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MASTER'S THESIS

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STROMOVÁ VLASTNOST NA VÍCE KARDINÁLECH
THE TREE PROPERTY AT MORE CARDINALS

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Prohlašuji, že jsem diplomovou práci vypracovala samostatně, že jsem řádně citovala všechny použité prameny a literaturu a že práce nebyla využita v rámci jiného vysokoškolského studia či k získání jiného nebo stejného titulu.

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Abstrakt: V této práci se zabýváme Aronszajnovými a speciálními Aronszajnovými stromy, jejich existencí a neexistencí. Zavádíme dnes nejběžněji užívanou definici speciálního Aronszajnova stromu a několik zobecnění této definice a zkoumáme vztahy mezi nimi. Dále se věnujeme stromové a slabé stromové vlastnosti, což je tvrzení, že na daném regulárním kardinálu κ neexistuje žádný Aronszajnův strom, respektive žádný speciální Aronszajnův strom. Definujeme a srovnáváme dva forcingy, Mitchellův a Gregorieffův, a následně je používáme k získání modelu, ve kterém máme (slabou) stromovou vlastnost na daném kardinálu. Nakonec ukážeme jak použít Mitchellův forcing ke konstrukci modelu, ve kterém máme (slabou) stromovou vlastnost na více kardinálech.

Klíčová slova: stromová vlastnost, slabá stromová vlastnost, Máhlův kardinál, slabě kompaktní kardinál, měřitelný kardinál, Aronszajnův strom, speciální Aronszajnův strom,

Abstract: In this thesis we study the Aronszajn and special Aronszajn trees, their existence and nonexistence. We introduce the most common definition of special Aronszajn tree and some of its generalizations and we examine the relations between them. Next we study the notions of the tree property and the weak tree property at a given regular cardinal κ . The tree property means that there are no Aronszajn trees at κ and the weak tree property means that there are no special Aronszajn trees at κ . We define and compare two forcings, the Mitchell forcing and the Grigorieff forcing, and we use them to obtain a model in which the (weak) tree property holds at a given cardinal. At the end, we show how to use the Mitchell forcing to construct a model in which the (weak) tree property holds at more than one cardinal.

Keywords: the tree property, the weak tree property, Mahlo cardinal, weakly compact cardinal, measurable cardinal, Aronszajn tree, special Aronszajn tree

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1 Introduction

We say that a regular cardinal κ has the (weak) tree property if there are no (special) Aronszajn trees at κ .¹ In 1930's, Nachman Aronszajn proved in ZFC that there is a special Aronszajn tree at ω_1 . Therefore ω_1 does not have the (weak) tree property. In 1949, Ernst Specker [Spe49] generalized Aronszajn's original result under an additional cardinality assumption. He proved that if $\kappa^{<\kappa} = \kappa$ then there exists a special Aronszajn tree at κ^+ . Hence if we want the (weak) tree property to hold at successor cardinal we need to violate GCH. On the other hand, if κ is strong limit and regular then κ has the tree property if and only if κ is weakly compact.

In any case, the tree property is connected with the existence of a weakly compact cardinal. In 1972, William Mitchell and Jack Silver [Mit72] proved that the tree property at ω_2 is consistent under the assumption that a weakly compact cardinal exists and the weak tree property at ω_2 is consistent under the assumption that a Mahlo cardinal exists. In the same paper they also proved that these large cardinal assumptions are necessary. In fact they proved it for an arbitrary cardinal κ^{++} , where κ is regular. In 1979, James Baumgartner and Richard Laver [BL79] showed that the tree property at ω_2 can be achieved by iterating the Sacks forcing for ω up to a weakly compact cardinal and in 1980, Akihiro Kanamori [Kan80] generalized this result to arbitrary κ^{++} , where κ is a regular cardinal. The method of Mitchell and Silver uses the fact that the Mitchell forcing is a projection of the product of a κ^+ -Knaster forcing and a κ^+ -closed forcing. On the other hand the method of Baumgartner and Laver uses the fact that the Sacks forcing has the fusion property. Actually only this property is crucial for the proof. Other forcings with the fusion property can be used the same way. In Chapter 4 we present both methods, the second modified for the Grigorieff forcing.

The Mitchell forcing can be used to get the tree property at two non-successive cardinals under the assumption of two weakly compact cardinals and to get the weak tree property at two successive cardinals under the assumption of two Mahlo cardinals. This can be generalized to obtain the tree property at every ω_{2n} , $0 < n < \omega$, under the assumption of ω -many weakly compact cardinals and the weak tree property at every ω_{n+1} , $0 < n < \omega$, under the assumption of ω -many Mahlo cardinals. We present this result in Chapter 5. However, Menachem Magidor showed that to get the tree property at two successive cardinals, at least a measurable cardinal is required, see [Abr83]. In 1983, Uri Abraham [Abr83] showed that this situation is consistent under the assumption of a supercompact cardinal and a weakly

¹Note that the definition of the weak tree property make sense only for successor cardinals, see Definition 3.21.

compact cardinal above it. In 1998, James Cummings and Matthew Foreman [CF98] generalized Abraham's result and showed that if we assume ω -many supercompact cardinals then the tree property at every cardinal ω_n , $1 < n < \omega$, is consistent.

The situation is much more complicated if we consider successor or even double successor of a singular strong limit cardinal κ . In 1996, Menachem Magidor and Saharon Shelah [MS96] proved under large cardinal assumptions that it is consistent that the tree property holds at κ^+ . In 1998, James Cummings and Matthew Foreman [CF98] showed that the tree property can hold at κ^{++} . Recently Itay Neeman [Nee] showed, again under large cardinal assumption, that it is consistent to have the tree property at every ω_n , $1 < n < \omega$ plus at $\aleph_{\omega+1}$. The question whether one can have the tree property at κ^+ and κ^{++} at the same time is still open.

2 Preliminaries

In this thesis we assume a general knowledge of the set theory, especially of forcing, large cardinals, elementary embeddings and trees.

Our notation is more or less standard. However, we use some concepts for which the notation has not been settled yet. Therefore we present our notation here and we give reference to the definitions where we first introduce these concepts.

• \mathbb{R}_κ	Definition 2.16
• M-special ω_1 -Aronszajn tree	Definition 3.7
• \mathcal{S} -special ω_1 -Aronszajn tree	Definition 3.11
• Non-Suslin ω_1 -Aronszajn tree	Definition 3.15
• The weak tree property	Definition 3.22
• M-special κ^+ -Aronszajn tree	Definition 3.26
• \mathcal{S} -special κ^+ -Aronszajn tree	Definition 3.31
• Non-Suslin κ^+ -Aronszajn tree	Definition 3.34

The notation of \mathcal{S} -special ω_1 -Aronszajn tree is motivated by the notation in [She98] and the notation of Non-Suslin ω_1 -Aronszajn tree is taken from [Han81].

The following lemmas are used very often throughout the entire thesis.

Lemma 2.1 (Easton's Lemma). *Assume $\mathbb{P}, \mathbb{Q} \in V$ are forcing notions, \mathbb{P} is κ -cc and \mathbb{Q} is κ -closed. Then the following holds:*

- (i) $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ is $\check{\kappa}$ -cc,
- (ii) $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ is $\check{\kappa}$ -distributive.

Lemma 2.2. *Suppose \mathbb{P} and \mathbb{Q} are forcing notions, $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then the following are equivalent:*

1. $G \times H$ is $\mathbb{P} \times \mathbb{Q}$ -generic over V ,
2. G is \mathbb{P} -generic over V and H is \mathbb{Q} -generic over $V[G]$,
3. H is \mathbb{Q} -generic over V and G is \mathbb{P} -generic over $V[H]$.

Furthermore, if these conditions hold, then $V[G \times H] = V[G][H] = V[H][G]$.

Lemma 2.3 (Silver's lifting lemma). *Let $j : M \rightarrow N$ be an elementary embedding between transitive models of ZFC. Let $\mathbb{P} \in M$ be a notion of forcing and G a \mathbb{P} -generic filter over M . Let H be $j(\mathbb{P})$ -generic over N . Then the following are equivalent:*

(i) $\forall p \in G \ j(p) \in H$.

(ii) *There exists an elementary embedding $j^* : M[G] \rightarrow N[H]$, such that $j^* \upharpoonright M = j$ and $j^*(G) = H$.*

2.1 Trees

As we are interested in the weak tree property and the tree property we present here some common definitions and basic lemmas about trees. These definitions and lemmas can be found in [Jec03].

Definition 2.4. We say that $(T, <)$ is a *tree* if $(T, <)$ is a partial order such that for each $t \in T$, the set $\{s \in T \mid s < t\}$ is wellordered by $<$.

Definition 2.5. We say that $S \subseteq T$ is a *subtree* of $(T, <)$ in the induced ordering $<$ if $\forall s \in S \ \forall t \in T (t < s \rightarrow t \in S)$.

Definition 2.6. Let T be a tree

- (i) If $t \in T$, then $ht(t, T) = ot(\{s \in T \mid s < t\})$ is *height of t* in T ;
- (ii) For each ordinal α , we define the α -*th level* of T as $T_\alpha = \{t \in T \mid ht(t) = \alpha\}$;
- (iii) The *height* of T , $ht(T)$, is the least α such that $T_\alpha = \emptyset$;
- (iv) $T \upharpoonright \alpha = \bigcup_{\beta < \alpha} T_\beta$ is a subtree of T of height α .

Definition 2.7. For a regular $\kappa \geq \omega$, T is called a κ -*tree* if T has height κ , and $|T_\alpha| < \kappa$ for each $\alpha < \kappa$.

Very often, κ -tree is isomorphic to a subtree of the full tree $({}^{<\kappa}\kappa, \subset)$. More precisely, this is the case whenever the κ -tree is normal. See the definition below.

Definition 2.8. A *normal κ -tree* is a tree T such that:

- (i) $ht(T) = \kappa$;
- (ii) $|T_\alpha| < \kappa$, for every $\alpha < \kappa$;
- (iii) $|T_0| = 1$;
- (iv) If $ht(s, T) = ht(t, T)$ is a limit ordinal, then $s = t$ if and only if $\{r \in T \mid r < s\} = \{r \in T \mid r < t\}$.

Note that the conditions (i) and (ii) ensure that a normal κ -tree is a κ -tree.

Lemma 2.9. *Let κ be a regular cardinal. Then every normal κ -tree is isomorphic to a subtree T' of the full tree $({}^{<\kappa}\kappa, \subset)$.*

Proof. We define by induction on $\alpha < \kappa$ isomorphisms $i_\alpha : T \upharpoonright \alpha \rightarrow T' \upharpoonright \alpha$ where $T' \upharpoonright \alpha$ is a subtree of ${}^{<\kappa}\kappa \upharpoonright \alpha$. Since $|T_\alpha| < \kappa$ for each $\alpha < \kappa$ by (ii) of Definition 2.8, there is a 1-1 function $g_\alpha : T_\alpha \rightarrow \kappa$.

Set $T'_0 = \{\emptyset\}$, by (iii) of Definition 2.8, $i_1(r) = \emptyset$ is an isomorphism between T_0 and T'_0 , where r is the unique root of T .

Suppose $i_\beta : T \upharpoonright \beta \rightarrow T' \upharpoonright \beta$ is constructed for each $\beta < \alpha$. First, if α is limit, set $i_\alpha = \bigcup_{\beta < \alpha} i_\beta$ and $T' \upharpoonright \alpha = \bigcup_{\beta < \alpha} T' \upharpoonright \beta$.

If $\alpha = \gamma + 1$ and γ is a successor, then we define i_α by extending i_γ setting for each $s \in T_\gamma$:

$$i_\alpha(s) = i_\gamma(t) \cup \{\langle \gamma, g_\gamma(s) \rangle\}, \quad (2.1)$$

where the node t is the immediate predecessor of s . Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) | s \in T_\gamma\}$.

If $\alpha = \gamma + 1$ and γ is limit, then we define i_α by extending i_γ setting for each $s \in T_\gamma$:

$$i_\alpha(s) = \bigcup \{i_\gamma(t) | t < s\}. \quad (2.2)$$

By (iv) of Definition 2.8, i_α is 1-1 and clearly it is also an isomorphism. Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) | s \in T_\gamma\}$.

At the end, set $T' = \bigcup_{\alpha < \kappa} T' \upharpoonright \alpha$ and $i = \bigcup_{\alpha < \kappa} i_\alpha$. □

Note that if we consider a successor cardinal κ^+ in the previous lemma, then the levels of the κ^+ -tree have size $\leq \kappa$. Hence we can strengthen the formulation of the previous lemma for successor cardinals in the following way.

Corollary 2.10. *Let κ be a cardinal. Every normal κ^+ -tree is isomorphic to a subtree T' of the full tree $({}^{<\kappa^+}\kappa, \subset)$.*

Proof. The proof is the same as before, the only difference is at successor ordinals. Since each $|T_\alpha| < \kappa^+$, so $|T_\alpha| \leq \kappa$ and we can take g_α to be a function from T_α to κ instead of to κ^+ . □

Definition 2.11. Let T be a tree. We say that B is a *branch* if it is a maximal chain in T .

Definition 2.12. Let κ be a regular cardinal. We say that a κ -tree T is a *Aronszajn tree* if it has no branch of size κ . We denote the class of all Aronszajn trees at κ as $\mathbb{A}(\kappa)$.

By König's Lemma, no ω -Aronszajn trees exist. On the other hand, by result of Aronszajn, there exists an ω_1 -Aronszajn tree. Moreover, if we assume GCH, then there exists a κ^+ -Aronszajn tree for each regular cardinal κ , by result of Specker [Spe49]. Therefore it is natural to ask, whether there can be a regular cardinal λ such that there are no λ -Aronszajn trees.

Definition 2.13. We say that a regular cardinal κ has the tree property, if there are no κ -Aronszajn trees.

The following lemma tells us that if we want to get the tree property at κ , it is enough to kill all normal κ -trees. The proof of the lemma is taken from [Jec03].

Lemma 2.14. *Let κ be a regular cardinal. If there exists a κ -Aronszajn tree, then there exists a normal κ -Aronszajn tree.*

Proof. Let T be a κ -Aronszajn tree. T has height κ and each level of T has size less than κ . We first choose one root $t \in T_0$ such that $|\{s \in T | s > t\}| = \kappa$. Let $T' = \{s \in T | s \geq t\}$, then T' satisfies condition (iii) from Definition 2.8.

Now we guarantee the condition (iv). Let $\alpha < \kappa$ be a limit ordinal and t be some node in T'_α . For every chain $C = \{s \in T' | s < t\}$ we add one extra node t_C such that $s < t_C$ for all $s \in C$ and $t_C < r$ for each r such that $r > s$ for all $s \in C$. Since for every chain we add one extra node to limit level, this new tree satisfies (iv). \square

There are two common strengthening of the notion of an Aronszajn tree. The first leads to a notion of a special Aronszajn tree, to which we dedicate an entire chapter later. The second leads to a notion of a Suslin tree.

Definition 2.15. Let κ be a regular cardinal. We say that κ -Aronszajn tree is Suslin, if it has no antichain of size κ . We denote the class of all Suslin trees at κ as $\mathbb{S}(\kappa)$.

The notion of an ω_1 -Suslin tree first appeared in connection with the Suslin problem of the characterization of the real line. Actually, in [Kur35] Kurepa showed that the original Suslin hypothesis (SH) can be formulated as the claim that there are no Suslin trees. For more details about Suslin hypothesis see [Jec03]. However, we consider Suslin trees because they are the opposite of special Aronszajn trees in the sense that no Aronszajn tree can be special and Suslin at the same time.

2.2 Dense Linear Order Without End Points

The structure of rational numbers \mathbb{Q} is countable dense linear order without end points. In this section we consider some common generalizations of \mathbb{Q} at higher

cardinals. The following definitions of \mathbb{Q}_κ and \mathbb{Q}_κ^* are taken from [Tod84]. In addition, we introduce our definition of a generalization of the real line for higher cardinals, because we want to generalize the concept of an \mathbb{R} -embeddable tree (see Definition 3.2) to higher cardinals.

Definition 2.16. Let κ be a regular cardinal. Then

$$\mathbb{Q}_\kappa^* = (\{f \in {}^\omega \kappa \mid \{n < \omega \mid f(n) \neq 0\} \text{ is finite}\} \setminus \{\bar{0}\}, <_{\text{lex}}); \quad (2.3)$$

$$\mathbb{Q}_\kappa = (\{f \in {}^\kappa 2 \mid \{\alpha < \kappa \mid f(\alpha) \neq 0\} < \kappa\} \setminus \{\bar{0}\}, <_{\text{lex}}); \quad (2.4)$$

$$\mathbb{R}_\kappa = (\{f \in {}^\kappa 2 \mid \{\alpha < \kappa \mid f(\alpha) = 0\} \text{ is cofinal in } \kappa\} \setminus \{\bar{0}\}, <_{\text{lex}}); \quad (2.5)$$

where $\bar{0}$ denote the sequence from κ to 2 such that $\bar{0}(\alpha) = 0$ for each $\alpha < \kappa$ and $<_{\text{lex}}$ is the lexicographical ordering. We sometimes write $<_{\mathbb{Q}_\kappa}$, $<_{\mathbb{Q}_\kappa^*}$, $<_{\mathbb{R}_\kappa}$ for the corresponding structure instead of $<_{\text{lex}}$ or just $<$ if it is not confusing.

Remark 2.17. Note that $\mathbb{Q}_\omega \cong \mathbb{Q} \cong \mathbb{Q}_\omega^*$. On the other hand, for $\kappa > \omega$, $\mathbb{Q}_\kappa \not\cong \mathbb{Q}_\kappa^*$, even if $|\mathbb{Q}_\kappa| = \kappa$. This holds, because \mathbb{Q}_κ^* does not contain any decreasing sequence of uncountable length. However, in \mathbb{Q}_κ there are decreasing sequences of length κ .

In this thesis we work mainly with \mathbb{Q}_κ because we managed to prove that it has some nice properties: we managed to generalize Kurepa's Theorem for \mathbb{Q}_κ and prove Lemma 2.19, which is very useful and plays the key role in proving Lemma 3.25. On the other hand, the main advantage of \mathbb{Q}_κ^* is that it always has size κ . When we work with \mathbb{Q}_κ we need to assume that $\kappa^{<\kappa} = \kappa$ for this.

The following lemma tells us that \mathbb{Q}_κ has the properties which we want from a generalization of \mathbb{Q} , with the exception that it does not have to have size κ .

Lemma 2.18. *The ordering \mathbb{Q}_κ is linear, dense, without endpoints and $|\mathbb{Q}_\kappa| = \kappa^{<\kappa}$.*

Proof. The ordering \mathbb{Q}_κ is clearly linear, without endpoints and also $|\mathbb{Q}_\kappa| = \kappa^{<\kappa}$. We verify just density.

Let $f < g$ in \mathbb{Q}_κ be given. Let α be the least ordinal such that $0 = f(\alpha) < g(\alpha) = 1$. Since $|\{\gamma < \kappa \mid f(\gamma) \neq 0\}| < \kappa$, there is the least $\beta > \alpha$ such that $f(\beta) = 0$. Let $h = f \upharpoonright \beta \cup \{\langle \beta, 1 \rangle\} \cup f \upharpoonright (\kappa \setminus (\beta + 1))$. Obviously, h satisfies $f < h < g$. \square

Lemma 2.19. (i) *Let $A = \langle f_\alpha \mid \alpha < \lambda \rangle$ be a decreasing sequence in \mathbb{Q}_κ , where λ is a limit ordinal such that $\omega \leq \lambda < \kappa$. Then A does not have the infimum in \mathbb{Q}_κ .*

(ii) *Let $B = \langle g_\alpha \mid \alpha < \lambda \rangle$ be an increasing sequence in \mathbb{Q}_κ where λ is a limit ordinal such that $\omega \leq \lambda < \kappa$. Then B has the supremum in \mathbb{Q}_κ .*

Proof. Ad (i). Let $A = \langle f_\alpha | \alpha < \lambda \rangle$ be given. Assume for contradiction that there is the infimum $f \in \mathbb{Q}_\kappa$ of A . Since $f \in \mathbb{Q}_\kappa$, there is $\beta_0 < \kappa$ such that for each $\beta \geq \beta_0$ $f(\beta) = 0$. Since $\lambda < \kappa$ and κ is regular, there is $\gamma_0 < \kappa$ such that for each $\gamma \geq \gamma_0$ and for each $\alpha < \lambda$ $f_\alpha(\gamma) = 0$. Let $\delta = \max\{\beta_0, \gamma_0\}$. We define $f^* = f \upharpoonright \delta \cup \{\langle \delta, 1 \rangle\} \cup \{\langle \beta, 0 \rangle | \beta > \delta\}$. Clearly, $f^* > f$. Since $f < f_\alpha$ for every $\alpha < \lambda$ and since $\delta = \max\{\beta_0, \gamma_0\}$, $f^* < f_\alpha$ for every $\alpha < \lambda$. This is a contradiction because we assume that f is the infimum of A .

Ad (ii). Let $B = \langle g_\alpha | \alpha < \lambda \rangle$ be given. We define supremum g by induction on $\beta < \kappa$.

For $\beta = 0$. Set

$$g(0) = \begin{cases} 1 & \text{if } \exists \alpha < \lambda (g_\alpha(0) = 1); \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $g \upharpoonright \beta$ is defined, then we define $g(\beta)$ as follows:

$$g(\beta) = \begin{cases} 1 & \text{if } \exists \alpha < \lambda \text{ such that } g_\alpha(\beta) = 1 \text{ and } g_\alpha \upharpoonright \beta + 1 > g \upharpoonright \beta \cup \langle \beta, 0 \rangle; \\ 0 & \text{otherwise.} \end{cases}$$

First note that g is in \mathbb{Q}_κ since κ is regular and $\lambda < \kappa$.

Now, we show that g is the supremum of B . It is obvious that $g_\alpha < g$ for every $\alpha < \lambda$. Hence it is enough to show that g is the least upper bound of B . Let $h < g$ be given. Then there is $\beta_0 < \kappa$ such that $h \upharpoonright \beta_0 = g \upharpoonright \beta_0$ and $0 = h(\beta_0) < g(\beta_0) = 1$. By definition of g there is α such that $g_\alpha \upharpoonright \beta_0 + 1 > g \upharpoonright \beta_0 \cup \langle \beta_0, 0 \rangle$. As $h \upharpoonright \beta_0 = g \upharpoonright \beta_0$ and $h(\beta_0) = 0$, $g \upharpoonright \beta_0 \cup \langle \beta_0, 0 \rangle = h \upharpoonright \beta_0 + 1$ and so $g_\alpha \upharpoonright \beta_0 + 1 > h \upharpoonright \beta_0 + 1$. Therefore $g_\alpha > h$. \square

Now, we present our generalization of Kurepa's Theorem for \mathbb{Q}_κ , but let us first recall the formulation of the original Kurepa's Theorem for ω and \mathbb{Q} .

Theorem 2.20. (Kurepa's Theorem) *Let $(E, <)$ be a partially ordered set. Then the following are equivalent:*

- (i) E is embeddable in \mathbb{Q} ;
- (ii) E is the union of at most ω -many antichains.

Lemma 2.21. (Generalized Kurepa's Theorem) *Assume $\kappa^{<\kappa} = \kappa$. Let $(E, <)$ be a partially ordered set. Then the following are equivalent:*

- (i) E is embeddable in \mathbb{Q}_κ ;

(ii) E is the union of at most κ -many antichains.

Proof. (i) \Rightarrow (ii) Let f be the embedding. Let $\{q_\alpha \mid \alpha < \kappa\}$ be an enumeration of \mathbb{Q}_κ . We define $A_\alpha = f^{-1}(q_\alpha)$ for each $q_\alpha \in \text{Rng}(f)$. Obviously, each A_α is an antichain since f is the embedding.

(ii) \Leftarrow (i) We assume that $\bigcup_{\alpha < \kappa} A_\alpha = E$, where each A_α is an antichain. Let $f : E \rightarrow \kappa$ be a function such that $A_\alpha = f^{-1}(\alpha)$. For $x \in E$ define $g(x)$ so that $g(x)(\alpha) = 1$ if and only if $\alpha \leq f(x)$ and $\{y \in E \mid y \leq x\} \cap A_\alpha \neq \emptyset$.

Notice that $\text{Rng}(g)$ is a subset of \mathbb{Q}_κ because $g(x)(\alpha) = 1$ implies that $\alpha \leq f(x)$, where $f(x) \in \kappa$.

Now, we check that g is an embedding. Assume that $x < y$ are in E and $x \in A_\alpha$, $y \in A_\beta$ for some $\beta \neq \alpha$. We distinguish two cases.

First suppose that $\alpha < \beta$. Then $g(x)(\alpha) = 1$ and also $g(y)(\alpha) = 1$ since $x < y$ and $x \in A_\alpha$. And for all $\gamma < \alpha$ if $g(x)(\gamma) = 1$ then $g(y)(\gamma) = 1$ and so $g(x) \upharpoonright \alpha \leq_{\text{lex}} g(y) \upharpoonright \alpha$. If $g(x) \upharpoonright \alpha <_{\text{lex}} g(y) \upharpoonright \alpha$, then $g(x) < g(y)$ and we are finished. If $g(x) \upharpoonright \alpha = g(y) \upharpoonright \alpha$, then we can continue as follows: for all $\gamma > \alpha$ it holds that $g(x)(\gamma) = 0$ since $\gamma > f(x)$. Hence $g(x)(\beta) = 0$ and $g(y)(\beta) = 1$; therefore $g(x) < g(y)$.

Next suppose that $\beta < \alpha$. Again for all $\gamma < \beta$, if $g(x)(\gamma) = 1$ then $g(y)(\gamma) = 1$ and so $g(x) \upharpoonright \beta \leq_{\text{lex}} g(y) \upharpoonright \beta$. Now, we show that $g(x)(\beta) = 0$ and $g(y)(\beta) = 1$. Assume for contradiction that $g(x)(\beta) = 1$. Then by definition of the function g , we know there exists $z \in A_\beta$ and $z \leq x$. Hence $z < y$ and this is a contradiction since there are two comparable elements in A_β . By the definition of g , $g(y)(\beta) = 1$ and so $g(x) < g(y)$. \square

Remark 2.22. Note that the assumption $\kappa^{<\kappa} = \kappa$ is necessary just in the proof of (i) \Rightarrow (ii). Note also that the proof for the case $\kappa = \omega$ is the proof of Kurepa's Theorem.

Partial orders from Lemma 2.21 have another useful characterization.

Lemma 2.23. *Let κ be regular and let $(E, <)$ be a partially ordered set. Then the following are equivalent:*

(i) E is the union of at most κ -many antichains;

(ii) there is $f : E \rightarrow \kappa$ such that if s, t are comparable in E , then $f(s) \neq f(t)$.

Proof. (i) \Rightarrow (ii) Since E is the union of at most κ -many antichains, $E = \bigcup_{\alpha < \kappa} A_\alpha$, where A_α is an antichain for each α . We define $f : E \rightarrow \kappa$ as follows: $f(s) = \alpha$ if and only if $s \in A_\alpha$. Clearly, if $s < t$ then s and t are in different antichains, hence $f(s) \neq f(t)$.

(ii) \Rightarrow (i) Let f be the function from the definition. Then $A_\alpha = f^{-1}(\alpha)$ is antichain for each $\alpha < \kappa$. \square

Now, we focus on the partial order \mathbb{R}_κ . We show that it has similar properties as \mathbb{R} .

Lemma 2.24. *The partial order \mathbb{R}_κ is*

(i) *linear, without endpoints;*

(ii) *\mathbb{Q}_κ is dense in \mathbb{R}_κ ;*

(iii) *Dedekind complete.*

Proof. It is easy to verify that \mathbb{R}_κ satisfies (i).

Ad (ii). Let $f <_{\mathbb{R}_\kappa} g$ in \mathbb{R}_κ be given. Let α be the least ordinal such that $0 = f(\alpha) < g(\alpha) = 1$. Since $\{\alpha < \kappa \mid f(\alpha) = 0\}$ is cofinal in κ , there is the least $\beta > \alpha$ such that $f(\beta) = 0$. Let $h = f \upharpoonright \beta \cup \{\langle \beta, 1 \rangle\} \cup \{\langle \gamma, 0 \rangle \mid \gamma > \beta\}$. Since $h(\alpha) = f(\alpha) < g(\alpha)$, $h <_{\mathbb{R}_\kappa} g$ and since $f(\beta) < h(\beta)$, $f <_{\mathbb{R}_\kappa} h$.

Ad (iii). It is enough to show that each increasing sequence with upper bound has the supremum. First note that each increasing sequence in \mathbb{R}_κ has cardinality at most $\kappa^{<\kappa}$ since \mathbb{Q}_κ is dense in \mathbb{R}_κ as we proved in the previous paragraph. Let $A = \langle f_\alpha \in \mathbb{R}_\kappa \mid \alpha < \lambda \rangle$ for some ordinal $\lambda \leq \kappa^{<\kappa}$ be given and let $f \in \mathbb{R}_\kappa$ be the upper bound of A . Let F_C be a choice function from $\mathcal{P}(\mathbb{Q}_\kappa)$ to \mathbb{Q}_κ . We define the sequence $A_{\mathbb{Q}_\kappa}$ in \mathbb{Q}_κ as follows:

$$A_{\mathbb{Q}_\kappa} = \langle g_\alpha \in \mathbb{Q}_\kappa \mid g_\alpha = F_C(\{q \in \mathbb{Q}_\kappa \mid f_\alpha < q < f_{\alpha+1}\}) \text{ and } \alpha < \lambda \rangle. \quad (2.6)$$

We show that $A_{\mathbb{Q}_\kappa}$ has the supremum g in \mathbb{R}_κ and that g is also the supremum of A in \mathbb{R}_κ . We define a function $g^* : \kappa \rightarrow 2$ by induction on $\beta < \kappa$.

For $\beta = 0$. Set

$$g^*(0) = \begin{cases} 1 & \text{if } \exists \alpha < \lambda (g_\alpha(0) = 1); \\ 0 & \text{otherwise.} \end{cases}$$

Let $g^* \upharpoonright \beta$ be defined, then we define $g^*(\beta)$ as follows:

$$g^*(\beta) = \begin{cases} 1 & \text{if } \exists \alpha < \lambda \text{ such that } g_\alpha(\beta) = 1 \text{ and } g_\alpha \upharpoonright \beta + 1 > g^* \upharpoonright \beta \cup \langle \beta, 0 \rangle; \\ 0 & \text{otherwise.} \end{cases}$$

Note that g^* may not be in \mathbb{R}_κ , but it holds that $g^* \neq \{\langle \alpha, 1 \rangle \mid \alpha < \kappa\}$ since the sequence has an upper bound in \mathbb{R}_κ .

Now, we need to show that g^* is the supremum of $A_{\mathbb{Q}_\kappa}$ in $(2^\kappa, <_{\text{lex}})$. However, the proof of this is the same as the proof of Lemma 2.19 (ii). Note that in the Lemma 2.19 (ii) we used the assumption that the sequence has length less than κ just for showing that the supremum is in \mathbb{Q}_κ .

As we mentioned earlier, g^* may not be in \mathbb{R}_κ , but note that $g^* \neq \{\langle \alpha, 1 \rangle \mid \alpha < \kappa\}$. If g^* is not in \mathbb{R}_κ , there is $\beta_0 < \kappa$ such that $g^*(\beta_0) = 0$ and $g^*(\beta) = 1$ for every $\beta > \beta_0$. Let $\bar{g} = g^* \upharpoonright \beta_0 \cup \{\langle \beta_0, 1 \rangle\} \cup \{\langle \beta, 0 \rangle \mid \beta > \beta_0\}$. Clearly $\bar{g} \in \mathbb{R}_\kappa$ and there is no function between g^* and \bar{g} in 2^κ . Now we define $g \in \mathbb{R}_\kappa$ by

$$g = \begin{cases} g^* & \text{if } g^* \in \mathbb{R}_\kappa; \\ \bar{g} & \text{otherwise.} \end{cases}$$

It is obvious that $g \in \mathbb{R}_\kappa$ and since g^* is the supremum of $A_{\mathbb{Q}_\kappa}$ in 2^κ , g is the supremum of $A_{\mathbb{Q}_\kappa}$ in \mathbb{R}_κ .

To finish the proof of the theorem, it suffices to show that g is also the supremum of A . The function g is clearly the upper bound of A . Now, we show that g is the least upper bound. Let $h < g$. Since g is the supremum of $A_{\mathbb{Q}_\kappa}$, there is $q \in A_{\mathbb{Q}_\kappa}$, such that $h < q$. But $q < r$ for some $r \in A$ by the definition of $A_{\mathbb{Q}_\kappa}$. Hence $h < r$. \square

2.3 Projection and Complete Embedding

In this section we define the concepts of projection and complete embedding, which are very useful for comparing forcing notions. We also present several definitions and facts concerning these concepts, which we will need in Chapter 4. For more details about projection see [Abr10] and about complete embedding see [Kun80].

Definition 2.25. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders. We say that a function $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a projection if

- (i) $\forall p, p' \in P (p \leq_P p' \rightarrow \pi(p) \leq_Q \pi(p'))$;
- (ii) $\forall p \in P \forall q \in Q (q \leq_Q \pi(p) \rightarrow \exists p' \in P (p' \leq_P p \text{ and } \pi(p') \leq q))$;
- (iii) $\pi'' P$ is dense in \mathbb{Q} .

Fact 2.26. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders. If there is a projection $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ and $D \subseteq Q$ is dense, then $\pi^{-1}'' D$ is dense in \mathbb{P} . Hence if G is a \mathbb{P} -generic over V , then $\langle \pi'' G \rangle = \{q \in Q \mid (\exists p \in G)(\pi(p) \leq q)\}$ is a \mathbb{Q} -generic filter over V .

Definition 2.27. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders and the map $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ be a projection. If H is a \mathbb{Q} -generic filter over V , then we define in $V[H]$ the following partially ordered set

$$\mathbb{P}/\mathbb{Q} = \{p \in P \mid \pi(p) \in G\} \quad (2.7)$$

with the induced ordering.

Fact 2.28. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders, the filter G be a \mathbb{P} -generic over V , the filter H be a \mathbb{Q} -generic over V and the map $\pi : P \rightarrow Q$ be a projection. Then filter G is also \mathbb{P}/\mathbb{Q} -generic over $V[H]$. The other direction is also true, if $G_0 \subseteq \mathbb{P}/\mathbb{Q}$ is \mathbb{P}/\mathbb{Q} -generic over $V[H]$, then G_0 is \mathbb{P} -generic over V .

Fact 2.29. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders and the map $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ be a projection. Then \mathbb{P} is forcing equivalent to the two-step iteration $\mathbb{Q} * \mathbb{P}/\mathbb{Q}$.

Definition 2.30. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders. We say that a function $i : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding if

- (i) $\forall p, p' \in P (p \leq_P p' \rightarrow \pi(p) \leq_Q \pi(p'))$;
- (ii) $\forall p, p' \in P (p \perp p' \leftrightarrow \pi(p) \perp \pi(p'))$;
- (iii) $\forall q \in Q \exists p \in P \forall p' \leq p (i(p') \parallel q)$.

Fact 2.31. Let $\mathbb{P} = (P, \leq_P)$ and $\mathbb{Q} = (Q, \leq_Q)$ be two partial orders, the filter H be a \mathbb{Q} -generic over V and the map $\pi : P \rightarrow Q$ be a complete embedding. Then $i^{-1}H$ is \mathbb{P} -generic over V and $V[i^{-1}H] \subseteq V[H]$.

3 Special Aronszajn Trees

Aronszajn tree was first constructed by Nachman Aronszajn and the construction can be found in [Kur35]. The constructed tree was actually a special Aronszajn tree. In this chapter we examine the definition of a special Aronszajn tree and its generalizations since in the following chapter we destroy all special Aronszajn trees at a given cardinal. Hence it is interesting to find out which other kinds of Aronszajn tree will be destroyed as well.

3.1 Special Aronszajn Trees of Height ω_1

Special Aronszajn trees of height ω_1 has more equivalent ways how to define them. Some of them can be naturally generalized. In this section we investigate these generalizations. In this section, when we talk about Aronszajn trees, we mean ω_1 -Aronszajn trees.

Definition 3.1. We say that an ω_1 -Aronszajn tree T is *special* if T is a union of countable many antichains. We denote the class of all special Aronszajn trees at ω_1 as $\mathbb{A}^{sp}(\omega_1)$.

Definition 3.2. Let κ be a regular cardinal, T be a κ -Aronszajn tree and $\mathbb{P} = \langle P, <_{\mathbb{P}} \rangle$ be a partially ordered set. We say that T is \mathbb{P} -embeddable if there is a function $f : T \rightarrow P$ such that $s <_T t \rightarrow f(s) <_{\mathbb{P}} f(t)$. We denote the class of all \mathbb{P} -embeddable trees at κ as $\mathbb{T}(\mathbb{P})(\kappa)$.

Lemma 3.3. *The following are equivalent for an ω_1 -Aronszajn tree T :*

- (i) T is special;
- (ii) There is $f : T \rightarrow \omega$ such that if s, t are comparable in T , then $f(s) \neq f(t)$;
- (iii) T is \mathbb{Q} -embeddable.

Proof. (iii) \Rightarrow (ii). Let f be the embedding and let i is any bijection between \mathbb{Q} and ω . Then $g = f \circ i$ is the desired function.

(ii) \Rightarrow (i). Let f be the function from the definition. Then $A_n = f^{-1}(n)$ is an antichain for each $n < \omega$.

(i) \Rightarrow (iii). This follows from Kurepa's Theorem. □

When we work with \mathbb{Q} -embeddable Aronszajn trees it is natural to consider also \mathbb{R} -embeddable Aronszajn trees and ask what is the connection between them. The following lemma tells us how to characterize \mathbb{R} -embeddable Aronszajn trees using \mathbb{Q} -embeddable Aronszajn trees. It was first proved in [Bau70], but the proof presented here is taken from [She98].

Lemma 3.4. *Let T be an ω_1 -tree. T is \mathbb{R} -embeddable if and only if $T^* = \bigcup_{\alpha < \omega_1} T_{\alpha+1}$ is \mathbb{Q} -embeddable.*

Proof. (\Rightarrow) Let T be \mathbb{R} -embeddable and $T^* = \bigcup_{\alpha < \omega_1} T_{\alpha+1}$. Let f be the embedding, $t \in T^*$ and let $s \in T$ be the immediate predecessor of t . We define $f' : T^* \rightarrow \mathbb{Q}$ as follows: $f'(t) = q$ where $q \in \mathbb{Q}$ such that $f(s) < q < f(t)$.

(\Leftarrow) Let $T^* = \bigcup_{\alpha < \omega_1} T_{\alpha+1}$ be \mathbb{Q} -embeddable and let f be the embedding. We define function $g : \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q}$ by induction on ω such that for each $q < q'$ it holds that $g(q)$ and $g(q')$ are disjoint open intervals. Moreover, for every $p \in g(q)$ it holds that $p < p'$ for all $p' \in g(q')$. Let $\mathbb{Q} = \{q_n | n < \omega\}$. We construct by induction embeddings $g_n : \{q_m | m \leq n\} \rightarrow \mathbb{Q} \times \mathbb{Q}$ and $g = \bigcup_{n < \omega} g_n$.

We set $g_0 = \{\langle q_0, \langle q_0^1, q_0^2 \rangle \rangle\}$ where q_0^1 and q_0^2 are arbitrary elements of \mathbb{Q} such that $q_0^1 < q_0^2$. Suppose we have constructed g_m and let $n = m + 1$. If $q_n > q_k$ for all $k < n$, then we set $g_n = g_m \cup \{\langle q_n, \langle q_n^1, q_n^2 \rangle \rangle\}$ where $q_n^1, q_n^2 > q_m^2$ for all $m < n$ and $q_n^1 < q_n^2$. If $q_n < q_m$ for all $m < n$, the construction is analogous.

If there are $k, l < n$ such that q_n is between q_k and q_l , then we use the density of \mathbb{Q} . Since $q_k^2 < q_l^1$, there is some r_0 such that $q_k^2 < r_0 < q_l^1$. Again by density, there is some r_1 between r_0 and q_l^1 . We set $g_n = g_m \cup \{\langle q_n, \langle r_0, r_1 \rangle \rangle\}$. Let $g = \bigcup_{n < \omega} g_n$.

Now, we define a function $i : \mathbb{Q} \rightarrow \mathbb{Q}$ by $i(q) = r$ where r is some element of $g(q)$. We define an embedding $f' : T \rightarrow \mathbb{R}$ as follows:

$$f'(t) = \begin{cases} i(f(t)) & \text{if } t \in T_{\alpha+1} \text{ for } \alpha < \omega_1; \\ \sup \{i(f(s)) | s < t \text{ and } s \in T_{\beta+1} \text{ and } \beta < \alpha\} & \text{otherwise.} \end{cases}$$

Now we need to check that the function f' is the embedding of T to \mathbb{R} . If $s < t$ and $s, t \in T^*$, then it is easy to see that $f'(s) < f'(t)$ because i is order-preserving. If $t \in T_\alpha$ for α limit, then $f'(s) < f'(t)$ since $f'(t)$ is the supremum. The only interesting case is $s \in T_\alpha$ for α limit and $t \in T_{\alpha+1}$. Then we need to show

$$f'(t) = i(f(t)) > \sup \{i(f(r)) | r < s \text{ and } r \in T_{\beta+1} \text{ and } \beta < \alpha\} = f'(s) \quad (3.1)$$

This follows from the construction of g . For every $r < s$ it holds that $i(f(r)) < q < i(f(t))$ where q is left boundary of $g(f(t))$. Hence

$$f'(s) = \sup \{i(f(r)) | r < s \text{ and } r \in T_{\beta+1} \text{ and } \beta < \alpha\} \leq q < i(f(t)) = f'(t). \quad (3.2)$$

□

As we showed in Lemma 2.14, if there exists an Aronszajn tree, then there exists a normal Aronszajn tree. The same holds for \mathbb{R} -embeddable Aronszajn trees.

Lemma 3.5. *If there exists an \mathbb{R} -embeddable ω_1 -Aronszajn tree, then there exists a normal \mathbb{R} -embeddable ω_1 -Aronszajn tree.*

Proof. Let T be an \mathbb{R} -embeddable ω_1 -Aronszajn tree. Since T is a ω_1 -Aronszajn tree, T has height ω_1 and each level of T has size less than ω_1 . First, we choose one root $t \in T_0$ such that $|\{s \in T \mid s > t\}| = \omega_1$. Let $T' = \{s \in T \mid s \geq t\}$. Since T' is a subtree of T , T' is \mathbb{R} -embeddable.

Now we guarantee the condition (iv) of Definition 2.8. Let S be equal to $\{\alpha < \omega_1 \mid \alpha \text{ is a limit ordinal}\}$. We define a relation of equivalence on T' by $s \sim t$ if and only if $\{u \in T' \mid u <_{T'} s\} = \{v \in T' \mid v <_{T'} t\}$ and $ht(s, T) \in S$. Let $T'' = T' / \sim$. The order in T'' is: $[s] <_{T''} [t]$ if $\exists s' \in [s]$ such that $s' < t$.

It is immediate to verify that T'' is a normal Aronszajn tree. We show that T'' is \mathbb{R} -embeddable. Since T' is \mathbb{R} -embeddable, there is an embedding $f : T' \rightarrow \mathbb{R}$. We define $g : T'' \rightarrow \mathbb{R}$ by

$$g([t]) = \begin{cases} f(t) & \text{if } ht(t, T') \notin S \\ \sup \{f(s) \mid s < t\} & \text{otherwise.} \end{cases}$$

It is easy to verify that g is an embedding. □

Remark 3.6. Compare the previous proof with the proof of Lemma 2.14. Note that the construction from Lemma 2.14 does not have to preserve \mathbb{R} -embeddability since we add new nodes to the tree and it can happen that it is impossible to extend the original embedding to these new nodes. For instance: Let T be an \mathbb{R} -embeddable ω_1 -tree and f be the witnessing embedding. It can happen that there is a node t such that $\sup(\{f(s) \mid s < t\}) = f(t)$, hence if we add the new node t^* between $\{s \in T \mid s < t\}$ and t as in the proof of Lemma 2.14, then we can not extend the original embedding to t^* .

Now, we introduce the concept of an M-special Aronszajn tree. Even if it not clear at first glance, this concept is related to the concept of \mathbb{R} -embeddable Aronszajn trees. As we showed in Corollary 2.10, each normal ω_1 -Aronszajn tree can be represented as a subtree of $(\omega^{<\omega_1}, \subset)$. Hence it is quite natural to ask what trees can be represented as a subtree of $(\{f \in \omega^{<\omega_1} \mid f \text{ is 1-1}\}, \subset)$.

Definition 3.7. We say that an ω_1 -Aronszajn tree T is *M-special* if T is isomorphic to the subtree of $\{s \in {}^{<\omega_1}\omega \mid s \text{ is 1-1}\}$. We denote the class of all M-special ω_1 -Aronszajn trees as $\mathbb{A}^{M-sp}(\omega_1)$.

Remark 3.8. We use the notation M-special to distinguish special Aronszajn trees defined by Mitchell in [Mit72] from now more established Definition 3.1. Note that

Mitchell's definition includes just normal trees in contrast to Definition 3.1. In this sense the notion of a special tree is more general than M-special. However, if we consider just normal trees then each special tree can be represented by an M-special tree.

The next lemma appears to be a part of the set-theoretic folklore.

Lemma 3.9. *If T is a normal special ω_1 -Aronszajn tree then T is M-special.*

Proof. If T is special then $T = \bigcup_{n < \omega} A_n$ where A_n is an antichain for each $n < \omega$. Since $|T_\alpha| \leq \omega$ for each $\alpha < \omega_1$, there is a 1-1 function $g_\alpha : T_\alpha \rightarrow \omega$.

We define by induction on $\alpha < \omega_1$ isomorphisms $i_\alpha : T \upharpoonright \alpha \rightarrow T' \upharpoonright \alpha$ where T' is a subtree of $\{s \in {}^{<\omega_1}(\omega \times \omega) \mid s \text{ is 1-1}\}$.

Set $T'_0 = \{\emptyset\}$ and $i_1(r) = \emptyset$, where r is the root of T . As we assume that T is normal, i_1 is an isomorphism between $T \upharpoonright 1$ and $T' \upharpoonright 1$.

Suppose that we have constructed $i_\beta : T \upharpoonright \beta \rightarrow T' \upharpoonright \beta$ for each $\beta < \alpha$. First, if α is limit, set $i_\alpha = \bigcup_{\beta < \alpha} i_\beta$ and $T' \upharpoonright \alpha = \bigcup_{\beta < \alpha} T' \upharpoonright \beta$.

If $\alpha = \gamma + 1$ and γ is a successor, then we define i_α by extending i_γ setting for each $s \in T_\gamma$:

$$i_\alpha(s) = i_\gamma(t) \cup \{\langle \gamma, \langle g_\gamma(s), n \rangle \rangle\}, \quad (3.3)$$

where the node t is the immediate predecessor of s and $s \in A_n$. Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) \mid s \in T_\gamma\}$. It is clear that each function in T'_γ is 1-1 since each two comparable nodes must be in different antichains.

If $\alpha = \gamma + 1$ and γ is limit, then we define i_α by extending i_γ setting for each $s \in T_\gamma$:

$$i_\alpha(s) = \bigcup \{i_\gamma(t) \mid t < s\}. \quad (3.4)$$

By (iv) of Definition 2.8, i_α is 1-1 and clearly it is also an isomorphism. Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) \mid s \in T_\gamma\}$. Again it is obvious that each function in T'_γ is 1-1 since it is a union of 1-1 functions with gradually increasing domains.

At the end, set $T' = \bigcup_{\alpha < \omega_1} T' \upharpoonright \alpha$ and $i = \bigcup_{\alpha < \omega_1} i_\alpha$. It is easy to see that the tree T' is isomorphic to a subtree of $\{s \in {}^{<\omega_1}\omega \mid s \text{ is 1-1}\}$ by any bijection between $\omega \times \omega$ and ω . Hence T is M-special. \square

Note that at limit step we use just the assumption that the tree is normal. Hence we can generalize this lemma to \mathbb{R} -embeddable trees. The proof of the implication from left to right can be found in [Dev72].

Lemma 3.10. *Let T be an ω_1 -Aronszajn tree. T is normal \mathbb{R} -embeddable if and only if T is M-special.*

Proof. (\Rightarrow) Let T be a normal \mathbb{R} -embeddable. Then $T^* = \bigcup_{\alpha < \omega_1} T_{\alpha+1}$ is \mathbb{Q} -embeddable and so $T^* = \bigcup_{n < \omega} A_n$ where A_n is an antichain for each n . The rest of the proof is the same as the proof of Lemma 3.9 since we used the antichains only in the successor step.

(\Leftarrow) Let T be \mathbb{M} -special. We define $f : T \rightarrow \mathbb{R}$ by setting $f(t) = \sum_{i=0}^{\infty} \frac{\mathcal{X}_{\text{Rng}(t)}(i)}{10^i}$, where \mathcal{X}_X is the characteristic function of set X . Since every node of T is a 1-1 function from some ordinal $\alpha < \omega_1$ to ω , if $s < t$ then $\text{Rng}(s) \subset \text{Rng}(t)$ and so there is $n < \omega$ such that $0 = \mathcal{X}_{\text{Rng}(s)}(n) < \mathcal{X}_{\text{Rng}(t)}(n) = 1$ and $\mathcal{X}_{\text{Rng}(s)} \upharpoonright n = \mathcal{X}_{\text{Rng}(t)} \upharpoonright n$. Hence $f(s) < f(t)$. \square

By Lemma 3.4, if the tree T is \mathbb{R} -embeddable then $T \upharpoonright S$ for $S = \{\alpha + 1 \mid \alpha < \omega_1\}$ is \mathbb{Q} -embeddable. So it is natural to introduce the concept of S -special for arbitrary unbounded subset of $S \subseteq \omega_1$. The following definition is from [She98].

Definition 3.11. Let S be an unbounded subset of ω_1 . We say that an ω_1 -tree T is S -special if $T \upharpoonright S$ is \mathbb{Q} -embeddable, where $T \upharpoonright S = \{t \in T \mid ht(t, T) \in S\}$ with the induced ordering. We say that an ω_1 -tree T is \mathcal{S} -special if there is S , an unbounded subset of ω_1 , such that T is S -special. We denote the class of all \mathcal{S} -special ω_1 -Aronszajn trees as $\mathbb{A}^{\mathcal{S}\text{-sp}}(\omega_1)$.

The following lemma from [DJ74] tells us that if we would consider S -special trees only for S closed unbounded subset of ω_1 , we would not get anything new.

Lemma 3.12. *Let C be a closed unbounded subset of ω_1 . If T is a C -special ω_1 -Aronszajn tree, then T is special.*

Proof. Let T be a C -special ω_1 -Aronszajn tree. Then $T \upharpoonright C = \bigcup_{n < \omega} A_n$, where each A_n is an antichain. Let $\{a_\alpha^n \mid \alpha < \omega_1\}$ be an enumeration of A_n for each $n < \omega$. Let $\{c_\alpha \mid \alpha < \omega_1\}$ be the monotone enumeration of C . For $\alpha < \omega_1$ and for $x \in T_{c_\alpha}$, we define $S_x = \{y \in T \upharpoonright c_{\alpha+1} \mid x <_T y\}$. Since each S_x is countable, let $\{s_m(x) \mid m < \omega\}$ be an enumeration of S_x . Set

$$A_{n,m} = \{s_m(a_\alpha^n) \mid \alpha < \omega_1\}. \quad (3.5)$$

Clearly, $A_{n,m}$ is an antichain of T for each $n, m < \omega$. Since C is closed unbounded, $T = \bigcup_{n < \omega} A_n \cup \bigcup_{n,m < \omega} A_{n,m}$. Hence T is special. \square

The following result, which can be found in [She98], tells us that if we have a model where all Aronszajn trees are S -special for some given unbounded subset of ω_1 , then in such model all Aronszajn trees are already special. As an easy corollary, we have that there is no model where all Aronszajn trees are \mathbb{R} -embeddable and there is a tree which is not special.

Lemma 3.13. *Let S be an unbounded subset of ω_1 . If every ω_1 -Aronszajn tree is S -special then every ω_1 -Aronszajn tree is special.*

Proof. Let $S = \{\alpha_\mu \mid \mu < \omega_1\}$ be an unbounded subset of ω_1 and T be a S -special ω_1 Aronszajn tree. We define a new tree

$$T' = \{\langle t, \beta \rangle \mid t \in T \text{ and } \beta < \alpha_{ht(t,T)} \text{ and } \forall s < t (\alpha_{ht(s,T)} < \beta)\}. \quad (3.6)$$

The tree T' is ordered by $<_{T'}$ as follows: $\langle t, \beta \rangle <_{T'} \langle s, \gamma \rangle$ if and only if $t < s$ or $(t = s \text{ and } \beta < \gamma)$. It is obvious that T' satisfies our definition of Aronszajn tree. Hence T' is S -special, i.e. $T' \upharpoonright S$ is special. Since T is isomorphic to $T' \upharpoonright S = \{\langle t, \alpha_{ht(t,T)} \rangle \mid t \in T\}$, T is special. \square

Corollary 3.14. *If each ω_1 -Aronszajn tree is \mathbb{R} -embeddable, then each ω_1 -Aronszajn tree is \mathbb{Q} -embeddable.*

Proof. Follows from previous lemma and Lemma 3.4. \square

Note that \mathcal{S} -special Aronszajn trees, including special, \mathbb{R} -embeddable and \mathbb{M} -special Aronszajn trees, are not Suslin in a strong sense. This means that every uncountable subset of such tree contains an uncountable antichain. See the following definition and lemma.

Definition 3.15. We say that an ω_1 -tree T is *non-Suslin* if every uncountable subset U of T contains an uncountable antichain. We denote the class of all non-Suslin Aronszajn trees at ω_1 as $\mathbb{A}^{NS}(\omega_1)$.

The name of non-Suslin trees is inspired by the fact that every non-Suslin tree is not Suslin. On the other hand, every tree that is not non-Suslin has a Suslin subtree, as follows from the next lemma that can be found in [Han81].

Lemma 3.16. *Let T be an ω_1 -Aronszajn tree. If T is not non-Suslin, then T has a subtree which is Suslin.*

Proof. Let T be an ω_1 -Aronszajn tree, which is not non-Suslin. Then there is a subset X of T such that $|X| = \omega_1$ and X does not contain antichain of size ω_1 . Let $T' = \{s \in T \mid (\exists t \in X)(s < t)\}$.

Now, we show that T' is Suslin. Assume for contradiction that $A \subseteq T'$ is an uncountable antichain. Then for any choice function $f : \mathcal{P}(X) \rightarrow X$, the set $\{f(\{s \in X \mid a \leq s\}) \mid a \in A\}$ has size ω_1 and it is an antichain in X . \square

Lemma 3.17. *Let T be an ω_1 -Aronszajn tree. If T is \mathcal{S} -special, then T is non-Suslin.*

Proof. Assume for contradiction that T is an \mathcal{S} -special ω_1 -Aronszajn tree which is not non-Suslin. By the previous lemma T has subtree T' which is Suslin. Since T is \mathcal{S} -special, T' is \mathcal{S} -special, too. Hence there is an unbounded subset S of ω_1 such that $T' \upharpoonright S = \bigcup_{n < \omega} A_n$, where A_n is an antichain for each n . By pigeon-hole principle, for some $n < \omega$ the size of A_n must be greater than ω . This contradicts the fact that T' is Suslin. \square

Now, we know that

$$\mathbb{A}^{sp}(\omega_1) \subseteq \mathbb{T}(\mathbb{R})(\omega_1) \subseteq \mathbb{A}^{\mathcal{S}\text{-}sp}(\omega_1) \subseteq \mathbb{A}^{NS}(\omega_1). \quad (3.7)$$

In the next section, we examine if these inclusions can be consistently proper.

3.1.1 The Existence of Special Aronszajn Trees of Height ω_1

The existence of special Aronszajn tree at ω_1 can be proved in ZFC and by Baumgartner's theorem published in [BMR70] it is consistent with ZFC that every Aronszajn tree at ω_1 is special, so $\mathbb{A}^{sp}(\omega_1) = \mathbb{T}(\mathbb{R})(\omega_1) = \mathbb{A}^{\mathcal{S}\text{-}sp}(\omega_1) = \mathbb{A}^{NS}(\omega_1)$ is consistent with ZFC. On the other hand, consistently, each inclusion can be proper.

The following theorem was first published in [Bau70] but the proof, which we present here, is based on [Dev72]. This theorem tells us that it is consistent that there is an Aronszajn tree which is M-special but not special. As a corollary we obtain that the first inclusion in (3.7) can be consistently proper.

Theorem 3.18. *Assume \diamond . Then there is a non-special Aronszajn tree which is a subtree of $\{s \in {}^{<\omega_1}\omega \mid s \text{ is 1-1}\}$.*

Proof. By \diamond there is a sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ such that $f_\alpha : \alpha \rightarrow \alpha$ for each α and for any function $f : \omega_1 \rightarrow \omega_1$ the set $\{\alpha < \omega_1 \mid f_\alpha = f \upharpoonright \alpha\}$ is stationary in ω_1 . We fix this sequence for the rest of the proof.

We construct the tree T and the function $\pi : T \rightarrow \omega_1$, which will code the tree in ω_1 , by induction on $\alpha < \omega_1$. For each α we require the following conditions:

- (T1) If $s \in T \upharpoonright \alpha$ then $|\omega \setminus \text{Rng}(s)| = \aleph_0$.
- (T2) If $s \in T \upharpoonright \alpha$ and $x \in [\omega \setminus \text{Rng}(s)]^{<\omega}$ then there is $s' \supseteq s$ on each higher level of $T \upharpoonright \alpha$ such that $\text{Rng}(s') \cap x = \emptyset$.
- ($\pi 0$) π_α is a 1-1 map from $T \upharpoonright \alpha$ to ω_1 such that $s \subset t \rightarrow \pi_\alpha(s) < \pi_\alpha(t)$.

Let $T_0 = \{\emptyset\}$. It is clear that T_0 satisfies both conditions.

Let $\alpha = \beta + 1$. Suppose $T \upharpoonright \beta + 1$ and $\pi_{\beta+1}$ are defined and they satisfy the conditions mentioned above. We want to construct level T_α . For each $s \in T_\beta$ we add

all one-point extensions $s \cup \langle \alpha, n \rangle$ of s such that $n \in \omega \setminus \text{Rng}(s)$. This is possible by (T1), which guarantees the existence of \aleph_0 such extensions. Since we add all such extensions of s , for each $x \in [\omega \setminus \text{Rng}(s)]^{<\omega}$ we can always find $t \in T_\alpha$ such that $s \subseteq t$ and $x \cap \text{Rng}(s) = \emptyset$ so $T \upharpoonright \alpha + 1$ satisfies (T2). Since $T \upharpoonright \beta + 1$ satisfies (T1), $T \upharpoonright \alpha + 1$ satisfies (T1), too. To obtain $\pi_{\alpha+1}$, we extend $\pi_{\beta+1}$ arbitrarily such that it satisfies the condition $(\pi 0)$.

Let α be limit. For each $\beta < \alpha$, suppose $T \upharpoonright \beta$ and π_β are defined and they satisfy the conditions mentioned above. Let $T'_\alpha = \bigcup_{\beta < \alpha} T_\beta$ and $\pi'_\alpha = \bigcup_{\beta < \alpha} \pi_\beta$. We must decide which α -branches of T'_α we put into T_α . We have to distinguish two cases. First, if f_α embeds $\pi'_\alpha''T'_\alpha$ to \mathbb{Q} , and $\text{Dom}(f_\alpha) = \pi'_\alpha''T'_\alpha$, then set

$$X_\alpha = \{(s, x) \mid s \in T'_\alpha \ \& \ x \in [\omega]^{<\omega} \ \& \ \text{Rng}(s) \cap x = \emptyset\}. \quad (3.8)$$

For $(s, x), (t, y)$ in X_α , we define $(s, x) \leq_\alpha (t, y)$ if and only if $s \subseteq t$ and $x \subseteq y$. This is a partial order on X_α . For each $q \in \mathbb{Q}$, set

$$\begin{aligned} \Delta_q^\alpha = \{ & (s, x) \in X_\alpha \mid f_\alpha(\pi'_\alpha(s)) \geq_{\mathbb{Q}} q \text{ or} \\ & (\forall (t, y) \in X_\alpha)((t, y) \geq_\alpha (s, x) \rightarrow f_\alpha(\pi'_\alpha(t)) <_{\mathbb{Q}} q)\}. \end{aligned} \quad (3.9)$$

It is easy to see that each Δ_q^α is cofinal in X_α . Let $(s, x) \in X_\alpha$ be given. If $f_\alpha(\pi'_\alpha(s)) \geq_{\mathbb{Q}} q$ then (s, x) is in Δ_q^α . If $f_\alpha(\pi'_\alpha(s)) <_{\mathbb{Q}} q$ and $\forall (t, y) \geq_\alpha (s, x)(f_\alpha(\pi'_\alpha(t)) <_{\mathbb{Q}} q)$ then (s, x) is in Δ_q^α , too. The last option is that $f_\alpha(\pi'_\alpha(s)) <_{\mathbb{Q}} q$ and there is $(t, y) \geq_\alpha (s, x)$ such that $f_\alpha(\pi'_\alpha(t)) \geq_{\mathbb{Q}} q$. Then by the definition of Δ_q^α , (t, y) is in Δ_q^α .

Let $s \in T'_\alpha$ and $x \in [\omega \setminus \text{Rng}(s)]^{<\omega}$. First, fix an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ with limit α and with $\alpha_0 = \text{length}(s)$. Let $g : \omega \rightarrow Q$ be a bijection. Let $s'_0 = s$ and $x'_0 = x$. By definition of X_α , (s'_0, x) is in X_α . As $\Delta_{g(0)}^\alpha$ is cofinal in X_α , we can find $(s_0, x_0) \geq_\alpha (s'_0, x'_0)$ in $\Delta_{g(0)}^\alpha$. By (T1) there is $m_0 \in \omega \setminus (\text{Rng}(s) \cup x_0)$. Let $x'_1 = x_0 \cup \{m_0\}$. By (T2) we can find $s'_1 \in T'_\alpha$ such that $s'_1 \supseteq s_0$, $\text{length}(s'_1) \geq \alpha_1$ and $\text{Rng}(s'_1) \cap x'_1 = \emptyset$. By definition of X_α , (s'_1, x'_1) is in X_α . As $\Delta_{g(1)}^\alpha$ is cofinal in X_α , hence we can find $(s_1, x_1) \geq_\alpha (s'_1, x'_1)$ in $\Delta_{g(1)}^\alpha$. Pick $m_1 \in \omega \setminus (\text{Rng}(s) \cup x_1)$, set $x_2 = x_1 \cup \{m_1\}$, and proceed inductively.

In the other case, if f_α does not embed $\pi'_\alpha''T'_\alpha$ to \mathbb{Q} , then we continue similar as before. Let $s \in T'_\alpha$ and $x \in [\omega \setminus \text{Rng}(s)]^{<\omega}$. First, fix an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ with limit α and with $\alpha_0 = \text{length}(s)$. Let $s_0 = s$ and $x_0 = x$. By (T1) there is $m_0 \in \omega \setminus (\text{Rng}(s) \cup x_0)$. Let $x_1 = x_0 \cup \{m_0\}$. By (T2) we can find $s_1 \in T'_\alpha$ such that $s_1 \supseteq s_0$, $\text{length}(s_1) \geq \alpha_1$ and $\text{Rng}(s_1) \cap x_1 = \emptyset$. Pick $m_1 \in \omega \setminus (\text{Rng}(s) \cup x_1)$, set $x_2 = x_1 \cup \{m_1\}$, and proceed inductively.

Let $s_x = \bigcup_{n < \omega} s_n$. Then s_x is an α sequence of natural numbers, which defines an α branch of T'_α . Let $T_\alpha = \{s_x \mid s \in T'_\alpha \ \& \ x \in [\omega \setminus \text{Rng}(s)]^{<\omega}\}$ and $T \upharpoonright \alpha + 1 = T'_\alpha \cup T_\alpha$.

It holds that $s_x \supseteq s$ and $\text{Rng}(s_x) \cap x = \emptyset$. This guarantees that condition (T2) is satisfied. Since $\{m_n | n < \omega\} \cap \text{Rng}(s_x) = \emptyset$, condition (T1) is satisfied. We can extend π'_α to $\pi_{\alpha+1}$ on $T \upharpoonright \alpha + 1$ arbitrarily such that it satisfies the condition $(\pi 0)$.

Finally, set $T = \bigcup_{\alpha < \omega_1} T_\alpha$ and $\pi = \bigcup_{\alpha < \omega_1} \pi_\alpha$. Then $\pi : T \rightarrow \omega_1$ is a 1-1 function such that $s \subseteq t \rightarrow \pi(s) <_{\omega_1} \pi(t)$.

For a contradiction, assume T is \mathbb{Q} -embeddable. Then there is a function f which embeds $\pi''T$ in \mathbb{Q} . Let

$$\begin{aligned} C = \{ \alpha < \omega_1 | \alpha \text{ is a limit ordinal and } \pi''T_\alpha = \pi'_\alpha''T'_\alpha \text{ and} \\ f \upharpoonright \alpha \text{ embeds } \pi'_\alpha''T'_\alpha \text{ in } \mathbb{Q} \text{ and} \\ (\forall s \in T'_\alpha)(\forall x \in [\omega \setminus \text{Rng}(s)]^{<\omega})(\forall q >_{\mathbb{Q}} f(\pi(s))) \\ ((\exists t \in T)(t \supseteq s \ \& \ \text{Rng}(t) \cap x = \emptyset \ \& \ f(\pi(t)) \geq_{\mathbb{Q}} q) \\ \rightarrow (\exists t' \in T'_\alpha)(t' \supseteq s \ \& \ \text{Rng}(t') \cap x = \emptyset \ \& \ f(\pi(t')) \geq_{\mathbb{Q}} q)) \}. \end{aligned} \quad (3.10)$$

It is easy to verify that C is closed unbounded subset of ω_1 . By \diamond , the set $\{\alpha \in \omega_1 | f \upharpoonright \alpha = f_\alpha\}$ is stationary so there is $\alpha \in C$ such that $f \upharpoonright \alpha = f_\alpha$. Let $t \in T_\alpha$. Let $q = f(\pi(t))$. By the construction of T , there is $(s, x) \in \Delta_q^\alpha$ such that $\text{Rng}(s) \cap x = \emptyset$ and $s \subset t$. Since f and π are order-preserving, $f(\pi(s)) <_{\mathbb{Q}} f(\pi(t)) = q$.

Since $f(\pi(s)) <_{\mathbb{Q}} q$ and $f(\pi(t)) \geq_{\mathbb{Q}} q$, by definition of C there exists $t' \in T'_\alpha$ such that $t' \supseteq s$, $\text{Rng}(t') \cap x = \emptyset$ and $f(\pi(t')) \geq_{\mathbb{Q}} q$. Note that (s, x) and (t', x) are in X_α and $(s, x) \leq_\alpha (t', x)$. Since (s, x) is in Δ_q^α , by (3.9) it must hold that $f_\alpha(\pi(s)) \geq_{\mathbb{Q}} q$. But $f_\alpha = f \upharpoonright \alpha$ and so $f(\pi(s)) \geq_{\mathbb{Q}} q$. This contradicts our earlier inequality $f(\pi(s)) <_{\mathbb{Q}} q$. \square

Corollary 3.19. *Assume \diamond . Then there is an \mathbb{R} -embeddable ω_1 -Aronszajn tree which is not special.*

Proof. By Lemma 3.10, every M-special ω_1 -Aronszajn tree is \mathbb{R} -embeddable. \square

The following lemma is a consequence of Theorem 3.18 and it shows us that the second inclusion in (3.7) can be consistently proper.

Lemma 3.20. *Assume \diamond . Then there is an ω_1 -Aronszajn tree, which is \mathcal{S} -special and it is not \mathbb{R} -embeddable.*

Proof. By Corollary 3.19 there is an ω_1 -Aronszajn tree which is \mathbb{R} -embeddable, but not \mathbb{Q} -embeddable. Let $\alpha < \omega_1$ be a limit ordinal and let $t \in T_\alpha$. For chain $C = \{s \in T | s < t\}$ we add a new node t_C such that $t_C < t$ and $t_C > s$ for all $s \in C$. Consider the tree T' which is created by adding such node for each limit node. Note that $\bigcup_{\alpha < \omega_1} T'_{\alpha+1} = T \setminus T_0$. Now, T' is not \mathbb{R} -embeddable since $\bigcup_{\alpha < \omega_1} T'_{\alpha+1}$

is not \mathbb{Q} -embeddable. But T' is S -special for $S = \{\alpha + 2 \mid \alpha < \omega_1\}$ since $T' \upharpoonright S = \bigcup_{\alpha < \omega_1} T_{\alpha+1} \setminus T_1$. \square

The claim that the last inclusion in (3.7) can be consistently proper is a consequence of the theorem published in [Sch14], which says that if ZFC is consistent, so is ZFC + SH + there is an Aronszajn tree T such that it is not \mathcal{S} -special. If SH holds, then by Lemma 3.16 each Aronszajn tree is non-Suslin. Therefore T is non-Suslin and it witnesses that ZFC + $\mathbb{A}^{\mathcal{S}\text{-sp}}(\omega_1) \neq \mathbb{A}^{NS}(\omega_1)$ is consistent.

The following picture illustrates our situation.

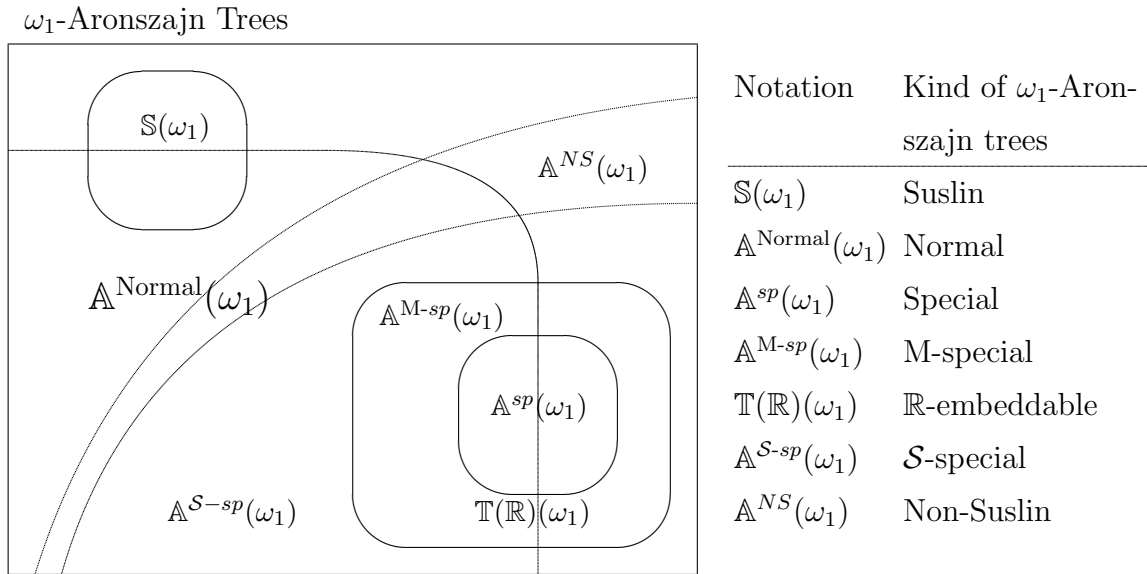


Figure 1: Description of the relations between various kinds of ω_1 -Aronszajn trees.

3.2 Higher Special Aronszajn Trees

In the previous section we have built the foundations for the investigation of special κ^+ -Aronszajn trees for any regular κ . We introduced the concept of special, \mathbb{R} -embeddable, M -special and \mathcal{S} -special ω_1 -Aronszajn trees. Now, we generalize these concepts to higher Aronszajn trees, which are in the center of our interest. When we talk about an Aronszajn tree in this section, we mean a κ^+ -Aronszajn tree for some regular cardinal $\kappa > \omega$.

Definition 3.21. Let κ be a cardinal. We say that κ^+ -Aronszajn tree T is *special* if T is a union of κ -many antichains. We denote the class of all special Aronszajn trees at κ^+ as $\mathbb{A}^{\text{sp}}(\kappa^+)$.

The notion of the weak tree property is connected to special Aronszajn trees in the same way as the tree property is connected to Aronszajn trees.

Definition 3.22. We say that a cardinal κ^+ has the weak tree property, if there are no special κ^+ -Aronszajn trees.

As in the previous section, the concept of a special Aronszajn tree has more equivalent definitions. However, we need to be careful when we talk about \mathbb{Q}_κ -embeddability, since this partial order does not have to have size κ .

Lemma 3.23. *Let κ be regular. The following are equivalent for a κ^+ -Aronszajn tree T :*

- (i) T is special;
- (ii) There is $f : T \rightarrow \kappa$ such that if s, t are comparable in T , then $f(s) \neq f(t)$.

Proof. The proof follows from Lemma 2.23. □

Lemma 3.24. *Assume $\kappa^{<\kappa} = \kappa$. Then κ^+ -Aronszajn tree T is special if and only if T is \mathbb{Q}_κ -embeddable.*

Proof. It follows from Lemma 2.21. □

Again as in the previous section, we can characterize \mathbb{R}_κ -embeddable Aronszajn trees using \mathbb{Q}_κ -embeddable Aronszajn trees. This is our generalization of Lemma 3.4.

Lemma 3.25. *Let κ be a regular cardinal. Let T be an κ^+ -tree. T is \mathbb{R}_κ -embeddable if and only if $T^* = \bigcup_{\alpha < \kappa^+} T_{\alpha+1}$ is \mathbb{Q}_κ -embeddable.*

Proof. (\Rightarrow) Let T be \mathbb{R}_κ -embeddable and $T^* = \bigcup_{\alpha < \kappa^+} T_{\alpha+1}$. Let f be the embedding, $t \in T^*$ and let $s \in T$ be the immediate predecessor of t . We define $f' : T^* \rightarrow \mathbb{Q}_\kappa$ as follows: $f'(t) = q$ where $q \in \mathbb{Q}_\kappa$ such that $f(s) < q < f(t)$.

(\Leftarrow) Let $T^* = \bigcup_{\alpha < \kappa^+} T_{\alpha+1}$ be \mathbb{Q}_κ -embeddable and let f be the embedding. We define function $g : \mathbb{Q}_\kappa \rightarrow \mathbb{Q}_\kappa \times \mathbb{Q}_\kappa$ by induction on κ such that for each $q < q'$ it holds that $g(q)$ and $g(q')$ are disjoint open intervals. Moreover, for every $p \in g(q)$ it holds that $p < p'$ for all $p' \in g(q')$. We enumerate \mathbb{Q}_κ by $\{q_\beta | \beta < \kappa\}$. The construction is quite similar to the construction from 3.4. We construct by induction on $\alpha < \kappa$ embeddings $g_\alpha : \{q_\beta | \beta \leq \alpha\} \rightarrow \mathbb{Q}_\kappa \times \mathbb{Q}_\kappa$ and $g = \bigcup_{\alpha < \kappa} g_\alpha$.

Set $g_0 = \{\langle q_0, \langle q_0^1, q_0^2 \rangle \rangle\}$.

Let $\alpha = \delta + 1$. We show just the most difficult case when q_α is between two sequences $A = \langle f_\beta | \beta < \lambda \rangle$ and $B = \langle h_\gamma | \gamma < \xi \rangle$, where A is increasing, B is decreasing, $\lambda, \xi < \kappa$ and for each $\beta < \lambda, \gamma < \xi$ we have $f_\beta, h_\gamma \in \text{Dom}(g_\delta)$. The proof for the other cases is similar. Now, we need to find something between $g_\delta'' A = \langle g_\delta(f_\beta) | \beta < \lambda \rangle$ and $g_\delta'' B = \langle g_\delta(h_\gamma) | \gamma < \xi \rangle$. Let

$$g_\delta'' A^2 = \langle g_\delta(f_\beta)^2 | \exists a \in \mathbb{Q}_\kappa \langle a, g_\delta(f_\beta)^2 \rangle = g_\delta(f_\beta) \text{ for } \beta < \lambda \rangle. \quad (3.11)$$

$$g''_\delta B^1 = \langle g_\delta(h_\gamma)^1 | \exists b \in \mathbb{Q}_\kappa \langle g_\delta(h_\gamma)^1, b \rangle = g_\delta(h_\gamma) \text{ for } \gamma < \xi \rangle. \quad (3.12)$$

Then $g''_\delta A^2$ is an increasing sequence of length $\lambda < \kappa$ and $g''_\delta B^1$ is a decreasing sequence of length $\xi < \kappa$. Hence by Lemma 2.19 the sequence $g''_\delta B^1$ does not have the infimum and the sequence $g''_\delta A^2$ has the supremum f . Since $g''_\delta B^1$ does not have the infimum, there is $q_2 > f$ such that $q_2 < g_\delta(h_\gamma)^1$ for each $\gamma < \xi$. By the density of \mathbb{Q}_κ there is some q_1 between f and q_2 . Set $g_\alpha = g_\delta \cup \{\langle q_\alpha, \langle q_1, q_2 \rangle \rangle\}$.

Let α be a limit ordinal. Then the proof is similar to the proof for α successor, but first we take $g_\alpha^* = \bigcup_{\delta < \alpha} g_\delta$. Then we continue as in the successor step with g_α^* instead of g_δ .

Let $g = \bigcup_{\alpha < \kappa} g_\alpha$.

Now, we define a function $i : \mathbb{Q}_\kappa \rightarrow \mathbb{Q}_\kappa$ by $i(q) = r$, where r is some element of $g(q)$. We define an embedding $f' : T \rightarrow \mathbb{R}_\kappa$ as follows:

$$f'(t) = \begin{cases} i(f(t)) & \text{if } t \in T_{\alpha+1} \text{ for } \alpha < \kappa^+; \\ \sup \{i(f(s)) | s < t \text{ and } s \in T_{\beta+1} \text{ and } \beta < \alpha\} & \text{otherwise.} \end{cases}$$

Now we need to check that the function f' is the embedding of T to \mathbb{R}_κ . If $s < t$ and $s, t \in T^*$, then it is easy to see that $f'(s) < f'(t)$ because i is order-preserving. If $t \in T_\alpha$ for α limit, then $f'(s) < f'(t)$ since $f'(t)$ is the supremum. The only interesting case is if $s \in T_\alpha$ for α limit and $t \in T_{\alpha+1}$. Then we need to show

$$f'(t) = i(f(t)) > \sup \{i(f(r)) | r < s \text{ and } r \in T_{\beta+1} \text{ and } \beta < \alpha\} = f'(s). \quad (3.13)$$

This follows from the construction of g . For every $r < s$ it holds that $i(f(r)) < q < i(f(t))$ where q is the left boundary of $g(f(t))$. Hence

$$f'(s) = \sup \{i(f(r)) | r < s \text{ and } r \in T_{\beta+1} \text{ and } \beta < \alpha\} \leq q < i(f(t)) = f'(t). \quad (3.14)$$

□

The generalization of the concept of an M-special Aronszajn tree at higher cardinals is of a particular interest to us since we use it in Chapter 4 for showing that the weak tree property is consistent at double successor of a regular cardinal under the assumption of Mahlo cardinal.

Definition 3.26. Let κ be a cardinal. We say that κ^+ -Aronszajn tree T is *M-special* if T is isomorphic to a subtree of $\{s \in {}^{<\kappa^+}\kappa | s \text{ is 1-1}\}$

The following lemma is a generalization of Lemma 3.9.

Lemma 3.27. *Let κ be a regular cardinal. If T is a normal special κ^+ -Aronszajn tree then T is M-special.*

Proof. If T is special then $T = \bigcup_{\xi < \kappa} A_\xi$ where A_ξ is an antichain for each $\xi < \kappa$. Since $|T_\alpha| \leq \kappa$ for each $\alpha < \kappa^+$, there is a 1-1 function $g_\alpha : T_\alpha \rightarrow \kappa$.

We define by induction on $\alpha < \kappa^+$ isomorphisms $i_\alpha : T \upharpoonright \alpha \rightarrow T' \upharpoonright \alpha$ where T' is a subtree of $\left\{ s \in {}^{<\kappa^+}(\kappa \times \kappa) \mid s \text{ is 1-1} \right\}$.

Set $T'_0 = \{\emptyset\}$ and $i_1(r) = \emptyset$, where r is the root of T . As we assume that T is normal, i_1 is an isomorphism between $T \upharpoonright 1$ and $T' \upharpoonright 1$.

Suppose that we have constructed $i_\beta : T \upharpoonright \beta \rightarrow T' \upharpoonright \beta$ for each $\beta < \alpha$. First, if α is limit, set $i_\alpha = \bigcup_{\beta < \alpha} i_\beta$ and $T' \upharpoonright \alpha = \bigcup_{\beta < \alpha} T' \upharpoonright \beta$.

If $\alpha = \gamma + 1$ and γ is a successor, then we define i_α by extending i_γ , setting for each $s \in T_\gamma$:

$$i_\alpha(s) = i_\gamma(t) \cup \{\langle \gamma, \langle g_\gamma(s), \xi \rangle \rangle\}, \quad (3.15)$$

where the node t is the immediate predecessor of s and $s \in A_\xi$. Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) \mid s \in T_\gamma\}$. It is clear that each function in T'_γ is 1-1 since each two comparable nodes must be in different antichains.

If $\alpha = \gamma + 1$ and γ is limit, then we define i_α by extending i_γ , setting for each $s \in T_\gamma$:

$$i_\alpha(s) = \bigcup \{i_\gamma(t) \mid t < s\}. \quad (3.16)$$

By (iv) of Definition 2.8, i_α is 1-1 and clearly it is also an isomorphism. Let $T' \upharpoonright \alpha = T' \upharpoonright \gamma \cup T'_\gamma$, where $T'_\gamma = \{i_\alpha(s) \mid s \in T_\gamma\}$. Again it is obvious that each function in T'_γ is 1-1 since it is a union of 1-1 functions with gradually increasing domains.

At the end, set $T' = \bigcup_{\alpha < \kappa^+} T' \upharpoonright \alpha$ and $i = \bigcup_{\alpha < \kappa^+} i_\alpha$. It is easy to see that the tree T' is isomorphic to a subtree of $\left\{ s \in {}^{<\kappa^+}\kappa \mid s \text{ is 1-1} \right\}$ by any bijection between $\kappa \times \kappa$ and κ . Hence T is M-special. \square

As in the case for ω_1 , note that at the limit step we used just the assumption that the tree is normal. Hence we can generalize this lemma to the following lemma. Note that for this we do not need the assumption $\kappa^{<\kappa} = \kappa$ since we use that the tree $\bigcup_{\alpha < \kappa^+} T_{\alpha+1}$ is special instead of \mathbb{Q}_κ -embeddable. We explicitly state this lemma here so it is clear that M-special trees are exactly those trees that are normal and whose successor levels form a special tree, as was the case at ω_1 .

Lemma 3.28. *Let κ be a regular cardinal. Let T be a κ^+ -Aronszajn tree. T is normal such that $T^* = \bigcup_{\alpha < \kappa^+} T_{\alpha+1}$ is special if and only if T is M-special.*

Proof. (\Rightarrow) Let $T^* = \bigcup_{\alpha < \kappa^+} T_{\alpha+1}$ is special. Then $T^* = \bigcup_{\xi < \kappa} A_\xi$ where A_ξ is antichain for each $\xi < \kappa$. The rest of the proof is the same as the proof of Lemma 3.27 since we used the antichains only in the successor step.

(\Leftarrow) Let T be an M-special tree. Then T is isomorphic to a subtree T' of $\left\{ s \in {}^{<\kappa^+}\kappa \mid s \text{ is 1-1} \right\}$ via i . We define $f : T^* \rightarrow \kappa$ by setting $f(t) = i(t)(\alpha)$ for

$ht(t, T) = \alpha + 1$. Let $s < t \in T^*$. Then $ht(s, T) = \beta + 1 < \alpha + 1 = ht(t, T)$. Since $i(s) \subset i(t)$, $i(s)(\beta) = i(t)(\beta)$. As $i(t)$ is 1-1, $i(t)(\beta) \neq i(t)(\alpha)$. Therefore $f(s) \neq f(t)$. \square

On the other hand, generalization of Lemma 3.10 requires the additional assumption that $\kappa^{<\kappa} = \kappa$ since we need to use Generalized Kurepa's Theorem.

Lemma 3.29. *Assume $\kappa^{<\kappa} = \kappa$. Let T be a κ^+ -Aronszajn tree. T is a normal \mathbb{R}_κ -embeddable tree if and only if T is M -special.*

Proof. It follows from Lemma 3.24 and Lemma 3.28. \square

Unlike in the previous section, here we are also interested in the question how the existence of one kind of special Aronszajn trees influences the existence of other kinds of special Aronszajn trees. The following lemma claims that if there are no M -special Aronszajn trees then there are no special Aronszajn trees at all.

Lemma 3.30. *Let κ be a regular cardinal. If there exists a special κ^+ -Aronszajn tree, then there exists an M -special Aronszajn tree.*

Proof. Let T be a special κ^+ -Aronszajn tree. We first add one root r such that $r < t$ for each $t \in T_0$. Now we guarantee the condition (iv). Let $\alpha < \kappa^+$ be a limit ordinal and t be some node in T_α . For every chain $C = \{s \in T \mid s < t\}$ we add one extra node t_C such that $s < t_C$ for all $s \in C$ and $t_C < r$ for each r such that $r > s$ for all $s \in C$. Since for every chain we add one extra node to the limit level, this new tree satisfies (iv). Denote this tree T' . This tree is normal and $T = \bigcup_{\alpha < \kappa^+} T'_{\alpha+1}$. By Lemma 3.28 the tree T' is M -special. \square

As in previous section it does make sense to introduce concept of \mathcal{S} -special Aronszajn trees.

Definition 3.31. Let κ be a regular cardinal and S be an unbounded subset of κ^+ . We say that the κ^+ -tree T is S -special if $T \upharpoonright S$ is special, where $T \upharpoonright S = \{t \in T \mid ht(t, T) \in S\}$ with the induced ordering. We say that a κ^+ -tree T is \mathcal{S} -special if there is S , an unbounded subset of κ^+ , such that T is S -special. We denote the class of all \mathcal{S} -special κ^+ -Aronszajn trees as $\mathbb{A}^{\mathcal{S}\text{-sp}}(\kappa^+)$.

The following lemmas are obvious generalizations of Lemmas 3.12 and 3.13.

Lemma 3.32. *Let C be a closed unbounded subset of κ^+ , where κ is a regular cardinal. If T is a C -special κ^+ -Aronszajn tree, then T is special.*

Proof. This is a simple generalization of Lemma 3.12. \square

Lemma 3.33. *Let κ be a regular cardinal and S be an unbounded subset of κ^+ . If every κ^+ -Aronszajn tree is S -special then every κ^+ -Aronszajn tree is special.*

Proof. Let $S = \{\alpha_\mu \mid \mu < \kappa^+\}$ be an unbounded subset of κ^+ and T be a S -special κ^+ -Aronszajn tree. We define a new tree

$$T' = \{\langle t, \beta \rangle \mid t \in T \text{ and } \beta < \alpha_{ht(t,T)} \text{ and } \forall s < t (\alpha_{ht(s,T)} < \beta)\}. \quad (3.17)$$

The tree T' is ordered by $<_{T'}$ as follows: $\langle t, \beta \rangle < \langle s, \gamma \rangle$ if and only if $t < s$ or $(t = s \text{ and } \beta < \gamma)$. It is obvious that T' satisfies our definition of Aronszajn tree. Hence T' is S -special, i.e. $T' \upharpoonright S$ is special. Since T is isomorphic to $T' \upharpoonright S = \{\langle t, \alpha_{ht(t,T)} \rangle \mid t \in T\}$, T is special. \square

Again, note that \mathcal{S} -special κ^+ -Aronszajn trees are not Suslin in a strong sense. This means that every subset of size κ^+ of such tree contains an antichain of size κ^+ . Hence we can generalize Definition 3.15 and Lemma 3.17.

Definition 3.34. Let κ be a regular cardinal and T be a κ^+ -Aronszajn tree. We say that T is *non-Suslin* if every subset U of T , which has size κ^+ , contains an antichain of size κ^+ . We denote the class of all non-Suslin Aronszajn trees at κ^+ as $\mathbb{A}^{NS}(\kappa^+)$.

The following lemmas are generalizations of Lemma 3.16 and Lemma 3.17.

Lemma 3.35. *Let κ be a regular cardinal and T be a κ^+ -Aronszajn tree. If T is not non-Suslin, then T has a subtree which is Suslin.*

Proof. Let T be a κ^+ -Aronszajn tree, which is not non-Suslin. Then there is a subset X of T such that $|X| = \kappa^+$ and X does not contain antichain of size κ^+ . Let $T' = \{s \in T \mid \exists t \in X (s < t)\}$.

Now, we show that T' is Suslin. Assume for contradiction that $A \subseteq T'$ is an antichain of size κ^+ . Then for any choice function $f : \mathcal{P}(X) \rightarrow X$, the set $\{f(\{s \in X \mid a \leq s\}) \mid a \in A\}$ has size κ^+ and it is an antichain in X . \square

Lemma 3.36. *Let κ be a regular cardinal and T be a κ^+ -Aronszajn tree. If T is \mathcal{S} -special, then T is non-Suslin.*

Proof. Let T be a κ^+ -Aronszajn tree which is not non-Suslin. By the previous lemma T has a subtree T' which is Suslin. Since T is \mathcal{S} -special, T' is \mathcal{S} -special, too. Hence there is unbounded subset S of κ^+ such that $T' \upharpoonright S = \bigcup_{\alpha < \kappa} A_\alpha$, where A_α is an antichain for each α . By pigeon-hole principle, for some $\alpha < \kappa$ the size of A_α must be greater than κ . This contradicts the fact that T is Suslin. \square

Lemma 3.37. *Let κ be a regular cardinal. If there is an \mathcal{S} -special κ^+ -Aronszajn tree then there is a special Aronszajn tree.*

Proof. Let T be an \mathcal{S} -special κ^+ -Aronszajn tree for some unbounded subset S of κ^+ . Then $T \upharpoonright S$ is a special Aronszajn tree. \square

The following theorem is only the summarization of what we have showed about the relative existence of different kinds of special Aronszajn trees. It tells us that the weak tree property at κ^+ is equivalent to the claim that there are no M-special κ^+ -Aronszajn trees and also to the claim that there are no \mathcal{S} -special κ^+ -Aronszajn trees.

Theorem 3.38. *Let κ be a regular. The following are equivalent*

- (i) $\mathbb{A}^{sp}(\kappa^+) = \emptyset$;
- (ii) $\mathbb{A}^M(\kappa^+) = \emptyset$;
- (iii) $\mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+) = \emptyset$.

Proof. Ad (i) \Leftrightarrow (ii). The claim from left to right follows from Lemma 3.28 and the converse follows from Lemma 3.30.

Ad (ii) \Leftrightarrow (iii). The claim from left to right follows from Lemma 3.37 and the converse follows from the definition of \mathcal{S} -special κ^+ -Aronszajn tree. \square

Now, we know that

$$\mathbb{A}^{sp}(\kappa^+) \subseteq \mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+) \subseteq \mathbb{A}^{NS}(\kappa^+) \text{ and } \mathbb{A}^M(\kappa^+) \subseteq \mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+). \quad (3.18)$$

Now, we examine if these inclusions can be consistently proper.

3.2.1 The Existence of Higher Special Aronszajn Trees

We are interested in special Aronszajn trees at successors of regular cardinals. While the existence of a special ω_1 -Aronszajn tree can be proved in ZFC, at higher cardinals we need some additional assumption, for example $\kappa^{<\kappa} = \kappa$ or weak square principle. The first one was used in construction by Specker in [Spe49] and the second one in the construction by Jensen in [Jen72]. On the other hand, it is possible to find a model with no special κ^+ -Aronszajn tree where $\kappa > \omega$ is regular, but this requires much stronger assumption. Throughout this section we assume that κ is a regular cardinal and $\kappa > \omega$.

Definition 3.39. $E_\kappa^{\kappa^+} = \{\alpha < \kappa^+ \mid cf(\alpha) = \kappa\}$

This theorem is our generalization of Theorem 3.18. As a corollary we obtain that the first inclusion in (3.18) can be consistently proper.

Theorem 3.40. *Assume $\kappa^{<\kappa} = \kappa$ and $\diamond_{\kappa^+}(E_{\kappa^+}^{\kappa^+})$. Then there is an M -special κ^+ -Aronszajn tree, which is not special.*

Proof. We follow the proof of Theorem 3.18. Again, we fix a diamond sequence $\langle f_\alpha \mid \alpha \in E_{\kappa^+}^{\kappa^+} \rangle$ for the rest of the proof and we generalize the induction assumption for κ^+ . For each $\alpha < \kappa^+$ we require the following conditions:

(T1) If $s \in T \upharpoonright \alpha$ then $|\kappa \setminus \text{Rng}(s)| = \kappa$.

(T2) If $s \in T \upharpoonright \alpha$ and $x \in [\kappa \setminus \text{Rng}(s)]^{<\kappa}$ then there is $s' \supseteq s$ on each higher level of $T \upharpoonright \alpha$ such that $\text{Rng}(s') \cap x = \emptyset$.

(π_0) π_α is a 1-1 map from $T \upharpoonright \alpha$ to κ^+ such that $s \subseteq t \rightarrow \pi_\alpha(s) < \pi_\alpha(t)$.

Let $T_0 = \{\emptyset\}$. If $\alpha = \beta + 1$ then for each $s \in T_\beta$ we add all one-point extensions of s by distinct $\gamma < \kappa$ which are not in $\text{Rng}(s)$ and we extend $\pi_{\beta+1}$ to $\pi_{\alpha+1}$ arbitrary such that satisfies condition (π_0).

Let α be limit. For each $\beta < \alpha$, suppose $T \upharpoonright \beta$ and π_β are defined and they satisfy the conditions mentioned above. Let $T'_\alpha = \bigcup_{\beta < \alpha} T_\beta$. We need to distinguish two cases. First, if α has cofinality less than κ then $T_\alpha = T'_\alpha$. We can add all possible sequences since $\kappa^{<\kappa} = \kappa$.

In the second case, if α has cofinality κ then we proceed as in Theorem 3.18. Again, we need to distinguish two cases: First, if f_α embeds $\pi'(\text{''}T'_\alpha)$ to \mathbb{Q}_κ and $\text{Dom}(f_\alpha) = \pi'_\alpha \text{''}T'_\alpha$, then set

$$X_\alpha = \{(s, x) \mid s \in T'_\alpha \ \& \ x \in [\kappa]^{<\kappa} \ \& \ \text{Rng}(s) \cap x = \emptyset\}. \quad (3.19)$$

For $(s, x), (t, y)$ in X_α , we define $(s, x) \leq_\alpha (t, y)$ if and only if $s \subseteq t$ and $x \subseteq y$. For each $q \in \mathbb{Q}_\kappa$, set

$$\Delta_q^\alpha = \{(s, x) \in X_\alpha \mid f_\alpha(\pi'_\alpha(s)) \geq_{\mathbb{Q}_\kappa} q \text{ or} \\ (\forall (t, y) \in X_\alpha)((t, y) \geq_\alpha (s, x) \rightarrow f_\alpha(\pi'_\alpha(t)) <_{\mathbb{Q}_\kappa} q)\}. \quad (3.20)$$

Let $s \in T'_\alpha$, $x \in [\kappa \setminus \text{Rng}(s)]^{<\kappa}$. Let $\langle \alpha_\gamma \mid \gamma < \kappa \rangle$ be cofinal in α with $\alpha_0 = \text{length}(s)$. Let $g : \kappa \rightarrow \mathbb{Q}_\kappa$ be a bijection. We define node $s_x \supset s$ by induction on $\gamma < \kappa$. If $\gamma < \kappa$ is a successor ordinal we can proceed as in the proof of Theorem 3.18. Let $\gamma < \kappa$ be limit. Since the size γ is less than κ we can take $s'_\gamma = \bigcup_{\beta < \gamma} s_\beta$ and $x'_\gamma = \bigcup_{\beta < \gamma} x_\beta$. As κ is regular, $|x'_\gamma| < \kappa$. Note that $\text{length}(s'_\gamma) \geq \alpha_\gamma$ and $\text{Rng}(s'_\gamma) \cap x'_\gamma = \emptyset$. Since (s'_γ, x'_γ) is in X_α and $\Delta_{g(\gamma)}^\alpha$ is cofinal in X_α , we can find $(s_\gamma, x_\gamma) \geq (s'_\gamma, x'_\gamma)$ in $\Delta_{g(\gamma)}^\alpha$.

In the other case, if f_α does not embed $\pi'(\text{''}T'_\alpha)$ to \mathbb{Q}_κ , then we proceed similar as before. Let $s \in T'_\alpha$, $x \in [\kappa \setminus \text{Rng}(s)]^{<\kappa}$. Let $\langle \alpha_\gamma \mid \gamma < \kappa \rangle$ be cofinal in α with

$\alpha_0 = \text{length}(s)$. We define node $s_x \supset s$ induction on $\gamma < \kappa$. If $\gamma < \kappa$ is a successor ordinal we can proceed as in the proof of Theorem 3.18. Let $\gamma < \kappa$ be limit. Since the size γ is less than κ , we can take $s_\gamma = \bigcup_{\beta < \gamma} s_\beta$ and $x_\gamma = \bigcup_{\beta < \gamma} x_\beta$. As κ is regular, $|x_\gamma| < \kappa$. Note that $\text{length}(s_\gamma) \geq \alpha_\gamma$ and $\text{Rng}(s_\gamma) \cap x_\gamma = \emptyset$.

Let $s_x = \bigcup_{\gamma < \kappa} s_\gamma$. As in Theorem 3.18 we define the level $T_\alpha = \{s_x \mid s \in T'_\alpha \text{ and } x \in [\kappa \setminus \text{Rng}(s)]^{<\kappa}\}$. It is easy to verify that $T \upharpoonright \alpha + 1 = T'_\alpha \cup T_\alpha$ satisfies the condition (T1) and (T2). Again, can extend π'_α to $\pi_{\alpha+1}$ on $T \upharpoonright \alpha + 1$ arbitrarily such that it satisfies the condition ($\pi 0$).

Finally, set $T = \bigcup_{\alpha < \kappa^+} T_\alpha$ and $\pi = \bigcup_{\alpha < \kappa^+} \pi_\alpha$. Then $\pi : T \rightarrow \kappa$ is a function such that $s \subseteq t \rightarrow \pi(s) < \pi(t)$.

For a contradiction assume that T is special. As we assume $\kappa^{<\kappa} = \kappa$, by Lemma 3.24 T is special if and only if T is \mathbb{Q}_κ -embeddable. Therefore there is a function f which embeds $\pi''T$ in \mathbb{Q}_κ . Let

$$\begin{aligned} C = \{ \alpha < \kappa^+ \mid \alpha \text{ is a limit ordinal and } \pi''T_\alpha = \pi'_\alpha''T'_\alpha \text{ and} \\ f \upharpoonright \alpha \text{ embeds } \pi'_\alpha''T'_\alpha \text{ in } \mathbb{Q}_\kappa \text{ and} \\ (\forall s \in T'_\alpha)(\forall x \in [\kappa \setminus \text{Rng}(s)]^{<\kappa})(\forall q >_{\mathbb{Q}_\kappa} f(\pi(s))) \\ ((\exists t \in T)(t \supseteq s \ \& \ \text{Rng}(t) \cap x = \emptyset \ \& \ f(\pi(t)) \geq_{\mathbb{Q}_\kappa} q) \\ \rightarrow (\exists t' \in T'_\alpha)(t' \supseteq s \ \& \ \text{Rng}(t') \cap x = \emptyset \ \& \ f(\pi(t')) \geq_{\mathbb{Q}_\kappa} q) \}. \end{aligned} \quad (3.21)$$

It is easy to verify that C is a closed unbounded subset of κ^+ . As we assume $\diamond_\kappa(E_\kappa^{\kappa^+})$, the set $\{ \alpha \in E_\kappa^{\kappa^+} \mid f \upharpoonright \alpha = f_\alpha \}$ is stationary, so there is $\alpha \in C$ such that $f \upharpoonright \alpha = f_\alpha$ and α has cofinality κ . Let $t \in T_\alpha$ and let $q = f(\pi(t))$. By the construction of T , there is $(s, x) \in \Delta_q^\alpha$ such that $\text{Rng}(s) \cap x = \emptyset$ and $s \subset t$. Since f and π are order-preserving, $f(\pi(s)) <_{\mathbb{Q}_\kappa} f(\pi(t)) = q$.

Since $f(\pi(s)) <_{\mathbb{Q}_\kappa} q$ and $f(\pi(t)) \geq_{\mathbb{Q}_\kappa} q$, by the definition of C there exists $t' \in T'_\alpha$ such that $t' \supseteq s$, $\text{Rng}(t') \cap x = \emptyset$ and $f(\pi(t')) \geq_{\mathbb{Q}_\kappa} q$. Note that $(s, x), (t', x)$ are in X_α and $(s, x) \leq_\alpha (t', x)$. Since (s, x) is in Δ_q^α and $f \upharpoonright \alpha = f_\alpha$, by (3.20) it must hold that $f_\alpha(\pi(s)) \geq_{\mathbb{Q}_\kappa} q$. But $f_\alpha = f \upharpoonright \alpha$ and so $f(\pi(s)) \geq_{\mathbb{Q}_\kappa} q$. This contradicts our earlier inequality $f(\pi(s)) <_{\mathbb{Q}_\kappa} q$. \square

Corollary 3.41. *Assume $\kappa^{<\kappa} = \kappa$ and $\diamond_{\kappa^+}(E_\kappa^{\kappa^+})$. Then there is an \mathbb{R}_κ -embeddable κ^+ -Aronszajn tree, which is not special.*

Proof. By Lemma 3.29, every M-special κ^+ -Aronszajn tree is \mathbb{R}_κ -embeddable. \square

Corollary 3.42. *Assume $\kappa^{<\kappa} = \kappa$ and $\diamond_{\kappa^+}(E_\kappa^{\kappa^+})$. Then there is an \mathcal{S} -special κ^+ -Aronszajn tree, which is not special.*

Proof. By Lemma 3.28, every M-special κ^+ -Aronszajn tree is \mathcal{S} -special for $S = \{ \alpha + 1 \mid \alpha < \kappa^+ \}$. \square

The next lemma is a straightforward generalization of Lemma 3.20 and tells us that the last inclusion in (3.18) can be consistently proper.

Lemma 3.43. *Assume $\kappa^{<\kappa} = \kappa$ and $\diamond_{\kappa^+}(E_{\kappa^+}^{\kappa^+})$. Then there is a κ^+ -Aronszajn tree, which is S -special for some S unbounded subset of κ^+ and it is not M -special and by our assumption it is not \mathbb{R}_{κ} -embeddable.*

Proof. The proof is the same as in Lemma 3.20. □

To show that the second inclusion in (3.18) can be consistently proper, i.e. that $\mathbb{A}^{S-sp} \neq \mathbb{A}^{NS}$, we need to introduce the notion of an ω -ascent path, which is due to Laver.

Definition 3.44. Let κ be a regular cardinal. We say that a κ^+ -Aronszajn tree T has the property of the ω -ascent path if there is a sequence $\langle x^\alpha \mid \alpha < \kappa^+ \rangle$ such that

- (i) for each $\alpha < \kappa^+$, x^α is a function from ω to T_α ;
- (ii) if $\alpha, \beta < \kappa$ with $\alpha < \beta$ then $\exists n \in \omega \forall m \geq n \ x_m^\alpha < x_m^\beta$.

If the tree T has a cofinal branch, then this branch is a 1-ascent path and it is obvious that T is not special. But Aronszajn trees do not have cofinal branches. Thus an ω -ascent path is a pseudo-branch with width ω which prevents the tree from being special.

The following theorem is due to Shelah ([SS88]), building on work of Laver and Todorćević.

Theorem 3.45. *Let $\kappa > \omega$ be a regular cardinal. Let T be a κ^+ -Aronszajn tree with the property of an ω -ascent path. Then T is not special.*

Proof. Assume for contradiction that there is a function $f : T \rightarrow \kappa$ such that f is 1-1 on chains. For $\alpha < \kappa^+$, $t \in T_\alpha$ and $\beta < \alpha$, let $pr_\beta(t)$ be the predecessor of t on T_β .

Fix $\gamma < \kappa$. For each $i < \omega$ and $\delta \in E_{\kappa^+}^{\kappa^+}$ there is $\alpha_\gamma(\delta, i)$ such that for every $\alpha \geq \alpha_\gamma(\delta, i)$

$$f(pr_\alpha(x_i^\delta)) \geq \gamma. \tag{3.22}$$

This holds since the set $\{s \in T \mid s < x_i^\delta\}$ is a chain with cofinality $\kappa > \omega$ and f is 1-1 on this chain. Since $\gamma < \kappa$, the argument follows.

Let $\alpha_\gamma(\delta) = \sup \{\alpha_\gamma(\delta, i) + 1 \mid i < \omega\}$. Since δ has cofinality κ , $\alpha_\gamma(\delta) < \delta$. Hence α_γ is a regressive function on the stationary set $E_{\kappa^+}^{\kappa^+}$, by Fodor's Lemma, there is some $S_\gamma \subseteq E_{\kappa^+}^{\kappa^+}$ and $\beta_\gamma < \kappa^+$ such that $\alpha_\gamma(\delta) = \beta_\gamma$ for $\delta \in S_\gamma$.

Now, we verify that for arbitrary $\gamma < \kappa$ the following holds:

$$\forall \beta \in [\beta_\gamma, \kappa^+) \exists i < \omega \ f(x_i^\beta) \geq \gamma \quad (3.23)$$

Let β be given and fix $\delta > \beta$ such that $\delta \in S_\gamma$. By the property of the ω -ascent path there is some $i < \omega$ such that $x_i^\beta <_T x_i^\delta$. Since $\beta \geq \beta_\gamma \geq \alpha_\gamma(\delta, i)$ and $x_i^\beta = pr_\beta(x_i^\delta)$, by (3.22) $f(x_i^\beta) \geq \gamma$.

Now we finish the proof. Let $\alpha^* = \sup \{\beta_\gamma \mid \gamma < \kappa\}$. It is obvious that $\alpha^* < \kappa^+$. Thus we can fix β such that $\alpha^* \leq \beta < \kappa^+$. Since (3.23) holds for each $\gamma < \kappa$, there is for every γ some $i_\gamma < \omega$ such that $f(x_{i_\gamma}^\beta) \geq \gamma$. By the pigeon-hole principle, there is some $i^* < \omega$ such that for unboundedly many $\gamma < \kappa$ it holds that $f(x_{i^*}^\beta) \geq \gamma$. This contradicts our assumption that f is a function from T to κ . \square

Remark 3.46. Note that no such argument can exist for ω_1 -trees since it is important for the proof that there is a regular cardinal between ω and κ^+ . This is the difference between the specialization forcing for ω_1 and for higher cardinals. In the case of higher cardinals, if T has an ω -ascent path, then any specialization forcing must collapse cardinals. On the other hand, specialization forcing for ω_1 -trees is ccc.

Corollary 3.47. *Let κ be a regular cardinal. Let T be a κ^+ -Aronszajn tree with the property of an ω -ascent path. Then T is not \mathcal{S} -special.*

Proof. Let $S \subseteq \kappa^+$ be an unbounded subset of κ^+ and $\langle x^\alpha \mid \alpha < \kappa^+ \rangle$ be an ω -ascent path. Then $\langle x^\alpha \mid \alpha < \kappa^+ \rangle \upharpoonright S$ is ω -ascent path for $T \upharpoonright S$ and by the previous theorem $T \upharpoonright S$ is not special. \square

Fact 3.48. *Let κ be a regular cardinal. Assume \square_κ . Then there is a non-Suslin κ^+ -Aronszajn tree with ω -ascent path.*

Proof. The construction of such tree can be found in [SS88]. \square

Hence we can conclude that the second inclusion in (3.18) can be consistently proper.

Corollary 3.49. *Let κ be a regular cardinal. Assume \square_κ . Then there is a non-Suslin κ^+ -Aronszajn T tree such that T is not \mathcal{S} -special.*

Proof. It follows from Corollary 3.47 and Fact 3.48. \square

Remark 3.50. Note that if we replace ω with an arbitrary regular cardinal $\lambda < \kappa$ in the definition of ω -ascent path, the proof of Theorem 3.45 does not change. Thus if κ^+ -Aronszajn tree T has the λ -ascent path for some regular $\lambda < \kappa$, then T is not \mathcal{S} -special.

The following picture illustrates our situation.

κ^+ -Aronszajn Trees

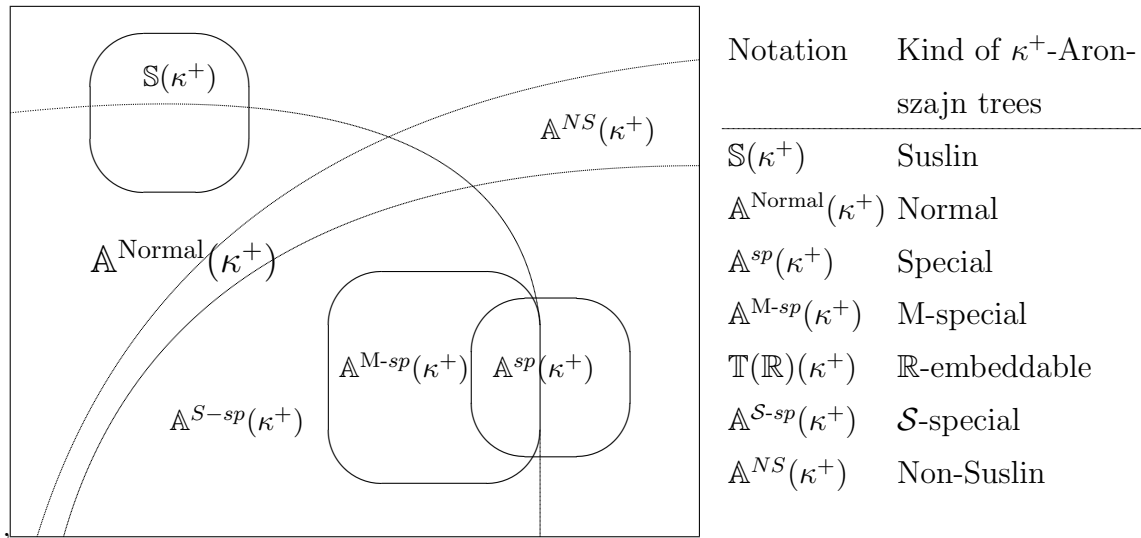


Figure 2: Description of the relations between various kinds of κ^+ -Aronszajn trees for κ a regular cardinal.

4 The Tree Property at One Cardinal

Recall the definitions of the tree property and of the weak tree property. Mitchell and Silver ([Mit72]) were the first to prove, under large cardinal assumption, that it is consistent for the double successor of a regular cardinal to have either of these properties. Later, Baumgartner and Laver ([BL79]) showed that the Sacks forcing can be used for the same result at ω_2 and Kanamori generalized this for arbitrary double successor of a regular cardinal in [Kan80]. In this section we present Mitchell and Silver's proof and the method of Baumgartner and Laver for the Grigorieff forcing.

4.1 Preliminaries

Here we present several *branching* lemmas. They state that forcings with some properties do not add new cofinal branches to some trees. This is useful when dealing with Aronszajn trees and the tree property.

The following lemma first appeared in [KT79] for a Suslin tree of height ω_1 . However, the generalization to a tree of height κ for $\kappa > \omega_1$ is obvious.

Lemma 4.1. *Let κ be a regular cardinal and \mathbb{P} be a κ -Knaster forcing notion. If T is a tree of height κ , then forcing with \mathbb{P} does not add any new cofinal branches to T .*

Actually, the assumption that the forcing \mathbb{P} is κ -Knaster can be weakened to the assumption that the forcing $\mathbb{P} \times \mathbb{P}$ is κ -cc.

Lemma 4.2. *Let κ be a regular cardinal and \mathbb{P} be a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ -cc. If T is a tree of height κ , then forcing with \mathbb{P} does not add any new cofinal branches to T .*

The next lemma was proved by Baumgartner in [Bau83] using the argument of Silver from [Sil71].

Lemma 4.3. *If $2^{\aleph_0} \geq \aleph_2$, T is an ω_2 -Aronszajn tree and \mathbb{P} is ω_1 -closed forcing, then in $V[\mathbb{P}]$, T has no cofinal branches.*

This lemma can be generalized to higher cardinals in the following way.

Lemma 4.4. *Let λ be a regular cardinal and T be a λ -tree. Let \mathbb{P} be κ^+ -closed forcing, where $2^\kappa \geq \lambda$, Then every cofinal branch through T in $V[\mathbb{P}]$ is already in V .*

This can be further generalized to the lemma which first appeared in [Ung12].

Lemma 4.5. *Let κ, λ be regular cardinals. Assume $2^\kappa \geq \lambda$. Let \mathbb{P} be κ^+ -cc and \mathbb{Q} be κ^+ -closed. Then \mathbb{Q} does not add cofinal branches to λ -trees in $V[\mathbb{P}]$.*

In fact, a stronger version of Lemma 4.5 easily follows from its proof.

Lemma 4.6. *Let κ, λ be regular cardinals. Assume $2^\kappa \geq \lambda$. Let ξ be a regular cardinal such that $\kappa < \xi \leq \lambda$. Let \mathbb{P} be ξ -cc and \mathbb{Q} be ξ -closed. Then \mathbb{Q} does not add cofinal branches to λ -trees in $V[\mathbb{P}]$.*

The following lemma is actually an easy observation and we use it in the proofs of the consistency of the weak tree property at more cardinals.

Lemma 4.7. *Let κ be a regular and T be an M -special κ^+ -Aronszajn tree. Let \mathbb{P} be a forcing notion. If \mathbb{P} preserves cardinals, then \mathbb{P} does not add cofinal branch to the tree T .*

Proof. Assume for contradiction that b is a cofinal branch in T in $V[\mathbb{P}]$. Then b is a 1-1 function from κ^+ to κ . Hence κ^+ is collapsed to κ . But this is a contradiction since we assumed that \mathbb{P} preserves cardinals. \square

4.2 Mitchell Forcing

In this section we define the Mitchell forcing, study its properties and use it to force the weak tree property and the tree property at double successor of a regular cardinal. This is a result of Mitchell and Silver from [Mit72]. The main advantage of the Mitchell forcing is that it is a projection of the product of two forcings, where the first has a good chain condition and the second is sufficiently closed.

Throughout this section we assume that κ, λ are regular cardinals and $\kappa < \lambda$.

Definition 4.8. The Cohen forcing $\text{Add}(\kappa, 1)$ is the collection of all functions $p : \kappa \rightarrow 2$, where $|\text{Dom}(p)| < \kappa$. The ordering is by reverse inclusion.

Definition 4.9. The Cohen forcing $\text{Add}(\kappa, \lambda)$ is the collection of all functions $p : \lambda \rightarrow 2$, where $|\text{Dom}(p)| < \kappa$. The ordering is by reverse inclusion.

We do not use the original definition of the Mitchell forcing. Instead, we work with more understandable and more common version from [Abr83].

Definition 4.10. The Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ is defined as follows: for an ordinal α , $\kappa < \alpha < \lambda$, let $\mathbb{Q}(\alpha)$ be an $\text{Add}(\kappa, \alpha)$ -name for the partially ordered set $\text{Add}(\kappa^+, 1)$.

The forcing $\mathbb{M}(\kappa, \lambda)$ is the collection of pairs (p, q) such that $p \in \text{Add}(\kappa, \lambda)$ and q is a function of cardinality less than κ^+ such that if $\alpha \in \text{Dom}(q)$ then $\kappa < \alpha < \lambda$ and $\emptyset \Vdash^{\text{Add}(\kappa, \alpha)} q(\alpha) \in \mathbb{Q}(\alpha)$.

$\mathbb{M}(\kappa, \lambda)$ is ordered by $(p, q) \leq (p', q')$ if and only if $p \leq_{\text{Add}(\kappa, \lambda)} p'$, $\text{Dom}(q') \subseteq \text{Dom}(q)$ and for all $\alpha \in \text{Dom}(q')$ $p \upharpoonright \alpha \Vdash q'(\alpha) \subseteq q(\alpha)$.

Now, we present the basic properties of the Mitchell forcing.

Lemma 4.11. $\mathbb{M}(\kappa, \lambda)$ is κ -closed.

Proof. Let $\xi < \kappa$ and $\langle (p_i, q_i) \mid i \in \xi \rangle$ be a decreasing sequence in $\mathbb{M}(\kappa, \lambda)$. Set $p = \bigcup_{i < \xi} p_i$. Since κ is regular, p is in $\text{Add}(\kappa, \lambda)$. Now, we define q as follows: $\text{Dom}(q) = \bigcup_{i < \xi} \text{Dom}(q_i)$. If α is in $\text{Dom}(q)$ and i_0 is the first i such that $\alpha \in \text{Dom}(q_i)$, then $p \upharpoonright \alpha \Vdash \langle q_i(\alpha) \mid i_0 \leq i < \xi \rangle$ is a decreasing sequence in $\mathbb{Q}(\alpha)$. Let $\alpha \in \text{Dom}(q)$ be given. We can define $q(\alpha)$ to be an $\text{Add}(\kappa, \alpha)$ -name such that $\emptyset \Vdash^{\text{Add}(\kappa, \alpha)} q(\alpha) \in \mathbb{Q}(\alpha)$ and $p \upharpoonright \alpha \Vdash q(\alpha) = \bigcup_{i_0 \leq i < \xi} q_i(\alpha)$. It is easy to verify that the pair (p, q) is in $\mathbb{M}(\kappa, \lambda)$ and that it is the lower bound of the given sequence. \square

Lemma 4.12. Let λ be an inaccessible cardinal. Then $\mathbb{M}(\kappa, \lambda)$ is λ -Knaster.

Proof. Let $A \subseteq \mathbb{M}(\kappa, \lambda)$ such that $|A| = \lambda$. Since $\text{Add}(\kappa, \lambda)$ is λ -Knaster, there is $Y \subseteq A$ such that $|Y| = \lambda$ and if $(p, q), (p', q') \in Y$, then $p \parallel p'$. Let $B' = \{q \mid (\exists p \in \text{Add}(\kappa, \lambda))((p, q) \in Y)\}$.

If $|B'| < \lambda$, then we fix for each $q \in B'$ the set $A_q = \{(p, q') \in \mathbb{M}(\kappa, \lambda) \mid (p, q') \in Y \text{ and } q' = q\}$. Then $Y = \bigcup_{q \in B'} A_q$ and if $q \neq q'$, then $A_q \cap A_{q'} = \emptyset$. Since $|B'| < \lambda$, there exists $q \in B'$ such that $|A_q| = \lambda$. It is easy to see that all elements of A_q are pairwise compatible.

If $|B'| = \lambda$, then set $B = \{\text{Dom}(q) \mid q \in B'\}$. We show that the set B has size λ . Let $X \subset \{\alpha \mid \kappa < \alpha < \lambda\}$ such that $|X| \leq \kappa$, we show that there are less than λ -many different conditions q with domain X . Let $\alpha \in X$ be given. Then $q(\alpha)$ is $\text{Add}(\kappa, \alpha)$ -name for a condition in $\text{Add}(\kappa^+, 1)^{V[\text{Add}(\kappa, \alpha)]}$. Since $\text{Add}(\kappa, \alpha)$ has size $\alpha^{<\kappa} = \mu < \lambda$, there are at most $\theta_\alpha = (\mu^\mu)^{\kappa^+}$ $\text{Add}(\kappa, \alpha)$ -nice names for conditions in $\text{Add}(\kappa^+, 1)$. Hence the number of conditions with domain X is $\prod_{\alpha \in X} \theta_\alpha = (\sup_{\alpha \in X} \theta_\alpha)^\kappa$ which is less than λ as λ is inaccessible. Therefore the set B has size λ .

By Δ -system lemma, there exist $C \subseteq B$ with $|C| = \lambda$, and $r \subseteq \lambda$ such that $|r| \leq \kappa$ and $\text{Dom}(p) \cap \text{Dom}(q) = r$ for every pair $\text{Dom}(p), \text{Dom}(q)$ from C . Let $A_1 = \{(p, q) \in B' \mid \text{Dom}(q) \in C\}$. As $|C| = \lambda$, it also holds that $|A_1| = \lambda$. Notice that if $(p, q), (p', q') \in A_1$ such that $q' \upharpoonright r = q \upharpoonright r$ then $(p, q) \parallel (p', q')$. Since p, p' are compatible, there is p^* such that $p^* \leq p$ and $p^* \leq p'$. It is easy to verify that $(p^*, q \cup q')$ witnesses the compatibility of (p, q) and (p', q') .

Now, we find $A_2 \subseteq A_1$ such that

$$\forall (p, q), (p', q') \in A_2 (q \upharpoonright r = q' \upharpoonright r) \quad (4.1)$$

and the size of A_2 is λ . This will be enough to conclude the proof. If $r = \emptyset$, then $A_2 = A_1$ is as required. Assume that $r \neq \emptyset$. As we showed above, since $|r| \leq \kappa$, there can be only α -many function satisfying the definition of $\mathbb{M}(\kappa, \lambda)$ with domain

r , for some $\alpha < \lambda$. Let $\langle f_i | i < \alpha \rangle$ be their enumeration. For each such f_i denote $A_{f_i} = \{q | q \upharpoonright r = f_i\}$. Then $A_1 = \bigcup_{i < \alpha} A_{f_i}$ where $i \neq j$ implies $A_{f_i} \cap A_{f_j} = \emptyset$. It follows that A_1 can be partitioned into α -many pieces. Hence there exists $i < \alpha$ such that $|A_{f_i}| = \lambda$. Set $A_2 = A_{f_i}$. \square

We show that there is a projection from the product of two forcings to the Mitchell forcing $\mathbb{M}(\kappa, \lambda)$, where the first forcing is $\text{Add}(\kappa, \lambda)$ and the second forcing is κ^+ -closed. Now, we define this second forcing.

Definition 4.13. We define $\mathbb{Q} = (Q(\kappa, \lambda), \leq_{\mathbb{Q}})$ as $Q(\kappa, \lambda) = \{(\emptyset, q) | (\emptyset, q) \in M(\kappa, \lambda)\}$ and the ordering $\leq_{\mathbb{Q}} = \leq_{M(\kappa, \lambda)} \upharpoonright Q(\kappa, \lambda)$.

Lemma 4.14. \mathbb{Q} is κ^+ -closed.

Proof. Let $\langle (\emptyset, q_\beta) \in Q(\kappa, \lambda) | \beta < \kappa \rangle$ be a decreasing sequence in \mathbb{Q} . We define a lower bound (\emptyset, q) of $\langle (\emptyset, q_\beta) \in Q(\kappa, \lambda) | \beta < \kappa \rangle$ as follows: $\text{Dom}(q) = \bigcup_{\beta < \kappa} \text{Dom}(q_\beta)$. If α is in $\text{Dom}(q)$ and β_0 is the first β such that α is in $\text{Dom}(q_\beta)$, then $\emptyset \Vdash \langle q_\beta(\alpha) | \beta_0 \leq \beta < \xi \rangle$ is a decreasing sequence in $\mathbb{Q}(\alpha)$. Let $\alpha \in \text{Dom}(q)$ be given. We can define $q(\alpha)$ to be an $\text{Add}(\kappa, \alpha)$ -name such that $\emptyset \Vdash q(\alpha) = \bigcup_{\beta_0 \leq \beta < \xi} q_\beta(\alpha)$. It is easy to see that (\emptyset, q) is in \mathbb{Q} since $\text{Dom}(q) \leq \kappa$ and the forcing $\text{Add}(\kappa^+, 1)$ is κ^+ -closed. \square

Lemma 4.15. Let a function $\pi : \text{Add}(\kappa, \lambda) \times \mathbb{Q} \rightarrow \mathbb{M}(\kappa, \lambda)$ be defined such that $\pi((p, (\emptyset, q))) = (p, q)$. Then π is a projection.

Proof. We need to verify the three conditions from the Definition 2.25.

Ad (i). Let $(p', (\emptyset, q')) \leq (p, (\emptyset, q))$. We want to show that $(p', q') \leq (p, q)$, i.e. $p' \leq p$, $\text{Dom}(q) \subseteq \text{Dom}(q')$ and $\forall \alpha \in \text{Dom}(q) p' \upharpoonright \alpha \Vdash q(\alpha) \subseteq q'(\alpha)$. By our assumption, $p' \leq p$, $\text{Dom}(q) \subseteq \text{Dom}(q')$ and $\emptyset \Vdash q(\alpha) \subseteq q'(\alpha)$. Hence it is easy to see that also $p' \upharpoonright \alpha \Vdash q(\alpha) \subseteq q'(\alpha)$ for $\alpha \in \text{Dom}(q)$.

Ad (ii). Let $(p', q') \leq \pi((p, (\emptyset, q))) = (p, q)$. We want to find a function q^* such that $(\emptyset, q^*) \leq (\emptyset, q)$ and $\pi((p', (\emptyset, q^*))) \leq (p', q')$. We define q^* as follows: $\text{Dom}(q^*) = \text{Dom}(q')$ and for all $\alpha \in \text{Dom}(q')$:

- If $\alpha \notin \text{Dom}(q)$, let $q^*(\alpha) = q'(\alpha)$.
- If $\alpha \in \text{Dom}(q)$, then we define $q^*(\alpha) \in V^{\text{Add}(\kappa, \alpha)}$ as
 - a) $p' \upharpoonright \alpha \Vdash q^*(\alpha) = q'(\alpha)$;
 - b) if $r \in \text{Add}(\kappa, \alpha)$ is incompatible with $p' \upharpoonright \alpha$, then $r \Vdash q^*(\alpha) = q(\alpha)$.

Hence $\emptyset \Vdash^{\text{Add}(\kappa, \alpha)} q(\alpha) \subseteq q^*(\alpha)$ for all $\alpha \in \text{Dom}(q)$. Since $\text{Dom}(q) \subseteq \text{Dom}(q') = \text{Dom}(q^*)$, we have $(p', (\emptyset, q^*)) \leq (p, (\emptyset, q))$ in $\text{Add}(\kappa, \lambda) \times \mathbb{Q}$. Now, by definition of projection π , $\pi(p', (\emptyset, q^*)) = (p', q^*)$. Hence it is enough to show that $(p', q^*) \leq (p', q')$. This is immediate since $\text{Dom}(q^*) = \text{Dom}(q')$ and by the definition of q^* , for each $\alpha \in \text{Dom}(q^*)$, $p' \restriction \alpha \Vdash q^*(\alpha) = q'(\alpha)$.

Ad (iii). It is easy to see that π is onto. Let $(p, q) \in \mathbb{M}(\kappa, \lambda)$, then $(p, (\emptyset, q)) \in \text{Add}(\kappa, \lambda) \times \mathbb{Q}$ and by the definition of π , $\pi((p, (\emptyset, q))) = (p, q)$. \square

Lemma 4.16. *There is a projection $\pi : \mathbb{M}(\kappa, \lambda) \rightarrow \text{Add}(\kappa, \lambda)$.*

Proof. We define the projection $\pi : \mathbb{M}(\kappa, \lambda) \rightarrow \text{Add}(\kappa, \lambda)$ as follows: For given $(p, q) \in \mathbb{M}(\kappa, \lambda)$, $\pi(p, q) = p$. It is obvious that π satisfies the conditions from Definition 2.25. \square

This means that the Cohen forcing is actually a subforcing of the Mitchell forcing. Therefore the following lemma tells us that each sequence of ordinals of length less than κ^+ is already added by a smaller forcing.

Lemma 4.17. *Assume $\kappa^{<\kappa} = \kappa$. Then all sets of ordinals in $V[\mathbb{M}(\kappa, \lambda)]$ of cardinality κ are in $V[\text{Add}(\kappa, \lambda)]$.*

Proof. By Lemma 4.15, there is a projection $\pi : \text{Add}(\kappa, \lambda) \times \mathbb{Q} \rightarrow \mathbb{M}(\kappa, \lambda)$, hence $V[\mathbb{M}(\kappa, \lambda)] \subseteq V[\text{Add}(\kappa, \lambda) \times \mathbb{Q}]$. Therefore it is enough to show that all sets of ordinals in $V[\text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of cardinality κ are in $V[\text{Add}(\kappa, \lambda)]$. Since $\kappa^{<\kappa} = \kappa$, $\text{Add}(\kappa, \lambda)$ is κ^+ -cc. By Lemma 4.14, \mathbb{Q} is κ^+ -closed. Hence, by Easton's lemma, \mathbb{Q} is κ^+ -distributive in $V[\text{Add}(\kappa, \lambda)]$, i.e. \mathbb{Q} does not add new sequences of ordinals of length less than κ^+ . Hence any sequence of ordinals in $V[\text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of cardinality κ is already in $V[\text{Add}(\kappa, \lambda)]$. \square

Corollary 4.18. *Assume $\kappa^{<\kappa} = \kappa$. Then κ^+ remains a cardinal in $V[\mathbb{M}(\kappa, \lambda)]$.*

Proof. Assume for contradiction that κ^+ is not a cardinal in $V[\mathbb{M}(\kappa, \lambda)]$. Then there is a function f from ξ onto κ^+ , for some $\xi < \kappa^+$. By the previous lemma, f is already in $V[\text{Add}(\kappa, \lambda)]$ and so κ^+ is not a cardinal in $V[\text{Add}(\kappa, \lambda)]$. Since $\text{Add}(\kappa, \lambda)$ is κ^+ -Knaster, κ^+ remains a cardinal in $V[\text{Add}(\kappa, \lambda)]$ and this is a contradiction. \square

Now, we know that all cardinals $\leq \kappa$ and cardinals $\geq \lambda$ are preserved by $\mathbb{M}(\kappa, \lambda)$. Moreover, κ^+ is preserved. Now, we focus on other properties of the final extension. We show that $2^\kappa = \lambda$ and that λ becomes the double successor of κ .

Lemma 4.19. *Let λ be an inaccessible cardinal. Then in $V[\mathbb{M}(\kappa, \lambda)]$, $2^\kappa = \lambda$.*

Proof. $2^\kappa \geq \lambda$ follows from Lemma 4.16. Now, we show that $2^\kappa \leq \lambda$. Since the forcing $\mathbb{M}(\kappa, \lambda)$ is λ -cc of size λ , which is inaccessible in V , $2^\kappa \leq \lambda$ easily follows by a common nice names argument. \square

Lemma 4.20. *Let ξ be an ordinal such that $\kappa < \xi < \lambda$. Then there is a projection $\pi : \mathbb{M}(\kappa, \lambda) \rightarrow \mathbb{M}(\kappa, \xi)$.*

Proof. We define the projection $\pi : \mathbb{M}(\kappa, \lambda) \rightarrow \mathbb{M}(\kappa, \xi)$ in the following way: for given $(p, q) \in \mathbb{M}(\kappa, \lambda)$, $\pi(p, q) = (p \upharpoonright \xi, q \upharpoonright \xi)$. It is easy to verify that π satisfies the conditions from Definition 2.25. \square

Lemma 4.21. *Assume $\kappa^{<\kappa} = \kappa$. Let λ be an inaccessible cardinal. Then $\lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa, \lambda)]$.*

Proof. By Lemma 4.12 all cardinals $\mu \geq \lambda$ are preserved by the Mitchell forcing and by Corollary 4.18 κ^+ is also preserved.

Now, we show that each cardinal ξ , $\kappa^+ < \xi < \lambda$, is collapsed to κ^+ . It is enough to show it for ξ regular. Let ξ be given. Then by Lemma 4.20 there is a projection $\pi' : \mathbb{M}(\kappa, \lambda) \rightarrow \mathbb{M}(\kappa, \xi + 1)$, so it suffices to show that the size of ξ in $V[\mathbb{M}(\kappa, \xi + 1)]$ is κ^+ .

Now, we show that there is a projection $\pi : \mathbb{M}(\kappa, \xi + 1) \rightarrow \text{Add}(\kappa, \xi) * \text{Add}(\kappa^+, 1)$. We define π as follows: for $(p, q) \in \mathbb{M}(\kappa, \xi + 1)$ let

$$\pi((p, q)) = \begin{cases} (p \upharpoonright \xi, q(\xi)) & \text{if } \xi \in \text{Dom}(q); \\ (p \upharpoonright \xi, \check{\emptyset}) & \text{otherwise.} \end{cases}$$

It is easy to verify that π preserves ordering and it is onto. We check only property (ii) of Definition 2.25. Assume that q' is defined on ξ and let $(p, \dot{r}) \leq \pi((p', q')) = (p' \upharpoonright \xi, q'(\xi))$. We want to find q^* such that $(p^*, q^*) \leq (p', q')$ and $\pi((p^*, q^*)) \leq (p, \dot{r})$, where $p^* = p \cup \{(\xi, p'(\xi))\}$. The condition q^* is defined as follows: $\text{Dom}(q^*) = \text{Dom}(q')$. Let $\alpha \in \text{Dom}(q^*)$. If $\alpha < \xi$, set $q^*(\alpha) = q'(\alpha)$. If $\alpha = \xi$, set $q^*(\alpha) = \dot{r}$.

Now we verify that q^* is the desired function. First, we show that $(p^*, q^*) \leq (p', q')$. Since $p \leq p' \upharpoonright \xi$ in $\text{Add}(\kappa, \xi)$, $p^* \leq p'$ in $\text{Add}(\kappa, \xi + 1)$. If $\beta < \xi$ and $\beta \in \text{Dom}(q^*)$, then $p \upharpoonright \beta \Vdash q'(\beta) = q^*(\beta)$ because $q^*(\beta) = q'(\beta)$. If $\beta = \xi$, then $p^* \upharpoonright \xi = p \Vdash q^*(\beta) \leq q'(\beta)$ since $q^*(\beta) = \dot{r}$ and $p \Vdash \dot{r} \leq q'(\xi)$. Therefore $(p^*, q^*) \leq (p', q')$.

Next we show that $\pi((p^*, q^*)) \leq (p, \dot{r})$. By the definition of π , $\pi((p^*, q^*)) = (p, q^*(\xi))$ and so it is easy to see that $(p, q^*(\xi)) \leq (p, \dot{r})$ since $q^*(\xi) = \dot{r}$.

If q' is not defined on ξ , then the proof is similar to the prof before, except we have to take $\text{Dom}(q^*) = \text{Dom}(q') \cup \{\xi\}$ and instead of $q(\xi)$ we consider $\check{\emptyset}$.

In $V[\text{Add}(\kappa, \xi)] \models 2^\kappa \geq \xi$ and in $V[\text{Add}(\kappa, \xi) * \text{Add}(\kappa^+, 1)] \models 2^\kappa = \kappa^+$, hence in $V[\text{Add}(\kappa, \xi) * \text{Add}(\kappa^+, 1)] \models |\xi| = \kappa^+$. Since there is a projection from $\mathbb{M}(\kappa, \xi + 1)$ to $\text{Add}(\kappa, \xi) * \text{Add}(\kappa^+, 1)$, $V[M(\kappa, \xi + 1)] \models |\xi| = \kappa^+$. \square

Corollary 4.22. *Assume $\kappa^{<\kappa} = \kappa$. The forcing \mathbb{Q} collapses cardinals between κ^+ and λ to κ^+ .*

Proof. Let $\xi, \kappa^+ < \xi < \lambda$, be given. In the proof of the previous lemma, we showed that each cardinal between κ^+ and λ is collapsed to κ^+ by $\mathbb{M}(\kappa, \lambda)$. Since there is a projection from $\text{Add}(\kappa, \lambda) \times \mathbb{Q}$ to $\mathbb{M}(\kappa, \lambda)$ by Lemma 4.15, $V[\mathbb{M}(\kappa, \lambda)] \subseteq V[\text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ and so $|\xi| = \kappa^+$ in $V[\text{Add}(\kappa, \lambda) \times \mathbb{Q}]$. As we assume $\kappa^{<\kappa} = \kappa$, the forcing $\text{Add}(\kappa, \lambda)$ is κ^+ -cc over V . As \mathbb{Q} is κ^+ -closed, $\text{Add}(\kappa, \lambda)$ is κ^+ -cc over $V[\mathbb{Q}]$ by Easton's Lemma and so preserves cardinals above κ^+ . Therefore the collapsing function had to be added by \mathbb{Q} . \square

As we showed above, there is a projection from $\mathbb{M}(\kappa, \lambda)$ to $\mathbb{M}(\kappa, \xi)$, where $\kappa < \xi < \lambda$. Therefore we can consider the Mitchell forcing as a two step iteration $\mathbb{M}(\kappa, \xi) * \mathbb{M}(\kappa, \lambda) / \mathbb{M}(\kappa, \xi)$ (see Fact 2.29). Moreover, in $V[\mathbb{M}(\kappa, \xi)]$ the forcing $\mathbb{M}(\kappa, \lambda) / \mathbb{M}(\kappa, \xi)$ behaves as the Mitchell forcing. This means that it is also a projection of the product of two forcings, where the first has a good chain condition and the second is sufficiently closed.

Lemma 4.23. *Let ξ be an ordinal such that $\kappa < \xi < \lambda$. In $V[\mathbb{M}(\kappa, \xi)]$ there is a κ -closed forcing \mathbb{Q}^* such that the partial order $\mathbb{M}(\kappa, \lambda) / \mathbb{M}(\kappa, \xi)$ is a projection of $\text{Add}(\kappa, [\xi, \lambda]) \times \mathbb{Q}^*$.*

Proof. Let G be $\mathbb{M}(\kappa, \xi)$ -generic over V and let us work in $V[G]$. Recall that $\mathbb{M}(\kappa, \lambda) / \mathbb{M}(\kappa, \xi) = \{(p, q) \in \mathbb{M}(\kappa, \lambda) \mid (p \restriction \xi, q \restriction \xi) \in G\}$. We define the forcing $\mathbb{Q}^* = \{(\emptyset, q) \in \mathbb{M}(\kappa, \lambda) \mid (\emptyset, q \restriction \xi) \in G\}$ with the induced ordering.

We show that the forcing \mathbb{Q}^* is κ^+ -closed. Let $\langle (\emptyset, q_\beta) \in \mathbb{Q}^* \mid \beta < \kappa \rangle$ be a decreasing sequence in \mathbb{Q}^* . We define a lower bound (\emptyset, q) of $\langle (\emptyset, q_\beta) \in \mathbb{Q}^* \mid \beta < \kappa \rangle$ as follows: $\text{Dom}(q) = \bigcup_{\beta < \kappa} \text{Dom}(q_\beta)$. If α is in $\text{Dom}(q)$ and β_0 is the first β such that α is in $\text{Dom}(q_\beta)$, then $\emptyset \Vdash \langle q_\beta(\alpha) \mid \beta_0 \leq \beta < \xi \rangle$ is a decreasing sequence in $\mathbb{Q}(\alpha)$. Let $\alpha \in \text{Dom}(q)$ be given. We can define $q(\alpha)$ to be an $\text{Add}(\kappa, \alpha)$ -name such that $\emptyset \Vdash q(\alpha) = \bigcup_{\beta_0 \leq \beta < \xi} q_\beta(\alpha)$. Now, we verify that q is in \mathbb{Q}^* . It is easy to see that $|\text{Dom}(q)| \leq \kappa$ and that $(\emptyset, q) \in \mathbb{M}(\kappa, \lambda)$. It remains to show that $(\emptyset, q \restriction \xi)$ is in G , but this follows from the assumption that, for each $\beta < \kappa$, $(\emptyset, q_\beta \restriction \xi) \in G$.

First we show that in $V[G]$ the forcing $\text{Add}(\kappa, [\xi, \lambda])$ is forcing equivalent to $\text{Add}(\kappa, \lambda) / \text{Add}(\kappa, \xi)$. The forcing $\text{Add}(\kappa, \lambda) / \text{Add}(\kappa, \xi)$ is already defined in $V[\pi^* G]$, where π^* is a projection from $\mathbb{M}(\kappa, \xi)$ to $\text{Add}(\kappa, \xi)$ such that for all $(p, q) \in \mathbb{M}(\kappa, \xi)$,

$\pi^*((p, q)) = p$. Hence $\pi^{**}G = \{p \in \text{Add}(\kappa, \xi) \mid (\exists(p', q') \in G)(p' = p)\}$ is $\text{Add}(\kappa, \xi)$ -generic filter over V . Normally we would have to take an upward closure of $\pi^{**}G$ to obtain the generic filter, but in this case $\pi^{**}G$ is already upward closed. Recall that the forcing $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)$ is defined as $\{p \in \text{Add}(\kappa, \lambda) \mid p \upharpoonright \xi \in \pi^{**}G\}$. By the definition of $\pi^{**}G$, this is equal to $\{p \in \text{Add}(\kappa, \lambda) \mid (\exists(p', q') \in G)(p' = p \upharpoonright \xi)\}$ in $V[G]$.

Now, in $V[\pi^{**}G]$, the forcing $\text{Add}(\kappa, [\xi, \lambda])$ is forcing equivalent to the forcing $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)$. This holds because both forcings $\text{Add}(\kappa, \xi) * \text{Add}(\kappa, [\xi, \lambda])$ and $\text{Add}(\kappa, \xi) * \text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)$ are forcing equivalent to $\text{Add}(\kappa, \lambda)$ in V .

Note that $\text{Add}(\kappa, [\xi, \lambda])^V = \text{Add}(\kappa, [\xi, \lambda])^{V[G]} = \text{Add}(\kappa, [\xi, \lambda])^{V[\pi^{**}G]}$ since both $\text{Add}(\kappa, \xi)$ and $\mathbb{M}(\kappa, \xi)$ are κ -closed. In addition, $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)^{V[G]}$ is equal to $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)^{V[\pi^{**}G]}$ since both are defined using only $\text{Add}(\kappa, \lambda)^V$ and $\pi^{**}G$. Therefore $\text{Add}(\kappa, [\xi, \lambda])$ is forcing equivalent to $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)$ in $V[G]$.

As we showed above, $\text{Add}(\kappa, [\xi, \lambda])$ is forcing equivalent to $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi)$ in $V[G]$, hence it suffices to find a projection from $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi) \times \mathbb{Q}^*$ to $\mathbb{M}(\kappa, \lambda)/\mathbb{M}(\kappa, \xi)$. We define the projection π as follows: $\pi((p, (\emptyset, q))) = (p, q)$. The function π is well defined, i.e. it is a function from $\text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi) \times \mathbb{Q}^*$ to $\mathbb{M}(\kappa, \lambda)/\mathbb{M}(\kappa, \xi)$: if $(p, (\emptyset, q)) \in \text{Add}(\kappa, \lambda)/\text{Add}(\kappa, \xi) \times \mathbb{Q}^*$, then $(\emptyset, q \upharpoonright \xi) \in G$ and there is q' such that $(p \upharpoonright \xi, q' \upharpoonright \xi) \in G$. It follows that $(p \upharpoonright \xi, q \upharpoonright \xi) \in G$.

Now, we need to check the conditions from Definition 2.25. It is easy to verify that π is onto and that it preserves ordering. We verify the condition (ii). Let $(p', q') \leq \pi((p, (\emptyset, q))) = (p, q)$. We want to find a function q^* such that $(\emptyset, q^*) \leq (\emptyset, q)$, $\pi((p', (\emptyset, q^*))) \leq (p', q')$, and $(\emptyset, q^* \upharpoonright \xi)$ is in G . We define q^* as follows: $\text{Dom}(q^*) = \text{Dom}(q')$ and for all $\alpha \in \text{Dom}(q')$:

- If $\alpha \notin \text{Dom}(q)$, let $q^*(\alpha) = q'(\alpha)$.
- If $\alpha \in \text{Dom}(q)$, then we define $q^*(\alpha) \in V^{\text{Add}(\kappa, \alpha)}$ as

(i) $p' \upharpoonright \alpha \Vdash q^*(\alpha) = q'(\alpha)$;

(ii) if $r \in \text{Add}(\kappa, \alpha)$ is incompatible with $p' \upharpoonright \alpha$, then $r \Vdash q^*(\alpha) = q(\alpha)$.

Hence $\emptyset \Vdash^{\text{Add}(\kappa, \alpha)} q(\alpha) \subseteq q^*(\alpha)$ for all $\alpha \in \text{Dom}(q)$. Since $\text{Dom}(q) \subseteq \text{Dom}(q') = \text{Dom}(q^*)$, we have $(p', (\emptyset, q^*)) \leq (p, (\emptyset, q))$. Now, by definition of projection π , $\pi(p', (\emptyset, q^*)) = (p', q^*)$. Hence it is enough to show that $(p', q^*) \leq (p', q')$. This is immediate since $\text{Dom}(q^*) = \text{Dom}(q')$ and by the definition of q^* , for each $\alpha \in \text{Dom}(q^*)$, $p' \upharpoonright \alpha \Vdash q^*(\alpha) = q'(\alpha)$. To finish the proof, we need to verify that $(\emptyset, q^* \upharpoonright \xi) \in G$. For α in $\text{Dom}(q^*)$, $p' \upharpoonright \alpha \Vdash q^*(\alpha) = q'(\alpha)$ by the definition of q^* . It follows that $(p', q^*) \geq (p', q')$, hence $(p' \upharpoonright \xi, q^* \upharpoonright \xi) \in G$. Since $(\emptyset, q^*) \geq (p', q^*)$, $(\emptyset, q^* \upharpoonright \xi) \in G$. \square

Now, we show how to use the Mitchell forcing to obtain the weak tree property or the tree property at the double successor of a given regular cardinal.

Theorem 4.24. *Assume GCH. Let κ be a regular cardinal. If there exists a Mahlo cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{M}(\kappa, \lambda)$ it holds that*

$$(i) \ 2^\kappa = \lambda = \kappa^{++};$$

(ii) κ^{++} has the weak tree property.

Proof. Ad (i). This follows from Lemma 4.19 and Lemma 4.21.

Ad (ii). Let G be an $\mathbb{M}(\kappa, \lambda)$ -generic over V . Suppose that T is an \mathbb{M} -special $\lambda = \kappa^{++}$ -Aronszajn tree in $V[G]$. Let $(p, q) \in G$ be such that $(p, q) \Vdash$ “ T is an \mathbb{M} -special κ^{++} -Aronszajn tree”. Then for each $\alpha < \lambda$, $T \upharpoonright \alpha$ has size at most κ^+ . Hence we can consider $T \upharpoonright \alpha$ as a subset of κ^+ . As such, $T \upharpoonright \alpha$ has a nice name $\pi_\alpha = \bigcup \{ \{ \check{\beta} \} \times A_\beta \mid \beta < \kappa^+ \text{ and } A_\beta \text{ is an antichain of } \mathbb{M}(\kappa, \lambda) \}$ and it holds that $\beta \in T \upharpoonright \alpha \leftrightarrow (\exists a \in A_\beta)(a \in G)$.

Now, we show that $T \upharpoonright \alpha$ is already in $V[G_\xi]$ for some $\xi < \lambda$, where $G_\xi = \{ (p \upharpoonright \xi, q \upharpoonright \xi) \mid (p, q) \in G \}$ is an $\mathbb{M}(\kappa, \xi)$ -generic filter because there is a projection from $\mathbb{M}(\kappa, \lambda)$ to $\mathbb{M}(\kappa, \xi)$ by Lemma 4.20 and because G_ξ is already upward closed. Set

$$X = \bigcup \{ \text{Dom}(p') \cup \text{Dom}(q') \mid (\exists \beta < \kappa^+)((p', q') \in A_\beta) \}. \quad (4.2)$$

The set X has size less than λ , since $\mathbb{M}(\kappa, \lambda)$ is λ -cc and the conditions of the Mitchell forcing are bounded in λ at both coordinates. Take $\xi = \sup \{ \beta < \lambda \mid \beta \in X \}$. Then $T \upharpoonright \alpha \in V[G_\xi]$ since π_α is also an $\mathbb{M}(\kappa, \xi)$ -nice name. Let \dot{T}_α be an $\mathbb{M}(\kappa, \lambda)$ -name for $T \upharpoonright \alpha$. Then we set

$$\sigma'(\alpha) = \min \left\{ \xi < \lambda \mid (\forall \alpha' < \alpha) (\exists \pi_{\alpha'}) (\pi_{\alpha'} \text{ is an } \mathbb{M}(\kappa, \xi)\text{-nice name and } (p, q) \Vdash \dot{T}_{\alpha'} = \pi_{\alpha'}) \right\}. \quad (4.3)$$

Then σ' is defined in V and it is easy to see that it is a continuous nondecreasing unbounded function from λ into λ . As we want to argue that σ' has a fixed point, we need σ' to be increasing. Hence we define $\sigma : \lambda \rightarrow \lambda$ by induction on $\alpha < \lambda$ as follows: if $\alpha = 0$ then $\sigma(\alpha) = \sigma'(\alpha)$. If $\alpha = \beta + 1$ then set

$$\sigma(\alpha) = \begin{cases} \sigma(\beta) + 1 & \text{if } \sigma'(\alpha) \leq \sigma(\beta); \\ \sigma'(\alpha) & \text{otherwise.} \end{cases}$$

If α is limit then $\sigma(\alpha) = \sup \{ \sigma(\beta) \mid \beta < \alpha \}$.

The function σ is continuous and increasing, hence there is a closed unbounded set of fixed points of σ . Moreover, σ still satisfies that $T \upharpoonright \alpha \in V[G_{\sigma(\alpha)}]$ since

$\sigma'(\alpha) \leq \sigma(\alpha)$ for each $\alpha < \lambda$. As λ is Mahlo in V , there is a fixed point δ of σ which is inaccessible in V . By Lemma 4.21, $|\delta| = \kappa^+$ in $V[G]$ and since T is a κ^{++} -Aronszajn tree in $V[G]$, there is a branch b of T of length δ in $V[G]$.

Now, we show that b is already in $V[G_\delta]$. Work in $V[G_\delta]$. $T \upharpoonright \delta$ is an M-special δ -Aronszajn tree. Moreover, $\delta = \kappa^{++}$ and $2^\kappa \geq \delta$. By Lemma 4.23, there is a projection from $\text{Add}(\kappa, [\delta, \lambda]) \times \mathbb{Q}^*$ to $\mathbb{M}(\kappa, \lambda)/\mathbb{M}(\kappa, \delta)$, where \mathbb{Q}^* is κ^+ -closed. Since $b \in V[G]$, $b \in V[G_\delta][H_1 \times H_2]$, where $H_1 \times H_2$ is $\text{Add}(\kappa, [\delta, \lambda]) \times \mathbb{Q}^*$ -generic over $V[G_\delta]$. Note that from the properties of product forcing, we have that $V[G_\delta][H_1 \times H_2] = V[G_\delta][H_2][H_1]$

In $V[G_\delta][H_2]$, note that $\text{Add}(\kappa, [\delta, \lambda])$ is κ^+ -Knaster since in $V[G_\delta][H_2]$, $\kappa^{<\kappa} = \kappa$ and we can apply the Δ -system argument. As δ is collapsed to κ^+ in $V[G_\delta][H_2][H_1]$ and $\text{Add}(\kappa, [\delta, \lambda])$ is κ^+ -Knaster in $V[G_\delta][H_2]$, δ has to be collapsed already in $V[G_\delta][H_2]$. Hence δ is an ordinal of cofinality κ^+ in $V[G_\delta][H_2]$. Let $T' = (T \upharpoonright \delta) \upharpoonright A$, where A is a cofinal subset of δ of size κ^+ . Note that T' need not be a κ^+ -tree, because $T \upharpoonright \delta$ is a κ^{++} -tree in $V[G_\delta]$, hence it is possible that T' has a level of size κ^+ . By Lemma 4.1, $\text{Add}(\kappa, [\delta, \lambda])$ does not add cofinal branches to the tree T' , hence it does not add cofinal branches to the tree $T \upharpoonright \delta$. Therefore $b \in V[G_\delta][H_2]$.

In $V[G_\delta]$, since $2^\kappa \geq \delta = \kappa^{++}$ and \mathbb{Q}^* is κ^+ -closed, we know by Lemma 4.4 that b could not be added by \mathbb{Q}^* . Hence b is in $V[G_\delta]$.

Work in $V[G_\delta]$. As $T \upharpoonright \delta$ is M-special, b is a 1-1 function from δ to κ^+ , so $M[G_\delta] \models \delta \leq \kappa^+$. Since G_δ is $M(\kappa, \delta)$ -generic over M and δ is inaccessible in M , $M[G_\delta] \models \delta = \kappa^{++}$ by Lemma 4.21. This is a contradiction.

Thus, we have proved that there are no M-special κ^{++} -Aronszajn trees in $V[G]$. By Theorem 3.38, κ^{++} has the weak tree property. \square

Remark 4.25. Note that by Theorem 3.38, there are no \mathcal{S} -special κ^{++} -Aronszajn trees in the extension in the previous theorem. On the other hand there could be a κ^{++} -Suslin tree. Assume that $V = L$ and that a Mahlo cardinal exists. Let λ be the least Mahlo cardinal. Then λ is not weakly compact and as we assume $V = L$, there is a λ -Suslin tree. For more details about this see [Dev84]. As in the previous theorem, we can force with $\mathbb{M}(\omega, \lambda)$ to obtain a model where $\lambda = \omega_2$ and the weak tree property holds at ω_2 . Since in the ground model there was a Suslin tree at λ and $\mathbb{M}(\omega, \lambda)$ is λ -Knaster, the λ -Suslin tree is preserved.

To keep things simple and clear we prove the next theorem under the assumption of a measurable cardinal. This assumption can be weakened to an existence of a weakly compact cardinal but the proof would be more technical and the technicalities could obscure the main ideas of the proof.

Theorem 4.26. *Assume GCH. Let κ be a regular cardinal. If there exists a measurable cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{M}(\kappa, \lambda)$ it holds that*

(i) $2^\kappa = \lambda = \kappa^{++}$;

(ii) κ^{++} has the tree property.

Proof. Ad (i). This follows from Lemma 4.19 and Lemma 4.21.

Ad (ii). Let G be an $\mathbb{M}(\kappa, \lambda)$ -generic over V . Since λ is measurable in V , there is an elementary embedding $j : V \rightarrow M$ with critical point λ and ${}^\lambda M \subseteq M$, where M is a transitive model of ZFC.

In M , the forcing $j(\mathbb{M}(\kappa, \lambda))$ is $\mathbb{M}(\kappa, j(\lambda))^M$ by the elementarity of j . Since $V_\lambda = V_\lambda^M$ and each condition in $\mathbb{M}(\kappa, \lambda)$ is bounded in V_λ , $\mathbb{M}(\kappa, j(\lambda))^M \upharpoonright \lambda = \mathbb{M}(\kappa, \lambda)^M = \mathbb{M}(\kappa, \lambda)^V$. Hence G is also $\mathbb{M}(\kappa, \lambda)^M$ -generic over M . By Lemma 4.20, there is a projection from $\mathbb{M}(\kappa, j(\lambda))$ to $\mathbb{M}(\kappa, \lambda)$ and we can define in $M[G]$ the forcing $\mathbb{M}(\kappa, j(\lambda))/\mathbb{M}(\kappa, \lambda)$. Since $\mathbb{M}(\kappa, j(\lambda))/\mathbb{M}(\kappa, \lambda)$ is definable in $M[G]$, it is definable in $V[G]$. Let H be $\mathbb{M}(\kappa, j(\lambda))/\mathbb{M}(\kappa, \lambda)$ -generic over $V[G]$, then H is $\mathbb{M}(\kappa, j(\lambda))/\mathbb{M}(\kappa, \lambda)$ -generic over $M[G]$ since $M[G] \subseteq V[G]$.

Work in $V[G][H]$. By Lemma 2.3, we can lift j to $j^* : V[G] \rightarrow M[G][H]$. Assume T is a λ -tree in $V[G]$. We show that T has a cofinal branch in $V[G]$. We can consider T as a subset of λ , so T has a nice name \dot{T} in V . \dot{T} has size λ since $\mathbb{M}(\kappa, \lambda)$ is λ -cc. As ${}^\lambda M \subseteq M$, \dot{T} is in M . Hence $T \in M[G]$. By elementarity of j^* , $j^*(T)$ is a $j^*(\lambda)$ -tree in $M[G][H]$ and since j^* is the identity below λ , $j^*(T) \upharpoonright \lambda = T$. As $j^*(T)$ is $j^*(\lambda)$ -tree in $M[G][H]$, it has branch b of length λ in $M[G][H]$. By Lemma 4.23, in $M[G]$ there is projection from $\text{Add}(\kappa, [\lambda, j^*(\lambda)]) \times \mathbb{Q}^*$ to $\mathbb{M}(\kappa, j^*(\lambda))/\mathbb{M}(\kappa, \lambda)$, where \mathbb{Q}^* is κ^+ -closed. Hence $M[G][H] \subseteq M[G][H_1 \times H_2]$, where $H_1 \times H_2$ is $\text{Add}(\kappa, [\lambda, j^*(\lambda)]) \times \mathbb{Q}^*$ -generic over $M[G]$. Therefore b is in $M[G][H_2][H_1]$.

In $M[G][H_2]$, note that $\text{Add}(\kappa, [\lambda, j^*(\lambda)])$ is κ^+ -Knaster since $\kappa^{<\kappa} = \kappa$. As λ is collapsed to κ^+ in $M[G][H_2][H_1]$ and $\text{Add}(\kappa, [\lambda, j^*(\lambda)])$ is κ^+ -Knaster in $M[G][H_2]$, λ has to be collapsed to κ^+ already in $M[G][H_2]$. Hence λ is an ordinal of cofinality κ^+ in $M[G][H_2]$. Let $T' = T \upharpoonright A$, where A is cofinal subset of λ of size κ^+ . Note that T' need not be a κ^+ -tree, because T is κ^{++} -tree in $M[G]$, hence it is possible that T' has a level of size κ^+ . By Lemma 4.1, $\text{Add}(\kappa, [\lambda, j^*(\lambda)])$ does not add cofinal branches to the tree T' , hence it does not add cofinal branches to the tree T . Therefore $b \in M[G][H_2]$.

In $V[G]$, since $2^\kappa \leq \lambda = \kappa^{++}$ and \mathbb{Q}^* is κ^+ -closed, we know by Lemma 4.4 that b could not be added by \mathbb{Q}^* . Therefore the branch b is already in $M[G]$ and so in $V[G]$. \square

4.3 Grigorieff forcing

In this section we show an alternative method to obtain the weak tree property and the tree property at double successor of a regular cardinal κ . This method was first used by Baumgartner and Laver in [BL79] for the case $\kappa = \omega$. Kanamori generalized this for arbitrary regular cardinal in [Kan80]. They used the Sacks forcing, taking advantage of its fusion property. We use the Grigorieff forcing instead of the Sacks forcing since it also has the fusion property and this property is crucial to the proofs of Baumgartner and Laver. Therefore, the proofs based on the Grigorieff forcing are pretty much the same as the proofs based on the Sacks forcing.

The Grigorieff forcing was first defined in [Gri71] by Grigorieff for $\kappa = \omega$, but the following definition is taken from [HV].

Definition 4.27. Let κ be a regular cardinal and let I be a normal ideal on κ extending the nonstationary ideal on κ . We define κ -Grigorieff forcing $\mathbb{G}_I(\kappa, 1) = (G_I(\kappa, 1), \leq)$ as

$$G_I(\kappa, 1) = \left\{ f : \kappa \rightarrow 2 \mid \text{Dom}(f) \in I \right\}, \quad (4.4)$$

where $f : \kappa \rightarrow 2$ denote a partial function from κ to 2. Ordering is by reverse inclusion, i.e. for $p, q \in G_I(\kappa, 1)$, $p \leq q$ if and only if $q \subseteq p$.

We show the chain condition and the closure of the Grigorieff forcing, we define a fusion sequence and show that each fusion sequence has the lower bound. We use these properties when dealing with the iteration of the Grigorieff forcing.

Lemma 4.28. *Assume $2^\kappa = \kappa^+$. Then the forcing $\mathbb{G}_I(\kappa, 1)$ is κ^{++} -cc.*

Proof. If $2^\kappa = \kappa^+$, then $|\mathbb{G}_I(\kappa, 1)| = \kappa^+$. Hence $\mathbb{G}_I(\kappa, 1)$ is κ^{++} -cc. \square

Lemma 4.29. *Let κ be a regular cardinal. If $\alpha < \kappa$ and $\langle p_\beta \mid \beta < \alpha \rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa, 1)$, then $p = \bigcup_{\beta < \alpha} p_\beta \in G_I(\kappa, 1)$. Hence $\mathbb{G}_I(\kappa, 1)$ is κ -closed.*

Proof. The proof is a direct consequence of our assumption about I since every normal ideal on κ extending the nonstationary ideal on κ is a κ -complete ideal. \square

Definition 4.30. Let κ be a regular cardinal. For $\alpha < \kappa$ and $p, q \in G_I(\kappa, 1)$ we define

$$p \leq_\alpha q \Leftrightarrow p \leq q \text{ and } \text{Dom}(p) \cap (\alpha + 1) = \text{Dom}(q) \cap (\alpha + 1). \quad (4.5)$$

We say that $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a *fusion sequence* if for every α , $p_{\alpha+1} \leq_\alpha p_\alpha$ and $p_\beta = \bigcup_{\alpha < \beta} p_\alpha$ for every limit $\beta < \kappa$.

Lemma 4.31. *Let κ be a regular cardinal. If $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a fusion sequence in $\mathbb{G}_I(\kappa, 1)$, then the union $p = \bigcup_{\alpha < \kappa} p_\alpha$ is a condition in $G_I(\kappa, 1)$ and $p \leq_\alpha p_\alpha$ for each $\alpha < \kappa$.*

Proof. It is sufficient to show that $\bigcup_{\alpha < \kappa} \text{Dom}(p_\alpha)$ is in I , or equivalently $\bigcap_{\alpha < \kappa} (\kappa \setminus \text{Dom}(p_\alpha))$ is in I^* , where I^* is the dual filter of I . Since I^* is a normal filter, $\Delta_{\alpha < \kappa}(\kappa \setminus \text{Dom}(p_\alpha))$ is in I^* and also the set $\{\beta < \kappa \mid \beta \text{ is a limit ordinal}\}$ is in I^* since I extend the nonstationary ideal on κ .

To finish the proof, it is enough to show that

$$\{\beta < \kappa \mid \beta \text{ is a limit ordinal}\} \cap \Delta_{\alpha < \kappa}(\kappa \setminus \text{Dom}(p_\alpha)) \subseteq \bigcap_{\alpha < \kappa} (\kappa \setminus \text{Dom}(p_\alpha)). \quad (4.6)$$

Let $\beta \in \{\delta < \kappa \mid \delta \text{ is a limit ordinal}\} \cap \Delta_{\alpha < \kappa}(\kappa \setminus \text{Dom}(p_\alpha))$ be given. Then for all $\gamma < \beta$, $\beta \notin \text{Dom}(p_\gamma)$. By the limit step of the definition of fusion sequence, $\beta \notin \text{Dom}(p_\beta)$. By (4.5), β is not in $\text{Dom}(p_\alpha)$ for each $\alpha \geq \beta$. Hence β is in $\bigcap_{\alpha < \kappa} (\kappa \setminus \text{Dom}(p_\alpha))$. \square

We define an iteration of the Grigorieff forcing of length λ with κ support. The definition of the iteration is standard. For more details see [Bau83].

Definition 4.32. Let κ be a regular cardinal and $\lambda > 0$ be an ordinal. Then we define $\mathbb{G}_I(\kappa, \lambda)$ by induction as follows:

- (i) The forcing $\mathbb{G}_I(\kappa, 1)$ is defined as in Definition 4.27.
- (ii) $\mathbb{G}_I(\kappa, \xi + 1) = \mathbb{G}_I(\kappa, \xi) * \dot{Q}_\xi$, where \dot{Q}_ξ is a $\mathbb{G}_I(\kappa, \xi)$ -name for the partial order $\mathbb{G}_I(\kappa, 1)$ as defined in the extension $V[\mathbb{G}_I(\kappa, \xi)]$.
- (iii) For limit ordinal ξ , $\mathbb{G}_I(\kappa, \xi)$ is the inverse limit of $\langle \mathbb{G}_I(\kappa, \zeta) \mid \zeta < \xi \rangle$ if $\text{cf}(\xi) \leq \kappa$ and the direct limit otherwise.

We consider $\mathbb{G}_I(\kappa, \lambda)$ as the collection of functions p such that for every $\xi < \lambda$, $p \upharpoonright \xi \Vdash_\xi p(\xi) \in \dot{Q}_\xi$ and $|\text{supp}(p)| \leq \kappa$. The ordering is defined as follows: for p, q in $\mathbb{G}_I(\kappa, \lambda)$, $p \leq q$ if and only if $\text{supp}(p) \supseteq \text{supp}(q)$ and for every $\xi \in \text{supp}(q)$, $p \upharpoonright \xi \Vdash_\xi p(\xi) \leq q(\xi)$.

Lemma 4.33. *Let κ be a regular cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Then $\mathbb{G}_I(\kappa, \lambda)$ has size λ and it is λ -Knaster.*

Proof. This follows from Proposition 7.13 in [Cum10]. \square

The following definitions, lemmas and theorem are motivated by paper [Kan80], where the same was made for the Sacks forcing. We define the notion of meet and use it to show that the iteration of the Grigorieff forcing is sufficiently closed and has the fusion property.

Definition 4.34. Let α be an ordinal. If $\{p_\beta | \beta < \alpha\} \subseteq \mathbb{G}_I(\kappa, \lambda)$, then the *meet* $p = \bigwedge_{\beta < \alpha} p_\beta$ is defined as follows:

$$\text{supp}(p) = \bigcup_{\beta < \alpha} \text{supp}(p_\beta) \text{ and } p \upharpoonright \gamma \Vdash p(\gamma) = \bigcup_{\beta < \alpha} p_\beta(\gamma) \text{ for } \gamma \in \text{supp}(p). \quad (4.7)$$

If $p \upharpoonright \gamma$ is not in $\mathbb{G}_I(\kappa, \gamma)$ for $\gamma \in \text{supp}(p)$ or $|\text{supp}(p)| > \kappa$, then $\bigwedge_{\beta < \alpha} p_\beta$ is left undefined.

Lemma 4.35. If $\alpha < \kappa$ and $\langle p_\beta | \beta < \alpha \rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa, \lambda)$, then $p = \bigwedge_{\beta < \alpha} p_\beta \in \mathbb{G}_I(\kappa, \lambda)$. Hence $\mathbb{G}_I(\kappa, \lambda)$ is κ -closed.

Proof. The proof is the consequence of the Theorem 2.5 in [Bau83]. \square

Definition 4.36. Let $p, q \in \mathbb{G}_I(\kappa, \lambda)$, $X \subseteq \lambda$ with $|X| < \kappa$ and $\alpha < \kappa$. Then we define

$$p \leq_{X, \alpha} q \Leftrightarrow p \leq q \text{ and } p \upharpoonright \xi \Vdash p(\xi) \leq_\alpha q(\xi) \text{ for all } \xi \in X. \quad (4.8)$$

We say that a sequence $(\langle p_\xi | \xi < \kappa \rangle, \langle X_\xi | \xi < \kappa \rangle)$ is a *fusion sequence* if it satisfies the following conditions:

- (i) $p_{\xi+1} \leq_{X_\xi, \xi} p_\xi$ and $p_\zeta = \bigwedge_{\xi < \zeta} p_\xi$ for every limit $\zeta < \kappa$;
- (ii) $|X_\xi| < \kappa$ and $X_\xi \subseteq X_{\xi+1}$ for every $\xi < \kappa$;
- (iii) $X_\zeta = \bigcup_{\xi < \zeta} X_\xi$ for every limit $\zeta < \kappa$ and $\bigcup_{\xi < \kappa} X_\xi = \bigcup_{\xi < \kappa} \text{supp}(p_\xi)$.

Lemma 4.37. Let κ be a regular cardinal and $\lambda > 0$ be an ordinal. If $(\langle p_\beta | \beta < \kappa \rangle, \langle X_\beta | \beta < \kappa \rangle)$ is a fusion sequence, then $p = \bigwedge_{\beta < \kappa} p_\beta$ is in $\mathbb{G}_I(\kappa, \lambda)$.

Proof. We prove the lemma by induction on $\xi \leq \lambda$ and we show that for each $\xi \leq \lambda$, $p \upharpoonright \xi \in \mathbb{G}_I(\kappa, \xi)$.

If $\xi = 0$, then $p(\xi)$ is in $\mathbb{G}_I(\kappa, 1)$ by Lemma 4.31.

If $\xi = \zeta + 1$, then we want to show that $p \upharpoonright \zeta \Vdash_\zeta p(\zeta) \in \dot{Q}_\zeta$. Since $p \upharpoonright \zeta \leq p_\beta \upharpoonright \zeta$ for all $\beta < \kappa$, it is clear that $p \upharpoonright \zeta \Vdash_\zeta \langle p_\beta(\zeta) | \beta < \kappa \rangle$ is a decreasing sequence in \dot{Q}_ζ . If ζ is not in $\text{supp}(p)$, then we are done, since $p \upharpoonright \zeta \Vdash_\zeta p(\zeta) = \dot{1} \in \dot{Q}_\zeta$. If $\zeta \in \bigcup_{\xi < \kappa} \text{supp}(p_\xi)$, then by the definition of meet, we know that $p \upharpoonright \zeta \Vdash p(\zeta) = \bigcup_{\beta < \kappa} p_\beta(\zeta)$. Since $\bigcup_{\beta < \kappa} X_\beta = \bigcup_{\beta < \kappa} \text{supp}(p_\beta)$, there is $\alpha < \kappa$ and X_α such that $\zeta \in X_\alpha$. As the sequence $\langle X_\beta | \beta < \kappa \rangle$ is increasing and $p \upharpoonright \zeta \leq p_\beta \upharpoonright \zeta$ for all $\beta < \kappa$, we have that $p \upharpoonright \zeta \Vdash p_{\beta+1}(\zeta) \leq_\beta p_\beta(\zeta)$ for all $\alpha \leq \beta \leq \kappa$. By Lemma 4.31, $p \upharpoonright \zeta \Vdash \bigcup_{\alpha < \beta < \kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$. Since $p \upharpoonright \zeta \Vdash_\zeta \langle p_\beta(\zeta) | \beta < \kappa \rangle$ is a decreasing sequence in \dot{Q}_ζ , $p \upharpoonright \zeta \Vdash \bigcup_{\alpha < \beta < \kappa} p_\beta(\zeta) = \bigcup_{\beta < \kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$.

If ξ is a limit ordinal, then $p \upharpoonright \xi$ is completely determined by $p \upharpoonright \zeta$, $\zeta < \xi$. If $\text{supp}(p \upharpoonright \xi)$ is not cofinal in ξ , then $p \upharpoonright \xi$ is in $\mathbb{G}_I(\kappa, \xi)$ since we considered just

inverse and direct limits. Suppose that $\text{supp}(p \upharpoonright \xi)$ is cofinal in ξ . Since $\text{supp}(p \upharpoonright \xi) = \bigcup_{\beta < \kappa} (\text{supp}(p_\beta \upharpoonright \xi))$, it must hold either that $\sup(\text{supp}(p_\beta \upharpoonright \xi)) = \xi$ for some β or that $\text{cf}(\xi) \leq \kappa$. In both cases $\mathbb{G}_I(\kappa, \xi)$ is the inverse limit of $\langle \mathbb{G}_I(\kappa, \zeta) \mid \zeta < \xi \rangle$ and since $p \upharpoonright \zeta \in \mathbb{G}_I(\kappa, \zeta)$ for all $\zeta < \xi$, $p \upharpoonright \xi$ is in $\mathbb{G}_I(\kappa, \xi)$. \square

The fusion property is used to show that κ^+ is preserved in the extension by $\mathbb{G}_I(\kappa, \lambda)$.

Fact 4.38. *Let κ be a regular cardinal. Assume that either κ is inaccessible or that \diamond_κ holds. Then $\mathbb{G}_I(\kappa, \lambda)$ preserves κ^+ .*

Proof. For the proof see [Kan80], where it is done for the Sacks forcing. The key of the proof is the fusion property, so it can be modified to the Grigorieff forcing. \square

We know that cardinals $\leq \kappa$ are preserved since we showed that $\mathbb{G}_I(\kappa, \lambda)$ is κ -closed. In addition, if λ is inaccessible, all cardinals $\geq \lambda$ are preserved as $\mathbb{G}_I(\kappa, \lambda)$ is λ -Knaster. Moreover, under additional assumptions κ^+ is preserved due to the fusion property. Now we show that the other cardinals are collapsed.

Lemma 4.39. *Let κ be a regular cardinal. Assume that either κ is inaccessible or that \diamond_κ holds. Let $\lambda > \kappa$ be an inaccessible cardinal. Then $V[\mathbb{G}_I(\kappa, \lambda)] \models \lambda = \kappa^{++}$.*

Proof. By Lemma 4.33 all cardinals $\mu \geq \lambda$ are preserved by the κ -support iteration of the Grigorieff forcing and by Fact 4.38, κ^+ is also preserved. Now, we show that each cardinal ξ , $\kappa^+ < \xi < \lambda$, is collapsed to κ^+ . It is enough to show this for ξ regular. Let ξ be given. Since in each step of the iteration $\mathbb{G}_I(\kappa, \xi)$ we add at least one set to the cardinal κ , $V[\mathbb{G}_I(\kappa, \xi)] \models 2^\kappa \geq |\xi|$. We work in $V[\mathbb{G}_I(\kappa, \xi)]$ and define a complete embedding i from $\text{Add}(\kappa^+, 1)$ to $\mathbb{G}_I(\kappa, [\xi, \xi + \kappa^+])$. This will mean that $|\xi|^{V[\mathbb{G}_I(\kappa, \xi + \kappa^+)]} = \kappa^+$.

We define i as follows: For $p \in \text{Add}(\kappa^+, 1)$, set $i(p) = q$ where q is define by:

(i) For $\alpha = \xi$, set

$$q(\alpha) = \begin{cases} \emptyset & \text{if } 0 \notin \text{Dom}(p); \\ \langle 0, 0 \rangle & \text{if } p(0) = 0; \\ \langle 0, 1 \rangle & \text{otherwise.} \end{cases}$$

(ii) For $\alpha > \xi$, set

$$q(\alpha) = \begin{cases} \check{\emptyset} & \text{if } \alpha \notin \text{Dom}(p); \\ \langle \check{0}, \check{0} \rangle & \text{if } p(\alpha) = 0; \\ \langle \check{0}, \check{1} \rangle & \text{otherwise,} \end{cases}$$

where $\check{0}$, $\check{0}$ and $\check{1}$ are canonical $\mathbb{G}_I(\kappa, [\xi, \alpha])$ -names for \emptyset , 0 and 1 . Note that q is a condition in $\mathbb{G}_I(\kappa, [\xi, \xi + \kappa^+])$, since $|\text{Dom}(p)| \leq \kappa$.

To finish the proof, we verify that i is the complete embedding, i.e. we verify the conditions from Definition 2.30.

Ad (i). Let $p \leq p'$ be conditions in $\text{Add}(\kappa^+, 1)$. Then $\text{Dom}(p') \subseteq \text{Dom}(p)$, hence $\text{supp}(i(p')) \subseteq \text{supp}(i(p))$. Let β be in $\text{supp}(i(p'))$. Since $p \leq p'$, $p(\beta) = p'(\beta)$.

Ad (ii). By the definition of i , $i(p) \upharpoonright \beta \Vdash i(p)(\beta) = \langle \check{0}, \check{0} \rangle = i(p')(\beta)$ if $p(\beta) = 0$ or $i(p) \upharpoonright \beta \Vdash i(p)(\xi) = \langle \check{0}, \check{1} \rangle = i(p')(\xi)$ if $p(\beta) = 1$. In either case $i(p) \upharpoonright \beta \Vdash i(p)(\beta) \leq i(p')(\beta)$.

Let p, p' be conditions in $\text{Add}(\kappa^+, 1)$. We want to show that $p \perp p' \Leftrightarrow i(p) \perp i(p')$.

(\Rightarrow) Let $i(p) \parallel i(p')$. We show that there is $r \in \text{Add}(\kappa^+, 1)$, $r \leq p$ and $r \leq p'$. If $\text{supp}(i(p)) \cap \text{supp}(i(p')) = \emptyset$, then we define r as follows: $\text{Dom}(r) = \text{supp}(i(p)) \cup \text{supp}(i(p'))$ and for $\alpha \in \text{Dom}(r)$

$$r(\alpha) = \begin{cases} 0 & \text{if } i(s)(\alpha) = \langle \check{0}, \check{0} \rangle; \\ 1 & \text{otherwise.} \end{cases}$$

where s is either p or p' depending on whether $\alpha \in \text{supp}(i(p))$ or $\alpha \in \text{supp}(i(p'))$.

If $\text{supp}(i(p)) \cap \text{supp}(i(p')) \neq \emptyset$, then we can define r the same way. It follows from the claim that if α is in $\text{supp}(i(p)) \cap \text{supp}(i(p'))$, then $i(p)(\alpha) = i(p')(\alpha)$. If not then $i(p)(\alpha)$ and $i(p')(\alpha)$ are incompatible.

(\Leftarrow) Let $p \parallel p'$. Then it follows from (i) that $i(p) \parallel i(p')$.

Ad (iii). Let $q \in \mathbb{G}_I(\kappa, [\xi, \xi + \kappa^+])$. We need to find $p \in \text{Add}(\kappa^+, 1)$ such that $(\forall p' \leq p)(i(p') \parallel q)$. We define p as follows: $\text{Dom}(p) = \text{supp}(q)$ and for $\alpha \in \text{Dom}(p)$

$$p(\alpha) = \begin{cases} 0 & \text{if } q \upharpoonright \alpha \Vdash \langle \check{0}, \check{0} \rangle \in q(\alpha); \\ 1 & \text{if } q \upharpoonright \alpha \Vdash \langle \check{0}, \check{1} \rangle \in q(\alpha). \end{cases}$$

It is easy to verify that p satisfies the desired conditions.

We know that in $V[\mathbb{G}_I(\kappa, \xi)] \models 2^\kappa \geq |\xi|$ and also that there is a complete embedding from $\text{Add}(\kappa^+, 1)$ to $\mathbb{G}_I(\kappa, [\xi, \xi + \kappa^+])$. Since $\text{Add}(\kappa^+, 1)$ collapses 2^κ to κ^+ , in $V[\mathbb{G}_I(\kappa, \xi)]$ also $\mathbb{G}_I(\kappa, [\xi, \xi + \kappa^+])$ collapses 2^κ to κ^+ . As a result, $V[\mathbb{G}_I(\kappa, \lambda)] \models |\xi| = \kappa^+$. \square

The following theorem tells us that the iteration of the Grigorieff forcing can be split into the two parts such that in the extension after iterating with the first part, the second part still has the nice properties of whole iteration.

Fact 4.40. *Let κ be a regular cardinal and $\alpha, \beta > 0$ be ordinals. Let $\dot{\mathbb{G}}_I(\kappa, \beta)$ be the $\mathbb{G}_I(\kappa, \alpha)$ -name for $\mathbb{G}_I(\kappa, \beta)$. Then \Vdash_α “ $\mathbb{G}_I(\kappa, [\alpha, \alpha + \beta])$ is forcing equivalent to $\dot{\mathbb{G}}_I(\kappa, \beta)$ ”.*

Proof. For the proof see [BL79], where this is done for the Sacks forcing. \square

Now we have almost everything we need to prove the main theorems of this section, the last thing we need to know is that this forcing does not add cofinal branches to the trees that are of our interest.

Fact 4.41. *Assume κ is an inaccessible cardinal or $\kappa = \omega$ and let $\lambda > 0$ be an ordinal. Then $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to κ^+ -trees, and more generally, if $\rho \geq \kappa$ is such that $2^\kappa > \rho$, then $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to the ρ^+ -trees.*

Proof. For the proof see [FH]. \square

Fact 4.42. *Assume $\omega_1 < \kappa = \xi^+$, $2^\xi = \xi^+$ and $\lambda > 0$ is an ordinal. Then $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to κ^+ -trees, and more generally, if $\rho \geq \kappa$ is such that $2^\kappa > \rho$, then $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to the ρ^+ -trees.*

Proof. For the proof see [FH]. \square

Remark 4.43. The assumption of $\kappa > \omega_1$ in Fact 4.42 is not essential. For $\kappa = \omega_1$ we need an additional assumption \diamond_{ω_1} . We do not need this assumption for $\kappa > \omega_1$ since we assume that $\kappa = \xi^+$ and $2^\xi = \xi^+$, which ensures \diamond_κ .

Now, we are ready to prove the main theorems of this section.

Theorem 4.44. *Assume GCH. Let κ be a regular cardinal. In the case of $\kappa = \omega_1$ assume in addition that \diamond_{ω_1} holds. If there exists a Mahlo cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{G}_I(\kappa, \lambda)$ it holds that*

(i) $2^\kappa = \lambda = \kappa^{++}$;

(ii) κ^{++} has the weak tree property.

Proof. Ad (i). It is easy to see that $2^\kappa \geq \lambda$. Now, we show that $2^\kappa \leq \lambda$. Since the forcing $\mathbb{G}_I(\kappa, \lambda)$ is λ -cc of size λ , which is inaccessible in V , $2^\kappa \leq \lambda$ easily follows by a common nice names argument. The equation $\lambda = \kappa^{++}$ follows from Lemma 4.39.

Ad (ii). Let G be a $\mathbb{G}_I(\kappa, \lambda)$ -generic over V . Suppose that T is an M-special $\lambda = \kappa^{++}$ -Aronszajn tree in $V[G]$. Let $q \in G$ be such that $q \Vdash$ “ T is an M-special κ^{++} -Aronszajn tree”. Then for each $\alpha < \lambda$, $|T \upharpoonright \alpha| = \kappa^+$. Hence we can consider $T \upharpoonright \alpha$ as a subset of κ^+ . As a subset of κ^+ , $T \upharpoonright \alpha$ has a nice name $\pi_\alpha = \bigcup \{ \{ \check{\beta} \} \times A_\beta \mid \beta < \kappa^+ \text{ and } A_\beta \text{ is an antichain of } \mathbb{G}_I(\kappa, \lambda) \}$ and it holds that $\beta \in T \upharpoonright \alpha \leftrightarrow (\exists a \in A_\beta)(a \in G)$.

Now we show that $T \upharpoonright \alpha$ is in $V[G_\xi]$ for some $\xi < \lambda$, where $G_\xi = \{p \upharpoonright \xi \mid p \in G\}$. Set $X = \bigcup \{ \text{supp}(p) \mid (\exists \beta < \kappa^+)(p \in A_\beta) \}$. The set X has size less than λ since

$\mathbb{G}_I(\kappa, \lambda)$ is λ -cc. Take $\xi = \sup \{\beta < \lambda \mid \beta \in X\}$. Then $T \restriction \alpha \in V[G_\xi]$ since π_α is also a $\mathbb{G}_I(\kappa, \xi)$ -nice name. Let \dot{T}_α be a $\mathbb{G}_I(\kappa, \lambda)$ -name for $T \restriction \alpha$. Set

$$\sigma'(\alpha) = \min \left\{ \xi < \lambda \mid (\forall \alpha' < \alpha) (\exists \pi_{\alpha'}) (\pi_{\alpha'} \text{ is } \mathbb{G}_I(\kappa, \xi)\text{-nice name and } q \Vdash \dot{T}_{\alpha'} = \pi_{\alpha'}) \right\}. \quad (4.9)$$

Then σ' is defined in V and it is easy to see that it is a continuous nondecreasing unbounded function from λ into λ . As we want to argue that σ' has a fixed point, we need σ' to be increasing. Hence we define $\sigma : \lambda \rightarrow \lambda$ by induction on $\alpha < \lambda$ as follows:

If $\alpha = 0$ then $\sigma(\alpha) = \sigma'(\alpha)$. If $\alpha = \beta + 1$ then set

$$\sigma(\alpha) = \begin{cases} \sigma(\beta) + 1 & \text{if } \sigma'(\alpha) \leq \sigma(\beta); \\ \sigma'(\alpha) & \text{otherwise.} \end{cases}$$

If α is limit then $\sigma(\alpha) = \sup \{\sigma(\beta) \mid \beta < \alpha\}$.

The function σ is continuous and increasing, hence there is a closed unbounded set of fixed points of σ . Moreover σ still satisfies that $T \restriction \alpha \in V[G_{\sigma(\alpha)}]$ since $\sigma'(\alpha) \leq \sigma(\alpha)$ for each $\alpha < \lambda$. As λ is Mahlo in V there is a fixed point δ of σ which is inaccessible in V .

Since T is a λ -Aronszajn tree in $V[G]$, there is branch b of T of length δ in $V[G]$. Now we show that b is already in $V[G_\delta]$. By Fact 4.40, the forcing $\mathbb{G}_I(\kappa, [\delta, \lambda])$ is forcing equivalent to $\mathbb{G}_I(\kappa, \lambda)$ as defined in $V[G_\delta]$. The forcing $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to δ -trees. This follows from Fact 4.42 in case κ is a successor and from Fact 4.41 in case κ is inaccessible. Therefore b is in $V[G_\delta]$.

Since T is M-special, b is a 1-1 function from δ to κ^+ , so $V[G_\delta] \models \delta \leq \kappa^+$. As G_δ is $M(\kappa, \delta)$ -generic over V and δ is inaccessible in V , $V[G_\delta] \models \delta = \kappa^{++}$. This is a contradiction. \square

The next theorem can be proved under the assumption of a weakly compact cardinal but we use a measurable cardinal for the same reasons as in Theorem 4.26.

Theorem 4.45. *Assume GCH. Let κ be a regular cardinal. If there exists a measurable cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{G}_I(\kappa, \lambda)$ it holds that*

- (i) $2^\kappa = \lambda = \kappa^{++}$;
- (ii) κ^{++} has the tree property.

Proof. Ad (i). It is easy to see that $2^\kappa \geq \lambda$. Now, we show that $2^\kappa \leq \lambda$. Since the forcing $\mathbb{G}_I(\kappa, \lambda)$ is λ -cc of size λ , which is inaccessible in V , $2^\kappa \leq \lambda$ easily follows by a common nice names argument. The equation $\lambda = \kappa^{++}$ follows from Lemma 4.39.

Ad (ii). Let G be a $\mathbb{G}_I(\kappa, \lambda)$ -generic over V . Since λ is measurable in V , there is an elementary embedding $j : V \rightarrow M$ with critical point λ and ${}^\lambda M \subseteq M$, where M is a transitive model of ZFC.

In M , the forcing $j(\mathbb{G}_I(\kappa, \lambda))$ is the iteration of $\mathbb{G}_I(\kappa, 1)$ of length $j(\lambda)$ with κ -support by the elementarity of j . The forcing $\mathbb{G}_I(\kappa, j(\lambda))^M$ is forcing equivalent to $\mathbb{G}_I(\kappa, \lambda)^M * \dot{\mathbb{G}}_I(\kappa, [\lambda, j(\lambda)])$. As j is identity below λ , $\mathbb{G}_I(\kappa, \alpha)^V = \mathbb{G}_I(\kappa, \alpha)^M$, for $\alpha < \lambda$ and since we take direct limit at λ , $\mathbb{G}_I(\kappa, \lambda)^V = \mathbb{G}_I(\kappa, \lambda)^M$. Hence G is also $\mathbb{G}_I(\kappa, \lambda)^M$ -generic over M .

Since the partial order $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)])$ is definable in $M[G]$, it is definable in $V[G]$. Let H be $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)])$ -generic over $V[G]$, then H is $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)])$ -generic over $M[G]$ since $M[G] \subseteq V[G]$.

Work in $V[G][H]$. By Lemma 2.3, we can lift j to $j^* : V[G] \rightarrow M[G][H]$. Assume T is a λ -tree in $V[G]$. We show that T has a cofinal branch in $V[G]$. We can consider T as a subset of λ , so T has a nice name \dot{T} in V . Since $|\dot{T}| \leq \lambda$ and ${}^\lambda M \subseteq M$, \dot{T} is in M . Hence $T \in M[G]$. By elementarity of j^* , $j^*(T)$ is a $j^*(\lambda)$ -tree in $M[G][H]$ and since j^* is the identity below λ , $j^*(T) \upharpoonright \lambda = T$. As T is $j^*(\lambda)$ -tree in $M[G][H]$, it has branch b of length λ in $M[G][H]$. By Fact 4.40, the forcing $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)])$ is forcing equivalent to $\mathbb{G}_I(\kappa, j(\lambda))$ as defined in $M[G]$. The forcing $\mathbb{G}_I(\kappa, j(\lambda))$ does not add cofinal branches to λ -trees. This follows from Fact 4.42 in case κ is a successor and from Fact 4.41 in case κ is inaccessible. Therefore b is in $M[G]$, hence in $V[G]$. \square

5 The Tree Property at More Cardinals

In the previous chapter we showed that the weak tree and the tree property can hold at double successor of a regular cardinal. Therefore, it is quite natural to ask whether the weak tree property or the tree property can hold at two or even more cardinals at the same time. Assuming two weakly compact cardinals or two Mahlo cardinals, it is not hard to generalize the methods of Section 4.2 to obtain a model where for instance $2^\omega = 2^{\omega_1} = \omega_2$ and $2^{\omega_2} = 2^{\omega_3} = \omega_4$ and both cardinals ω_2 and ω_4 has the tree property or the weak tree property, respectively. However the naive approach fails if one assumes two weakly compact cardinals and tries to force the tree property at two successive cardinals. Magidor showed that this is not just a technical problem, he showed that the assumption of the tree property at two successive cardinals implies that there exists a model with a measurable cardinal (see [Abr83]). On the other hand, in the case of the weak tree property the assumption of two Mahlo cardinals is sufficient.

The results presented in this chapter are implicit in [Mit72].

5.1 The Weak Tree Property

Here, we focus on the weak tree property. We show that the weak tree property can hold at two successive cardinals under the assumption of two Mahlo cardinals. Then we extend this result to ω -many successive cardinals under the assumption of ω -many Mahlo cardinals. Before we get to the proofs, we show the generalization of the Mitchell forcing.

Throughout this section we assume that κ, ξ, λ are regular cardinals and $\kappa < \xi < \lambda$.

The following definition can be found in [Ung].

Definition 5.1. The forcing $\mathbb{M}(\kappa, \xi, \lambda)$ is defined as follows: for an ordinal $\alpha, \xi < \alpha < \lambda$, let $\mathbb{Q}(\alpha)$ be an $\text{Add}(\kappa, \alpha)$ -name for the partially ordered set $\text{Add}(\kappa^+, 1)$.

The forcing $\mathbb{M}(\kappa, \xi, \lambda)$ is the collection of pairs (p, q) such that $p \in \text{Add}(\kappa, \lambda)$ and q is a function of cardinality less than ξ such that if $\alpha \in \text{Dom}(q)$ then $\xi < \alpha < \lambda$ and $\emptyset \Vdash^{\text{Add}(\kappa, \alpha)} "q(\alpha) \in \mathbb{Q}(\alpha)"$.

The $\mathbb{M}(\kappa, \xi, \lambda)$ is ordered by $(p, q) \leq (p', q')$ if and only if $p \leq_{\text{Add}(\kappa, \lambda)} p'$, $\text{Dom}(q') \subseteq \text{Dom}(q)$ and for all $\alpha \in \text{Dom}(q')$ $p \restriction \alpha \Vdash q'(\alpha) \subseteq q(\alpha)$.

Remark 5.2. Note that the Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ defined in Chapter 4 is equal to $\mathbb{M}(\kappa, \xi, \lambda)$ for $\xi = \kappa^+$.

The following lemmas, corollaries and definition are direct and obvious generalizations of lemmas, corollaries and definition in Section 4.2. Therefore we skip the proofs.

Lemma 5.3. $\mathbb{M}(\kappa, \xi, \lambda)$ is κ -closed.

Lemma 5.4. Let λ be an inaccessible cardinal. Then $\mathbb{M}(\kappa, \xi, \lambda)$ is λ -Knaster.

Definition 5.5. We define $\mathbb{Q} = (Q(\kappa, \xi, \lambda), \leq_{\mathbb{Q}(\kappa, \xi, \lambda)})$, as follows: $Q(\kappa, \xi, \lambda) = \{(\emptyset, q) \mid (\emptyset, q) \in M(\kappa, \xi, \lambda)\}$ and the ordering $\leq_{\mathbb{Q}} = \leq_{\mathbb{M}(\kappa, \xi, \lambda)} \upharpoonright Q(\kappa, \xi, \lambda)$.

Lemma 5.6. \mathbb{Q} is ξ -closed.

Lemma 5.7. Let $\pi : \text{Add}(\kappa, \lambda) \times \mathbb{Q}$ be a function such that $\pi((p, (\emptyset, q))) = (p, q)$. Then π is a projection.

Lemma 5.8. There is projection $\pi : \mathbb{M}(\kappa, \xi, \lambda) \rightarrow \text{Add}(\kappa, \lambda)$.

Lemma 5.9. Assume $\kappa^{<\kappa} = \kappa$. Then all set of ordinals in $V[\mathbb{M}(\kappa, \xi, \lambda)]$ of cardinality less than ξ are in $V[\text{Add}(\kappa, \lambda)]$.

Lemma 5.10. Assume $\kappa^{<\kappa} = \kappa$. Then ξ remains a cardinal in $\mathbb{M}(\kappa, \xi, \lambda)$.

Lemma 5.11. Let λ be an inaccessible cardinal. Then in $V[\mathbb{M}(\kappa, \xi, \lambda)]$, $2^\kappa = \lambda$.

Lemma 5.12. Let μ be an ordinal, $\xi < \mu < \lambda$. Then there is a projection $\pi : \mathbb{M}(\kappa, \xi, \lambda) \rightarrow \mathbb{M}(\kappa, \xi, \mu)$.

Lemma 5.13. Assume $\kappa^{<\kappa} = \kappa$. Let λ be an inaccessible cardinal. Then $\lambda = \xi^+$ in $V[\mathbb{M}(\kappa, \xi, \lambda)]$.

Corollary 5.14. Assume $\kappa^{<\kappa} = \kappa$. The forcing \mathbb{Q} collapses cardinals between ξ and λ to ξ .

Lemma 5.15. For all $\mu, \xi < \mu < \lambda$, in $V[\mathbb{M}(\kappa, \xi, \mu)]$ the forcing $\mathbb{M}(\kappa, \xi, \lambda)/\mathbb{M}(\kappa, \xi, \mu)$ is a projection of $\text{Add}(\kappa, [\mu, \lambda]) \times \mathbb{Q}^*$, where \mathbb{Q}^* is ξ -closed.

Remark 5.16. Recall we assume $\kappa < \xi < \lambda$ are regular cardinals. If λ is a weakly compact or Mahlo cardinal, then $\mathbb{M}(\kappa, \xi, \lambda)$ forces that $2^\kappa = \lambda = \xi^+$ and that the tree property or the weak tree property holds at λ . The first follows immediately from the previous lemmas and the second from the fact that the proofs of Theorem 4.26 and Theorem 4.24 remain correct if we substitute κ^+ by a regular cardinal ξ between κ and λ as these are the only two properties of κ^+ we used in the proofs.

As we showed above, $\mathbb{M}(\kappa, \xi, \lambda)$ has the nice property that each sequence of ordinals of length less than ξ added by $\mathbb{M}(\kappa, \xi, \lambda)$ is already added by $\text{Add}(\kappa, \lambda)$. For the proof we used the chain condition of $\text{Add}(\kappa, \lambda)$ in the ground model. As we want to force the weak tree property at more cardinals, we need to consider the product of Mitchell forcings. The following lemma tells us what we need to preserve this property in products.

Lemma 5.17. *Assume $\kappa^{<\kappa} = \kappa$. Let \mathbb{R} be a forcing notion. If \mathbb{R} is ξ -Knaster, all sets of ordinals in $V[\mathbb{R} \times \mathbb{M}(\kappa, \xi, \lambda)]$ of cardinality less than ξ are in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$.*

Proof. By Lemma 5.7, there is a projection from $\text{Add}(\kappa, \lambda) \times \mathbb{Q}$ to $\mathbb{M}(\kappa, \xi, \lambda)$, where \mathbb{Q} is ξ -closed. Hence $V[\mathbb{R} \times \mathbb{M}(\kappa, \xi, \lambda)] \subseteq V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$, so it suffices to show that all sets of ordinals in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of cardinality less than ξ are in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$. As we assume $\kappa^{<\kappa} = \kappa$, the forcing $\text{Add}(\kappa, \lambda)$ is κ^+ -Knaster and as $\xi \geq \kappa^+$, it is ξ -cc. Since \mathbb{R} is ξ -Knaster, $\mathbb{R} \times \text{Add}(\kappa, \lambda)$ is ξ -cc. Hence, by Easton's Lemma, \mathbb{Q} is ξ -distributive in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$, i.e. \mathbb{Q} does not add any new sequences of ordinals of length less than ξ . Therefore each sequence of ordinals in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of length less than ξ is already in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$. \square

Before we present the main proofs, let us explain the motivation for using the forcing $\mathbb{M}(\kappa, \xi, \lambda)$. The naive approach to obtain a model with the weak tree property at two successive cardinals would be to use the forcing $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_1) = \mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \kappa^{++}, \lambda_1)$, where λ_0 and λ_1 are Mahlo cardinals such that $\lambda_1 > \lambda_0 > \kappa$. However, due to the reasons described below, this is not a good approach. Therefore we use $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$. This forcing is more suitable to force the weak tree property at κ^{++} and κ^{+++} at the same time since $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is a projection of $\text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}_1$, where \mathbb{Q}_1 is λ_0 -closed. This means that we can work in $V[\mathbb{Q}_1]$ since λ_0 is still Mahlo here and $\mathbb{M}(\kappa, \kappa^+, \lambda_0)^V = \mathbb{M}(\kappa, \kappa^+, \lambda_0)^{V[\mathbb{Q}_1]}$. If we try to force with the first forcing, the forcing $\mathbb{M}(\kappa^+, \kappa^{++}, \lambda_1)$ is a projection of $\text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}'_1$, where \mathbb{Q}'_1 is just κ^{++} -closed, so we do not know how to continue. The situation where we use that \mathbb{Q}_1 is λ_0 -closed appears more than once in the proof. In addition, note that in $V[\mathbb{M}(\kappa, \kappa^+, \lambda_0)]$, we need to collapse cardinals only above λ_0 since all cardinals between κ^+ and λ_0 have already been collapsed by $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$.

Lemma 5.18. *Assume GCH. Let κ be a regular cardinal and let $\lambda_1 > \lambda_0 > \kappa$ be inaccessible cardinals. Then each cardinal $\xi \leq \kappa$ or $\xi \geq \lambda_1$ is preserved by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$. Moreover, κ^+ and λ_0 are preserved.*

Proof. Let $\xi \leq \kappa$. Then ξ is preserved since $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is κ -closed. Let $\xi \geq \lambda_1$. Since $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is λ_1 -Knaster and $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -Knaster, $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is λ_1 -cc, so ξ is preserved.

Now, we show that κ^+ is preserved by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$. Since there is a projection from $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ to $\text{Add}(\kappa, \lambda_0) \times \mathbb{Q}_0$, where \mathbb{Q}_0 is κ^+ -closed, it suffices to show that κ^+ is preserved by $\text{Add}(\kappa, \lambda_0) \times \mathbb{Q}_0 \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$. As $\text{Add}(\kappa, \lambda_0)$ is κ^+ -Knaster, κ^+ is still a cardinal in $V[\text{Add}(\kappa, \lambda_0)]$ and by Easton's Lemma $\mathbb{Q}_0 \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is κ^+ -distributive in $V[\text{Add}(\kappa, \lambda_0)]$. Therefore κ^+ remains a cardinal in $V[\text{Add}(\kappa, \lambda_0)][\mathbb{Q}_0 \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)]$.

Now, we verify that λ_0 is preserved by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$. Since there is a projection from $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ to $\text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}_1$, where \mathbb{Q}_1 is λ_0 -closed, it is enough to show that λ_0 is preserved by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}_1$. In $V[\mathbb{Q}_1]$, λ_0 is a cardinal since \mathbb{Q}_1 is λ_0 -closed and by Easton's Lemma, $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1)$ is λ_0 -cc. Therefore λ_0 is a cardinal in $V[\mathbb{Q}_1][\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1)]$. \square

Now, we have proved everything we need to prove the main theorems of this section. Both Theorems 5.19 and 5.22 are implicit in Mitchell's paper [Mit72], but we use a construction due to Unger from [Ung].

Theorem 5.19. *Assume GCH. Let κ be a regular cardinal. If there exist Mahlo cardinals $\lambda_1 > \lambda_0 > \kappa$, then in the generic extension by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ it holds that*

(i) $2^\kappa = \lambda_0 = \kappa^{++}$ and $2^{\kappa^+} = \lambda_1 = \kappa^{+++}$;

(ii) κ^{++} and κ^{+++} have the weak tree property.

Proof. Ad (i). $2^\kappa \geq \lambda_0$ holds because there is a projection from $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ to $\text{Add}(\kappa, \lambda_0)$. Now, we show that $2^\kappa \leq \lambda_0$. Since $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ is κ^+ -closed, each sequence of ordinals of length less than κ^+ is in $V[\mathbb{M}(\kappa, \kappa^+, \lambda_0)]$ and by Lemma 5.9, it is in $V[\text{Add}(\kappa, \lambda_0)]$. Since this forcing is κ^+ -cc of size λ_0 , which is inaccessible in V , $2^\kappa \leq \lambda_0$ easily follows by a common nice names argument.

The proof of $2^{\kappa^+} = \lambda_1$ is similar to the proof before. $2^{\kappa^+} \geq \lambda_1$ follows from the fact that there is a projection from $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ to $\text{Add}(\kappa^+, \lambda_1)$. Now, we show that $2^{\kappa^+} \leq \lambda_1$. Since $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -Knaster, we can use Lemma 5.17 and since $\kappa^+ < \lambda_0$, each sequence of length $\leq \kappa^+$ is in $V[\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1)]$. As $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1)$ is λ_0 -cc and it has size λ_1 , which is inaccessible in V , $2^{\kappa^+} \leq \lambda_1$ easily follows by a common nice names argument.

The equalities $\kappa^{++} = \lambda_0$ and $\kappa^{+++} = \lambda_1$ follow from Lemma 5.13 and Lemma 5.18.

Ad (ii). We show that a) κ^{++} has the weak tree property and b) κ^{+++} has the weak tree property.

Ad a) Assume for contradiction that $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ adds an M-special $\kappa^{++} = \lambda_0$ -Aronszajn tree T . By Lemma 5.7, T is also added by $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}_1$, where \mathbb{Q}_1 is λ_0 -closed. Let $G_0 \times H_1 \times H'_1$ be $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, \lambda_1) \times \mathbb{Q}_1$ -generic over V .

Consider $W = V[H'_1]$. Since \mathbb{Q}_1 is λ_0 -closed in V and $\lambda_0 > \kappa^{++}$, $\text{Add}(\kappa^+, \lambda_1)$ is still κ^{++} -Knaster and κ^+ -closed. In addition, $V_{\lambda_0} = W_{\lambda_0}$. As $V_{\lambda_0} = W_{\lambda_0}$ and conditions in $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ are bounded in V_{λ_0} , $\mathbb{M}(\kappa, \kappa^+, \lambda_0)^V = \mathbb{M}(\kappa, \kappa^+, \lambda_0)^W$. Moreover, λ_0 is Mahlo in W because λ_0 -closed forcings preserve stationary sets in λ_0 .

As in Theorem 4.24, we can define in W a continuous increasing function σ from λ_0 into λ_0 such that $T \restriction \alpha \in W[G_0^{\sigma(\alpha)}][H_1^{\sigma(\alpha)}]$, where $G_0^{\sigma(\alpha)}$ is equal to $\{(p \restriction \sigma(\alpha), q \restriction \sigma(\alpha)) \mid (p, q) \in G_0\}$ and $H_1^{\sigma(\alpha)}$ is equal to $\{p \restriction \sigma(\alpha) \mid p \in H_1\}$. As λ_0 is Mahlo in W there is a fixed point δ of σ which is inaccessible in W .

In final extension $W[G_0 \times H_1]$, δ is collapsed to κ^+ , so $T \restriction \delta$ has a branch b of length δ . Now, we show that b is in $W[G_0^\delta \times H_1^\delta]$. We do this by showing that the forcing with $\mathbb{M}(\kappa, \kappa^+, \lambda_0)/\mathbb{M}(\kappa, \kappa^+, \delta) \times \text{Add}(\kappa^+, [\delta, \lambda_1])$ over $W[G_0^\delta \times H_1^\delta]$ could not add branch b . It is important to note that the forcing $\mathbb{M}(\kappa, \kappa^+, \lambda_0)/\mathbb{M}(\kappa, \kappa^+, \delta)$ is defined in $W[G_0^\delta]$ and the forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is defined in $W[H_1^\delta]$.

In $W[G_0^\delta \times H_1^\delta]$, $\delta = \kappa^{++}$ and $2^\kappa \geq \delta$ since G_0^δ is $\mathbb{M}(\kappa, \kappa^+, \delta)$ -generic over W and $\mathbb{M}(\kappa, \kappa^+, \delta) \times \text{Add}(\kappa^+, \delta)$ is δ -cc. As $\delta = \kappa^{++}$, $T \restriction \delta$ is an M-special δ -Aronszajn tree. By Lemma 4.23, there is a projection from $\text{Add}(\kappa, [\delta, \lambda_0]) \times \mathbb{Q}_0^*$ to $\mathbb{M}(\kappa, \kappa^+, \lambda_0)/\mathbb{M}(\kappa, \kappa^+, \delta)$, where \mathbb{Q}_0^* is κ^+ -closed. Since $b \in W[G_0][H_1]$, $b \in W[G_0^\delta \times H_1^\delta][H_1^{[\delta, \lambda_1]} \times F_1 \times F_2]$, where $H_1^{[\delta, \lambda_1]} \times F_1 \times F_2$ is $(\text{Add}(\kappa^+, [\delta, \lambda_1]) \times \text{Add}(\kappa, [\delta, \lambda_0]) \times \mathbb{Q}_0^*)$ -generic over $W[G_0^\delta \times H_1^\delta]$. We reorganize the forcing as follows: $W[G_0^\delta \times H_1^\delta][H_1^{[\delta, \lambda_1]} \times F_2 \times F_1]$.

First note that $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is κ^+ -distributive and δ -cc over $W[G_0^\delta][H_1^\delta]$. The forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is defined in $W[H_1^\delta]$. Now in $W[H_1^\delta]$, $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is δ -Knaster since $\alpha^{<\kappa^+} < \delta$ for all $\alpha < \delta$, so we can use the Δ -system argument. Since $\mathbb{M}(\kappa, \kappa^+, \delta)$ is δ -cc over $W[H_1^\delta]$, the forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is δ -cc over $W[G_0^\delta \times H_1^\delta]$. Now, we focus on the distributivity. The forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is κ^+ -closed in $W[H_1^\delta]$. Since there is a projection from $\text{Add}(\kappa, \delta) \times \mathbb{Q}_0$ to $\mathbb{M}(\kappa, \kappa^+, \delta)$, where \mathbb{Q}_0 is κ^+ -closed, it is enough to show that $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is κ^+ -distributive over $W[\text{Add}(\kappa, \delta) \times \mathbb{Q}_0 \times H_1^\delta]$. Since the forcing $\text{Add}(\kappa^+, \delta)$ is κ^+ -closed in W , \mathbb{Q}_0 is κ^+ -closed in $W[H_1^\delta]$ and so $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is κ^+ -closed in $W[H_1^\delta \times \mathbb{Q}_0]$. By Easton's Lemma, $\text{Add}(\kappa, \delta)$ is κ^+ -cc in $W[H_1^\delta \times \mathbb{Q}_0]$ and therefore again by Easton's Lemma $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is κ^+ -distributive in $W[\text{Add}(\kappa, \delta) \times \mathbb{Q}_0 \times H_1^\delta]$ and so in $W[G_0^\delta \times H_1^\delta]$. Since in $W[G_0^\delta \times H_1^\delta]$ $\delta = \kappa^{++}$, the forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ preserves cardinals over $W[G_0^\delta \times H_1^\delta]$ and by Lemma 4.7 this means that it can not add a cofinal branch to an M-special Aronszajn tree. Therefore it does not add the branch b to $T \restriction \delta$.

Now, we show that the forcing \mathbb{Q}_0^* is κ^+ -closed in $W[G_0^\delta \times H_1^\delta][H_1^{[\delta, \lambda_1]}]$. The forcing \mathbb{Q}_0^* is defined in $W[G_0^\delta]$ and by Lemma 5.15 it is κ^+ -closed in $W[G_0^\delta]$. First note that $\text{Add}(\kappa^+, \delta) * \text{Add}(\kappa^+, [\delta, \lambda_1])$ is forcing equivalent to $\text{Add}(\kappa^+, \lambda_1)$. Next we show it is κ^+ -distributive in $W[G_0^\delta]$. Since there is a projection from $\text{Add}(\kappa, \lambda_0) \times \mathbb{Q}_0$ to $\mathbb{M}(\kappa, \kappa^+, \delta)$, it suffices to show that $\text{Add}(\kappa^+, \lambda_1)$ is κ^+ -distributive in $W[\text{Add}(\kappa, \lambda_0) \times \mathbb{Q}_0]$. As $\text{Add}(\kappa^+, \lambda_1)$ and \mathbb{Q}_0 are κ^+ -closed in W , $\text{Add}(\kappa^+, \lambda_1)$ is κ^+ -closed in $W[\mathbb{Q}_0]$. By Easton's Lemma, $\text{Add}(\kappa, \lambda_0)$ is κ^+ -cc in $W[\mathbb{Q}_0]$. Again by Easton's Lemma, $\text{Add}(\kappa^+, \lambda_1)$ is κ^+ -distributive in $W[\text{Add}(\kappa, \lambda_0) \times \mathbb{Q}_0]$. Now, we

know that $\text{Add}(\kappa^+, \lambda_1)$ is κ^+ -distributive and \mathbb{Q}_0^* is κ^+ -closed in $W[G_0^\delta]$, hence \mathbb{Q}_0^* is κ^+ -closed in $W[G_0^\delta \times H_1^\delta][H_1^{\delta, \lambda_1}]$. In $W[G_0^\delta \times H_1^\delta][H_1^{\delta, \lambda_1}]$, $\delta = \kappa^{++}$ and $2^\kappa \geq \delta$ since $\text{Add}(\kappa^+, [\delta, \lambda_1])$ preserves cardinals over $W[G_0^\delta \times H_1^\delta]$. Hence we can apply Lemma 4.4, so b can not be added by \mathbb{Q}_0^* .

In $W[G_0^\delta \times H_1^\delta][H_1^{\delta, \lambda_1} \times F_2]$, note that $\text{Add}(\kappa, [\delta, \lambda_0])$ is κ^+ -Knaster since $\kappa^{<\kappa} = \kappa$. In $W[G_0^\delta \times H_1^\delta][H_1^{\delta, \lambda_1} \times F_2][F_1]$, δ is collapsed to κ^+ and since $\text{Add}(\kappa, [\delta, \lambda_0])$ is κ^+ -Knaster in $W[G_0^\delta][H_1^\delta][H_1^{\delta, \lambda_1} \times F_2]$, δ has to be collapsed already in the extension $W[G_0^\delta][H_1^\delta][H_1^{\delta, \lambda_1} \times F_2]$. Let $T' = (T \upharpoonright \delta) \upharpoonright A$, where A is cofinal subset of δ of size κ^+ . Then T' has height κ^+ and by Lemma 4.1, $\text{Add}(\kappa, [\delta, \lambda_0])$ does not add cofinal branches to the tree T' , hence it does not add cofinal branches to the tree $T \upharpoonright \delta$.

We showed that in $W[G_0 \times H_1]$ $T \upharpoonright \delta$ has a cofinal branch b and that the intermediate forcings could not add this branch. That means that b is already in $W[G_0^\delta \times H_1^\delta]$. This is a contradiction since $T \upharpoonright \delta$ is an Aronszajn tree in $W[G_0^\delta \times H_1^\delta]$.

Ad b) Now we verify that κ^{+++} has the weak tree property. Assume for contradiction that $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ adds an \mathbb{M} -special $\kappa^{+++} = \lambda_1$ -tree T . Let $G_0 \times G_1$ be $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \lambda_1)$ -generic over V .

As in Theorem 4.24, we can define in V a continuous unbounded function σ from λ_1 into λ_1 such that $T \upharpoonright \alpha \in W[G_0^{\sigma(\alpha)} \times G_1^{\sigma(\alpha)}]$, where $G_0^{\sigma(\alpha)}$ and $G_1^{\sigma(\alpha)}$ are the corresponding restrictions of the generic filters G_0 and G_1 . Let δ be an inaccessible fixed point of σ such that $\lambda_0 < \delta < \lambda_1$ in V .

In $V[G_0 \times G_1]$, δ is collapsed to κ^{++} and so $T \upharpoonright \delta$ has a branch b of length δ . Note that since $\delta > \lambda_0$, $G_0^\delta = G_0$. Now, we show that b is in $V[G_0 \times G_1^\delta]$. We do this by showing that the forcing $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)/\mathbb{M}(\kappa^+, \lambda_0, \delta)$ could not add branch b . The forcing $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)/\mathbb{M}(\kappa^+, \lambda_0, \delta)$ is defined in $V[G_1^\delta]$.

In $V[G_0 \times G_1^\delta]$, $\delta = \kappa^{+++}$ and $2^{\kappa^+} \geq \delta$ since G_1^δ is $\mathbb{M}(\kappa^+, \lambda_0, \delta)$ -generic over V and $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \mathbb{M}(\kappa^+, \lambda_0, \delta)$ is δ -cc in V . As $\delta = \kappa^{+++}$, $T \upharpoonright \delta$ is an \mathbb{M} -special δ -Aronszajn tree. By Lemma 4.23, there is a projection from $\text{Add}(\kappa^+, [\delta, \lambda_1]) \times \mathbb{Q}_1^*$ to $\mathbb{M}(\kappa^+, \lambda_0, \lambda_1)/\mathbb{M}(\kappa^+, \lambda_0, \delta)$, where \mathbb{Q}_1^* is λ_0 -closed. Since $b \in V[G_0 \times G_1]$, $b \in V[G_0 \times G_1^\delta][F_1 \times F_2]$, where $F_1 \times F_2$ is $\text{Add}(\kappa^+, [\delta, \lambda_1]) \times \mathbb{Q}_1^*$ -generic over $V[G_0 \times G_1^\delta \times H_2^\delta]$. We reorganize the forcing as follows: $V[G_0 \times G_1^\delta][F_2 \times F_1]$.

Now we show that the forcing $\mathbb{Q}_1^* \times \text{Add}(\kappa^+, [\delta, \lambda_1])$ can not add branch b to the tree $T \upharpoonright \delta$.

Work in $V[G_1^\delta]$. Now, we show that the forcing $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -cc in $V[G_1^\delta]$. The forcing $\mathbb{M}(\kappa^+, \lambda_0, \delta)$ is a projection of $\text{Add}(\kappa^+, \delta) \times \mathbb{Q}_1$, where \mathbb{Q}_1 is λ_0 -closed. Hence it is enough to show that $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -cc in $V[\mathbb{Q}_1 \times \text{Add}(\kappa^+, \delta)]$. The forcing \mathbb{Q}_1 is λ_0 -closed, hence $V_{\lambda_0} = V[\mathbb{Q}_1]_{\lambda_0}$. Since the conditions of $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ are bounded in V_{λ_0} , $\mathbb{M}(\kappa, \kappa^+, \lambda_0)^V = \mathbb{M}(\kappa, \kappa^+, \lambda_0)^{V[\mathbb{Q}_1]}$ and therefore it is λ_0 -Knaster in $V[\mathbb{Q}_1]$. As $\kappa^+ < \lambda_0$, $\text{Add}(\kappa^+, \delta)$ is still κ^{++} -cc in $V[\mathbb{Q}_1]$. Hence $\text{Add}(\kappa^+, \delta) \times$

$\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -cc in $V[\mathbb{Q}_1]$ and so $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -cc in $V[\mathbb{Q}_1 \times \text{Add}(\kappa^+, \delta)]$. Hence it is λ_0 -cc in $V[G_1^\delta]$.

The forcing \mathbb{Q}_1^* is defined in $V[G_1^\delta]$ and it is λ_0 -closed in $V[G_1^\delta]$. Since G_1^δ is $\mathbb{M}(\kappa^+, \lambda_0, \delta)$ -generic, $2^{\kappa^+} \geq \delta$ and $\delta = \lambda_0^+$. Therefore we can apply Lemma 4.6 with $V[G_1^\delta]$ as the ground model. Hence \mathbb{Q}_1^* can not add the branch b to $T \upharpoonright \delta$ over $V[G_1^\delta][G_0] = V[G_0 \times G_1^\delta]$.

Next we show that $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -cc in $V[G_0 \times G_1^\delta][F_2]$. The forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is defined in $V[G_1^\delta]$, where for each $\alpha < \lambda_0$, $\alpha^\kappa < \lambda_0$ and so by Δ -system argument, the forcing $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -Knaster in $V[G_1^\delta]$. In fact notice that \mathbb{Q}_1^* does not add any new sequences of ordinals of length less than λ_0 , so the same Δ -system argument shows that $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -Knaster in $V[G_1^\delta][F_2]$. As we showed above, $\mathbb{M}(\kappa, \kappa^+, \lambda_0)$ is λ_0 -cc in $V[G_1^\delta]$. Therefore, by Easton's Lemma, it is λ_0 -cc in $V[G_1^\delta][F_2]$. Hence the forcing $\mathbb{M}(\kappa, \kappa^+, \lambda_0) \times \text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -cc in $V[G_1^\delta][F_2]$ and therefore $\text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -cc in $V[G_1^\delta][F_2][G_0] = V[G_0 \times G_1^\delta][F_2]$. Since $\text{Add}(\kappa^+, [\delta, \lambda_1]) \times \text{Add}(\kappa^+, [\delta, \lambda_1])$ is forcing equivalent to $\text{Add}(\kappa^+, [\delta, \lambda_1])$, $\text{Add}(\kappa^+, [\delta, \lambda_1]) \times \text{Add}(\kappa^+, [\delta, \lambda_1])$ is λ_0 -cc in $V[G_0 \times G_1^\delta][F_2]$. The cardinal δ is collapsed to the λ_0 in $V[G_0 \times G_1^\delta][F_2]$. Therefore δ is an ordinal of cofinality λ_0 in $V[G_0 \times G_1^\delta][F_2]$. Let $T' = (T \upharpoonright \delta) \upharpoonright A$, where A is a cofinal subset of δ of size λ_0 . By Lemma 4.2, $\text{Add}(\kappa^+, [\delta, \lambda_1])$ does not add cofinal branches to the tree T' , hence it could not add the branch b to the tree $T \upharpoonright \delta$.

We showed that in $V[G_0 \times G_1]$ $T \upharpoonright \delta$ has a cofinal branch b and that the intermediate forcings could not add this branch. That means that b is already in $V[G_0 \times G_1^\delta]$. This is a contradiction since $T \upharpoonright \delta$ is an Aronszajn tree in $V[G_0 \times G_1^\delta]$. \square

Remark 5.20. Note that the same method can not be applied to obtain the tree property at two successive cardinals. Here we made use of the fact that κ -closed forcings do not destroy Mahlo cardinal κ . This does not hold for weakly compact cardinal since even $\text{Add}(\kappa, 1)$ may destroy the weak compactness of κ .

Note also that this construction can not be used for the Grigorieff forcing since it uses specific properties of the generalized Mitchell forcing, especially the projection to a product where one factor is highly closed.

Now, we extend the result of the previous theorem to ω -many successive cardinals. We start our construction with ω and we show that the weak tree property can hold at every ω_n for n such that $1 < n < \omega$, under the assumption of ω -many Mahlo cardinals. However, a straightforward modification of the construction allows us to start at arbitrary regular κ instead of ω and force the weak tree property at every κ^{+n} , for n such that, $1 < n < \omega$, under the assumption of ω -many Mahlo cardinals above κ .

Lemma 5.21. *Suppose that GCH holds. Denote $\lambda_0 = \omega$ and $\lambda_1 = \omega_1$. Assume that there exists an increasing sequence of inaccessible cardinals $\langle \lambda_n | 1 < n < \omega \rangle$ with the supremum λ . Let $\prod_{n < \omega} \mathbb{M}_n$ denote the full support product of forcings $\mathbb{M}_n = \mathbb{M}(\lambda_n, \lambda_{n+1}, \lambda_{n+2})$ for $n < \omega$. Then for each n it holds that λ_n is preserved by $\prod_{n < \omega} \mathbb{M}_n$. Moreover each cardinal $\xi \geq \lambda$ is preserved.*

Proof. We do not need to show that λ_0 is preserved since $\lambda_0 = \omega$. Let $n < \omega$ be given. We show that λ_{n+1} is preserved. Note that $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in V and $\prod_{i > n} \mathbb{M}_i$ is λ_{n+1} -closed in V . By Lemma 5.15, there is a projection from $\text{Add}(\lambda_n, \lambda_{n+2}) \times \mathbb{Q}_n$ to \mathbb{M}_n . Hence it is enough to show that λ_{n+1} is preserved by $\mathbb{P}_1 \times \mathbb{P}_2$, where $\mathbb{P}_1 = \prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \lambda_{n+2})$ and $\mathbb{P}_2 = \mathbb{Q}_n \times \prod_{i > n} \mathbb{M}_i$. Since the forcing \mathbb{P}_1 is λ_{n+1} -cc, λ_{n+1} remains a cardinal in $V[\mathbb{P}_1]$. By Easton's Lemma \mathbb{P}_2 is λ_{n+1} -distributive in $V[\mathbb{P}_1]$, so $\mathbb{P}_1 \times \mathbb{P}_2$ preserves λ_{n+1} as a cardinal.

The cardinal λ is preserved since it is the supremum of $\langle \lambda_n | n < \omega \rangle$ and by the previous paragraph, each λ_n is preserved.

The claim that each cardinal $\xi > \lambda$ is preserved follows from the fact that $\prod_{n < \omega} \mathbb{M}_n$ is λ^+ -cc since it has size λ . \square

Theorem 5.22. *Suppose GCH holds. Denote $\lambda_0 = \omega$ and $\lambda_1 = \omega_1$. Assume that there exists an increasing sequence of inaccessible cardinals $\langle \lambda_n | 1 < n < \omega \rangle$ with the supremum λ . Let $\prod_{n < \omega} \mathbb{M}_n$ denote the full support product of forcings $\mathbb{M}_n = \mathbb{M}(\lambda_n, \lambda_{n+1}, \lambda_{n+2})$ for $n < \omega$. Then in the generic extension by $\prod_{n < \omega} \mathbb{M}_n$ for each $n < \omega$ it holds that*

(i) $2^{\lambda_n} = \lambda_{n+2} = \lambda_{n+1}^+ = \omega_{n+2}$ and $\lambda = \aleph_\omega$;

(ii) λ_{n+2} has the weak tree property.

Proof. Ad (i). Let $n < \omega$ be given. The inequality $2^{\lambda_n} \geq \lambda_{n+2}$ follows from the fact that there is projection from \mathbb{M}_n to $\text{Add}(\lambda_n, \lambda_{n+2})$. Now, we show that $2^{\lambda_n} \leq \lambda_{n+2}$. Since $\prod_{i > n} \mathbb{M}_i$ is λ_{n+1} closed, each sequence of ordinals of length less than λ_{n+1} is in $\prod_{i \leq n} \mathbb{M}_i$. As \mathbb{M}_{n-1} is λ_{n+1} -Knaster and for each $i < n - 1$ the size of \mathbb{M}_i is less than λ_{n+1} , $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -Knaster and we can use Lemma 5.17 to obtain that each sequence of length less than λ_{n+1} is in $V[\prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \lambda_{n+2})]$. Since $\prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \lambda_{n+2})$ is λ_{n+1} -cc and it has size λ_{n+2} , which is inaccessible in V , $2^{\lambda_n} \leq \lambda_{n+2}$ easily follows by a common nice names argument.

The equalities $\lambda_{n+2} = \lambda_{n+1}^+ = \omega_{n+2}$ and $\lambda = \aleph_\omega$ follow from Lemma 5.21 and Lemma 5.13.

Ad (ii). For contradiction assume that the forcing $\prod_{n < \omega} \mathbb{M}_n$ adds an M-special λ_{n+2} -Aronszajn tree T . Then T is added by $\prod_{i \leq n+2} \mathbb{M}_i$ since the forcing $\prod_{i > n+2} \mathbb{M}_i$ is λ_{n+3} -closed. As T is λ_{n+2} -tree and $\prod_{i \leq n+1} \mathbb{M}_i$ is λ_{n+3} -Knaster, as we showed

above, we can use Lemma 5.17 to obtain that T is already added by $\prod_{i \leq n+1} \mathbb{M}_i \times \text{Add}(\lambda_{n+2}, \lambda_{n+4})$. By Lemma 5.7, there is a projection from $\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_{n+1}$ to \mathbb{M}_{n+1} , where \mathbb{Q}_{n+1} is λ_{n+2} -closed. Hence T is also added by $\prod_{i \leq n} \mathbb{M}_i \times \text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_{n+1} \times \text{Add}(\lambda_{n+2}, \lambda_{n+4})$. Let $G_{i < n} \times G_n \times H_{n+1}^1 \times H_{n+1}^2 \times H_{n+2}$ be a $\prod_{i \leq n} \mathbb{M}_i \times \text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_{n+1} \times \text{Add}(\lambda_{n+2}, \lambda_{n+4})$ -generic filter over V .

Consider $W = V[H_{n+1}^2 \times H_{n+2}]$. Since $\mathbb{Q}_{n+1} \times \text{Add}(\lambda_{n+2}, \lambda_{n+4})$ is λ_{n+2} -closed in V and $\lambda_{n+2} > \lambda_{n+1}$, $\text{Add}(\lambda_{n+1}, \lambda_{n+3})$ is still λ_{n+1}^+ -Knaster and λ_{n+1} -closed in W . In addition $V_{\lambda_{n+2}} = W_{\lambda_{n+2}}$. As $V_{\lambda_{n+2}} = W_{\lambda_{n+2}}$ and $\lambda_{n+2} > \lambda_{n+1}$, $(\prod_{i < n} \mathbb{M}_i)^V = (\prod_{i < n} \mathbb{M}_i)^W$. As $V_{\lambda_{n+2}} = W_{\lambda_{n+2}}$ and the conditions of \mathbb{M}_n are bounded in $V_{\lambda_{n+2}}$, $\mathbb{M}_n^V = \mathbb{M}_n^W$. Moreover, λ_{n+2} is still Mahlo, because λ_{n+2} -closed forcings preserve stationary sets in λ_{n+2} .

As in Theorem 4.24, we can define in W a continuous unbounded function σ from λ_{n+2} into λ_{n+2} such that $T \upharpoonright \alpha \in W[G_{i < n}^{\sigma(\alpha)} \times G_n^{\sigma(\alpha)} \times (H_{n+1}^1)^{\sigma(\alpha)}]$, where $G_{i < n}^{\sigma(\alpha)}$, $G_n^{\sigma(\alpha)}$ and $(H_{n+1}^1)^{\sigma(\alpha)}$ are the corresponding restrictions of the generic filters $G_{i < n}$, G_n and H_{n+1}^1 . Let δ be an inaccessible fixed point of σ such that $\lambda_{n+1} < \delta < \lambda_{n+2}$ in W .

Work in $W[G_{i < n} \times G_n \times H_{n+1}^1]$. The cardinal δ is collapsed to λ_{n+1} and so $T \upharpoonright \delta$ has a branch b of length δ . Note that since $\delta > \lambda_{n+1}$, $G_{i < n}^\delta = G_{i < n}$. Now, we show that b is in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. We do this by showing that the forcing $\mathbb{M}_n/\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta) \times \text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ could not add the branch b . It is important to note that the forcing $\mathbb{M}_n/\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ is defined in $W[G_n^\delta]$ and the forcing $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is defined in $W[(H_{n+1}^1)^\delta]$.

In $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$, $\delta = \lambda_{n+1}^+$ and $2^{\lambda_n} \geq \delta$ since G_n^δ is $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ -generic filter and $\prod_{i < n} \mathbb{M}_i \times \mathbb{M}(\lambda_n, \lambda_{n+1}, \delta) \times \text{Add}(\lambda_{n+1}, \delta)$ is δ -cc in W . As $\delta = \lambda_{n+1}^+$, $T \upharpoonright \delta$ is M-special δ -Aronszajn tree.

By Lemma 4.23, there is a projection from the product $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}]) \times \mathbb{Q}_n^*$ to $\mathbb{M}_n/\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$, where \mathbb{Q}_n^* is λ_{n+1} -closed. Since $b \in W[G_{i < n} \times G_n \times H_{n+1}^1]$, $b \in W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta][[(H_{n+1}^1)^{[\delta, \lambda_{n+3}]} \times F_n^1 \times F_n^2]]$, where $(H_{n+1}^1)^{[\delta, \lambda_{n+3}]} \times F_n^1 \times F_n^2$ is $(\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}]) \times \text{Add}(\lambda_n, [\delta, \lambda_{n+2}]) \times \mathbb{Q}_n^*)$ -generic over $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. We reorganize the forcing as follows: $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta][[(H_{n+1}^1)^{[\delta, \lambda_{n+3}]} \times F_n^2 \times F_n^1]]$.

Now, we show that $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is λ_{n+1} -distributive and δ -cc in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. First, we focus on the distributivity. The forcing $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc and the forcing $\text{Add}(\lambda_{n+1}, \delta)$ is λ_{n+1} -closed in W . In addition, by Lemma 5.7, there is a projection from $\text{Add}(\lambda_n, \delta) \times \mathbb{Q}_n$ to $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$, where \mathbb{Q}_n is λ_{n+1} -closed. Hence it suffices to show that $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is λ_{n+1} -distributive in $W[\text{Add}(\lambda_{n+1}, \delta) \times \mathbb{Q}_n \times \prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \delta)]$.

Since $\text{Add}(\lambda_{n+1}, \delta) \times \mathbb{Q}_n$ and $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ are both λ_{n+1} -closed in W , $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is λ_{n+1} -closed in $W[\text{Add}(\lambda_{n+1}, \delta) \times \mathbb{Q}_n]$. As $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc

and $\text{Add}(\lambda_n, \delta)$ is λ_{n+1} -Knaster in W , $\prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \delta)$ is λ_{n+1} -cc in W . By Easton's Lemma $\prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \delta)$ is λ_{n+1} -cc in $W[\text{Add}(\lambda_{n+1}, \delta) \times \mathbb{Q}_n]$. Again, by Easton's Lemma, $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is λ_{n+1} -distributive in $W[\text{Add}(\lambda_{n+1}, \delta) \times \mathbb{Q}_n \times \prod_{i < n} \mathbb{M}_i \times \text{Add}(\lambda_n, \delta)]$. Therefore it is λ_{n+1} -distributive in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$.

Next we focus on the chain condition of the forcing $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$. The forcing $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is defined in $W[(H_{n+1}^1)^\delta]$ and by Δ -system argument it is δ -Knaster in $W[(H_{n+1}^1)^\delta]$. The forcing $\prod_{i < n} \mathbb{M}_i \times \mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ is δ -cc and the forcing $\text{Add}(\lambda_{n+1}, \delta)$ is δ -Knaster in W , so $\prod_{i < n} \mathbb{M}_i \times \mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ is δ -cc in $W[(H_{n+1}^1)^\delta]$. Therefore $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is δ -cc in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. Now, we know that $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is λ_{n+1} -distributive and δ -cc in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. Since $\delta = \lambda_{n+1}^+$ in $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$, the forcing $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ preserves cardinals over $W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. By Lemma 4.7 it can not add the branch b to the M-special tree $T \upharpoonright \delta$.

Now work in $W[G_n^\delta \times (H_{n+1}^1)^\delta][(H_{n+1}^1)^{\delta, \lambda_{n+3}}] = W_0$. We show that $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in W_0 and that the forcing \mathbb{Q}_n^* is λ_{n+1} -closed in W_0 . First note that $\text{Add}(\lambda_{n+1}, \delta) * \text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ is forcing equivalent to $\text{Add}(\lambda_{n+1}, \lambda_{n+3})$. We want to show that the forcing $\text{Add}(\lambda_{n+1}, \lambda_{n+3})$ is λ_{n+1} -distributive in $W[G_n^\delta]$. However, this easily follows from the fact that it is λ_{n+1} -closed in W and from the fact that $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ is a projection of a λ_{n+1} -closed and a λ_{n+1} -cc forcing.

Next we verify that \mathbb{Q}_n^* is λ_{n+1} -closed in W_0 . As $\text{Add}(\lambda_{n+1}, \lambda_{n+3})$ is λ_{n+1} -distributive and \mathbb{Q}_n^* is λ_{n+1} -closed in $W[G_n^\delta]$, \mathbb{Q}_n^* is λ_{n+1} -closed in W_0 .

Now, we focus on the chain condition of $\prod_{i < n} \mathbb{M}_i$ in W_0 . The forcing $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in W . Since there is a projection from $\mathbb{Q}_n \times \text{Add}(\lambda_n, \lambda_{n+2})$ to $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$, it suffices to show that $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in the extension $W[\mathbb{Q}_n \times \text{Add}(\lambda_n, \lambda_{n+2}) \times (H_{n+1}^1)^\delta][(H_{n+1}^1)^{\delta, \lambda_{n+3}}]$. As the forcing $\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_n$ is λ_{n+1} -closed in W , by Easton's Lemma the forcing $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in $W[\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_n]$. Now note that we can use Δ -system Lemma in $W[\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_n]$ and show that $\text{Add}(\lambda_n, \lambda_{n+2})$ is λ_{n+1} -Knaster in $W[\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_n]$. Therefore $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in $W[\text{Add}(\lambda_{n+1}, \lambda_{n+3}) \times \mathbb{Q}_n \times \text{Add}(\lambda_n, \lambda_{n+2})]$ and so in W_0 .

Now, we have almost everything to conclude that \mathbb{Q}_n^* does not add branch b to $T \upharpoonright \delta$ over $W_0[G_{< n}]$. In $W[G_n^\delta \times (H_{n+1}^1)^\delta]$, $\delta = \lambda_{n+1}^+$ and $2^{\lambda_n} \geq \delta$ since G_n^δ is $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta)$ -generic and the forcing $\mathbb{M}(\lambda_n, \lambda_{n+1}, \delta) \times \text{Add}(\lambda_{n+1}, \delta)$ is δ -cc. Since the forcing $\text{Add}(\lambda_{n+1}, [\delta, \lambda_{n+3}])$ preserves cardinals over $W[G_n^\delta \times (H_{n+1}^1)^\delta]$, $\delta = \lambda_{n+1}^+$ and $2^{\lambda_n} \geq \delta$ in W_0 . As $\prod_{i < n} \mathbb{M}_i$ is λ_{n+1} -cc in W_0 and the forcing \mathbb{Q}_n^* is λ_{n+1} -closed in W_0 , the assumptions of Lemma 4.6 are satisfied and so \mathbb{Q}_n^* does not add branch b to $T \upharpoonright \delta$ over $W_0[G_{< n}] = W[G_n^\delta \times (H_{n+1}^1)^\delta][(H_{n+1}^1)^{\delta, \lambda_{n+3}}][G_{< n}] = W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta][(H_{n+1}^1)^{\delta, \lambda_{n+3}}]$.

Let $W_1 = W[G_{i < n} \times G_n^\delta \times (H_{n+1}^1)^\delta][(H_{n+1}^1)^{\delta, \lambda_{n+3}} \times F_n^2]$. Now, we show that the

forcing $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -cc in W_1 . Since in W_0 it holds for each $\alpha < \lambda_{n+1}$ that $\alpha^{<\lambda_n} < \lambda_{n+1}$, the forcing $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -Knaster in W_0 . In fact notice that \mathbb{Q}_n^* does not add any new sequences of ordinals of length less than λ_{n+1} , so the same Δ -system argument shows that $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -Knaster in $W_0[F_n]$. As we showed above, the forcing $\prod_{i<n} \mathbb{M}_i$ is λ_{n+1} -cc in W_0 . By Easton's Lemma, it is λ_{n+1} -cc in $W_0[F_n^2]$. Hence the forcing $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -cc in $W_0[F_n^2][G_{i<n}] = W[G_{i<n} \times G_n^\delta \times (H_{n+1}^1)^\delta][[(H_{n+1}^1)^\delta]^{[\delta, \lambda_{n+3}]} \times F_n^2]$.

Now, we show that $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ could not add the branch b over W_1 . As $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}]) \times \text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is forcing equivalent to $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$, it is also λ_{n+1} -cc in W_1 . In W_1 , note that the cardinal δ is collapsed to λ_{n+1} since it is collapsed in $W_1[F_1]$, but $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -cc over W_1 . Therefore δ has to be collapsed in W_1 . Hence δ is an ordinal with a cofinal subset of size λ_{n+1} . Let $T' = (T \upharpoonright \delta) \upharpoonright A$, where A is a cofinal subset of δ of size λ_{n+1} . As $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}]) \times \text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ is λ_{n+1} -cc, by Lemma 4.2, $\text{Add}(\lambda_n, [\delta, \lambda_{n+2}])$ does not add cofinal branches to the tree T' , hence it could not add the branch b to the tree $T \upharpoonright \delta$.

We showed that in $W[G_{i<n} \times G_n \times H_{n+1}^1]$ $T \upharpoonright \delta$ has a cofinal branch b and that the intermediate forcings could not add this branch. That means that b is already in $W[G_{<n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. This is a contradiction since $T \upharpoonright \delta$ is an Aronszajn tree in $W[G_{<n} \times G_n^\delta \times (H_{n+1}^1)^\delta]$. \square

5.2 The Tree Property

While the consistency of the weak tree property at two successive cardinals is provable from the assumption of two Mahlo cardinals, the consistency of the tree property at two successive cardinals is not provable from two weakly compact cardinals by the result of Magidor (see [Abr83]). However, this assumption is enough to show the consistency of the tree property at two non-successive cardinals. Here, we present the result that the tree property can hold at two non-successive cardinals and then we generalize this for ω -many non-successive cardinals.

We use the assumption of measurable cardinals. This assumption can be weakened to an existence of weakly compact cardinals but the proof would be more technical and the technicalities could obscure the main ideas of the proofs.

Before we present the proof of the consistency of the tree property at two non-successive cardinals, let us examine the naive attempt. Let $\lambda_1 > \lambda_0$ be two weakly compact cardinals or in our case two measurable cardinals. Try to force with $\mathbb{M}(\omega, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$. Then it is fairly easy to see that in the final extension $2^\omega = 2^{\omega_1} = \omega_2$ and $2^{\omega_2} = 2^{\omega_3} = \omega_4$ and that each ω_2 -tree is already added by $\mathbb{M}(\omega, \lambda_0) \times \text{Add}(\lambda_0, 1)$. We already know that $\mathbb{M}(\omega, \lambda_0)$ forces the tree property at

ω_2 . However, we do not know how $\text{Add}(\lambda_0, 1)$ affects $\mathbb{M}(\omega, \lambda_0)$. Therefore we need to deal with the forcing $\text{Add}(\lambda_0, 1)$. This forcing is sufficiently closed, so $\mathbb{M}(\omega, \lambda_0)^V = \mathbb{M}(\omega, \lambda_0)^{V[\text{Add}(\lambda_0, 1)]}$. The only difference is that we do not know whether λ_0 is still weakly compact or measurable in the generic extension by $\text{Add}(\lambda_0, 1)$. Therefore we need to start more carefully. The problem can be fixed by using forcing which ensures that the forcing $\text{Add}(\lambda_0, 1)$ preserves the compactness or in our case the measurability. We call this forcing the *preparation forcing*. It does not matter if we are dealing with weakly compact or measurable cardinal, the preparation forcing is the same. Now, we define this forcing and examine its properties.

Definition 5.23. Let $\alpha \geq 1$ be an ordinal and \mathbb{R}_α be an iteration of length α . We say that \mathbb{R}_α is an iteration with *Easton support* if the following holds: for every limit ordinal $\beta \leq \alpha$, \mathbb{R}_β is a direct limit if β is regular and inverse limit otherwise.

Definition 5.24. Let $\alpha \geq 1$ be an ordinal. We define the forcing notion \mathbb{P}_α to be an iteration of length α with Easton support where we force with the Cohen forcing $\text{Add}(\xi, 1)$ at every inaccessible $\xi < \alpha$ and with the trivial forcing otherwise.

Note that for λ inaccessible the forcing $\mathbb{P}_{\lambda+1} * \text{Add}(\lambda, 1)$ is forcing equivalent to $\mathbb{P}_{\lambda+1}$ since $\mathbb{P}_{\lambda+1} * \text{Add}(\lambda, 1) = \mathbb{P}_\lambda * \text{Add}(\lambda, 1) * \text{Add}(\lambda, 1)$ and the two step iteration of Cohen forcing $\text{Add}(\lambda, 1) * \text{Add}(\lambda, 1)$ is forcing equivalent to one Cohen $\text{Add}(\lambda, 1)$. Now we need to show that forcing $\mathbb{P}_{\lambda+1}$ for λ measurable preserves the measurability of λ .

Here, we mention a few lemmas without proofs. We use them later in the proof of Theorem 5.32, which tells us that the forcing $\mathbb{P}_{\lambda+1}$ for λ measurable preserves the measurability of λ . The lemmas are taken from [Cum10].

Definition 5.25. Let $j : M \rightarrow N$ be an elementary embedding and let $\mathbb{P} \in M$. A *master condition for j and \mathbb{P}* is a condition q in $j(\mathbb{P})$ such that for every dense set $D \subset \mathbb{P}$ with $D \in M$, there is condition $p \in D$ such that q is compatible with $j(p)$. Moreover, if $q \leq j(p)$, we say that q is *strong master condition for j and \mathbb{P}* .

Note that if q is a master condition for j and \mathbb{P} , and H is a $j(\mathbb{P})$ -generic over N such that $q \in H$, then $j^{-1}H$ generates a \mathbb{P} -generic filter G over M such that $j''G \subseteq H$. Hence we can use Lemma 2.3 and lift the elementary embedding j . Moreover, if q is a strong master condition, then we can define $G = \{p \in \mathbb{P} \mid q \leq j(p)\}$. The filter G is \mathbb{P} -generic over M such that $j^{-1}H = G$ for each $j(\mathbb{P})$ -generic filter H over N , such that $q \in H$. Again, we can use Lemma 2.3 and lift the elementary embedding j .

Lemma 5.26. *Let M, N be inner models of ZFC such that $M \subseteq N$. Let $N \models \text{“}\kappa \text{ is a regular cardinal”}$. Then $N \models \text{“}{}^{<\kappa}M \subseteq M \text{ if and only if } N \models \text{“}{}^{<\kappa}On \subseteq M \text{”}$.*

Lemma 5.27. *Let M, N be inner models of ZFC such that $M \subseteq N$. Let $N \models “\kappa$ is a regular cardinal” and let $N \models {}^{<\kappa}M \subseteq M$. Let $\mathbb{P} \in M$ be a notion of forcing. If $M \models “\mathbb{P}$ is κ -closed”, then $N \models “\mathbb{P}$ is κ -closed”.*

Lemma 5.28. *Let M, N be inner models of ZFC such that $M \subseteq N$ and let $\mathbb{P} \in M$ be a notion of forcing. If $N \models {}^{<\kappa}M \subseteq M$, $N \models “\mathbb{P}$ is κ -cc” and G is \mathbb{P} -generic over N , then $N[G] \models {}^{<\kappa}M[G] \subseteq M[G]$.*

Lemma 5.29. *Let M be an inner model and \mathbb{P} is a forcing notion. If, in V , \mathbb{P} is κ -closed and has no more than κ many antichains in M , then there is $G \in V$, G is \mathbb{P} -generic over M . Moreover, if $p \in \mathbb{P}$, then we can find G in V such that $p \in G$.*

Lemma 5.30. *Let δ be Mahlo. Then \mathbb{P}_δ is δ -cc with size δ and $\mathbb{P}_{\delta+1}$ is δ^+ -cc with size δ .*

If \mathbb{Q}_α is an iteration of length α and $\beta < \alpha$, then forcing \mathbb{Q}_α is forcing equivalent to \mathbb{Q}_β followed by $(\beta-\alpha)$ -iteration defined in $V[G_\beta]$. We denote this forcing $\mathbb{Q}_{\beta,\alpha}$. For the definition and more details about this, see Chapter 5 of [Bau83].

Lemma 5.31. *Let κ be a regular cardinal and let $\delta < \kappa$ be Mahlo. If λ is the least inaccessible greater than δ then in $V[\mathbb{P}_{\delta+1}]$, $\mathbb{P}_{\delta+1,\kappa}$ is λ -closed.*

The proof of the following theorem is a simplified version of the proof for violating GCH at a measurable cardinal from [Cum10]. The idea behind these proofs belongs to Silver, who was the first to use the master condition to show that GCH can fail at a measurable cardinal.

Theorem 5.32. *Assume that GCH holds in V and κ is a measurable cardinal. Then κ is measurable in $V[\mathbb{P}_{\kappa+1}]$.*

Proof. Since κ is measurable in V , there is an elementary embedding $j : V \rightarrow M$ with critical point κ and ${}^\kappa M \subseteq M$, where M is an inner model of ZFC. We denote G_κ to be a \mathbb{P}_κ -generic filter over V and g_κ to be an $\text{Add}(\kappa, 1)$ -generic filter over $V[G_\kappa]$.

Now we focus on \mathbb{P}_κ . By elementarity of j , $j(\mathbb{P}_\kappa)$ is an iteration with Easton support of length $j(\kappa)$, which adds one subset to every inaccessible cardinal $\xi < \kappa$. As κ is inaccessible in M , $j(\mathbb{P}_\kappa)_\kappa$ is direct limit. Since $V_\kappa^M = V_\kappa$ and $j(\mathbb{P}_\kappa)_\kappa$ is direct limit, $j(\mathbb{P}_\kappa)_\kappa = \mathbb{P}_\kappa$. By the definition of direct limit, for each p in \mathbb{P}_κ there is $\alpha < \kappa$ such that for all β if $\alpha < \beta < \kappa$ then $p(\beta) = \emptyset$. By elementarity of j , for each $j(p)$ in $j(\mathbb{P}_\kappa)$ there exists $\alpha < \kappa$ such that for all β if $\alpha < \beta < j(\kappa)$ then $j(p)(\beta) = \emptyset$. Therefore $j(p) \upharpoonright \kappa = p$ and $j(p)(\alpha) = \emptyset$ for each α such that $\kappa \leq \alpha < j(\kappa)$.

Now, we need to build $\mathbb{P}_{\kappa,j(\kappa)}$ -generic over $M[G_\kappa]$ which is in $V[G_\kappa * g_\kappa]$. In $V[G_\kappa * g_\kappa]$, we already have a generic filter for the first stage of $\mathbb{P}_{\kappa,j(\kappa)}$. Hence we

want to find $\mathbb{P}_{\kappa+1, j(\kappa)}$ -generic over $M[G_\kappa * g_\kappa]$ which is in $V[G_\kappa * g_\kappa]$. As a fact we use that $V[G_\kappa * g_\kappa] \models$ “ $\mathbb{P}_{\kappa+1, j(\kappa)}$ is κ^+ -closed” and $V[G_\kappa * g_\kappa] \models$ “ $\mathbb{P}_{\kappa+1, j(\kappa)}$ has at most κ^+ maximal antichains in $M[G_\kappa * g_\kappa]$ ”. Therefore by Lemma 5.29 there is a filter $H \in V[G_\kappa * g_\kappa]$ such that it is $\mathbb{P}_{\kappa+1, j(\kappa)}$ -generic over $M[G_\kappa][g_\kappa]$ which is in $V[G_\kappa][g_\kappa]$. So we can lift the elementary embedding to $j : V[G_\kappa] \rightarrow M[G_{j(\kappa)}]$, where $G_{j(\kappa)} = G_\kappa * g_\kappa * H$.

We show that in $V[G_\kappa * g_\kappa]$, ${}^\kappa M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$. By Lemma 5.30, $\mathbb{P}_{\kappa+1}$ is κ -cc. Hence, by Lemma 5.28, $V[G_\kappa * g_\kappa] \models {}^\kappa M[G_\kappa * g_\kappa] \subseteq M[G_\kappa * g_\kappa]$. By Lemma 5.26, this is equivalent to $V[G_\kappa * g_\kappa] \models {}^\kappa \text{On} \subseteq M[G_\kappa * g_\kappa]$. Since $M[G_\kappa * g_\kappa] \subseteq M[G_{j(\kappa)}]$, $V[G_\kappa * g_\kappa] \models {}^\kappa \text{On} \subseteq M[G_{j(\kappa)}]$. Again we use Lemma 5.26 and we have $V[G_\kappa * g_\kappa] \models {}^\kappa M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$.

Now we look at $\text{Add}(\kappa, 1)$. For each $p \in \text{Add}(\kappa, 1)$, $j(p) = p$ since $j \upharpoonright \kappa = \text{id}$. Let $r = \bigcup g_\kappa$, then r is in $M[G_{j(\kappa)}]$ because g_κ is in $M[G_{j(\kappa)}]$. Since r is a function from κ to 2, $r \in j(\text{Add}(\kappa, 1))$. Note that r is a strong master condition for j and $\text{Add}(\kappa, 1)$ since for each $p \in g_\kappa$, $r \leq p = j(p)$.

Now, we claim that in $V[G_\kappa * g_\kappa]$, $j(\text{Add}(\kappa, 1))$ is κ^+ -closed and has at most κ^+ maximal antichains in $M[G_{j(\kappa)}]$. In $M[G_{j(\kappa)}]$, it holds that $j(\text{Add}(\kappa, 1))$ is $j(\kappa)$ -closed by elementarity. Since $V \models |j(\kappa)| = \kappa^+$, then, by Lemma 5.27, $j(\text{Add}(\kappa, 1))$ is κ^+ -closed in $V[G_\kappa * g_\kappa]$. Now, we show that $j(\text{Add}(\kappa, 1))$ has at most κ^+ maximal antichains. First note that $V[G_\kappa] \models$ “ $\text{Add}(\kappa, 1)$ has at most κ^+ maximal antichain”. Since $j(\text{Add}(\kappa, 1))$ is κ^+ -cc in $V[G_\kappa]$, we can consider each antichain as a subset of κ . By Lemma 5.30, \mathbb{P}_κ has size κ and it is κ -cc. Since we assume GCH, there are only $\kappa^\kappa = \kappa^+$ -many nice names for subsets of κ in V . As in $V[G_\kappa]$ it holds that $\text{Add}(\kappa, 1)$ has at most κ^+ maximal antichains, by elementarity in $M[G_{j(\kappa)}]$, it holds that $j(\text{Add}(\kappa, 1))$ has at most $j(\kappa^+)$ maximal antichains. Since $V \models |j(\kappa^+)| = \kappa^+$, $V[G_\kappa * g_\kappa] \models$ “ $j(\text{Add}(\kappa, 1))$ has κ^+ maximal antichains in $M[G_{j(\kappa)}]$.”

Therefore we can use Lemma 5.29 and find a generic filter h over $M[G_{j(\kappa)}]$ such that $h \in V[G_\kappa * g_\kappa]$ and $r \in h$. As $r \in h$, $j''g_\kappa \subseteq h$. Hence we can lift the elementary embedding $j : V[G_\kappa] \rightarrow M[G_{j(\kappa)}]$ to $j^* : V[G_\kappa * g_\kappa] \rightarrow M[G_{j(\kappa)} * h]$. Since we built $G_{j(\kappa)} * h$ in $V[G_\kappa * g_\kappa]$, κ is still measurable in $V[G_\kappa * g_\kappa]$. \square

Again, as in the case of the weak tree property we need to generalize Lemma 4.15 for the product.

Lemma 5.33. *Assume $\kappa^{<\kappa} = \kappa$. Let \mathbb{R} be a forcing notion. If \mathbb{R} is κ^+ -cc, all sets of ordinals in $V[\mathbb{R} \times \mathbb{M}(\kappa, \lambda)]$ of cardinality less than κ^+ are in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$.*

Proof. By Lemma 4.15, there is a projection from $\text{Add}(\kappa, \lambda) \times \mathbb{Q}$ to $\mathbb{M}(\kappa, \lambda)$, where \mathbb{Q} is κ^+ -closed. Hence $V[\mathbb{R} \times \mathbb{M}(\kappa, \lambda)] \subseteq V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ and so it suffices to show that all set of ordinals in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of cardinality less than κ^+ are

in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$. As we assume $\kappa^{<\kappa} = \kappa$, the forcing $\text{Add}(\kappa, \lambda)$ is κ^+ -Knaster. Since \mathbb{R} is κ^+ -cc, $\mathbb{R} \times \text{Add}(\kappa, \lambda)$ is κ^+ -cc. Hence, by Easton's Lemma, \mathbb{Q} is κ^+ -distributive in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$, i.e. \mathbb{Q} does not add any new sequences of ordinals of length less than κ^+ . Therefore each sequence of ordinals in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda) \times \mathbb{Q}]$ of length less than κ^+ is already in $V[\mathbb{R} \times \text{Add}(\kappa, \lambda)]$. \square

Lemma 5.34. *Assume GCH. Let κ be a regular cardinal and $\lambda_1 > \lambda_0 > \kappa$ be inaccessible cardinals. Then each cardinal $\xi \leq \kappa$ or $\xi \geq \lambda_1$ is preserved by $\mathbb{P}_{\lambda_1} * (\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1))$. Moreover, κ^+ , λ_0 and λ_0^+ are preserved.*

Proof. It is common knowledge that \mathbb{P}_{λ_1} preserves cardinals and also it is easy to see that GCH still holds in $V[\mathbb{P}_{\lambda_1}]$. Hence it is enough to show that $(\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1))$ preserves the desired cardinals.

Let $\xi \leq \kappa$. Then ξ is preserved since $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$ is κ -closed. Let $\xi \geq \lambda_1$. Since $\mathbb{M}(\lambda_0, \lambda_1)$ is λ_1 -Knaster and $\mathbb{M}(\kappa, \lambda_0)$ is λ_0 -Knaster, $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$ is λ_1 -cc. Therefore ξ is preserved.

Now, we show that κ^+ , λ_0 and λ_0^+ is preserved. Since $\mathbb{M}(\kappa, \lambda_0)$ is λ_0 -Knaster, λ_0 remains cardinal in $V[\mathbb{M}(\kappa, \lambda_0)]$. By Easton's Lemma, $\mathbb{M}(\lambda_0, \lambda_1)$ is λ_0 -distributive in $V[\mathbb{M}(\kappa, \lambda_0)]$, hence λ_0 is still cardinal in $V[\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)]$.

Next we show that κ^+ and λ_0^+ are preserved. By Corollary 4.18, κ^+ is preserved by $\mathbb{M}(\kappa, \lambda_0)$ and by Easton's Lemma the forcing $\mathbb{M}(\lambda_0, \lambda_1)$ is λ_0 -distributive in $V[\mathbb{M}(\kappa, \lambda_0)]$. Since $\lambda_0 > \kappa^+$, κ^+ remains cardinal in $V[(\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1))]$. Now, we focus on λ_0^+ . By Corollary 4.18, λ_0^+ is preserved by $\mathbb{M}(\lambda_0, \lambda_1)$ and by Easton's Lemma the forcing $\mathbb{M}(\kappa, \lambda_0)$ is λ_0 -cc in $V[\mathbb{M}(\lambda_0, \lambda_1)]$. Therefore λ_0 remains cardinal in $V[(\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1))]$. \square

Now, we are ready to prove the main theorems of this section.

Theorem 5.35. *Assume GCH. Let κ be a regular cardinal. If there exist measurable cardinals $\lambda_1 > \lambda_0 > \kappa$, then in the generic extension by $\mathbb{P}_{\lambda_1} * (\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1))$ it holds that*

(i) $2^\kappa = \lambda_0 = \kappa^{++}$ and $2^{\kappa^{++}} = \lambda_1 = \kappa^{+4}$;

(ii) κ^{++} and κ^{+4} have the tree property.

Proof. Ad (i). $2^\kappa \geq \lambda_0$ follows from the fact that there is a projection from $\mathbb{M}(\kappa, \lambda_0)$ to $\text{Add}(\kappa, \lambda_0)$. Now, we show that $2^\kappa \leq \lambda_0$. Since $\mathbb{M}(\lambda_0, \lambda_1)$ is λ_0 -closed, each sequence of ordinals of length less than κ^+ is in $V[\mathbb{M}(\kappa, \lambda_0)]$ and by Lemma 4.17, it is in $V[\text{Add}(\kappa, \lambda_0)]$. Since this forcing is κ^+ -cc of size λ_0 , which is inaccessible in V , $2^\kappa \leq \lambda_0$ easily follows by a common nice names argument.

The proof of $2^{\lambda_0} = \lambda_1$ is similar to the proof before. $2^{\lambda_0} \geq \lambda_1$ follows from the fact that there is projection from $\mathbb{M}(\lambda_0, \lambda_1)$ to $\text{Add}(\lambda_0, \lambda_1)$. Now, we show that $2^{\lambda_0} \leq \lambda_1$. Since $\mathbb{M}(\kappa, \lambda_0)$ is λ_0 -cc, we can use Lemma 5.33 and so each sequence of length $\leq \lambda_0$ is in $V[\mathbb{M}(\kappa, \lambda_0) \times \text{Add}(\lambda_0, \lambda_1)]$. As $\mathbb{M}(\kappa, \lambda_0) \times \text{Add}(\lambda_0, \lambda_1)$ is λ_0^+ -cc and it has size λ_1 , which is inaccessible in V , $2^{\lambda_0} \leq \lambda_1$ easily follows by a common nice names argument.

The equalities $\kappa^{++} = \lambda_0$ and $\kappa^{+4} = \lambda_1$ follow from Lemma 4.21 and Lemma 5.34. Ad (ii). The tree property at κ^{+4} follows immediately from the Theorem 4.26.

We now prove that the tree property holds at κ^{++} . Let F be \mathbb{P}_{λ_1} -generic over V and $G_0 \times G_1$ be $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$ -generic over $V[F]$. Assume that T is λ_0 -tree in $V[F][G_0 \times G_1]$. Since T is λ_0 -tree and $\mathbb{M}(\kappa, \lambda_0)$ is λ_0 -cc, by Lemma 5.33 T is in $V[F][G_0][H_1]$, where H_1 is $\text{Add}(\lambda_0, \lambda_1)$ -generic over $V[F][G_0]$. As T is λ_0 -tree in $V[F][G_0][H_1]$ and $\mathbb{M}(\kappa, \lambda_0) \times \text{Add}(\lambda_0, \lambda_1)$ is λ_0^+ -cc, T has a name in $V[F]^{\mathbb{M}(\kappa, \lambda_0) \times \text{Add}(\lambda_0, \lambda_1)}$ of size at most λ_0 . Hence T is already in $V[F][G_0][H_1^\xi]$, where ξ is an ordinal of size λ_0 in $V[F]$ and $H_1^\xi = \{p \upharpoonright \xi \mid p \in H_1\}$ is $\text{Add}(\lambda_0, \lambda_1) \upharpoonright \xi$ -generic over $V[F][G_0]$. Since ξ is an ordinal of size λ_0 , the forcing $\text{Add}(\lambda_0, \lambda_1) \upharpoonright \xi$ is forcing equivalent to $\text{Add}(\lambda_0, 1)$.

Work in $V[F][H_1^\xi]$. As the forcing $\text{Add}(\lambda_0, 1)$ is λ_0 -closed, $V[F]_{\lambda_0} = V[F][H_1^\xi]_{\lambda_0}$. Since conditions of the forcing $\mathbb{M}(\kappa, \lambda_0)^{V[F]}$ are bounded in $V[F]_{\lambda_0}$, $\mathbb{M}(\kappa, \lambda_0)^{V[F][H_1^\xi]} = \mathbb{M}(\kappa, \lambda_0)^{V[F]}$. Now, we need to show that λ_0 is still measurable in $V[F][H_1^\xi]$. First we verify that $\mathbb{P}_{\lambda_1} * \text{Add}(\lambda_0, 1)$ is forcing equivalent to \mathbb{P}_{λ_1} . Note that \mathbb{P}_{λ_0+1} is forcing equivalent to $\mathbb{P}_{\lambda_0+1} * \text{Add}(\lambda_0, 1)$ since $\mathbb{P}_{\lambda_0+1} * \text{Add}(\lambda_0, 1) = \mathbb{P}_{\lambda_0} * \text{Add}(\lambda_0, 1) * \text{Add}(\lambda_0, 1)$ and the two step iteration of Cohen forcing $\text{Add}(\lambda_0, 1) * \text{Add}(\lambda_0, 1)$ is forcing equivalent to one Cohen $\text{Add}(\lambda_0, 1)$. By Lemma 5.31, the forcing $\mathbb{P}_{\lambda_0+1, \lambda_1}$ is in $V[\mathbb{P}_{\lambda_0+1}]$ at least λ_0^+ -closed, so $\mathbb{P}_{\lambda_1} * \text{Add}(\lambda_0, 1)$ is forcing equivalent to $\mathbb{P}_{\lambda_0+1} * \text{Add}(\lambda_0, 1) * \mathbb{P}_{\lambda_0+1, \lambda_1}$. Therefore $\mathbb{P}_{\lambda_1} * \text{Add}(\lambda_0, 1)$ is forcing equivalent to \mathbb{P}_{λ_1} . Now, we can show that λ_0 is still measurable in $V[F][H_1^\xi]$. By Theorem 5.32, the forcing \mathbb{P}_{λ_0+1} preserves measurability of λ_0 and by Lemma 5.31 the forcing $\mathbb{P}_{\lambda_0+1, \lambda_1}$ is λ_0^+ -closed in $V[\mathbb{P}_{\lambda_0+1}]$, λ_0 is still measurable in $V[F][H_1^\xi]$. Hence we can continue with $V[F][H_1^\xi]$ as the ground model as in Theorem 4.26. \square

Remark 5.36. Let κ be a regular cardinal and $\lambda_1 > \lambda_0 > \kappa$ be weakly compact cardinals. Compare the two forcings $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$ and $\mathbb{G}_I(\kappa, \lambda_0) \times \mathbb{G}_I(\lambda_0, \lambda_1)$. As we showed in Theorem 5.35, the forcing $\mathbb{M}(\kappa, \lambda_0) \times \mathbb{M}(\lambda_0, \lambda_1)$ forces the tree property at $\kappa^{++} = \lambda_0$ and at $\kappa^{+4} = \lambda_1$. When we verified the tree property at λ_0 , we used the fact that a λ_0 -tree is added already by $\mathbb{M}(\kappa, \lambda_0) \times \text{Add}(\lambda_0, 1)$. Therefore we can prepare the model in such a way that we first force with the preparation forcing which ensures that λ_0 is still measurable. In the case of the Grigorieff forcing we do not know how to ensure that λ_0 -trees are added by some subforcing of $\mathbb{G}_I(\lambda_0, \lambda_1)$.

Hence this construction can not be repeated for the Grigorieff forcing.

Lemma 5.37. *Suppose GCH holds. Denote $\lambda_0 = \omega$. Assume that there exists an increasing sequence of inaccessible cardinals $\langle \lambda_n \mid 0 < n < \omega \rangle$ with the supremum λ . Let $\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ denote the full support product of forcings $\mathbb{M}(\lambda_n, \lambda_{n+1})$ for $n < \omega$. Then for each n it holds that λ_n and λ_n^+ are preserved by $\mathbb{P}_\lambda * \prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$.*

Proof. It is common knowledge that \mathbb{P}_λ preserves cardinals and also it is easy to see that GCH still holds in $V[\mathbb{P}_\lambda]$. Hence it is enough to show that $\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ preserves the desired cardinals.

λ_0 is trivially preserved since $\lambda = \omega$. The cardinal $\lambda_0^+ = \omega_1$ is preserved since by Corollary 4.18 it is preserved by $\mathbb{M}(\lambda_0, \lambda_1)$ and by Easton's Lemma the forcing $\prod_{0 < n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ is λ_1 -distributive in $V[\mathbb{M}(\lambda_0, \lambda_1)]$. Let $n < \omega$ be given. Now we show that λ_{n+1} is preserved. Since $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+1} -cc, λ_{n+1} is still cardinal in $V[\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})]$. As $\prod_{i > n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+1} -closed in V , by Easton's Lemma it is λ_{n+1} -distributive in $V[\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})]$ and therefore λ_{n+1} remains cardinal in $V[\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})]$.

Next we show that λ_{n+1}^+ is preserved. Note that $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+1} -cc in V and $\prod_{i > n+1} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+2} -closed in V . By Lemma 4.15, there is a projection from $\text{Add}(\lambda_{n+1}, \lambda_{n+2}) \times \mathbb{Q}_{n+1}$ to $\mathbb{M}(\lambda_{n+1}, \lambda_{n+2})$, where \mathbb{Q}_{n+1} is λ_{n+1}^+ -closed. Now, we show that λ_{n+1}^+ remains cardinal in $V[\mathbb{P}_1 \times \mathbb{P}_2]$, where $\mathbb{P}_1 = \prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_i) \times \text{Add}(\lambda_{n+1}, \lambda_{n+2})$ and $\mathbb{P}_2 = \mathbb{Q}_{n+1} \prod_{i > n+1} \mathbb{M}(\lambda_i, \lambda_{i+1})$. Since \mathbb{P}_1 is λ_{n+1}^+ -cc λ_{n+1}^+ remains cardinal in $V[\mathbb{P}_1]$. By Easton's Lemma \mathbb{P}_2 is λ_{n+1} -distributive in $V[\mathbb{P}_1]$, so λ_{n+1}^+ remains cardinal in $V[\mathbb{P}_1 \times \mathbb{P}_2]$.

The cardinal λ is preserved since it is the supremum of λ_n for $n < \omega$, and by the previous paragraph each λ_n is preserved.

The claim that each cardinal $\xi > \lambda$ is preserved follows from the fact that the size of $\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ is λ . \square

Theorem 5.38. *Suppose GCH holds. Denote $\lambda_0 = \omega$. Assume that there exists an increasing sequence of measurable cardinals $\langle \lambda_n \mid 0 < n < \omega \rangle$ with the supremum λ . Let $\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ denote the full support product of forcings $\mathbb{M}(\lambda_n, \lambda_{n+1},)$ for $n < \omega$. Then in the generic extension by $\prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ for each $n < \omega$ it holds that*

(i) $2^{\lambda_n} = \lambda_{n+1} = \lambda_n^{++} = \omega_{2(n+1)}$ and $\lambda = \aleph_\omega$,

(ii) λ_{n+1} has the tree property.

Proof. Ad (i). Let $n < \omega$ be given. The inequality $2^{\lambda_n} \geq \lambda_{n+1}$ follows from the fact that there is a projection from $\mathbb{M}(\lambda_n, \lambda_{n+1})$ to $\text{Add}(\lambda_n, \lambda_{n+1})$. Now, we show

that $2^{\lambda_n} \leq \lambda_{n+1}$. Since $\prod_{i>n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+1} -closed, each sequence of ordinals of length less than λ_{n+1} is in $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})$. Since $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_n -cc, we can use Lemma 5.33 and so each sequence of length $\leq \lambda_n$ is in $V[\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1}) \times \text{Add}(\lambda_n, \lambda_{n+1})]$. As $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1}) \times \text{Add}(\lambda_n, \lambda_{n+1})$ is λ_n^+ -cc and it has size λ_{n+1} , which is inaccessible in V , $2^{\lambda_n} \leq \lambda_{n+1}$ easily follows by a common nice names argument.

The equalities $\lambda_{n+1} = \lambda_n^{++} = \omega_{2(n+1)}$ and $\lambda = \aleph_\omega$ follow from Lemma 4.21 and Lemma 5.37.

Ad (ii). Suppose that $\mathbb{P}_\lambda * \prod_{n < \omega} \mathbb{M}(\lambda_n, \lambda_{n+1})$ adds a λ_{n+1} -tree T . Then T is added by $\mathbb{P}_\lambda * \prod_{i \leq n+1} \mathbb{M}(\lambda_i, \lambda_{i+1})$ since the forcing $\prod_{i > n+1} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+2} -closed. Let F be \mathbb{P}_λ -generic over V and $G_{i < n} \times G_n \times G_{n+1}$ be $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1}) \times \mathbb{M}(\lambda_{n+1}, \lambda_{n+2})$ -generic over $V[F]$. As T is a λ_{n+1} -tree and $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_{n+1} -cc in $V[F]$, by Lemma 5.33, T is already in $V[F][G_{i < n} \times G_n][H_{n+1}]$, where H_{n+1} is $\text{Add}(\lambda_{n+1}, \lambda_{n+2})^{V[F]}$ -generic over $V[F][G_{i \leq n}]$.

Since T is a λ_{n+1} -tree in $V[F][G_{i < n} \times G_n][H_{n+1}]$ and the forcing $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1}) \times \text{Add}(\lambda_{n+1}, \lambda_{n+2})$ is λ_{n+1}^+ -cc in $V[F]$, T has a $\prod_{i \leq n} \mathbb{M}(\lambda_i, \lambda_{i+1}) \times \text{Add}(\lambda_{n+1}, \lambda_{n+2})$ -nice name in $V[F]$ of size at most λ_{n+1} . Hence T is already in $V[F][G_{i < n} \times G_n][H_{n+1}^\xi]$, where ξ is an ordinal of size λ_{n+1} in $V[F]$ and $H_{n+1}^\xi = \{p \upharpoonright \xi \mid p \in H_{n+1}\}$ is $\text{Add}(\lambda_{n+1}, \xi)$ -generic over $V[F][G_{i < n} \times G_n]$. Since ξ is an ordinal of size λ_{n+1} , the forcing $\text{Add}(\lambda_{n+1}, \xi)$ is forcing equivalent to $\text{Add}(\lambda_{n+1}, 1)$.

Work in $V[F][H_{n+1}^\xi]$. As the forcing $\text{Add}(\lambda_{n+1}, 1)$ is λ_{n+1} -closed, $V[F]_{\lambda_{n+1}} = V[F][H_{n+1}^\xi]_{\lambda_{n+1}}$. Since the forcing $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ has size less than λ_{n+1} , the forcing $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})^{V[F]} = \prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})^{V[F][H_{n+1}^\xi]}$ and since conditions of forcing $\mathbb{M}(\lambda_n, \lambda_{n+1})$ are bounded in $V[F]_{\lambda_{n+1}}$, $\mathbb{M}(\lambda_n, \lambda_{n+1})^{V[F]} = \mathbb{M}(\lambda_n, \lambda_{n+1})^{V[F][H_{n+1}^\xi]}$. Now, we need to show that λ_{n+1} is still measurable in $V[F][H_1^\xi]$. First we verify that $\mathbb{P}_\lambda * \text{Add}(\lambda_{n+1}, 1)$ is forcing equivalent to \mathbb{P}_λ . Note that $\mathbb{P}_{\lambda_{n+1}+1}$ is forcing equivalent to $\mathbb{P}_{\lambda_{n+1}+1} * \text{Add}(\lambda_{n+1}, 1)$ since $\mathbb{P}_{\lambda_{n+1}+1} * \text{Add}(\lambda_{n+1}, 1) = \mathbb{P}_{\lambda_{n+1}} * \text{Add}(\lambda_{n+1}, 1) * \text{Add}(\lambda_{n+1}, 1)$ and the two step iteration of Cohen forcing $\text{Add}(\lambda_{n+1}, 1) * \text{Add}(\lambda_{n+1}, 1)$ is forcing equivalent to one Cohen $\text{Add}(\lambda_{n+1}, 1)$. By Lemma 5.31, the forcing $\mathbb{P}_{\lambda_{n+1}+1, \lambda}$ is in $V[\mathbb{P}_{\lambda_{n+1}+1}]$ at least λ_{n+1}^+ -closed, so $\mathbb{P}_\lambda * \text{Add}(\lambda_{n+1}, 1)$ is forcing equivalent to $\mathbb{P}_{\lambda_{n+1}+1} * \text{Add}(\lambda_{n+1}, 1) * \mathbb{P}_{\lambda_{n+1}+1, \lambda}$. Therefore $\mathbb{P}_\lambda * \text{Add}(\lambda_{n+1}, 1)$ is forcing equivalent to \mathbb{P}_λ . Now, we can show that λ_{n+1} is still measurable in $V[F][H_1^\xi]$. By Theorem 5.32, the forcing $\mathbb{P}_{\lambda_{n+1}+1}$ preserves measurability of λ_{n+1} and by the Lemma 5.31 the forcing $\mathbb{P}_{\lambda_{n+1}+1, \lambda}$ is λ_{n+1}^+ -closed in $V[\mathbb{P}_{\lambda_{n+1}+1}]$. Therefore λ_{n+1} is still measurable in $V[F][H_{n+1}^\xi]$.

Let $V[F][H_{n+1}^\xi] = W$, now we can continue similarly way as in the proof of Theorem 4.26. Since λ_{n+1} is measurable in W , there is an elementary embedding $j : W \rightarrow M$ with critical point λ_{n+1} and $\lambda_{n+1} M \subseteq M$, where M is a transitive model of ZFC.

In M , the forcing $j(\mathbb{M}(\lambda_n, \lambda_{n+1}))$ is $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))^M$ by the elementarity of j . Since $W_{\lambda_{n+1}} = W_{\lambda_{n+1}}^M$ and each condition in $\mathbb{M}(\lambda_0, \lambda_1)$ is bounded in $W_{\lambda_{n+1}}$, $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))^M \upharpoonright \lambda_{n+1} = \mathbb{M}(\lambda_n, \lambda_{n+1})^M = \mathbb{M}(\lambda_n, \lambda_{n+1})^W$. We also know that $j(\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})) = \prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})^M = \prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})^W$ since it has size λ_n and $\lambda_n < \lambda_{n+1}$ in W . Therefore the filter $G_{i < n} \times G_n$ is also $(\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})) \times \mathbb{M}(\lambda_n, \lambda_{n+1})^M$ -generic over M .

By Lemma 4.20, there is a projection from $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))$ to $\mathbb{M}(\lambda_n, \lambda_{n+1})$ and we can define in $M[G_n]$ the forcing $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))/\mathbb{M}(\lambda_n, \lambda_{n+1})$. Since the ordering $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))/\mathbb{M}(\lambda_n, \lambda_{n+1})$ is definable in $M[G_n]$, it is also definable in $W[G_n]$. Let H_n be $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))/\mathbb{M}(\lambda_n, \lambda_{n+1})$ -generic over $W[G_{i < n} \times G_n]$, then H_n is $\mathbb{M}(\lambda_n, j(\lambda_{n+1}))/\mathbb{M}(\lambda_n, \lambda_{n+1})$ -generic over $M[G_{i < n} \times G_n]$ since $M[G_{i < n} \times G_n] \subseteq W[G_{i < n} \times G_n]$.

Work in $W[G_{i < n} \times G_n][H_n]$. By Lemma 2.3, we can lift j to $j^* : W[G_{i < n} \times G_n] \rightarrow M[G_{i < n} \times G_n][H_n]$. Now, we show that our λ_{n+1} -tree T has a cofinal branch in $W[G_{i < n} \times G_n]$. We can consider T as some subset of λ_{n+1} and so T has a nice name \dot{T} in W . Since $|\dot{T}| \leq \lambda_{n+1}$ and $\lambda_{n+1}M \subseteq M$, \dot{T} is in M . Hence $T \in M[G_{i < n} \times G_n]$. By elementarity of j^* , $j^*(T)$ is a $j^*(\lambda_{n+1})$ -tree in $M[G_{i < n} \times G_n][H_n]$ and since j^* is the identity below λ_{n+1} , $j^*(T) \upharpoonright \lambda_{n+1} = T$. As $j^*(T)$ is $j^*(\lambda_{n+1})$ -tree in $M[G_{i < n} \times G_n][H_n]$, it has branch b of length λ_{n+1} in $M[G_{i < n} \times G_n][H_n]$.

By Lemma 4.23, in $M[G_n]$ there is a projection from $\text{Add}(\lambda_n, [\lambda_{n+1}, j^*(\lambda_{n+1})]) \times \mathbb{Q}_n^*$ to $\mathbb{M}(\lambda_n, j^*(\lambda_{n+1}))/\mathbb{M}(\lambda_n, \lambda_{n+1})$, where \mathbb{Q}_n^* is λ_n^+ -closed. Therefore $M[G_{i < n} \times G_n][H_n] \subseteq M[G_{i < n} \times G_n][H_n^1 \times H_n^2]$, where $H_n^1 \times H_n^2$ is $\text{Add}(\lambda_n, [\lambda_{n+1}, j^*(\lambda_{n+1})]) \times \mathbb{Q}_n^*$ -generic over $M[G_{i < n} \times G_n]$. Therefore b is in $M[[G_{i < n} \times G_n][H_n^2][H_n^1]$.

Note that in $M[G_{i < n} \times G_n][H_n^2]$ $\text{Add}(\lambda_n, [\lambda_n, j^*(\lambda_{n+1})])$ is λ_n^+ -Knaster since it holds that $\lambda_n^{<\lambda_n} = \lambda_n$. In $M[G_{i < n} \times G_n][H_n^2][H_n^1]$ it holds that λ_{n+1} is collapsed to λ_n^+ . As the forcing $\text{Add}(\kappa, [\lambda, j^*(\lambda)])$ is λ_n^+ -Knaster in $M[G_{i < n} \times G_n][H_n^2]$, λ_{n+1} has to be collapsed to λ_n^+ already in $M[G_{i < n} \times G_n][H_n^2]$. Therefore λ_{n+1} is an ordinal of cofinality λ_n^+ in $M[G_{i < n} \times G_n][H_n^2]$. Let $T' = T \upharpoonright A$, where A is a cofinal subset of λ_{n+1} of size λ_n^+ . By Lemma 4.1, $\text{Add}(\lambda_n, [\lambda_{n+1}, j^*(\lambda_{n+1})])$ does not add cofinal branches to the tree T' , hence it does not add the branch b to the tree T . Therefore $b \in M[G_{i < n} \times G_n][H_n^2]$.

In $M[G_n]$, since $\mathbb{M}(\lambda_n, \lambda_{n+1})$ is λ_n -closed, it holds that $(\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1}))^M = (\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1}))^{M[G_n]}$ and therefore $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_n -cc. Since $2^{\lambda_n} \geq \lambda_{n+1}$, $\prod_{i < n} \mathbb{M}(\lambda_i, \lambda_{i+1})$ is λ_n -cc and \mathbb{Q} is λ_n^+ -closed, the assumptions of Lemma 4.5 are satisfied and so b is already in $M[G_n \times G_{i < n}]$ and so in $V[G_{i < n} \times G_n]$. \square

As in the case of the weak tree property, this result can be extended to an arbitrary regular κ and force the weak tree property at every κ^{+2n} , for n such that $0 < n < \omega$, under the assumption of ω -many weakly compact cardinals above κ .

6 Conclusion

In this thesis we studied the tree and the weak tree property at a given regular cardinal κ . The tree property means that there are no Aronszajn trees at κ and the weak tree property means that there are no special Aronszajn trees at κ .

First we considered special κ^+ -Aronszajn trees for regular κ and we examined generalizations of definitions of a special Aronszajn tree. We introduced the concepts of M-special, \mathcal{S} -special and non-Suslin Aronszajn trees and showed that

$$\mathbb{A}^{sp}(\kappa^+) \subseteq \mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+) \subseteq \mathbb{A}^{NS}(\kappa^+) \text{ and } \mathbb{A}^M(\kappa^+) \subseteq \mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+); \quad (6.1)$$

where $\mathbb{A}^{sp}(\kappa^+)$, $\mathbb{A}^M(\kappa^+)$, $\mathbb{A}^{\mathcal{S}\text{-}sp}(\kappa^+)$, $\mathbb{A}^{NS}(\kappa^+)$ denote the classes of all special, M-special, \mathcal{S} -special and non-Suslin κ^+ -Aronszajn trees. The first inclusion follows immediately from the definition of a special Aronszajn tree and the definition of an \mathcal{S} -special Aronszajn trees, the second from Lemma 3.36 and the last inclusion follows from Lemma 3.28. We also showed that each of these inclusions can be consistently proper. This can be found in Section 3.2 as Corollary 3.42, Corollary 3.49 and Lemma 3.43. In connection with the weak tree property we showed (Theorem 3.38) that there are no special Aronszajn trees if and only if there are no M-special Aronszajn trees if and only if there are no \mathcal{S} -special Aronszajn trees.

Next we examined the Mitchell forcing and the Grigorieff forcing. Both of these forcings can be used to show that it is consistent that the weak tree property or the tree property holds at the double successor of a regular cardinal κ , under large cardinals assumptions. The Mitchell forcing uses the fact that it is a projection of the product of two forcings, as we showed in Lemma 4.15, where the first has a good chain condition and the second is sufficiently closed. On the other hand, the Grigorieff forcing uses the fact that it has the fusion property, as we showed in Lemma 4.37. These properties were crucial for showing that κ^+ is preserved by these forcings and also for the proofs of the main theorems in Chapter 4. These theorems state that using either the Mitchell forcing or the Grigorieff forcing we can get a model, where the weak tree property holds at κ^{++} , under the assumption of a Mahlo cardinal (Theorem 4.24 and Theorem 4.44), and a model, where the tree property holds at κ^{++} , under the assumption of a weakly compact cardinal (Theorem 4.26 and Theorem 4.45). Actually, we used the assumption of the existence of a measurable cardinal instead of a compact cardinal, but the weakening to the existence of a weakly compact cardinal poses no problem. Theorem 4.24 and Theorem 4.26 were first proved by Mitchell and Silver in [Mit72]. Theorem 4.44 and Theorem 4.45 were first proved by Baumgartner and Laver for the case $\kappa = \omega$ in [BL79] and later generalized for an arbitrary regular cardinal by Kanamori in [Kan80].

At the end, we focused on the weak tree property and the tree property at more cardinals. The method of Mitchell and Silver can be quite easily generalized to get a model where the weak tree property holds at ω -many successive cardinals, under the assumption of ω -many Mahlo cardinals, and also to get a model where the tree property holds at ω -many non-successive cardinals, under the assumption of ω -many weakly compact cardinals. We presented this result in Chapter 5 as Theorem 5.22 and Theorem 5.38. Again we assumed the existence of ω -measurable cardinals instead of weakly compact cardinals, but as in the previous case the weakening to the assumption of ω -many weakly compact cardinals poses no problem.

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