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DIPLOMOVÁ PRÁCE

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Zobecněné obyčejné diferenciální rovnice v metrických prostorech

Katedra matematické analýzy

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I would like to thank my advisor, prof. RNDr. Jan Malý DrSc., for everything he has done for me. Especially for being strict when necessary, yet leaving me enough creative freedom, for always pointing me in the right direction and always giving me the right perspective to look at a given problem.

Prohlašuji, že jsem tuto diplomovou práci vypracoval samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

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Název práce: Zobecněné obyčejné diferenciální rovnice v metrických prostorech

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Abstrakt: Cílem této práce je vybudovat základy zobecněných obyčejných diferenciálních rovnic v metrických prostorech. Diferenciální rovnice v metrických prostorech již byly studovány dříve, avšak zvolený přístup není schopen zahrnout obecnější druhy integrálních rovnic. Práce nabízí definici, která kombinuje obecnost metrických prostorů se silou Kurzweilových zobecněných obyčejných diferenciálních rovnic. Dále prezentujeme věty o jednoznačnosti a existenci, které poskytují nové výsledky i v kontextu euklidovských prostorů.

Klíčová slova: Zobecněné obyčejné diferenciální rovnice, metrické prostory, křivky, Henstock-Kurzweilův integrál.

Title: Generalized ordinary differential equations in metric spaces

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Abstract: The aim of this thesis is to build the foundations of generalized ordinary differential equation theory in metric spaces. While differential equations in metric spaces have been studied before, the chosen approach cannot be extended to include more general types of integral equations. We introduce a definition which combines the added generality of metric spaces with the strength of Kurzweil's generalized ordinary differential equations. Additionally, we present existence and uniqueness theorems which offer new results even in the context of Euclidean spaces.

Keywords: Generalized ordinary differential equations, metric spaces, curves, Henstock-Kurzweil integral.

Introduction

Let $f : \mathbb{R}^n \times (a,b) \to \mathbb{R}^n$ be a continuous function and consider the standard ordinary differential equation $x' = f(x,t)$. To solve it in the classical sense means finding an interval $I \subset (a,b)$ and a function $x : I \to \mathbb{R}^n$ such that the derivative $x'(t)$ exists for all $t \in I$ and $x'(t) = f(x(t),t)$. Given an initial condition $x(t_0) = x_0$ it is equivalent with

$$x(t) = x_0 + (R)\int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s. \tag{1}$$

The next step in the evolution of ordinary differential equations is Carathéodory's approach, based on using the Lebesgue integral on the right side of (1) i.e.

$$x(t_2) - x(t_1) = (\mathrm{L})\int_{t_1}^{t_2} f(x(s),s)\,\mathrm{d}s. \tag{2}$$

While studying continuous dependence of (2) on a parameter, J. Kurzweil realized that all relevant information can be extracted from the function

$$F(x,t) = \int_{t_0}^{t} f(x,s)\,\mathrm{d}s. \tag{3}$$

In order to describe the notion of a solution of the differential equation (2) by means of the function (3) he introduced the concept of the generalized Perron integral in his paper [1]. Surprisingly, this new construction of the Perron integral was based on Riemann sums. The same construction was independently, and for unrelated reasons, discovered by R. Henstock.

Let us interpret the classical ordinary diferential equation as the problem to find a function u which behaves infinitesimally as $t \mapsto u(\tau) + f(u(\tau),\tau)(t-\tau)$ near each $\tau \in (a,b)$. J. Kurzweil used his generalized Perron integral to continue this notion and investigate a very general problem. Given a function $F : \mathbb{R}^n \times [a,b] \times [a,b] \to \mathbb{R}^n$, he looked for a function $u : [a,b] \to \mathbb{R}^n$ satisfying the "tangent behaviour"

$$u(t) \sim u(\tau) + F(u(\tau),\tau,t) - F(u(\tau),\tau,\tau) \quad \text{for} \quad t \to \tau. \tag{4}$$

This problem is referred to as GODE (generalized ordinary differential equation). J. Kurzweil, together with J. Jarník, S. Schwabik and M. Tvrdý, continued developing this theory, even extending its methods into Banach spaces, as seen in [3], [2]. The use of generalized Perron integral allows to handle a much wider class of problems than the Carathéodory approach.

Any attempt to study ordinary differential equations in the context of metric spaces struggles against the lack of linear structure. The concept of derivative has no sense for functions with values in a metric space. However, if we normalize the function $F(x, \tau, t)$ from (4) by requiring $F(x, \tau, \tau) = x$, we can avoid addition in (4) and observe that the problem to find a function u which behaves infinitesimally as $F(u(\tau), \tau, \cdot)$ near each $\tau \in [a, b]$ can be posed in metric spaces.

Using a completely different notation and terminology, a similar approach has been actually used to study differential equations (also called "mutational equations") in metric spaces, see e.g. [4, 5, 6] by J.-P. Aubin, [7, 8] by T. Lorenz and [9] by J. Tabor. However, this theory has been developed only in the classical or Carathéodory setting.

So far, the theory of GODEs has not been considered in metric spaces. The reason for this is that the concept of a solution is based on integration, and therefore requires linear structure of the target. Our idea is to modify the concept of solution so that the new definition has a good sense in metric spaces. This enables us to build a theory which has the power to include both the theory of GODEs and the current theories in metric spaces. Our main achievements are uniqueness and existence theorems which show that existing results of the GODE theory can be carried over into metric spaces and even offer some improvements.

The first chapter contains notation and a recollection of standard theorems. The second chapter presents the basics of the strong Kurzweil-Henstock integral and generalized ordinary differential equations. The third chapter describes methods of extending the definition of a generalized ordinary differential equation into metric spaces. The fourth and fifth chapter deal with uniqueness and existence of a solution respectively. The last chapter deals with comparison of provided and existing theorems in the context of Euclidean spaces.

Preliminaries

We will use the following standard symbols:

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^+$	The set of positive real numbers.
$\mathbb{R}^+_0$	The set of nonnegative real numbers.
$f : X \to Y$	f is a mapping from X to Y.
$A \subset B$	The set A is a subset of the set B.
$A \cap B$	The intersection of sets A and B.
$A \cup B$	The union of sets A and B.
$x \in M$	x is an element of M.
$\phi(X)$	The image of the set X under the mapping ϕ.
$\phi^{-1}(X)$	The preimage of the set X under the mapping ϕ.
$\mathcal{B}(x, r)$	Closed ball with the center x and radius r.
$\mathcal{U}(x, r)$	Open ball with the center x and radius r.
$\mathcal{C}(K, L)$	The space of continuous mappings between K and L.
$\varnothing$	The empty set.
$f \circ g$	The composition mapping $x \longmapsto f(g(x))$.
$\lvert x \rvert$	Absolute value of $x \in \mathbb{R}$.

For $x \in \mathbb{R}$ we say that x is positive if $x > 0$ and nonnegative if $x \geq 0$. A function $f : \mathbb{R} \to \mathbb{R}$ is increasing if $f(s) < f(t)$ for $s < t$ and nondecreasing if $f(s) \leq f(t)$ for $s < t$. For a real function $h : \mathbb{R} \to \mathbb{R}$ we will use $h(x+)$ to denote the limit of h at the point $x \in \mathbb{R}$ from the right, if it exists. By $\mathrm{D}^+ f(\tau)$ we denote the upper right derivative of the function f at the point $\tau \in \mathbb{R}$ i.e.

$$\mathrm{D}^+ f(\tau) = \limsup_{t \to \tau_+} \frac{f(t) - f(\tau)}{t - \tau}$$

Theorem (Arzela-Ascoli)**.** *Let (K, ϱ) and (L, σ) be metric spaces, let K be compact and let $\mathcal{M}$ be a subset of $\mathcal{C}(K, L)$. Then $\mathcal{M}$ is relatively compact with respect to the topology of uniform convergence if*
(**A1**) *The set $\mathcal{M}$ is uniformly equicontinuous i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that all $f \in \mathcal{M}$ and $x, y \in K$ with $\varrho(x, y) < \delta$ satisfy $\sigma(f(x), f(y)) < \varepsilon$.*
(**A2**) *The set $\{f(x)\,;\; f \in M\}$ has a compact closure for every $x \in K$.*

Theorem (Carathéodory)**.** *Let the function $F : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ satisfy the following conditions:*
(**C1**) *The function $x \mapsto f(x, t)$ is continuous for almost all $t \in [a, b]$.*

(**C2**) *The function* $t \mapsto f(x,t)$ *is measurable for all* $x \in \mathbb{R}^n$*.*

(**C3**) *There exists a Lebesgue integrable function* $g : [a,b] \to \mathbb{R}_0^+$ *such that*

$$\|f(x,t)\| \le g(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and almost all } t \in [a,b].$$

Then for every $x_0 \in \mathbb{R}^n$ *and every* $t_0 \in [a,b]$ *there exists* $\Delta > 0$ *and a function* $x : [a,b] \cap [t_0 - \Delta, t_0 + \Delta] \to \mathbb{R}^n$ *such that*

$$x(t) = x_0 + (\mathrm{L}) \int_{t_0}^{t} f(x(s), s)\, \mathrm{d}s \quad \text{for} \quad t \in [a,b] \cap [t_0 - \Delta, t_0 + \Delta].$$

1. Current Theory

1.1 SKH Integral

The term partition of $[a,b] \subset \mathbb{R}$ will stand for any collection of closed intervals and tags $\{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ such that $t_0 = a$, $t_k = b$ and $\tau_i \in [t_{i-1},t_i]$. Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real function defined on $[a,b]$. A partition of $[a,b]$ is called δ-fine if for each $i = 1,\dots,k$ it satisfies $[t_{i-1},t_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]$.

Lemma 1.1 (Cousin). *For every $\delta : [a,b] \to \mathbb{R}^+$ the set of all δ-fine partitions of $[a,b]$ is nonempty.*

Proof. There exist finitely many $\tau_1,\dots,\tau_n$ such that

$$[a,b] \subset \bigcup_{i=1}^{n} \mathcal{U}(\tau_i,\delta(\tau_i)).$$

Choose a minimal subset of $\tau_1,\dots,\tau_n$ such that the intervals $\mathcal{U}(\tau_i,\delta(\tau_i))$ still cover the whole interval $[a,b]$ and rename it to $\tau_1 < \ldots < \tau_m$.

From the minimality of the cover we can observe that for every $i = 1,\dots,m-1$ we have $\mathcal{U}(\tau_i,\delta(\tau_i)) \cap \mathcal{U}(\tau_{i+1},\delta(\tau_{i+1})) \neq \varnothing$. Otherwise, every point of the interval $(\tau_i + \delta(\tau_i), \tau_{i+1} - \delta(\tau_{i+1}))$ would have to be covered by a ball containing either $\mathcal{U}(\tau_i,\delta(\tau_i))$ or $\mathcal{U}(\tau_{i+1},\delta(\tau_{i+1}))$, therefore contradicting the minimality of the cover.

We set $t_0 = a$, $t_{m+1} = b$ and choose $t_i \in \mathcal{U}(\tau_i,\delta(\tau_i)) \cap \mathcal{U}(\tau_{i+1},\delta(\tau_{i+1})) \cap (\tau_i,\tau_{i+1})$ arbitrarily. The resulting partition is obviously δ-fine. $\square$

Definition 1.2 (J. Kurzweil). A function $f : [a,b] \to \mathbb{R}^n$ is called KH integrable (Kurzweil-Henstock) over $[a,b]$ if there exists $A \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}^+$ such that every δ-fine partition $\{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ of $[a,b]$ satisfies

$$\|A - \sum_{i=1}^{k} f(\tau_i)(t_i - t_{i-1})\| < \varepsilon.$$

We say that A is the definite KH integral of f over $[a,b]$ and use the symbol

$$(\text{HK}) \int_a^b f(x)\,\mathrm{d}x.$$

A function $u : [a,b] \to \mathbb{R}^n$ will be called an indefinite KH integral of f on $[a,b]$ if

$$u(t) - u(s) = (\text{HK}) \int_s^t f(x)\,\mathrm{d}x \quad \text{for} \quad [s,t] \subset [a,b].$$

While this might seem like a simple modification of the Riemann definition, the resulting integral is equivalent to the Perron integral. The following version, called Strong Kurzweil-Henstock integral, is better suited to dealing with values in more abstract spaces. In contrast with common habits, we no longer limit ourselves to closed intervals as domains for the indefinite integral.

In the rest of this chapter E will denote a normed linear space and $I \subset \mathbb{R}$ will denote an interval.

Definition 1.3. A function $u : I \to E$ is called an indefinite SKH integral of $f : I \to E$ if for every $\varepsilon > 0$ and every $[S, T] \subset I$ there exists $\delta : [S, T] \to \mathbb{R}^+$ such that for every δ-fine partition $\{[t_{i-1}, t_i], \tau_i\}_{i=1}^k$ of $[S, T]$ we have

$$\sum_{i=1}^{k} \|u(t_i) - u(t_{i-1}) - f(\tau_i)(t_i - t_{i-1})\| < \varepsilon.$$

We define the definite SKH integral of f over $[S, T] \subset I$ as

$$(\mathrm{SKH}) \int_S^T f(x)\, \mathrm{d}x = u(T) - u(S).$$

Obviously, any SKH integrable function is KH integrable. The reverse is only true in case of $E = \mathbb{R}^n$ and is called the Saks-Henstock Lemma. However, for the definition of the generalized ordinary differential equations we will need an even more general type of integration. The idea is that instead of integrating with respect to an additive set function like $(t_i - t_{i-1})$ or, in the Stieltjes case, $g(t_i) - g(t_{i-1})$, we use a fully nonadditive expression.

Definition 1.4. A function $u : I \to E$ is called an indefinite SKH integral of $U : I \times I \to E$ if for every $\varepsilon > 0$ and every $[S, T] \subset I$ there exists $\delta : [S, T] \to \mathbb{R}^+$ such that for every δ-fine partition $\{[t_{i-1}, t_i], \tau_i\}_{i=1}^k$ of $[S, T]$ we have

$$\sum_{i=1}^{k} \|u(t_i) - u(t_{i-1}) - U(\tau_i, t_i) + U(\tau_i, t_{i-1})\| < \varepsilon.$$

We define the definite SKH integral of U over $[S, T] \subset I$ as

$$(\mathrm{SKH}) \int_S^T \mathrm{D}_t\, U(\tau, t) = u(T) - u(S).$$

In this case, the integrand is referred to as a function of coupled variables in order to call attention to their fundamentally different roles. As we can easily observe, a function $u : I \to E$ is an indefinite SKH integral of $f : I \to E$ if it is an indefinite SKH integral of $U(\tau, t) = f(\tau)\, t$ in the sense of coupled variables.

1.2 MC Integral

The monotonically controlled (MC for short) integral was introduced in [10] by J. Malý and H. Bendová. Their aim was to build the foundations of integral theory at the generality of Perron integral while using unexpectedly simple definitions and proofs. They prove that the MC integral of a function with respect to a Lebesgue-Stieltjes measure is equivalent to the corresponding SKH integral. In this section we generalize the defnition of the MC integral and the equivalence result to the case of coupled variables, which will often prove useful.

Definition 1.5. A function $u : I \to E$ is called an indefinite MC integral of $U : I \times I \to E$ if there exists an increasing function $\xi : I \to \mathbb{R}$, called control function of (U, u) on I, such that

$$\lim_{t\to\tau,\, t\in I} \frac{\|u(t) - u(\tau) - U(\tau, t) + U(\tau, \tau)\|}{\xi(t) - \xi(\tau)} = 0 \quad \text{for} \quad \tau \in I.$$

We define the definite MC integral of U over $[S, T] \subset I$ as

$$(\mathrm{MC}) \int_S^T \mathrm{D}_t\, U(\tau, t) = u(T) - u(S).$$

Note that if ξ is a control function of (U, u) on I, $\alpha > 0$ and $\zeta : I \to \mathbb{R}$ is a nondecreasing function, then $\alpha\xi + \zeta$ is also a control function of (U, u) on I.

Lemma 1.6. *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be sequences of real numbers such that $a_k < b_k$ for every $k \in \mathbb{N}$. Let $(a_k) \searrow a$, $(b_k) \nearrow b$ and let $u : (a, b) \to E$ be an indefinite MC integral of $U : (a, b) \times (a, b) \to E$ on (a_k, b_k) for each $k \in \mathbb{N}$. Then u is an indefinite MC integral of U on (a, b) and the control function of (U, u) on (a, b) can be assumed to be bounded.*

Proof. For each $k \in \mathbb{N}$ let γ_k be a control function of (U, u) on (a_{k+1}, b_{k+1}). Since γ_k is bounded on (a_k, b_k), we may assume that $0 \le \gamma_k \le 1$ on (a_k, b_k). We define the functions ζ_k as

$$\zeta_k(x) = \begin{cases} 0, & x \le a_k, \\ \gamma_k(\tau), & a_k < \tau < b_k, \\ 1, & x \ge b_k. \end{cases}$$

The bounded control function of (U, u) on (a, b) is then found in the form

$$\zeta = \sum_{k=1}^{\infty} 2^{-k} \zeta_k.$$

□

Theorem 1.7 (Equivalence). *A function $u : I \to E$ is an indefinite MC integral of $U : I \times I \to E$ on I if and only if it is an indefinite SKH integral of U on I.*

Proof. First, we assume that $I = [a, b]$ and that u is an indefinite MC integral of U on $[a, b]$ with a control function satisfying $0 \leq \xi \leq 1$.

Choose $\varepsilon > 0$. Then for each $\tau \in [a, b]$ there exists $\delta(\tau) > 0$ such that every $t \in (\tau - \delta(\tau), \tau + \delta(\tau)) \cap [a, b]$ satisfies

$$\|u(t) - u(\tau) - U(\tau, t) + U(\tau, \tau)\| < \varepsilon|\xi(t) - \xi(\tau)|.$$

For a $\delta/2$-fine partition $\{[t_{i-1}, t_i], \tau_i\}_{i=1}^k$ we can see that

$$\begin{aligned}\|u(t_i) - u(\tau_i) - U(\tau_i, t_i) + U(\tau_i, \tau_i)\| &< \varepsilon(\xi(t_i) - \xi(\tau_i)),\\ \|u(\tau_i) - u(t_{i-1}) + U(\tau_i, t_{i-1}) - U(\tau_i, \tau_i)\| &< \varepsilon(\xi(\tau_i) - \xi(t_{i-1})).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\sum_{i=1}^{k} \|u(t_i) - u(t_{i-1}) - U(\tau_i, t_i) + U(\tau_i, t_{i-1})\|\\ &< \varepsilon \sum_{i=1}^{k} (\xi(t_i) - \xi(\tau_i) + \xi(\tau_i) - \xi(t_{i-1})) = \varepsilon(\xi(b) - \xi(a)) \leq \varepsilon.\end{aligned}$$

Let u be an indefinite SKH integral of U on $[a, b]$. For $k \in \mathbb{N}$ let $\delta_k : [a, b] \to \mathbb{R}^+$ correspond to $\varepsilon_k = 2^{-k}$. For $\tau \in (a, b]$ we define $P_k(\tau)$ as the set of all δ_k-fine partitions of $[a, \tau]$ and

$$\xi_k(\tau) = \sup\{\sum_{P} \|u(t_i) - u(t_{i-1}) - U(\tau_i, t_i) + U(\tau_i, t_{i-1})\|\,;\ P \in P_k(\tau)\},$$

$$\xi(\tau) = \tau + \sum_{k=1}^{\infty} k\ \xi_k(\tau).$$

Choose $\varepsilon > 0$ and find $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$. It follows that for $\tau \in (a, b]$ and $t \in (\tau - \delta_{k_0}(\tau), \tau)$ we have

$$\begin{aligned}&\|u(t) - u(\tau) - U(\tau, t) + U(\tau, \tau)\| \leq \xi_{k_0}(\tau) - \xi_{k_0}(t)\\ &\leq \sum_{k=1}^{\infty} \frac{k}{k_0}(\xi_k(\tau) - \xi_k(t)) \leq \frac{1}{k_0}(\xi(\tau) - \xi(t)) < \varepsilon(\xi(\tau) - \xi(t)).\end{aligned}$$

Similarly, for $t \in (\tau, \tau + \delta_{k_0}(\tau))$ we have

$$\|u(t) - u(\tau) - U(\tau, t) + U(\tau, \tau)\| \leq \xi_{k_0}(t) - \xi_{k_0}(\tau) < \varepsilon(\xi(t) - \xi(\tau)).$$

If I is not necessarily closed, we easily write it as a union of an increasing sequence of closed intervals and use Lemma 1.6. □

1.3 Generalized Ordinary Differential Equations

Definition 1.8. Let $F : E \times [a,b] \times [a,b] \to E$ be an arbitrary mapping. We say that a function $u : [a,b] \to E$ is a solution of the equation $x' = \mathrm{D}_t\, F(x,\tau,t)$ on $[a,b]$ if

$$u(t) = u(s) + (\mathrm{SKH}) \int_s^t \mathrm{D}_t\, F(u(\tau),\tau,t) \quad \text{for} \quad [s,t] \subset [a,b]. \tag{1.1}$$

However, when working with the definition, we will find it more convenient to interpret (1.1) in terms of indefinite SKH integral i.e. for every $\varepsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}^+$ such that every δ-fine partition P of $[a,b]$ satisfies

$$\sum_P ||u(t_i) - u(t_{i-1}) - F(u(\tau_i),\tau_i,t_i) + F(u(\tau_i),\tau_i,t_{i-1})|| < \varepsilon. \tag{1.2}$$

It can always be arranged for the function F to satisfy $F(x,\tau,\tau) = x$ without changing the equation $x' = \mathrm{D}_t\, F(x,\tau,t)$. This is done by considering the function $\tilde{F}(x,\tau,t) = x + F(x,\tau,t) - F(x,\tau,\tau)$, since then we will have

$$\tilde{F}(u(\tau_i),\tau_i,t_{i-1}) - \tilde{F}(u(\tau_i),\tau_i,t_i) = F(u(\tau_i),\tau_i,t_{i-1}) - F(u(\tau_i),\tau_i,t_i).$$

This process is called normalization and is one of the main ideas behind extending generalized ordinary differential equations into metric spaces. It is what allows us to interpret the equation as

$$u(t) \sim F(u(\tau),\tau,t) \quad \text{for} \quad t \to \tau.$$

To be more specific, it will allow us to express the sum (1.2) in terms of distance of two elements by using the identity $u(\tau) = F(u(\tau),\tau,\tau)$.

Lemma 1.9. *Let $F : E \times [a,b] \times [a,b] \to E$ satisfy $F(x,\tau,\tau) = x$. A function $u : [a,b] \to E$ is a solution of the equation $x' = \mathrm{D}_t\, F(x,\tau,t)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}^+$ such that every δ-fine partition P of $[a,b]$ satisfies*

$$\sum_P \Big(||u(t_i) - F(u(\tau_i),\tau_i,t_i)|| + ||u(t_{i-1}) - F(u(\tau_i),\tau_i,t_{i-1})|| \Big) < \varepsilon. \tag{1.3}$$

Proof. Fix $\varepsilon > 0$ and find $\delta : [a,b] \to \mathbb{R}^+$ such that (1.2) holds for every δ-fine partition $P = \{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ of $[a,b]$. However, if $\{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ is δ-fine, then the partition

$$\{[s_{j-1},s_j],\sigma_j\}_{j=1}^{2k} = \{\ldots,[t_{i-1},\tau_i],\tau_i,[\tau_i,t_i],\tau_i,\ldots\}$$

is also δ-fine, and therefore

$$\sum_{i=1}^k \Big(||u(t_i) - u(\tau_i) - F(u(\tau_i),\tau_i,t_i) + F(u(\tau_i),\tau_i,\tau_i)|| \\ + ||u(\tau_i) - u(t_{i-1}) - F(u(\tau_i),\tau_i,\tau_i) + F(u(\tau_i),\tau_i,t_{i-1})|| \Big) < \varepsilon.$$

By using $u(\tau_i) = F(u(\tau_i), \tau_i, \tau_i)$ we obtain

$$\sum_{i=1}^{k} \Big(||u(t_i) - F(u(\tau_i), \tau_i, t_i)|| + ||u(t_{i-1}) - F(u(\tau_i), \tau_i, t_{i-1})|| \Big) < \varepsilon.$$

Let us now assume that for every $\varepsilon > 0$ there exists $\delta : [a, b] \to \mathbb{R}^+$ such that every δ-fine partition P of $[a, b]$ satisfies (1.3). Then we have

$$\begin{aligned}
&\sum_P ||u(t_i) - u(t_{i-1}) - F(u(\tau_i), \tau_i, t_i) + F(u(\tau_i), \tau_i, t_{i-1})|| \\
\le &\sum_P \Big(||u(t_i) - u(\tau_i) - F(u(\tau_i), \tau_i, t_i) + F(u(\tau_i), \tau_i, \tau_i)|| \\
&\qquad + ||u(\tau_i) - u(t_{i-1}) - F(u(\tau_i), \tau_i, \tau_i) + F(u(\tau_i), \tau_i, t_{i-1})|| \Big) \\
= &\sum_P \Big(||u(t_i) - F(u(\tau_i), \tau_i, t_i)|| + ||u(t_{i-1}) - F(u(\tau_i), \tau_i, t_{i-1})|| \Big) < \varepsilon.
\end{aligned}$$

As this was the desired result, the proof is finished. □

Many works concerned with generalized ordinary differential equation theory only study the equation $x' = \mathrm{D}_t\, G(x, t)$, defined by

$$u(t) = u(s) + (\text{SKH}) \int_s^t \mathrm{D}_t\, G(u(\tau), t) \quad \text{for} \quad [s, t] \subset [a, b].$$

The reason for this is that a large class of GODEs (see [2]) can be simplified by putting

$$G(x, t) = (\text{SKH}) \int_a^t \mathrm{D}_s\, F(x, \sigma, s).$$

However, this type of equation offers no analogy to normalization. As an example, consider Carathéodory's equation

$$x(\nu) - x(\sigma) = (\text{L}) \int_\sigma^\nu f(x(s), s)\, \mathrm{d}s.$$

It is equivalent with $x' = \mathrm{D}_t\, G(x, t)$, where

$$G(x, t) = (\text{L}) \int_{t_0}^t f(x, s)\, \mathrm{d}s$$

for t_0 fixed (see [3], Theorem 5.14), while the normalized form is

$$F(x, \tau, t) = x + (\text{L}) \int_\tau^t f(x, s)\, \mathrm{d}s.$$

Note that the previously mentioned choice $F(x, \tau, t) = f(x, \tau)\, t$ and its normalized form $F(x, \tau, t) = x + f(x, \tau)(t - \tau)$ are equivalent with

$$x(\nu) - x(\sigma) = (\text{SKH}) \int_\sigma^\nu f(x(s), s)\, \mathrm{d}s,$$

as seen, among other options, in Theorem 17.3 of [2].

2. Solutions in metric spaces

In the remainder of the text X always denotes a metric space. We adopt the convention that $|x - y|$ stands for the distance between x and y.

Definition 2.1 (J. Malý and B. Skovajsa)**.** Let $I \subset \mathbb{R}$ be an interval and assume that $F : X \times I \times I \to X$ satisfies $F(x, \tau, \tau) = x$. We say that $u : I \to X$ is a solution of $x' = \mathrm{D}_t F(x, \tau, t)$ on I if

$$\text{(SKH)} \int_S^T \mathrm{D}_t \, |u(t) - F(u(\tau), \tau, t)| = 0 \quad \text{for} \quad [S, T] \subset I. \tag{2.1}$$

Equivalently, the indefinite SKH integral of $U(\tau, t) = |u(t) - F(u(\tau), \tau, t)|$ is equal to zero on I.

As a trivial consequence of the equivalence theorem we have that substituting the MC integral into 2.1 results in an equivalent definition.

We expand the definition of a solution into more practical forms:

(i) We say that $u : (a, b) \to X$ is a solution of $x' = \mathrm{D}_t F(x, \tau, t)$ on (a, b) if there exists an increasing function $\xi : (a, b) \to \mathbb{R}$ such that

$$\lim_{t \to \tau} \frac{|u(t) - F(u(\tau), \tau, t)|}{\xi(t) - \xi(\tau)} = 0 \quad \text{for} \quad \tau \in (a, b). \tag{2.2}$$

In this case we will refer to ξ as control function of (F, u).

(ii) Alternatively, we say that $u : [a, b] \to X$ is a solution of $x' = \mathrm{D}_t F(x, \tau, t)$ on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta : [a, b] \to \mathbb{R}^+$ such that every δ-fine partition P of $[a, b]$ satisfies

$$\sum_P \Big(|u(t_i) - F(u(\tau_i), \tau_i, t_i)| + |u(t_{i-1}) - F(u(\tau_i), \tau_i, t_{i-1})| \Big) < \varepsilon. \tag{2.3}$$

Proof of (i). The form (2.2) is a direct interpretation of (2.1). The indefinite MC integral of $|u(t) - F(u(\tau), \tau, t)|$ being equal to zero on (a, b) is defined by the property

$$\lim_{t \to \tau} \frac{\Big| |u(t) - F(u(\tau), \tau, t)| - |u(\tau) - F(u(\tau), \tau, \tau)| \Big|}{\xi(t) - \xi(\tau)} = 0.$$

However, we assumed F to be normalized, which means $u(\tau) = F(u(\tau), \tau, \tau)$. Therefore, $|u(\tau) - F(u(\tau), \tau, \tau)| = 0$ and we obtain (2.2). □

Proof of (ii). Let us assume that the indefinite SKH integral of $|u(t)-F(u(\tau),\tau,t)|$ is zero on $[a,b]$ i.e. for every $\varepsilon>0$ there exists $\delta:[a,b]\to\mathbb{R}^+$ such that for every δ-fine partition $\{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ of $[a,b]$ we have

$$\sum_{i=1}^{k}\Big||u(t_i)-F(u(\tau_i),\tau_i,t_i)|-|u(t_{i-1})-F(u(\tau_i),\tau_i,t_{i-1})|\Big|<\varepsilon.$$

However, if $\{[t_{i-1},t_i],\tau_i\}_{i=1}^k$ is δ-fine, then the partition

$$\{[s_{j-1},s_j],\sigma_j\}_{j=1}^{2k}=\{\dots,[t_{i-1},\tau_i],\tau_i,[\tau_i,t_i],\tau_i,\dots\}$$

is also δ-fine, and therefore

$$\sum_{j=1}^{2k}\Big||u(s_j)-F(u(\sigma_j),\sigma_j,s_j)|-|u(s_{j-1})-F(u(\sigma_j),\sigma_j,s_{j-1})|\Big|<\varepsilon. \tag{2.4}$$

For $j=2i-1$ we have

$$\begin{aligned}|u(s_j)-F(u(\sigma_j),\sigma_j,s_j)|&=|u(\tau_i)-F(u(\tau_i),\tau_i,\tau_i)|=0,\\|u(s_{j-1})-F(u(\sigma_j),\sigma_j,s_{j-1})|&=|u(t_{i-1})-F(u(\tau_i),\tau_i,t_{i-1})|.\end{aligned} \tag{2.5}$$

Similarly, for $j=2i$ we have

$$\begin{aligned}|u(s_j)-F(u(\sigma_j),\sigma_j,s_j)|&=|u(t_i)-F(u(\tau_i),\tau_i,t_i)|,\\|u(s_{j-1})-F(u(\sigma_j),\sigma_j,s_{j-1})|&=|u(\tau_i)-F(u(\tau_i),\tau_i,\tau_i)|=0.\end{aligned} \tag{2.6}$$

By substituting (2.5) and (2.6) back into (2.4) we get (2.3). The reverse implication is trivial. □

We can now observe that on normed linear spaces our definition is equivalent to Kurzweil's as a trivial consequence of (2.3) and Lemma 1.9.

This can also be easily achieved by using (2.2). Let E be a normed linear space and let $F:E\times(a,b)\times(a,b)\to E$ satisfy $F(x,\tau,\tau)=x$. We can see that $u:(a,b)\to E$ is an indefinite MC integral of $F(u(\tau),\tau,t)$, and therefore a solution of $x'=\mathrm{D}_t\,F(x,\tau,t)$, if and only if there exists an increasing function $\xi:(a,b)\to\mathbb{R}$ such that

$$\lim_{t\to\tau}\frac{||u(t)-u(\tau)-F(u(\tau),\tau,t)+F(u(\tau),\tau,\tau)||}{\xi(t)-\xi(\tau)}=0\quad\text{for}\quad\tau\in(a,b).$$

However, this is equivalent with (2.2) due to $u(\tau)=F(u(\tau),\tau,\tau)$.

3. Uniqueness

Our focus when dealing with uniqueness and existence of a solution will be to generalize theorems from [3] by S. Schwabik. We restrict our attention to the continuous case. Ultimately, we hope to treat solutions in the space of regulated functions. However, this leads to additional technical complications which will be pursued in future research.

Definition 3.1. Let X be a metric space, $I \subset \mathbb{R}$ an interval and $x : I \to X$ a solution of $x' = \mathrm{D}_t F(x, \tau, t)$ on $[\tau, \tau+\delta] \subset I$ with $x(\tau) = x_\tau$. We say that x is locally unique in the future at the point (x_τ, τ) if for any solution $y : I \to X$ of $x' = \mathrm{D}_t F(x, \tau, t)$ on $[\tau, \tau+\eta]$ such that $y(\tau) = x_\tau$ there exists $\zeta > 0$ such that $y(t) = x(t)$ for $t \in [\tau, \tau+\zeta]$.

Definition 3.2. We say that $G : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$ if

$$||G(x, t_2) - G(x, t_1)|| \leq |h(t_2) - h(t_1)| \tag{3.1}$$

$$||G(x, t_2) - G(x, t_1) - G(y, t_2) + G(y, t_1)|| \leq \omega(||x - y||)\, |h(t_2) - h(t_1)| \tag{3.2}$$

for $(x, t_2), (x, t_1), (y, t_2), (y, t_1) \in \Omega$, where $\Omega = \mathcal{U}_R \times (a, b)$, $h : [a, b] \to \mathbb{R}$ is nondecreasing and $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and nondecreasing with $\omega(0) = 0$ and $\omega(\nu) > 0$ for $\nu > 0$. The symbol $\mathcal{U}_R$ stands for $\{x \in \mathbb{R}^n\,;\ \|x\| < R\}$.

In the case of uniqueness, our motivation comes from the following theorem:

Theorem 3.3 (S. Schwabik, [3], page 122)**.** *Let $G : \Omega \to \mathbb{R}^n$ belong to $\mathcal{F}(\Omega, h, \omega)$, where the function h is nondecreasing and continuous from the left, $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous, nondecreasing, $\omega(0) = 0$ and for $\nu > 0$ we have $\omega(\nu) > 0$ and*

$$\lim_{r \to 0_+} \int_r^\nu \frac{1}{\omega(s)}\, \mathrm{d}s = +\infty. \tag{3.3}$$

Then every solution $x : (a, b) \to \mathbb{R}^n$ of $x' = \mathrm{D}_t G(x, t)$ such that

$$x(\tau) + \lim_{t \to \tau_+} G(x(\tau), t) - G(x(\tau), \tau) \in \mathcal{U}_R$$

is locally unique in the future at $(x(\tau), \tau)$.

Lemma 3.4. *Let $f : (a, b) \to \mathbb{R}$ be a real function which at every point $x \in (a, b)$ satisfies*

(T1) $\mathrm{D}^+ f(x) \geq 0$,

(T2) $\limsup_{h \to 0_+} f(x - h) \leq f(x)$.

Then f is nondecreasing on (a, b).

Proof. See [12], page 135. □

Theorem 3.5. *Let X be a metric space, let $F : X \times (a,b) \times (a,b) \to X$ fulfil $F(x,\tau,\tau) = x$ and let us assume the following:*
(**U1**) *The function $t \mapsto F(x,\tau,t)$ is continuous for every $x \in X$ and $\tau \in (a,b)$.*
(**U2**) *There exists a nondecreasing and continuous function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\omega(0) = 0$ and for $\nu > 0$ we have $\omega(\nu) > 0$ and*

$$\lim_{r\to 0_+} \int_r^\nu \frac{1}{\omega(s)} \, \mathrm{d}s = +\infty. \tag{3.4}$$

(**U3**) *There exists $\xi : (a,b) \to \mathbb{R}$ increasing such that all $x, y \in X$ and all $\tau \in (a,b)$ satisfy*

$$\liminf_{t\to\tau_+} \frac{|F(x,\tau,t) - F(y,\tau,t)| - |x-y|}{\xi(t) - \xi(\tau)} \le \omega(|x-y|). \tag{3.5}$$

Then every solution of $x' = \mathrm{D}_t\, F(x,\tau,t)$ on (a,b) is locally unique in the future.

Proof. Let u, v be two solutions and $\alpha, \beta \in (a,b)$ such that $\alpha < \beta$, $u(\alpha) = v(\alpha)$ and $u(\beta) \neq v(\beta)$. We also assume that ξ already controls both (F,u) and (F,v) and that for $t > s$ we have $\xi(t) - \xi(s) \ge t - s$. Condition (**U1**) implies that any solution of $x' = \mathrm{D}_t\, F(x,\tau,t)$ is continuous. Indeed, for a fixed $\tau \in (a,b)$ and $\varepsilon > 0$ we can choose $\delta_1 > 0$ such that $|F(u(\tau),\tau,t) - u(\tau)| < \varepsilon$ for $|t-\tau| < \delta_1$. Then, we choose $\delta_2 \in (0,\delta_1)$ such that $|u(t) - F(u(\tau),\tau,t)| < \varepsilon\,|\xi(t) - \xi(\tau)|$ for $|t - \tau| < \delta_2$. Since ξ is increasing, we can choose $\delta_3 \in (0,\delta_2)$ such that $|\xi(t) - \xi(\tau)| < \xi(\tau+) - \xi(\tau-) + 1$ for $|t-\tau| < \delta_3$. Finally, we obtain

$$\begin{aligned} |u(t) - u(\tau)| &\le |u(t) - F(u(\tau),\tau,t)| + |F(u(\tau),\tau,t) - u(\tau)| \\ &< \varepsilon(\xi(\tau+) - \xi(\tau-) + 2). \end{aligned}$$

Since the solutions are continuous, we can achieve $u \neq v$ everywhere on (α,β) by simple modification of α. Now fix $\nu > 0$ and define $\Delta(t) = |u(t) - v(t)|$ and

$$\Phi(r) = \int_r^\nu \frac{1}{\omega(s)} \, \mathrm{d}s.$$

Consider the function $t \mapsto \Phi(\Delta(t)) + \xi(t)$ on (α,β). It is well defined everywhere on (α,β), since $\Delta > 0$ on (α,β). Since ξ is bounded on $[\alpha,\beta]$ and $\Delta(\alpha) = 0$, we use (3.4) to easily deduce that

$$\lim_{t\to\alpha_+} \Phi(\Delta(t)) + \xi(t) = +\infty. \tag{3.6}$$

We now employ Lemma 3.4 to show that $t \mapsto \Phi(\Delta(t)) + \xi(t)$ is nondecreasing on (α,β), which obviously contradicts (3.6). Since ξ is increasing and $\Phi \circ \Delta$ is continuous, we can easily see that their sum satisfies (**T2**). The rest of the proof will focus on showing that all $\tau \in (\alpha,\beta)$ satisfy

$$\mathrm{D}^+(\Phi(\Delta(\tau)) + \xi(\tau)) \ge 0.$$

We notice that Φ is decreasing and for every $s > 0$ we have

$$\Phi'(s) = -\frac{1}{\omega(s)} \tag{3.7}$$

We fix $\tau \in (\alpha, \beta)$ and distinguish two cases:

1) For every $\zeta > 0$ there exists $t \in (\tau, \tau + \zeta)$ such that $\Delta(t) \leq \Delta(\tau)$. This implies $\Phi(\Delta(t)) + \xi(t) \geq \Phi(\Delta(\tau)) + \xi(\tau)$, and therefore $\mathrm{D}^+(\Phi(\Delta(\tau)) + \xi(\tau)) \geq 0$.

2) There exists $\zeta > 0$ such that for every $t \in (\tau, \tau + \zeta)$ we have $\Delta(t) > \Delta(\tau)$. We notice that

$$\Phi'(\Delta(\tau)) = \lim_{s \to \Delta(\tau)} \frac{\Phi(s) - \Phi(\Delta(\tau))}{s - \Delta(\tau)} = \lim_{t \to \tau_+} \frac{\Phi(\Delta(t)) - \Phi(\Delta(\tau))}{\Delta(t) - \Delta(\tau)} \tag{3.8}$$

Fix $\varepsilon > 0$. We combine (3.7) and (3.8) in order to find $\delta \in (0, \zeta)$ such that for every $t \in (\tau, \tau + \delta)$ we have

$$\Phi(\Delta(t)) - \Phi(\Delta(\tau)) \geq (-\frac{1}{\omega(\Delta(\tau))} - \varepsilon)(\Delta(t) - \Delta(\tau)) \tag{3.9}$$

Since ξ controls both (F, u) and (F, v), we can find $\eta \in (0, \delta)$ such that all $t \in (\tau, \tau + \eta)$ satisfy

$$\frac{|u(t) - F(u(\tau), \tau, t)|}{\xi(t) - \xi(\tau)} < \varepsilon, \quad \frac{|v(t) - F(v(\tau), \tau, t)|}{\xi(t) - \xi(\tau)} < \varepsilon. \tag{3.10}$$

Now, fix $\sigma \in (0, \eta)$ and use (3.5) to find $T \in (\tau, \tau + \sigma)$ such that

$$\frac{|F(u(\tau), \tau, T) - F(v(\tau), \tau, T)| - \Delta(\tau)}{\xi(T) - \xi(\tau)} \leq \omega(\Delta(\tau)) + \varepsilon. \tag{3.11}$$

We define $E_u(t) := |u(t) - F(u(\tau), \tau, t)|$, $E_v(t) := |v(t) - F(v(\tau), \tau, t)|$ and notice that $\Delta(t) \leq |F(u(\tau), \tau, t) - F(v(\tau), \tau, t)| + E_u(t) + E_v(t)$. Applying this to (3.11) results in

$$\frac{\Delta(T) - \Delta(\tau) - E_u(T) - E_v(T)}{\xi(T) - \xi(\tau)} \leq \omega(\Delta(\tau)) + \varepsilon.$$

We add (3.10) and obtain

$$\frac{\Delta(T) - \Delta(\tau)}{\xi(T) - \xi(\tau)} \leq \omega(\Delta(\tau)) + 3\,\varepsilon.$$

In combination with (3.9) we have

$$\Phi(\Delta(T)) - \Phi(\Delta(\tau)) \geq -(\frac{1}{\omega(\Delta(\tau))} + \varepsilon)(\omega(\Delta(\tau)) + 3\,\varepsilon)(\xi(T) - \xi(\tau)),$$

$$\frac{\Phi(\Delta(T)) - \Phi(\Delta(\tau))}{\xi(T) - \xi(\tau)} \geq -(1 + \varepsilon C(\tau)),$$

$$\frac{\Phi(\Delta(T)) - \Phi(\Delta(\tau)) + \xi(T) - \xi(\tau)}{\xi(T) - \xi(\tau)} \geq -\varepsilon C(\tau).$$

We recall that $\xi(t) - \xi(s) \geq t - s$ for $t > s$, and therefore

$$\frac{\Phi(\Delta(T)) - \Phi(\Delta(\tau)) + \xi(T) - \xi(\tau)}{T - \tau} \geq -\varepsilon C(\tau).$$

Since such T will be in every right neighbourhood of τ, we finally arrive at $\mathrm{D}^+(\Phi(\Delta(\tau)) + \xi(\tau)) \geq 0$ and the proof is finished. □

4. Existence

In the case of existence, our aim is to generalize the following theorem:

Theorem 4.1 (S. Schwabik, [3], page 114)**.** *Let $G : \Omega \to \mathbb{R}^n$ belong to $\mathcal{F}(\Omega, h, \omega)$ and let $(x_\tau, \tau) \in \Omega$ satisfy*

$$x_\tau + \lim_{t \to \tau_+} G(x_\tau), t) - G(x_\tau, \tau) \in \mathcal{U}_R.$$

Then there exists $\Delta > 0$ and a solution $x : [\tau - \Delta, \tau + \Delta] \to \mathbb{R}^n$ of $x' = \mathrm{D}_t\, G(x, t)$ on $[\tau - \Delta, \tau + \Delta]$ with $x(\tau) = x_\tau$.

Once again, we add assumptions of continuity to avoid additional technical complications. As previously mentioned, this is a temporary restriction, since our ultimate goal is to find solutions among regulated functions.

Lemma 4.2. *Let the functions $u_k : [a, b] \to X$, $k \in \mathbb{N}$, be continuous on $[a, b]$ and assume that for every $\varepsilon > 0$ and every $\tau \in [a, b]$ there exists $k_0 \in \mathbb{N}$ and $\eta > 0$ such that for $k \in \mathbb{N}, k \geq k_0$ we have that $|t - \tau| < \eta$ implies $|u_k(t) - u_k(\tau)| < \varepsilon$. Then the functions $\{u_k\}_{k \in \mathbb{N}}$ are uniformly equicontinuous.*

Proof. Fix $\varepsilon > 0$. Since each of the functions u_k is uniformly continuous, we find the corresponding $\delta_k > 0$. First, we notice that for every $m \in \mathbb{N}$ the functions $\{u_k\}_{k \leq m}$ are uniformly equicontinuous, as we can set $\delta := \min\{\delta_1, \dots, \delta_m\}$.

We can cover the interval $[a, b]$ by the system $\{\mathcal{U}(\tau_j, \eta(\tau_j)/2)\,;\ j = 1, \dots, J\}$. Every τ_j corresponds to a certain $k_j \in \mathbb{N}$ from the assumption on functions u_k. We set $k_0 := \max\{k_j\,;\ j = 1, \dots, J\}$ and $\delta := \min\{\eta(\tau_j)/2\,;\ j = 1, \dots, J\}$. We already know that the functions u_k are uniformly equicontinuous for $k \leq k_0$. For $k \geq k_0$ and $t, s \in [a, b]$ such that $|t - s| < \delta$ there exists $m \in \{1, \dots, J\}$ such that $t \in \mathcal{U}(\tau_m, \eta(\tau_m)/2)$, and therefore $s \in \mathcal{U}(\tau_m, \eta(\tau_m))$. Since $k \geq k_0 \geq k_m$, we obtain

$$|u_k(t) - u_k(s)| \leq |u_k(t) - u_k(\tau_m)| + |u_k(\tau_m) - u_k(s)| < 2\varepsilon.$$

□

Theorem 4.3. *Let X be a metric space in which every closed ball is compact, let $F : X \times [a, b] \times [a, b] \to X$ satisfy $F(x, \tau, \tau) = x$ and let us assume the following:*
(**E1**) *For every $\tau, t \in [a, b]$ the function $x \mapsto F(x, \tau, t)$ is continuous.*
(**E2**) *There exists $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ increasing and continuous such that $\zeta(0) = 0$ and for every $x \in X$ and every $\tau, t, s \in [a, b]$ we have*

$$|F(x, \tau, t) - F(x, \tau, s)| < \zeta(|t - s|).$$

(**E3**) *For every* $y \in X$ *and* $\tau, t \in [a,b]$ *there exists* $z \in X$ *such that* $F(z,\tau,t) = y$.
(**E4**) *There exists* $\xi : [a,b] \to \mathbb{R}$ *increasing such that for every* $\varepsilon > 0$ *there exists* $\eta : [a,b] \to \mathbb{R}^+$ *such that all* $\tau, \sigma \in [a,b]$, *all* $t, s \in \mathcal{U}(\tau, \eta(\tau)) \cap \mathcal{U}(\sigma, \eta(\sigma))$ *and all* $x, y \in X$ *with* $|x - y| < \max\{\eta(\tau), \eta(\sigma)\}$ *satisfy*

$$|F(x,\tau,t) - F(y,\sigma,t)| < |F(x,\tau,s) - F(y,\sigma,s)| + \varepsilon|\xi(t) - \xi(s)|.$$

Then for every $x_0 \in X$ *and* $\tau_0 \in [a,b]$ *there exists a solution* $u : [a,b] \to X$ *of* $x' = \mathrm{D}_t\, F(x,\tau,t)$ *on* $[a,b]$ *with* $u(\tau_0) = x_0$.

Proof. **Step 1**: Find a solution candidate. Fix $x \in X$ and a partition $\{[t_{i-1}, t_i], \tau_i\}$ of $[a,b]$. We set $v(a) := x$. For $t \in [t_{i-1}, t_i]$ let $v(t) := F(z, \tau_i, t)$, where $z \in X$ is chosen so that $F(z, \tau_i, t_{i-1}) = v(t_{i-1})$. This can always be arranged due to condition (**E3**). For fixed $\delta : [a,b] \to \mathbb{R}^+$ any function constructed from a δ-fine partition will also be called δ-fine. For arbitrary $\tau \in [a,b]$ and δ-fine partition $\{[t_{i-1}, t_i], \tau_i\}$ with $\tau \in [t_m, t_{m+1}]$ the partition

$$\{\ldots [t_{m-1}, t_m], \tau_m, [t_m, \tau], \tau, [\tau, t_{m+1}], \tau, [t_{m+1}, t_{m+2}], \tau_{m+1}, \ldots\}$$

is also δ-fine. Therefore, we can begin the construction from $\tau \in [a,b]$ and build analogously on the left side.

Set $\varepsilon_k := 2^{-k}$, find the corresponding $\eta_k : [a,b] \to \mathbb{R}^+$ from condition (**E4**) and put

$$\gamma_k(\tau) := \min\{2^{-k}, \eta_1(\tau)/2, \ldots, \eta_k(\tau)/2, \zeta^{-1}(\eta_1(\tau)/2), \ldots, \zeta^{-1}(\eta_k(\tau)/2)\}. \quad (4.1)$$

Since Lemma 1.1 ensures the existence of γ_k-fine partitions P_k of $[a,b]$, we can use these partitions to construct γ_k-fine approximate solutions $u_k : [a,b] \to X$ with $u_k(\tau_0) = x_0$.

We now verify the assumptions of Lemma 4.2. The most important fact to remember is that the approximate solutions u_k consist of segments of the form $t \mapsto F(u_k(\tau_i), \tau_i, t)$ for $t \in [t_{i-1}, t_i]$.

Fix $\tau \in [a,b)$ and $\varepsilon > 0$. First, find $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varepsilon$ and $k_1 \geq k_0$ such that $2\,\zeta(2^{-k_1+1}) < \eta_{k_0}(\tau)$. Then, find $\beta > 0$ such that $2\,\zeta(\beta) < \eta_{k_0}(\tau) - 2\,\zeta(2^{-k_1+1})$ and $\alpha > 0$ such that for $t \in (\tau, \tau + \alpha)$ we have

$$|\xi(\tau+) - \xi(t)| < \min\{1/6, \eta_{k_0}(\tau) - 2\,\zeta(2^{-k_1+1}) - 2\,\zeta(\beta)\}. \quad (4.2)$$

Set $\delta := \min\{\alpha, \beta, \eta_{k_0}(\tau), \zeta^{-1}(\varepsilon/6)\}$.

Fix $t \in (\tau, \tau+\delta(\tau))$ and $k \geq k_1$. Since $\delta \leq \zeta^{-1}(\varepsilon/6)$, we notice that

$$\begin{aligned} |u_k(t) - u_k(\tau)| &\leq |u_k(t) - F(u_k(\tau), \tau, t)| + |F(u_k(\tau), \tau, t) - u_k(\tau)| \\ &\leq |u_k(t) - F(u_k(\tau), \tau, t)| + \frac{\varepsilon}{6}. \end{aligned} \tag{4.3}$$

The second inequality of (4.3) holds by $u_k(\tau) = F(u_k(\tau), \tau, \tau)$ and (**E2**). Note that hereafter we will be using the uniform continuity of $F(x, \tau, t)$ in the third variable, i.e. condition (**E2**), without explicit mention.

We finish the estimate (4.3) by dealing with $|u_k(t) - F(u_k(\tau), \tau, t)|$.

Find $j, m \in \mathbb{N}, j \leq m$ such that for the corresponding intervals $[t_{j-1}, t_j] \in P_k$ and $[t_{m-1}, t_m] \in P_k$ we have $\tau \in [t_{j-1}, t_j)$ and $t \in [t_{m-1}, t_m]$. The necessity to forbid $\tau = t_j$ comes from the possibility of τ being a point of discontinuity of the function ξ. The case $j = m$ is trivial, since $t \in [t_{j-1}, t_j]$ implies $|u_k(t) - u_k(\tau)| < \varepsilon/6$. Now, assume that $m \geq j+1$ and use induction by i to proceed. Our aim will be to show that if

$$|F(u_k(\tau), \tau, s) - u_k(s)| < |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(s) - \xi(t_j)| \tag{4.4}$$

holds for all $s \in [t_{i-1}, t_i] \cap [\tau, t]$, then it holds for all $s \in [t_i, t_{i+1}] \cap [\tau, t]$, where $i = j+1, \ldots, m-1$.

First, we need to verify that (4.4) holds for $s \in [t_j, t_{j+1}] \cap [\tau, t]$. We recall that (4.1) ensures that u_k are 2^{-n} fine for $n \leq k$. Together with $k \geq k_1$ we have $|t_j - \tau| \leq 2^{-k_1+1}$, and therefore $|u_k(\tau) - u_k(t_j)| \leq \zeta(2^{-k_1+1})$. Similarly, $|u_k(t_j) - u_k(\tau_{j+1})| \leq \zeta(2^{-k_1+1})$. Since k_1 was chosen so that $2\,\zeta(2^{-k_1+1}) < \eta_{k_0}(\tau)$, we obtain $|u_k(\tau) - u_k(\tau_{j+1})| < \eta_{k_0}(\tau)$. We also recall that u_k are constructed to be $\eta_n/2$-fine for $n \leq k$, in particular $\eta_{k_0}/2$-fine. Together with

$$|\tau - s| \leq |\tau - t| < \delta \leq \eta_{k_0}(\tau)$$

we have that $[t_j, s] \subset \mathcal{U}(\tau, \eta_{k_0}(\tau)) \cap \mathcal{U}(\tau_{j+1}, \eta_{k_0}(\tau_{j+1}))$. This allows us to use condition (**E4**) with $F(u_k(\tau_{j+1}), \tau_{j+1}, s) = u_k(s)$ and $F(u_k(\tau_{j+1}), \tau_{j+1}, t_j) = u_k(t_j)$ to obtain

$$|F(u_k(\tau), \tau, s) - u_k(s)| < |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(s) - \xi(t_j)|.$$

Now, assume that $j+2 \leq i \leq m-1$ and that (4.4) holds for all $s \in [t_{i-1}, t_i] \cap [\tau, t]$. Since $i < m$, we have $[t_{i-1}, t_i] \subset [\tau, t]$. This means we can use (4.4) with $s = t_i$ to obtain

$$|F(u_k(\tau), \tau, t_i) - u_k(t_i)| < |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(t_i) - \xi(t_j)|.$$

Fix $s \in [t_i, t_{i+1}] \cap [\tau, t]$ and note that $[t_i, s] \subset \mathcal{U}(\tau, \eta_{k_0}(\tau)) \cap \mathcal{U}(\tau_{i+1}, \eta_{k_0}(\tau_{i+1}))$. In order to justify the use of condition (**E4**), we compute

$$\begin{aligned}
|u_k(\tau) - u_k(\tau_{i+1})| &\le |u_k(\tau) - F(u_k(\tau), \tau, t_i)| + |F(u_k(\tau), \tau, t_i) - u_k(t_i)| \\
&\quad + |u_k(t_i) - u_k(\tau_{i+1})| \\
&\le \zeta(\delta) + |F(u_k(\tau), \tau, t_i) - u_k(t_i)| + \zeta(2^{-k_1+1}) \\
&\le \zeta(\delta) + \zeta(2^{-k_1+1}) + |F(u_k(\tau), \tau, t_j) - u_k(t_j)| \\
&\quad + \varepsilon_{k_0}|\xi(t_i) - \xi(t_j)| \\
&\le \zeta(\delta) + \zeta(2^{-k_1+1}) + |F(u_k(\tau), \tau, t_j) - u_k(\tau)| \\
&\quad + |u_k(\tau) - u_k(t_j)| + \varepsilon_{k_0}|\xi(t_i) - \xi(t_j)| \\
&\le 2\,\zeta(\delta) + 2\,\zeta(2^{-k_1+1}) + \varepsilon_{k_0}|\xi(t_i) - \xi(t_j)| \\
&\le 2\,\zeta(\delta) + 2\,\zeta(2^{-k_1+1}) + |\xi(t_i) - \xi(\tau+)|
\end{aligned}$$

However, $2\,\zeta(\delta) + 2\,\zeta(2^{-k_1+1}) + |\xi(t_i) - \xi(\tau+)| < \eta_{k_0}(\tau)$ due to (4.2). Therefore, we can use (**E4**) again to get

$$\begin{aligned}
|F(u_k(\tau), \tau, s) - u_k(s)| &< |F(u_k(\tau), \tau, t_i) - u_k(t_i)| + \varepsilon_{k_0}|\xi(s) - \xi(t_i)| \\
&< |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(t_i) - \xi(t_j)| + \varepsilon_{k_0}|\xi(s) - \xi(t_i)| \\
&= |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(s) - \xi(t_j)|.
\end{aligned}$$

This completes the induction step and by setting $s = t$ we obtain the estimate

$$|F(u_k(\tau), \tau, t) - u_k(t)| < |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0}|\xi(t) - \xi(t_j)|. \tag{4.5}$$

We recall that we have

$$|u_k(t) - u_k(\tau)| \le |u_k(t) - F(u_k(\tau), \tau, t)| + \frac{\varepsilon}{6}$$

We combine this with $|\xi(t) - \xi(\tau+)| < 1/6$, $\varepsilon_{k_0} < \varepsilon$ and (4.5) to obtain

$$|u_k(t) - u_k(\tau)| \le |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \frac{2\,\varepsilon}{6}.$$

We finish by using

$$|F(u_k(\tau), \tau, t_j) - u_k(t_j)| \le |F(u_k(\tau), \tau, t_j) - u_k(\tau)| + |u_k(\tau) - u_k(t_j)| < \frac{2\,\varepsilon}{6}.$$

Thus, it becomes clear that $|u_k(t) - u_k(\tau)| < \varepsilon$. An analogous procedure can be done on the left neighbourhood of every $\tau \in (a, b]$, and therefore $\{u_k\}_{k\in\mathbb{N}}$ satisfy the assumptions of Lemma 4.2. As a result, they are uniformly equicontinuous i.e. there exists $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ increasing and continuous such that $\omega(0) = 0$ and for every $t, s \in [a, b]$ and every $k \in \mathbb{N}$ we have

$$|u_k(t) - u_k(s)| < \omega(|t - s|).$$

Set $\delta := \omega^{-1}(1)$ and find $M \in \mathbb{N}$ such that $M \geq (b-a)/\delta$. Then we can see that $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(x_0, M)$. Note that we will be using the uniform equicontinuity of the solutions with the continuity module ω without further mention.

The Arzela-Ascoli theorem now ensures the existence of a subsequence of $\{u_k\}$ uniformly converging to $u : [a,b] \to X$.

Step 2: We now prove that u solves $x' = \mathrm{D}_t\, F(x,\tau,t)$. For an arbitrary partition P of $[a,b]$ we can see that

$$\begin{aligned}
&\sum_P \Big(|u(t_{i-1}) - F(u(\tau_i),\tau_i,t_{i-1})| + |u(t_i) - F(u(\tau_i),\tau_i,t_i)|\Big)\\
&= \sum_P \Big(|\lim_{k\to\infty} u_k(t_{i-1}) - \lim_{k\to\infty} F(u_k(\tau_i),\tau_i,t_{i-1})|\\
&\qquad + |\lim_{k\to\infty} u_k(t_i) - \lim_{k\to\infty} F(u_k(\tau_i),\tau_i,t_i)|\Big)\\
&= \sum_P \Big(\lim_{k\to\infty} |u_k(t_{i-1}) - F(u_k(\tau_i),\tau_i,t_{i-1})| + \lim_{k\to\infty} |u_k(t_i) - F(u_k(\tau_i),\tau_i,t_i)|\Big)\\
&= \lim_{k\to\infty} \sum_P \Big(|u_k(t_{i-1}) - F(u_k(\tau_i),\tau_i,t_{i-1})| + |u_k(t_i) - F(u_k(\tau_i),\tau_i,t_i)|\Big).
\end{aligned}$$

Therefore, it suffices to show that for every $\varepsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}^+$ and $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}, k \geq k_0$ and every δ-fine partition P of $[a,b]$ we have

$$\sum_P |u_k(t_{i-1}) - F(u_k(\tau_i),\tau_i,t_{i-1})| + |u_k(t_i) - F(u_k(\tau_i),\tau_i,t_i)| < \varepsilon.$$

Fix $k \in \mathbb{N}$ and recall the construction of u_k from P_k. For $\tau \in [a,b]$ we distinguish four cases:

(i) There exists $\alpha > 0$ and $i \in \mathbb{N}$ such that $[\tau-\alpha, \tau+\alpha] \subset [t_{i-1}, t_i]$.
(ii) For $\tau = a$ there exists $\alpha > 0$ such that $[a, a+\alpha] \subset [t_0, t_1]$.
(iii) For $\tau = b$ there exists $\alpha > 0$ and $i \in \mathbb{N}$ such that $[b-\alpha, b] \subset [t_{i-1}, t_i]$.
(iv) There exists $i \in \mathbb{N}$ such that $t_i = \tau$. Therefore, we can find $\alpha > 0$ such that

$$[\tau-\alpha, \tau] \subset [t_{i-1}, t_i] \quad \text{and} \quad [\tau, \tau+\alpha] \subset [t_i, t_{i+1}].$$

For fixed $k \in \mathbb{N}$ and $\tau \in [a,b]$ we will denote the corresponding α by $\alpha(k,\tau)$.

Fix $\varepsilon > 0$ and find $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varepsilon/(\xi(b) - \xi(a))$. Fix $\tau \in [a,b]$ and let the function η correspond to $\varepsilon_{k_0} = 2^{-k_0}$. Find $k_1 \geq k_0$ such that $\zeta(2^{-k_1}) < \eta(\tau)$. For $h \geq 0$ define $\beta(h) := \eta(\tau) - \zeta(2^{-k_1}) - \omega(h)$. This function is decreasing, continuous and $\beta(0) = \eta(\tau) - \zeta(2^{-k_1}) > 0$. These properties imply the existence of $h_0 \in (0, \eta(\tau))$ such that $\beta(h) > 0$ for $h \in [0, h_0]$. Define

$$\delta(\tau) = \min\{h_0, \alpha(k_0,\tau), \alpha(k_0+1,\tau), \ldots, \alpha(k_1,\tau)\}.$$

Now, fix $k \geq k_0$ and $r, s \in [a, b]$ such that $\tau - \delta(\tau) \leq r \leq \tau \leq s \leq \tau + \delta(\tau)$.

First, let $k_0 \leq k \leq k_1$. Since $\delta(\tau) \leq \alpha(k, \tau)$, there exists $\tau_j \in P_k$ such that $u_k(\tau) = F(u_k(\tau_j), \tau_j, \tau)$ and $u_k(s) = F(u_k(\tau_j), \tau_j, s)$. We recall that (4.1) yields $|\tau - \tau_j| \leq \zeta^{-1}(\eta(\tau_j)/2)$, and therefore $|u_k(\tau) - u_k(\tau_j)| < \eta(\tau_j)$. We also recall that $|\tau - s| \leq \delta(\tau) < \eta(\tau)$ and $\tau, s \in \mathcal{B}(\tau_j, \eta(\tau_j)/2) \subset \mathcal{U}(\tau_j, \eta(\tau_j))$. Condition (**E4**) now implies

$$\begin{aligned}|F(u_k(\tau), \tau, s) - F(u_k(\tau_j), \tau_j, s)| < {} & |F(u_k(\tau), \tau, \tau) - F(u_k(\tau_j), \tau_j, \tau)| \\ & + \varepsilon_{k_0} |\xi(s) - \xi(\tau)|.\end{aligned}$$

By using $F(u_k(\tau), \tau, \tau) = u_k(\tau) = F(u_k(\tau_j), \tau_j, \tau)$ and $F(u_k(\tau_j), \tau_j, s) = u_k(s)$ we get

$$|F(u_k(\tau), \tau, s) - u_k(s)| < \varepsilon_{k_0} |\xi(s) - \xi(\tau)|.$$

An analogous procedure on the left neighbourhood of τ gives us

$$|F(u_k(\tau), \tau, r) - u_k(r)| < \varepsilon_{k_0} |\xi(\tau) - \xi(r)|.$$

Now, let $k \geq k_1$. Again, the condition $|\tau - t| < \eta(\tau)$ is obviously satisfied for $t \in [\tau, s]$ and the use of condition (**E4**) is only limited by $|u_k(\tau) - u_k(\tau_i)|$ for $\tau_i \in P_k$. However, unlike Step 1, we do not need the conclusion of condition (**E4**) to hold on $[t_{i-1}, t_i]$ in order to verify its assumptions on $[t_i, t_{i+1}]$. Find $j, m \in \mathbb{N}$ such that $\tau \in [t_{j-1}, t_j]$ and $s \in [t_{m-1}, t_m]$ and note that

$$\begin{aligned}
&|u_k(\tau) - u_k(\tau_j)| \leq \zeta(2^{-k_1}) < \eta(\tau),\\
&|u_k(\tau) - u_k(\tau_{j+1})| \leq \omega(|\tau - \tau_{j+1}|) < \eta(\tau),\\
&|u_k(\tau) - u_k(\tau_{j+2})| \leq \omega(|\tau - \tau_{j+2}|) < \eta(\tau),\\
&\qquad\vdots\\
&|u_k(\tau) - u_k(\tau_{m-1})| \leq \omega(|\tau - \tau_{m-1}|) < \eta(\tau),\\
&|u_k(\tau) - u_k(\tau_m)| \leq |u_k(\tau) - u_k(s)| + |u_k(s) - u_k(\tau_m)|\\
&\qquad \leq \omega(|\tau - s|) + \zeta(2^{-k_1}) < \eta(\tau).
\end{aligned}$$

Persistent use of condition (**E4**) gives us

$$\begin{aligned}
|F(u_k(\tau), \tau, s) - u_k(s)| &< |F(u_k(\tau), \tau, t_{m-1}) - u_k(t_{m-1})| + \varepsilon_{k_0} |\xi(s) - \xi(t_{m-1})|,\\
|F(u_k(\tau), \tau, t_{m-1}) - u_k(t_{m-1})| &< |F(u_k(\tau), \tau, t_{m-2}) - u_k(t_{m-2})|\\
&\quad + \varepsilon_{k_0} |\xi(t_{m-1}) - \xi(t_{m-2})|,\\
&\vdots\\
|F(u_k(\tau), \tau, t_{j+1}) - u_k(t_{j+1})| &< |F(u_k(\tau), \tau, t_j) - u_k(t_j)| + \varepsilon_{k_0} |\xi(t_{j+1}) - \xi(t_j)|,\\
|F(u_k(\tau), \tau, t_j) - u_k(t_j)| &< |F(u_k(\tau), \tau, \tau) - u_k(\tau)| + \varepsilon_{k_0} |\xi(t_j) - \xi(\tau)|\\
&= \varepsilon_{k_0} |\xi(t_j) - \xi(\tau)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}|F(u_k(\tau),\tau,s)-u_k(s)| &< \varepsilon_{k_0}\,|\xi(s)-\xi(t_{m-1})|+\ldots+\varepsilon_{k_0}\,|\xi(t_j)-\xi(\tau)|\\ &= \varepsilon_{k_0}\,|\xi(s)-\xi(\tau)|.\end{aligned}$$

For any δ-fine partition $\{[s_{i-1},s_i],\sigma_i\}_{i=1}^n$ of $[a,b]$ and $k\geq k_0$ we obtain

$$\sum_{i=1}^n |u_k(s_{i-1})-F(u_k(\sigma_i),\sigma_i,s_{i-1})|+|u(s_i)-F(u_k(\sigma_i),\sigma_i,s_i)|$$
$$< 2^{-k_0}\sum_{i=1}^n |\xi(s_{i-1})-\xi(\sigma_i)|+|\xi(\sigma_i)-\xi(s_i)| = 2^{-k_0}(\xi(b)-\xi(a))<\varepsilon,$$

and the proof is finished. □

5. Euclidean Case

First of all, we present an example that demonstrates the difference between the generalized and standard ordinary differential equation theory. Consider the functions

$$G(x,t) = \mathfrak{C}(t),$$
$$F(x,\tau,t) = x + \mathfrak{C}(t) - \mathfrak{C}(\tau),$$

on the domain $\mathbb{R}\times[0,1]$ and $\mathbb{R}\times[0,1]\times[0,1]$ respectively, where $\mathfrak{C}(t)$ is the Cantor stair function. These functions satisfy the assumptions of theorems 3.3, 3.5, 4.1 and 4.3 and solutions of the associated GODEs are vertical translations of the Cantor function itself. However, it is not covered by Carathéodory's approach (much less by the classical theory), as the solutions are not absolutely continuous.

Uniqueness

Let us recall the uniqueness theorem 3.3. Our main contribution to the theory of generalized ordinary differential equations on Euclidean spaces is that we can replace condition (3.1) (bounded variation) with continuity of $t \mapsto G(x,t)$ at every $x \in \mathcal{U}_R$. To this end, we verify the assumptions of Theorem 3.5 for the function $F(x,\tau,t) = x + G(x,t) - G(x,\tau)$. Indeed, if $t \mapsto G(x,t)$ is continuous, then the function $t \mapsto x + G(x,t) - G(x,\tau)$ is also continuous. Conditions (3.3) and (3.4) are identical and require no further comment. Notice that

$$\|F(x,\tau,t) - F(y,\tau,t)\| = \|x - y + G(x,t) - G(x,\tau) + G(y,t) - G(y,\tau)\|.$$

Therefore,

$$\|F(x,\tau,t) - F(y,\tau,t)\| - \|x-y\| \le \|G(x,t) - G(x,\tau) + G(y,t) - G(y,\tau)\|.$$

If (3.2) holds, then we have

$$\|G(x,t) - G(x,\tau) + G(y,t) - G(y,\tau)\| \le \omega(||x-y||)\,|h(t) - h(\tau)|,$$

and therefore

$$\liminf_{t\to\tau_+} \frac{\|F(x,\tau,t) - F(y,\tau,t)\| - \|x-y\|}{h(t)-h(\tau)} \le \omega(\|x-y\|).$$

Another notable difference between Theorem 3.3 and Theorem 3.5 is that we have managed to weaken (3.2) to a local version.

Existence

Similarly to uniqueness, we can replace bounded variation in Theorem 4.1 with continuity of $t \mapsto G(x,t)$. Once again, we verify the assumptions of Theorem 4.3 for $F(x,\tau,t) = x + G(x,t) - G(x,\tau)$. We notice that

$$\begin{aligned} \|F(x,\tau,t) - F(y,\tau,t)\| &= \|x + G(x,t) - G(x,\tau) - y - G(y,t) + G(y,\tau)\| \\ &\leq \|x-y\| + \|G(x,t) - G(x,\tau) - G(y,t) + G(y,\tau)\| \\ &\leq \|x-y\| + \omega(\|x-y\|)\,|h(t) - h(\tau)|. \end{aligned}$$

Condition (**E1**) is therefore satisfied. Since Theorem 4.1 only gives local existence, we can confine the problem to a compact set K around the initial condition. As a result, $F(x,\tau,t)$ is uniformly continuous on $K \times [a,b] \times [a,b]$, and therefore satisfies (**E2**). The condition (**E3**) is then a simple result of topological degree theory, as for $|t-\tau|$ small enough the mapping $x \mapsto F(x,\tau,t)$ is uniformly close to the identity mapping $x \mapsto F(x,\tau,\tau)$.

Trading bounded variation for continuity might seem impractical at first. However, consider the following existence theorem from [11] by R. Henstock. We will show that while it is not contained in Theorem 4.1, it does follow from Theorem 4.3.

Theorem 5.1. *Assume that $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ satisfies the following conditions:*
(**H1**) *The function $x \mapsto f(x,t)$ is continuous for almost all $t \in [a,b]$.*
(**H2**) *The function $t \mapsto f(x,t)$ is SKH integrable over $[a,b]$ for every $x \in \mathbb{R}^n$.*
(**H3**) *There exists $S \subset \mathbb{R}^n$ compact and $\delta : [a,b] \to \mathbb{R}^+$ such that all δ-fine partitions $\{\alpha = \alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k = \beta\}$ of $[\alpha,\beta] \subset [a,b]$ and all functions $w : [a,b] \to \mathbb{R}^n$ satisfy*

$$\sum_{i=1}^{k} f(w(\tau_i),\tau_i)(\alpha_i - \alpha_{i-1}) \in S.$$

Then for every $v \in \mathbb{R}^n$ and $\tau \in [a,b]$ there exists $y : [a,b] \to \mathbb{R}^n$ such that

$$y(t) = v + (\mathrm{SKH})\int_\tau^t f(y(s),s)\,\mathrm{d}s \quad \textit{for} \quad t \in [a,b].$$

We prove that (**H1**) – (**H3**) for $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ imply (**E1**) – (**E4**) for

$$F(x,\tau,t) = x + (\mathrm{SKH})\int_\tau^t f(x,s)\,\mathrm{d}s.$$

Since condition (**H3**) ensures that the solution stays in a compact set around the initial condition (v,τ), we can limit ourselves to studying the function f on $\mathcal{B}_R := \{x \in \mathbb{R}^n\,;\ \|x\| \leq R\}$ for $R > 0$ sufficiently large. We make use of the following decomposition theorem ([3], page 78).

Theorem 5.2. *A function $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ satisfies* (**H1**) – (**H3**) *if and only if $f(x,t) = g(t) + h(x,t)$, where $g : [a,b] \to \mathbb{R}^n$ is SKH integrable over $[a,b]$ and $h : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ satisfies the Carathéodory conditions* (**C1**) – (**C3**).

As an immediate result of this theorem we have that $x \mapsto F(x,\tau,t)$ is continuous, since the Carathéodory conditions are exactly what we need to use the Lebesgue dominated convergence theorem for $s \mapsto h(x_n, s)$ with $x_n \to x$ and get

$$\lim_{n\to\infty} \int_\tau^t h(x_n,s)\,\mathrm{d}s = \int_\tau^t h(x,s)\,\mathrm{d}s.$$

Since $F(x,\tau,t)$ is obviously continuous in τ and t and $\mathcal{B}_R$ is compact, we easily get condition (**E2**). Therefore, the functions $x \mapsto F(x,\tau,t)$ are uniformly close to the identity mapping for $|\tau - t|$ small enough and from the topological degree theory we get (**E3**). In order to verify condition (**E4**), we apply the following theorem ([3], page 135).

Theorem 5.3. *Let $h : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ satisfy* (**C1**) – (**C3**) *on $\mathcal{B}_R \times [a,b]$. Then there exists $p : [a,b] \to \mathbb{R}$ Lebesgue integrable and $\omega : [0,2R] \to \mathbb{R}_0^+$ continuous and increasing with $\omega(0) = 0$ such that*

$$||G(x,t_2) - G(x,t_1) - G(y,t_2) + G(y,t_1)|| \le \omega(||x-y||) \int_{t_1}^{t_2} p(s)\,\mathrm{d}s$$

for $x, y \in \mathcal{B}_R$ and $t_1, t_2 \in [a,b]$, where $G(x,t)$ is defined as

$$G(x,t) = \int_{t_0}^t h(x,s)\,\mathrm{d}s.$$

We notice that

$$F(x,\tau,t) = x + G(x,t) - G(x,\tau) + (\mathrm{SKH}) \int_\tau^t g(s)\,\mathrm{d}s,$$

and therefore

$$F(x,\tau,t_2) - F(x,\tau,t_1) = G(x,t_2) - G(x,t_1) + (\mathrm{SKH}) \int_{t_1}^{t_2} g(s)\,\mathrm{d}s$$
$$F(y,\sigma,t_1) - F(y,\sigma,t_2) = G(y,t_1) - G(y,t_2) + (\mathrm{SKH}) \int_{t_2}^{t_1} g(s)\,\mathrm{d}s$$

We arrive at the result

$$\|F(x,\tau,t_2) - F(x,\tau,t_1) - F(y,\sigma,t_2) + F(y,\sigma,t_1)\| \le \omega(\|x-y\|)\,|\xi(t_2) - \xi(t_1)|$$

where ξ is is defined as

$$\xi(t) = \int_a^t |p(s)|\,\mathrm{d}s.$$

For any $\varepsilon > 0$ we can find $\eta > 0$ such that $\omega(\eta) < \varepsilon$. Since ω is increasing, $\|x-y\| < \eta$ implies $\omega(\|x-y\|) < \varepsilon$. We finish by applying $|\ \|a\| - \|b\|\ | \leq \|a-b\|$ to get that for $\|x-y\| < \eta$ we have

$$\|F(x,\tau,t_2) - F(y,\sigma,t_2)\| \leq \|F(x,\tau,t_1) - F(y,\sigma,t_1)\| + \varepsilon|\xi(t_2) - \xi(t_1)|$$

Therefore, $F(x,\tau,t)$ satisfies (**E4**) and the case of Theorem 5.1 is covered by Theorem 4.3.

To see that Theorem 5.1 theorem is not contained in Theorem 4.1, consider the following example:

$$H(x) = \begin{cases} x^2\cos(\pi/x^2) & 0 < x \leq 1, \\ 0 & x = 0. \end{cases}$$

The derivative

$$H'(x) = \begin{cases} 2x\cos(\pi/x^2) + (2\pi/x)\sin(\pi/x^2) & 0 < x \leq 1, \\ 0 & x = 0, \end{cases}$$

exists everywhere on the interval $[0,1]$. Since H' is SKH integrable, Theorem 5.1 can be applied to the the case $f(x,t) = H'(t)$ due to Theorem 5.2. However, the corresponding function

$$G(x,t) = (\mathrm{SKH})\int_0^t H'(t) = H(t)$$

does not satisfy the assumptions of Theorem 4.1, since H is not of bounded variation.

While Theorem 5.1 is able to handle functions of unbounded variation, it remains within the bounds of the standard ordinary differential equation theory. It allows for the solution to be an indefinite SHK integral of the right hand side, but not in the sense of coupled variables. Therefore, the example involving the Cantor function is not covered.

Thus, we can see that Theorem 4.1 and Theorem 5.1 are incomparable. However, our existence theorem 4.3 contains both a large portion of Theorem 4.1 and the entire Theorem 5.1.

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