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Systems of Morphisms over Gödel Fuzzy Logic

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Declaration of unaided work

Hereby I declare that this thesis is my own unaided work and that I used only the sources listed in references exclusively.

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Abstract

This work introduces some very basic concepts of category theory as built up over first-order predicate Gödel fuzzy logic (with crisp identity and the delta operator). A fuzzy variation of a classical concept of a category is considered. Then several systems of morphisms loosely based on the crisp categories Rel and Set are defined and examined. Accordingly, all the systems under consideration consist of fuzzy sets as objects and various kinds of binary fuzzy relations as morphisms. Our approach is a logic-based graded generalization of crisp (classical) category-theoretical approaches to fuzzy sets, which have been initiated by Goguen.

Keywords

Systems of morphisms, Fuzzy categories, Fuzzy sets, Fuzzy relations, Gödel fuzzy logic

Contents

Chapter 1

Introduction

After more than seventy years of development, category theory has established an undisputable position in contemporary mathematics. Concepts such as categories, functors, natural transformations etc. first emerged in the works of Eilenberg and Mac Lane in the early 1940's in connection with algebraic topology [14], [15]. In its early years, concepts of category theory had been used only by specialists in specific areas of mathematical fields such as just mentioned algebraic topology, algebraic geometry and homological algebra. Particularly due to the pioneering work of Lawvere [28], [29], [30] from the 1960's and early 70's category theory has come into the interest of many mathematicians, logicians and other scientists and it has been no longer a privilege of just a few specialists. Its simple but at the same time powerful language, illustrative way of expressing and high level of abstraction make category theory a very suitable tool for studying mathematical structures and their mutual relationships. It provides a conceptual framework for the whole mathematics, and thus can be regarded as an alternative to set theory, as categories naturally occur in every mathematical field and also in the related fields such as theoretical computer science, theoretical physics, logic etc.

The term 'fuzzy logic' emerged in the train of the concept of fuzzy set, which was introduced by Zadeh in 1965 [43]. The main purpose of logic, in general, is to study the notion of (logical) consequence. What is specific for fuzzy logic, in contrast to the classical one, is that the propositions (sentences), among which the relation of consequence is considered, can be true to some degree only. Thus, fuzzy logic can be understood as a logic of comparative truth. Usually, these truth degrees are represented by the elements of the real unit interval $[0, 1]$, but also more general structures may be used for the representation. In this work we will be dealing with the well-known variant of the first-order predicate fuzzy logic (with crisp identity and the delta operator) called Gödel fuzzy logic, symbolized by $\mathrm{G}\forall_{\Delta=}$. It is one of the substructural logics in the family of the so called (continuous) t-norm fuzzy logics. It was implicitly defined by Kurt Gödel in 1932 [19] in connection with a study of intuitionistic logic. It should be noted that Gödel's aim was not to study the relation of consequence involving uncertainty or vagueness, he only needed some suitable kind of infinitely many-valued logic for his purposes of examination of intuitionistic logic. Gödel logic was extensively studied by Dummett [13], and therefore is also known as Gödel–Dummett logic.

The aim of this work is to study some very fundamental concepts of category theory in case of a formal logic in a background of the theory is just $G\forall_{\Delta=}$. Our motivation for doing this is led by a conviction that building up any part of mathematics over some suitable kind of fuzzy logic, or non-classical logic in general, can generally lead up to interesting and useful results. However the state of the art of formal fuzzy logic had not made it possible to use fuzzy logic as a formal background for mathematics until the development of first- and higher-order fuzzy logic. Since its origins, fuzzy logic has been primarily used by engineers and computer scientists for applications in branches like control theory, artificial intelligence, system engineering, many subfields of computer science etc. and it has to be said that it has achieved a great success in these branches. But for decades fuzzy logic had not been the subject of interest for most mathematicians or logicians (as an example of honorable exception can serve the work of Gottwald [21], [22]). Their unconcern was mainly caused by the arbitrariness, methodologically confused and generally chaotic state of the field of fuzzy logic at the time. A breakthrough came in the 1990's. Especially thanks to the famous work of Hájek [23], in which he has made some important 'design choices', the area of fuzzy logic has become much more well-arranged and methodologically clearer than before. Since that time, (deductive) fuzzy logic, or fuzzy logic in the narrow sense as distinguished by Zadeh in [43], has been the subject of intensive research and has already reached the point of its development, in which it is possible and reasonable to use it as a formal framework for axiomatic mathematical theories. Thanks to this development of formal fuzzy logic it is now possible (and necessary) to distinguish *traditional fuzzy mathematics*, as the one that deals with fuzzy sets and other fuzzy concepts, but which still relies on the classical logic in its background, from *formal fuzzy mathematics*, the one that re-builds classical-mathematical concepts on the background of some formal fuzzy logic. The latter approach has been developed mostly in the last decade mainly due to the work of Běhounek, Bodenhofer, Cintula and others, see e.g. [2], [3], [4], [5].

Finally, let us give a brief note on the classical category-theoretical approaches to fuzzy sets, which on the one hand have inspired us to start the fuzzy-logical approach to categories, but which on the other hand are fundamentally different from our approach. Shortly after Zadeh's pioneering work the significant generalizations have been made by Goguen [17], [18]. Firstly, he considered more general structures than the real unit interval $[0, 1]$ as the target structures of characteristic functions corresponding to fuzzy sets (in fact, the idea of doing this has been marked in [43] already, but as late as in [17] it has been fully realized). Secondly, he considered fuzzy sets as objects and suitable kind of fuzzy relations as morphisms in such a way that together they constituted a category. Since then, many categories of fuzzy sets have been defined and examined. Let us mention especially the works of Higgs [24], [25], Höhle [26], [27], Solovjovs [34], [35], [36] and Stout [37], [38], [39]. All these approaches essentially differ from ours with the intent that the formal logic in the background of the theory is classical. Thus the categories, as well as all the category-theoretical concepts, obtained by these approaches are crisp; they only involve fuzzy sets as objects and certain kind of fuzzy relations or fuzzy functions as morphisms.

Significant generalization of these approaches and overall progress in fuzzification of entire categories have been achieved by Šostak in [40], [41], [42]. In his work, Šostak considers the concept of a fuzzy category, in which elements of some crisp universe are objects of the category to some degree only and similarly for morphisms. He also considers fuzzy analogues of some classical (crisp) category-theoretical concepts like a subcategory, a quotient category, a terminal object, an initial object, a monomorphism, an epimorphism, a functor etc. Thus Šostak's approach can be seen as a 'next-level fuzzification', as opposed to the formely mentioned approaches, as it allows us to obtain entire fuzzy categories (and not just crisp categories of fuzzy sets) and other fuzzy counterparts of crisp category-theoretical concepts. But our approach differs even from this one. The difference consists at least in the starting points and thus in the methodological characters of the two approaches, as our starting point and thus the entire approach can be seen as the (fuzzy) logical one. We start out by explicitly stating that in our case $G\forall_{\Delta=}$ plays the role of the formal logical background, within which some category-theoretical concepts are being defined and examined, or in other words some parts of category theory are being rebuilt up. So in our case the process of 'natural' fuzzification of category-theoretical concepts has been achieved by a different approach than in Šostak's case.

The body of the work is divided into two main parts. In the first part (Preliminaries) the necessary fuzzy-logical, fuzzy set-theoretical and category-theoretical basics are presented. By fuzzy-logical basics we mean especially a presentation of Gödel fuzzy logic. By fuzzy set-theoretical basics we mean a presentation of fuzzy analogues, now being defined over $G\forall_{\Delta=}$, of some very fundamental set-theoretical concepts like the empty set, a singleton, a relation, inclusion, the Cartesian product etc. Finally, by category-theoretical basics we mean presentation of the most fundamental concepts of the theory like these of a category, a terminal object, an initial object, a product, a coproduct etc. Reader familiar with these fields can go through that part quickly or can go straight to the next main part (Systems of morphisms), which is devoted to the examination of several systems of morphisms building up against the backdrop of $G\forall_{\Delta=}$. In all the considered systems fuzzy sets play the role of objects and thus these systems primarily differ in terms of determination of their morphisms. We say 'systems of morphisms' instead of categories, because in the general case it is not in any way guaranteed that these systems satisfy, in $G\forall_{\Delta=}$ internalized, axioms for categories and thus that they actually constitute categories. Hence the title of this work is 'Systems of morphisms over Gödel fuzzy logic'.

Chapter 2

Preliminaries

This section is intended as a brief introduction to Gödel fuzzy logic, fuzzy set-theoretical basics and category-theoretical basics. Only very basic concepts and motivations, which are necessary for the following chapter, are presented. For readers interested in these fields more deeply we recommend e.g. [12], [23], [33] for the subject of fuzzy logics, [3], [4], [5], [32] as an introduction to fuzzy set (class) theory and [1], [20], [31] for the subject of category theory.

2.1 Gödel fuzzy logic

Gödel fuzzy logic can be seen as a product of at least two different approaches or motivations. On the one hand it can be seen as the extension of *intuitionistic logic* and on the other hand there is a class of *continuous t-norm fuzzy logics*, in which it plays a prominent role, along with Łukasiewicz logic and product logic. For some details about these two approaches see remark 2.1.5 below. Let us naturally start by the propositional version of Gödel fuzzy logic, usually denoted by G.

Definition 2.1.1 A *language of the propositional Gödel fuzzy logic* G contains symbols for countably many *propositional variables* (or *atoms*) $p_1, p_2, p_3, \ldots$ and *logical connectives* &, $\to$ and $\bot$ (and auxiliary symbols for brackets). *Propositional formulas* are defined as usual: each propositional variable is a propositional formula (atomic propositional formula), $\bot$ is a propositional formula, if φ and ψ are propositional formulas, then $(\varphi \mathbin{\&} \psi)$ and $(\varphi \to \psi)$ are propositional formulas. Other (derived) logical connectives can be defined as follows:

- *disjunction* $(\varphi \vee \psi)$ is defined as $(((\varphi \to \psi) \to \psi) \mathbin{\&} ((\psi \to \varphi) \to \varphi))$
- *negation* $\neg\varphi$ is defined as $(\varphi \to \bot)$
- *equivalence* $(\varphi \leftrightarrow \psi)$ is defined as $((\varphi \to \psi) \mathbin{\&} (\psi \to \varphi))$
- *truth* $\top$ is defined as $\neg\bot$

From now on let us accept the convention common in formal (fuzzy) logic about omitting unnecessary brackets, given by the agreement that $\neg$ takes precedence over & and $\vee$, which whereas take precedence over $\to$ and $\leftrightarrow$.

Definition 2.1.2 An *evaluation of propositional variables* is a mapping $e\colon \mathrm{Var} \longrightarrow [0,1]$, where Var is a set of propositional variables. An *evaluation of propositional formulas* (of the logic G) is a mapping $e_*\colon \mathrm{Form(G)} \longrightarrow [0,1]$, where Form(G) is a set of propositional formulas of logic G, such that for each $\varphi, \psi \in \mathrm{Form(G)}$ and each $p \in \mathrm{Var}$ the following hold:

- $e(\varphi \mathbin{\&} \psi) = \min(e(\varphi), e(\psi))$
- $e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$, where $x \Rightarrow y = 1$ iff $x \leq y$ and $x \Rightarrow y = y$ otherwise, for each $x, y \in [0,1]$
- $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$
- $e(\neg\varphi) = e(\varphi) \Rightarrow 0$
- $e(\bot) = 0$
- $e(\top) = 1$

Definition 2.1.3 We say that a formula φ is a *tautology* of the logic G (more precisely a $[0,1]_{\mathrm{G}}$*-tautology*) if $e(\varphi) = 1$ holds for each $e\colon \mathrm{Var} \longrightarrow [0,1]$.

Definition 2.1.4 The *axiomatic system* of the logic G consists of the following axioms:

- (BL1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- (BL4) $\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$
- (BL5a) $(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
- (BL5b) $(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
- (BL6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
- (BL7) $\bot \to \varphi$
- (G) $\varphi \to \varphi \mathbin{\&} \varphi$

The deduction rule is *modus ponens*:

- (MP) From φ and $\varphi \to \psi$ infer ψ

Remark 2.1.5 The axiomatic system of G is presented here as an extension of the so called *basic logic* BL, which has been introduced by Hájek in [23]. Basic logic constitutes a common axiomatization for all continuous t-norm fuzzy logics. The original axiomatic system of BL consisted of the above axioms (BL1)–(BL7) plus the following two:

- (BL2) $\varphi \mathbin{\&} \psi \to \varphi$
- (BL3) $\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$

These two were later shown to be redundant in [10] and [9] respectively. On the other hand G can also be introduced as an extension of intuitionistic logic, see e.g. [23] pp. 98. Axioms of intuitionistic logic are the following ones:

- (I1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- (I2) $\varphi \to \varphi \vee \psi$
- (I3) $\psi \to \varphi \vee \psi$
- (I4) $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
- (I5) $\varphi \mathbin{\&} \psi \to \varphi$
- (I6) $\varphi \mathbin{\&} \psi \to \psi$
- (I7) $(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \mathbin{\&} \psi))$
- (I8) $(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
- (I9) $(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
- (I10) $\varphi \mathbin{\&} \neg\varphi \to \psi$
- (I11) $(\varphi \to \psi \mathbin{\&} \neg\psi) \to \neg\varphi$

G can be obtained by adding (BL6), or alternatively $(\varphi \to \psi) \vee (\psi \to \varphi)$ (*prelinearity*), to (I1)–(I11).

It is easy to check that each axiom of G is a $[0,1]_{\mathrm{G}}$-tautology and that if φ and $\varphi \to \psi$ are $[0,1]_{\mathrm{G}}$-tautologies, then ψ is as well. As a consequence, each formula provable in G is a $[0,1]_{\mathrm{G}}$-tautology. In other words, the just mentioned calculus (also denoted by G) is *sound* w.r.t. the $[0,1]$-semantics given above.

Remark 2.1.6 The semantics we have seen so far, in which the real unit interval $[0,1]$ represents the set of truth values and which interprets logical connectives as above, is called the *standard semantics* (of propositional Gödel fuzzy logic). If we take an algebraic view and consider a structure $\mathbf{G} = \langle [0,1], \min, \max, \Rightarrow, 0, 1\rangle$, then $\mathbf{G}$ is called a *standard* G-*algebra*. Standard G-algebras are special cases of more general structures called G-*algebras*, which are defined as follows.

Definition 2.1.7 A partially ordered set $\mathbf{L} = \langle \mathrm{L}, \leq \rangle$ with the least element $0_{\mathbf{L}}$, the greatest element $1_{\mathbf{L}}$ and together with the operations of minimum $\min(x,y)$, maximum $\max(x,y)$ and residuation $x \Rightarrow y$, defined as $x \Rightarrow y = 1_{\mathbf{L}}$ for $x \leq y$ and $x \Rightarrow y = y$ otherwise, is called a G-*algebra*. We say that a G-algebra is *linear* (or a G-*chain*) if its ordering is linear, i.e. if $x \leq y$ or $y \leq x$ holds for each $x, y \in \mathrm{L}$.

A natural generalization of a concept of an evaluation (of propositional variables and formulas) and a tautology to the general case of G-algebras is obvious. As a set of truth values is now taken the domain of a given G-algebra $\mathbf{L}$ and logical connectives are represented by its operations. Let us give a precise definition.

Definition 2.1.8 Let $\mathbf{L} = \langle \langle \mathrm{L}, \leq \rangle, \min, \max, \Rightarrow, 0_{\mathbf{L}}, 1_{\mathbf{L}} \rangle$ be a given G-algebra. An $\mathbf{L}$-*evaluation* of propositional variables is any mapping $e\colon \mathrm{Var} \longrightarrow \mathrm{L}$. This extends uniquely to all propositional formulas in obvious way:

- $e(\varphi \mathbin{\&} \psi) = \min(e(\varphi), e(\psi))$
- $e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$
- $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$
- $e(\neg\varphi) = e(\varphi) \Rightarrow 0_{\mathbf{L}}$
- $e(\bot) = 0_{\mathbf{L}}$
- $e(\top) = 1_{\mathbf{L}}$

A formula φ is an **L**-*tautology* if $e(\varphi) = 1_{\mathbf{L}}$ for each **L**-evaluation e. Moreover φ is called a G-*tautology* if it is an **L**-tautology for each G-algebra **L**.

By a *theory* we mean an arbitrary set of formulas (in a given language). A *proof* in a theory T over the logic G is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each of them is either an axiom of the logic G or is a member of the set T (i.e. a special axiom) or is derived from some preceding formulas by the deduction rule of modus ponens. The fact that φ is provable from T in the logic G is denoted by $\mathrm{T} \vdash_{\mathrm{G}} \varphi$. If **L** is a G-algebra, T a theory, and e an **L**-evaluation, then we say that e is an **L**-*model* of T if $e(\varphi) = 1_{\mathbf{L}}$ holds for each axiom $\varphi \in \mathrm{T}$. The proof of the following theorem can be found in [23] (Theorem 4.2.17, pp. 101) with reference to [13].

Theorem 2.1.9 (*Strong completeness* of G) Let T be a theory and let $\varphi \in \mathrm{Form(G)}$. Then the following three statements are equivalent:

- $\mathrm{T} \vdash_{\mathrm{G}} \varphi$
- $e(\varphi) = 1_{\mathbf{L}}$ for every G-chain **L** and every **L**-model e of T
- $e(\varphi) = 1_{\mathbf{L}}$ for every G-algebra **L** and every **L**-model e of T

Let us now consider the useful extension of the language of the propositional logic G by a single unary connective Δ, called *Baaz delta*. Δ is semantically characterized as follows. For each G-chain **L** and each $x \in \mathrm{L}$ (i.e. truth value of some propositional formula), $\Delta(x) = 1_{\mathbf{L}}$ iff $x = 1_{\mathbf{L}}$ and $\Delta(x) = 0_{\mathbf{L}}$ iff $x < 1_{\mathbf{L}}$.

Definition 2.1.10 The language of the logic G_Δ is the language of the logic G plus a single unary connective Δ. The axioms of the logic G_Δ are the axioms of G plus the following ones:

- (Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
- (Δ2) $\Delta(\varphi \vee \psi) \to \Delta\varphi \vee \Delta\psi$
- (Δ3) $\Delta\varphi \to \varphi$
- (Δ4) $\Delta\varphi \to \Delta\Delta\varphi$
- (Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

The deduction rules of G_Δ are modus ponens and *necessitation*:

- (Nec) From φ infer $\Delta\varphi$

Definition 2.1.11 A G_Δ-*algebra* is a structure $\mathbf{L} = \langle\langle \mathrm{L}, \leq\rangle, \min, \max, \Rightarrow, 0_\mathbf{L}, 1_\mathbf{L}, \Delta\rangle$ such that the following hold:

- $\langle\langle \mathrm{L}, \leq\rangle, \min, \max, \Rightarrow, 0_\mathbf{L}, 1_\mathbf{L}\rangle$ is a G-algebra
- $\max(\Delta x, (\Delta x \Rightarrow 0_\mathbf{L})) = 1_\mathbf{L}$
- $\Delta(\max(x, y)) \leq \max(\Delta x, \Delta y)$
- $\Delta x \leq x$
- $\Delta x \leq \Delta\Delta x$
- $\Delta(x \Rightarrow y) \leq \Delta x \Rightarrow \Delta y$
- $\Delta 1_\mathbf{L} = 1_\mathbf{L}$

Strong completeness of the logic G_Δ w.r.t. G_Δ-algebras is also shown in [23] (Theorem 4.2.17, pp. 101).

Let us now turn our attention to the predicate first-order version of Gödel fuzzy logic with crisp identity and the Δ operator, usually denoted by $G\forall_{\Delta=}$. Let us start more generally again.

Definition 2.1.12 A *multi-sorted predicate language* is a tuple $\mathcal{L} = \langle \mathbf{S}, \mathbf{P}, \mathbf{F}, \mathbf{A}\rangle$, where $\mathbf{S}$ is a non-empty set of *sorts* of countably many *variables*, $\mathbf{P}$ is a set of *predicate* symbols, $\mathbf{F}$ is a set of *function* symbols and $\mathbf{A}$ is an *arity function* assigning a sequence of sorts $(s_1, \ldots, s_n)$, for $n \geq 0$ to each predicate symbol and a sequence of sorts $(s_1, \ldots, s_n, s_{n+1})$, for $n \geq 0$ to each function symbol. Function symbols of arity (s_1) are called *object constants* of sort (s_1). Predicate and function symbols are collectively called *nonlogical* or *special* symbols. On the other hand, the symbols $x^s, y^s, z^s, \ldots$ for variables of each sort s, *logical connectives* $\&, \rightarrow, \bot$ and *quantifiers* $\forall$ and $\exists$, are called *logical* symbols.

More complex syntactical structures (well-formed strings of symbols) are composed from symbols of the given predicate language in the following manner.

Definition 2.1.13 Each variable of sort s is a *term* of sort s and if $t_1, \ldots, t_n$ are terms of respective sorts $s_1, \ldots, s_n$ and $\mathrm{F} \in \mathbf{F}$ is a function symbol of arity $(s_1, \ldots, s_n, s_{n+1})$, then $\mathrm{F}(t_1, \ldots, t_n)$ is a term of sort s_{n+1}. *Atomic formulas* are of the form $\mathrm{P}(t_1, \ldots, t_n)$, where $\mathrm{P} \in \mathbf{P}$ is a predicate symbol of arity $(s_1, \ldots, s_n)$ and $t_1, \ldots, t_n$ are terms of respective sorts $s_1, \ldots, s_n$. Each atomic formula is a *formula* and if φ and ψ are formulas and x is a variable, then $(\varphi \,\&\, \psi), (\varphi \rightarrow \psi), \Delta\varphi, \bot, (\forall x)\varphi$ and $(\exists x)\varphi$ are formulas as well. Other (derived) logical connectives can be defined in the same manner as in the propositional case, see definition 2.1.1. We say that an occurrence of a variable x in a formula φ is *bound* if it is in the scope of some quantifier over x; otherwise it is called *free*. A formula φ is called a *sentence* if all occurrences of all variables in φ are bound. Finally, a term t is *substitutable* for the variable x in a formula $\varphi(x)$ if t is of the same sort as x and no variable occurring in t becomes bound in $\varphi(t)$. By $\mathrm{Form}(\mathcal{L})$ we denote the set of all (predicate) formulas over a given language

$\mathcal{L}$ and by Sent($\mathcal{L}$) we denote the set of all sentences over a given language $\mathcal{L}$.

Let us recall that in our case the symbol $=$ for crisp identity is considered as a logical symbol, which means that it can occur in formulas even though it is not listed among the symbols of a given language. Also, we will see that the symbol is treated in kind of a particular way. In the rest of this section, let $\mathcal{L} = \langle \mathbf{S}, \mathbf{P}, \mathbf{F}, \mathbf{A} \rangle$ be an arbitrary predicate language and $\mathbf{B} = \langle \langle \mathrm{B}, \leq \rangle, \min, \max, \Rightarrow, 0_{\mathbf{B}}, 1_{\mathbf{B}}, \Delta \rangle$ a $\mathrm{G}_{\boldsymbol{\Delta}}$-chain.

Definition 2.1.14 Let $\mathcal{L}$ be a predicate language and let $\mathbf{B}$ be a $\mathrm{G}_{\boldsymbol{\Delta}}$-chain. Then a $\mathbf{B}$-*structure* for $\mathcal{L}$ is a triple $\mathbf{M} = \langle (\mathrm{M}_s)_{s \in \mathbf{S}}, (\mathrm{P}_{\mathbf{M}})_{\mathrm{P} \in \mathbf{P}}, (\mathrm{F}_{\mathbf{M}})_{\mathrm{F} \in \mathbf{F}} \rangle$, where each M_s is a non-empty set, called a *domain* of sort s, $\mathrm{P}_{\mathbf{M}}$ is an n-ary $\mathbf{B}$-fuzzy relation on respective domains $\mathrm{M}_{s_1}, \ldots, \mathrm{M}_{s_n}$, i.e. $\mathrm{P}_{\mathbf{M}} \colon \mathrm{M}_{s_1} \times \ldots \times \mathrm{M}_{s_n} \longrightarrow \mathbf{B}$ for each predicate symbol $\mathrm{P} \in \mathbf{P}$ of arity $(s_1, \ldots, s_n)$ and $\mathrm{F}_{\mathbf{M}}$ is an n-ary function $\mathrm{F}_{\mathbf{M}} \colon \mathrm{M}_{s_1} \times \ldots \times \mathrm{M}_{s_n} \longrightarrow \mathrm{M}_{s_{n+1}}$ for each function symbol $\mathrm{F} \in \mathbf{F}$ of arity $(s_1, \ldots, s_n, s_{n+1})$. Fuzzy relations $\mathrm{P}_{\mathbf{M}}$ and functions $\mathrm{F}_{\mathbf{M}}$ (for each $\mathrm{P} \in \mathbf{P}$ and $\mathrm{F} \in \mathbf{F}$, respectively) are called *realizations* (of respective symbols in a given structure).

Definition 2.1.15 Let $\mathbf{M}$ be a $\mathbf{B}$-structure. Then an $\mathbf{M}$-*evaluation* is a mapping e assigning to each variable of each sort s an element from M_s. Let an $\mathbf{M}$-evaluation e be given. Then by $e(x/a)$ we mean the $\mathbf{M}$-evaluation which maps x of sort s to $a \in \mathrm{M}_s$ and coincides with e on all the other variables.

Definition 2.1.16 Let $\mathbf{M}$ be a $\mathbf{B}$-structure and e an $\mathbf{M}$-evaluation. Then *values* of the terms and *truth values* of the formulas in a $\mathbf{B}$-structure $\mathbf{M}$ for an $\mathbf{M}$-evaluation e are defined as follows:

- $\| x \|^{\mathbf{B}}_{\mathbf{M},e} = e(x)$
- $\| \mathrm{F}(t_1, \ldots, t_n) \|^{\mathbf{B}}_{\mathbf{M},e} = \mathrm{F}_{\mathbf{M}}(\| t_1 \|^{\mathbf{B}}_{\mathbf{M},e}, \ldots, \| t_n \|^{\mathbf{B}}_{\mathbf{M},e})$, for each $\mathrm{F} \in \mathbf{F}$
- $\| t_1 = t_2 \|^{\mathbf{B}}_{\mathbf{M},e} = 1_{\mathbf{B}}$ if $\| t_1 \|^{\mathbf{B}}_{\mathbf{M},e} = \| t_2 \|^{\mathbf{B}}_{\mathbf{M},e}$; $\| t_1 = t_2 \|^{\mathbf{B}}_{\mathbf{M},e} = 0_{\mathbf{B}}$ otherwise
- $\| \mathrm{P}(t_1, \ldots, t_n) \|^{\mathbf{B}}_{\mathbf{M},e} = \mathrm{P}_{\mathbf{M}}(\| t_1 \|^{\mathbf{B}}_{\mathbf{M},e}, \ldots, \| t_n \|^{\mathbf{B}}_{\mathbf{M},e})$, for each $\mathrm{P} \in \mathbf{P}$
- $\| \varphi \,\&\, \psi \|^{\mathbf{B}}_{\mathbf{M},e} = \min(\| \varphi \|^{\mathbf{B}}_{\mathbf{M},e}, \| \psi \|^{\mathbf{B}}_{\mathbf{M},e})$
- $\| \varphi \to \psi \|^{\mathbf{B}}_{\mathbf{M},e} = \| \varphi \|^{\mathbf{B}}_{\mathbf{M},e} \Rightarrow \| \psi \|^{\mathbf{B}}_{\mathbf{M},e}$
- $\| \Delta\varphi \|^{\mathbf{B}}_{\mathbf{M},e} = 1_{\mathbf{B}}$ if $\| \varphi \|^{\mathbf{B}}_{\mathbf{M},e} = 1_{\mathbf{B}}$; $\| \Delta\varphi \|^{\mathbf{B}}_{\mathbf{M},e} = 0_{\mathbf{B}}$ otherwise
- $\| \bot \|^{\mathbf{B}}_{\mathbf{M},e} = 0_{\mathbf{B}}$
- $\| \forall x^s \varphi \|^{\mathbf{B}}_{\mathbf{M},e} = \inf_{a \in M_s} \{ \| \varphi \|^{\mathbf{B}}_{\mathbf{M},e(x^s/a)} \}$
- $\| \exists x^s \varphi \|^{\mathbf{B}}_{\mathbf{M},e} = \sup_{a \in M_s} \{ \| \varphi \|^{\mathbf{B}}_{\mathbf{M},e(x^s/a)} \}$

If the infimum or supremum does not exist, then we take the values of the respective formulas as undefined. We say that a structure $\mathbf{M}$ is *safe* if all the needed infima and suprema exist, i.e. if $\| \varphi \|^{\mathbf{B}}_{\mathbf{M},e}$ is defined for each formula φ and each $\mathbf{M}$-evaluation e.

Definition 2.1.17 Let $\mathbf{M}$ be a safe $\mathbf{B}$-structure and let φ be a formula. Then the *truth value* of φ in $\mathbf{M}$ is defined as follows:

- $\| \varphi \|^{\mathbf{B}}_{\mathbf{M}} = \inf\{\| \varphi \|^{\mathbf{B}}_{\mathbf{M},e} \mid e \text{ an } \mathbf{M}\text{-evaluation}\}$

We say that a formula φ is *valid* in a **B**-structure **M**, denoted by $\mathbf{M} \models \varphi$, if $\| \varphi \|^{\mathbf{B}}_{\mathbf{M}} = 1_{\mathbf{B}}$ for each **M**-evaluation e. Finally, we say that a structure **M** is a *model* of a theory T (a set of sentences in a given language, i.e. $\mathrm{T} \subseteq \mathrm{Sent}(\mathcal{L})$), denoted by $\mathbf{M} \models \mathrm{T}$, if $\mathbf{M} \models \varphi$ holds for each $\varphi \in \mathrm{T}$.

Definition 2.1.18 The *axiomatic system* of the predicate first-order Gödel fuzzy logic with crisp identity and Δ operator consists of the following axioms:

- (G_Δ) All formulas resulting from the axioms of propositional Gödel fuzzy logic with Δ by substituting arbitrary $\mathcal{L}$-formulas for propositional variables
- ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ
- ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ
- ($\forall$2) $(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi)$, where x is not free in ψ
- ($\exists$2) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$, where x is not free in ψ
- ($\forall$3) $(\forall x)(\psi \vee \varphi) \to (\psi \vee (\forall x)\varphi)$, where x is not free in ψ
- (=1) $x = x$
- (=2) $x = y \to (\varphi(x) \to \varphi(y))$, where y is substitutable for x in φ

The deduction rules are modus ponens, necessitation and *generalization*:

- (Gen) From φ infer $(\forall x)\varphi$

Finally, let us present the strong completeness theorem for the logic $\mathrm{G}\forall_{\Delta=}$. For $\mathrm{G}\forall_{\Delta}$ the proof can be found again in [23] (Theorem 5.2.9, pp. 122), where a general result for a first-order version of an arbitrary schematic extension of the basic logic BL is considered. For the extension by equality cf. [11] (section 4.2) and [12] (section 5.3).

Theorem 2.1.19 (*Strong completeness* of $\mathrm{G}\forall_{\Delta=}$) Let T be a theory and let $\varphi \in \mathrm{Form}(\mathcal{L})$. Then the following three statements are equivalent:

- $\mathrm{T} \vdash_{\mathrm{G}\forall_{\Delta=}} \varphi$
- $\mathbf{M} \models \varphi$ for every G_Δ-chain **B** and every safe **B**-model **M** of T
- $\mathbf{M} \models \varphi$ for the standard G_Δ-chain **B** and every **B**-model **M** of T

Let us finish this part by giving a sample of formulas provable in $\mathrm{G}\forall_{\Delta=}$. The proofs can either be found in [16] and [23] or are easy corollaries of theorems proven there. Since theorem 2.1.20 will be used very frequently throughout the following chapter, we will only refer to its numbered formulas.

Theorem 2.1.20 Let $\mathcal{L}$ be a predicate language and let $\varphi, \psi, \chi, \delta, \gamma, \nu$ be $\mathcal{L}$-formulas such that φ does not contain x', χ does not contain x freely and let t be a term substitutable for x in φ and for both x and y in ψ. Then the following formulas are provable in $\mathrm{G}\forall_{\Delta=}$:

- (T1) $\varphi \to (\psi \to \varphi)$
- (T2) $(\varphi \to (\psi \to \delta)) \to (\psi \to (\varphi \to \delta))$
- (T3a) $\varphi \to \varphi$
- (T3b) $\varphi \leftrightarrow \varphi$
- (T4a) $(\varphi \to \psi) \to (\varphi \,\&\, \delta \to \psi \,\&\, \delta)$
- (T4b) $(\varphi \to \psi) \to (\varphi \vee \delta \to \psi \vee \delta)$
- (T4c) $(\varphi \to \psi) \to (\varphi \,\&\, \delta \to \psi)$
- (T5) $\varphi \,\&\, (\varphi \to \psi) \to \psi$
- (T6) $\varphi \to (\psi \to \varphi \,\&\, \psi)$
- (T7) $(\varphi \to \psi) \vee (\psi \to \varphi)$
- (T8) $((\varphi \to \delta) \,\&\, (\psi \to \delta)) \to (\varphi \vee \psi \to \delta)$
- (T9) $(\varphi \to \psi) \,\&\, (\delta \to \gamma) \to ((\psi \to \delta) \to (\varphi \to \gamma))$
- (T10) $(\varphi \to \psi) \,\&\, (\delta \to \gamma) \to (\varphi \,\&\, \delta \to \psi \,\&\, \gamma)$
- (T11) $\varphi \,\&\, \psi \to \varphi \vee \psi$
- (T12a) $\varphi \,\&\, (\psi \vee \delta) \leftrightarrow (\varphi \,\&\, \psi) \vee (\varphi \,\&\, \delta)$
- (T12b) $\varphi \vee (\psi \,\&\, \delta) \leftrightarrow (\varphi \vee \psi) \,\&\, (\varphi \vee \delta)$
- (T13a) $(\varphi \,\&\, \psi) \,\&\, \delta \leftrightarrow \varphi \,\&\, (\psi \,\&\, \delta)$
- (T13b) $(\varphi \vee \psi) \vee \delta \leftrightarrow \varphi \vee (\psi \vee \delta)$
- (T14a) $\varphi \,\&\, \psi \leftrightarrow \psi \,\&\, \varphi$
- (T14b) $\varphi \vee \psi \leftrightarrow \psi \vee \varphi$
- (T15) $\varphi \leftrightarrow \varphi \,\&\, \varphi$
- (T16) $\varphi \to \neg\neg\varphi$
- (T17) $\neg\varphi \to (\varphi \to \psi)$
- (T18) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
- (T19a) $\neg(\varphi \,\&\, \psi) \leftrightarrow \neg\varphi \vee \neg\psi$
- (T19b) $\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \,\&\, \neg\psi$
- (T20a) $\varphi \,\&\, \top \leftrightarrow \varphi$
- (T20b) $\varphi \vee \top \leftrightarrow \top$
- (T21a) $\varphi \,\&\, \bot \leftrightarrow \bot$
- (T21b) $\varphi \vee \bot \leftrightarrow \varphi$

- (T=1) $x = y \vee \neg(x = y)$
- (T=2a) $x = y \to y = x$
- (T=2b) $x = y \leftrightarrow y = x$
- (T=3) $x = y \ \&\ y = z \to x = z$
- (T=4) $x = y \ \&\ \varphi(x) \to \varphi(y)$
- (T=5) $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$
- (T=6) $(\exists x)(x = t \ \&\ \varphi(x)) \leftrightarrow \varphi(t)$
- (T=7) $(\forall x)(\exists y)(x = y)$

- (TΔ1) $\Delta(\varphi \ \&\ \psi) \leftrightarrow \Delta\varphi \ \&\ \Delta\psi$
- (TΔ2) $(\forall x)\Delta\varphi \leftrightarrow \Delta(\forall x)\varphi$
- (TΔ3) $(\exists x)\Delta\varphi \to \Delta(\exists x)\varphi$

- (T∀1a) $\chi \leftrightarrow (\forall x)\chi$
- (T∀1b) $\chi \leftrightarrow (\exists x)\chi$
- (T∀2a) $(\forall x)\varphi(x) \leftrightarrow (\forall x')\varphi(x')$
- (T∀2b) $(\exists x)\varphi(x) \leftrightarrow (\exists x')\varphi(x')$
- (T∀3a) $(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$
- (T∀3b) $(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$
- (T∀4) $(\exists x)(\forall y)\varphi \to (\forall y)(\exists x)\varphi$
- (T∀5a) $(\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$
- (T∀5b) $(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$
- (T∀6a) $(\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$
- (T∀6b) $(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$
- (T∀7) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
- (T∀8) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
- (T∀9) $(\forall x)\varphi \ \&\ (\exists x)\psi \to (\exists x)(\varphi \ \&\ \psi)$
- (T∀10a) $(\forall x)(\chi \ \&\ \varphi) \leftrightarrow \chi \ \&\ (\forall x)\varphi$
- (T∀10b) $(\exists x)(\chi \ \&\ \varphi) \leftrightarrow \chi \ \&\ (\exists x)\varphi$

- (T∀11a) $(\forall x)(\varphi \mathbin{\&} \psi) \leftrightarrow (\forall x)\varphi \mathbin{\&} (\forall x)\psi$
- (T∀11b) $(\exists x)(\varphi \mathbin{\&} \psi) \to (\exists x)\varphi \mathbin{\&} (\exists x)\psi$
- (T∀12) $(\forall x)\varphi \vee (\forall x)\psi \to (\forall x)(\varphi \vee \psi)$
- (T∀13) $(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$
- (T∀14) $(\exists x)(\chi \vee \varphi) \leftrightarrow \chi \vee (\exists x)\varphi$
- (T∀15a) $(\forall x)(\varphi \to (\forall y)(\psi \to \nu)) \leftrightarrow (\forall y)(\psi \to (\forall x)(\varphi \to \nu))$
- (T∀15b) $(\exists x)(\varphi \mathbin{\&} (\exists y)(\psi \mathbin{\&} \nu)) \leftrightarrow (\exists y)(\psi \mathbin{\&} (\exists x)(\varphi \mathbin{\&} \nu))$
- (T∀16) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$
- (T∀17) $\neg(\exists x)\varphi \leftrightarrow (\forall x)\neg\varphi$
- (T∀18) $\psi(t,t) \to (\exists x)\psi(x,t)$

The following theorem will also be useful in most of proofs in the following chapter. Thus, as in the previous case, we will use it with no explicit reference to its number. The theorem holds in any extension of the basic logic BL, see [12] (Theorem 5.1.5).

Theorem 2.1.21 (*Intersubstitutivity*) Let χ be a formula and let χ' be a formula resulting from χ by replacing some occurences of its subformula φ by a formula ψ. Then the following holds:

- (Cong) $\varphi \leftrightarrow \psi \vdash_{\mathrm{G}\forall_\Delta} \chi \leftrightarrow \chi'$

2.2 Fuzzy set-theoretical basics

Crisp sets can be identified with their *characteristic functions*, i.e. functions from some crisp universe (or domain) U to the set $\{0,1\}$ of truth values. A natural generalization of this idea, which has been done by Zadeh in [43], consists in the consideration of the real unit interval $[0,1]$ as a structure of truth values. This has led to the concept of (traditional) *fuzzy set*, i.e. a function from some crisp universe U to the real unit interval $[0,1]$. Later on, Goguen [17], [18] generalized Zadeh's idea by considering more general structures of truth values. Usually, various kinds of lattices are considered as targets of characteristic functions, which are identified with fuzzy sets. So in the classical (traditional) case, i.e. on the background of classical logic, a fuzzy set A is (identified with) a mapping $A\colon \mathrm{U} \longrightarrow \mathrm{L}$, where L is a lattice of truth values. Similarly, a (traditional n-ary) *fuzzy relation* is a mapping $R\colon \mathrm{U}^n \longrightarrow \mathrm{L}$.

But let us once again emphasize that our approach is essentially different from the traditional one as the logic in the background of the theory is fuzzy, in our case $\mathrm{G}\forall_{\Delta=}$. So all the concepts within the theory, except those which are defined as crisp, become fuzzy too. So one of the important and crucial features of fuzzy set theory built over a fuzzy logic is that there is no need to directly refer to truth values anymore and thus to have variables for truth values in its language, as opposed to the case of traditional fuzzy mathematics. For example, the degree to which x is an element of a fuzzy set A is simply expressed by the atomic formula $x \in A$. Thus most of the expressions look essentially the same as in the classical case, i.e. when set theory is built over classical logic.

A detailed examination of fundamental concepts of set theory as built up over G$\forall$ can be found in [2]. For our purposes we will consider a G$\forall_{\Delta=}$-variant of Fuzzy Class Theory introduced in [5]. Fuzzy Class Theory (FCT for short) can be characterized as Henkin-style higher-order fuzzy logic. In our case it is a theory in multi-sorted G$\forall_{\Delta=}$. We will denote the theory by GFCT. So let us give a brief overview of some necessary basic concepts of GFCT in this section.

Definition 2.2.1 The language of GFCT contains:

- Variables for *atomic objects* from some crisp universe U, denoted by $x, y, z, \ldots, x_1, x_2, \ldots, x', x'', \ldots$
- Variables for *fuzzy sets of atomic objects*, also called fuzzy sets of the first order, denoted by $A, B, C, \ldots$
- Variables for *fuzzy sets of fuzzy sets of atomic objects*, called fuzzy sets of the second order, denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, A^{(2)}, B^{(2)}, C^{(2)}, \ldots$
- Etc., in general variables for *fuzzy sets of the n-th order*, denoted by $A^{(n)}, B^{(n)}, C^{(n)}, \ldots$ Each sort of variables of the n-th order also subsumes (for the subsumption of sorts see [5]) sorts for *k-tuples of variables of the n-th order*, for each $k \in \mathbf{N}$.
- The binary *membership* predicate symbol $\in$ between objects of n-th order and $(n+1)$-st order.
- *Comprehension terms* $\{x \mid \varphi(x)\}$ of order $(n+1)$, for any formula $\varphi(x)$ of GFCT and any variable x of order n.
- Terms $\langle x_1, \ldots, x_k \rangle$ of order n for *k-tuples* of objects $x_1, \ldots, x_k$ of order n.

The axiomatic system of GFCT consists (besides the logical axioms of G$\forall_{\Delta=}$) of the following axioms:

- (The *comprehension* axioms) $x \in \{y \mid \varphi(y)\} \leftrightarrow \varphi(x)$, for each variables x, y of the same order and each formula φ possibly containing additional variables of any orders besides y.
- (The *extensionality* axioms) $(\forall x)\Delta(x \in A \leftrightarrow x \in B) \to A = B$, for variables A, B of each order.
- (The *tuple-identity* axioms) $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \to x_1 = y_1 \;\&\; \ldots \;\&\; x_k = y_k$, for each $k \geq 0$ and each order of the variables.

The comprehension axioms express the fact that each property definable by a formula φ of GFCT delimits a fuzzy set $\{x \mid \varphi(x)\}$ (of order $(n+1)$, where n is the order of x). Then $A = \{x \mid \varphi(x)\}$ means that $\Delta(x \in A \leftrightarrow \varphi(x))$ holds for all x. GFCT can alternatively be formulated with the comprehension axioms of the form $(\exists A)\Delta(\forall x)(x \in A \leftrightarrow \varphi(x))$. The above-introduced comprehension terms then arise as the *Skolem functions* of these axioms, for details see [7] (Theorem 5.1.7).

The *intended models* of GFCT, whose construction ensures the consistency of GFCT, are the systems of all fuzzy sets and all k-ary fuzzy relations (for each $k \in \mathbf{N}$) of all finite orders over a given crisp universe U and a G_Δ-chain $\mathbf{B}$. More specifically, fuzzy sets of the first order are then all (crisp) functions from U to $\mathbf{B}$, fuzzy sets of the second order are all (crisp) functions from the set of all fuzzy sets of the first order to $\mathbf{B}$ and so on for fuzzy sets of arbitrary order. If the real unit interval $[0,1]$ is taken as the set of truth values, the models are called *standard models*. Models containing all fuzzy sets, i.e. fuzzy sets of all orders, are called *full models*. Full standard models are called *Zadeh models*, since they correspond exactly to Zadeh's original notion of fuzzy set, see [43]. For a more detailed characterization of models of FCT we refer reader to section 5.5.2 of [7].

For atomic objects, i.e. elements of the crisp universe U, $x = y$ holds in a model if and only if x and y denote the same object in the universe U of the model. For fuzzy sets of arbitrary orders, $A = B$ holds if and only if their characteristic functions are identical. Right to left implication of this equivalence is exactly what the axiom of extensionality expresses, while the converse implication follows immediately from the axioms of identity. Finally, the identity of membership degrees $x \in A$ and $y \in B$ is expressible by the formula $\Delta(x \in A \leftrightarrow y \in B)$. Identical atomic objects, fuzzy sets and membership degrees are freely intersubstitutable within formulas of GFCT.

Further on we will use common abbreviations concerning comprehension terms like $\{\langle x,y\rangle \mid \varphi(x,y)\}$ meaning $\{z \mid (\exists x,y)(z = \langle x,y\rangle \mathbin{\&} \varphi(z))\}$, $\{x \in A \mid \varphi(x)\}$ meaning $\{x \mid x \in A \mathbin{\&} \varphi(x)\}$ and $\{A \subseteq B \mid \varphi(A)\}$ meaning $\{A \mid A \subseteq B \mathbin{\&} \varphi(A)\}$. Let us also remark that within the whole work the words 'set' and 'class' can be freely interchanged.

Definition 2.2.2 The elementary fuzzy set constants, operations, relations between fuzzy sets and properties of fuzzy sets are in GFCT defined as follows:

- (*empty set*) $\emptyset \stackrel{\mathrm{df}}{=} \{x \mid \bot\}$
- (*universal class*) $V \stackrel{\mathrm{df}}{=} \{x \mid \top\}$
- (*singleton*) $\{a\} \stackrel{\mathrm{df}}{=} \{x \mid x = a\}$
- (*intersection*) $A \cap B \stackrel{\mathrm{df}}{=} \{x \mid x \in A \mathbin{\&} x \in B\}$
- (*union*) $A \cup B \stackrel{\mathrm{df}}{=} \{x \mid x \in A \vee x \in B\}$
- (*intersection of a class of classes*) $\bigcap \mathcal{A} \stackrel{\mathrm{df}}{=} \{x \mid (\exists A)(A \in \mathcal{A} \mathbin{\&} x \in A)\}$
- (*union of a class of classes*) $\bigcup \mathcal{A} \stackrel{\mathrm{df}}{=} \{x \mid (\forall A)(A \in \mathcal{A} \rightarrow x \in A)\}$
- (*complement*) $\backslash A \stackrel{\mathrm{df}}{=} \{x \mid \neg(x \in A)\}$
- (*difference*) $A \setminus B \stackrel{\mathrm{df}}{=} \{x \mid x \in A \mathbin{\&} \neg(x \in B)\}$
- (*inclusion*) $A \subseteq B \stackrel{\mathrm{df}}{\equiv} (\forall x)(x \in A \rightarrow x \in B)$

- (*bi-inclusion*) $A \approxeq B \stackrel{\text{df}}{\equiv} (\forall x)(x \in A \leftrightarrow x \in B)$
- (*power set*) $\mathrm{Pow}(A) \stackrel{\text{df}}{=} \{X \mid X \subseteq A\}$
- (*crispness*) $\mathrm{Crisp}(A) \stackrel{\text{df}}{\equiv} (\forall x)\Delta(x \in A \vee \neg(x \in A))$
- (*fuzziness*) $\mathrm{Fuzzy}(A) \stackrel{\text{df}}{\equiv} \neg\mathrm{Crisp}(A)$
- (*E-extensionality*) $\mathrm{Ext}_E(A) \stackrel{\text{df}}{\equiv} (\forall x, y)(\langle x, y\rangle \in E \;\&\; x \in A \to y \in A)$

Definition 2.2.3 The elementary fuzzy relational operations are in GFCT defined as follows:

- (*Cartesian product*) $A \times B \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid x \in A \;\&\; x \in B\}$
- (*domain*) $\mathrm{Dom}(R) \stackrel{\text{df}}{=} \{x \mid (\exists y)(\langle x, y\rangle \in R)\}$
- (*range*) $\mathrm{Rng}(R) \stackrel{\text{df}}{=} \{y \mid (\exists x)(\langle x, y\rangle \in R)\}$
- (*composition*) $R \circ S \stackrel{\text{df}}{=} \{\langle x, z\rangle \mid (\exists y)(\langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in S)\}$
- (*inverse relation*) $R^{-1} \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid \langle y, x\rangle \in R\}$
- (*identity relation* on A) $\mathrm{Id}_A \stackrel{\text{df}}{=} \{\langle x, x'\rangle \mid x \in A \;\&\; x' \in A \;\&\; x = x'\}$

Definition 2.2.4 The elementary properties of fuzzy relations are in GFCT defined as follows:

- (*reflexivity* on A) $\mathrm{Refl}_A(R) \stackrel{\text{df}}{\equiv} (\forall x)(x \in A \to \langle x, x\rangle \in R)$
- (*irreflexivity* on A) $\mathrm{Irrefl}_A(R) \stackrel{\text{df}}{\equiv} (\forall x)(x \in A \to \neg(\langle x, x\rangle \in R))$
- (*symmetry* on A) $\mathrm{Sym}_A(R) \stackrel{\text{df}}{\equiv} (\forall x, y)(x \in A \;\&\; y \in A \to (\langle x, y\rangle \in R \to \langle y, x\rangle \in R))$
- (*asymmetry* on A) $\mathrm{ASym}_A(R) \stackrel{\text{df}}{\equiv} (\forall x, y)(x \in A \;\&\; y \in A \to \neg(\langle x, y\rangle \in R \;\&\; \langle y, x\rangle \in R))$
- (*transitivity* on A) $\mathrm{Trans}_A(R) \stackrel{\text{df}}{\equiv} (\forall x, y, z)(x \in A \;\&\; y \in A \;\&\; z \in A \to (\langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in R \to \langle x, z\rangle \in R))$
- (*totality* on A) $\mathrm{Total}_A(R) \stackrel{\text{df}}{\equiv} (\forall x)(x \in A \to (\exists y)(y \in B \;\&\; \langle x, y\rangle \in R))$, where $R \subseteq A \times B$
- (*surjectivity* on B) $\mathrm{Sur}_B(R) \stackrel{\text{df}}{\equiv} (\forall y)(y \in B \to (\exists x)(x \in A \;\&\; \langle x, y\rangle \in R))$, where $R \subseteq A \times B$
- (*E-functionality*) $\mathrm{Fnc}_E(R) \stackrel{\text{df}}{\equiv} (\forall x, y, y')(\langle x, y\rangle \in R \;\&\; \langle x, y'\rangle \in R \to \langle y, y'\rangle \in E)$
- (*E-injectivity*) $\mathrm{Inj}_E(R) \stackrel{\text{df}}{\equiv} (\forall x, x', y)(\langle x, y\rangle \in R \;\&\; \langle x', y\rangle \in R \to \langle x, x'\rangle \in E)$

- (*E-extensionality*) $\mathrm{Ext}_E(R) \stackrel{\mathrm{df}}{\equiv} (\forall x, x', y)(\langle x, x'\rangle \in E \;\&\; \langle x, y\rangle \in R \rightarrow \langle x', y\rangle \in R)$

The concept of identity relation as well as the concepts of reflexivity, irreflexivity, symmetry, asymmetry, transitivity, totality and surjectivity are introduced here as relativized to a (fuzzy) set A. Special cases of these concepts are given by $A = V$, in which we let the index V out, i.e. we write just Id instead of Id_V, for example. An important special case of E-functionality and E-injectivity is the one where E is the identity relation, i.e. $\mathrm{Fnc}_{\mathrm{Id}}(R) \stackrel{\mathrm{df}}{\equiv} (\forall x, y, y')(\langle x, y\rangle \in R \;\&\; \langle x, y'\rangle \in R \rightarrow y = y')$ and $\mathrm{Inj}_{\mathrm{Id}}(R) \stackrel{\mathrm{df}}{\equiv} (\forall x, x', y)(\langle x, y\rangle \in R \;\&\; \langle x', y\rangle \in R \rightarrow x = x')$, respectively.

Taking E-functionality together with E-injectivity, totality and surjectivity as requirements to binary fuzzy relations into consideration we obtain fuzzy generalization of the important classical concept of *bijection* (or *bijective function*).

Definition 2.2.5 In GFCT we define the following compound property of binary fuzzy relation $R \subseteq A \times B$ as follows:

- (*E-bijection*) $\mathrm{Bi}_E(R) \stackrel{\mathrm{df}}{\equiv} \mathrm{Fnc}_E(R) \;\&\; \mathrm{Inj}_E(R) \;\&\; \mathrm{Total}(R) \;\&\; \mathrm{Sur}(R)$

As in the previous definiton probably the most important special case of E-bijection is the one concerning Id-functionality and Id-injectivity of R, i.e. $\mathrm{Bi}_{\mathrm{Id}}(R)$.

Let us recall that a crisp binary relation $R \subseteq X \times X$ is called an *equivalence relation* on a set X if it is reflexive, symmetric and transitive on X. A fuzzy counterpart of the concept of an equivalence relation is called a *similarity* (on a fuzzy set X).

Definition 2.2.6 The following compound property of a binary fuzzy relation $R \subseteq X \times X$ is in GFCT defined as follows:

- (*similarity*) $\mathrm{Sim}_X(R) \stackrel{\mathrm{df}}{\equiv} \mathrm{Refl}_X(R) \;\&\; \mathrm{Sym}_X(R) \;\&\; \mathrm{Trans}_X(R)$

2.3 Category-theoretical basics

Categories may be considered, from certain point of view, as universes for specific kinds of mathematical discourse. All these universes have something in common. Generally they are determined by two specifications. Firstly, it has to be specified what constitutes objects of the universe of discourse and secondly what constitutes morphisms (often called simply arrows) that link these objects. Morphisms must preserve a certain kind of stucture, which is significant for objects of a given universe. For example, a universe of discourse of linear algebra consists of vector spaces representing objects and linear transformations (i.e. homomorphisms of vector spaces) representing structure-preserving morphisms between vector spaces, a universe of discourse of topology consists of topological spaces as objects and continuous maps as morphisms between them and so on. So let us start by giving a precise general definition of these universes of discourse–categories. The following definitions are taken from [1], [20] and [31].

Definition 2.3.1 A *category* $\mathcal{C}$ is a triple $\mathcal{C} = \langle \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}(\_,\_), \circ \rangle$ consisting of:

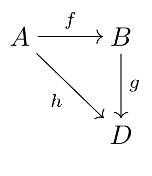
- A class $\mathrm{Ob}(\mathcal{C})$, whose members are called $\mathcal{C}$*-objects* and are usually denoted by $A, B, C, \ldots$
- For each pair (A, B) of $\mathcal{C}$-objects a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose members are called $\mathcal{C}$*-morphisms* from A to B and are usually denoted by $f, g, h, \ldots,$ $R, S, T, \ldots$ The sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$, for each pair (A, B) of $\mathcal{C}$-objects, are pairwise disjoint.
- A binary operation $\circ$ from $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C)$ to $\mathrm{Hom}_{\mathcal{C}}(A, C)$, for each $A, B, C \in \mathrm{Ob}(\mathcal{C})$, called a *composition* of morphisms. If f and g are composable $\mathcal{C}$-morphisms, i.e. if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, then their composition (an element of $\mathrm{Hom}_{\mathcal{C}}(A, C)$) is denoted by $f \circ g$. The operation satisfies the following two conditions:
  - (i) (*associativity*) If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ are (composable) $\mathcal{C}$-morphisms, then $(f \circ g) \circ h = f \circ (g \circ h)$ holds.
  - (ii) (*existence of identities*) For each $\mathcal{C}$-object B, there exists a $\mathcal{C}$-morphism $1_B \in \mathrm{Hom}_{\mathcal{C}}(B, B)$, called the *identity morphism* on B, such that $f \circ 1_B = f$ and $1_B \circ g = g$ hold for all $\mathcal{C}$-morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then A is called the *domain* of f, denoted by $A = \mathrm{dom}(f)$, and B is called the *codomain* of f, denoted by $B = \mathrm{cod}(f)$. The class of all $\mathcal{C}$-morphisms, denoted by $\mathrm{Hom}(\mathcal{C})$, is defined as the union of all the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$, for each $A, B \in \mathrm{Ob}(\mathcal{C})$, i.e. $\mathrm{Hom}(\mathcal{C}) = \bigcup_{A,B \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 2.3.2 Let us give an example of two categories, on which systems of morphisms examined in chapter 3 are loosely based.

- Category Rel consists of sets as objects and binary relations as morphisms, i.e. $\mathrm{Hom}_{\mathrm{Rel}}(A, B) = \{R \mid R \subseteq A \times B\}$, for each pair of objects.
- Category Set consists of sets as objects and set functions as morphisms, i.e. $\mathrm{Hom}_{\mathrm{Set}}(A, B) = \{R \mid R \subseteq A \times B \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(R)\}$, for each pair of objects.

One of the most significant features of category theory consists in its simple, illustrative and thus in some sense expressively effective language. Definitions, theorems and other statements are usually expressed in a graphical form of (commutative) diagrams consiting of objects and morphisms between them. Thus, for example, the fact that $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is expressed by $f\colon A \longrightarrow B$ or even better by $A \xrightarrow{\;f\;} B$. The fact that $h = f \circ g$ can be expressed by the following picture - *diagram*



along with a statement that this diagram *commutes*. In the following definitions we will need to express the fact that there exists exactly one morphism such that some condition holds. The uniqueness of such morphism will be graphically expressed by a dashed arrow $A \dashrightarrow^{f} B$.

Also a more global viewpoint, in which categories themselves are treated as 'objects' in a kind of a 'metacategory', can be considered. In this case 'morphisms' between categories are called functors. Let us be more precise.

Definition 2.3.3 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then a mapping F is called a *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ if it assigns to each $\mathcal{C}$-object A a $\mathcal{D}$-object $F(A)$ and to each $\mathcal{C}$-morphism $f\colon A \longrightarrow B$ a $\mathcal{D}$-morphism $F(f)\colon F(A) \longrightarrow F(B)$ in such a way that the following conditions hold:

- F preserves composition of morphisms: $F(f \circ g) = F(f) \circ F(g)$, for all composable $\mathcal{C}$-morphisms f, g
- F preserves identity morphisms: $F(1_A) = 1_{F(A)}$ for each $\mathcal{C}$-object A

Before we start to present definitions of category-theoretical concepts, it is useful to mention here an important feature of categories. For each category $\mathcal{C}$ there exists its *dual* (or *opposite*) category $\mathcal{C}^{\text{op}} = \langle \text{Ob}(\mathcal{C}^{\text{op}}), \text{Hom}_{\mathcal{C}^{\text{op}}}, \circ^{\text{op}}, 1^{\text{op}} \rangle$, given by the following conditions:

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$, for each $A, B \in \text{Ob}(\mathcal{C}^{\text{op}})$
- $f \circ^{\text{op}} g = g \circ f$, for each $f, g \in \text{Hom}_{\mathcal{C}^{\text{op}}}$
- $1_A^{\text{op}} = 1_A$, for each $A \in \text{Ob}(\mathcal{C}^{\text{op}})$

Thanks to this fact, for each concept or construction described by a statement (in the language of categories) Σ there naturally exists its dual concept or construction described by a statement Σ^{op}, which is obtained from the original statement by replacing 'dom' by 'cod' (and conversely) and '$h = f \circ g$' by '$h = g \circ f$', which means that all morphisms and composites referred to by Σ are of the opposite direction in Σ^{op}. Dual concepts or constructions are usually denoted by the same name as the original concept but with the prefix 'co'. Thus we have for example dual concepts like 'cone' and 'cocone', 'limit' and 'colimit', 'product' and 'coproduct', 'equalizer' and 'coequalizer' etc. On the other hand there are several dual concepts not following that rule, for example 'monomorphism' and 'epimorphism', 'initial object' and 'terminal object', 'pullback' and 'pushout' etc.

Definition 2.3.4 Let $f\colon A \longrightarrow B$ be a morphism. We say that f is an *isomorphism* if there exists a morphism $g\colon B \longrightarrow A$, such that $f \circ g = 1_A$ and $g \circ f = 1_B$ hold. Isomorphisms are usually denoted by $f\colon A \cong B$.

Remark 2.3.5 A morphism $g\colon B \longrightarrow A$ from the previous definition can be at most one in fact. Thus, when it exists, it is called the *inverse* of $f\colon A \longrightarrow B$, and denoted by $f^{-1}\colon B \longrightarrow A$. It is defined by the conditions $f \circ f^{-1} = 1_A$ and

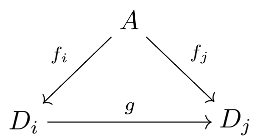
$f^{-1} \circ f = 1_B$.

We say that two objects are isomorphic, denoted by $A \cong B$, if there exists an isomorphism between them. Isomorphic objects can be considered as 'essentially the same', i.e. having exactly the same structure recognizable by means of particular mathematical theory. So isomorphic objects are virtually indistinguishable in terms of that theory. For example, isomorphic groups look essentially the same from the point of view of theory of groups, they differ only by 'names' of their elements. Thus the most interesting and important properties and constructions examined within the particular theory are those, which are invariant with respect to isomorphisms of that theory. So most of the concepts and constructions of category theory are not defined or performed uniquely in fact, but 'uniquely up to isomorphism', which is fully sufficient and even more desirable.

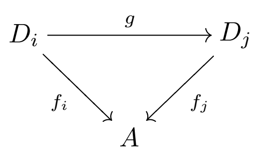
In category theory, there is a family of definitions and constructions, which are based on the same specific condition. The entity in question, which is defined or constructed, has the certain property canonically, which now means that any other entity with such property factors through it. In this situation, we say that the entity in question has a *universal property* or that it is *universal* amongst all the other entities with such property. But even before we present several definitions of this kind, we need to introduce important concepts of a diagram in a category and its limit.

Definition 2.3.6 Let $\mathcal{I}$ and $\mathcal{C}$ be categories. Then a *diagram* in a category $\mathcal{C}$ is a functor $D: \mathcal{I} \longrightarrow \mathcal{C}$, where a category $\mathcal{I}$ is called an *index category* (or a *scheme*) of a diagram D. In words, a diagram in a category $\mathcal{C}$ is simply a collection of $\mathcal{C}$-objects $A_i, A_j, A_k, \ldots$, together with some $\mathcal{C}$-morphisms $f: A_i \longrightarrow A_j$ between some of these objects. Thus there can possibly be more then one morphism between a given pair of objects, possibly none.

Definition 2.3.7 Let $D: \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Then a *cone* over (for) a diagram D consists of a $\mathcal{C}$-object A, called a vertex, together with a $\mathcal{C}$-morphism $f_i: A \longrightarrow D_i$ for each object D_i in D, such that $f_i \circ g = f_j$ holds, whenever $g: D_i \longrightarrow D_j$ is a morphism in the diagram D, i.e. such that

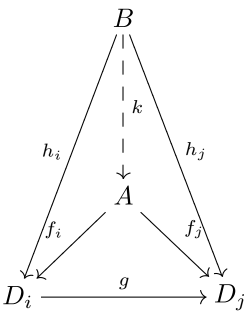


commutes. Cones over a diagram D are usually denoted by $\{f_i: A \longrightarrow D_i\}$. Dually, a *cocone* under a diagram D consists of a C-object A together with a C-morphism $f_i: D_i \longrightarrow A$ for each object D_i in D, such that $g \circ f_j = f_i$, whenever $g: D_i \longrightarrow D_j$ is a morphism in the diagram D, i.e. such that

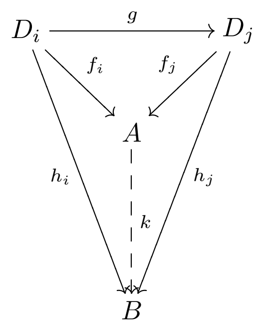


commutes. Cocones under a diagram D are usually denoted by $\{f_i\colon D_i \longrightarrow A\}$.

Definition 2.3.8 Let $D\colon \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram in a category $\mathcal{C}$ and let $\{f_i\colon A \longrightarrow D_i\}$ be a cone over D. We say that $\{f_i\colon A \longrightarrow D_i\}$ is a *limit* of a diagram D if for any other cone $\{h_i\colon B \longrightarrow D_i\}$ over D there exists exactly one morphism $k\colon B \longrightarrow A$ such that $k \circ f_i = h_i$ holds for all $i \in I$, i.e. such that



commutes. Dually, let $D\colon \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram in a category $\mathcal{C}$ and let $\{f_i\colon D_i \longrightarrow A\}$ be a cocone under D. We say that $\{f_i\colon D_i \longrightarrow A\}$ is a *colimit* of a diagram D if for any other cocone $\{h_i\colon D_i \longrightarrow B\}$ over D there exists exactly one morphism $k\colon A \longrightarrow B$ such that $f_i \circ k = h_i$ holds for all $i \in I$, i.e. such that



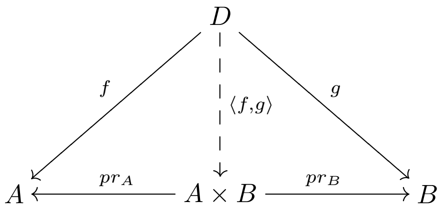
commutes.

Limits and colimits are determined uniquely up to isomorphism.

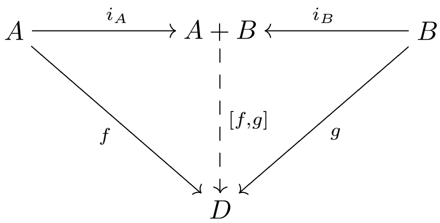
Let us now present several definitions, in which universal property plays the crucial role. All of these are in fact limits or colimits of very simple diagrams.

Definition 2.3.9 Let T be an object of a category $\mathcal{C}$. We say that T is a *terminal object* of $\mathcal{C}$ if for each $\mathcal{C}$-object A there is exactly one morphism from A to T. Dually, we say that a $\mathcal{C}$-object I is an *initial object* of $\mathcal{C}$ if for each $\mathcal{C}$-object A there is exactly one morphism from I to A. Such objects which are both terminal and initial are called *zero objects*. In other words, a terminal object is a limit and an initial object is a colimit of an empty diagram, i.e. of a diagram such that the index category consists of no objects (and thus no morphisms between them).

Definition 2.3.10 Let A and B be $\mathcal{C}$-objects. Then a *product* of A and B is a $\mathcal{C}$-object $A \times B$ together with a pair of $\mathcal{C}$-morphisms $pr_A\colon A \times B \longrightarrow A$ and $pr_B\colon A \times B \longrightarrow B$ such that for any other $\mathcal{C}$-object D together with a pair of $\mathcal{C}$-morphisms $f\colon D \longrightarrow A$ and $g\colon D \longrightarrow B$ there exists exactly one $\mathcal{C}$-morphism $\langle f, g\rangle\colon D \longrightarrow A \times B$ such that $\langle f, g\rangle \circ pr_A = f$ and $\langle f, g\rangle \circ pr_B = g$ hold, i.e. such that



commutes. Dually, let A and B be $\mathcal{C}$-objects. Then a *coproduct* (or a *sum*) of A and B is a $\mathcal{C}$-object $A+B$ together with a pair of $\mathcal{C}$-morphisms $i_A\colon A \longrightarrow A+B$ and $i_B\colon B \longrightarrow A+B$ such that for any other $\mathcal{C}$-object D together with a pair of $\mathcal{C}$-morphisms $f\colon A \longrightarrow D$ and $g\colon B \longrightarrow D$ there exists exactly one $\mathcal{C}$-morphism $[f, g]\colon A + B \longrightarrow D$ such that $i_A \circ [f, g] = f$ and $i_B \circ [f, g] = g$ hold, i.e. such that



commutes. In other words, product is a limit and coproduct is a colimit of a diagram such that the index category consists of just two unconnected objects.

Chapter 3

Systems of morphisms

In this main chapter we present and examine several systems of morphisms (candidates for categories, or protocategories) as built up over GFCT (see section 2.2). We present only very basic GFCT-based analogues of systems and concepts, which are well-known in the classical setting, i.e. when built up over classical first-order logic. Since our background logic $G\forall_{\Delta=}$ is weaker then the classical one, we must review even the very basic and well-known classical proofs in order to make sure whether they work also in GFCT or not.

The chapter is devided into four sections. In the first section an initial definition of a system of morphisms is presented, as well as several auxiliary lemmas, which are used subsequently in the following sections. In the second and the third section two sorts of systems of morphisms (loosely based on the crisp categories Rel and Set, respectively) are presented and examined. In the fourth and final section we present some results obtained by a more detailed examination of one of the systems presented in the second section.

3.1 Initial steps

The following definition of a system of morphisms serves as a basis on which particular systems in the following sections are examined.

Definition 3.1.1 A *system of morphisms* $\mathcal{C}$ consists of:

- A crisp set $\mathrm{POb}(\mathcal{C})$ of potential objects.
- A fuzzy set $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{POb}(\mathcal{C})$ of objects, usually denoted by $A, B, C, \ldots$
- A crisp set $\mathrm{PHom}(\mathcal{C})$ of potential morphisms.
- A fuzzy set $\mathrm{Hom}_{\mathcal{C}}(f, A, B) \subseteq \mathrm{PHom}(\mathcal{C}) \times \mathrm{POb}(\mathcal{C}) \times \mathrm{POb}(\mathcal{C})$ of morphisms from A to B, for each pair $A, B \in \mathrm{POb}(\mathcal{C})$ of potential objects, usually denoted by $f, g, h, \ldots, R, S, T, \ldots$
- A ternary fuzzy relation $\circ_{\mathcal{C}} \subseteq \mathrm{PHom}(\mathcal{C}) \times \mathrm{PHom}(\mathcal{C}) \times \mathrm{PHom}(\mathcal{C})$ of composition of potential morphisms.

In contrast to classical categories we do not put down any other requirements to just defined systems of morphisms. Properties which correspond to the definitional requirements of a classical category (see definition 2.3.1) will be proven for some systems; for others they may hold in various degrees.

Remark 3.1.2 For the sake of possibility of comparison let us present (in a slightly adjusted form due to differencies in denotation) the definition of a crucial concept of an L-fuzzy category as defined by Šostak in [42]. In the following definition the truth degrees structure L always denotes a GL-monoid, for details see e.g. [42], pp. 409.

- An *L-fuzzy category* $\mathcal{C}$ ([42], pp. 411) consists of:
  - A class $\mathrm{Ob}(\mathcal{C})$ of potential objects.
  - An L-fuzzy subclass ω of $\mathrm{Ob}(\mathcal{C})$, i.e. a mapping $\omega\colon \mathrm{Ob}(\mathcal{C}) \longrightarrow L$.
  - A class $\mathrm{Hom}(\mathcal{C}) = \bigcup\{\mathrm{Hom}_{\mathcal{C}}(A, B) \mid A, B \in \mathrm{Ob}(\mathcal{C})\}$ of pairwise disjoint sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of potential objects $A, B \in \mathrm{Ob}(\mathcal{C})$. The members of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are called potential morphisms from A to B and the members of $\mathrm{Hom}(\mathcal{C})$ are called potential morphisms of the category $\mathcal{C}$.
  - An L-fuzzy subclass μ of $\mathrm{Hom}(\mathcal{C})$, i.e. a mapping $\mu\colon \mathrm{Hom}(\mathcal{C}) \longrightarrow L$, such that if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then $\mu(f) \leq \omega(A) \wedge \omega(B)$.
  - A composition $\circ$ of morphisms is defined, i.e. for each triple A, B, C of objects there exists a mapping $\circ\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$, such that the following axioms are satisfied:
    (i) preservation of morphisms: $\mu(f \circ g) \geq \mu(f) * \mu(g)$
    (ii) associativity: if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, then $f \circ (g \circ h) = (f \circ g) \circ h$ holds
    (iii) existence of identities: for each $A \in \mathrm{Ob}(\mathcal{C})$ there exists an identity morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that $\mu(1_A) = \omega(A)$ and for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$, all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and all $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ it holds $1_A \circ f = f$ and $g \circ 1_A = g$.

  Let $\alpha \in L$ and let $\mathrm{Ob}^\alpha(\mathcal{C}) = \{A \in \mathrm{Ob}(\mathcal{C}) \mid \omega(A) \geq \alpha\}$, $\mathrm{Hom}^\alpha(\mathcal{C}) = \{f \in \mathrm{Hom}(\mathcal{C}) \mid \mu(f) \geq \alpha\}$ and $\mathrm{Hom}^\alpha_{\mathcal{C}}(A, B) = \{f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid \mu(f) \geq \alpha\}$. The elements of $\mathrm{Ob}^\alpha(\mathcal{C})$ will be reffered to as α-objects of the L-fuzzy category $\mathcal{C}$, while the elements of $\mathrm{Hom}^\alpha(\mathcal{C})$ as α-morphisms of the L-fuzzy category $\mathcal{C}$.

Let us now present several auxiliary lemmas (provable in GFCT), which will be used in the following sections. From now on, when writing formulas, proofs and other expressions we will, for the sake of simplicity, use abbreviations common in classical mathematics, or in traditional fuzzy mathematics like $(\forall x \in A)\varphi$ instead of $(\forall x)(x \in A \rightarrow \varphi)$, $(\exists x \in A)\varphi$ instead of $(\exists x)(x \in A \mathbin{\&} \varphi)$, $x \notin A$ instead of $\neg(x \in A)$ (similarly for other binary predicates), Ax instead of $x \in A$, $Rx_1 \ldots x_n$ instead of $\langle x_1, \ldots, x_n\rangle \in R$, $RSxz$ instead of $\langle x, z\rangle \in (R \circ S)$ etc. Also the common convention concerning quantifiers will be used, i.e. $(\forall x_1) \ldots (\forall x_n)\varphi$ abbreviated by $(\forall x_1, \ldots, x_n)\varphi$ and similarly for $\exists$. When writing proofs let us use the following graphical convention:

$(n)\ \varphi$

$(n+1) \Rightarrow \psi$,

which means that GFCT proves $\varphi \to \psi$, thus in particular by MP if GFCT proves φ, then GFCT also proves ψ. Similarly for $\Leftrightarrow$, meaning that GFCT proves $\varphi \leftrightarrow \psi$.

Lemma 3.1.3 If GFCT proves φ, then GFCT proves $(\forall x \in A)\varphi$.
Proof:

(1) φ; proven–assumption

(2) $\varphi \to (x \in A \to \varphi)$; (T1)

(3) $x \in A \to \varphi$; by (MP) on (1) and (2)

(4) $(\forall x)(x \in A \to \varphi)$; by (Gen) on (3)

(5) $\Leftrightarrow (\forall x \in A)\varphi$; by conventional abbreviation concerning general quantification

Q.E.D.

Lemma 3.1.4 If GFCT proves $Ax \leftrightarrow Bx$, then GFCT proves $A = B$.
Proof:

(1) $Ax \leftrightarrow Bx$; proven–assumption

(2) $\Delta(Ax \leftrightarrow Bx)$; by (Nec) on (1)

(3) $(\forall x)\Delta(Ax \leftrightarrow Bx)$; by (Gen) on (2)

(4) $(\forall x)\Delta(Ax \leftrightarrow Bx) \to A = B$; the axiom of extensionality

(5) $A = B$; by (MP) on (3) and (4)

Q.E.D.

Lemma 3.1.5 If GFCT proves $\varphi(x) \vee \neg\varphi(x)$, then GFCT proves the crispness of a set $A = \{y \mid \varphi(y)\}$.
Proof:

(1) $\varphi(x) \vee \neg\varphi(x)$; proven–assumption

(2) $\Leftrightarrow x \in \{y \mid \varphi(y)\} \vee x \notin \{y \mid \varphi(y)\}$; by the comprehension axiom, (Cong)

(3) $\Leftrightarrow x \in A \vee x \notin A$; by $A = \{y \mid \varphi(y)\}$ and (T=5)

(4) $\Delta(x \in A \vee x \notin A)$; by (Nec) on (3)

(5) $(\forall x)\Delta(x \in A \vee x \notin A)$; by (Gen) on (4)

(6) $\Leftrightarrow$ Crisp(A); by definition of Crisp(A)

Q.E.D.

- Among special instances of lemma 3.1.5 are the following three cases concerning the empty set, a singleton and a complement of a set.

  Lemma 3.1.5.1 Crisp($\emptyset$), Crisp($\{a\}$) and Crisp($\backslash A$) are provable in GFCT.
  Proof:

  (1) $\bot \to \bot$; (BL7)
  (2) $\Leftrightarrow \neg\bot$; by definition of $\neg$
  (3) $\Leftrightarrow \bot \vee \neg\bot$; by (T21b)
  (4) $\Leftrightarrow x \in \emptyset \vee x \notin \emptyset$; by definition of $\emptyset$, (Cong)
  (5) $\Leftrightarrow$ Crisp($\emptyset$); by lemma 3.1.5

  q.e.d.

  (6) $x = a \vee x \neq a$; (T=1)
  (7) $\Leftrightarrow x \in \{a\} \vee x \notin \{a\}$; by definition of $\{a\}$, (Cong)
  (8) $\Leftrightarrow$ Crisp($\{a\}$); by lemma 3.1.5

  q.e.d.

  (9) $x \in A \ \& \ x \notin A \to \bot$; (I10)
  (10) $\Leftrightarrow \neg(x \in A \ \& \ x \notin A)$; by definition of $\neg$
  (11) $\Leftrightarrow x \notin A \vee \neg(x \notin A)$; by (T19a)
  (12) $\Leftrightarrow x \in \backslash A \vee x \notin \backslash A$; by definition of $\backslash A$, (Cong)
  (13) $\Leftrightarrow$ Crisp($\backslash A$); by lemma 3.1.5

  q.e.d.

  Q.E.D.

Lemma 3.1.6 Crisp(A) $\to (Ax \to \Delta Ax)$ is provable in GFCT.
Proof:

- First, we need to show one obvious auxiliary fact (lemma 3.1.6.1):

  (1) $\Delta(\neg\varphi) \to \neg\varphi$; ($\Delta$3)
  (2) $\Leftrightarrow \Delta(\neg\varphi) \to (\varphi \to \bot)$; by definition of $\neg$, (Cong)
  (3) $\Rightarrow \Delta(\neg\varphi) \ \& \ \varphi \to \bot$; by (BL5b)
  (4) $\bot \to \Delta(\neg\varphi) \ \& \ \varphi$; (BL7)
  (5) $\Delta(\neg\varphi) \ \& \ \varphi \leftrightarrow \bot$; by (T6) from (3) and (4) and by definition of $\leftrightarrow$

  q.e.d.

(1) Crisp(A) $\to$ Crisp(A); (T3a)

(2) $\Leftrightarrow$ Crisp(A) $\to (\forall x)\Delta(Ax \vee \neg Ax)$; by definition of Crisp(A), (Cong)

(3) $\Rightarrow$ Crisp(A) $\to (\Delta Ax \vee \Delta\neg Ax)$; by ($\forall$1), ($\Delta$2) and (BL1)

(4) $\Rightarrow$ Crisp(A) & $Ax \to (\Delta Ax \vee \Delta\neg Ax) \ \& \ Ax$; by (T4a)

(5) $\Leftrightarrow$ Crisp(A) & $Ax \to ((Ax \ \& \ \Delta Ax) \vee (Ax \ \& \ \Delta\neg Ax))$; by (T12a), (Cong)

(6) $\Leftrightarrow$ Crisp(A) & $Ax \to ((Ax \mathbin{\&} \Delta Ax) \vee \bot)$; by lemma 3.1.6.1, (Cong)

(7) $\Leftrightarrow$ Crisp(A) & $Ax \to Ax \mathbin{\&} \Delta Ax$; by (T21b), (Cong)

(8) $\Rightarrow$ Crisp(A) & $Ax \to \Delta Ax$; by (BL2), (BL1)

(9) $\Rightarrow$ Crisp(A) $\to (Ax \to \Delta Ax)$; by (BL5a)

- Moreover, the opposite inner implication can be proven:

(10) $\Delta Ax \to Ax$; (Δ3)

(11) $(\Delta Ax \to Ax) \to$ (Crisp(A) $\to (\Delta Ax \to Ax)$); (T1)

(12) Crisp(A) $\to (\Delta Ax \to Ax)$; by (MP) from (10) and (11)

(13) Crisp(A) $\to ((\Delta Ax \to Ax) \mathbin{\&} (Ax \to \Delta Ax))$; by (I7) from (12) and (9)

(14) $\Leftrightarrow$ Crisp(A) $\to (\Delta Ax \leftrightarrow Ax)$; by definition of $\leftrightarrow$, (Cong)

Q.E.D.

Lemma 3.1.7 $\emptyset \subseteq A$ is provable in GFCT.
Proof:

(1) $\bot \to Ax$; (BL7)

(2) $(\forall x)(\bot \to Ax)$; by (Gen) on (1)

(3) $\Leftrightarrow (\forall x)(x \in \emptyset \to Ax)$; by definition of $\emptyset$, (Cong)

(4) $\Leftrightarrow \emptyset \subseteq A$; by definition of $\subseteq$

Q.E.D.

Lemma 3.1.8 $A \subseteq \emptyset \to A = \emptyset$ is provable in GFCT.
Proof:

(1) $A \subseteq \emptyset \to A \subseteq \emptyset$; (T3a)

(2) $\Leftrightarrow A \subseteq \emptyset \to (\forall x)(Ax \to x \in \emptyset)$; by definition of $\subseteq$, (Cong)

(3) $\Rightarrow A \subseteq \emptyset \to (Ax \to x \in \emptyset)$; by ($\forall$1), (BL1)

(4) $\Leftrightarrow A \subseteq \emptyset \to x \in \backslash A$; by definitions of $\emptyset$, $\neg$ and $\backslash A$, (Cong)

(5) $\Leftrightarrow A \subseteq \emptyset \to x \in \backslash A$ & Crisp($\backslash A$); by lemma 3.1.5.1, (T20a), (Cong)

(6) $\Rightarrow A \subseteq \emptyset \to \Delta(x \in \backslash A)$; by (BL5b) from lemma 3.1.6, (BL1)

(7) $\Leftrightarrow A \subseteq \emptyset \to \Delta(Ax \to x \in \emptyset)$; by definitions of $\backslash A$, $\neg$ and $\emptyset$, (Cong)

(8) $\bot \to Ax$; (BL7)

(9) $\Leftrightarrow x \in \emptyset \to Ax$; by definition of $\emptyset$, (Cong)

(10) $\Delta(x \in \emptyset \to Ax)$; by (Nec) from (9)

(11) $A \subseteq \emptyset \to \Delta(Ax \to x \in \emptyset) \mathbin{\&} \Delta(x \in \emptyset \to Ax)$; from (7) and (10) by (T20a), (Cong)

(12) $\Leftrightarrow A \subseteq \emptyset \to \Delta(Ax \leftrightarrow x \in \emptyset)$; by (T$\Delta$1), by definition of $\leftrightarrow$, (Cong)

(13) $(\forall x)(A \subseteq \emptyset \to \Delta(Ax \leftrightarrow x \in \emptyset))$; by (Gen) on (12)

(14) $\Leftrightarrow A \subseteq \emptyset \to (\forall x)\Delta(Ax \leftrightarrow x \in \emptyset)$; by (T$\forall$5a)

(15) $\Rightarrow A \subseteq \emptyset \to A = \emptyset$; by the axiom of extensionality, (BL1)

Q.E.D.

Lemma 3.1.9 $\emptyset \times B = \emptyset$ is provable in GFCT.
Proof:

(1) $(\emptyset \times B)xy \to (\emptyset \times B)xy$; (T3a)

(2) $\Leftrightarrow (\emptyset \times B)xy \to x \in \emptyset \mathbin{\&} By$; by definition of $\times$, (Cong)

(3) $\Leftrightarrow (\emptyset \times B)xy \to \bot \mathbin{\&} By$; by definition of $\emptyset$, (Cong)

(4) $\Leftrightarrow (\emptyset \times B)xy \to \bot$; by (T21a), (Cong)

(5) $\Leftrightarrow (\emptyset \times B)xy \to \langle x, y\rangle \in \emptyset$; by definition of $\emptyset$, (Cong)

(6) $(\forall x, y)((\emptyset \times B)xy \to \langle x, y\rangle \in \emptyset)$; by (Gen) on (5)

(7) $\Leftrightarrow \emptyset \times B \subseteq \emptyset$; by definition of $\subseteq$

(8) $\Rightarrow \emptyset \times B = \emptyset$; by lemma 3.1.8

(9) $B \times \emptyset = \emptyset$ is proven similarly using commutativity of &

Q.E.D.

Lemma 3.1.10 $A = \emptyset \mathbin{\&} A = \{a\} \to \bot$ is provable in GFCT.
Proof:

(1) $A = \emptyset \mathbin{\&} A = \{a\} \to A = \emptyset \mathbin{\&} A = \{a\}$; (T3a)

(2) $\Rightarrow A = \emptyset \mathbin{\&} A = \{a\} \to \{a\} = \emptyset$; by (T=2b), (Cong), (T=3), (BL1)

(3) $\{a\} = \emptyset \to (x \in \{a\} \to x \in \emptyset)$; (=2)

(4) $\{a\} = \emptyset \to (\forall x)(x \in \{a\} \to x \in \emptyset)$; by (Gen), (T$\forall$5a) on (3)

(5) $A = \emptyset \mathbin{\&} A = \{a\} \to (\forall x)(x \in \{a\} \to x \in \emptyset)$; by (BL1) from (2) and (4)

(6) $\Leftrightarrow A = \emptyset \mathbin{\&} A = \{a\} \to (\forall x)(x = a \to \bot)$; by definitions of $\{a\}$ and $\emptyset$, (Cong)

(7) $\Leftrightarrow A = \emptyset \mathbin{\&} A = \{a\} \to (\forall x)\neg(x = a)$; by definition of $\neg$, (Cong)

(8) $\Leftrightarrow A = \emptyset \mathbin{\&} A = \{a\} \to \neg(\exists x)(x = a)$; by (T$\forall$17), (Cong)

(9) $\Leftrightarrow A = \emptyset \mathbin{\&} A = \{a\} \to (\exists x)(x = a) \mathbin{\&} \neg(\exists x)(x = a)$; by (T=7), (T20a), (Cong)

(10) $\Rightarrow A = \emptyset \;\&\; A = \{a\} \to \bot$; by (I10), (BL1)

(11) Moreover, by (BL7) we get the opposite implication $\bot \to A = \emptyset \;\&\; A = \{a\}$ and thus we get that $A = \emptyset \;\&\; A = \{a\} \leftrightarrow \bot$ is provable in GFCT.

Q.E.D.

Lemma 3.1.11 $R \subseteq A \times B \;\&\; S \subseteq B \times C \to R \circ S \subseteq A \times C$ is provable in GFCT.

Proof:

(1) $R \subseteq A \times B \;\&\; S \subseteq B \times C \to R \subseteq A \times B \;\&\; S \subseteq B \times C$; (T3a)

(2) $\Leftrightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (\forall x, y)(Rxy \to (A \times B)xy) \;\&\; (\forall y, z)(Syz \to (B \times C)yz)$; by definition of $\subseteq$, (Cong)

(3) $\Leftrightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (\forall x, y, z)((Rxy \to Ax \;\&\; By) \;\&\; (Syz \to By \;\&\; Cz))$; by (T$\forall$10a), (T$\forall$11a), definition of $\times$, (Cong)

(4) $\Rightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (Rxy \to Ax \;\&\; By) \;\&\; (Syz \to By \;\&\; Cz)$; by ($\forall$1), (BL1)

(5) $\Rightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (Rxy \;\&\; Syz \to Ax \;\&\; By \;\&\; Cz)$; by (T10), (T15), (Cong)

(6) $\Rightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (By \;\&\; Rxy \;\&\; Syz \to Ax \;\&\; By \;\&\; Cz)$; by (T4a), (T14a), (T15), (Cong), (BL1)

(7) $\Rightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (By \;\&\; Rxy \;\&\; Syz \to Ax \;\&\; Cz)$; by (BL5b), (BL2), (BL1), (BL5a)

(8) $(\forall x, y, z)(R \subseteq A \times B \;\&\; S \subseteq B \times C \to (By \;\&\; Rxy \;\&\; Syz \to Ax \;\&\; Cz))$; by (Gen) on (7)

(9) $\Leftrightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (\forall x, z)((\exists y)(By \;\&\; Rxy \;\&\; Syz) \to Ax \;\&\; Cz)$; by (T$\forall$5a), (T$\forall$3a), (T$\forall$6a), (Cong)

(10) $\Leftrightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to (\forall x, z)(RSxz \to (A \times C)xz)$; by definitions of $\circ$, $\times$, (Cong)

(11) $\Leftrightarrow R \subseteq A \times B \;\&\; S \subseteq B \times C \to R \circ S \subseteq A \times C$; by definition of $\subseteq$, (Cong)

- Moreover, the Δ-version (lemma 3.1.11Δ) of the previous result will be used in the next chapter:

(12) $\Delta(R \subseteq A \times B \;\&\; S \subseteq B \times C \to R \circ S \subseteq A \times C)$; by (Nec) on (11)

(13) $\Rightarrow \Delta(R \subseteq A \times B) \;\&\; \Delta(S \subseteq B \times C) \to \Delta(R \circ S \subseteq A \times C)$; by ($\Delta$5), (T$\Delta$1), (Cong)

Q.E.D.

Lemma 3.1.12 $A \approxeq B \;\&\; B \subseteq C \to A \subseteq C$ is provable in GFCT.

Proof:

(1) $A \approxeq B \;\&\; B \subseteq C \to A \approxeq B \;\&\; B \subseteq C$; (T3a)

(2) $\Leftrightarrow A \approxeq B \;\&\; B \subseteq C \rightarrow (\forall x)(Ax \leftrightarrow Bx) \;\&\; (\forall x)(Bx \rightarrow Cx)$; by definitions of $\approxeq$, $\subseteq$, (Cong)

(3) $\Leftrightarrow A \approxeq B \;\&\; B \subseteq C \rightarrow (\forall x)((Ax \rightarrow Bx) \;\&\; (Bx \rightarrow Ax) \;\&\; (Bx \rightarrow Cx))$; by definition of $\leftrightarrow$, (T$\forall$11a), (Cong)

(4) $\Rightarrow A \approxeq B \;\&\; B \subseteq C \rightarrow (Ax \rightarrow Bx) \;\&\; (Bx \rightarrow Cx)$; by ($\forall$1), (T14a), (I6), (Cong), (BL1)

(5) $\Rightarrow A \approxeq B \;\&\; B \subseteq C \rightarrow (Ax \rightarrow Cx)$; by (BL5b) on (BL1), (BL1)

(6) $(\forall x)(A \approxeq B \;\&\; B \subseteq C \rightarrow (Ax \rightarrow Cx))$; by (Gen) on (5)

(7) $\Leftrightarrow A \approxeq B \;\&\; B \subseteq C \rightarrow A \subseteq C$; by (T$\forall$5a), by definition of $\subseteq$, (Cong)

Q.E.D.

Lemma 3.1.13 $R \approxeq R' \;\&\; T \approxeq R \circ S \rightarrow T \approxeq R' \circ S$ is provable in GFCT.
Proof:

(1) $(Rxy \rightarrow R'xy) \rightarrow (Rxy \rightarrow R'xy)$; (T3a)

(2) $\Rightarrow (Rxy \rightarrow R'xy) \rightarrow (By \;\&\; Rxy \;\&\; Syz \rightarrow By \;\&\; R'xy \;\&\; Syz)$; by (T4a), (T13a), (T14a), (BL1)

(3) $(\forall y)(Rxy \rightarrow R'xy) \rightarrow ((\exists y)(By \,\&\, Rxy \,\&\, Syz) \rightarrow (\exists y)(By \,\&\, R'xy \,\&\, Syz))$; by (Gen) on (2), by (T$\forall$7), (T$\forall$8), (BL1)

(4) $\Leftrightarrow (\forall y)(Rxy \rightarrow R'xy) \rightarrow (RSxz \rightarrow R'Sxz)$; by definition of $R \circ S$, (Cong)

(5) $(\forall x, y)(Rxy \rightarrow R'xy) \rightarrow (\forall x, z)(RSxz \rightarrow R'Sxz)$; by (Gen) on (4), by (T$\forall$5a), (T$\forall$7), (Cong)

(6) $\Leftrightarrow R \subseteq R' \rightarrow R \circ S \subseteq R' \circ S$; by definition of $\subseteq$

(7) $(RSxz \rightarrow R'Sxz) \;\&\; (Txz \rightarrow RSxz) \rightarrow (Txz \rightarrow RSxz) \;\&\; (RSxz \rightarrow R'Sxz)$; by definition of $\leftrightarrow$ and (I5) from (T14a)

(8) $\Rightarrow (RSxz \rightarrow R'Sxz) \;\&\; (Txz \rightarrow RSxz) \rightarrow (Txz \rightarrow R'Sxz)$; by (BL5b) on (BL1), (BL1)

(9) $(\forall x, z)(RSxz \rightarrow R'Sxz) \;\&\; (\forall x, z)(Txz \rightarrow RSxz) \rightarrow (\forall x, z)(Txz \rightarrow R'Sxz)$; by (Gen) on (8), by (T$\forall$7), (T$\forall$11a), (Cong)

(10) $\Rightarrow R \circ S \subseteq R' \circ S \rightarrow (T \subseteq R \circ S \rightarrow T \subseteq R' \circ S)$; by definition of $\subseteq$ and (BL5a)

(11) $R \subseteq R' \;\&\; T \subseteq R \circ S \rightarrow T \subseteq R' \circ S$; by (BL1) from (6) and (10), (BL5b)

(12) $R' \subseteq R \;\&\; R \circ S \subseteq T \rightarrow R' \circ S \subseteq T$; proven similarly as (11)

(13) $R \approxeq R' \;\&\; T \approxeq R \circ S \rightarrow T \approxeq R' \circ S$; by (T6) from (11) and (12), by definitions of $\subseteq$, $\leftrightarrow$ and $\approxeq$, by (T$\forall$10a), (T$\forall$11a), (Cong)

Q.E.D.

Lemma 3.1.14 Sim($\approxeq$) is provable in GFCT.
Proof:

(i) checking reflexivity of $\approxeq$:

(1) $Ax \leftrightarrow Ax$; (T3b)

(2) $(\forall x)(Ax \leftrightarrow Ax)$; by (Gen) on (1)

(3) $\Leftrightarrow A \approxeq A$; by definition of $\approxeq$

q.e.d.

(ii) checking symmetry of $\approxeq$:

(4) $(Ax \leftrightarrow Bx) \rightarrow (Ax \leftrightarrow Bx)$; (T3a)

(5) $\Leftrightarrow (Ax \leftrightarrow Bx) \rightarrow (Bx \leftrightarrow Ax)$; by definition of $\leftrightarrow$, (T14a), (Cong)

(6) $(\forall x)(Ax \leftrightarrow Bx) \rightarrow (\forall x)(Bx \leftrightarrow Ax)$; by (Gen) on (5), (T∀7)

(7) $\Leftrightarrow A \approxeq B \rightarrow B \approxeq A$; by definition of $\approxeq$

q.e.d.

(iii) checking transitivity of $\approxeq$:

(8) $(Ax \leftrightarrow Bx)$ & $(Bx \leftrightarrow Cx) \rightarrow (Ax \leftrightarrow Bx)$ & $(Bx \leftrightarrow Cx)$; (T3a)

(9) $\Leftrightarrow (Ax \leftrightarrow Bx)$ & $(Bx \leftrightarrow Cx) \rightarrow (Ax \rightarrow Bx)$ & $(Bx \rightarrow Cx)$ & $(Cx \rightarrow Bx)$ & $(Bx \rightarrow Ax)$; by definition of $\leftrightarrow$, (T13a), (T14a), (Cong)

(10) $(Ax \rightarrow Bx)$ & $(Bx \rightarrow Cx) \rightarrow (Ax \rightarrow Cx)$; by (BL5b) from (BL1)

(11) $(Cx \rightarrow Bx)$ & $(Bx \rightarrow Ax) \rightarrow (Cx \rightarrow Ax)$; by (BL5b) from (BL1)

(12) $(Ax \rightarrow Bx)$ & $(Bx \rightarrow Cx)$ & $(Cx \rightarrow Bx)$ & $(Bx \rightarrow Ax) \rightarrow (Ax \rightarrow Cx)$ & $(Cx \rightarrow Ax)$; by (T6) and (T10) from (10) and (11)

(13) $(Ax \leftrightarrow Bx)$ & $(Bx \leftrightarrow Cx) \rightarrow (Ax \leftrightarrow Cx)$; by (BL1) from (9) and (12), by definition of $\leftrightarrow$, (Cong)

(14) $(\forall x)(Ax \leftrightarrow Bx)$ & $(\forall x)(Bx \leftrightarrow Cx) \rightarrow (\forall x)(Ax \leftrightarrow Cx)$; by (Gen) on (13), (T∀7), (T∀11a), (Cong)

(15) $\Leftrightarrow A \approxeq B$ & $B \approxeq C \rightarrow A \approxeq C$; by definition of $\approxeq$

q.e.d.

(16) $(\forall A)(A \approxeq A)$ & $(\forall A, B)(A \approxeq B \rightarrow B \approxeq A)$ & $(\forall A, B, C)(A \approxeq B$ & $B \approxeq C \rightarrow A \approxeq C)$; by (T6), (Gen), (T∀10a), (T∀11a) and (Cong) from (3), (7) and (15)

(17) $\Leftrightarrow \mathrm{Sim}(\approxeq)$; by definitions of $\mathrm{Refl}(R)$, $\mathrm{Symm}(R)$, $\mathrm{Trans}(R)$ and $\mathrm{Sim}(R)$

Q.E.D.

Associativity of composition of fuzzy relations (as defined in definition 2.2.3) is a well-known result within the field of fuzzy mathematics and its proof, which we present here just for the sake of completeness, can be found e.g. in [8], pp. 278, proof of Theorem 6.15.

Lemma 3.1.15 $(\forall R \subseteq A \times B, S \subseteq B \times C, T \subseteq C \times D)((R \circ S) \circ T = R \circ (S \circ T))$ is provable in GFCT.

Proof:

(1) $((R \circ S) \circ T)xw \leftrightarrow ((R \circ S) \circ T)xw$; (T3b)

(2) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (\exists z)(Cz \mathbin{\&} (R \circ S)xz \mathbin{\&} Tzw)$; by definition of $(R \circ S) \circ T$, (Cong)

(3) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (\exists z)(Cz \mathbin{\&} (\exists y)(By \mathbin{\&} Rxy \mathbin{\&} Syz) \mathbin{\&} Tzw)$; by definition of $R \circ S$, (Cong)

(4) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (\exists z)(\exists y)(Cz \mathbin{\&} By \mathbin{\&} Rxy \mathbin{\&} Syz \mathbin{\&} Tzw)$; by (T∀10b), (Cong)

(5) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (\exists y)(By \mathbin{\&} Rxy \mathbin{\&} (\exists z)(Cz \mathbin{\&} Syz \mathbin{\&} Tzw))$; by (T∀3b), (T∀10b), (Cong)

(6) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (\exists y)(By \mathbin{\&} Rxy \mathbin{\&} (S \circ T)yw)$; by definition of $S \circ T$, (Cong)

(7) $\Leftrightarrow ((R \circ S) \circ T)xw \leftrightarrow (R \circ (S \circ T))xw$; by definition of $R \circ (S \circ T)$, (Cong)

(8) $(R \circ S) \circ T = R \circ (S \circ T)$; by lemma 3.1.4

(9) $(\forall R \subseteq A \times B, S \subseteq B \times C, T \subseteq C \times D)((R \circ S) \circ T = R \circ (S \circ T))$; by lemma 3.1.3 using $R \in \mathrm{Pow}(A \times B) \overset{\mathrm{df}}{\equiv} R \subseteq A \times B$

Q.E.D.

Lemma 3.1.16 $(\forall F, G)(\mathrm{Fnc}_{\mathrm{Id}}(F) \mathbin{\&} \mathrm{Fnc}_{\mathrm{Id}}(G) \to \mathrm{Fnc}_{\mathrm{Id}}(F \circ G))$ is provable in GFCT.

Proof:

(1) $\mathrm{Fnc}_{\mathrm{Id}}(F) \to \mathrm{Fnc}_{\mathrm{Id}}(F)$; (T3a)

(2) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (Fxy \mathbin{\&} Fxy' \to y = y')$; by definition of Id-functionality, (Cong), (∀1), (BL1)

(3) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (By \mathbin{\&} Fxy \mathbin{\&} Gyz \mathbin{\&} Fxy' \to y = y' \mathbin{\&} Gyz)$; by (T4a), (T4c), (BL1), (T13a), (T14a), (Cong)

(4) $\mathrm{Fnc}_{\mathrm{Id}}(F) \to (\forall y)(By \mathbin{\&} Fxy \mathbin{\&} Gyz \mathbin{\&} Fxy' \to y = y' \mathbin{\&} Gyz)$; by (Gen) on (3), (T∀5a), (Cong)

(5) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to ((\exists y)(By \mathbin{\&} Fxy \mathbin{\&} Gyz) \mathbin{\&} Fxy' \to (\exists y)(y = y' \mathbin{\&} Gyz))$; by (T∀8), (T∀10b), (Cong), (BL1)

(6) $\Leftrightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (FGxz \mathbin{\&} Fxy' \to Gy'z)$; by definition of $F \circ G$, (T=6), (Cong), (BL1)

(7) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (FGxz \;\&\; Fxy' \;\&\; Gy'z' \to Gy'z \;\&\; Gy'z')$; by (T4a), (BL1)

(8) $\mathrm{Fnc}_{\mathrm{Id}}(F) \to (\forall y')(FGxz \;\&\; Fxy' \;\&\; Gy'z' \to Gy'z \;\&\; Gy'z')$; by (Gen) on (7), (T∀5a), (Cong)

(9) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (FGxz \;\&\; (\exists y')(Fxy' \;\&\; Gy'z') \to (\exists y')(Gy'z \;\&\; Gy'z'))$; by (T∀8), (T∀10b), (Cong), (BL1)

(10) $\Leftrightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \to (FGxz \;\&\; FGxz' \to (\exists y')(Gy'z \;\&\; Gy'z'))$; by definition of $F \circ G$, (Cong)

(11) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; FGxz \;\&\; FGxz' \to (\exists y')(Gy'z \;\&\; Gy'z')$; by (BL5b)

(12) $\mathrm{Fnc}_{\mathrm{Id}}(G) \to \mathrm{Fnc}_{\mathrm{Id}}(G)$; (T3a)

(13) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(G) \to ((\exists y')(Gy'z \;\&\; Gy'z') \to z = z')$; by definition of Id-functionality, (∀1), (T∀6a), (Cong), (BL1)

(14) $\mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \;\&\; FGxz \;\&\; FGxz' \to (\exists y')(Gy'z \;\&\; Gy'z') \;\&\; ((\exists y')(Gy'z \;\&\; Gy'z') \to z = z')$; by (T6) from (11) and (13), (T10), (BL1)

(15) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \;\&\; FGxz \;\&\; FGxz' \to z = z'$; by (T5), (BL1)

(16) $\Rightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \to (FGxz \;\&\; FGxz' \to z = z')$; by (BL5a)

(17) $\mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \to (\forall x, z, z')(FGxz \;\&\; FGxz' \to z = z')$; by (Gen) on (16), (T∀5a), (Cong)

(18) $\Leftrightarrow \mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \to \mathrm{Fnc}_{\mathrm{Id}}(F \circ G)$; by definition of Id-functionality, (Cong)

- Moreover, the Δ version (lemma 3.1.16Δ) of the previous result will be used in the next chapter:

(19) $\Delta(\mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \mathrm{Fnc}_{\mathrm{Id}}(G) \to \mathrm{Fnc}_{\mathrm{Id}}(F \circ G))$; by (Nec) on (18)

(20) $\Rightarrow \Delta\mathrm{Fnc}_{\mathrm{Id}}(F) \;\&\; \Delta\mathrm{Fnc}_{\mathrm{Id}}(G) \to \Delta\mathrm{Fnc}_{\mathrm{Id}}(F \circ G)$; by (Δ5), (TΔ1), (Cong)

Q.E.D.

Lemma 3.1.17 $\mathrm{Fnc}_{\mathrm{Id}}(\mathrm{Id}_A)$ is provable in GFCT.

Proof:

(1) $Ax \;\&\; Ax' \;\&\; x = x' \;\&\; Ax \;\&\; Ax'' \;\&\; x = x'' \to Ax \;\&\; Ax' \;\&\; x = x' \;\&\; Ax \;\&\; Ax'' \;\&\; x = x''$; (T3a)

(2) $\Leftrightarrow \mathrm{Id}_A xx' \;\&\; \mathrm{Id}_A xx'' \to Ax \;\&\; Ax' \;\&\; x = x' \;\&\; Ax \;\&\; Ax'' \;\&\; x = x''$; by definition of Id_A, (Cong)

(3) $\Rightarrow \mathrm{Id}_A xx' \;\&\; \mathrm{Id}_A xx'' \to x = x' \;\&\; x = x''$; by (T13a), (T14a), (Cong), (I5), (BL1)

(4) $\Rightarrow \mathrm{Id}_A xx' \;\&\; \mathrm{Id}_A xx'' \to x' = x''$; by (T=2b), (Cong), (T=3), (BL1)

(5) $(\forall x, x', x'')(\mathrm{Id}_A xx' \;\&\; \mathrm{Id}_A xx'' \to x' = x'')$; by (Gen) on (4)

(6) $\Leftrightarrow \mathrm{Fnc}_{\mathrm{Id}}(\mathrm{Id}_A)$; by definition of Id-functionality

Q.E.D.

Lemma 3.1.18 $R \subseteq A \times B \to R \approxeq R \circ \mathrm{Id}_B$ is provable in GFCT.

Proof:

(1) $R \subseteq A \times B \to R \subseteq A \times B$; (T3a)

(2) $\Rightarrow R \subseteq A \times B \to (Rxy \to Ax \,\&\, By)$; by definitions $\subseteq$ and $\times$, (Cong), ($\forall$1), (BL1)

(3) $(Rxy \to Ax \,\&\, By) \to (Rxy \,\&\, Rxy \to Ax \,\&\, By \,\&\, Rxy)$; (T4a)

(4) $R \subseteq A \times B \to (Rxy \to Ax \,\&\, By \,\&\, Rxy)$; from (2) and (3) by (BL1), (T15), (Cong)

(5) $Rxy \,\&\, Ax \,\&\, By \to Rxy$; (I5)

(6) $R \subseteq A \times B \to (Rxy \leftrightarrow Ax \,\&\, By \,\&\, Rxy)$; from (4) and (5) by (T20a), by definition of $\leftrightarrow$, (Cong)

(7) $R \subseteq A \times B \to (Rxy \leftrightarrow By \,\&\, Rxy)$; proven similarly as (6) using $Rxy \to By$

(8) $(\exists y')(y' = y \,\&\, By' \,\&\, By \,\&\, Rxy') \leftrightarrow By \,\&\, Rxy$; (T=6), (T15), (Cong)

(9) $R \subseteq A \times B \to (Rxy \leftrightarrow (\exists y')(By' \,\&\, Rxy' \,\&\, y' = y \,\&\, By' \,\&\, By))$; from (7) and (8) by (Cong), (T13a), (T14a), (T15)

(10) $\Leftrightarrow R \subseteq A \times B \to (Rxy \leftrightarrow (\exists y')(By' \,\&\, Rxy' \,\&\, \mathrm{Id}_B y'y))$; by definition of Id_B, (Cong)

(11) $\Leftrightarrow R \subseteq A \times B \to (Rxy \leftrightarrow (R \circ \mathrm{Id}_B)xy)$; by definition of $R \circ \mathrm{Id}_B$, (Cong)

(12) $R \subseteq A \times B \to (\forall x, y)(Rxy \leftrightarrow (R \circ \mathrm{Id}_B)xy)$; by (Gen), (T$\forall$5a), (Cong) from (11)

(13) $\Leftrightarrow R \subseteq A \times B \to R \approxeq R \circ \mathrm{Id}_B$; by definition of $\approxeq$, (Cong)

(14) $R \subseteq A \times B \to R \approxeq \mathrm{Id}_A \circ R$; proven similarly, using $R \subseteq A \times B \to (Rxy \leftrightarrow Ax \,\&\, Rxy)$

- Moreover, the following two versions of the previous result will be used in the next chapter:
- Lemma $3.1.18^{\Delta}$:

(15) $\Delta(R \subseteq A \times B \to R \approxeq R \circ \mathrm{Id}_B)$; by (Nec) on (13)

(16) $\Rightarrow \Delta(R \subseteq A \times B) \to R \approxeq R \circ \mathrm{Id}_B$; by ($\Delta$5), ($\Delta$3), (BL1)

- Lemma $3.1.18^{\Delta}_{\Delta}$:

(17) $\Delta(R \subseteq A \times B \to R \approxeq R \circ \mathrm{Id}_B)$; by (Nec) on (13)

(18) $\Rightarrow \Delta(R \subseteq A \times B) \to R = R \circ \mathrm{Id}_B$; by ($\Delta$5), by definition of $\approxeq$, (TΔ2), (Cong), by the axiom of extensionality, (BL1)

Q.E.D.

3.2 Systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$

We start out our examinations by considering four basic systems of morphisms based on the crisp category Rel of crisp sets and crisp binary relations (see example 2.3.2). Accordingly, all the four systems consist of fuzzy sets as (potential) objects and binary fuzzy relations between them as potential morphisms. Let us give a precise definition.

Definition 3.2.1 The four basic systems of morphisms based on the crisp category Rel are defined as follows:

- The (crisp) set of potential objects is the universal class, which is also the set of objects, i.e. $\mathrm{POb}(\mathrm{FRel}^{(\Delta)}_{(\Delta)}) \stackrel{\mathrm{df}}{=} \mathrm{Ob}(\mathrm{FRel}^{(\Delta)}_{(\Delta)}) \stackrel{\mathrm{df}}{=} V$
- The (crisp) set of potential morphisms is the set of all binary fuzzy relations, i.e. subsets of $V \times V$, i.e. $\mathrm{PHom}(\mathrm{FRel}^{(\Delta)}_{(\Delta)}) \stackrel{\mathrm{df}}{=} \mathrm{Pow}(V \times V)$
- The system FRel is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FRel}}(R, A, B) \stackrel{\mathrm{df}}{\equiv} R \subseteq A \times B$
  - $\circ_{\mathrm{FRel}}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T \approxeq R \circ S$
- The system FRel^{Δ} is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FRel}^{\Delta}}(R, A, B) \stackrel{\mathrm{df}}{\equiv} \Delta(R \subseteq A \times B)$
  - $\circ_{\mathrm{FRel}^{\Delta}}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T \approxeq R \circ S$
- The system FRel_{Δ} is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FRel}_{\Delta}}(R, A, B) \stackrel{\mathrm{df}}{\equiv} R \subseteq A \times B$
  - $\circ_{\mathrm{FRel}_{\Delta}}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T = R \circ S$
- The system $\mathrm{FRel}^{\Delta}_{\Delta}$ is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FRel}^{\Delta}_{\Delta}}(R, A, B) \stackrel{\mathrm{df}}{\equiv} \Delta(R \subseteq A \times B)$
  - $\circ_{\mathrm{FRel}^{\Delta}_{\Delta}}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T = R \circ S$

Let us make a convention that whenever possible (i.e. whenever clear which system is under consideration) we will write $\circ(R, S, T)$ and $\mathrm{Hom}(R, A, B)$ instead of for example $\circ_{\mathrm{FRel}}(R, S, T)$ and $\mathrm{Hom}_{\mathrm{FRel}}(R, A, B)$ respectively. Let us also stress the importance of distinguishing different meanings of the symbol $\circ$. Prefixally written $\circ(R, S, T)$ means the ternary fuzzy relation, while infixally written $R \circ S$ means the binary operation (from definition 2.2.3).

The following remark expresses several natural and fundamental properties of $\circ(\_, \_, \_)$ and $\mathrm{Hom}(\_, \_, \_)$, which can be expected to hold in some of the systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$. Proofs of or counter-examples to particular statements of the remark w.r.t. all the four systems are given in the following propositions.

Remark 3.2.2

(i) $(\forall R,S)(\exists T)\circ(R,S,T)$

(ii) $(\forall R,S,T,T')(\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow T\approxeq T')$

(iii) $(\forall R,S,T)(\circ(R,S,T)\ \&\ \mathrm{Hom}(R,A,B)\ \&\ \mathrm{Hom}(S,B,C)\rightarrow \mathrm{Hom}(T,A,C))$

(iv) $(\forall R,S,T,U,V,W)(\circ(R,S,U)\ \&\ \circ(S,T,V)\rightarrow(\circ(U,T,W)\leftrightarrow\circ(R,V,W)))$

(v) $(\forall B)(\exists S)(\forall R,T)(\mathrm{Hom}(S,B,B)\ \&\ \mathrm{Hom}(R,A,B)\ \&\ \mathrm{Hom}(T,B,C)\rightarrow\circ(R,S,R)\ \&\ \circ(S,T,T))$

For the sake of clarity, the statements of the previous remark are presented in a simplified form. In fact all quantifiers from the statements (i)–(v) of the previous remark are relativized to the crisp set $\mathrm{PHom}(\mathrm{FRel}^{(\Delta)}_{(\Delta)})$ except for the first quantifier of (v), which is relativized to the crisp set $\mathrm{POb}(\mathrm{FRel}^{(\Delta)}_{(\Delta)})$.

Proposition 3.2.3 $(\forall R,S)(\exists T)\circ(R,S,T)$ is provable in GFCT w.r.t. all the four systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$.

Proof:

(1) $R\circ S\approxeq R\circ S$; reflexivity of $\approxeq$ (lemma 3.1.14)

(1Δ) $R\circ S=R\circ S$; (=1)

(2) $\Leftrightarrow\circ(R,S,R\circ S)$; by definition of $\circ_{\mathrm{FRel}}$ (and $\circ_{\mathrm{FRel}^\Delta}$) or $\circ_{\mathrm{FRel}_\Delta}$ (and $\circ_{\mathrm{FRel}^\Delta_\Delta}$)

(3) $\circ(R,S,R\circ S)\rightarrow(\exists T)\circ(R,S,T)$; (∃1)

(4) $(\exists T)\circ(R,S,T)$; by (MP) from (2) and (3)

(5) $(\forall R,S)(\exists T)\circ(R,S,T)$; by (Gen) on (4)

Q.E.D.

Proposition 3.2.4 $(\forall R,S,T,T')(\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow T\approxeq T')$ is provable in GFCT w.r.t. all the four systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$.

Proof:

- FRel and FRel^Δ:

(1) $\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow\circ(R,S,T)\ \&\ \circ(R,S,T')$; (T3a)

(2) $\Leftrightarrow\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow T\approxeq R\circ S\ \&\ T'\approxeq R\circ S$; by definition of $\circ_{\mathrm{FRel}}$ (and $\circ_{\mathrm{FRel}^\Delta}$), (Cong)

(3) $\Rightarrow\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow T\approxeq T'$; by symmetry and transitivity of $\approxeq$ (lemma 3.1.14), (Cong), (BL1)

(4) $(\forall R,S,T,T')(\circ(R,S,T)\ \&\ \circ(R,S,T')\rightarrow T\approxeq T')$; by (Gen) on (3)

q.e.d.

- FRel_Δ and $\mathrm{FRel}^\Delta_\Delta$:

(1) $\circ(R,S,T)$ & $\circ(R,S,T') \to \circ(R,S,T)$ & $\circ(R,S,T')$; (T3a)

(2) $\Leftrightarrow \circ(R,S,T)$ & $\circ(R,S,T') \to T = R\circ S$ & $T' = R\circ S$; by definition of $\circ_{\mathrm{FRel}_\Delta}$ (and $\circ_{\mathrm{FRel}_\Delta^\Delta}$), (Cong)

(3) $\Rightarrow \circ(R,S,T)$ & $\circ(R,S,T') \to T = T'$; by (T=2b), (T=3), (Cong), (BL1)

(4) $\Rightarrow \circ(R,S,T)$ & $\circ(R,S,T') \to T \approxeq T'$; by (T=5), (BL1)

(5) $(\forall R,S,T,T')(\circ(R,S,T)$ & $\circ(R,S,T') \to T \approxeq T')$; by (Gen) on (4)

(6) Moreover, from (3) we see that a stronger version of 3.2.2(ii) (obtained by replacing $\approxeq$ by $=$) is provable in GFCT w.r.t. the systems FRel_Δ and $\mathrm{FRel}_\Delta^\Delta$. The fact that it does not hold even w.r.t. the systems FRel and FRel^Δ can be demonstrated as follows. Let us consider a one-element model, i.e. $U = \{a\}$ and let the only non-zero values be as follows: $\| Aa \| = \| Raa \| = \| Saa \| = \| Taa \| = 1$ and let $\| T'aa \| = 0.5$, then $\| T \approxeq R\circ S$ & $T' \approxeq R\circ S \to T = T' \| = 0$.

q.e.d.

Q.E.D.

Proposition 3.2.5 $(\forall R,S,T)(\circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(T,A,C))$ is provable in GFCT w.r.t. the systems FRel, FRel_Δ and $\mathrm{FRel}_\Delta^\Delta$ and does not hold w.r.t. the system FRel^Δ.

Proof:

- FRel:

(1) $\circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$& $\mathrm{Hom}(S,B,C)$; (T3a)

(2) $\Rightarrow \circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \circ(R,S,T)$ & $\mathrm{Hom}(R\circ S,A,C)$; by (I6), (BL1), lemma 3.1.11, (T4a), (T13a), (T14a), (T15), (Cong)

(3) $\Leftrightarrow \circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to T \approxeq R\circ S$ & $R\circ S \subseteq A\times C$; by definitions of $\circ_{\mathrm{FRel}}$ and $\mathrm{Hom}_{\mathrm{FRel}}(R\circ S,A,C)$, (Cong)

(4) $\Rightarrow \circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(T,A,C)$; by lemma 3.1.12, (BL1), by definition of $\mathrm{Hom}_{\mathrm{FRel}}(T,A,C)$, (Cong)

(5) $(\forall R,S,T)(\circ(R,S,T)$ & $\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(T,A,C))$; by (Gen) on (4)

q.e.d.

- FRel_Δ:

(1) $R\circ S = T \to ((\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(R\circ S,A,C)) \to (\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(T,A,C)))$; (=2)

(2) $\Leftrightarrow R\circ S = T \to ((R \subseteq A\times B$ & $S \subseteq B\times C \to R\circ S \subseteq A\times C) \to (\mathrm{Hom}(R,A,B)$ & $\mathrm{Hom}(S,B,C) \to \mathrm{Hom}(T,A,C)))$; by definition of FRel_Δ-morphisms, (Cong)

(3) $\Rightarrow (R \subseteq A \times B \ \& \ S \subseteq B \times C \to R \circ S \subseteq A \times C) \to (R \circ S = T \to (\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C)))$; by (T2)

(4) $R \subseteq A \times B \ \& \ S \subseteq B \times C \to R \circ S \subseteq A \times C$; lemma 3.1.11

(5) $R \circ S = T \to (\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C))$; by (MP) from (3) and (4)

(6) $\Rightarrow R \circ S = T \ \& \ \mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C)$; by (BL5b)

(7) $\Leftrightarrow \circ(R, S, T) \ \& \ \mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C)$; by (T=2b), by definition of $\circ_{\mathrm{FRel}_\Delta}$, (Cong)

(8) $(\forall R, S, T)(\circ(R, S, T) \ \& \ \mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C))$; by (Gen) on (7)

q.e.d.

- $\mathrm{FRel}^\Delta_\Delta$:

Proven similarly as in the case of FRel_Δ using lemma 3.1.11Δ.

q.e.d.

- FRel^Δ (counter-example):

In any standard model of GFCT, let the only non-zero values be as follows: $\| Ax \| = \| By \| = \| Cz \| = \| Rxy \| = \| Syz \| = 0.5$ and let $\| Txz \| = 0.6$, then $\| T \approxeq R \circ S \ \& \ \Delta(R \subseteq A \times B) \ \& \ \Delta(S \subseteq B \times C) \to \Delta(T \subseteq A \times C) \| = 0$.

q.e.d.

Q.E.D.

Proposition 3.2.6 $(\forall R, S, T, U, V, W)(\circ(R, S, U) \ \& \circ(S, T, V) \to (\circ(U, T, W) \leftrightarrow \circ(R, V, W)))$ is provable w.r.t. all the four systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$.
Proof:

- FRel_Δ and $\mathrm{FRel}^\Delta_\Delta$:

(1) $U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \to U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V$; (T3a)

(2) $\Rightarrow U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \to V = S \circ T \ \& \ W = R \circ V$; by (I6), (BL1)

(3) $V = S \circ T \ \& \ W = R \circ V \to W = R \circ (S \circ T)$; (T=4)

(4) $U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \to W = R \circ (S \circ T)$; by (BL1) from (2) and (3)

(5) $\Leftrightarrow U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \to W = (R \circ S) \circ T$; by associativity of $\circ$ (lemma 3.1.15)

(6) $\Rightarrow U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \rightarrow U = R \circ S \ \& \ W = (R \circ S) \circ T$; by (T4a), (T13a), (T14a), (Cong)

(7) $U = R \circ S \ \& \ W = (R \circ S) \circ T \rightarrow W = U \circ T$; (T=4)

(8) $U = R \circ S \ \& \ V = S \circ T \ \& \ W = R \circ V \rightarrow W = U \circ T$; by (BL1) from (6) and (7)

(9) $\Rightarrow U = R \circ S \ \& \ V = S \circ T \rightarrow (W = R \circ V \rightarrow W = U \circ T)$; by (BL5a)

(10) $U = R \circ S \ \& \ V = S \circ T \rightarrow (W = U \circ T \rightarrow W = R \circ V)$; proven similarly, starting with $U = R \circ S \ \& \ V = S \circ T \ \& \ W = U \circ T \rightarrow U = R \circ S \ \& \ V = S \circ T \ \& \ W = U \circ T$

(11) $U = R \circ S \ \& \ V = S \circ T \rightarrow (W = R \circ V \leftrightarrow W = U \circ T)$; by (I7) from (9) and (10), by definition of $\leftrightarrow$, (Cong)

(12) $\Leftrightarrow \circ(R, S, U) \ \& \ \circ(S, T, V) \rightarrow (\circ(U, T, W) \leftrightarrow \circ(R, V, W))$; by definition of $\circ_{\mathrm{FRel}_\Delta}$ (and $\circ_{\mathrm{FRel}_\Delta^\Delta}$), (Cong)

(13) $(\forall R, S, T, U, V, W)(\circ(R, S, U) \ \& \ \circ(S, T, V) \rightarrow (\circ(U, T, W) \leftrightarrow \circ(R, V, W)))$; by (Gen) on (12)

q.e.d.

- FRel and FRel^Δ:

  Proven similarly as in the previous case of FRel_Δ and $\mathrm{FRel}_\Delta^\Delta$ using lemma 3.1.13 instead of (T=4).

q.e.d.

Q.E.D.

Proposition 3.2.7 $(\forall B)(\exists S)(\forall R, T)(\mathrm{Hom}(S, B, B) \ \& \ (\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(T, B, C) \rightarrow \circ(R, S, R) \ \& \ \circ(S, T, T)))$ is provable in GFCT w.r.t. the systems FRel, FRel^Δ and $\mathrm{FRel}_\Delta^\Delta$.

Proof:

- FRel:

(1) $R \subseteq A \times B \rightarrow R \approxeq R \circ \mathrm{Id}_B$; lemma 3.1.18

(2) $T \subseteq B \times C \rightarrow T \approxeq \mathrm{Id}_B \circ T$; lemma 3.1.18

(3) $\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(T, B, C) \rightarrow \circ(R, \mathrm{Id}_B, R) \ \& \ \circ(\mathrm{Id}_B, T, T)$; from (1) and (2) by (T6) and by definitions of $\mathrm{Hom}_{\mathrm{FRel}}(\_, \_, \_)$, $\circ_{\mathrm{FRel}}$, (Cong)

(4) $\Leftrightarrow \mathrm{Hom}(\mathrm{Id}_B, B, B) \ \& \ (\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(T, B, C) \rightarrow \circ(R, \mathrm{Id}_B, R) \ \& \ \circ(\mathrm{Id}_B, T, T))$; by definition of Id_B and (T20a)

(5) $(\forall B)(\exists S)(\forall R, T)(\mathrm{Hom}(S, B, B) \ \& \ (\mathrm{Hom}(R, A, B) \ \& \ \mathrm{Hom}(T, B, C) \rightarrow \circ (R, S, R) \ \& \ \circ(S, T, T)))$; by (Gen) and ($\exists$1) from (4)

q.e.d.

- FRel^Δ:

Proven similarly as in the case of FRel, using lemma $3.1.18^{\Delta}$ and $\Delta\mathrm{Hom}(\mathrm{Id}_B, B, B)$.

q.e.d.

- $\mathrm{FRel}^{\Delta}_{\Delta}$:

  Proven similarly as in the case of FRel, using lemma $3.1.18^{\Delta}_{\Delta}$ and $\Delta\mathrm{Hom}(\mathrm{Id}_B, B, B)$.

q.e.d.

- The proof of the form used in the previous three cases cannot be used in the case of FRel_{Δ} as the analogue of lemma 3.1.18 required for this case is not valid.

Q.E.D.

3.3 Systems $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$

In this section we consider the four systems of morphisms based on the crisp category Set. The category Set consists of crisp sets as objects and crisp functions as morphisms (see example 2.3.2). So the considered systems consist of fuzzy sets as (potential) objects and Id-functions as morphisms. Systems $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$ can be regarded as certain variants of the previously introduced systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$ in the sense that all have fuzzy sets as objects and morphisms of $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$ are those morphisms of $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$, which in addition are Id-functions. Therefore we examined systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$ in prior to $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$.

Definition 3.3.1 The four basic systems of morphisms based on the crisp category Set are defined as follows:

- The (crisp) set of potential objects is the universal class, which is also the set of objects, i.e. $\mathrm{POb}(\mathrm{FSet}^{(\Delta)}_{(\Delta)}) \overset{\mathrm{df}}{=} \mathrm{Ob}(\mathrm{FSet}^{(\Delta)}_{(\Delta)}) \overset{\mathrm{df}}{=} V$
- The (crisp) set of potential morphisms is the set of all binary fuzzy relations, i.e. subsets of $V \times V$, i.e. $\mathrm{PHom}(\mathrm{FSet}^{(\Delta)}_{(\Delta)}) \overset{\mathrm{df}}{=} \mathrm{Pow}(V \times V)$
- The system FSet is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FSet}}(R, A, B) \overset{\mathrm{df}}{\equiv} R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(F)$
  - $\circ_{\mathrm{FRel}}(R, S, T) \overset{\mathrm{df}}{\equiv} T \approxeq R \circ S$
- The system FSet^{Δ} is defined as follows:
  - $\mathrm{Hom}_{\mathrm{FSet}^{\Delta}}(R, A, B) \overset{\mathrm{df}}{\equiv} \Delta(R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(F))$
  - $\circ_{\mathrm{FSet}^{\Delta}}(R, S, T) \overset{\mathrm{df}}{\equiv} T \approxeq R \circ S$
- The system FSet_{Δ} is defined as follows:

- $\mathrm{Hom}_{\mathrm{FSet}_\Delta}(R, A, B) \stackrel{\mathrm{df}}{\equiv} R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(F)$
- $\circ_{\mathrm{FRel}^\Delta}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T = R \circ S$

- The system $\mathrm{FSet}_\Delta^\Delta$ is defined as follows:

  - $\mathrm{Hom}^{\mathrm{FSet}_\Delta^\Delta}(R, A, B) \stackrel{\mathrm{df}}{\equiv} \Delta(R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(F))$
  - $\circ_{\mathrm{FSet}_\Delta^\Delta}(R, S, T) \stackrel{\mathrm{df}}{\equiv} T = R \circ S$

Let us now consider the same statements as in remark 3.2.2 but w.r.t. the systems $\mathrm{FSet}_{(\Delta)}^{(\Delta)}$.

Proposition 3.3.2 $(\forall R, S)(\exists T)\circ(R, S, T)$ is provable in GFCT w.r.t. all the four systems $\mathrm{FSet}_{(\Delta)}^{(\Delta)}$.

Proof:

Similar to the proof of proposition 3.2.3 because composition of morphisms is defined in exactly the same way in the systems $\mathrm{FSet}_{(\Delta)}^{(\Delta)}$ as in the systems $\mathrm{FRel}_{(\Delta)}^{(\Delta)}$.

Q.E.D.

Proposition 3.3.3 $(\forall R, S, T, T')(\circ(R, S, T) \ \&\ \circ(R, S, T') \to T \approxeq T')$ is provable in GFCT w.r.t. all the four systems $\mathrm{FSet}_{(\Delta)}^{(\Delta)}$.

Proof:

Similar to the proof of proposition 3.2.4 because composition of morphisms is defined in exactly the same way in the systems $\mathrm{FSet}_{(\Delta)}^{(\Delta)}$ as in the systems $\mathrm{FRel}_{(\Delta)}^{(\Delta)}$.

Q.E.D.

Proposition 3.3.4 $(\forall R, S, T)(\circ(R, S, T) \ \&\ \mathrm{Hom}(R, A, B) \ \&\ \mathrm{Hom}(S, B, C) \to \mathrm{Hom}(T, A, C))$ is provable in GFCT w.r.t. the systems FSet_Δ and $\mathrm{FSet}_\Delta^\Delta$ and does not hold w.r.t. the systems FSet and FSet^Δ.

Proof:

- FSet_Δ:

  Proven similarly as in the case of FRel_Δ in proposition 3.2.5, moreover using lemma 3.1.16.

  q.e.d.

- $\mathrm{FSet}_\Delta^\Delta$:

  Proven similarly as in the case of $\mathrm{FRel}_\Delta^\Delta$ in proposition 3.2.5, moreover using lemma 3.1.16Δ.

  q.e.d.

- FSet (counter-example):

In any standard model of GFCT, let the only non-zero values be as follows: $\| Ax \| = \| By \| = \| Cz \| = \| Rxy \| = \| Rxy' \| = \| Syz \| = \| Txz \| = \| y = y' \| = 1$, let $\| Syz' \| = \| z = z' \| = 0$, and let $\| Txz' \| = 0.5$, then $\| T \approxeq R \circ S \ \&\ R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(R) \ \&\ S \subseteq B \times C \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(S) \rightarrow T \subseteq A \times C \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(T) \| = 0$.

q.e.d.

- FSet^{Δ} (counter-example):

  Similarly as in the case of FRel^{Δ} in proposition 3.2.5, in addition let $\| Rxy' \| = \| Syz' \| = 0.5$, $\| y = y' \| = \| z = z' \| = 1$ and $\| Txz' \| = 0.6$, then $\| T \approxeq R \circ S \ \&\ \Delta(R \subseteq A \times B \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(R)) \ \&\ \Delta(S \subseteq B \times C \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(S)) \rightarrow \Delta(T \subseteq A \times C \ \&\ \mathrm{Fnc}_{\mathrm{Id}}(T)) \| = 0$.

q.e.d.

Q.E.D.

Proposition 3.3.5 $(\forall R, S, T, U, V, W)(\circ(R, S, U) \ \&\ \circ(S, T, V) \rightarrow (\circ(U, T, W) \leftrightarrow \circ(R, V, W)))$ is provable w.r.t. all the four systems $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$.

Proof:

Similar to the proof of proposition 3.2.6 because composition of morphisms is defined in exactly the same way in the systems $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$ as in the systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$.

Q.E.D.

Proposition 3.3.6 $(\forall B)(\exists S)(\forall R, T)(\mathrm{Hom}(S, B, B) \ \&\ \mathrm{Hom}(R, A, B) \ \&\ \mathrm{Hom}(T, B, C) \rightarrow \circ(R, S, R) \ \&\ \circ(S, T, T))$ is provable in GFCT w.r.t. the systems FSet, FSet^{Δ} and $\mathrm{FSet}^{\Delta}_{\Delta}$.

Proof:

Similar to the proof of proposition 3.2.7, in addition using lemma 3.1.17 in the case of FSet and its Δ-version (obtained simply by (Nec) on (6) in the proof of lemma 3.1.17) in the cases of FSet^{Δ} and $\mathrm{FSet}^{\Delta}_{\Delta}$.

q.e.d.

- As in the case of proposition 3.2.7, the proof of the previous form cannot be used in the case of FSet_{Δ}.

Q.E.D.

3.4 (Co)limits in FRel_{Δ}

In this final section we present several results obtained by a more datailed examination of the system FRel_{Δ}. These results concern limits and colimits of the most simple digrams within the system FRel_{Δ}. Because within our initial classical paradigm, the category Rel, it is more intuitive to start with colimits (within the respective limit-colimit pair), becuse at least these which we are interested in are the same as those in the category Set, we will proceed in the same way by starting our examination with colimits.

But first of all, fuzzy correlates of the respective classical category-theoretical concepts (see section 2.3) must be defined. Because a purpose of this section is to give just primary results, which can potentially be modified later, we decided to choose the most natural a straightforward way of defining these fuzzy correlates, which means that we simply reinterpreted (in GFCT) the classically-defined notions. Let us present these concepts by more formal expressions (by formulas rather than by diagrams, in contrast to classical category-theoretical manner) at this place, in order to make them more transparent from the point of view of formal (fuzzy) logic.

Definition 3.4.1 Let us fix an arbitrary fuzzy system of morphisms $\mathcal{C}$. In accordance with the classical definitions (see section 2.3) we define the following elementary fuzzy category-theoretical notions as follows:

- (*fuzzy terminal object*) $\mathrm{Term}_{\mathcal{C}}(A) \overset{\mathrm{df}}{\equiv} (\forall B)(\exists f)(\forall g)(\mathrm{Hom}_{\mathcal{C}}(f, B, A)\ \&\ (\mathrm{Hom}_{\mathcal{C}}(g, B, A) \to f = g))$
- (*fuzzy initial object*) $\mathrm{Init}_{\mathcal{C}}(A) \overset{\mathrm{df}}{\equiv} (\forall B)(\exists f)(\forall g)(\mathrm{Hom}_{\mathcal{C}}(f, A, B)\ \&\ (\mathrm{Hom}_{\mathcal{C}}(g, A, B) \to f = g))$
- (*fuzzy product*) $\mathrm{Prod}_{\mathcal{C}}(A \times B, pr_A, pr_B) \overset{\mathrm{df}}{\equiv} (\forall C, f, g)(\mathrm{Hom}_{\mathcal{C}}(f, C, A)\ \&\ \mathrm{Hom}_{\mathcal{C}}(g, C, B) \to (\exists h)(\forall h')(\mathrm{Hom}_{\mathcal{C}}(h, C, A \times B)\ \&\ \circ(h, pr_A, f)\ \&\ \circ(h, pr_B, g)\ \&\ (\mathrm{Hom}_{\mathcal{C}}(h', C, A \times B)\ \&\ \circ(h', pr_A, f)\ \&\ \circ(h', pr_B, g) \to h = h')))$
- (*fuzzy coproduct*) $\mathrm{CoProd}_{\mathcal{C}}(A + B, i_A, i_B) \overset{\mathrm{df}}{\equiv} (\forall C, f, g)(\mathrm{Hom}_{\mathcal{C}}(f, A, C)\ \&\ \mathrm{Hom}_{\mathcal{C}}(g, B, C) \to (\exists h)(\forall h')(\mathrm{Hom}_{\mathcal{C}}(h, A + B, C)\ \&\ \circ(i_A, h, f)\ \&\ \circ(i_B, h, g)\ \&\ (\mathrm{Hom}_{\mathcal{C}}(h', A + B, C)\ \&\ \circ(i_A, h', f)\ \&\ \circ(i_B, h', g) \to h = h')))$

Because within the whole section we work with just the system FRel_Δ we will omit its designation from indexes of all concepts occurring within this section.

The empty set is the only initial object in the crisp category Rel, as well as in Set. The following proposition claims that the empty set is also the fuzzy initial object in the system FRel_Δ. Let us show that whenever $\mathrm{Hom}(R, \emptyset, B)$, then $R = \emptyset$, from which as a simple consequence we get that $\mathrm{Init}(\emptyset)$ holds.

Proposition 3.4.2 $\mathrm{Hom}(R, \emptyset, B) \leftrightarrow R = \emptyset$ is provable in GFCT. As a consequence, $\mathrm{Init}(\emptyset)$ is provable in GFCT.

Proof:

(1) $R \subseteq \emptyset \to R = \emptyset$; lemma 3.1.8

(2) $\Leftrightarrow R \subseteq \emptyset \times B \to R = \emptyset$; by lemma 3.1.9, (T=5), (Cong)

(3) $\Leftrightarrow \mathrm{Hom}(R, \emptyset, B) \to R = \emptyset$; by defintion of $\mathrm{Hom}_{\mathrm{FRel}_\Delta}$, (Cong)

(4) $\emptyset \subseteq \emptyset \times B \to (R = \emptyset \to R \subseteq \emptyset \times B)$; by (T2) from (=2)

(5) $\emptyset \subseteq \emptyset \times B$; lemma 3.1.7

(6) $R = \emptyset \to R \subseteq \emptyset \times B$; by (MP) from (4) and (5)

(7) $\Leftrightarrow R = \emptyset \to \mathrm{Hom}(R, \emptyset, B)$; by defintion of $\mathrm{Hom}_{\mathrm{FRel}_\Delta}$, (Cong)

(8) $\mathrm{Hom}(R, \emptyset, B) \leftrightarrow R = \emptyset$; from (3) and (7) by (T6) and by definition of $\leftrightarrow$

(9) $(\forall R)(\mathrm{Hom}(R, \emptyset, B) \to R = \emptyset)$; by (Gen) on (3)

(10) $\Leftrightarrow (\exists S)(S = \emptyset \ \&\ (\forall R)(\mathrm{Hom}(R, \emptyset, B) \to R = S))$; by (T=6)

(11) $\Leftrightarrow (\exists S)(\mathrm{Hom}(S, \emptyset, B) \ \&\ (\forall R)(\mathrm{Hom}(R, \emptyset, B) \to R = S))$; by (8), (Cong)

(12) $(\forall B)(\exists S)(\mathrm{Hom}(S, \emptyset, B) \ \&\ (\forall R)(\mathrm{Hom}(R, \emptyset, B) \to R = S))$; by (Gen) and (T∀10a) on (11)

(13) $\Leftrightarrow \mathrm{Init}(\emptyset)$; by definition of $\mathrm{Init}(\_)$

Q.E.D.

Dually, the empty set is a also the only terminal object in the category Rel (not so in Set, where it is any singleton!). By checking the previous proof we immediately get the following analogue of proposition 3.4.2.

Proposition 3.4.3 $\mathrm{Hom}(R, A, \emptyset) \to R = \emptyset$ is provable in GFCT. As a consequence, $\mathrm{Term}(\emptyset)$ is provable in GFCT.

Proof:

Similar to the proof of proposition 3.4.2.

Q.E.D.

In the category Rel, as well as in Set, the coproducts of two sets are isomorphic to their disjoint union so let us consider this situation within the system FRel_Δ. Let A, B be two fuzzy sets and let us define $A' \stackrel{\mathrm{df}}{=} \{\langle x, y\rangle \mid Ax \ \&\ y \in \{0\}\} = A \times \{0\}$ and $B' \stackrel{\mathrm{df}}{=} \{\langle x, y\rangle \mid Bx \ \&\ y \in \{1\}\} = B \times \{1\}$, where the constants 0 and 1 stand for (within the context of this section) the sets $\emptyset$ and $\{\emptyset\}$, respectively. We claim that A' and B' are then disjoint.

Proposition 3.4.4 Let A', B' be as defined above. Then $A' \cap B' = \emptyset$ is provable in GFCT.

Proof:

(1) $\langle x, y\rangle \in A' \cap B' \leftrightarrow \langle x, y\rangle \in A' \cap B'$; (T3b)

(2) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow \langle x, y\rangle \in A' \ \&\ \langle x, y\rangle \in B'$; by definition of $\cap$, (Cong)

(3) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow Ax \ \&\ y \in \{0\} \ \&\ Bx \ \&\ y \in \{1\}$; by definitions of A' and B', (Cong)

(4) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow Ax \ \&\ y = \emptyset \ \&\ Bx \ \&\ y = \{\emptyset\}$; by definitions of singleton, 0 and 1, (Cong)

(5) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow Ax \ \&\ Bx \ \&\ \bot$; by lemma 3.1.10, (Cong)

(6) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow \bot$; by (T21a), (Cong)

(7) $\Leftrightarrow \langle x, y\rangle \in A' \cap B' \leftrightarrow \langle x, y\rangle \in \emptyset$; by definition of $\emptyset$, (Cong)

(8) $A' \cap B' = \emptyset$; by lemma 3.1.4

Q.E.D.

So let us define the *disjoint union* of two fuzzy sets A and B as $A + B \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid \langle x, y\rangle \in A' \vee \langle x, y\rangle \in B'\} = A' \cup B'$. Associated with the disjoint union are two fuzzy relations $i_A \subseteq A \times (A + B)$ and $i_B \subseteq B \times (A + B)$, called the *injections*, defined as $i_A \stackrel{\text{df}}{=} \{\langle x, \langle x, y\rangle\rangle \mid Ax \ \&\ y = 0\}$ and $i_B \stackrel{\text{df}}{=} \{\langle x, \langle x, y\rangle\rangle \mid Bx \ \&\ y = 1\}$, respectively. Now suppose some other fuzzy set C together with two fuzzy relations $R \subseteq A \times C$ and $S \subseteq B \times C$ be given. We claim there is a fuzzy relation $T \subseteq (A + B) \times C$ such that $i_A \circ T = R$ and $i_B \circ T = S$ hold.

Proposition 3.4.5 Let $T \subseteq (A + B) \times C$ be a fuzzy relation defined as $T \stackrel{\text{df}}{=} \{\langle\langle x, y\rangle, z\rangle \mid (Rxz \ \&\ y = 0) \vee (Sxz \ \&\ y = 1)\}$. Then $i_A \circ T = R \ \&\ i_B \circ T = S$ is provable in GFCT.

Proof:

(1) $(i_A \circ T)xz \leftrightarrow (i_A \circ T)xz$; (T3b)

(2) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)(\langle x, \langle x, y\rangle\rangle \in i_A \ \&\ \langle\langle x, y\rangle, z\rangle \in T)$; by definition of $i_A \circ T$, (Cong)

(3) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)((Ax \ \&\ y = 0) \ \&\ ((Rxz \ \&\ y = 0) \vee (Sxz \ \&\ y = 1)))$; by definitions of i_A and T, (Cong)

(4) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)((Ax \ \&\ y = 0 \ \&\ Rxz \ \&\ y = 0) \vee (Ax \ \&\ y = 0 \ \&\ Sxz \ \&\ y = 1))$; by (T12a), (Cong)

(5) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)((Ax \ \&\ Rxz \ \&\ y = 0) \vee (Ax \ \&\ Sxz \ \&\ \bot))$; by (T15), lemma 3.1.10, (Cong)

(6) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)((Ax \ \&\ Rxz \ \&\ y = 0) \vee \bot)$; by (T21a), (Cong)

(7) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow (\exists y)(Ax \ \&\ Rxz \ \&\ y = 0)$; by (T21b), (Cong)

(8) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow Ax \ \&\ Rxz \ \&\ (\exists y)(y = 0)$; by (T$\forall$10b), (Cong)

(9) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow Ax \ \&\ Rxz$; by ($\forall$1) from (T=7) (existence of $\emptyset$), (T20a), (Cong)

(10) $\Leftrightarrow (i_A \circ T)xz \leftrightarrow Rxz$; by $R \subseteq A \times C$, (Cong)

(11) $i_A \circ T = R$; by lemma 3.1.4

(12) $i_B \circ T = S$; proven similarly using existence of $\{\emptyset\}$ and $S \subseteq B \times C$

Q.E.D.

Thus we have just proven that there is $T \subseteq (A+B) \times C$ such that R factors through T and i_A and S factors through T and i_B. Moreover, we claim the uniqueness of such fuzzy relation T.

Proposition 3.4.6 Let $T' \subseteq (A+B) \times C$ be a fuzzy relation such that $i_A \circ T' = R$ and $i_B \circ T' = S$. Then $T' = T$ and consequently $\text{CoProd}(A + B, i_A, i_B)$ is provable in GFCT.

Proof:

(1) $\langle\langle x, y\rangle, z\rangle \in T' \leftrightarrow \langle\langle x, y\rangle, z\rangle \in T'$; (T3b)

(2) $\Leftrightarrow \langle\langle x, y\rangle, z\rangle \in T' \leftrightarrow (y = 0 \;\&\; (i_A \circ T')xz) \vee (y = 1 \;\&\; (i_B \circ T')yz)$; by definitions of i_A and i_B, (Cong)

(3) $\Leftrightarrow \langle\langle x, y\rangle, z\rangle \in T' \leftrightarrow (y = 0 \;\&\; Rxz) \vee (y = 1 \;\&\; Syz)$; by $i_A \circ T' = R$ and $i_B \circ T' = S$

(4) $\Leftrightarrow \langle\langle x, y\rangle, z\rangle \in T' \leftrightarrow \langle\langle x, y\rangle, z\rangle \in T$; by definition of T, (Cong)

(5) $T' = T$; by lemma 3.1.4

- The rest of the proof proceeds in a similar way to the proof of proposition 3.4.2.

Q.E.D.

Dually, in the category Rel the disjoint union of two sets is also a product of the two objects (not so in Set, where it is their Cartesian product!), so let us consider this situation within the system FRel_Δ. Again let A and B be two fuzzy sets. Dually to the injections, there are two certain fuzzy relations $pr_A \subseteq (A + B) \times A$ and $pr_B \subseteq (A + B) \times B$, called the *projections*, which are defined as $pr_A \stackrel{\text{df}}{=} \{\langle\langle x, y\rangle, x\rangle \mid Ax \;\&\; y = 0\}$ and $pr_B \stackrel{\text{df}}{=} \{\langle\langle x, y\rangle, x\rangle \mid Bx \;\&\; y = 1\}$, respectively. Now suppose some other fuzzy set C together with two fuzzy relations $R \subseteq C \times A$ and $S \subseteq C \times B$ be given. We claim there is a fuzzy relation $T \subseteq C \times (A + B)$ such that $T \circ pr_A = R$ and $T \circ pr_B = S$ hold.

Proposition 3.4.7 Let $T \subseteq C \times (A + B)$ be a fuzzy relation defined as $T \stackrel{\text{df}}{=} \{\langle z, \langle x, y\rangle\rangle \mid (Rzx \;\&\; y = 0) \vee (Szx \;\&\; y = 1)\}$. Then $T \circ pr_A = R \;\&\; T \circ pr_B = S$ is provable in GFCT.

Proof:

Similar to the proof of proposition 3.4.5.

Q.E.D.

Again, we claim the uniqueness of such fuzzy relation T.

Proposition 3.4.8 Let $T' \subseteq C \times (A+B)$ be a fuzzy relation such that $T' \circ pr_A = R$ and $T' \circ pr_B = S$. Then $T' = T$ and consequently $\text{Prod}(A + B, pr_A, pr_B)$ is provable in GFCT.

Proof:

Similar to the proof of proposition 3.4.6.

Q.E.D.

Chapter 4

Conclusion

The goal of this work was to define and examine several systems of morphisms built up over formal fuzzy logic. Specifically, GFCT, i.e. fuzzy class theory over the logic $G\forall_{\Delta=}$, serves as our foundational theory. This effort of examination of fuzzy correlates of classical categories–paradigms can be broadly described as consisting of the two main steps:

- First, fuzzy correlates of classical category-theoretical concepts must be defined.
- Second, in accordance with these correlates, fuzzy analogues of particular classical categories are introduced and examined.

Let us be more specific about the way this work deals with these tasks.

Concerning the first step, we did not define a fuzzy analogue of a classical concept of a category. Instead we introduced a looser (weaker) concept of a fuzzy system of morphisms. To be more precise, at first instance (definition 3.1.1) we did not require any specific properties from a (ternary fuzzy) relation of composition of (potential) morphisms. Potential suitable properties of such systems of morphisms (their resemblance with the classical concept of a category) are then proved as theorems for particular systems of morphisms.

Concerning the second task, two sorts of systems of morphisms (with four various systems in each sort) have been examined in the work, systems $\mathrm{FRel}^{(\Delta)}_{(\Delta)}$ (section 3.2) and $\mathrm{FSet}^{(\Delta)}_{(\Delta)}$ (section 3.3), respectively. The crisp categories Rel and Set served as the classical paradigms for our examination. Several requirements concerning composition of (potential) morphisms and fuzzy sets of (potential) morphisms between two (potential) objects have been formulated (remark 3.2.2) and examined w.r.t. all the eight systems. Let us stress that for example associativity of composition of morphisms has been proven for each of our systems, while the existence of composition-neutral morphism has been proven just for some of them. Finally (section 3.4), fuzzy correlates of the classical concepts of a terminal object, an initial object, a product and a coproduct were defined and examined w.r.t. one particular system.

Due to restricted and elementary character of the work (proceeding from the very foundations with all the necessary preliminaries included) we were able to consider only very fundamental fuzzy category-theoretical conceps and (limitedly) examine just several very simple systems of morphisms. A wide range of (fuzzy) category-theoretical topics has been completely left untouched and thus there is almost unlimited space for potential subsequent research in the field. Therefore in conclusion let us give a few hints concerning the most interesting ways (in the author's opinion), in which subsequent research in the field could possibly proceed:

- To consider other various definitions of systems of morphism or even give a suitable definition of a fuzzy category.
- To consider systems (possible fuzzy categories), in which (potential) objects are not plain fuzzy sets (as in our case), but fuzzy sets equipped with some additional structure, while (potential) morphisms are the respective structure preserving fuzzy relations (functions). Particularly, fuzzy preorders, fuzzy posets, fuzzy magmas, fuzzy groups etc. could be considered as (potential) objects in these systems and thus fuzzy analogues of crisp categories as Ord, Mag, Grp etc. could be defined and examined.
- To examine (variants of) fuzzy correlates of other classical category-theoretical concepts like these of a functor (faithful, full, forgetful etc.), adjunction, a complete category, a topos etc.
- Finally, background fuzzy logics different from $G\forall_{\Delta =}$ can be considered.

Bibliography

[1] J. Adámek, H. Herrlich, G. E. Strecker. *Abstract and Concrete Categories: The Joy of Cats.*, Wiley, New York, 1990.

[2] L. Běhounek. *Teorie množin v Gödelově fuzzy logice.*, Master thesis, Charles University in Prague, Faculty of Arts, 2002.

[3] L. Běhounek. *Logical Foundations of Fuzzy Mathematics.*, PhD thesis, Charles University in Prague, Faculty of Arts, 2009.

[4] L. Běhounek, U. Bodenhofer, P. Cintula. *Relations in Fuzzy Class Theory: Initial Steps.*, Fuzzy Sets and Systems 159, pp. 1729-1772, 2008.

[5] L. Běhounek, P. Cintula. *Fuzzy Class Theory.*, Fuzzy Sets and Systems 154, pp. 34-55, 2006.

[6] L. Běhounek, P. Cintula. *Fuzzy Class Theory: A Primer v1.0.*, Technical report No. 939, Institute of Computer Science, Academy of Sciences of the Czech Republic, 2006.

[7] L. Běhounek, P. Cintula, P. Hájek. *Introduction to Mathematical Fuzzy Logic.*, In: P. Cintula, P. Hájek, C. Noguera (eds.), Handbook of Mathematical Fuzzy Logic, College Publications, pp. 1-101, 2011.

[8] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles.*, Kluwer Academic/Plenum Publishers, New York, 2002.

[9] K. Chvalovský. *On the Independence of Axioms in BL and MTL.*, Fuzzy Sets and Systems 197, pp. 123-129, 2012.

[10] P. Cintula. *Short Note: On the Redundancy of Axiom (A3) in BL and MTL.*, Soft Computing, Vol. 9, No. 12, pp. 942, 2005.

[11] P. Cintula, P. Hájek. *Triangular Norm Based Predicate Fuzzy Logics.*, Fuzzy Sets and Systems 161, pp. 311-346, 2010.

[12] P. Cintula, P. Hájek, C. Noguera. *Handbook of Mathematical Fuzzy Logic.*, College Publications, 2011.

[13] M. Dummett. *A Propositional Calculus with Denumerable Matrix.*, Journal of Symbolic Logic 24, pp. 97-106, 1959.

[14] S. Eilenberg, S. Mac Lane. *Group Extensions and Homology.*, Annals of Mathematics 43, pp. 757-831, 1942.

[15] S. Eilenberg, S. Mac Lane. *General Theory of Natural Equivalences.*, Transactions of the American Mathematical Society 58, pp. 231-294, 1945.

[16] F. Esteva, L. Godo. *Monoidal T-norm Based Logic: Towards a Logic for Left-continuous T-norms.*, Fuzzy Sets and Systems 124(3), pp. 271-288, 2001.

[17] J. A. Goguen. *L-Fuzzy Sets.*, Journal of Mathematical Analysis and Applications 18, pp. 145-174, 1967.

[18] J. A. Goguen. *Concept Representation in Natural and Artifical Languages: Axioms, Extensions, and Applications for Fuzzy Sets.*, International Journal of Man-Machine Studies 6, pp. 513-516, 1974.

[19] K. Gödel. *Zum intuitionitischen Aussagenkalkül.*, Anzeiger Akademie der Wissenschaften Wien, Math.-naturwissensch. Klasse 69, pp. 65-66, 1932.

[20] R. Goldblatt. *Topoi: The Categorial Analysis of Logic.*, Dover Publications, New York, 2006.

[21] S. Gottwald. *Mehrwertige Logik.*, Akademie-Verlag, Berlin, 1988.

[22] S. Gottwald. *Fuzzy Sets and Fuzzy Logic.*, Vieweg, Weisbaden, 1993.

[23] P. Hájek. *Metamathematics of Fuzzy Logic.*, Kluwer Academic Publishers, 1998.

[24] D. Higgs. *A Category Approach to Boolean-Valued Set Theory.*, Preprint, University of Waterloo, 1973.

[25] D. Higgs. *Injectivity in the Topos of Complete Heyting Algebra Valued Sets.*, Canadian Journal of Mathematics 36, pp. 550-568, 1984.

[26] U. Höhle. *M-valued Sets and Sheaves over Integral Commutative CL-monoids.*, S. E. Rodabaugh et al. (eds.), Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, pp. 33-72, 1992.

[27] U. Höhle. *GL-quantales: Q-valued Sets and Their Singletons.*, Studia Logica 61, pp. 123-148, 1998.

[28] F. W. Lawvere. *An Elementary Theory of the Category of Sets.*, Proceedings of the National Academy of Sciences USA 52, pp. 1506-1511, 1964.

[29] F. W. Lawvere. *The Category of Categories as a Foundation for Mathematics.*, Proceedings of the Conference on Categorical Algebra, La Jolla, Springer-Verlag, New York, pp. 1-21, 1966.

[30] F. W. Lawvere. Introduction and ed. for *Toposes, Algebraic Geometry and Logic.*, Lecture Notes in Mathematics, Vol. 274, Springer-Verlag, pp. 1-12, 1972.

[31] S. Mac Lane. *Categories for the Working Mathematician.*, 2nd edition, Springer-Verlag, New York, 1998.

[32] V. Novák. *Fuzzy Sets and Their Applications.*, Adam Hilger, Bristol, 1989.

[33] V. Novák, I. Perfilieva, J. Močkoř. *Mathematical Principles of Fuzzy Logic.*, Kluwer Academic Publishers, 1999.

[34] S. A. Solovjovs. *On the Category Set(JCPos).*, Fuzzy Sets and Systems 157, pp. 459-465, 2006.

[35] S. A. Solovjovs. *Categories of Lattice-valued Sets as Categories of Arrows.*, Fuzzy Sets and Systems 157, pp. 843-854, 2006.

[36] S. A. Solovjovs. *On a Generalization of Goguen's Category Set(L).*, Fuzzy Sets and Systems 158, pp. 367-385, 2007.

[37] L. N. Stout. *Topoi and Categories of Fuzzy Sets.*, Fuzzy Sets and Systems 12, pp. 169-184, 1984.

[38] L. N. Stout. *A Survey of Fuzzy Sets and Topos Theory.*, Fuzzy Sets and Systems 42, pp. 3-14, 1991.

[39] L. N. Stout. *Categories of Fuzzy Sets with Values in a Quantale or Projectale.*, U. Höhle and E. P. Klement (eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, pp. 219-234, 1995.

[40] A. Šostak. *Towards the Concept of a Fuzzy Category.*, Acta Univ. Latviensis (Ser. Math.) 562, pp. 85-94, 1991.

[41] A. Šostak. *On a Concept of a Fuzzy Category.*, In: 14th Linz Seminar on Fuzzy Set Theory: Non-Classical Logics and Their Applications, Linz, Austria, pp. 63-66, 1992.

[42] A. Šostak. *Fuzzy Categories Versus Categories of Fuzzily Structured Sets: Elements of the Category of Fuzzy Categories.*, In: MathematikArbeitspapiere 48: Categorical Methods in Algebra and Topology, Universität Bremen, pp. 407-437, 1997.

[43] L. A. Zadeh. *Fuzzy Sets.*, Information and Control 8, pp. 338-353, 1965.