Report on the PhD thesis of Marek Cúth

Separable reduction theorems, systems of projections and retractions

The dissertation of Mgr. Marek Cúth consists of four chapters, each containing a separate article already published or accepted for a publication (all of them in well-recognized mathematical journals). The first three chapters contain results obtained by Mgr. Cúth, while the last chapter is a joint work with his supervisor Ondřej Kalenda. One has to admit that, besides the aforementioned papers, Mgr. Cúth is a co-author of 6 other articles/preprints (according to arXiv.org), which makes his research record quite impressive, taking into account the very early stage of his mathematical career.

The general theme of the thesis is the problem of detecting certain properties of general Banach spaces from their separable subspaces. This is called *separable reduction*: in order to check certain property one works with separable subspaces instead of the entire space. More precisely, given a property $\mathfrak{P}$ of Banach spaces, we say that $\mathfrak{P}$ is *separably determined* if a Banach space X has property $\mathfrak{P}$ whenever there exists a "rich enough" family $\mathcal{F}$ consisting of separable subspaces of X such that each $F \in \mathcal{F}$ has property $\mathfrak{P}$. The usual meaning of being "rich enough" is that the family should be cofinal among all separable subspaces and σ-complete, that is, closed under the closures of countable increasing chains. Actually, this concept has already been studied by several authors (started around 2000 by Borwein and Moors) under the name *rich family*. The last chapter of the presented dissertation studies this concept thoroughly.

The author also considers Banach spaces that can be described by means of projectional skeletons, which encode relevant information in separable subspaces. This is a concept I introduced and studied few years ago, inspired by several earlier works on non-separable Banach spaces. The family $\mathcal{F}$ of all images of a projectional skeleton is (just by definition) rich. Actually, there is an open question whether every rich family of 1-complemented separable subspaces of a given Banach space comes from a projectional skeleton. Anyway, studying projectional skeletons leads to analyzing certain "sufficiently

closed" separable subspaces. Here one encounters a very useful and important tool from logic: the method of elementary submodels, based on the Löwenheim-Skolem theorem for first-order logic. I am absolutely convinced (also from personal communications) that Mgr. Cúth is extremely skilled in applying this method in functional analysis. This is also proved by the fact that some of his new results on separable reductions immediately received attention of the Banach space theory community, resulting in the following two citations:

1. M. Fabian, A. Ioffe, *Separable reduction in the theory of Fréchet subdifferentials*, Set-Valued Var. Anal. 21 (2013) 661–671

2. L. Veselý, L. Zajíček, *On differentiability of convex operators*, J. Math. Anal. Appl. 402 (2013) 12–22

One has to stress out that Veselý and Zajíček actually *use* the results and some arguments from the first two works of Marek Cúth (the second one joint with Martin Rmoutil, not included in the PhD thesis), where the method of elementary submodels has been introduced for proving separable reduction theorems.

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At this point, based on the information collected above, I could already conclude my report with no further discussion, since there is no doubt that Mgr. Marek Cúth deserves the PhD degree, even discarding his achievements not included in the PhD thesis. However, in order to make my report more convincing and to argue with any possible hesitations about the final decision, below I discuss some details from each chapter separately.

Chapter 1: *Separable reduction theorems by the method of elementary submodels*

In this work the author gives his own presentation of the method of elementary submodels, which is easy to understand for mathematicians not very familiar with logic. He shows how several properties of functions are reflected to a separable subspace induced by a fixed countable elementary submodel. By this way, he gives a new and elegant proof of the result of Zajíček that the property of "being Fréchet differentiable" is separably determined. Next, he extends to non-separable spaces two known results (one of Zajíček and one of Lindenstrauss, Preiss) on differentiability of certain Lipschitz functions, just by showing that the negations of these results reflect to separable subspaces.

Chapter 2: *Noncommutative Valdivia compacta*

Here the author studies skeletons of retractions on compact Hausdorff topological spaces, aiming at new characterizations of Corson compact spaces. Among the main results, I find Theorems 2.2.1 and 2.2.2 most interesting. They say in particular that if K is not a Corson compact then there is an equivalent norm on the Banach space $C(K)$ such that X has no 1-projectional skeleton and also some continuous image of K has no skeleton of retractions. This is a significant extension of the results of O. Kalenda concerning Valdivia compacta. Let us recall that K is Valdivia compact if and only if it has a commutative skeleton of retractions—justifying the title of this chapter. One of the main tools in this work is, of course, the method of elementary submodels.

Chapter 3: *Simultaneous projectional skeletons*

In this work the author essentially extends the results of M. Valdivia on simultaneous resolutions of the identity. In order to achieve this, the author creates a new concept of a *simultaneous projectional skeleton*. As one of the applications, a new interesting (and highly nontrivial) characterization of Asplund spaces is given: Namely, a Banach space is Asplund if and only if its dual has 1-projectional skeleton with respect to every equivalent norm.

Chapter 4: *Rich families and elementary submodels*

Here the author (jointly with his supervisor O. Kalenda) makes a deep study of the interactions between two concepts: *rich families of separable subspaces* and the method of elementary submodels applied to Banach spaces. More precisely, they ask whether these two methods are equivalent when applied to separable reductions. The main result (Theorem 4.3.3) is that the method of elementary submodels is "not weaker" than the method of using rich families. This is done by a smart analysis of Skolem functions (the essential tool from the proof of Löwenheim-Skolem theorem). The main result, which is of meta-mathematical nature, actually belongs to logic. The authors apply it for showing that every projectional skeleton can be "simplified", assuming that it is indexed by the images of the projections. For me, the most exciting part of this work is the open problem, whether separable reductions obtained by the methods of rich families and by elementary submodels are equivalent in any Banach space. The authors prove it in case where the Banach space is of density $\aleph_1$ or has a minimal fundamental system.

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Critical remarks: In this particular case, I find the idea of gluing published articles a little bit misfortune. The reason is that all the four papers deal with a similar topic, have the same general ideas and methods, therefore they can easily be compiled into a nice homogeneous text. For readers interested in the subject, this could possibly be a good important, and pleasant to read, source of knowledge. Moreover, in the present form, the author incidentally hides the development of his research work, as the current state-of-the-art of the topic has changed rapidly, from the time just before his first work until now. This is important especially when the author of the thesis solves some of the problems posed in his previous works.

The remarks above do not influence my general high opinion about the contents of Cúth's dissertation.

Conclusion: Summarizing, Mgr. Marek Cúth has completed an excellent PhD thesis, presenting strong results, already published in very good journals and having some visible impact. This evidently shows his ability of working, as an independent researcher, in various topics of abstract analysis and related areas. I am absolutely convinced that the presented dissertation fulfills all the conditions for obtaining a PhD degree.

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