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Faculty of Mathematics and Physics

Bachelor's Thesis

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Automorphism Groups of Geometrically Represented Graphs

Computer Science Institute

Supervisor

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I hereby declare that I have written all text of this thesis on my own.

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I am also grateful to my family, especially my parents, who supported me during my studies.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Prague, May 23, 2014 Peter Zeman

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Kľúčové slová: grupy automorfizmov, intervalové grafy
Abstrakt:

V tejto práci skúmame grupy automorfizmov grafov s veľmi silnou štruktúrou. Pravdepodobne jeden z prvých výsledkov v tomto smere je Jordanova charakterizácia triedy grúp automorfizmov stromov $\mathcal{T}$ z roku 1869.

Prekvapivo, grupy automorfizmov prienikových grafov boli študované iba veľmi málo. Aj pre veľmi pochopené triedy prienikových grafov, je štruktúra ich grúp automorfizmov neznáma. Hlavná otázka, ktorou sa zaoberáme je, či sa z dobrej znalosti reprezentácií prienikového grafu geometrických objektov dá zrekonštruovať jeho grupa automorfizmov. V práci skúmame hlavne intervalové grafy.

Intervalové grafy sú prienikové grafy intervalov na reálnej osi. Sú jednou z najstarších a najviac študovaných tried prienikových grafov. Náš hlavný výsledok hovorí, že trieda grúp automorfizmov intervalových grafov $\mathcal{I}$ je rovnaká ako trieda grúp automorfizmov stromov $\mathcal{T}$. Navyše ukazujeme postup ako pre daný intervalový graf skonštruovať strom s rovnakou grupou automorfizmov a tak isto obrátene, pre daný strom skonštruujeme intervalový graf.

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Abstract:

In this thesis, we are interested in automorphism groups of classes of graphs with a very strong structure. Probably the first nontrivial result in this direction is from 1869 due to Jordan. He gave a characterization of the class $\mathcal{T}$ of the automorphism groups of trees.

Surprisingly, automorphism groups of intersection-defined classes of graphs were studied only briefly. Even for deeply studied classes of intersection graphs the structure of their automorphism groups is not well known. We study the problem of reconstructing the automorphism group of a geometric intersection graph from a good knowledge of the structure of its representations. We mainly deal with interval graphs.

Interval graphs are intersection graphs of intervals on the real line. They are one of the oldest and most studied classes of geometric intersection graphs. Our main result is that the class $\mathcal{T}$ is the same as the class $\mathcal{I}$ of the automorphism groups of interval graphs. Moreover, we show for an interval graph how to find a tree with the same automorphism group, and vice versa.

1 Introduction

An automorphism of a graph X is a permutation of its vertices such that two vertices are adjacent if and only if their images are connected with an edge. The group $\mathrm{Aut}(X)$ of all such permutations is called the *automorphism group of* X. A graph X *represents* a group G if $\mathrm{Aut}(X)$ is isomorphic to G.

Most graphs are *asymmetric*, that is, they have *no other* automorphisms aside the identity (see e.g. [16]). However, many graphs arising from various algebraic, topological and combinatorial applications have non-trivial automorphism groups, which makes the study of the automorphism groups of graphs important.

Complexity Theory Motivation. The study of the automorphism groups of graphs is also motivated by problems in computational complexity theory. A long-standing open problem in the complexity theory is whether there exists an algorithm that can test isomorphism of finite algebraic structures in polynomial time. All such algebraic structures can be encoded by graphs in polynomial time [21, 33]. Therefore, it suffices to solve the isomorphism problem for graphs.

Problem:	$\textsc{GraphIso}(X_1, X_2)$
Input:	Graphs X_1 and X_2.
Question:	Is X_1 isomorphic to X_2?

The problem $\textsc{GraphIso}(X_1, X_2)$ is very important in the complexity theory. It is one of the few computational problems that are known to belong to NP, but are not known whether they are solvable in polynomial time and are also not know to be NP-complete. For many special classes of graphs, such as trees, planar graphs, interval graphs, $\textsc{GraphIso}(X_1, X_2)$ is known to be solvable in polynomial time. At the same time, there exists a strong theoretical evidence against NP-completeness of $\textsc{GraphIso}(X_1, X_2)$. It is known that it belongs to the low hierarchy of the class NP [36], which implies that it is not NP-complete unless the polynomial-time hierarchy collapses to its second level. For basic concepts in the complexity theory we refer to [1]. One of the most famous results concerning $\textsc{GraphIso}(X)$ is that it can be solved in polynomial time for graphs of bounded degree [28].

The graph isomorphism problem is closely related to a fundamental computational problem in algebraic graph theory. It is the problem of finding a generating set of the automorphism group of a graph.

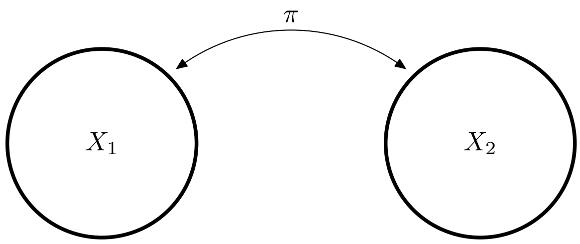


Figure 1.1: Suppose that we are given two connected graphs X_1 and X_2. We set X to be the disjoint union of X_1 and X_2 and find the generating set of $\mathrm{Aut}(X)$. If the generating set contains a permutation π that swaps X_1 and X_2, then X_1 and X_2 are isomorphic. If X_1 and X_2 are disconnected, then we set X to be the disjoint union of their complements, since the automorphism group of a graph is isomorphic the automorphism group of its complement.

Problem:	GraphAut(X)
Input:	A graph X.
Output:	Generating permutations for $\mathrm{Aut}(X)$.

Problem GraphIso(X_1, X_2) has a polynomial time reduction to GraphAut(X). The reduction is shown in Figure 1.1. On the other hand, GraphAut(X) can be solved by solving GraphIso(X_1, X_2) at most $\mathcal{O}(n^4)$ times [30].

1.1 Graphs With a Strong Structure

A famous result, known as Frucht's theorem [13], claims that every finite group is isomorphic to the automorphism group of some finite graph. We are interested in automorphism groups of classes of graphs with a very strong structure.

Probably the first nontrivial result in this direction is from 1869 due to Jordan [23]. He gave a characterization (see Theorem 2.6) of the class $\mathcal{T}$ of the automorphism groups of trees. It says that we can get the automorphism groups of trees from the trivial group by a sequence of two operations: the direct product and the wreath product with a symmetric group. The direct product constructs automorphisms that act independently on non-isomorphic subtrees, while the wreath product constructs automorphisms that permute isomorphic subtrees.

Another class of graphs with understood automorphism groups are planar graphs. Babai gave a characterization in 1973 [2]. He reduces a planar graph to a 3-connected planar graph for which the automorphism group can be determined [39]. He proceeds in such way that he is able to construct the automorphism group of the original planar graph using group products.

Geometric Intersection Graphs. We can assign geometric objects to the vertices of a graph and encode its edges by intersections of these objects. More formally, an intersection representation $\mathcal{R}$ of X is a collection of sets $\{R_x : x \in V(X)\}$ such that $R_x \cap R_y \neq \emptyset$ if and only if $xy \in E(X)$. Every graph can be represented in this way [29]. Therefore, to obtain interesting classes of graphs, the sets R_x are usually some specific geometric objects. The most famous classes of geometric intersection graphs include interval graphs, circle graphs, circular-arc graphs, permutation graphs and function graphs.

The problem of characterizing the intersection graphs of families of sets having some geometrical property is an interesting problem and is often motivated by real world applications. Sometimes even an application gives an intersection representation. Many hard combinatorial problems can be often solved efficiently on geometric intersection graphs. Another reason for considering an intersection representation of a graph is that it can provide much better visualisation of the graph and therefore, possibly a much better understanding of the structure of the graph. For example, the structure of the graph in Figure 1.2 is more clear from its interval representation. For more information about intersection graph theory see for example [37, 32, 17].

Surprisingly, automorphism groups of intersection-defined classes of graphs were studied only briefly. Even for very deeply studied classes of intersection graphs the structure of their automorphism groups is not well known. In this area, the mostly studied are classical graph-theoretic properties (the chromatic number, forbidden graph characterization, and so on) or the complexity of the recognition problem.

We study the problem of reconstructing the automorphism group of a geometric intersection graph from a good knowledge of the structure of its representations. In this thesis, we deal mainly with interval graphs.

Interval Graphs. Interval graphs are intersection graphs of intervals on the real line. They are one of the oldest and most studied classes of graphs, first introduced by Hajós [19] in 1957.

An *interval representation* $\mathcal{R}$ of a graph X is a set of closed intervals $\{I_x : x \in V(X)\}$ such that $xy \in E(X)$ if and only if $I_x \cap I_y \neq \emptyset$. In other words, an edge of X is represented by an intersection of intervals. A graph X is an *interval graph* if there exists an interval representation $\mathcal{R}$ of X. Figure 1.2 shows an example.

One of the reasons why interval graphs were studied quite extensively is that they have real world applications, for example in biology. Benzer [3] showed a direct relation between interval graphs and the arrangement of genes in the chromosome. Mutations correspond to a damaged segment on a chromosome. Each mutation can damage a different set of genes. At that time, the only information that could be gathered was the set of deformities caused by a mutation. We can form a graph by making each mutation into a vertex and adding an edge between two vertices if the

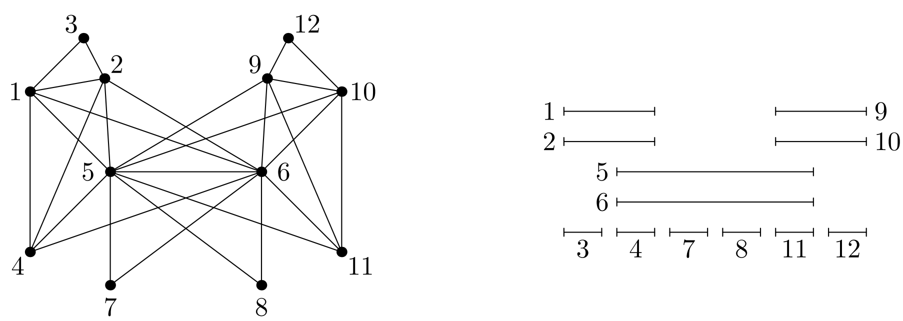


Figure 1.2: An interval graph and one of its interval representations.

mutations share a common deformity. Benzer found that a graph formed in this way from an experiment with mutations is an interval graph. This was considered a strong evidence supporting the theory that genes are arranged in a simple linear fashion. Interval graphs have also many other applications (see for example [34, 38]).

Interval graphs have also many useful theoretical properties and nice mathematical characterizations. In many cases, very hard computational problems are polynomially solvable for interval graphs. These problems include graph isomorphism, maximum clique, k-coloring, maximum independent set, and so on.

1.2 Results of This Thesis

In this thesis, we study the automorphism groups of interval graphs. The structure of their representations is already very well understood due to Booth and Lueker [4].

In 1981 Cobourn and Booth [8] designed a linear-time algorithm that computes generating automorphisms of automorphism group of an interval graph. Our result gives an explicit description of these automorphism groups in terms of group products, so also from the algorithmic point of view we get a better information about the groups. Moreover, our description of the automorphism groups of interval graphs is much more detailed and shows the relation between the structure of all representations of an interval graph and its automorphism group.

Let $\mathcal{I}$ be the class of finite groups that are isomorphic to the automorphism group of some interval graph and let $\mathcal{T}$ be the class of finite groups that are isomorphic to the automorphism groups of some finite tree. Our main result is the following theorem.

Theorem 1.1. *The class $\mathcal{I}$ of the automorphism groups of interval graphs is the same as the class $\mathcal{T}$ of the automorphism groups of trees. For each interval graph X, there exists a tree T such that* $\mathrm{Aut}(X)$ *is isomorphic to* $\mathrm{Aut}(T)$*, and vice versa.*

This is surprising because the class INT of finite interval graphs and the class TREE of finite trees are very different graph classes. The intersection INT $\cap$ TREE are exactly the graphs called CATERPILLARS. Those are the trees having a path P such that all vertices are within distance at most one of P. Automorphism groups of CATERPILLARS are very restricted compared to interval graphs and trees; see Proposition 3.12.

Another important classes of graphs related to INT are the classes AT-FREE and CHOR. The first one is the class of *asteroidal triple-free* graphs. Three vertices of a graph form an *asteroidal triple* if every two of them are connected by a path avoiding

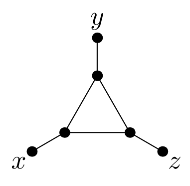


Figure 1.3: A graph that is not a tree and contains an asteroidal triple (x, y, z).

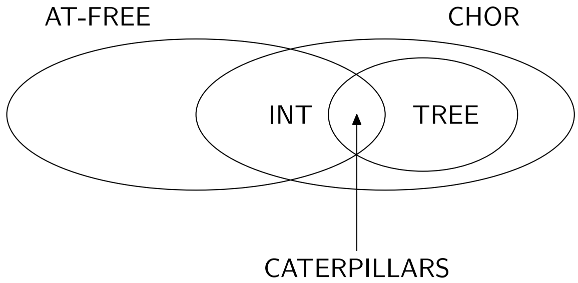


Figure 1.4: The inclusions between the described classes of graphs.

the neighbourhood of the third. A graph is asteroidal triple-free if it does not contain any asteroidal triple. The class CHOR is the class of *chordal graphs*. A chordal graph is a graph that *does not* contain an induced cycle of length four or more. Another characterization of chordal graphs says that chordal graphs are intersection graphs of subtrees of a tree [15], which is a generalization of interval graphs (if the tree is a path, then we get an interval graph). It is well known that a graph is an interval graph if and only if it is in AT-FREE $\cap$ CHOR [26].

The problem GraphIso(X_1, X_2) is polynomially reducible to testing isomorphism of chordal graphs [27]. Moreover, the reduction shows that for an arbitrary graph there exists a chordal graph with the same automorphism group. So, chordal graphs are universal for automorphism groups and the structural study of their groups is finished.

The equality of $\mathcal{I}$ and $\mathcal{T}$ was already mentioned by Hanlon [20] in his paper about counting interval graphs. However, his paper lacks an explanation or a proof of this result. Moreover, for an interval graph X, finding a tree T such that $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(T)$ is stated as an open problem. We can solve this problem easily using our description of the class $\mathcal{I}$. Therefore our understanding of the structure of $\mathcal{I}$ is much deeper. We are also able to find for a tree an interval graph with the same group of automorphisms.

Our characterization of the class $\mathcal{I}$ is based on the Jordan's characterization (see Theorem 2.6) of the class $\mathcal{T}$. We add a third operation, the semidirect product with $\mathbb{Z}_2$, which corresponds to a reflection symmetry of a part of an interval representation. Then we prove the equality of $\mathcal{I}$ and $\mathcal{T}$. We show that this third operation can be replaced by a sequence of the first two operations.

2 Preliminaries

In Section 2.1, we describe some basic concepts of group theory that are essential for the main result. For a comprehensive treatment of the basics of group theory, see for example [35, 10], for a visual treatment of group theory, see [5]. In Section 2.2 we give a definition of PQ-trees and *modified* PQ-trees. Each PQ-tree is a data structure which captures all possible representations of an interval graph.

Notation. We use X and Y to denote graphs. The set of the vertices of a graph X is denoted by $V(X)$ and the set of the edges by $E(X)$. The remaining letters like G and H are used to denote groups.

We assume that the reader is familiar with the basic properties of groups. The following notation is used for the standard groups:

- $\mathbb{S}_n$ is the *symmetric group* whose elements are n-element permutations,
- $\mathbb{D}_n$ is the *dihedral group* whose elements are symmetries of the regular n-gon, including both rotations and reflections,
- $\mathbb{Z}_n$ is the *cyclic group* whose elements are integers $0, \ldots, n-1$ and the operation is addition modulo n.

We define an equivalence relation $\sim_{TW}$ on the vertices of an interval graph X where $x \sim_{TW} y$ means that x and y belong to precisely the same maximal cliques, or in other words, they have precisely the same neighbourhoods. If two vertices x and y are in $\sim_{TW}$ we say that they are *twin* vertices. The equivalence classes of $\sim_{TW}$ are called *twin classes*. Twin vertices are usually not interesting in the study of geometric intersection graphs. However, they need to be considered for automorphism groups.

2.1 Group Products

In algebra, group products are used to decompose large groups into smaller ones. Consider for example the well know puzzle called Rubik's Cube. The Rubik's Cube group is the set G of all *cube moves* on the Rubik's Cube. The cardinality of G is given by

$$|G| = 43\ 252\ 003\ 274\ 489\ 856\ 000.$$

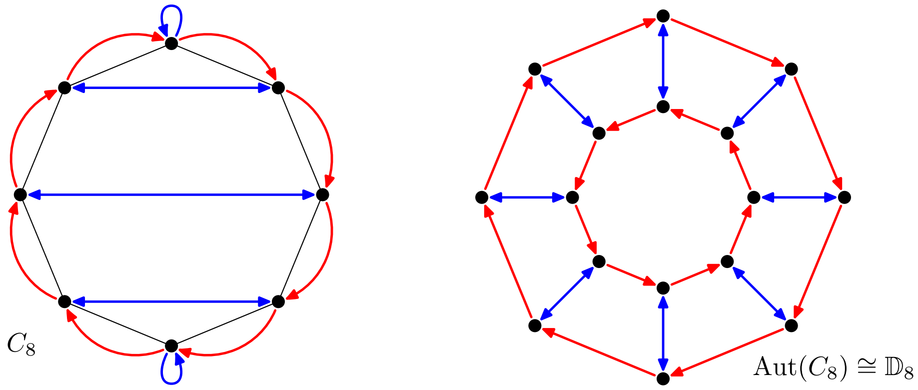


Figure 2.1: The cycle graph C_8 with the action of $\mathrm{Aut}(C_8)$ on its vertices and a Cayley graph of $\mathrm{Aut}(C_8)$. Note that $\mathrm{Aut}(C_8)$ is isomorphic to $\mathbb{D}_8$. It is generated by two automorphisms: the rotation symmetry (depicted by the red arrows); the reflection symmetry (depicted by the blue arrows).

The Rubik's Cube group is large and its structure is not obvious. Using group products, one can derive that the group is isomorphic to

$$(\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}) \rtimes ((\mathbb{A}_8 \times \mathbb{A}_{12}) \rtimes \mathbb{Z}_2),$$

where $\mathbb{A}_n$ is the group of all even n-element permutations. From this, the structure of the Rubik's Cube group is much more clear.

Here, we explain two basic group theoretic methods for constructing larger groups from smaller ones, namely *direct product* and *semidirect product*. We show how these group operations can be used to construct automorphism groups of graphs. At the end of this section, we prove Jordan's characterization of the class $\mathcal{T}$.

Inspired by [5], we use *Cayley graphs* to visualize groups. Cayley graphs were actually invented by Cayley [6] for this purpose and now they also play an important role in combinatorial and geometric group theory. A Cayley graph is a colored oriented graph that depicts the abstract structure of a group. Suppose that G is a group and S is a generating set. The Cayley graph (G, S) is a graph constructed as follows:

- The elements of G correspond one-to-one to the vertices.
- Each generator $s \in S$ is represented by a unique colour $c(s)$.
- For any $g \in G$ and $s \in S$, the there is a directed edge (g, gs) of colour $c(s)$.

Figure 2.1 and Figure 2.2 show examples of graphs and Cayley graphs of their automorphism groups.

2.1.1 Direct Product

The direct product $G \times H$ of groups G and H with operations $\cdot_G$ and $\cdot_H$, respectively, is the set of pairs (g, h) where $g \in G$ and $h \in H$ with operation defined componentwise:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2).$$

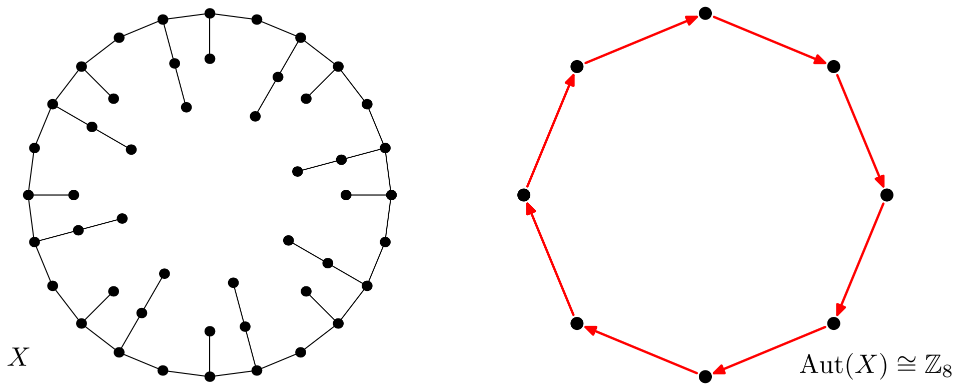


Figure 2.2: A graph with the automorphism group isomorphic to the group $\mathbb{Z}_8$ and a Cayley graph of $\mathbb{Z}_8$. A graph like this one has only the rotation symmetries as automorphisms. Therefore, its automorphism group is isomorphic to a subgroup of $\mathrm{Aut}(C_8)$. We note that the Frucht's theorem is proved in a similar way. One has to use some gadgets to encode the oriented edges and colours of a Cayley graph.

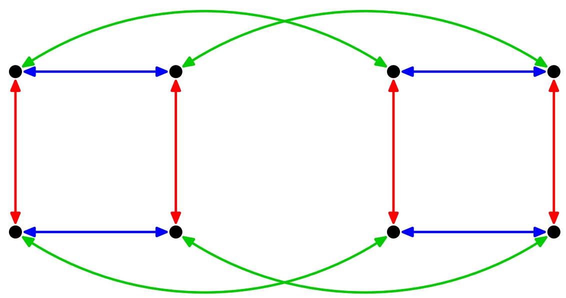
When there is no confusion we simply write $(g_1 \cdot g_2, h_1 \cdot h_2)$ or $(g_1 g_2, h_1 h_2)$. The direct product of n groups is defined similarly.

Suppose that we have the direct product $G_1 \times \cdots \times G_n$ of groups $G_1, \ldots, G_n$. We can define a homomorphism $\pi\colon G_1 \times G_2 \times \cdots \times G_n \to G_2, \times \cdots \times G_n$ by

$$\pi\big((g_1, g_2, \ldots, g_n)\big) = (g_2, \ldots, g_n).$$

The kernel $\mathrm{Ker}(\pi)$ is clearly isomorphic to G_1. Therefore, G_1 is a normal subgroup of $G_1 \times \cdots \times G_n$. Analogously, each G_i is a normal subgroup of $G_1 \times \cdots \times G_n$. On the other hand, the semidirect product, discussed in Section 2.1.2, takes two groups G and H and constructs a larger group such that only G is a normal subgroup.

Example 2.1. This figure shows a Cayley graph of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the group contains two copies of $\mathbb{Z}_2 \times \mathbb{Z}_2$, with the corresponding elements connected according to the pattern of $\mathbb{Z}_2$.



Direct product can be used to construct automorphism groups of graphs that are disconnected and their connected components pairwise are non-isomorphic. In this case, the automorphism group of a graph X is a direct product of the automorphism groups of its connected components $X_1, \ldots, X_k$:

$$\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_k).$$

This is because each automorphism acts independently on each component X_i.

2.1.2 Semidirect Product

However, if we want to construct the automorphism group of a disconnected graph which has some isomorphic connected components, direct product *is not* sufficient because the automorphisms that permute the isomorphic components are not included in the direct product.

Example 2.2. The automorphism group of the graph X is isomorphic to $\mathbb{S}_3 \times \mathbb{Z}_2$, but the automorphism group of the graph Y *is not* $\mathbb{Z}_2 \times \mathbb{Z}_2$. The direct product does not include the automorphisms which swap the components. The automorphism group of Y is not even $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ because, for example, swapping the components and swapping the vertices of the left component do not commute.

If we construct a larger group from some groups G and H using the direct product, then both G and H are normal subgroups of the resulting group. The motivation for the semidirect product is to construct a group from G and H for which G does not have to be a normal subgroup.

The direct product $G \times H$ contains identical copies of G, with corresponding elements connected according to the pattern of H, as shown in Example 2.1. In the semidirect product of the groups G and H, the group H also determines a pattern according to which some copies of G are connected, however, those copies of G do not need to be all identical.

First, we explain a special case. The semidirect product of the group G with its automorphism group $\mathrm{Aut}(G)$, denoted by

$$G \rtimes \mathrm{Aut}(G).$$

The elements are all pairs (g, f) such that $g \in G$ and $f \in \mathrm{Aut}(G)$. The operation is defined in the following way:

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1 \cdot f_1(g_2), f_1 \cdot f_2).$$

Note that $G \rtimes \mathrm{Aut}(G)$ with the operation defined like this forms a group. It is straightforward to see that the identity element is $(1, 1)$ and that the inverse of the element (g, f) is the element $(f^{-1}(g^{-1}), f^{-1})$.

We can think of it as all possible isomorphic copies of G connected according to the pattern of $\mathrm{Aut}(G)$. The element (g_1, f_1) is in the isomorphic copy G_1 of G which we get by applying the automorphism f_1 on G. Multiplying (g_1, f_1) by $(g_2, 1)$ corresponds to a movement inside G_1. Multiplying (g_1, f_1) by $(1, f_2)$ corresponds to a movement from G_1 to another isomorphic copy of G.

In general, the semidirect product is defined for any two groups G and H, and a homomorphism $\varphi\colon H \to \mathrm{Aut}(G)$, denoted by

$$G \rtimes_\varphi H.$$

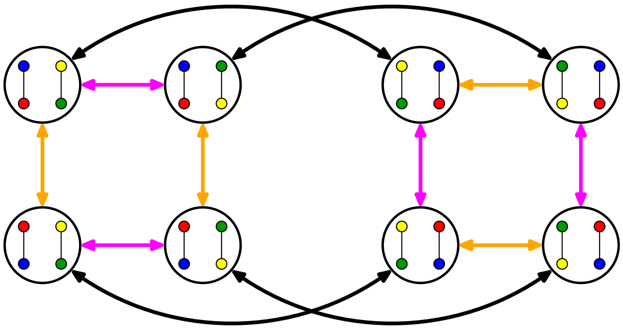


Figure 2.3: A Cayley graph of $\mathrm{Aut}(Y)$ where Y is from Example 2.2. The generators are the following: the permutation that swaps the left component and fixes the right component (orange); the permutation that swaps the right component and fixes the left component (purple); the permutation that swaps the components (black).
The subgroup of $\mathrm{Aut}(Y)$ which acts on the components independently and does not swap them, corresponds to the isomorphic copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is on the left in the Cayley graph. Swapping the components with the black automorphism changes the orange automorphism changes the orange automorphism to the purple automorphism, and vice versa. In other words, swapping the vertices of some component does no commute with swapping the components. Therefore, the group $\mathrm{Aut}(Y)$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

It is the set of all pairs (g, h) such that $g \in G$ and $h \in H$. The operation is defined similarly to the operation defined on $G \rtimes \mathrm{Aut}(G)$:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot \varphi(h_1)(g_2), h_1 \cdot h_2).$$

Again, it is quite straightforward to check that $G \rtimes_\varphi H$ is a group. We can think of the homomorphism φ as if it assigns an isomorphic copy of G to each element of the group H. The isomorphic copies of G are then connected according to the pattern of the group H. We write $G \rtimes H$ when there is no danger of confusion.

Example 2.3. Dihedral group $\mathbb{D}_8$ is equal to $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$. Figure 2.1 shows a Cayley graph of $\mathbb{D}_8$ (on the right). The elements of the two isomorphic copies of $\mathbb{Z}_8$ are connected according to the pattern of $\mathbb{Z}_2$.

Example 2.4. Let Y be the graph from Example 2.2. The group $\mathrm{Aut}(Y)$ is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. Figure 2.3 shows a Cayley graph of $\mathrm{Aut}(Y)$. The elements of the two isomorphic copies of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are connected according to the pattern of $\mathbb{Z}_2$.

Wreath Product. The group $G \wr \mathbb{S}_n$ is the wreath product of a group G with $\mathbb{S}_n$.[1] It is a shorthand for the semidirect product $G^n \rtimes_\varphi \mathbb{S}_n$, where $\varphi\colon \mathrm{Aut}(G^n) \to \mathbb{S}_n$ is a homomorphism defined by

$$\varphi(\pi) = \text{the automorphism that maps } (g_1, \dots, g_n) \text{ to } (g_{\pi(1)}, \dots, g_{\pi(n)}).$$

The reason for defining the wreath product is that it occurs quite often in the study of the automorphism groups of graphs.

[1] For the purposes of this thesis, it is sufficient to define $G \wr \mathbb{S}_n$. In general, the wreath product can be defined for any groups G and H.

The following two theorems are due to Jordan [23]. Theorem 2.5 shows how to construct the automorphism groups of a disconnected graph using group products. Theorem 2.6 gives a characterization of the class of the automorphism groups of trees in terms of direct and wreath products. We also prove these theorems because the ideas are used later in Chapter 3.

Theorem 2.5 (Automorphism groups of disconnected graphs)**.** *If $X_1, \dots, X_n$ are pairwise non-isomorphic connected graphs and X is the disjoint union of k_i copies of X_i, for $i = 1, \dots, n$, then*

$$\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(X_n) \wr \mathbb{S}_{k_n}.$$

Proof. First, we deal with a special case. Suppose that X consists *only* of k isomorphic copies of X_1, denoted by $Y_1, \dots, Y_{k_1}$. Each automorphism α of X can be encoded by a $(k_1 + 1)$-tuple

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k_1}, \pi),$$

where $(\alpha_1, \dots, \alpha_{k_1}) \in \mathrm{Aut}(Y_1)^{k_1}$ and $\pi \in \mathbb{S}_{k_1}$. The automorphism $(\alpha_1, \dots, \alpha_{k_1}, \pi)$ first acts on each component using $(\alpha_1, \dots, \alpha_{k_1})$, and then it permutes the components according to the permutation π.

The action of $\alpha \cdot \beta = (\alpha_1, \alpha_2, \dots, \alpha_{k_1}, \pi) \cdot (\beta_1, \beta_2, \dots, \beta_{k_1}, \rho)$ can be described in the following way. First, α acts independently on each Y_j using α_j, then it permutes them by π, then β acts independently on each Y_j and then it permutes them by ρ. We want to represent this as one automorphism acting first independently on each Y_j, and then premuting them. The problem is that π does not commute with the action of β_j. Therefore, to swap them, we have to let $\beta_{\pi(j)}$ act on Y_j. Figure 2.4 shows an example. So, the operation on $\mathrm{Aut}(X)$ can be defined by

$$\begin{aligned}\alpha \cdot \beta &= (\alpha_1, \alpha_2, \dots, \alpha_{k_1}, \pi) \cdot (\beta_1, \beta_2, \dots, \beta_{k_1}, \rho)\\ &= (\alpha_1 \cdot \beta_{\pi(1)}, \alpha_2 \cdot \beta_{\pi(2)}, \dots, \alpha_{k_1} \cdot \beta_{\pi(k_1)}, \pi \circ \rho).\end{aligned}$$

In other words, we get

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(Y_1)^{k_1} \rtimes_\varphi \mathbb{S}_{k_1} = \mathrm{Aut}(Y_1) \wr \mathbb{S}_{k_1}$$

where $\varphi \colon \mathbb{S}_{k_1} \to \mathrm{Aut}(\mathrm{Aut}(Y_1)^n)$ is the homomorphism defined by

$$\varphi(\pi) = \text{the automorphism that maps } (\alpha_1, \alpha_2, \dots, \alpha_{k_1}) \text{ to } (\alpha_{\pi(1)}, \alpha_{\pi(2)}, \dots, \alpha_{\pi(k_1)}).$$

Now we consider the general case. No automorphism of X can swap a copy of X_i with a copy of X_j because they are non-isomorphic. Therefore, each automorphism acts independently on the isomorphic copies of each X_i, so to get $\mathrm{Aut}(X)$ is the direct product of all $\mathrm{Aut}(X_i) \wr \mathbb{S}_{k_i}$. □

Theorem 2.6 (Jordan, 1869)**.** *A finite group G is isomorphic to an automorphism group of a finite tree tree if and only if $G \in \mathcal{T}$, where the class $\mathcal{T}$ of finite groups is defined inductively as follows:*

(a) $\{1\} \in \mathcal{T}$.

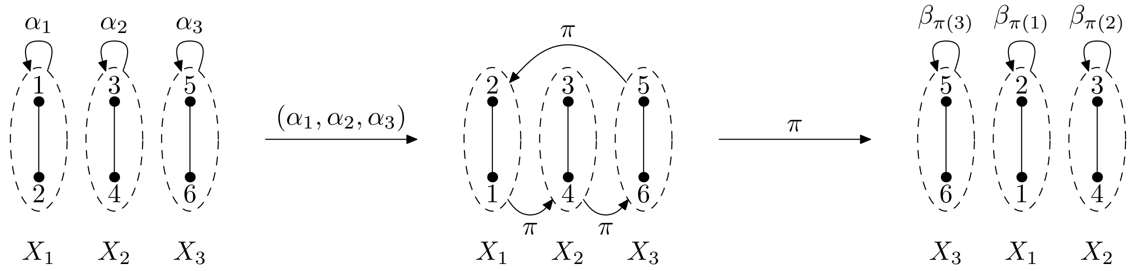


Figure 2.4: The figure shows a graph X with isomorphic components X_1, X_2, X_3, and the action of the automorphism encoded by the quadruple $(\alpha_1, \alpha_2, \alpha_3, \pi)$ such that $(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2) \times \mathrm{Aut}(X_3)$ and $\pi \in \mathbb{S}_3$. The automorphism α_1 swaps the vertices 1 and 2, and α_2, α_3 are identities. Suppose that $(\beta_1, \beta_2, \beta_3, \rho)$ encodes an automorphism of X. Each β_j has to act on the correct component. This is achieved by letting $\beta_{\pi(1)}$ act on X_1, $\beta_{\pi(2)}$ on X_2, and $\beta_{\pi(3)}$ on X_3.

(b) If $G_1, G_2 \in \mathcal{T}$, then $G_1 \times G_2 \in \mathcal{T}$.

(c) If $G \in \mathcal{T}$ and $n \geq 2$, then $G \wr \mathbb{S}_n \in \mathcal{T}$.

Proof. Every tree has a center, which is either a vertex, or an edge. If the center is an edge, then we subdivide the edge. This does not change the automorphism group. The center of a tree is fixed by every automorphism. Therefore, deleting the root does not change the automorphism group of the tree. So the problem of determining automorphism groups of trees can be reduced to rooted trees.

First, we construct for each $G \in \mathcal{T}$ a rooted tree T such that $\mathrm{Aut}(T) \cong G$.

- Let $G_1, G_2 \in \mathcal{T}$ and let T_1, T_2 be rooted trees such that $\mathrm{Aut}(T_1) \cong G_2$ and $\mathrm{Aut}(T_2) \cong G_2$. We construct the tree T by attaching the roots of T_1 and T_2 to a new root r. If $G_1 \cong G_2$ we further subdivide one of the newly created edges. Clearly, we get $\mathrm{Aut}(T) \cong G_1 \times G_2$.

- If $G \in \mathcal{T}$ and T_1 is a rooted tree such that $\mathrm{Aut}(T_1) \cong G$, then we construct T by attaching n copies of T_1 to the same root. By Theorem 2.5 $\mathrm{Aut}(T) \cong G \wr \mathbb{S}_n$.

Now, it remains to prove the converse. For each rooted tree T, the group $\mathrm{Aut}(T)$ is in the class $\mathcal{T}$. If T is a rooted tree containing only one vertex, then clearly $\mathrm{Aut}(T) \in \mathcal{T}$. Otherwise, we delete the root and get a forest of rooted trees $T_1, \dots, T_n$. We determine the automorphism group of each T_i recursively and use Theorem 2.5 to construct the group $\mathrm{Aut}(T)$. It follows that $\mathrm{Aut}(T) \in \mathcal{T}$. $\square$

2.2 Tree Representations of Interval Graphs

In this section, we briefly explain PQ-trees and show how they relate to interval graphs. Then we introduce a modified version of PQ-trees which we use in Chapter 3 to characterize the automorphism groups of interval graphs.

2.2.1 PQ-trees

PQ-trees were invented by Booth and Lueker [4] for the purpose of solving the *consecutive ordering problem*. For a set S and restricting sets $R_1, \ldots, R_k$, the task is to find a linear ordering of S such that every R_i appears consecutively in it as one block.

Example 2.7. Consider the set $S = \{a, b, c, d, e\}$ and the restricting sets $R_1 = \{a, b\}$, $R_2 = \{c, d, e\}$ and $R_3 = \{b, c\}$. The orderings *abcde*, *abced*, *decba* and *edcba* are the only feasible orderings of U, any other ordering violates some restriction. For instance, the ordering $a\underline{b}d\underline{c}e$ violates R_3.

A PQ-tree is a rooted tree designed for solving the consecutive ordering problem efficiently. In addition to that, they store *all* feasible orderings of the set S.

The leaves of the tree correspond one-to-one to the elements of S. The inner nodes are of two types: The *P-nodes* and the *Q-nodes*. We assume that each P-node has at least two children and that each Q-node has at least three children. For every inner node, the order of its children is fixed.

The *frontier* of a PQ-tree T is a permutation of the set S obtained by ordering the leaves of T from left to right. The frontier of T represents one ordering of S.

To obtain all feasible orderings of S we can modify T by applying a finite sequence of the following two *equivalence transformations*:

- Arbitrarily permute the children of a P-node.
- Reverse the order of the children of a Q-node.

The PQ-tree obtained from T by applying a finite sequence of equivalence transformations ε is denoted by T_ε. A PQ-tree T' is *equivalent* with T if one can be obtained from the other using a finite sequence of equivalence transformations. Each sequence of equivalence transformations encodes a permutation of $V(T)$, the nodes of T.

Booth and Lueker [4] proved that a PQ-tree exists for every instance of the consecutive ordering problem and it can be constructed in a linear time. Figure 2.5 shows all equivalent PQ-trees representing, all feasible orderings of the set S, for the instance of Example 2.7, with P-nodes are denoted by circles and Q-nodes by rectangles.

PQ-trees and Interval Graphs. The following characterization of interval graphs is given by Fulkerson and Gross [14]. It shows the relation between interval graphs and the consecutive ordering problem.

Lemma 2.8 (Fulkerson and Gross)**.** *A graph X is an interval graph if and only if there exists an ordering of the maximal cliques $\mathcal{C}(X)$ such that for every vertex $x \in V(X)$, the maximal cliques containing x appear consecutively in it.*

Proof. Let $\{I_x : x \in X\}$ be an interval representation of X and let $C_1, \ldots, C_k$ be the maximal cliques. By Helly's Theorem, the intersection $\bigcap_{x \in C_i} I_x$ is non-empty, and

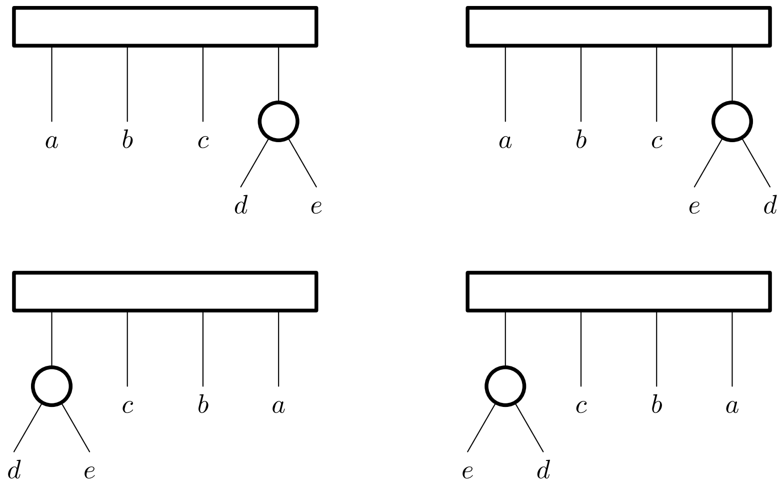


Figure 2.5: Four PQ-trees that represent all feasible orderings of the instance of Example 2.7, the circles are P-nodes and the rectangles are Q-nodes.

therefore there exist a point c_i in it. The ordering of $c_1, \dots, c_k$ from left to right gives the required ordering.

Given an ordering of the maximal cliques $C_1, \dots, C_k$, we place points $c_1, \dots, c_k$ in this ordering on the real line. To each vertex v, we assign the minimal interval I_x such that $c_i \in I_x$ if and only if $x \in C_i$. We obtain a valid interval representation $\{I_x : x \in V(X)\}$ of X. $\square$

Recognition of interval graphs in linear time was an open problem, first solved by Booth and Lueker [4] using PQ-trees. By Lemma 2.8, the problem of recognizing interval graphs can be simply reduced to the consecutive ordering problem. To test whether a graph X is an interval graph, let S to be the set of all maximal cliques $\mathcal{C}(X)$. For each vertex x, we define a restricting set $R_x = \{C \in \mathcal{C}(X) : x \in C\}$. Lemma 2.8 says that X is an interval graph if and only if there exist a linear ordering of S such that every R_x appears consecutively in it. The algorithm for solving the consecutive ordering problem constructs a PQ-tree T such that the frontier of T gives one possible consecutive ordering of $\mathcal{C}(X)$. We get all possible orderings of $\mathcal{C}(X)$ by applying sequences of equivalence transformations. Figure 2.6 shows an example of an interval graphs and a PQ-tree representing it.

2.2.2 MPQ-trees

A modified PQ-tree (MPQ-tree) is basically a PQ-tree with some additional information about the twin vertices. MPQ-trees were first mentioned by Korte and Möhring [25], they used them to show simpler linear-time recognition algorithm for interval graphs than the one of Booth and Lueker. MPQ-trees were used by Coulborn and Booth [8] to design a linear-time algorithm for computing a set of generator of the automorphism group of an interval graph, however, they mention them only implicitly.

Suppose that T is a PQ-tree representing an interval graph X. To obtain an MPQ-tree M from T we assign sets, called *sections*, to the nodes of T. Leafs and

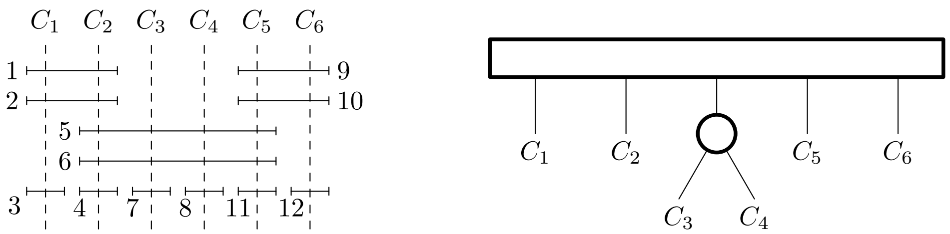


Figure 2.6: An interval graph and a PQ-tree which represents one consecutive ordering of its maximal cliques. We can get all other possible orderings by applying the equivalence transformations on the PQ-tree.

P-nodes have assigned only one section, while Q-nodes have one section for each of its children. We assign the sections to the nodes of T in the following way:

- For every leaf L, the section $\mathrm{sec}(L)$ contains those vertices of X that are only in the maximal clique represented by L, and no other maximal cliques.
- For every P-node P, the section $\mathrm{sec}(P)$ contains those vertices of X that are in all maximal cliques represented by the leaves of the subtree of P, and no other maximal cliques.
- For every Q-node Q and its children $Q_1, \dots, Q_n$, the section $\mathrm{sec}_i(Q)$ contains those vertices of X that are in the maximal cliques represented by the leaves of the subtree of Q_i *and also* some other Q_j, but are not in any other maximal clique represented by a leaf that is not in the subtree of Q. We denote the union $\mathrm{sec}_1(Q) \cup \cdots \cup \mathrm{sec}_n(Q)$ by $\mathrm{sec}(Q)$.

Figure 2.7 shows an example of an MPQ-tree.

If x is a vertex of an interval graph X and M is an MPQ-tree representing X, then N_x denotes the node of M such that $x \in \mathrm{sec}(N_x)$. The following lemma shows that the MPQ-tree M captures the structure of the graph X.

Lemma 2.9. *For any two vertices x and y of X there is an edge between x and y if an only if the nodes $N(x)$ and $N(y)$ lie on a path from the root of M to some leaf.*

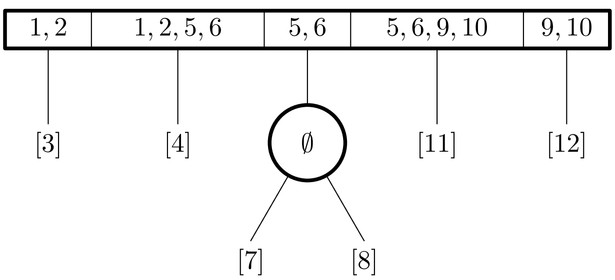


Figure 2.7: An MPQ-tree that represents the interval graph from Figure 2.6. The twin vertices belong to the same sections of the MPQ-tree.

Proof. If $xy \in E(X)$, then there exists a maximal clique L of X such that $x, y \in L$. Since L is one of the leaves of M, from the definition of MPQ-trees we have that N_x and N_y lie on the path from the root of M to L.

If N_x and N_y lie on the same path from the root of M to some leaf L, then by the definition we have that $x, y \in L$. Since L is a maximal clique of X, it follows that $xy \in E(X)$. ☐

Corollary 2.10. *Vertices $x, y \in V(X)$ that are in the same sections of an MPQ-tree M for the interval graph X belong to the same twin classes of X, i.e., $x \sim_{TW} y$.*

3 Automorphism Groups of Interval Graphs

In this chapter, we derive a characterization of the class $\mathcal{I}$ of the automorphism groups of finite interval graphs. We show that it is equal to the class $\mathcal{T}$ of the automorphism groups of finite trees. Finally, we show how to construct for an interval graph X a tree T such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(T)$, and vice versa.

3.1 Automorphisms Groups of PQ-trees

Here, we give a definition of an automorphism of a PQ-tree and an MPQ-tree that represent an interval graph X. We show that the automorphism group of the PQ-tree is isomorphic to a subgroup of $\mathrm{Aut}(X)$. Further, the additional information in the MPQ-tree makes its automorphism group isomorphic to $\mathrm{Aut}(X)$.

Automorphism Groups of PQ-trees. Let T be a PQ-tree representing an interval graph X. We define each *symmetric* sequence of equivalence transformations to be an automorphism of T. More formally, a sequence of equivalence transformations $\varepsilon\colon V(T) \to V(T)$ is an *automorphism of* T if there exists a permutation $\alpha\colon V(X) \to V(X)$ of the vertices of X such that after replacing each leaf L in T_ε with $\alpha(L)$ we get T. We say that α *cancels* ε. Figure 3.1 shows an example.

Lemma 3.1. *Automorphisms of a PQ-tree T representing X form a group.*

Proof. Suppose that ε_1 and ε_2 are automorphisms of T and α_1 cancels ε_1, and α_2 cancels ε_2. The composition $\alpha_2 \circ \alpha_1$ cancels $\varepsilon_1 \cdot \varepsilon_2$, so $\varepsilon_1 \cdot \varepsilon_2$ is also an automorphism of the PQ-tree T. The inverse of an automorphism ε can be constructed similarly. $\square$

We denote the group of automorphisms of a PQ-tree T representing X by $\mathrm{Aut}(T)$. The following lemma shows that a permutation which cancels an automorphism of T is an automorphism of X.

Lemma 3.2. *If ε is an automorphism of a PQ-tree T representing X and α cancels ε, then α is an automorphism of X.*

Proof. Let $x, y \in V(X)$ be two vertices. The vertices x and y are adjacent if and only if they are contained in some maximal clique. The permutation α induces a permutation

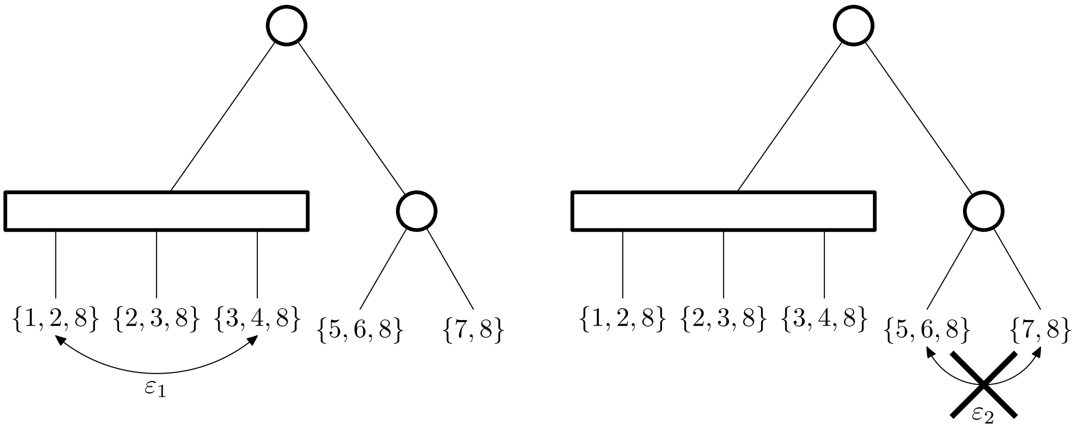


Figure 3.1: The equivalence transformation ε_1 on the left is the *only* automorphism of the PQ-tree. For example the transformation ε_2 on the right is not an automorphism because there is no permutation α of the vertices such that $\alpha(\{7,8\}) = \{5,6,8\}$.

of the maximal cliques $\mathcal{C}(X)$, since it cancels ε. So, $\alpha(x)$ and $\alpha(y)$ are in the same maximal clique if and only if x and y are in the same maximal clique. ☐

By Lemma 3.2 each automorphism ε of T induces at least one automorphism of X. The next lemma shows that each automorphism of X induces a unique automorphism of T.

Lemma 3.3. *If α is an automorphism of X, then there exists a unique automorphism ε of T that reorders $\mathcal{C}(X)$ in the same way as α.*

Proof. The PQ-tree T stores all possible orderings of the maximal cliques $\mathcal{C}(X)$. Therefore, there exists an automorphism ε of T such that it reorders the maximal cliques in the same way as α. It is indeed an automorphism, since α^{-1} cancels ε. The automorphism ε is unique because there is only one possible reordering of the maximal cliques induced by α. ☐

Multiple automorphisms of X can reorder $\mathcal{C}(X)$ in the same way. If there exists a twin class of size greater that one, then some automorphisms of X reorder $\mathcal{C}(X)$ in the same way, but permute the twin class differently. We define a mapping $\phi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(T)$ by

$$\phi(\alpha) = \varepsilon$$

where ε the unique equivalence transformation of T that gives the same reordering of $\mathcal{C}(X)$ as α. According to Lemma 3.2 and Lemma 3.3, the mapping ϕ is well defined and surjective. It is straightforward to see that *ϕ is a homomorphism*. Moreover, ϕ is a quotient homomorphism, that is, it is possible that two automorphisms of X are mapped by ϕ to the same automorphism of T.

In general, the automorphism group of a PQ-tree T representing X is not isomorphic to the automorphism group of X. An automorphism $\alpha \in \mathrm{Aut}(X)$ is in $\mathrm{Ker}(\phi)$ if it only swaps vertices x, y that belong to the same twin classes. By the first isomorphism theorem, we get

$$\mathrm{Aut}(T) \cong \frac{\mathrm{Aut}(G)}{\mathrm{Ker}(\phi)}.$$

If $\mathrm{Ker}(\phi)$ is nontrivial, then $\mathrm{Aut}(T)$ is not isomorphic to $\mathrm{Aut}(X)$. In the following text we show that an MPQ-tree representing X captures the whole $\mathrm{Aut}(X)$.

Automorphism Groups of MPQ-trees. Here we give a definition of an automorphism group of an MPQ-tree. Let M be an MPQ-tree representing an interval graphs X and let T be the underlying PQ-tree.

An *automorphism of a P-node* P is a permutation of the set $\{x \in V(X) : x \in \mathrm{sec}(P)\}$ of vertices of X. We denote the automorphism group of the node P by $\mathrm{Aut}(P)$. The automorphism gourp of P is isomorphic to $\mathbb{S}_k$. The automorphism group $\mathrm{Aut}(L)$ of a leaf L is defined in a similar way.

An *automorphism of a Q-node* Q is a permutation of some set of vertices of X that belong to the same sections of Q. More formally, if $V_1, \dots, V_\ell$ are the subsets of $V(X)$ such that the vertices in each V_i belong to the same sections of Q, then an automorphism of the Q-node Q is a ℓ-tuple $(\pi_1, \dots, \pi_\ell)$ where π_i is a permutation of the set V_i. The automorphisms of the node Q form the group $\mathrm{Aut}(Q)$ with the operation defined componentwise.

Example 3.4. The automorphism group of the Q-node in Figure 2.7 is isomorphic to $\mathbb{S}_2 \times \mathbb{S}_2 \times \mathbb{S}_2$. This is because the sets $V_1 = \{1, 2\}$, $V_2 = \{5, 6\}$ and $V_3 = \{9, 10\}$ are the subsets of the vertex set of the graph represented by the MPQ-tree such that the vertices in each V_i belong to the same sections of the Q-node. The automorphism group of each leaf is the trivial group.

Let $N_1, \dots, N_k$ be the nodes of M. Each group $\mathrm{Aut}(N_i)$ is isomorphic to a symmetric group if N_i is a P-node or a leaf and if N_i is a Q-node, then $\mathrm{Aut}(N_i)$ is isomorphic to a direct product of symmetric groups. Therefore, also the group $\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_k)$ is isomorphic to a direct product of symmetric groups.

An *automorphism of the MPQ-tree* M is a $(k+1)$-tuple $(\nu_{N_1}, \dots, \nu_{N_k}, \varepsilon)$ where ν_{N_i} is an automorphism of the node N_i and ε is an automorphism of the underlying PQ-tree T. Figure 3.2 shows an example of an automorphism of an MPQ-tree.

Lemma 3.5. *The automorphisms of M form the group $\mathrm{Aut}(M)$ with the operation defined as follows*

$$(\mu_{N_1}, \dots, \mu_{N_k}, \delta) \cdot (\nu_{N_1}, \dots, \nu_{N_k}, \varepsilon) = (\mu_{N_1} \cdot \nu_{\delta(N_1)}, \dots, \mu_{N_k} \cdot \nu_{\delta(N_k)}, \delta \cdot \varepsilon).$$

Proof. The automorphism $(\mu_{N_1}, \dots, \mu_{N_k}, \delta)$ first acts on each node N_i by μ_{N_i} and then it permutes the nodes according to the equivalence transformation δ. Therefore, the automorphism $\nu_{\delta(N_i)}$ of the node $\delta(N_i)$ has to be composed with μ_{N_i}. $\square$

By Lemma 3.5, it follows that the group $\mathrm{Aut}(M)$ is a semidirect product of $\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_k)$ and $\mathrm{Aut}(T)$. More formally,

$$\mathrm{Aut}(M) \cong \big(\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_k)\big) \rtimes_\psi \mathrm{Aut}(T)$$

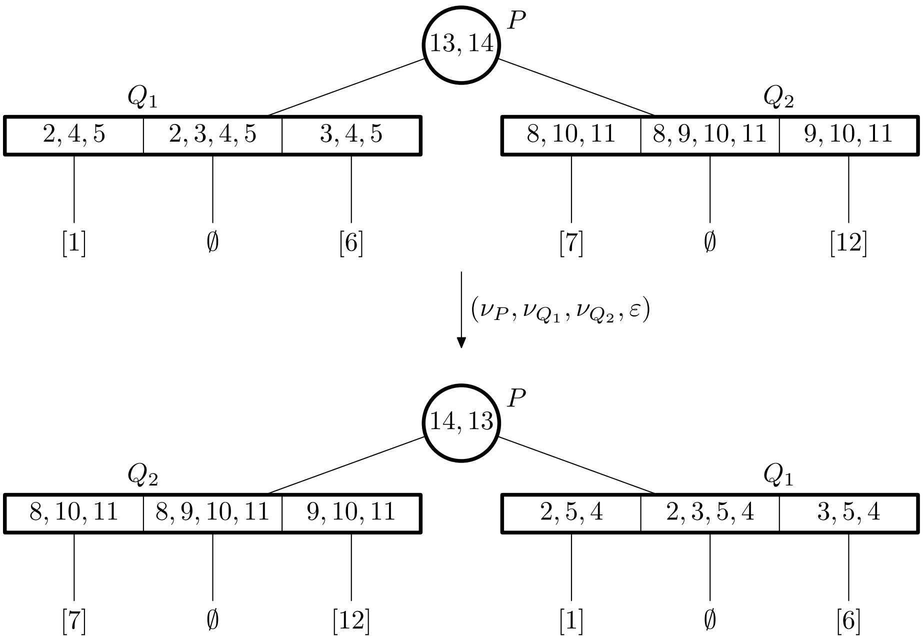


Figure 3.2: One automorphism $(\nu_P, \nu_{Q_1}, \nu_{Q_2}, \varepsilon)$ of an MPQ-tree. The automorphism ν_P of the node P swaps the vertices 13 and 14, the automorphism ν_{Q_1} of the node Q_1 is the identity automorphism, the automorphism ν_{Q_2} of the node Q_2 swaps the vertices 4 and 5, and the automorphism ε is an automorphism of the underlying PQ-tree.

where $\psi\colon \mathrm{Aut}(T) \to \mathrm{Aut}(\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_k))$ is the homomorphism defined as

$$\psi(\varepsilon) = \text{the automorphism that maps } (\nu_{N_1}, \dots, \nu_{N_k}) \text{ to } (\nu_{\varepsilon(N_1)}, \dots, \nu_{\varepsilon(N_k)}).$$

Proposition 3.6. *The automorphism group of M is isomorphic to the automorphism group of X.*

Proof. Let M be an MPQ-tree representing an interval graph X and let $N_1, \dots, N_k$ be the nodes of M. We fix some consecutive ordering on the maximal cliques $\mathcal{C}(X)$ (see Lemma 2.8) and we also fix an ordering $<_{TW}$ on each twin class.

Suppose that α is an automorphism of X. Then α can be decomposed into $\alpha_1 \circ \alpha_2$ such that α_1 only permutes those vertices of X that are in the same twin class, and α_2 permutes the maximal cliques $\mathcal{C}(X)$ in the same way as α and preserves the ordering $<_{TW}$ on each equivalence class of $\sim_{TW}$. The decomposition is shown in Figure 3.3. The permutation α_1 can be uniquely identified with an element of the group $\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_k)$ and the permutation α_2 can be uniquely identified with an element of the group $\mathrm{Aut}(T)$. Therefore, the permutation α can be uniquely identified with the automorphism $(\nu_{N_1}, \dots, \nu_{N_k}, \varepsilon)$.

If $(\nu_{N_1}, \dots, \nu_{N_k}, \varepsilon)$ is an automorphism of M, then the k-tuple $(\nu_{N_1}, \dots, \nu_{N_k})$ can be uniquely identified with an automorphism α_1 of X such that it does not change the ordering of the maximal cliques. There exists a unique automorphism α_2 of X that preserves the ordering $<_{TW}$, and permutes the maximal cliques $\mathcal{C}(X)$ in the same way as ε. So, the automorphism $(\nu_{N_1}, \dots, \nu_{N_k}, \varepsilon)$ of M can be uniquely identified with the decomposition $\alpha = \alpha_1 \circ \alpha_2$.

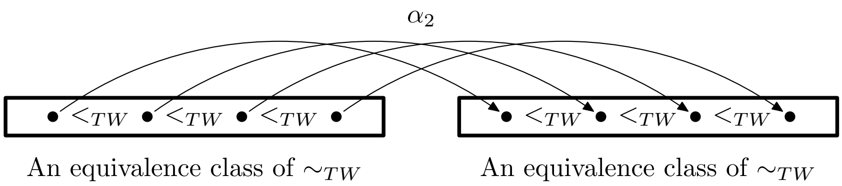


Figure 3.3: The permutation α_1 permutes the vertices in the equivalence class on the left and the permutation α_2 preserves the ordering $<_{TW}$.

We can define a bijective mapping $\phi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(M)$ by

$$\phi(\alpha) = (\nu_{N_1}, \ldots, \nu_{N_k}, \varepsilon)$$

where $\nu_{N_1}, \ldots, \nu_{N_k}, \varepsilon$ are as above. It is straightforward to check that ϕ is an isomorphism. $\square$

3.2 Characterization of the Automorphism Groups

In this section we derive a characterization of the class $\mathcal{I}$ of the automorphism groups of interval graphs, and prove that it is equal to the class $\mathcal{T}$ of the automorphism groups of trees. We use an MPQ-tree to represent an interval graph. To determine the automorphism group of the MPQ-tree we distinguish the case when the root is a P-node, and when the root is a Q-node. A similar analysis to the one in Theorem 2.6 can be done in the P-node case. The most important is Lemma 3.9 which deals with the Q-node case. Finally, we show that each group in $\mathcal{I}$ can be built inductively from the trivial group using two group products.

Suppose that X is an interval graph. Let M be an MPQ-tree representing X and let T be the underlying PQ-tree. From the previous section we have that

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(M) \cong F \rtimes_\psi \mathrm{Aut}(T)$$

where F is the direct product of the automorphism groups of the nodes of M. It corresponds to the automorphisms of X that preserve the ordering of the maximal cliques, that is, automorphisms that permute only twin vertices. Clearly F is isomorphic to a subgroup of $\mathrm{Aut}(X)$. Each automorphism of X can perform two operations: (1) permute the twin vertices; (2) change the consecutive ordering of the maximal cliques $\mathcal{C}(X)$. Those two operations are not commutative.

To obtain the automorphism group of X, we just need to determine $\mathrm{Aut}(M)$. For this, we use a similar technique as Jordan used for the automorphisms group of a trees; see Theorem 2.6. We distinguish two cases: (1) the root of M is a P-node; (2) the root of M is a Q-node. First, we prove a lemma that claims that the class $\mathcal{I}$ is closed under the direct product.

Lemma 3.7. *If $G_1, G_2 \in \mathcal{I}$, then also $G_1 \times G_2 \in \mathcal{I}$. In other words, let X_1, X_2 be interval graphs such that* $\mathrm{Aut}(X_1) \cong G_1$ *and* $\mathrm{Aut}(X_2) \cong G_2$. *Then there exists a graph X such that* $\mathrm{Aut}(X) \cong G_1 \times G_2$.

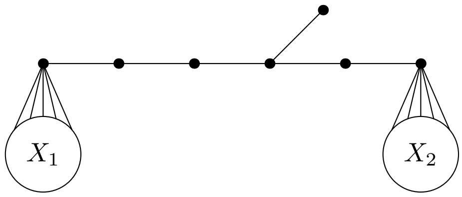


Figure 3.4: Two interval graphs are attached to an asymmetric path. The automorphism group is $\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2)$.

Proof. We just use the disjoint union of X_1 and X_2. In the case that X_1 and X_2 are isomorphic, we further attach them to an asymmetric path; see Figure 3.4. Since the asymmetric path is an interval graph, it follows that the whole graph X is an interval graph. In both cases , we get $\mathrm{Aut}(X) = G_1 \times G_2$. $\square$

The following two lemmas deal with the P-node case and with the Q-node case, respectively.

Lemma 3.8 (The P-node case)**.** *Suppose that the root of M is a P-node P with* $\mathrm{sec}(P) = \{x_1, \dots, x_\ell\}$. *If $M_1, \dots, M_n$ are pairwise non-isomorphic MPQ-trees, and the subtrees of P consist of k_i isomorphic copies of M_i, $i = 1, \dots, k$, then*

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(M_1) \wr \mathbb{S}_{k_1} \times \mathrm{Aut}(M_2) \wr \mathbb{S}_{k_2} \times \cdots \times \mathrm{Aut}(M_n) \wr \mathbb{S}_{k_n} \times \mathbb{S}_\ell.$$

Proof. The group $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(M') \times \mathbb{S}_\ell$ where M' is MPQ-tree obtained from M by removing the section. If we remove the root of M', the resulting graph is disconnected with each subtree corresponding to one connected component. By Theorem 2.5 the group $\mathrm{Aut}(M')$ is isomorphic to the direct product of the wreath products $\mathrm{Aut}(M_i) \wr \mathbb{S}_{k_i}$. $\square$

In the next lemma we deal with a Q-node in the root of M. A Q-node Q is *symmetric* if there exists an automorphism of M that reverses the order of the subtrees of Q. Otherwise, Q is *asymmetric*.

Lemma 3.9 (The Q-node case)**.** *Suppose that the root of M is a Q-node Q.*

(1) If Q is asymmetric, then $\mathrm{Aut}(M)$ *is isomorphic to a direct product of groups from the class $\mathcal{I}$.*

(2) If Q is symmetric, then

$$\mathrm{Aut}(M) \cong (G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2,$$

where $G_1, G_2, G_3 \in \mathcal{I}$, $G_1 \cong G_3$ and $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(G_1 \times G_2 \times G_3)$ *is the homomorphism defined by*

$\varphi(0) =$ *the identity automorphism,*
$\varphi(1) =$ *the automorphism that maps* (g_1, g_2, g_3) *to* (g_3, g_2, g_1).

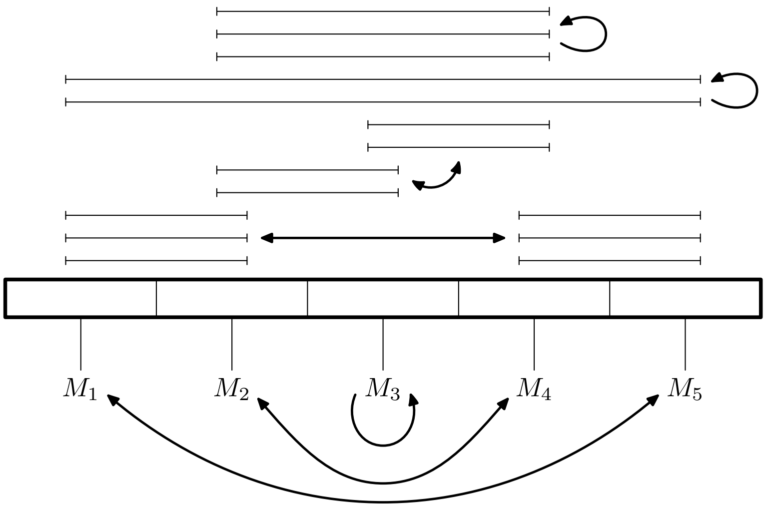


Figure 3.5: An example of a symmetric node. The arrows represent the action of an automorphism that reverses the order of the subtrees of the Q-node.

Proof. Let Q be asymmetric. If $M_1, \ldots, M_n$ are the subtrees of Q, then

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(M_1) \times \mathrm{Aut}(M_2) \times \cdots \times \mathrm{Aut}(M_n) \times \mathrm{Aut}(Q).$$

Each subtree corresponds to an interval graph, so each $\mathrm{Aut}(M_i)$ is in the class $\mathcal{I}$ by Proposition 3.6. Since every symmetric group is in $\mathcal{I}$, it follows from Lemma 3.7 that also $\mathrm{Aut}(Q)$ is in $\mathcal{I}$.

If Q is symmetric, then $\mathrm{Aut}(M)$ contains automorphisms that reverse the subtrees of Q. In this case, the group

$$\mathrm{Aut}(M_1) \times \cdots \times \mathrm{Aut}(M_n) \times \mathrm{Aut}(Q)$$

is the subgroup of $\mathrm{Aut}(M)$ that preserves the ordering of the subtrees. The idea is to write this subgroup as a direct product of three groups $G_1 \times G_2 \times G_3$ such that $G_1 \cong G_3$. The group G_1 corresponds to the automorphisms that act on the left side of the tree, G_2 to the automorphisms that act in the middle, and G_3 to the automorphisms that act on the right side. The reason for this is that reversing the subtrees of Q swaps the left side with the right side. In other words, it swaps the action of G_1 and G_3. Therefore, the group $\mathrm{Aut}(M)$ can be expressed as the semidirect product of $G_1 \times G_2 \times G_3$ with $\mathbb{Z}_2$. An example is shown in Figure 3.5.

The twin classes of X that are in Q are symmetric. This means that if $\{x_1, \ldots, x_\ell\}$ is a twin class and $x_1, \ldots, x_\ell$ belong to $\mathrm{sec}_i(Q), \ldots, \mathrm{sec}_{i+k}(Q)$, then there exist vertices $y_1, \ldots, y_\ell$ belonging to the sections $\mathrm{sec}_{n-i-k+1}(Q), \ldots \mathrm{sec}_{n-i+1}(Q)$. This can be seen in Figure 3.5. If $i = n-i-k+1$ and $i+k = n-i+1$, then $\{x_1, \ldots, x_\ell\}$ is in the middle and is fixed by each automorphism that reverses the order of the subtrees of Q.

We can write $\mathrm{Aut}(Q)$ as a direct product of three subgroups $H_1 \times H_2 \times H_3$. The subgroup H_1 permutes the vertices in the twin classes that are in the left part of Q, H_2 permutes the middle and H_3 permutes the right side of Q. The automorphism group of the tree in Figure 3.5 is isomorphic to

$$\Big(\underbrace{\mathrm{Aut}(M_1) \times \mathrm{Aut}(M_2) \times \overbrace{\mathbb{S}_3 \times \mathbb{S}_2}^{H_1}}_{G_1} \times \underbrace{\mathrm{Aut}(M_3) \times \overbrace{\mathbb{S}_2 \times \mathbb{S}_3}^{H_2}}_{G_2}$$
$$\times \underbrace{\overbrace{\mathbb{S}_2 \times \mathbb{S}_3}^{H_3} \times \mathrm{Aut}(M_4) \times \mathrm{Aut}(M_5)}_{G_3}\Big) \rtimes_\varphi \mathbb{Z}_2.$$

More formally, H_1 is the subgroup of $\mathrm{Aut}(Q)$ that permutes the sets $\{x_1, \dots, x_n\}$ such that $x_1, \dots, x_\ell \in \mathrm{sec}_i(Q), \dots, \mathrm{sec}_{i+k}(Q)$ and $i < n - i - k + 1$, H_2 is the subgroup of $\mathrm{Aut}(Q)$ that permutes the sets for which $i = n - i - k + 1$, and H_3 permutes the sets for which $i > n - i - k + 1$. It holds that $\mathrm{Aut}(Q) = H_1 \times H_2 \times H_3$.

We define

$$G_1 = \mathrm{Aut}(M_1) \times \cdots \times \mathrm{Aut}(M_{\lfloor n/2 \rfloor}) \times H_1,$$

and

$$G_3 = H_3 \times \mathrm{Aut}(M_{\lceil n/2 \rceil + 1}) \times \cdots \times \mathrm{Aut}(M_n).$$

Note that $G_1 \cong G_3$. The group G_2 is defined according to the parity of n. If n is odd, then $G_2 = \mathrm{Aut}(M_{\lceil n/2 \rceil}) \times H_2$. Otherwise, $G_2 = H_2$. Clearly, G_1, G_2, G_3 belong to the class $\mathcal{I}$, since each $\mathrm{Aut}(M_i)$ and each H_i is in $\mathcal{I}$. It is straightforward to see that the subgroup of $\mathrm{Aut}(M)$ that preserves the ordering of the subtrees of Q is $G_1 \times G_2 \times G_3$.

Each automorphism of M is can be encoded by a quadruple (g_1, g_2, g_3, z) where $(g_1, g_2, g_3) \in G_1 \times G_2 \times G_3$ and $z \in \mathbb{Z}_2$. The automorphism (g_1, g_2, g_3, z) first acts on the subtrees of M and the twin classes in Q by (g_1, g_2, g_3). The value z represents whether the ordering of the subtrees is reversed ($z = 1$) or preserved ($z = 0$).

The action of the composition

$$(g_1, g_2, g_3, z_1) \cdot (k_1, k_2, k_3, z_2)$$

of two automorphism of M can be described in the following way. Let us consider the special case when $z_1 = 1$ and $z_2 = 0$. First, (g_1, g_2, g_3, z_1) acts on the subtrees and the twin classes using (g_1, g_2, g_3). Then the ordering of the subtrees is reversed, since $z_1 = 1$. So, (k_1, k_2, k_3) acts on M such that k_1 acts on the right side and k_3 acts on the left side. This means that we have to compose g_1 with k_3, and g_3 with k_1. Therefore,

$$(g_1, g_2, g_3, z_1) \cdot (k_1, k_2, k_3, z_2) = (\underline{g_1 \cdot k_3}, g_2 \cdot k_2, \underline{g_3 \cdot k_1}, z_1).$$

In general,

$$(g_1, g_2, g_3, z_1) \cdot (k_1, k_2, k_3, z_2) = \begin{cases} (g_1 \cdot k_1, g_2 \cdot k_2, g_3 \cdot k_3, z_1 + z_2) & \text{if } z_1 = 0 \\ (\underline{g_1 \cdot k_3}, g_2 \cdot k_2, \underline{g_3 \cdot k_1}, z_1 + z_2) & \text{if } z_1 = 1 \end{cases}$$

Formally, $\mathrm{Aut}(M)$ is isomorphic to the semidirect product $(G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2$, where $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(G_1 \times G_2 \times G_3)$ is the homomorphism defined as in the statement of this lemma. $\square$

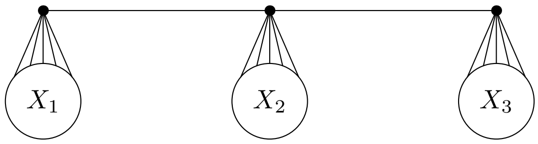


Figure 3.6: Interval graphs attached to a path.

Lemma 3.8 and Lemma 3.9 suggest that the class $\mathcal{I}$ is closed under the direct products, the wreath products with $\mathbb{S}_n$ and the semidirect products of direct products with $\mathbb{Z}_2$. Lemma 3.10 gives a characterization of $\mathcal{I}$ in terms of group products.

Lemma 3.10. *A finite group G is isomorphic to an automorphism group of a finite interval graph if and only if $G \in \mathcal{I}$, where the class $\mathcal{I}$ of finite groups is defined inductively as follows:*

(a) $\{1\} \in \mathcal{I}$.

(b) If $G_1, G_2 \in \mathcal{I}$, then $G_1 \times G_2 \in \mathcal{I}$.

(c) If $G \in \mathcal{I}$ and $n \geq 2$, then $G \wr \mathbb{S}_n \in \mathcal{I}$.

(d) If $G_1, G_2, G_3 \in \mathcal{I}$ and $G_1 \cong G_3$, then

$$(G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2 \in \mathcal{I},$$

where $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(G_1 \times G_2 \times G_3)$ is the homomorphism defined by

$$\begin{aligned} &\varphi(0) = \textit{the identity automorphism},\\ &\varphi(1) = \textit{the automorphism that maps } (g_1, g_2, g_3) \textit{ to } (g_3, g_2, g_1). \end{aligned}$$

Proof. Clearly $\{1\} \in \mathcal{I}$. We prove that the class $\mathcal{I}$ is closed under the operations (b), (c) and (d). From Lemma 3.7 we have that $\mathcal{I}$ is closed under the operation (b).

For (c), let $G \in \mathcal{I}$ and $n \geq 2$. Then there exists an interval graph X_1 such that $\mathrm{Aut}(X_1) \cong G$. We construct an interval graph X by taking the disjoint union of n copies of X_1. From Theorem 2.5 we have that $G \wr \mathbb{S}_n \cong \mathrm{Aut}(X)$. Therefore, $\mathcal{I}$ is closed under the operation (c).

It remains to prove (d). If $G_1, G_2, G_3 \in \mathcal{I}$ and $G_1 \cong G_3$, then there exist interval graphs X_1, X_2, X_3 such that $G_1 \cong \mathrm{Aut}(X_1), G_2 \cong \mathrm{Aut}(X_2), G_3 \cong \mathrm{Aut}(X_3)$. Without a loss of generality we assume that $X_1 \cong X_3$. We construct an interval graph X by attaching X_1, X_2, X_3 to a path, as shown in Figure 3.6. It is straightforward to see that $(G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2 \cong \mathrm{Aut}(X)$. Therefore, $\mathcal{I}$ is closed under the operation (d).

For the converse, we want to show that each interval graph X has $\mathrm{Aut}(X)$ isomorphic to a group $G \in \mathcal{I}$. Let M be an MPQ-tree representing X. According to Lemma 3.8 and Lemma 3.9, we have that $\mathrm{Aut}(M) \in \mathcal{I}$ and by Proposition 3.6 also $\mathrm{Aut}(X) \in \mathcal{I}$. ∎

Theorem 3.11. *The class $\mathcal{I}$ is the same as the class $\mathcal{T}$.*

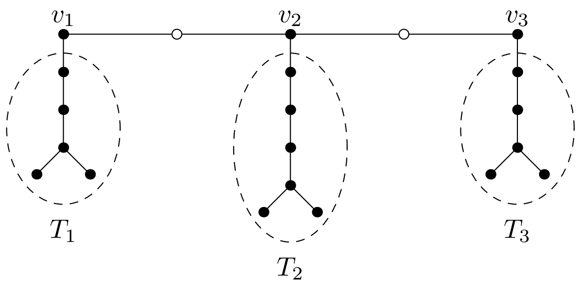


Figure 3.7: Trees attached to a path by their roots. The automorphism group of the tree is not isomorphic to $\big(\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2) \times \mathrm{Aut}(T_3)\big) \rtimes_\varphi \mathbb{Z}_2$. We fix this by subdividing the edges v_1v_2 and v_2v_3.

Proof. We show that the class $\mathcal{T}$ of the automorphism groups of trees is closed under the operation (d) from the statement of Lemma 3.10.

Suppose that $G_1, G_2, G_3 \in \mathcal{T}$ and $G_1 \cong G_3$, then there exist trees T_1, T_2, T_3 such that $G_1 \cong \mathrm{Aut}(T_1), G_2 \cong \mathrm{Aut}(T_2), G_3 \cong \mathrm{Aut}(T_3)$. Without a loss of generality we assume that $T_1 \cong T_3$. We construct a tree T by attaching T_1, T_2, T_3 on a path by a vertex, as shown in Figure 3.7. If is not true that $(G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2 \cong \mathrm{Aut}(T)$, then we fix this by subdividing the edges, as shown in Figure 3.7. $\square$

3.3 On Equality of The Automorphism Groups

We proved that the class $\mathcal{I}$ of the automorphism groups of finite interval graphs is the same as the class $\mathcal{T}$ of the automorphism groups of finite trees. Natural problems is to find for each interval graph X a tree T such that the automorphism group of X is isomorphic to the automorphism group of T, and vice versa. Here, we solve these problems which also gives an alternative proof of Theorem 3.11.

From Interval Graphs to Trees. For an interval graph X we construct a tree T such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(T)$. Let M be an MPQ-tree representing X. From Proposition 3.6 we have that $\mathrm{Aut}(M) \cong \mathrm{Aut}(X)$. We construct the tree T inductively.

If the root of M is a P-node P, then we use Lemma 3.8. Let $M_1, \dots, M_n$ be the subtrees of P as in the statement of the lemma and let $T_1, \dots, T_n$ be trees such that $\mathrm{Aut}(T_i) \cong \mathrm{Aut}(M_i)$. We construct T by attaching the trees $T_1, \dots, T_n$ to a new vertex v by their roots. In case the section $\mathrm{sec}(P)$ is nonempty, then we attach $|\mathrm{sec}(P)|$ edges to the vertex v (we subdivide them if necessary).

If the root of M is a Q-node Q, then we use Lemma 3.9. Let $M_1, \dots, M_n$ be the subtrees of Q, and let $T_1, \dots, T_n$ be trees such that $\mathrm{Aut}(T_i) \cong \mathrm{Aut}(M_i)$, and let T_S be a tree such that $\mathrm{Aut}(T_S) \cong \mathrm{Aut}(Q)$. If Q is asymmetric, then we construct T by attaching $T_1, \dots, T_n, T_S$ to an asymmetric path (and subdivide edges if necessary). If Q is symmetric, then we construct T similarly as in Theorem 3.11.

From Trees to Interval Graphs. For a tree T we construct an interval graph X such that $\mathrm{Aut}(T) \cong \mathrm{Aut}(X)$. The idea is to place the intervals so that they copy the pattern of the given tree T, as shown in Figure 3.8. We assume that T is a rooted

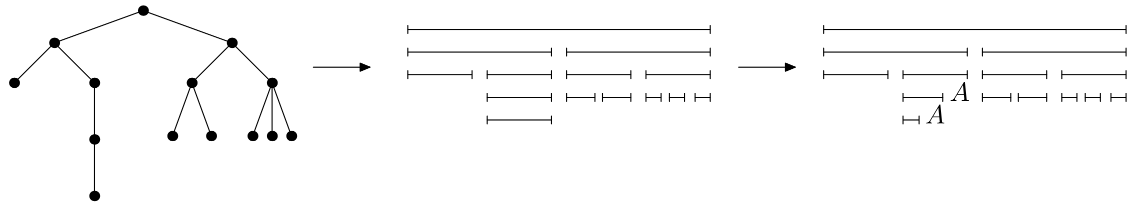


Figure 3.8: First, we place the intervals according to the pattern of the tree. The automorphism group of the constructed interval graph is isomorphic to $\mathbb{S}_3 \times \mathbb{S}_2 \times \mathbb{S}_3$. However, the automorphism group of the tree is $\mathbb{S}_2 \times \mathbb{S}_3$. We fix this by adding copies of an asymmetric path A (an example of an asymmetric path is shown in Figure 3.4), which has the automorphism group isomorphic to the trivial group.

tree, let r be the root and let $c_1, \dots, c_n$ be its children. We choose an interval R to represent the root r. Then we choose an interval C_i for each of its children so that $C_i \cap C_j = \emptyset$ and $C_i \subseteq R$. We recursively construct the subtrees of each child c_i.

If T contains some vertices with only one child, then $\mathrm{Aut}(T)$ is a subgroup of the automorphism groups of the constructed interval graph. The issue is that the construction creates twin vertices that can be permuted. This can be fixed by adding asymmetric paths, as in Figure 3.8.

Automorphism Groups of Caterpillar Graphs. Here, we apply the results from the previous chapter to characterize the automorphism groups of CATERPILLARS. Those are the trees for which removing the leaves produces a path P. We call this path a *central path.*

Proposition 3.12. *Let X be a caterpillar graph and let P be the central path.*

(1) If no automorphism swaps the path P, then the group $\mathrm{Aut}(X)$ *is isomorphic to a direct product of symmetric groups.*

(2) If there exists an automorphism of X that swaps the path P, then

$$\mathrm{Aut}(X) \cong (G_1 \times G_2 \times G_3) \rtimes_\varphi \mathbb{Z}_2,$$

where G_2 is isomorphic to $\mathbb{S}_k$, $G_1 \cong G_3$ are isomorphic to a direct product of symmetric groups, and φ is the homomorphism defined as in Lemma 3.9.

Proof. The root of an MPQ-tree M representing a caterpillar graph X is a Q-node. All twin classes are trivial, since X is a atree. Each child of the root is either a P-node, or a leaf. All children of every P-node are leaves. If there exist an automorphism that swaps the central path P, then the root is symmetric, otherwise it is asymmetric. We apply Lemma 3.8 and Lemma 3.9 to determine the automorphism group of the MPQ-tree M. $\square$

4 Conclusions

We conclude by describing three open problems concerning important intersection-defined classes of graphs, namely circle graphs, function graphs and circular-arc graphs. The structure of their automorphism group is currently unknown. We know that those classes of geometric intersection graphs have different automorphism groups than trees, since all of them contain the graph C_4. The automorphism group of C_4 is isomorphic to the dihedral group $\mathbb{D}_4$ which does not belong to $\mathcal{T}$.

Circle Graphs. Circle graphs are intersection graphs of chords of a circle. They were first considered by Even and Itai [11] in the study of stack sorting techniques. The structure of all representations of circle graphs is described in [7].

A *circle representation* $\mathcal{R}$ of a graph X is a set of chords $\{C_x : x \in V(X)\}$ such that $xy \in E(X)$ if and only if the chords C_x and C_y intersect. A graph X is a circle graph if there exists a circle representation $\mathcal{R}$ of X. Figure 4.1 shows an example of circle graph and its circle representation.

Problem 4.1. *What is the class of the automorphism groups of circle graphs?*

Function Graphs. A representation of a function graph assigns a continuous function $f\colon [0,1] \to \mathbb{R}$ to every vertex of the graph. Edges are represented by intersections of those functions. The class of permutation graphs, which is a subclass of function graphs, contains graphs that can be represented in the same way, but by linear func-

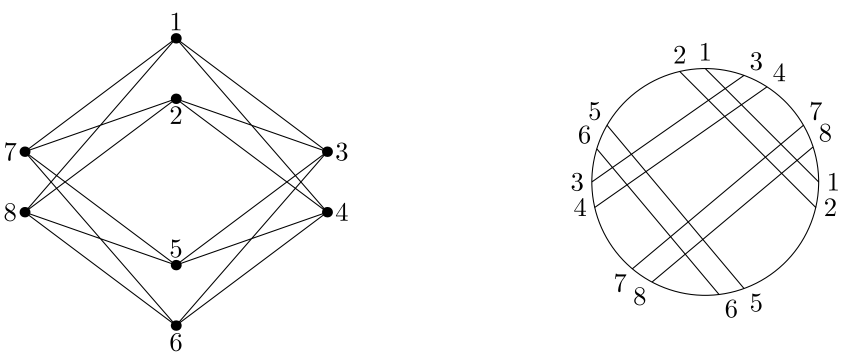


Figure 4.1: A circle graph and on of its circle representations. The automorphism group of the graph is isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{D}_4$.

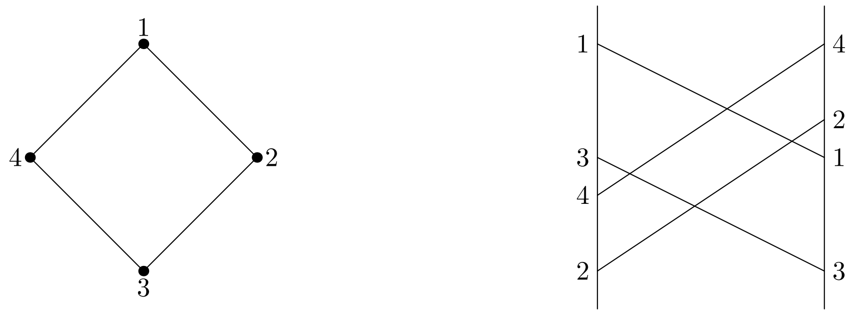


Figure 4.2: A permutation graph and one of its representations. The automorphism group of the graph is isomorphic to $\mathbb{D}_4$.

tions. Figure 4.2 shows an example of a permutation graph. The structure of all representations of function is described in [24].

Function graphs are the complements of so-called *comparability graphs* [18]. A comparability graph is a graph of some partial ordering. In other words, comparability graphs are graphs of which edges can be oriented transitively. Permutation graphs are exactly the intersection of function and comparability graphs [12].

Problem 4.2. *What is the class of the automorphism groups of function graphs?*

Circular-arc Graphs. Circular-arc graphs are intersection graphs of arcs of a circle. Figure 4.3 shows an example of a circular-arc graph. They are a natural generalization of interval graphs. If there exists a point of the circle that is not covered by an arc, then the circle can be cut at that point and stretched to a line, which yields an interval representation.

Surprisingly, the class of circular-arc graphs is very different from the class of interval graphs. The main difference is that in the case of circular-arc graphs, the maximal cliques do not behave nicely. A circular-arc graph can have exponential number of maximal cliques.

Generalizing some of the results known for interval graphs to the class of all circular-arc graphs is a challenging problem. McConnel [31] solved the recognition problem for circular-arc graphs in linear time. However, no polynomial-time isomorphism test for circular-arc graphs is currently known. For some time it seemed that the problem is solved since Hsu [22] claimed to have a polynomial-time algorithm, but only recently it was proved he dealt incorrectly with one case [9].

Problem 4.3. *What is the class of the automorphism groups of circular-arc graphs?*

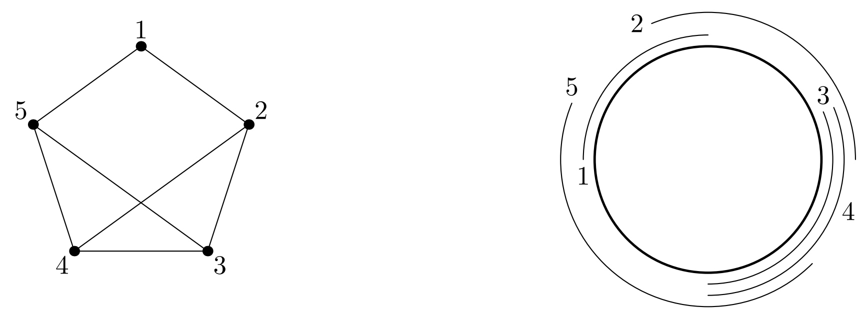


Figure 4.3: A circular-arc graph and one of its representations. The automorphism group of the graph is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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