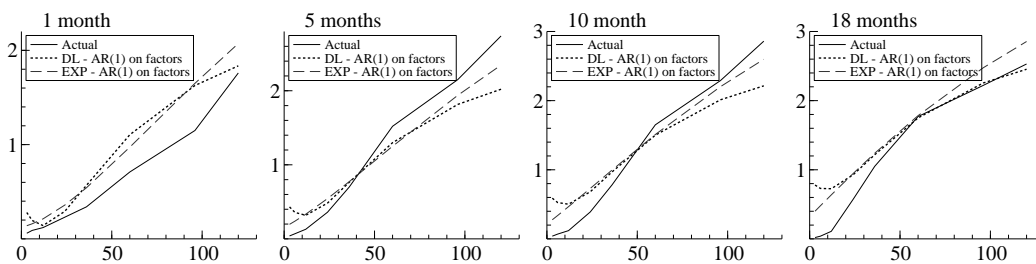


Figure 5.15: USD - forecasted yield curves. Tenors on horizontal axis.

Source: Author's calculations.

our model is better in the medium range of 5 and 10 months. On the other hand, in the short horizon and the long horizon, the Diebold-Li model provides better results comparable to the simple AR(1) applied only to yields.

Chapter 6

Conclusion

This bachelor thesis aims on an overall improvement of the original Nelson-Siegel model reinterpreted by Diebold and Li (2006). The decision to use exactly this relatively parsimonious Diebold-Li approach was influenced mainly by the intention to test whether this simple model, potentially slightly updated, can be efficiently applied to present data of different currency regions.

In Chapter 4, following the mentioned motivation, we introduce the alternative model in order to determine exact λ for each data set. Of course, we only provide one possible option which is not in general proven as the most efficient, nevertheless we build it on straightforward measurement method and very powerful optimization method, which might ensure very strong results. At this point, the only issue which deserves discussion is a starting value for the BFGS optimization process. Automatically, inspired by the Diebold-Li framework, we take the value equal to 0.0609 without exact explanation. To support a meaningfulness of this idea, based on our secondary experiments, this value roughly corresponds to the values maximizing the medium term factors for given sets of maturities. Moreover, with respect to the factor loadings, quite extravagant values ensure totally unreasonable results. Therefore, we can conclude that, without any additional computations, it is possible to use the value 0.0609.

Unfortunately, the results of fitting and forecasting listed in Chapter 5 are not generally consistent and do not directly recommend any model as the best one. When focusing on fitting yield curves, a predicative accuracy depends on a value of factor λ , since our update consists mainly in providing a method for lambda determination. In general, for shorter maturities, our alternative model, especially the one with the unit weighting, is a solid rival to the original Diebold-Li approach. Moreover, for longer maturities, from the maturity of 7

years, our model, in this case the one with exponential weighting, provides substantially better results. It can be said, that this conclusion holds for all three currency regions, except for the USD mid-range maturities. Note that the newly obtained lambda factors are all at least lower than the original value equal to 0.0609.

Concerning the out-of-sample forecasting, the best accuracy is shown predominantly by the simple first-order autoregressive model applied to interest rates. The Diebold-Li approach based on extrapolating model coefficients is significantly better only in case of the US Dollar from 3 to 18 months ahead. In this case, the model is the most efficient when applied to the estimates obtained via our alternative approach with the exponential weighting of mean squared errors. For other currencies all considered applications of Diebold-Li framework predicts comparably among themselves, but slightly worse than the AR(1) applied to yields. We attribute this mainly to the persistency of AR(1) models applied to level, slope and curvature which causes inability to properly capture the development of interest rates becoming in later years even negative.

To summarize our findings, the principle of alternative lambda determination appears to be quite rational. By improving the predicative accuracy of longer maturities it consequently improves the overall model performance. Moreover, at least for some currencies, it ensures more precise estimates of future rates. All in all, we evaluate this modified Diebold-Li framework, in its prior version, reasonably applicable to present data, although it is not perfectly suitable for all currency regions. On the other, currently when interest rates are driven by too many factors, as we mention in the previous paragraph, it is a truly tough task to provide just one universal approach for everything. Therefore, when dealing with a problem of proper model selection for fitting or forecasting yield curves, still we may suggest to use the Diebold-Li framework with exponentially weighted MSEs merely for fitting purposes, but for exact forecasting it might be better to use likewise the straightforward application of AR(1) to interest rates. Nevertheless, if we strongly appeal to precision, it is possibly better to go a little bit further into details. At this point, as an auxiliary factor, we can use some real bond market volatility indicator. Concerning this thesis, our findings connected with bond market volatilities for all the currency regions show an interesting correlation. For illustration, we compare a data retrieved with permission of Bloomberg via Bloomberg Professional Service platform with relative difference in errors between our models. Firstly, recall that for the US Dollar and the British Pound the Diebold-Li model and

the AR(1) applied to yields provide similar predicative performance, whereas for the Euro the latter AR(1) is substantially better. Secondly, as we can see in Table 6.1, the Euro bond market is far the most volatile compared to the US Dollar and the British Pound.

Maturity	EUR	USD	GBP
1-3 years	-	65	76
3-5 years	81	54	62
5-7 years	122	45	53
7-10 years	184	38	41
ALL>1 year	160	32	37

Table 6.1: Bond market volatilities in %
Source: Bloomberg.

Therefore, by connecting these two findings, we can conclude that for more volatile markets the Diebold-Li approach provides less efficiency in predicting future values. As a more appropriate substitute we would choose the AR(1) model applied to yields.

In conclusion, this thesis contributes to the topic of yield curve modelling by revaluing the Nelson-Siegel model in the more or less outdated version. The introduced alternatives, appealing to an overall improvement and incorporating sound methodology, give generally positive signs of success in application to present data.

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Appendix A

In-Sample Fitting of Yield Curves: OLS Residuals

In the following tables we list average values of OLS residuals along with their standard deviations for each maturity and model. The residuals are computed as differences between observed and predicted yields. Consequent deviation from these means is captured by values of standard deviation provided in brackets.

A.1 EUR

tenor	DL		EXP		UNIT	
3M	-0.0525	(0.0511)	0.0045	(0.0783)	-0.0015	(0.073)
6M	0.0002	(0.0254)	0.0001	(0.0258)	-0.0011	(0.0247)
9M	0.0318	(0.0346)	-0.0036	(0.0405)	-0.0013	(0.0403)
1Y	0.0478	(0.0314)	-0.0062	(0.0505)	-0.0015	(0.0476)
2Y	0.0288	(0.0257)	-0.0069	(0.0454)	-0.0004	(0.0376)
3Y	-0.0246	(0.0252)	0.005	(0.0185)	0.0066	(0.0194)
5Y	-0.0604	(0.0387)	0.0441	(0.0652)	0.034	(0.0605)
7Y	-0.1078	(0.0547)	-0.0823	(0.0334)	-0.0899	(0.0375)
10Y	0.1367	(0.0592)	0.0453	(0.0217)	0.0553	(0.0222)

Table A.1: EUR - means and standard deviations of residuals by maturities and type of model.

Source: Author's calculations.

A.2 GBP

tenor	DL		EXP		UNIT	
3M	0.0039	(0.0764)	0.0344	(0.0832)	0.0276	(0.0801)
6M	-0.0007	(0.0198)	0.0013	(0.0185)	0.0001	(0.0182)
9M	-0.0033	(0.0546)	-0.0188	(0.0529)	-0.0163	(0.0528)
1Y	-0.0055	(0.0617)	-0.0305	(0.0655)	-0.0256	(0.0630)
2Y	0.0035	(0.0253)	-0.0156	(0.0270)	-0.0099	(0.0221)
3Y	0.0091	(0.0464)	0.0168	(0.0379)	0.0175	(0.0402)
5Y	0.0043	(0.0551)	0.0406	(0.0547)	0.0332	(0.0468)
7Y	-0.0484	(0.0919)	-0.0397	(0.0820)	-0.0433	(0.0869)
10Y	0.0371	(0.1066)	0.0117	(0.0566)	0.0167	(0.0671)

Table A.2: GBP - means and standard deviations of residuals by maturities and type of model.

Source: Author's calculations.

A.3 USD

tenor	DL		EXP		UNIT	
3M	-0.055	(0.0619)	-0.0132	(0.0837)	-0.0201	(0.0787)
6M	0.0372	(0.0448)	0.0296	(0.0429)	0.0291	(0.0441)
1Y	0.0402	(0.0744)	-0.0124	(0.0713)	-0.0057	(0.0709)
2Y	0.0218	(0.035)	-0.0126	(0.0539)	-0.005	(0.0471)
3Y	-0.0319	(0.0379)	-0.0108	(0.0249)	-0.0098	(0.0261)
5Y	-0.0278	(0.0635)	0.0514	(0.0786)	0.0403	(0.0738)
7Y	0.0799	(0.0705)	-0.0633	(0.0610)	-0.0693	(0.0647)
10Y	0.0953	(0.0887)	0.0313	(0.0505)	0.0405	(0.0565)

Table A.3: USD - means and standard deviations of residuals by maturities and type of model.

Source: Author's calculations.

Appendix B

Out-of-Sample Forecasting: Autoregressive Coefficients

B.1 EUR

	<i>AR coef.</i>	DL	EXP	UNIT	<i>tenor</i>	AR(y): γ	AR(y): δ
L	$\hat{\gamma}$	0.1826 (1.5090)	0.6941 (2.1248)	0.5826 (2.0433)	3M	-0.0163 (0.2191)	0.9981 (0.1024)
	$\hat{\delta}$	0.9487 (0.3838)	0.8553 (0.4597)	0.8709 (0.4171)	6M	-0.0179 (0.2272)	0.9985 (0.1032)
S	$\hat{\gamma}$	-0.0515 (0.6206)	-0.0952 (0.8011)	-0.0739 (0.8558)	9M	-0.0187 (0.2342)	0.9984 (0.1043)
	$\hat{\delta}$	0.9741 (0.2448)	0.9730 (0.2318)	0.9761 (0.2251)	1Y	-0.0195 (0.2408)	0.9984 (0.1057)
C	$\hat{\gamma}$	-0.1521 (0.6388)	-0.2259 (0.4806)	-0.2096 (0.4496)	2Y	-0.0203 (0.2716)	0.9977 (0.1133)
	$\hat{\delta}$	0.9474 (0.1943)	0.8989 (0.1628)	0.9103 (0.1481)	3Y	-0.0220 (0.3142)	0.9982 (0.1246)
					5Y	-0.0213 (0.4423)	0.9982 (0.1578)
					7Y	0.0019 (0.6372)	0.9916 (0.2065)
				10Y	0.0193 (0.9240)	0.9876 (0.2706)	

Table B.1: EUR - average estimated AR(1) coefficients along with average standard errors in brackets.

Source: Author's calculations.

B.2 GBP

	<i>AR coef.</i>	DL	EXP	UNIT	<i>tenor</i>	AR(y): $\hat{\gamma}$	AR(y): $\hat{\delta}$
L	$\hat{\gamma}$	1.4192 (3.4378)	1.4342 (3.4177)	1.4330 (3.4020)	3M	-0.0421 (0.5945)	1.000 (0.1444)
	$\hat{\delta}$	0.7027 (0.7169)	0.7007 (0.7111)	0.7006 (0.7070)	6M	-0.0390 (0.5887)	0.9992 (0.1435)
S	$\hat{\gamma}$	-0.0406 (0.2846)	-0.0420 (0.2886)	-0.0415 (0.2909)	9M	-0.0343 (0.5917)	0.9977 (0.1444)
	$\hat{\delta}$	0.9924 (0.1259)	0.9931 (0.1255)	0.9929 (0.1252)	1Y	-0.0290 (0.6018)	0.9963 (0.1469)
C	$\hat{\gamma}$	-0.1323 (0.2991)	-0.1332 (0.2935)	-0.1329 (0.2904)	2Y	-0.0076 (0.6889)	0.9915 (0.1663)
	$\hat{\delta}$	0.9341 (0.1096)	0.9293 (0.1091)	0.9311 (0.1088)	3Y	0.0155 (0.8141)	0.9871 (0.1933)
					6Y	0.0740 (1.1312)	0.9760 (0.2610)
					7Y	0.1845 (1.5809)	0.9536 (0.3563)
				10Y	0.3644 (2.3014)	0.9159 (0.5101)	

Table B.2: GBP - average estimated AR(1) coefficients along with average standard errors in brackets.

Source: Author's calculations.

B.3 USD

	<i>AR coef.</i>	DL	EXP	UNIT	<i>tenor</i>	AR(y): γ	AR(y): δ
L	$\hat{\gamma}$	0.1479 (1.1955)	0.6162 (1.8917)	0.3165 (1.5654)	3M	-0.0102 (0.2433)	0.9886 (0.0858)
	$\hat{\delta}$	0.9622 (0.2594)	0.8939 (0.3649)	0.9361 (0.2666)	6M	-0.0110 (0.2571)	0.9889 (0.0876)
S	$\hat{\gamma}$	-0.0573 (0.5047)	-0.1625 (0.6372)	-0.0772 (0.7114)	1Y	-0.0070 (0.2662)	0.9868 (0.0900)
	$\hat{\delta}$	0.9788 (0.1717)	0.9676 (0.1693)	0.9813 (0.1527)	2Y	0.0018 (0.3003)	0.9842 (0.0970)
C	$\hat{\gamma}$	-0.1464 (0.4991)	-0.1539 (0.3476)	-0.1735 (0.3433)	3Y	0.0106 (0.3407)	0.9826 (0.1056)
	$\hat{\delta}$	0.9565 (0.1487)	0.9170 (0.1172)	0.9274 (0.0869)	5Y	0.0277 (0.4583)	0.9805 (0.1298)
					7Y	0.0551 (0.6072)	0.9760 (0.1591)
					10Y	0.0826 (0.8127)	0.9718 (0.1998)

Table B.3: USD - average estimated AR(1) coefficients along with average standard errors in brackets.

Source: Author's calculations.