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DOCTORAL THESIS

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Separable reduction theorems, systems of projections and retractions

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Separabilní redukce, systémy projekcí a retrakcí

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Abstrakt: Tato práce sestává ze čtyř odborných článků. V prvním článku zkoumáme, zda jsou některé vlastnosti množin (funkcí) separabilně určené. K tomu používáme tzv. "metodu elementárních submodelů". Ve druhém článku zobecňujeme některé výsledky týkající se Valdiviových kompaktů (ekvivalentně prostorů s komutativním retrakčním skeletonem) do kontextu prostorů s retrakčním skeletonem (ne nutně komutativním). Ve třetím článku se dále věnujeme prostorům s projekčním (resp. retrakčním) skeletonem. Za určitých podmínek dokazujeme existenci "simultánních projekčních skeletonů" a tento výsledek dále používáme k dalšímu poznávání struktury prostorů s projekčním (resp. retrakčním) skeletonem. Ve čtvrtém článku podrobněji analyzujeme metodu elementárních submodelů a porovnáváme ji s "metodou bohatých familií".

Klíčová slova: separabilí redukce, projekční skeleton, retrakční skeleton, elementární submodely

Title: Separable reduction theorems, systems of projections and retractions

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Abstract: This thesis consists of four research papers. In the first paper we study whether certain properties of sets (functions) are separably determined. In our results we use the "method of elementary submodels". In the second paper we generalize some results concerning Valdivia compacta (equivalently spaces with a commutative retractional skeleton) to the context of spaces with a retractional skeleton (not necessarily commutative). The third paper further studies the structure of spaces with a projectional (resp. retractional) skeleton. Under certain conditions we prove the existence of a "simultaneous projectional skeleton" and we use this result to prove other statements concerning the structure of spaces with a projectional (resp. retractional) skeleton. In the last paper we study the method of elementary submodels in a greater detail and we compare it with the "method of rich families".

Keywords: separable reduction theorems, projectional skeleton, retractional skeleton, elementary submodels

Contents

Introduction

The theses consists of this introductory section and the following four papers:

- *Separable reduction theorems by the method of elementary submodels*,
  Fund. Math. 219 (2012), 191–222 ([4]).
- *Noncommutative valdivia compacta*,
  Comment. Math. Univ. Carolinae 55 (2014), 53–72 ([5]).
- *Simultaneous projectional skeletons*,
  J. Math. Anal. Appl. 411 (2014), 19–29 ([6]).
- (with O. Kalenda) *Rich families and elementary submodels*,
  accepted in Cent. Eur. J. Math. (2014) ([8]).

Each paper constitutes one chapter. The papers are presented in their original form, only the list of references is moved to the end of this thesis. Let us now briefly introduce the topics treated in this thesis.

The general theme tyeing the papers together is the study of the structure of nonseparable Banach spaces.

For the study of nonseparable Banach spaces, construction of a separable subspace with certain property is sometimes important. It enables us to transfer properties from smaller (separable) spaces to larger ones. One of the important approaches is the "separable reduction". By a separable reduction we usually mean the possibility to extend the validity of a statement from separable spaces to the nonseparable setting without knowing the proof of the statement in the separable case. The proof of separable reduction theorems depends on a "separable determination". This notion is used in the literature in different ways. Here, a statement ϕ concerning a nonseparable Banach space X is considered to be *separably determined* if

$$\text{The statement } \phi \text{ holds in } X \iff \forall F \in \mathcal{F}: \text{ The statement } \phi \text{ holds in } F,$$

where $\mathcal{F}$ is a sufficiently large family of separable subspaces of X; typically, for any separable subspace of X there is a bigger subspace from $\mathcal{F}$. Note that in the literature sometimes a statement is considered to be "separably determined" if only the implication from the left to the right above holds. Although in applications one makes the final deduction using just one separable subspace, it is convenient to know that the family $\mathcal{F}$ is large in order to join finitely many arguments together. Hence, given a statement ϕ, we are trying to construct a large family of separable subspaces $\mathcal{F}$ so that the above holds for ϕ. There are several approaches to this. One of them is the concept of "rich families", another concept is the set-theoretical "method of elementary submodels". In the first chapter, we use the last mentioned method of elementary submodels. We make a systematic study of the use of this method to obtain separable determination theorems concerning properties of subsets of and functions on normed or Banach spaces. Using this method, we prove many previously known results and several new ones. To mention one of them, we extend the validity of Zajíček's result [46, Proposition 3.3] from spaces with separable dual to the general Asplund spaces.[1]

Another important approach when transferring properties from separable subspaces to the nonseparable setting is the decomposition of a nonseparable Banach space into smaller pieces (complemented subspaces). There is a hope that if we glue them together, their properties will be preserved by the nonseparable Banach space we started with. One possible concept of such a decomposition is a *projectional skeleton* introduced by W. Kubiś in [33]. In the next two chapters we study the structure of the spaces with a projectional skeleton.

Spaces with a projectional skeleton are more general than Plichko spaces, but closely connected with them. Similarly, in [34] there has been introduced a class of compact spaces with a retractional skeleton and it has been observed in [34] and [33] that those spaces are more general than Valdivia compacta, but they share a lot of properties with them. Motivated by the above, we wanted to see how many properties are preserved and we have generalized some results concerning Valdivia compacta and 1-Plichko spaces. In the second chapter we present four main theorems (Theorems 2.2.1, 2.2.2, 2.2.5 and 2.2.6), generalizing results previously known for Valdivia compact spaces. Not all the arguments for Valdivia compacta could be moved to the new context, therefore some new ideas and concepts had to be found. We also state a short list of open questions, which are of independent interest.

In the third chapter, we solve one of the questions formulated in the second chapter. In order to do so, we prove under certain conditions the existence of a simultaneous projectional skeleton for certain subspaces of $C(K)$ spaces. This generalizes a result on simultaneous projectional resolutions of identity proved by M. Valdivia [43]. We collect some consequences of this result. In particular we give a new characterization of Asplund spaces using the notion of projectional skeleton (Theorem 3.5.3): *A Banach space X is Asplund if and only if the dual space has a 1-projectional skeleton after every renorming of X if and only if the bidual unit ball has a retractional skeleton after every renorming of X.* In particular, this gives an answer to [29, Question 1] (let us just note that the answer has already been known to O. Kalenda before and it has been contained in one of his unpublished remarks; using our results we were able to extend his idea from the context of "commutative" skeletons to the general case).

In the last chapter we further investigate the method of elementary submodels studied already in the first chapter. We compare two methods of proving separable reduction theorems – the method of rich families and the method of elementary submodels. We show that any result proved using rich families holds also when formulated with elementary submodels and the converse is true in spaces with fundamental minimal system and in spaces of density $\aleph_1$. It seems to be an open problem whether the converse holds in general. We also apply our results to show that a projectional skeleton may be without loss of generality indexed by ranges of its projections.

[1] In the first chapter there are also mentioned some problems. Some of them have already been solved or at least partially solved. First, the relationship between the method of rich families and the method of elementary submodels mentioned in the third section is further studied in the fourth chapter. Next, the problem mentioned just after Theorem 1.4.16 is solved in [9], where it is proved that σ-porosity is separably determined.

1. Separable reduction theorems by the method of elementary submodels

(Published in Fund. Math., 219 (2012), 191-222.)

Abstract: We simplify the presentation of the method of elementary submodels and we show that it can be used to simplify proofs of existing separable reduction theorems and to obtain new ones. Given a nonseparable Banach space X and either a subset $A \subset X$ or a function f defined on X, we are able for certain properties to produce a separable subspace of X which determines whether A or f has the property in question. Such results are proved for properties of sets: of being dense, nowhere dense, meager, residual or porous, and for properties of functions: of being continuous, semicontinuous or Fréchet differentiable. Our method of creating separable subspaces enables us to combine results, so we easily get separable reductions of properties such as being continuous on a dense subset, Fréchet differentiable on a residual subset, etc. Finally, we show some applications of separable reduction theorems and demonstrate that some results of Zajíček, Lindenstrauss and Preiss hold in the nonseparable setting as well.

1.1 Introduction

The method of elementary submodels is a set-theoretical method which can be used in various branches of mathematics. A. Dow [12] illustrated the use of this method in topology, W. Kubiś [33] used it in functional analysis, namely to construct projections on Banach spaces. In the present work we slightly simplify and specify the method of elementary submodels from [33] and we study whether this method can be used to prove separable reduction theorems which have not been proved by other (more standard) methods.

In this way we prove the following two results. First, we show that porosity is a separably determined property. Second, we extend the validity of Zajíček's result [46, Proposition 3.3] from spaces with separable dual to general Asplund spaces.

It seems that the main advantages of the concept of elementary submodels are the following:

- any finite number of results may be combined,
- the results may be used for more than one space at a time (having two spaces X and Y which are dependent on each other in some way, we can use results for X and Y and combine them).

Thus, the real strength of this method is revealed when we prove enough results to combine them together.

The structure of the paper is as follows: First we introduce elementary submodels and prove some general results. Then we point out how this method is connected with the question of separable subspaces. Next, we collect properties of sets and functions which are separably determined. Finally, we produce two extensions of the results in [46] and [39] using the method of elementary submodels.

Below we recall the most relevant notions, definitions and notations.

We denote by ω the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$, by $\mathbb{R}_+$ the interval $(0, \infty)$, and $\mathbb{Q}_+$ stands for $\mathbb{R}_+ \cap \mathbb{Q}$. Whenever we say that a set is countable, we mean that it is either finite, or infinite and countable. If f is a mapping then we denote by $\operatorname{Rng} f$ the range of f and by $\operatorname{Dom} f$ its domain. By writing $f : X \to Y$ we mean that f is a mapping with $\operatorname{Dom} f = X$ and $\operatorname{Rng} f \subset Y$. By the symbol $f \restriction_Z$ we denote the restriction of the mapping f to the set Z. The closure (resp. interior) of a set A are denoted by $\overline{A}$ (resp. $\operatorname{Int}(A)$); the interior relative to a subspace Y is denoted by $\operatorname{Int}_Y(A)$.

If $\langle X, \rho \rangle$ is a metric space, we denote by $B(x, r)$ the open ball $\{y \in X;\ \rho(x, y) < r\}$. We shall consider normed linear spaces over the field of real numbers (but many results hold for complex spaces as well). If X is a normed linear space and $A \subset X$, we denote by $\operatorname{conv} A$ the convex hull of A, by $\overline{A}^w$ the weak closure of A and by $\operatorname{span} A$ the linear span of A. Moreover S_X is the unit sphere $\{x \in X;\ \|x\| = 1\}$, and X^* stands for the (continuous) dual space of X. We denote by $\mathcal{C}(K)$ the space of continuous functions on a compact Hausdorff space K.

1.2 Elementary submodels

In this section we describe the method of creating countable sets with certain properties using elementary submodels. First, we define what elementary submodels are. Next, we show how countable sets with certain properties can be created using those elementary submodels. This method is based on the set-theoretical Theorem 1.2.2. This is a combination of the Reflection Theorem and the Löwenheim–Skolem Theorem. We refer the reader to Kunen's book [36], where further details can be found.

The idea to use this method in functional analysis comes from the Kubiś's article [33]. Some of the following results are based on this article and slightly modified to our situation (namely Lemma 1.2.6 and Propositions 1.2.10, 1.3.2, 1.3.6, 1.3.7).

Let us first recall some definitions:

Let N be a fixed set and ϕ a formula in the language of ZFC. Then the *relativization of ϕ to N* is the formula ϕ^N which is obtained from ϕ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each "$\exists x$" by "$\exists x \in N$".

For example, if

$$\phi := (\forall x)\ (\forall y)\ (\exists z)\ (x \in z\ \wedge\ y \in z)$$

and $N = \{a, b\}$, then the relativization of ϕ to N is

$$\phi^N = (\forall x \in N)\ (\forall y \in N)\ (\exists z \in N)\ (x \in z\ \wedge\ y \in z).$$

It is clear that ϕ is satisfied, but ϕ^N is not.

If $\phi(x_1, \ldots, x_n)$ is a formula with all free variables shown (i.e. a formula whose free variables are exactly $x_1, \ldots, x_n$) then ϕ *is absolute for* N if and only if

$$(\forall a_1, \ldots, a_n \in N)(\phi^N(a_1, \ldots, a_n) \leftrightarrow \phi(a_1, \ldots, a_n)).$$

A list of formulas, $\phi_1, \ldots, \phi_n$, is said to be *subformula closed* if every subformula of a formula in the list is also contained in the list.

Any formula of set theory can be written using the symbols $\in$, $=$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\exists$, (,), [,] and symbols for variables. Let us assume a subformula closed list of formulas $\phi_1, \ldots, \phi_n$ is written in this way. Then it is not difficult to show that the absoluteness of $\phi_1, \ldots, \phi_n$ for N means that those formulas do not create any new sets in N. This result is contained in the following lemma (a proof can be found in [36, Lemma IV.7.3]):

Lemma 1.2.1. *Let N be a set and $\phi_1, \ldots, \phi_n$ a subformula closed list of formulas (only containing $\in, =, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \exists, (,), [,]$ and symbols for variables). Then the following are equivalent:*

(i) *$\phi_1, \ldots, \phi_n$ are absolute for N*

(ii) *Whenever ϕ_i is of the form $(\exists x)(\phi_j(x, y_1, \ldots y_l))$ (with all free variables shown), then*

$$(\forall y_1, \ldots y_l \in N)\,[(\exists x)(\phi_j(x, y_1, \ldots, y_l)) \rightarrow (\exists x \in N)(\phi_j(x, y_1, \ldots, y_l))]$$

The method of elementary submodels is mainly based on the following set-theoretical theorem (a proof can be found in [36, Theorem IV.7.8]).

Theorem 1.2.2. *Let $\phi_1, \ldots, \phi_n$ be any formulas and X any set. Then there exists a set $M \supset X$ such, that*

$$(\phi_1, \ldots, \phi_n \text{ are absolute for } M) \quad \wedge \quad (|M| \leq \max(\omega, |X|)).$$

Since the set from the previous theorem will often be used, the following definition is useful.

Definition 1.2.3. Let $\phi_1, \ldots, \phi_n$ be any formulas and let X be any countable set. Let $M \supset X$ be a countable set such that $\phi_1, \ldots, \phi_n$ are absolute for M. Then we say that M *is an elementary submodel for $\phi_1, \ldots, \phi_n$ containing X*, and write $M \prec (\phi_1, \ldots, \phi_n;\ X)$. The relation between X, $\phi_1, \ldots, \phi_n$ and M is often called *the elementarity of M*.

Using Lemma 1.2.1 it is easy to see that the countable union of a monotone sequence of elementary submodels is also an elementary submodel.

Lemma 1.2.4. *Let $\varphi_1, \ldots, \varphi_n$ be a subformula closed list of formulas and let X be any countable set. Let $\{M_k\}_{k\in\omega}$ be a sequence of sets satisfying*

(i) $M_i \subset M_j, \quad i \leq j$,

(ii) $(\forall k \in \omega)\ [M_k \prec (\varphi_1, \ldots, \varphi_n;\ X)]$.

Set $M := \bigcup_{k\in\omega} M_k$. Then also $M \prec (\varphi_1, \ldots, \varphi_n;\ X)$.

Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown and let M be some elementary submodel for ϕ. To use the absoluteness of ϕ for M efficiently, we need to know that many sets are elements of M. The reason is that for $a_1, \ldots, a_n \in M$ we have $\phi(a_1, \ldots, a_n)$ if and only if $\phi^M(a_1, \ldots, a_n)$. Therefore, it is our first aim to force the elementary submodel M to contain many different objects. Let us see a simple example how it can be achieved.

Example 1.2.5. Consider the following formulas:

$$\varphi_1(x,a) := (\forall z)(z \in x \leftrightarrow (z \in a \ \vee \ z = a)), \qquad \varphi_2(a) := (\exists x)(\varphi_1(x,a)).$$

Then for any $M \prec (\varphi_1, \varphi_2;\ \emptyset)$ we have $a \cup \{a\} \in M$ whenever $a \in M$.

Proof. Fix an $a \in M$. Then $\varphi_2(a)$ is satisfied (the set of x satisfying $\varphi_1(x,a)$ is $a \cup \{a\}$). By the absoluteness of φ_2 for M there exists an $x \in M$ satisfying $\varphi_1^M(x,a)$. Fix such an $x \in M$. Then $\varphi_1^M(x,a)$ holds. Therefore, using the absoluteness of φ_1, $\varphi_1(x,a)$ is satisfied as well. But the only possibility for $\varphi_1(x,a)$ to be satisfied is that $x = a \cup \{a\}$; hence $a \cup \{a\} \in M$. □

The preceding example can be generalized. Using the following lemma we can force an elementary submodel M to contain all the required objects created (uniquely) from elements of M.

Lemma 1.2.6. *Let $\phi(y, x_1, \dots, x_n)$ be a formula with all free variables shown and let X be a countable set. Let M be a fixed set, $M \prec (\phi, (\exists y)(\phi(y, x_1, \dots, x_n));\ X)$ and let $a_1, \dots, a_n \in M$ be such that there exists only one set u satisfying $\phi(u, a_1, \dots, a_n)$. Then $u \in M$.*

Proof. By the absoluteness of the formula $\exists y \phi(y, x_1, \dots, x_n)$, there exists $y_0 \in M$ satisfying $\phi^M(y_0, a_1, \dots, a_n)$. By the absoluteness of ϕ, for this $y_0 \in M$ the formula $\phi(y_0, a_1, \dots, a_n)$ holds. But such y_0 is unique and therefore $u = y_0 \in M$. □

Let us see how one can force M to contain its finite subsets and natural numbers.

Proposition 1.2.7. *Consider the following formulas:*

$$\begin{aligned}
\varphi_1 &:= (\forall z)(z \in x \leftrightarrow z \neq z),\\
\varphi_{1E} &:= (\exists x)(\varphi_1(x)),\\
\varphi_2 &:= (\forall z)(z \in x \leftrightarrow (z \in u \ \vee \ z = v)),\\
\varphi_{2E} &:= (\exists x)(\varphi_2(x, u, v)).
\end{aligned}$$

Let X be a nonempty countable set. Then

(i) *if $M \prec (\varphi_1, \varphi_{1E};\ X)$, then $\emptyset \in M$;*

(ii) *if $M \prec (\varphi_2, \varphi_{2E};\ X)$, then for every $u, v \in M$ is $u \cup \ \{v\} \in M$;*

(iii) *if $M \prec (\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E};\ X)$, then $\omega \subset M$;*

(iv) *if $M \prec (\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E};\ X)$, then for every finite set $s \subset M$ is $s \in M$.*

Proof. (i) and (ii) follow immediately from Lemma 1.2.6; (iii) follows from (i) and (ii) by induction on n; (iv) follows from (i) and (ii) by induction on the cardinality of s. □

It would be laborious and pointless to use only the basic language of set theory. For example, we often write $x < y$ as a shortcut for the formula $\varphi(x, y, <)$ with all free variables shown. Therefore, in the following text we use this extended language of set theory as we are used to. We shall also use the following convention.

Convention. Whenever we say

for any suitable elementary submodel M (the following holds...),

we mean that

there exists a list of formulas $\phi_1, \ldots, \phi_n$ and a countable set Y such that for every $M \prec (\phi_1, \ldots, \phi_n;\ Y)$ (the following holds...).

By using this terminology we hide the information about the formulas $\phi_1, \ldots, \phi_n$ and the set Y. This is, however, not important in applications.

Remark 1.2.8. Let us have sentences $T_1(a), \ldots, T_n(a)$. Assume that whenever we fix an $i \in \{1, \ldots, n\}$, then for any suitable elementary submodel M_i the sentence $T_i(M_i)$ is satisfied. Then it is easy to verify that for any suitable elementary submodel M the sentence

$$T_1(M) \wedge \ldots \wedge T_n(M)$$

is satisfied (it suffices to combine all the lists of formulas and all the sets from the convention above). In other words, we are able to combine any finite number of results we have proved using the method of elementary submodels. This includes all the theorems starting with "For any suitable elementary submodel M the following holds:".

Let us give some more results about suitable elementary submodels.

Proposition 1.2.9. *For any suitable elementary submodel M the following holds: Let f be a function such that $f \in M$. Then*

(i) $\operatorname{Dom} f \in M$,

(ii) $\operatorname{Rng} f \in M$,

(iii) $(\forall x \in M \cap \operatorname{Dom} f)(f(x) \in M)$.

Proof. Fix an elementary submodel M for formulas marked $(*)$ in the proof below and all their subformulas. Let $f \in M$ be a function. Then $\operatorname{Dom} f$ is the object uniquely defined by the following formula (the same for all functions f; f is a free variable in this formula):

$$(\exists D)(\forall x)(x \in D \leftrightarrow (\exists y)(f(x) = y)). \qquad (*)$$

By Lemma 1.2.6, $\operatorname{Dom} f \in M$. Similarly, $\operatorname{Rng} f \in M$ as it is the object uniquely defined by the formula

$$(\exists R)(\forall y)(y \in R \leftrightarrow (\exists x)(f(x) = y)). \qquad (*)$$

For (iii) we use (i) and the absoluteness of the formula

$$(\forall x \in \operatorname{Dom} f)(\exists y)(f(x) = y). \qquad (*)$$

□

The proofs in the following text often begin in the same way. To avoid unnecessary repetitions, by saying "Fix a $(*)$-elementary submodel M [containing $A_1, \ldots, A_n$]" we will understand the following:

"Consider the formulas $\varphi_1, \varphi_{1E}, \varphi_2, \varphi_{2E}$ from Proposition 1.2.7 and all the formulas marked $(*)$ in all the preceding proofs (and all their subformulas). Add to them formulas marked $(*)$ in the proof below (and all their subformulas). Denote such a list of formulas by $\phi_1, \ldots, \phi_n$. Fix a countable set X containing the sets ω, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$ and all the common operations and relations on real numbers $(+, -, \cdot, :, <)$. Fix an elementary submodel M for formulas $\phi_1, \ldots, \phi_n$ containing X [such that $A_1, \ldots, A_n \in M$]".

Thus, any $(*)$-elementary submodel M is suitable for all the preceding theorems, propositions and lemmas from this paper, making it possible to use all these results for M.

Using this new agreement, let us prove another proposition.

Proposition 1.2.10. *For any suitable elementary submodel M the following holds:*

(i) *Let S be a finite set. Then*
$$S \in M \leftrightarrow S \subset M.$$

(ii) *Let S be a countable set. Then*
$$S \in M \rightarrow S \subset M.$$

(iii) *For every natural number $n > 0$ and for arbitrary sets $a_0, \ldots, a_n$,*
$$a_0, \ldots, a_n \in M \leftrightarrow \langle a_0, \ldots, a_n \rangle \in M.$$

(iv) *If $A, B \in M$, then $A \cap B \in M$, $B \setminus A \in M$ and $A \cup B \in M$.*

Proof. Fix a $(*)$-elementary submodel M. To prove (ii), let $S \in M$ be a countable set. If $S = \emptyset$, then $S \subset M$. If $S \neq \emptyset$, then
$$(\exists f)\,(f \text{ is a function from } \omega \text{ onto } S). \qquad (*)$$
Thus, by the absoluteness of the formula above, there exists $f \in M$ satisfying
$$(f \text{ is a function from } \omega \text{ onto } S)^M.$$
Fix one such f. Then, using the absoluteness of the formula "f is a function from ω onto S", f is a function from ω onto S. Because f is a function with $\operatorname{Rng} f = S$ and $\operatorname{Dom} f = \omega \subset M$, by Proposition 1.2.9, $S \subset M$.

Let us prove that (i) holds. If $S \in M$ is finite, then $S \subset M$ by (ii). If $S \subset M$ is finite, then $S \in M$ by Proposition 1.2.7.

(iii) follows easily from (i) by induction on $n \in \omega, n \geq 1$. It is enough to realize, that $\langle a_0, a_1 \rangle = \{a_0, \{a_0, a_1\}\}$ and $\langle a_0, \ldots, a_n \rangle = \langle \langle a_0, \ldots, a_{n-1} \rangle, a_n \rangle$.

Suppose we have sets $A, B \in M$. Then, by Lemma 1.2.6 and the absoluteness of the formulas (and their subformulas)
$$(\exists C)(\forall x)(x \in C \leftrightarrow x \in A \wedge x \in B), \qquad (*)$$
$$(\exists D)(\forall x)(x \in D \leftrightarrow x \in B \wedge x \notin A), \qquad (*)$$
$$(\exists E)(\forall x)(x \in E \leftrightarrow x \in A \vee x \in B), \qquad (*)$$
(iv) holds. □

1.3 Elementary submodels in the context of normed linear spaces

Now we are prepared for some more concrete results concerning mostly metric spaces or normed linear spaces. Before we proceed, let us propose the following agreements.

If $\langle X, \rho\rangle$ is a metric space (resp. $\langle X, +, \cdot, \|\cdot\|\rangle$ is a normed linear space) and M an elementary submodel, then by saying M *contains* X (or by writing $X \in M$) we mean that $\langle X, \rho\rangle \in M$ (resp. $\langle X, +, \cdot, \|\cdot\|\rangle \in M$). If A is a set, then by saying that an elementary submodel M contains A we mean that $A \in M$.

If X is a topological space and M an elementary submodel, then we denote by X_M the set $\overline{X \cap M}$.

Proposition 1.3.1. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ be a metric space. If M contains X, then $B(x, r) \in M$ whenever $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$.*

Proof. Fix a $(*)$-elementary submodel M containing X. Let $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$. Then $B(x, r)$ is the object uniquely defined by the formula

$$(\exists U)(\forall z)(z \in U \;\leftrightarrow\; z \in X \wedge \rho(x, z) < r). \qquad (*)$$

Thus, by Lemma 1.2.6, $B(x, r) \in M$. □

The idea of the following proposition comes from [33].

Proposition 1.3.2. *For any suitable elementary submodel M the following holds: Let X be a normed linear space. If M contains X and a set $A \subset X$, then:*

(i) $\overline{\operatorname{span}(A) \cap M}$ *is a closed separable linear subspace of X.*

(ii) $\overline{\operatorname{conv}(A) \cap M}$ *is a convex set.*

(iii) *If A is convex, then $\overline{A \cap M} = \overline{A \cap M}^w$.*

In particular, X_M is separable subspace of X and $X_M = \overline{X \cap M}^w$.

Proof. Fix a $(*)$-elementary submodel M containing X and A. By Proposition 1.2.10, $\mathbb{Q} \subset M$ and $\langle \mathbb{R}, +, -, \cdot, :, < \rangle \in M$.

The elementary submodel M contains the functions $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$. Consequently (by Proposition 1.2.9), $X \cap M$ is a $\mathbb{Q}$-linear subspace of X. Therefore (i) and (ii) hold. Assertion (iii) follows easily from (ii) because for convex sets the weak and the norm closures coincide. □

Given a Banach space X, a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y, we are able to get a family of sets

$$\mathcal{M}(X) := \{X_M;\ M \prec (\phi_1, \dots, \phi_n;\ Y)\}.$$

By choosing $\phi_1, \dots, \phi_n$ and Y suitably, it is possible to force $\mathcal{M}(X)$ to be a family of closed separable subspaces of X having some specific properties. One can easily join finite number of arguments (lists of formulas) and get another family of separable subspaces having the same properties as the original family and perhaps even some more.

In [40] similar families of closed separable subspaces are used to get separable reduction theorems. Those families are called rich. This concept has been originally introduced in [3] by Borwein and Moors. For further applications of this method, see for example [41], where more references may be found.

Definition 1.3.3. Let X be a Banach space. A family $\mathcal{R}$ of separable subspaces of X is called *rich* if

(i) for every increasing sequence R_i in $\mathcal{R}$, $\overline{\bigcup_{i\in\omega} R_i}$ belongs to $\mathcal{R}$, and

(ii) each separable subspace of X is contained in an element of $\mathcal{R}$.

A connection between the notion of rich families and elementary submodels is described in the following lemma.

Lemma 1.3.4. *Let X be a Banach space. Then there exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y such that for every countable set Z and every list of formulas $\varphi_1, \dots, \varphi_k$ such that $\phi_1, \dots, \phi_n, \varphi_1, \dots, \varphi_k$ is subformula closed, the family*

$$\mathcal{M} := \{M;\ M \prec (\phi_1, \dots, \phi_n, \varphi_1, \dots, \varphi_k;\ Y \cup Z)\}$$

satisfies the following conditions:

(i) *the set $\{X_M;\ M \in \mathcal{M}\}$ is a family of closed separable subspaces of X,*

(ii) *For every increasing sequence $\{M_i\}_{i\in\omega} \subset \mathcal{M}$ of elementary submodels,*

$$\bigcup_{i\in\omega} M_i \in \mathcal{M} \quad \text{and} \quad \overline{\bigcup_{i\in\omega} X_{M_i}} = X_{\bigcup_{i\in\omega} M_i}.$$

(iii) *For every separable subspace V of X there exists $M \in \mathcal{M}$ such that $V \subset X_M$.*

Proof. The existence of $\phi_1, \dots, \phi_n$ and Y such that $\{X_M;\ M \in \mathcal{M}\}$ is a family of closed separable subspaces follows from Proposition 1.3.2 above. For (ii), fix an increasing sequence M_i of elementary submodels from the assumption. Then (by Lemma 1.2.4) it is enough to show that $\overline{\bigcup_{i\in\omega} X_{M_i}} = X_{\bigcup_{i\in\omega} M_i}$. One inclusion follows from the fact that $\bigcup_{i\in\omega} X_{M_i} \subset \overline{\bigcup_{i\in\omega} X \cap M_i} = X_{\bigcup_{i\in\omega} M_i}$. The opposite one holds, because $\bigcup_{i\in\omega} X \cap M_i \subset \bigcup_{i\in\omega} \overline{X \cap M_i} = \bigcup_{i\in\omega} X_{M_i}$. Thus, $X_{\bigcup_{i\in\omega} M_i} = \overline{\bigcup_{i\in\omega} X \cap M_i} \subset \overline{\bigcup_{i\in\omega} X_{M_i}}$. For (iii), take any separable subspace V of X and a countable set D dense set in V. Then taking $M \prec (\phi_1, \dots, \phi_n, \varphi_1, \dots, \varphi_k;\ Y \cup Z \cup D)$, we conclude that $V \subset X_M$. □

It is not known to the author whether those two approaches (rich families and elementary submodels) to the separable reduction theorems are equivalent in some way. It seems that the method using elementary submodels is slightly stronger, as it is able to work with more than one space at a time (see Lemmas 1.3.6, 1.3.7 and 1.6.15), while the method of rich families concerns one space.

In [33] there is introduced a slightly different method of getting the elementary submodels M. It is proved there that in the case of some classical Banach spaces (namely $\ell_p(\Gamma)$ and $\mathcal{C}(K)$) it is possible to describe the subspace X_M. Slightly modifying the ideas from [33], the same results hold in our case as well.

Definition 1.3.5. Let Γ be a set. Then we denote by suppt_Γ the mapping which maps $x \in \mathbb{R}^\Gamma$ to $\mathrm{suppt}_\Gamma(x) = \{\alpha \in \Gamma;\ x(\alpha) \neq 0\}$.

Proposition 1.3.6. *For any suitable elementary submodel M the following holds: Let $X = \ell_p(\Gamma)$, where $1 \leq p < \infty$ and Γ is an arbitrary set. If M contains X, suppt_Γ and Γ, then*

$$X_M = \{x \in X;\ \mathrm{suppt}_\Gamma(x) \subset M\}.$$

Consequently, X_M can be identified with $\ell_p(\Gamma \cap M)$.

Proof. Fix a $(*)$-elementary submodel M containing X, suppt_Γ, Γ. Denote by A the set on the right-hand side above. For every $x \in X \cap M$ the set $\mathrm{suppt}_\Gamma(x)$ is countable. Thus, by Propositions 1.2.9 and 1.2.10, $\mathrm{suppt}_\Gamma(x) \subset M$ and $x \in A$. We have proved that $X \cap M \subset A$. From the obvious fact that A is a closed set we have $X_M \subset A$. On the other hand, if $x \in A$ then arbitrarily close to x we can find $y \in A$ such that $s = \mathrm{suppt}_\Gamma(y) \subset M$ is finite and $y(\alpha) \in \mathbb{Q}$ for $\alpha \in s$. Thus, using Proposition 1.2.10, we have $s \in M$ and $y \restriction_s \in M$ (because $y \restriction_s = \bigcup_{\alpha \in s}\{\langle \alpha, y(\alpha)\rangle\}$). Using the absoluteness of the formula

$$(\exists z \in X)(z \restriction_s = y \restriction_s \ \wedge z \restriction_{\Gamma \setminus s} = 0), \tag{$*$}$$

we have $y \in M$. Hence $x \in \overline{X \cap M} = X_M$. □

Given a compact space K and an arbitrary elementary submodel M we define the following equivalence relation $\sim_M$ on K:

$$x \sim_M y \quad \leftrightarrow \quad (\forall f \in \mathcal{C}(K) \cap M)(f(x) = f(y)).$$

We shall write $K/_M$ instead of $K/_{\sim_M}$ and we shall denote by q^M the canonical quotient map. It is not hard to check that $K/_M$ is a compact Hausdorff space.

Observe that we can identify the spaces $\{\varphi \circ q^M;\ \varphi \in \mathcal{C}(K/_M)\}$ and $\mathcal{C}(K/_M)$. Indeed, define

$$F(\varphi) := \varphi \circ q^M, \quad \varphi \in \mathcal{C}(K/_M).$$

It is obvious that F is an isometric mapping from $\mathcal{C}(K/_M)$ onto $\{\varphi \circ q^M;\ \varphi \in \mathcal{C}(K/_M)\}$.

Lemma 1.3.7. *For any suitable elementary submodel M the following holds: Let K be a compact space and $X = \mathcal{C}(K)$. Let $\cdot$ denote the pointwise product of functions in $\mathcal{C}(K)$. If M contains X, $\cdot$ and K, then*

$$X_M = \{\varphi \circ q^M;\ \varphi \in \mathcal{C}(K/_M)\}.$$

Consequently, we can identify X_M with the space $\mathcal{C}(K/_M)$, where $K/_M$ is a metrizable compact space.

Proof. Fix a $(*)$-elementary submodel M containing X, $\cdot$, K. Denote by Y the set on the right-hand side above. For a given function $f \in \mathcal{C}(K) \cap M$ we define

$$\varphi([x]_M) := f(x), \quad x \in K.$$

It is easy to verify that φ is a continuous function. Consequently, $f \in Y$ and $X_M \subset Y$.

For the proof of the reverse inclusion, let us identify X_M with a subspace of $\mathcal{C}(K/_M)$. Then, by Propositions 1.3.2 and 1.2.9, X_M is a closed subspace closed under the operation $\cdot$. From the definition of $\sim_M$ it follows that X_M separates points in $K/_M$. Using the absoluteness of the formula

$$(\forall c \in \mathbb{R})(\exists f \in X)(\forall x \in K : f(x) = c), \qquad (*)$$

M contains every constant rational function; thus, X_M contains all the constant functions. From the Stone-Weierstrass theorem $X_M = \mathcal{C}(K/_M)$.

Since $X_M = \mathcal{C}(K/_M)$ is a separable space, $K/_M$ is metrizable. □

1.4 Properties of sets

Let us consider the following situation. Let X be a normed linear space. We would like to recognize whether a given set $A \subset X$ has a property (P). For every separable subspace $V_0 \subset X$ we would like to find a closed separable subspace $V \supset V_0$ such that A has the property (P) in X if and only if $A \cap V$ has the property (P) in the subspace V.

Using the method of elementary submodels, it is enough to show that for any suitable elementary submodel M (dependent only on the space X and perhaps also on the set A), the set A has the property (P) if and only if $A \cap X_M$ has the property (P) in X_M.

Let us prove some results for the properties of being dense and having empty interior.

Proposition 1.4.1. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ be a metric space and $A, S \subset X$. If M contains X, A and S, then*

$$\operatorname{Int}_S (A \cap S) \neq \emptyset \leftrightarrow \operatorname{Int}_{S\cap X_M} (A \cap S \cap X_M) \neq \emptyset,$$

$$A \cap S \textit{ is dense in } S \leftrightarrow A \cap S \cap X_M \textit{ is dense in } S \cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and S. By Proposition 1.2.10, $A^C \in M$ whenever $A \in M$. Since A is dense in X if and only if A^C has empty interior in X, it is enough to show the first equivalence.

If $A \cap S$ has nonempty interior in S, then there exists a ball in S, which is a subset of $A \cap S$. Thus,

$$(\exists x \in S)(\exists r \in \mathbb{R}_+)(\forall y \in S)(y \in B(x, r) \to y \in A). \qquad (*)$$

In the preceding formula we use the abbreviation $y \in B(x, r)$ for $y \in X \wedge \rho(y, x) <_{\mathbb{R}} r$. Free variables in the preceding formula are $\mathbb{R}_+, X, \rho, <_{\mathbb{R}}, A, S$. Those are contained in M. This allows us to use the elementarity of M (i.e. the absoluteness of the preceding formula for M). Thus, we find $x \in S \cap M$ and $r \in \mathbb{R}_+ \cap M$ such that $((\forall y \in S)(z \in B(x, r) \to z \in A))^M$. By elementarity again, $B(x, r) \cap S$ is a subset of $A \cap S$. Consequently, $B(x, r) \cap S \cap X_M \subset A \cap S \cap X_M$. Since $x \in B(x, r) \cap S \cap X_M$, we have proved that $A \cap S \cap X_M$ contains a nonempty open set in $S \cap X_M$.

Conversely, assume that $\operatorname{Int}_{S\cap X_M} (A \cap S \cap X_M) \neq \emptyset$. Then

$$(\exists x \in S \cap X_M)(\exists r \in \mathbb{R}_+)(B(x, r) \cap S \cap X_M \subset A \cap S).$$

Take $q \in (0, r/2) \cap \mathbb{Q}_+$ and $x_0 \in X \cap M$ such that $\rho(x, x_0) < q$. Then

$$(B(x_0, q) \cap S \cap X_M) \subset (B(x, r) \cap S \cap X_M) \subset A \cap S.$$

The statement $B(x_0, q) \cap S \cap M \subset A \cap S$ can be written in the following way:

$$(\forall y \in S \cap M)\ (\rho(y, x_0) < q \to y \in A \cap S).$$

Therefore, using the absoluteness of

$$(\forall y \in S)\ (\rho(y, x_0) < q \to y \in A \cap S), \tag{$*$}$$

we can see that $B(x_0, q) \cap S \subset A \cap S$. But the point x is in $B(x_0, q) \cap S$. Consequently, $\mathrm{Int}_S\,(A \cap S) \neq \emptyset$. □

Another set property which is separably determined is that of being nowhere dense.

Proposition 1.4.2. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho\rangle$ be a metric space, $G \subset X$ an open set and $A \subset X$. If M contains X, A and G, then*

$$A \cap G \text{ is nowhere dense in } G \leftrightarrow A \cap G \cap X_M \text{ is nowhere dense in } G \cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and G. By Proposition 1.2.10, $C \cap B \in M$ whenever $C, B \in M$. It is well known, that $E \subset G$ is nowhere dense in G if and only if it is nowhere dense in X (see [37, p. 71]). Consequently, it is enough to prove the proposition for $G = X$.

It is well known that set A is nowhere dense in a metric space X if and only if the following formula holds:

$$(\forall x \in X)(\forall r \in \mathbb{R}_+)(\exists y \in X)(\exists s \in \mathbb{R}_+)(B(y, s) \subset B(x, r) \setminus A).$$

It is easy to check that this is equivalent to

$$(\forall x \in X)(\forall r \in \mathbb{Q}_+)(\exists y \in X)(\exists s \in \mathbb{Q}_+)(B(y, s) \subset B(x, r) \setminus A). \tag{$*$)(1.1}$$

All the free variables in the preceding formula are elements of M.

Let us prove the implication from the right to the left first. If A is not nowhere dense in X, then

$$(\exists x \in X)(\exists r \in \mathbb{Q}_+)(\forall y \in X)(\forall s \in \mathbb{Q}_+)(B(y, s) \not\subset B(x, r) \setminus A). \tag{$*$}$$

Using the elementarity of M, there exist $x \in X \cap M$ and $r \in \mathbb{Q}_+$ such that

$$(\forall y \in X)(\forall s \in \mathbb{Q}_+)(B(y, s) \not\subset B(x, r) \setminus A). \tag{1.2}$$

Choose an arbitrary $y \in X_M$, $s \in \mathbb{Q}_+$ and find $y_0 \in X \cap M$ such that $\rho(y, y_0) < \frac{1}{2}s$. Then $B(y_0, \frac{1}{2}s) \subset B(y, s)$. From (1.2),

$$(\exists z \in X)\,(z \in B(y_0, \tfrac{1}{2}s) \setminus (B(x, r) \setminus A)). \tag{$*$}$$

Using the elementarity of M, we may fix $z \in X \cap M$ satisfying the formula above. Thus, for given $y \in X_M$ and $s \in \mathbb{Q}_+$ we have found $z \in X \cap M$ satisfying

$$z \in \quad B(y_0, 1/2s) \setminus (B(x,r) \setminus A) \quad \subset \quad B(y,s) \setminus (B(x,r) \cap X_M \setminus A).$$

Consequently,

$$B(y,s) \cap X_M \not\subseteq (B(x,r) \cap X_M) \setminus A.$$

The negation of $(*)$(1.1) holds in X_M; thus, $A \cap X_M$ is not nowhere dense in X_M.

For the proof of the converse, let A be nowhere dense in X. Choose any $x \in X_M$ and $r \in \mathbb{Q}_+$. Pick $x_0 \in X \cap M$ satisfying $\rho(x, x_0) < r/2$. Then $B(x_0, r/2) \subset B(x,r)$. For the point x_0 and the number $r/2$ find $y \in X$ and $s \in \mathbb{Q}_+$ as in formula $(*)$(1.1). Using the elementarity of M, we may assume that $y \in X \cap M$. Consequently,

$$B(y,s) \;\subset\; B(x_0, r/2) \setminus A \;\subset\; B(x,r) \setminus A.$$

Formula $(*)$(1.1) is satisfied in X_M; thus, $A \cap X_M$ is nowhere dense in X_M. □

It is natural to ask whether the property of being meager is separably determined. One implication is easy:

Proposition 1.4.3. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \textit{ is meager in } X \to A \cap X_M \textit{ is meager in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X and A. Let $\{R_n\}_{n\in\omega}$ be a family of nowhere dense sets such that $A \subset \bigcup_{n\in\omega} R_n$. Then

$$\begin{aligned}(\exists\varphi)(&\varphi \text{ is a function with } \operatorname{Dom}\varphi = \omega, \varphi(n) \text{ is a nowhere dense subset of } X\\ &\text{for every } n \in \omega, \text{ and } A \subset \bigcup_{n\in\omega} \varphi(n)).\end{aligned} \qquad (*)$$

Using the elementarity of M, we find $\varphi \in M$ satisfying the formula above. Consequently, by Proposition 1.2.9, $\varphi(n) \in M$ for every $n \in \omega$.

By Proposition 1.4.2, the set $\varphi(n) \cap X_M$ is nowhere dense in X_M for every $n \in \omega$. Moreover, $A \cap X_M \subset \bigcup_{n\in\omega}(\varphi(n) \cap X_M)$. Therefore, $A \cap X_M$ is meager in X_M. □

For the converse to the implication of the preceding proposition, we need to add some assumptions. Let us first recall what it means to be somewhere meager.

Definition 1.4.4. Let X be a metric space and $A \subset X$. If there are $x \in X$ and $r > 0$ such that $B(x,r) \cap A$ is meager in X, we say that A *is somewhere meager in* X.

We will need the following easy and well-known fact.

Lemma 1.4.5. *Let X be a complete metric space and let $A \subset X$ have the Baire property. Then*

$$X \setminus A \textit{ is not meager } \leftrightarrow A \textit{ is somewhere meager in } X.$$

With the help of this lemma we can prove a converse to the implication of Proposition 1.4.3. First, we need a result for the properties of having the Baire property and being somewhere meager.

Proposition 1.4.6. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ is somewhere meager in } X \to A \cap X_M \text{ is somewhere meager in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and A, and assume that A is somewhere meager. By Propositions 1.2.10 and 1.3.1, $B(x,r) \in M$ whenever $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$, and $C \cap B \in M$ whenever $C, B \in M$.

Because A is somewhere meager, the following formula holds:

$$(\exists x \in X)(\exists r \in \mathbb{R}_+)(B(x,r) \cap A \text{ is meager in } X). \qquad (*)$$

Using the elementarity of M, we find $x \in X \cap M$ and $r \in \mathbb{R}_+ \cap M$ such that $B(x,r) \cap A$ is meager in X. Since $B(x,r) \cap A \in M$, by Proposition 1.4.3, $B(x,r) \cap A \cap X_M$ is meager in X_M. □

Proposition 1.4.7. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ has the Baire property in } X \to A \cap X_M \text{ has the Baire property in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and A and assume that A has the Baire property. Then

$$(\exists D)(\exists P)(D \text{ is a } G_\delta \text{ subset in } X,\ P \text{ is meager subset in } X, \text{ and } A = D \cup P). \qquad (*)$$

Using the elementarity of M, we find $D, P \in M$ satisfying the formula above. By Proposition 1.4.3, $P \cap X_M$ is meager in X_M. Consequently, $A \cap X_M$ is the union of the G_δ set $D \cap X_M$ and the meager set $P \cap X_M$. □

Finally, we prove a converse of Proposition 1.4.3 under additional assumptions.

Theorem 1.4.8. *For any suitable elementary submodel M the following holds: Let X be a complete metric space, $G \subset X$ an open set and $A \subset X$ a set with the Baire property. If M contains X, G and A, then*

$$A \cap G \text{ is meager in } G \leftrightarrow A \cap G \cap X_M \text{ is meager in } G \cap X_M,$$

$$A \cap G \text{ is residual in } G \leftrightarrow A \cap G \cap X_M \text{ is residual in } G \cap X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and G. By Proposition 1.2.10, $B \cap C \in M$ and $B^C \in M$ whenever $B, C \in M$. It is well known that a set $D \subset G$ is meager in X if and only if it is meager in G (see [37, p. 83]). Thus, it is sufficient to prove the first equivalence for $G = X$.

The implication from left to right follows from Proposition 1.4.3. For the converse, assume that A is not meager in X. Then, by Lemma 1.4.5, A^C is somewhere meager in X. Thus, by Proposition 1.4.6, $A^C \cap X_M$ is somewhere meager in X_M. Hence, by Propositions 1.4.7 and 1.4.5, $A \cap X_M$ is not meager in X_M. □

Let us find out whether the property of sets of being porous is separably determined. We use the following definition from [44].

Definition 1.4.9. Let X be a metric space, $A \subset X$, $x \in X$ and $R > 0$. Then we define $\gamma(x, R, A)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus A$.
Further, we define the *upper porosity of A at x in X* as

$$\overline{p}_X(A, x) := 2 \limsup_{R \to 0^+} \frac{\gamma(x, R, A)}{R},$$

and the *lower porosity of A at x in X* as

$$\underline{p}_X(A, x) := 2 \liminf_{R \to 0^+} \frac{\gamma(x, R, A)}{R}.$$

When it is clear which space X we mean, we often say *upper* (resp. *lower*) *porosity of A at x* and write $\overline{p}(A, x)$ (resp. $\underline{p}(A, x)$).

We say that A is *upper porous* (resp. *lower porous*, *c-upper porous*, *c-lower porous*) at x if $\overline{p}(A, x) > 0$ (resp. $\underline{p}(A, x) > 0$, $\overline{p}(A, x) \geq c$, $\underline{p}(A, x) \geq c$).

We say that A is *upper porous* (resp. *lower porous*, *c-upper porous*, *c-lower porous*) if A is upper porous (resp. lower porous, c-upper porous, c-lower porous) at each $y \in A$. We say that A is *σ-upper porous* (resp. *σ-lower porous*) if it is a countable union of upper porous (resp. lower porous) sets.

Definition 1.4.10. Let $\langle X, \rho \rangle$ be a metric space and $A \subset X$. Then $d(x, A) := \inf\{\rho(x, a);\ a \in A\}$ for $x \in X$.

The following lemma is probably well known, but I have not found any reference.

Lemma 1.4.11. *Let $\langle X, \rho \rangle$ be a metric space, $A \subset X$ and $x \in A$. Set*

$$p_1(A, x) := \limsup_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u, A)}{R}, \qquad p_2(A, x) := \liminf_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u, A)}{R}.$$

Then $p_1(A, x) \leq \overline{p}(A, x) \leq 2p_1(A, x)$ and $p_2(A, x) \leq \underline{p}(A, x) \leq 2p_2(A, x)$.

Proof. To show $\overline{p}(A, x) \leq 2p_1(A, x)$ and $\underline{p}(A, x) \leq 2p_2(A, x)$, it is sufficient to prove that $\gamma(x, R, A) \leq \sup_{u \in B(x,R)} \mathrm{d}(u, A)$ for every $R > 0$. Choose some $R > 0$, $r \geq 0$ and $z \in X$ satisfying $B(z, r) \subset B(x, R) \setminus A$. We would like to find $u \in B(x, R)$ such that $r \leq \mathrm{d}(u, A)$. But it is easy to check that $u = z$ satisfies those conditions.

Now we will prove that $\overline{p}(A, x) \geq p_1(A, x)$ and $\underline{p}(A, x) \geq p_2(A, x)$. Take any $R > 0$ and $u \in B(x, R)$ and notice that then $\mathrm{d}(u, A) \leq \gamma(x, 2R, A)$.
Indeed, put $r = \mathrm{d}(u, A)$ and $z = u$. Then for every $y \in B(z, r)$ we have

$$\rho(u, y) = \rho(z, y) < r = \mathrm{d}(u, A),$$

so $y \notin A$. Moreover (using the fact that $r = \mathrm{d}(u, A) < R$, since $x \in A$ and so $u \in B(x, R)$),

$$\rho(y, x) \leq \rho(y, z) + \rho(z, x) < r + R < 2R.$$

Thus, $B(z,r) \subset B(x,2R) \setminus A$ and $\mathrm{d}(u,A) \le \gamma(x,2R,A)$.

As an immediate consequence we get

$$2\limsup_{R\to 0^+} \frac{\gamma(x,2R,A)}{2R} \ge p_1(A,x), \qquad 2\liminf_{R\to 0^+} \frac{\gamma(x,2R,A)}{2R} \ge p_2(A,x).$$

Now it is easy to check that also $\overline{p}(A,x) \ge p_1(A,x)$ and $\underline{p}(A,x) \ge p_2(A,x)$. □

The following two propositions show that the first implication about porous sets holds.

Proposition 1.4.12. *For any suitable elementary submodel M the following holds: Let $\langle X,\rho\rangle$ be a metric space. If M contains X and a set $A \subset X$, then*

$$A \textit{ is not upper porous in } X \to A \cap X_M \textit{ is not upper porous in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and A. The set A is upper porous in X if and only if the following formula holds:

$$(\forall x \in A)(\exists m \in \mathbb{Q}_+)(\forall R_0 > 0)(\exists R \in (0,R_0))(\gamma(x,R,A) > Rm).$$

This formula is equivalent to

$$(\forall x \in A)(\exists m \in \mathbb{Q}_+)(\forall R_0 > 0)(\exists R \in (0,R_0))(\exists r > Rm)(\exists z \in X)$$
$$(B(z,r) \subset B(x,R) \setminus A).$$

Notice that this last formula is equivalent to one where we take only rational numbers R_0, R and r. Indeed, it is obvious that we may consider only rational numbers R_0. Take any $x \in A$ choose $m \in \mathbb{Q}_+$ as in the formula above, and pick $R_0 \in \mathbb{Q}_+$. Then

$$(\exists R \in (0,R_0))(\exists r > Rm)(\exists z \in X)(B(z,r) \subset B(x,R) \setminus A).$$

Fix $R \in (0,R_0)$, $r > Rm$ and $z \in X$ as in the formula above. If we take a rational number $R_q \in (R, \min\{R_0, \frac{r}{m}\})$, then $B(z,r) \subset B(x,R_q) \setminus A$. Thus, R may be without loss of generality considered to be rational. Having now the rational $R \in (0,R_0)$, real $r > Rm$ and $z \in X$ such that $B(z,r) \subset B(x,R) \setminus A$, take a rational $r_q \in (Rm,r)$. Then $B(z,r_q) \subset B(x,R) \setminus A$. Consequently, r may be without loss of generality considered to be rational.

We have seen that A is not upper porous in X if and only if the following formula holds:

$$(\exists x \in A)(\forall m \in \mathbb{Q}_+)(\exists R_0 \in \mathbb{Q}_+)(\forall R \in (0,R_0) \cap \mathbb{Q}_+)(\forall r \in (Rm,\infty) \cap \mathbb{Q}_+)$$
$$(\forall z \in X)\ (B(z,r) \not\subseteq B(x,R) \setminus A). \qquad (*)(1.3)$$

Thus, when A is not upper porous in X we can choose $x \in A$ as in $(*)(1.3)$. Using the elementarity of M, we may assume that $x \in M$. Now, fix $m \in \mathbb{Q}_+$ and pick $R_0 \in \mathbb{Q}_+$ as in $(*)(1.3)$. Fix $R \in (0,R_0) \cap \mathbb{Q}_+$, $r \in (Rm,\infty) \cap \mathbb{Q}_+$ and $z \in X_M$. Then take $r' \in (Rm,r) \cap \mathbb{Q}$ and $z_0 \in X \cap M$ such that $\rho(z,z_0) < r - r'$. Thus, $B(z_0,r') \subset B(z,r)$. Then

$$(\exists y \in X)\ (y \in B(z_0,r')\ \setminus\ (B(x,R) \setminus A)). \qquad (*)$$

For r' and z_0 we can find (using the elementarity of M) point $y \in M$ such that

$$y \in B(z_0,r')\ \setminus\ (B(x,R) \setminus A) \quad \subset \quad B(z,r)\ \setminus\ (B(x,R) \setminus A).$$

Consequently, $(*)(1.3)$ is satisfied in X_M so $A \cap X_M$ is not upper porous in X_M. □

Proposition 1.4.13. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X and a set $A \subset X$, then*

$$A \text{ is not lower porous in } X \to A \cap X_M \text{ is not lower porous in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and A. If A is not lower porous, then as in the proof of Proposition 1.4.12,

$$\begin{aligned}(\exists x \in A)(\forall m \in \mathbb{Q}_+)(\forall R_0 \in \mathbb{Q}_+)(\exists R \in (0, R_0))(\forall r \in (Rm, \infty) \cap \mathbb{Q}_+)\\ (\forall z \in X)\ (B(z,r) \nsubseteq B(x,R) \setminus A).\end{aligned} \qquad (*)(1.4)$$

Using the elementarity of M, choose $x \in A \cap M$ as in the formula above. Then fix $m, R_0 \in \mathbb{Q}_+$ and find $R \in (0, R_0)$ such that

$$(\forall r \in (Rm, \infty) \cap \mathbb{Q}_+)(\forall z \in X)(B(z,r) \nsubseteq B(x,R) \setminus A).$$

Using the elementarity of M we may assume that $R \in M$. Now choose any $r \in (Rm, \infty) \cap \mathbb{Q}_+$ and $z \in X_M$. Then find $r' \in (Rm, r) \cap \mathbb{Q}$ and $z_0 \in B(z, r - r') \cap M$. Thus, $B(z_0, r') \subset B(z, r)$. Then

$$(\exists y \in X)\ (y \in B(z_0, r')\ \setminus\ (B(x,R) \setminus A)). \qquad (*)$$

For r' and z_0 we can find (using the elementarity of M) a point $y \in M$ such that $y \in B(z_0, r') \setminus (B(x,R) \setminus A)$. Consequently,

$$X_M \cap B(z,r) \nsubseteq B(x,R) \setminus A.$$

Thus, $(*)(1.4)$ is satisfied in X_M and so $A \cap X_M$ is not lower porous in X_M. □

To see that the converse holds we will follow the ideas of [40, p. 42]. The following result is proved there for a rich family of subspaces (in the case where X is a Banach space). We give the proof for spaces constructed from elementary submodels (which holds even in the case of metric spaces).

Lemma 1.4.14. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho \rangle$ be a metric space and $f : X \to \mathbb{R}$ a function. If M contains X and f, then for every $R > 0$ and $x \in X_M$,*

$$\sup_{u \in B(x,R)} f(u) = \sup_{u \in B(x,R) \cap X_M} f(u).$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and f. Fix $x \in X_M$ and $R > 0$. To verify that $\sup_{u \in B(x,R)} f(u) \le \sup_{u \in B(x,R) \cap X_M} f(u)$ (the other inequality is obvious), take any $S \in \mathbb{Q}_+$ satisfying $S < \sup_{u \in B(x,R)} f(u)$. Then there exists $u \in B(x,R)$ such that $S < f(u)$. Now, find $R_q, \varepsilon \in \mathbb{Q}_+$ such that $R_q < R$ and $\rho(u,x) < R_q - \varepsilon$. Pick some $x_0 \in B(x, \varepsilon/2) \cap M$. Then $u \in B(x_0, R_q - \varepsilon/2)$ and using the absoluteness of the formula

$$(\exists u \in X)(\rho(u, x_0) < R_q - \varepsilon/2\ \wedge\ S < f(u)), \qquad (*)$$

there exists $u \in B(x_0, R_q - \varepsilon/2) \cap M \subset B(x,R) \cap M$ such that $S < f(u)$. Consequently, $S < \sup_{u \in B(x,R) \cap X_M} f(u)$. □

Proposition 1.4.15. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X, $A \subset X$ and $d(\cdot, A)$, then for every $x \in A \cap X_M$,*

$$A \textit{ is lower porous at } x \rightarrow A \cap X_M \textit{ is lower porous at } x \textit{ in the space } X_M,$$

$$A \textit{ is upper porous at } x \rightarrow A \cap X_M \textit{ is upper porous at } x \textit{ in the space } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and $d(\cdot, A)$, and fix some $x \in A \cap X_M$ such that A is c-upper porous at x for some rational $c > 0$. Thus, by Lemmas 1.4.11 and 1.4.14,

$$\begin{aligned} c \leq \overline{p}_X(A, x) \qquad &\leq 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R)} \frac{\mathrm{d}(u, A)}{R} = 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R) \cap X_M} \frac{\mathrm{d}(u, A)}{R} \\ &\leq 2 \limsup_{R \to 0^+} \sup_{u \in B(x,R) \cap X_M} \frac{\mathrm{d}(u, A \cap X_M)}{R} \leq 2\overline{p}_{X_M}(A \cap X_M, x). \end{aligned}$$

Consequently, $A \cap X_M$ is $c/2$-upper porous at x in X_M. The result for lower porosity follows similarly. □

Corollary 1.4.16. *For any suitable elementary submodel M the following holds: Let X be a metric space. If M contains X, $A \subset X$ and $\mathrm{d}(\cdot, A)$, then*

$$A \textit{ is lower porous in } X \leftrightarrow A \cap X_M \textit{ is lower porous in } X_M,$$

$$A \textit{ is upper porous in } X \leftrightarrow A \cap X_M \textit{ is upper porous in } X_M,$$

$$A \textit{ is } \sigma\textit{-lower porous in } X \rightarrow A \cap X_M \textit{ is } \sigma\textit{-lower porous in } X_M,$$

$$A \textit{ is } \sigma\textit{-upper porous in } X \rightarrow A \cap X_M \textit{ is } \sigma\textit{-upper porous in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, A and $d(\cdot, A)$. Then the porosity results follow from Propositions 1.4.12, 1.4.13 and 1.4.15. The σ-porosity results are then obtained as in the proof of Proposition 1.4.3 using the absoluteness of the following two formulas:

$$\begin{aligned} &(\exists \varphi)(\varphi \text{ is function with } \operatorname{Dom} \varphi = \omega, \varphi(n) \text{ are lower porous subsets of } X \\ &\qquad \text{for every } n \in \omega,\ A \subset \bigcup_{n \in \omega} \varphi(n)). \end{aligned} \tag{$*$}$$

$$\begin{aligned} &(\exists \varphi)(\varphi \text{ is function with } \operatorname{Dom} \varphi = \omega, \varphi(n) \text{ are upper porous subsets of } X \\ &\qquad \text{for every } n \in \omega,\ A \subset \bigcup_{n \in \omega} \varphi(n)). \end{aligned} \tag{$*$}$$

□

The author does not know whether the converse implications of the preceding result about σ-porosity hold as well.

1.5 Properties of functions

Suppose X is a normed linear space and f a function defined on X. The aim of this section is to study the properties (P) of f which are "separably determined". To be more concrete, we want to find a closed separable subspace X_M such that for every $x \in X_M$,

$$f \text{ has property } (P) \text{ at } x \leftrightarrow f \restriction_{X_M} \text{ has property } (P) \text{ at } x.$$

Using the method of elementary submodels it is possible to combine the results about functions with those about sets.

The first property we are interested in is continuity.

Definition 1.5.1. Let $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ be metric spaces, $G \subset X$ an open subset and $f : G \to Y$ a function. Then we denote by $C(f)$ the set of points where f is continuous.

Theorem 1.5.2. *For any suitable elementary submodel M the following holds: Let $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ be metric spaces, $G \subset X$ open subset and $f : G \to Y$ a function. If M contains X, f and Y, then $C(f) \in M$ and for every $x \in X_M \cap G$,*

$$f \text{ is continuous at } x \leftrightarrow f \restriction_{X_M} \text{ is continuous at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y and f. Then $G \in M$, since $G = \operatorname{Dom} f$. Now, $C(f)$ is uniquely defined by the formula

$$(\exists C)(\forall z)(z \in C \leftrightarrow z \in G \wedge f \text{ is continuous at } z); \tag{$*$}$$

hence $C(f) \in M$. Let us prove the desired equivalence. The left-to-right implication holds for every subspace of X. Conversely, suppose that f is not continuous at $x \in X_M \cap G$. Then we can find $k \in \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})(\exists y, z \in G)\,[y, z \in B(x, 1/n) \quad \wedge \quad \sigma(f(y), f(z)) > 1/k]\,. \tag{1.5}$$

Fix $n \in \mathbb{N}$ and $x_0 \in B(x, 1/2n) \cap M$. As $B(x_0, 1/2n)$ is open set containing x, there exists $l \in \mathbb{N}$ such that $B(x, 1/l) \subset B(x_0, 1/2n)$. By (1.5), there are $y, z \in G$ satisfying

$$y, z \in B(x, 1/l) \quad \wedge \quad \sigma(f(y), f(z)) > 1/k.$$

Consequently,

$$(\exists y, z \in G)\,(y, z \in B(x_0, 1/2n) \quad \wedge \quad \sigma(f(y), f(z)) > 1/k)\,. \tag{$*$}$$

All the free variables in this formula are in M, so by the elementarity of M and the fact that $B(x_0, 1/2n) \subset B(x, 1/n)$, there are $y, z \in G \cap M$ such that

$$y, z \in B(x, 1/n) \quad \wedge \quad \sigma(f(y), f(z)) > 1/k. \tag{1.6}$$

We have just shown that for each $n \in \mathbb{N}$ we can find $y, z \in G \cap M$ satisfying (1.6). Consequently, $f \restriction_{X_M}$ is not continuous at x. □

Having proved that property (P) of f (continuity in this case) is separably determined, for the set $A := \{x : f \text{ has the property } (P) \text{ at } x\}$ the following holds:

$$A \cap X_M = \{x : f \restriction_{X_M} \text{ has property } (P) \text{ at } x\}.$$

If $A \in M$, we can combine results about set properties and function properties. In particular, if $A \in M$, by Proposition 1.4.1 there exists a closed separable subspace X_M such that

$$\{x : f \text{ has property } (P) \text{ at } x\} \text{ is dense in } X \leftrightarrow \\ \{x : f \restriction_{X_M} \text{ has property } (P) \text{ at } x\} \text{ is dense in } X_M.$$

Thus, an immediate consequence of the preceding theorem and results about separably determined set properties is the following.

Corollary 1.5.3. *For any suitable elementary submodel M the following holds: Let X and Y be metric spaces, $G \subset X$ open subset and $f : G \to Y$ a function. Suppose that X is complete. If M contains X, Y and f, then*

$$C(f) \text{ is dense in } G \leftrightarrow C(f \restriction_{X_M}) \text{ is dense in } G \cap X_M,$$

$$C(f) \text{ is nowhere dense in } G \leftrightarrow C(f \restriction_{X_M}) \text{ is nowhere dense in } G \cap X_M,$$

$$C(f) \text{ is meager in } G \leftrightarrow C(f \restriction_{X_M}) \text{ is meager in } G \cap X_M,$$

$$C(f) \text{ is residual } G \leftrightarrow C(f \restriction_{X_M}) \text{ is residual in } G \cap X_M,$$

$$C(f)^C \text{ is upper porous in } X \leftrightarrow C(f \restriction_{X_M})^C \text{ is upper porous in } X_M,$$

$$C(f)^C \text{ is lower porous in } X \leftrightarrow C(f \restriction_{X_M})^C \text{ is lower porous in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y and f. Then $G \in M$, because $G = \operatorname{Dom} f$. It is well known that $C(f)$ is a G_δ set [37, pp. 207-208]. From the preceding theorem, $C(f) \cap X_M = C(f \restriction_{X_M})$. Therefore, the result is an immediate consequence of Propositions 1.4.1, 1.4.2, 1.4.12, 1.4.13 and Theorems 1.4.8, 1.5.2. □

The next property we examine is lower (or upper) semicontinuity. Let us recall the definition in metric spaces.

Definition 1.5.4. Let X be a metric space, $G \subset X$ open subset, $f : G \to [-\infty, \infty]$ a function and $x \in G$. If for every sequence $\{x_n\}_{n\in\omega} \subset G$,

$$x_n \to x \quad \text{implies} \quad \liminf_{n\to\infty} f(x_n) \geq f(x),$$

then we say that f is *lower semicontinuous* (*lsc*) at x.

If $-f$ is *lsc* at x, we say that f is *upper semicontinuous* (*usc*) at x.

The following lemma will be used to prove that the lower (and upper) semicontinuity is a separably determined property.

Lemma 1.5.5. *Let X be a metric space, $G \subset X$ open subset, $f : G \to [-\infty, \infty]$ a function and $x \in G$. Then f is lsc at x if and only if for every $c \in \mathbb{Q} \cap (-\infty, f(x))$ there exists $n \in \mathbb{N}$ such that $f[B(x, 1/n) \cap G] \subset (c, \infty]$.*

Proof. We may assume that $f(x) > -\infty$ (if $f(x) = -\infty$, then the lemma is obvious).
"$\Rightarrow$" Suppose there exists $c \in \mathbb{Q} \cap (-\infty, f(x))$ and $\{x_n\}_{n\in\omega} \subset G$ such that $x_n \in B(x, 1/n)$, but $f(x_n) \leq c$. Then $x_n \to x$, but $\liminf_{n\to\infty} f(x_n) \leq c < f(x)$. Thus, f is not *lsc* at x.

"$\Leftarrow$" First, assume that $f(x) < \infty$. Fix $\varepsilon > 0$, $c \in \mathbb{Q} \cap (f(x) - \varepsilon, f(x))$ and a sequence $\{x_n\}_{n\in\omega} \subset G$ with $x_n \to x$. Then there exists $k \in \mathbb{N}$ such that $f[B(x, 1/k) \cap G] \subset (c, \infty]$. Next, there exists n_0 such that $x_n \in B(x, 1/k)$ for every $n \geq n_0$. Consequently, $f(x_n) > c > f(x) - \varepsilon$ for every $n \geq n_0$; hence, $\liminf_{n\to\infty} f(x_n) \geq f(x) - \varepsilon$. As ε could be arbitrarily small, we have $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

In the case that $f(x) = \infty$, we fix $K \in \mathbb{N}$, $c \in \mathbb{Q} \cap (K, \infty)$ and a sequence $\{x_n\}_{n\in\omega} \subset G$ with $x_n \to x$. As above it follows that $\liminf_{n\to\infty} f(x_n) \geq K$. □

Proposition 1.5.6. *For any suitable elementary submodel M the following holds: Let X be a metric space, $G \subset X$ an open subset and $f : G \to [-\infty, \infty]$ a function. If M contains X and f, then for every $x \in X_M \cap G$,*

$$f \text{ is lsc at } x \leftrightarrow f\restriction_{X_M} \text{ is lsc at } x.$$

Proof. Immediately from the definition it is obvious that the left-to-right implication holds for any subspace of X. Fix a $(*)$-elementary submodel M containing $X$and f and assume that f is not *lsc* at $x \in X_M \cap G$. Then, by Lemma 1.5.5, there exists $c \in \mathbb{Q} \cap (-\infty, f(x))$ such that for every $n \in \mathbb{N}$ there exists $y \in B(x, 1/n) \cap G$ such that $f(y) \leq c$. Choose any $n \in \mathbb{N}$ and $x_0 \in B(x, 1/2n) \cap M$. Then $B(x_0, 1/2n) \subset B(x, 1/n)$ is an open set containing x, so there exists $l \in \mathbb{N}$ such that $B(x, 1/l) \subset B(x_0, 1/2n)$. For such an $l \in \mathbb{N}$ there exists $y \in B(x, 1/l) \cap G$ such that $f(y) \leq c$. Consequently,

$$(\exists y \in B(x_0, 1/2n) \cap G)(f(y) \leq c). \qquad (*)$$

Using the elementarity of M, we find $y \in B(x_0, 1/2n) \cap G \cap M \subset B(x, 1/n) \cap G \cap M$ such that $f(y) \leq c$. For any $n \in \mathbb{N}$ we have found $y \in B(x, 1/n) \cap G \cap X_M$ such that $f(y) \leq c$. By Lemma 1.5.5, $f\restriction_{X_M}$ is not *lsc* at x. □

Corollary 1.5.7. *For any suitable elementary submodel M the following holds: Let X be a metric space, $G \subset X$ open subset and $f : G \to [-\infty, \infty]$ a function. Let $-$ denote the operation which maps every function $h : G \to [-\infty, \infty]$ to $-h$. If M contains X and f and $-$, then for every $x \in X_M \cap G$,*

$$f \text{ is usc at } x \leftrightarrow f\restriction_{X_M} \text{ is usc at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, f and $-$. Then $-f \in M$, thus it is enough to use the preceding proposition. □

The last function property examined in this article is Fréchet differentiability. We use the following definition.

Definition 1.5.8. Let X and Y be normed linear spaces, $G \subset X$ an open subset, $f : G \to Y$ a function and $x \in G$.

(i) If there exists a continuous linear operator $A : X \to Y$ such that

$$\lim_{u\to x} \frac{f(u) - f(x) - A(u - x)}{\|u - x\|} = 0,$$

then we say that *function f is Fréchet differentiable at x*. We denote by $D(f)$ the set of the points at which f is Fréchet differentiable.

(ii) For $c, \varepsilon, \delta > 0$ we define $D(f, c, \varepsilon, \delta)$ as the set of all $x \in G$ satisfying

$$\left\| \frac{f(y+tv) - f(y)}{t} - \frac{f(y) - f(y-hv)}{h} \right\| \le \varepsilon$$

whenever

$$v \in X, \quad \|v\| = 1, \quad t > 0, \quad h > 0, \quad y \in B(x, \delta), \quad y - hv \in B(x, \delta),$$
$$y + tv \in B(x, \delta) \quad \text{and} \quad \min(t, h) > c\|y - x\|.$$

The following relationship between sets $D(f, c, \varepsilon, \delta)$ and Fréchet differentiability is shown in [45].

Lemma 1.5.9. *Let X be a normed linear space, $G \subset X$ an open subset and Y a Banach space. Let $f : G \to Y$ be a function. Then f is Fréchet differentiable at a point $x \in G$ if and only if f is continuous at x and $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} D(f, 1/n, 1/n, 1/k)$.*

Using this lemma, it is shown in [45] that the property "to be Fréchet differentiable" is separably determined. Let us prove a similar result using the method of elementary submodels.

Theorem 1.5.10. *For any suitable elementary submodel M the following holds: Let X be a normed linear space, $G \subset X$ an open subset and Y a Banach space. Let $f : G \to Y$ be a function. If M contains X, f and Y, then $D(f) \in M$ and for every $x \in X_M \cap G$,*

$$f \text{ is Fréchet differentiable at } x \leftrightarrow f\restriction_{X_M} \text{ is Fréchet differentiable at } x.$$

Proof. Fix a $(*)$-elementary submodel M containing X, Y, f. $D(f)$ is the object uniquely defined by the formula

$$(\exists D)(\forall z)(z \in D \leftrightarrow z \in D \wedge f \text{ is Fréchet differentiable at } z), \tag{$*$}$$

hence $D(f) \in M$. Fix a point $x \in X_M \cap G$. Then, by Theorem 1.5.2, f is continuous at x if and only if $f\restriction_{X_M}$ is continuous at x. Thus, using Lemma 1.5.9, it is sufficient to check that

$$x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} D(f, 1/n, 1/n, 1/k) \leftrightarrow x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} D(f\restriction_{X_M}, 1/n, 1/n, 1/k).$$

The left-to-right implication is obvious (it holds for every subspace of X). Conversely, assume that $x \notin \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} D(f, 1/n, 1/n, 1/k)$.
Fix $n \in \mathbb{N}$ satisfying $x \notin \bigcup_{k \in \mathbb{N}} D(f, 1/n, 1/n, 1/k)$. Then for every $k \in \omega$,

$$(\exists v \in X,\ \|v\| = 1)(\exists t, h > 0)(\exists y \in X)$$
$$\left(\begin{matrix} y \in B(x, 1/k), \quad y - hv \in B(x, 1/k), \quad y + tv \in B(x, 1/k), \\ \min(t, h) > \frac{1}{n}(\|y - x\| + 0), \quad \left\| \frac{f(y+tv) - f(y)}{t} - \frac{f(y) - f(y-hv)}{h} \right\| > \frac{1}{n} \end{matrix} \right). \tag{$*$}$$

Pick some v, t, h and y as in the formula above and find $\eta \in \mathbb{Q}_+$ such that

$$\|y - x\| < \frac{1}{k} - 2\eta, \quad \|y - hv - x\| < \frac{1}{k} - 2\eta,$$
$$\|y + tv - x\| < \frac{1}{k} - 2\eta, \quad \min(t, h) > \frac{1}{n}(\|y - x\| + 2\eta).$$

Further, take $x_0 \in B(x, \eta) \cap M$. Then

$$\|y - x_0\| \leq \|y - x\| + \|x - x_0\| < \frac{1}{k} - \eta, \quad \|y - hv - x_0\| < \frac{1}{k} - \eta,$$
$$\|y + tv - x_0\| < \frac{1}{k} - \eta, \quad \frac{1}{n}(\|y - x_0\| + \eta) \leq \frac{1}{n}(\|y - x\| + 2\eta) < \min(t, h).$$

Using the elementarity of M we get the existence of $v \in X \cap M$ with $\|v\| = 1$, $t, h \in \mathbb{R}_+ \cap M$ and $y \in X \cap M$ such that:

$$y \in B(x_0, 1/k - \eta) \subset B(x, 1/k), \quad y - hv \in B(x_0, 1/k - \eta) \subset B(x, 1/k),$$
$$y + tv \in B(x_0, 1/k - \eta) \subset B(x, 1/k), \quad \min(t, h) > \frac{1}{n}(\|y - x_0\| + \eta) > \frac{1}{n}\|y - x\|,$$
$$\left\| \frac{f(y + kv) - f(y)}{k} - \frac{f(y) - f(y - hv)}{h} \right\| > \frac{1}{n}.$$

Consequently, $x \notin \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} D(f \restriction_{X_M}, 1/n, 1/n, 1/k)$. □

We would like to combine this result with Theorem 1.4.8, stating that being a residual subset is separably determined property for sets with Baire property in complete metric spaces. The following result comes from [45].

Theorem 1.5.11. *Let X be a normed linear space, $G \subset X$ open subset, and let Y be a Banach space. Let $f : G \to Y$ be a function. Then $D(f)$ is an $F_{\sigma\delta}$ set.*

Using this result we immediately get the following corollary (obviously, even more is true, as in the case of continuity).

Corollary 1.5.12. *For any suitable elementary submodel M the following holds: Let X, Y be Banach spaces, $G \subset X$ open subset and $f : G \to Y$ a function. If M contains X, Y and f, then*

$$D(f) \text{ is dense in } G \leftrightarrow D(f \restriction_{X_M}) \text{ is dense in } G \cap X_M,$$

$$D(f) \text{ is residual in } G \leftrightarrow D(f \restriction_{X_M}) \text{ is residual in } G \cap X_M.$$

1.6 Applications

In this last section we give two applications of the theorems proved above. Both extend the validity of already known theorems to the nonseparable setting. In the first case we take up the result proved in [46, Proposition 3.3] for spaces with separable dual. The method of elementary submodels will allow us to prove that the same theorem holds in general Asplund spaces. The second application will extend the result proved in [39, Theorem 4.8] for $\mathcal{C}(K)$ spaces with K countable compact and for subspaces of c_0 to the case of $\mathcal{C}(K)$ spaces with K a general scattered compact space and to subspaces of $c_0(\Gamma)$ with Γ possibly uncountable.

Separable reductions of the results mentioned above have already been examined using the method of rich families (for the concept of rich families see the Section 1.3). In the first case Zajíček [46, Theorem 5.2] only achieved to prove a weaker variant of the theorem in Asplund spaces. In the second case, the separable reduction to the subspaces of $c_0(\Gamma)$ easily follows using the work of J. Lindenstrauss, D. Preiss and J. Tišer [40, Corollary 5.6.2] and the result of Zajíček [46, Theorem 4.7]. The extension to spaces $\mathcal{C}(K)$ with K scattered compact can be achieved using the result of Gorak [19, proof of Theorem 2.1] and the above mentioned results by Lindenstrauss, Preiss, Tišer and Zajíček.

Remark 1.6.1. Various combinations of the above proved theorems may be considered to be applications as well. For example, by Corollary 1.5.12 the following holds: Let X, Y be Banach spaces, $f : X \to Y$ a function. Then for every separable subspace $V \subset X$ there exists a closed separable subspace $W \subset X$ with $V \subset W$ such that f is Fréchet differentiable on a residual set if and only if $f \restriction_W$ is Fréchet differentiable on a residual set in W.

Let us now discuss the first application.

L. Zajíček proved in [46] a result included as Theorem 1.6.5 below. This theorem was proved for spaces with separable dual. We will use the method of elementary submodels to get the same result for Asplund spaces.

In the following, unless stated otherwise, X will be a Banach space. The equality $X = X_1 \oplus \ldots \oplus X_n$ means that X is the direct sum of non-trivial closed linear subspaces $X_1, \ldots, X_n$ and the corresponding projections $P_i : X \to X_i$ are continuous.

Recall that X is an Asplund space if each continuous convex real-valued function on X is Fréchet differentiable at each point of X except on a first category set; it si known that X is an Asplund space if and only if Y^* is separable for every separable subspace $Y \subset X$.

We will need the following well-known fact (see [46]).

Lemma 1.6.2. *Let X be a Banach space, $0 \neq u \in X$, and suppose $X = W \oplus \operatorname{span}\{u\}$. Then the mapping $w \in W \mapsto w + \mathbb{R}u \in X/\operatorname{span}\{u\}$ is a linear homeomorphism.*

The following definition is used in the theorem from [46].

Definition 1.6.3. Let f be a real-valued function defined on an open subset G of a Banach space X.

(i) We say that f is generically Fréchet differentiable on G if the set $D(f)$ of points where f is Fréchet differentiable is residual in G.

(ii) We say that f is strictly differentiable at $a \in G$ if there exists $x^* \in X^*$ such that

$$\lim_{(x,y)\to(a,a),x\neq y} \frac{f(y) - f(x) - x^*(y-x)}{\|y-x\|} = 0.$$

(iii) We say that f is *essentially smooth* (*esm* for short) *on the line* $L = a + \mathbb{R}v$ (where $a \in X$, $0 \neq v \in X$) if the function $\phi(t) := f(a + tv)$ is strictly differentiable at a.e.

point of its domain. (Obviously, the definition is correct: it does not depend on the choice of a or v).

(iv) We say that a line L is parallel to v (where $0 \neq v \in X$) if there exists $a \in X$ such that $L = a + \mathbb{R}v$.

(v) We say that f is *essentially smooth on a generic line parallel to* $0 \neq v \in X$ if f is essentially smooth on all lines parallel to v, except a first category set of lines in the factor space $X/\operatorname{span}\{v\}$.

Remark 1.6.4. Let X be a normed linear space, $G \subset X$ open subset, $f : G \to \mathbb{R}$ function, Y a subspace of X and $a, v \in Y$, $v \neq 0$. Consider the line $L = a + \mathbb{R}v$. Then it follows immediately from the definition above that $L \subset Y$ and that f is essentially smooth on L if and only if $f\restriction_Y$ is.

The theorem proved in [46, Proposition 3.3] reads as follows.

Theorem 1.6.5. *Let $X = X_1 \oplus \ldots \oplus X_n$ be a Banach space with separable dual X^*. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Suppose that, for each $1 \leq i \leq n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable on G.*

Using the concept of rich families, it is proved in [46, Theorem 5.2] that this result holds under slightly stronger assumptions even in the case of nonseparable Asplund spaces. Using the method of elementary submodels we will prove that the conclusion of Theorem 1.6.5 holds in exactly the same form in nonseparable Asplund spaces.

Let us start with the following lemma.

Lemma 1.6.6. *For any suitable elementary submodel M the following holds: Let X be a normed linear space, $X = X_1 \oplus \ldots \oplus X_n$. Let $P_1, \ldots, P_n$ be the corresponding projections onto subspaces $X_1, \ldots, X_n$. If M contains X and $P_1, \ldots, P_n$, then*

$$X_M = P_1(X_M) \oplus \ldots \oplus P_n(X_M).$$

Proof. Fix a $(*)$-elementary submodel M containing $X$and $P_1, \ldots, P_n$. Then, by Proposition 1.2.9, $P_i(X \cap M) \subset X \cap M$ for each $i \in \{1, \ldots, n\}$. From the continuity of projections $P_1, \ldots, P_n$ it follows that $P_i(X_M) \subset X_M$ for each $i \in \{1, \ldots, n\}$. Consequently, $X_M = P_1(X_M) \oplus \ldots \oplus P_n(X_M)$. □

Theorem 1.6.7. *Let $X = X_1 \oplus \ldots \oplus X_n$ be an Asplund space. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Suppose that, for each $1 \leq i \leq n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable on G.*

Proof. Let $P_1, \ldots, P_n$ be the continuous projections onto the subspaces $X_1, \ldots, X_n$. By Corollary 1.5.12, Propositions 1.4.1, 1.2.10, 1.3.2 and Lemma 1.6.6, there exist formulas $\varphi_1, \ldots, \varphi_l$ and a countable set Y such that for the set

$$Z := \{X, f, P_1, \ldots, P_n, D_1, \ldots, D_n, S_{X_1}, \ldots, S_{X_n}, Y\}$$

and for every elementary submodel $M \prec (\varphi_1, \ldots, \varphi_l;\ Z)$ it is true that:

(P1) Every countable set $S \in M$ is a subset of M.

(P2) $X_M = P_1(X_M) \oplus \ldots \oplus P_n(X_M)$.

(P3) Whenever sets $A, S \subset X$ are in M, then

$$A \cap S \text{ is dense in } S \leftrightarrow A \cap S \cap X_M \text{ is dense in } S \cap X_M.$$

(P4) $D(f)$ is residual in $G \leftrightarrow D(f \restriction_{X_M})$ is residual in $G \cap X_M$.

(P5) X_M is separable subspace of X.

Without loss of generality we may assume that the list of formulas $\varphi_1, \ldots, \varphi_l$ is subformula closed. Notice that for every subspace N of X satisfying $N = P_1(N) \oplus \ldots \oplus P_n(N)$ we have $S_{X_i} \cap N = S_{P_i(N)}$. Indeed,

$$S_{X_i} \cap N = S_X \cap X_i \cap N = S_X \cap X_i \cap P_i(N) = S_X \cap P_i(N) = S_{P_i(N)}.$$

Let us define inductively a sequence $\{M_k\}_{k\in\omega}$ of elementary submodels:

- For $k = 0$ choose any elementary submodel $M_0 \prec (\varphi_1, \ldots, \varphi_n;\ Z)$.
- Whenever M_k is defined, we pick for every $i \in \{1, \ldots, n\}$ a countable subset $C_{k,i}$ of $D_i \cap X_{M_k}$ dense in $S_{P_i(X_{M_k})} = S_{X_i} \cap X_{M_k}$. Then, for every $v \in C_{k,i}$, it follows from the assumptions and Lemma 1.6.2 that the set $\{a \in G;\ f \text{ is } esm \text{ on the line } a+\mathbb{R}v\}$ is residual. Consequently, there exists a G_δ dense subset $G_{k,v}$ such that f is *esm* on each line parallel to v, intersecting $G_{k,v}$.
  Now we let M_{k+1} to be an elementary submodel for the formulas $\varphi_1, \ldots, \varphi_l$ containing $\{Z, C_{k,1}, \ldots, C_{k,n}, M_k, \{G_{k,v}\}_{v\in\bigcup_{i=1}^n C_{k,i}}\}$.

Finally, we define $M := \bigcup_{k\in\omega} M_k$. Then, by Lemma 1.2.4, $M \prec (\varphi_1, \ldots, \varphi_n;\ Z)$. Therefore, (P1)–(P5) hold for M.

We need to verify that for the space X_M and the function $f \restriction_{X_M}$ the conditions of Theorem 1.6.5 are satisfied. Then, by (P4), f is generically Fréchet differentiable on G.

Since X is an Asplund space, $(X_M)^*$ is separable. Obviously, $f \restriction_{X_M}$ is locally Lipschitz. By (P2), $X_M = P_1(X_M) \oplus \ldots \oplus P_n(X_M)$. For $i \in \{1, \ldots, n\}$ we define $C_i := \bigcup_{k\in\omega} C_{k,i}$. Let us verify that this set is dense in $S_{P_i(X_M)} = S_{X_i} \cap X_M$.

Fix $\varepsilon > 0$ and $y \in S_{X_i} \cap X_M = S_{X_i} \cap \overline{\bigcup_{k\in\omega}(X \cap M_k)}$. Then find some $y_0 \in B(y, \varepsilon/3) \cap \bigcup_{k\in\omega}(X \cap M_k)$ and take $k \in \omega$ such that $y_0 \in X \cap M_k$. Then $y_0/\|y_0\| \in X_{M_k} \cap S_{X_i}$. Furthermore,

$$\begin{aligned}\left\|\frac{y_0}{\|y_0\|} - y\right\| &\le \left\|\frac{y_0}{\|y_0\|} - y_0\right\| + \|y_0 - y\| = |1 - \|y_0\|| + \|y_0 - y\| \\ &= |\|y\| - \|y_0\|| + \|y_0 - y\| \le 2\|y_0 - y\| < 2\varepsilon/3.\end{aligned}$$

Since $C_{k,i}$ is dense in $S_{X_i} \cap X_{M_k}$, there exists $c_{k,i} \in C_{k,i} \subset C_i$ such that $\|c_{k,i} - y_0/\|y_0\|\| < \varepsilon/3$. Consequently,

$$\|c_{k,i} - y\| \le \left\|c_{k,i} - \tfrac{y_0}{\|y_0\|}\right\| + \left\|\tfrac{y_0}{\|y_0\|} - y\right\| < \varepsilon.$$

Notice that, by (P1), $C_i \subset M$ for every $i \in \{1,\dots,n\}$. It remains to show that for every $i \in \{1,\dots,n\}$ and $v \in C_i$ the set

$$R_v := \{a \in G \cap X_M;\ f\restriction_{X_M} \text{ is } esm \text{ on the line } a + \mathbb{R}v\}$$

is residual in X_M.

Fix $v \in C_i$ and find $k \in \omega$ such that $v \in C_{k,i}$. Then $R_v \supset G_{k,v} \cap X_M$. As $G_{k,v} \in M$, using (P3), $G_{k,v} \cap X_M$ is dense G_δ set in X_M. Consequently, R_v is residual in X_M. □

The second application extends [39, Theorem 4.8], recalled here as Theorem 1.6.11 below. This theorem was proved for $\mathcal{C}(K)$ spaces where K is a countable compact space, and for subspaces of c_0. We will use the method of elementary submodels to get the same result for $\mathcal{C}(K)$ spaces where K is a scattered compact space and for subspaces of $c_0(\Gamma)$ for Γ possibly uncountable.

Recall that a set $A \subset T$ (where T is an arbitrary topological space) is called *scattered* if every nonempty subset of A has an isolated point. It is well known that a continuous image of a scattered compact is scattered and that a metrizable scattered compact space is countable (see [14, Lemmas 14.20 and 14.21]). Using those two well-known facts we get easily the following.

Lemma 1.6.8. *Let K, L be compact spaces with K scattered and L metrizable, and let $f : K \to L$ be a continuous mapping onto L. Then L is a countable set.*

Recall that a Banach space Y is said to have *the Radon-Nikodým property* (RNP) if every Lipschitz function $f : \mathbb{R} \to Y$ is differentiable almost everywhere (or equivalently every such f has a point of differentiability; see [39]).

The result of J. Lindenstrauss and D. Preiss uses the notion of Γ-null sets. Therefore, let us give some basic notations. For further information about this notion see [40, Chapter 5].

Let X be a Banach space and let $T := [0,1]^{\mathbb{N}}$ be endowed with the product topology and product Lebesgue measure $\mathscr{L}^{\mathbb{N}}$. We denote by $\Gamma(X)$ the space of continuous mappings

$$\gamma : T \to X$$

having continuous partial derivatives $D_j\gamma$ (we consider one-sided derivatives at points where the jth coordinate is 0 or 1). We equip $\Gamma(X)$ with the topology generated by the seminorms

$$\|\gamma\|_\infty = \sup_{t\in T} \|\gamma(t)\| \quad \text{and} \quad \|\gamma\|_k = \sup_{t \in T} \|D_k\gamma(t)\|, k \geq 1.$$

Equivalently, this topology may be defined by the seminorms

$$\|\gamma\|_{\leq k} = \max\{\|\gamma\|_\infty, \|\gamma\|_1, \dots, \|\gamma\|_k\}.$$

The space $\Gamma(X)$ with this topology is a Fréchet space; in particular it is a Polish space whenever X is separable.

We also define $\Gamma_n(X) = \mathcal{C}^1([0,1]^n, X)$ and consider the norm $\|\cdot\|_{\leq n}$ on this space. Notice that $\Gamma_n(X)$ is a subspace of $\Gamma(X)$ in the sense that functions depending on the first n coordinates only are naturally identified with functions from $\Gamma_n(X)$.

A Borel subset $A \subset X$ is called Γ-null if the set $\{\gamma \in \Gamma(X);\ \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is residual in $\Gamma(X)$.

The following two lemmas come from [40, Lemma 5.3.2 and Lemma 5.4.1].

Lemma 1.6.9. *Whenever (X_n) is an increasing sequence of subspaces of X whose union is dense in X, then $\bigcup_{n=1}^{\infty} \Gamma_n(X_n)$ is dense in $\Gamma(X)$.*

Lemma 1.6.10. *Let A be a Borel subset of a Banach space X.*
Then the set $\{\gamma \in \Gamma(X);\ \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is Borel.

The result from [39, Theorem 4.8] is as follows.

Theorem 1.6.11. *The following spaces have the property that every Lipschitz mapping of them into space with the RNP is Fréchet differentiable everywhere except on a Γ-null set: $\mathcal{C}(K)$ for countable compact K and subspaces of c_0.*

Let us first focus on the set property of being Γ-null. To see that it is separably determined, we give the following lemmas.

Lemma 1.6.12. *Let X be a finite dimensional Banach space and let $\{x_1, \ldots, x_n\}$ be a basis of X. Then for every $k \in \omega$,*

$$\Gamma_k(X) = \{\textstyle\sum_{i=1}^{n} \gamma_i x_i;\ \gamma_i \in \Gamma_k(\mathbb{R})\}.$$

Proof. For every $k \in \omega$, $\gamma \in \Gamma_k(X)$ and $t \in [0,1]^k$ there are unique numbers $\gamma_1(t), \ldots, \gamma_n(t)$ such that $\gamma(t) = \sum_{i=1}^{n} \gamma_i(t) x_i$. It is easy to verify that for every $i \in \{1, \ldots, n\}$ the mapping γ_i is an element of $\Gamma_k(\mathbb{R})$ and that $D_j\gamma(t) = \sum_{i=1}^{n} D_j\gamma_i(t) x_i$ whenever $j \in \{1, \ldots, k\}$ and $t \in [0,1]^k$. Thus, $\Gamma_k(X) = \{\sum_{i=1}^{n} \gamma_i x_i;\ \gamma_i \in \Gamma_k(\mathbb{R})\}$. ☐

Lemma 1.6.13. *Let X be a separable Banach space with a countable dense set D. Then*

$$\Gamma(X) = \overline{\{\textstyle\sum_{i=1}^{n} \gamma_i x_i;\ \gamma_i \in \Gamma_n(\mathbb{R}), x_i \in D, n \in \mathbb{N}\}}.$$

Proof. Let N be either the dimension of X if it is finite, or $N = \mathbb{N}$ if X is infinite-dimensional. Then take a countable linearly dense set $\{x_n\}_{n \in N} \subset D$ which is linearly independent. Denote by X_n the subspace $\operatorname{span}\{x_i; i \leq n\}$. Then, by the preceding lemma and Lemma 1.6.9, the set $\{\sum_{i=1}^{n} \gamma_i x_i;\ \gamma_i \in \Gamma_n(\mathbb{R}), n \in \mathbb{N}\}$ is dense in $\Gamma(X)$. ☐

Remark 1.6.14. The preceding lemma holds even in the case when X is nonseparable (with uncountable set $D := X$). This is because the range of every $\gamma \in \Gamma(X)$ is separable. Thus, considering that $\gamma \in \Gamma(\overline{\operatorname{span}}\{\operatorname{Rng} \gamma\})$, we may use the result for separable spaces.

Lemma 1.6.15. *For any suitable elementary submodel M the following holds: Let X be a Banach space. If M contains X and $\{\Gamma_n(X)\}_{n=1}^{\infty}$, then*

$$\overline{\Gamma(X) \cap M} = \Gamma(X_M)$$

Proof. Fix a $(*)$-elementary submodel M containing X, $\{\Gamma_n(\mathbb{R})\}_{n=1}^{\infty}$ and $\{\Gamma_n(X)\}_{n=1}^{\infty}$ (it is not necessary to mention the set $\{\Gamma_n(\mathbb{R})\}_{n=1}^{\infty}$ in the assumptions of the lemma as it does not depend on the space X - see Convention on the page 8). Then, by Proposition 1.2.9, $\Gamma(X) \cap M \subset \Gamma(X_M)$; consequently, $\overline{\Gamma(X) \cap M} \subset \Gamma(X_M)$.

For the other inclusion, denote, for every $n \in \mathbb{N}$,

$$A_n := \left\{ \sum_{i=1}^{n} \gamma_i x_i;\ \gamma_i \in \Gamma_n(\mathbb{R}), x_i \in X \cap M \right\}.$$

Using the preceding lemma, it is sufficient to show that for every $n \in \mathbb{N}$, $A_n \subset \overline{\Gamma(X) \cap M}$. Fix $n \in \mathbb{N}$. Using the absoluteness of the formula (for every $n \in \mathbb{N}$ the formula is the same - what does change is the free variable $\Gamma_n(\mathbb{R})$ in it)

$$(\exists D)(D \text{ is countable and dense in } \Gamma_n(\mathbb{R})), \tag{$*$}$$

we may find a countable set $D \in M$ such that D is dense in $\Gamma_n(\mathbb{R})$. Moreover, whenever $\gamma_0 \in \Gamma(R) \cap M$ and $x_0 \in X \cap M$, then $\gamma_0 x_0$ is the function uniquely defined by the formula

$$(\exists f \in \Gamma_n(X))(\forall t \in [0,1]^n)(f(t) = \gamma_0(t)x_0); \tag{$*$}$$

consequently, $\gamma_0 x_0 \in M$. As the space $\Gamma(X) \cap M$ is $\mathbb{Q}$-linear, it follows that $\{\sum_{i=1}^n \gamma_i x_i;\ \gamma_i \in D, x_i \in X \cap M\} \subset \overline{\Gamma(X) \cap M}$. It is easy to verify that this subset of $\overline{\Gamma(X) \cap M}$ is dense in A_n. □

Remark 1.6.16. The preceding lemma is of independent interest. Observe that combining it with the results from the previous sections one finds that, for every suitable elementary submodel and for every set $A \subset \Gamma(X)$ contained in M, the set A is dense (resp. nowhere dense) in $\Gamma(X)$ if and only if $A \cap \Gamma(X_M)$ is dense (resp. nowhere dense) in $\Gamma(X_M)$. When A has the Baire property, then the same equivalence holds for the residuality of A. This result gives us separable subspaces with properties that were not achieved in [40] using the method of rich families (see [40, Lemma 5.6.1]).

Corollary 1.6.17. *For any suitable elementary submodel M the following holds: Let X be a Banach space. If M contains X, $\{\Gamma_n(X)\}_{n=1}^\infty$ and a Borel set A, then*

$$A \textit{ is } \Gamma\textit{-null in } X \leftrightarrow A \cap X_M \textit{ is } \Gamma\textit{-null in } X_M.$$

Proof. Fix a $(*)$-elementary submodel M containing X, $\{\Gamma_n(X)\}_{n=1}^\infty$ and a Borel set A. Then, in view of Lemmas 1.6.10 and 1.6.15, the set $\{\gamma \in \Gamma(X);\ \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A) = 0\}$ is residual in $\Gamma(X)$ if and only if $\{\gamma \in \Gamma(X_M);\ \mathscr{L}^{\mathbb{N}}\gamma^{-1}(A \cap X_M) = 0\}$ is residual in $\Gamma(X_M)$. □

Using the preceding results, we can give the promised extension of Theorem 1.6.11.

Theorem 1.6.18. *The following spaces have the property that every Lipschitz function of them into space with the RNP is Fréchet differentiable everywhere except on a Γ-null set: $\mathcal{C}(K)$ for K scattered compact and subspaces of $c_0(\Gamma)$, where Γ is an arbitrary set.*

Proof. Suppose we have a space X as in the assumptions (either $X = \mathcal{C}(K)$ for K scattered compact, or $X \subset c_0(\Gamma)$), a Banach space Y with RNP and a Lipschitz function $f : X \to Y$. Using the preceding corollary and Theorem 1.5.10, choose an elementary submodel M satisfying:

- X_M is a separable subspace of X.
- f is Fréchet differentiable everywhere except on a Γ-null set in X if and only if $f\restriction_{X_M}$ is Fréchet differentiable everywhere except on a Γ-null set in X_M.

If $X = \mathcal{C}(K)$, then (using Lemma 1.3.7) choose M such that in addition $X_M = \mathcal{C}(K/_M)$, where $K/_M$ is metrizable compact and a continuous image of K. By Lemma 1.6.8, $K/_M$ is a countable compact space. Hence, by Theorem 1.6.11, $f\restriction_{X_M}$ is Fréchet differentiable everywhere except on a Γ-null set in X_M. Therefore, f is Fréchet differentiable everywhere except on a Γ-null set.

If $X = c_0(\Gamma)$, then X_M is a separable subspace of X, so X_M is a subspace of c_0. Then, using the same arguments as above, f is Fréchet differentiable everywhere except on a Γ-null set. □

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2. Noncommutative Valdivia compacta

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Abstract: We prove some generalizations of results concerning Valdivia compact spaces (equivalently spaces with a commutative retractional skeleton) to the spaces with a retractional skeleton (not necessarily commutative). Namely, we show that the dual unit ball of a Banach space is Corson provided the dual unit ball of every equivalent norm has a retractional skeleton. Another result to be mentioned is the following. Having a compact space K, we show that K is Corson if and only if every continuous image of K has a retractional skeleton.

We also present some open problems in this area.

2.1 Introduction

It is well known that separable Banach spaces have many nice properties. In particular, any separable Banach space admits an equivalent norm which is locally uniformly convex (see e.g. [11, Theorem II.2.6]) and any separable Banach space admits a Markushevich basis (see e.g. [20, Theorem 1.22]). Nonseparable Banach spaces need not have those properties. As an example we may take the space ℓ_∞ which does not admit any equivalent locally uniformly convex norm and it does not admit a Markushevich basis either (see e.g. [11, Theorem II.7.10] and [20, Theorem 5.12]). However, some nonseparable Banach space share those properties of separable ones. For example, any Hilbert space has a locally uniformly convex norm and admits a Markushevich basis.

Having certain nonseparable Banach space, sometimes it is useful to decompose it into smaller pieces (subspaces). There is a hope that if we glue them together, their properties will be preserved by the nonseparable Banach space we started with.

One possible concept of such a decomposition is a *projectional resolution of the identity* (PRI, for short - see e.g. [20, Definition 3.35]). However, in a Banach space of density larger than $\aleph_1$, the existence of a PRI does not tell us much about the structure of the space. There are some ways to solve this problem. One of them is the concept of a *projectional generator* (PG, for short). This is a technical tool from which the existence of a PRI follows (see e.g. [20, Theorem 3.42]). Moreover, the existence of a PG has consequences also for the structure of the space (see e.g. [20, Theorem 5.44]).

Nevertheless, the concept of a PG is not completely satisfactory as it is quite technical. It seems that the right notion is that of a *projectional skeleton* introduced by W.Kubiś in [33]. The existence of a 1-projectional skeleton implies the existence of a PRI and it has some consequences for the structure of the space (see e.g. [33, Corollary 25 and Proposition 26]). Moreover, this notion is not so technical as the concept of a projectional generator.

Spaces with a projectional skeleton are more general than Plichko spaces, but closely connected with them. Similarly, in [34] there has been introduced a class of compact spaces with a retractional skeleton and it has been observed in [34] and [33] that those spaces are more general than Valdivia compacta, but they share a lot of properties with them.

Motivated by the above, we wanted to see how many properties are preserved and we have generalized some results concerning Valdivia compacta and 1-Plichko spaces.

Namely, we show that the dual unit ball of a Banach space is Corson provided the dual unit ball of every equivalent norm has a retractional skeleton. This generalizes the result contained in [26]. Another result to be mentioned is the following. Having a compact space K which is a continuous image of a space with a retractional skeleton, we show that the dual unit ball of $\mathcal{C}(K)$ is Corson whenever the dual unit ball of every subspace of $\mathcal{C}(K)$ has a retractional skeleton. This generalizes the result from [27].

Proofs of these main results are analogous to the proofs from [20] and [27]. We only had to use some conclusions from [33], [22] and apply them. However, three times we had to come with another approach when proving some auxiliary results (see Lemma 2.3.5, 2.4.9 and 2.4.10).

Nonetheless, for some statements concerning Valdivia compact spaces we were unable to give similar results concerning spaces with a retractional skeleton. Some of those problems are formulated at the end of this article.

Below we recall the most relevant notions, definitions and notations.

We denote by ω the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$. Whenever we say that a set is countable, we mean that the set is either finite or infinite and countable. If f is a mapping then we denote by $\operatorname{Rng} f$ the range of f and by $\operatorname{Dom} f$ the domain of f.

Let T be a topological space. The closure of a set A we denote by $\overline{A}$. We say that $A \subset T$ is *countably closed* if $\overline{C} \subset A$ for every countable $C \subset A$. A topological space T is a *Fréchet-Urysohn space* if for every $A \subset T$ and every $x \in \overline{A}$ there is a sequence $x_n \in A$ with $x_n \to x$.

All compact spaces are assumed to be Hausdorff. Let K be a compact space. By $\mathcal{C}(K)$ we denote the space of continuous functions on K. $P(K)$ stands for the space of probability measures with the w^*–topology (the w^*–topology is taken from the representation of $P(K)$ as a compact subset of $(\mathcal{C}(K)^*, w^*)$).

Let Γ be a set. We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \ |\{\gamma \in \Gamma : \ x(\gamma) \neq 0\}| \leq \omega\}$. Given a compact K, $A \subset K$ is called a Σ-*subset* of K if there is a homeomorphic embedding $h : K \to [0,1]^\kappa$ such that $A = h^{-1}[\Sigma(\kappa)]$. A compact space K is *Corson compact* if K is a Σ-subset of K. A compact space K is *Valdivia compact* if there exists a dense Σ-subset of K.

We shall consider Banach spaces over the field of real numbers (but many results hold for complex spaces as well). If X is a Banach space and $A \subset X$, we denote by $\operatorname{conv} A$ the convex hull of A. B_X is the unit ball in X (i.e. the set $\{x \in X; \ \|x\| \leq 1\}$). X^* stands for the (continuous) dual space of X. For a set $A \subset X^*$ we denote by $\overline{A}^{w^*}$ the *weak** closure of A.

A set $D \subset X^*$ is r-*norming* if

$$\|x\| \leq r. \sup\{|x^*(x)| : \ x^* \in D \cap B_{X^*}\}.$$

We say that a set $D \subset X^*$ is norming if it is r-norming for some $r \geq 1$.

Recall that a Banach space X is called *Plichko* (resp. 1-*Plichko*) if there are a linearly dense set $M \subset X$ and a norming (resp. 1-norming) set $D \subset X^*$ such that for every $x^* \in D$ the set $\{m \in M : x^*(m) \neq 0\}$ is countable.

Definition 2.1.1. A *projectional skeleton* in a Banach space X is a family of projections $\{P_s\}_{s\in\Gamma}$, indexed by an up-directed partially ordered set Γ, such that

(i) $X = \bigcup_{s\in\Gamma} P_s X$ and each $P_s X$ is separable,

(ii) $s \leq t \Rightarrow P_s = P_s \circ P_t = P_t \circ P_s$,

(iii) given $s_0 < s_1 < \cdots$ in Γ, $t = \sup_{n\in\omega} s_n$ exists and $P_t X = \overline{\bigcup_{n\in\omega} P_{s_n} X}$.

We shall say that $\{P_s\}_{s\in\Gamma}$ is an *r-projectional skeleton* if it is a projectional skeleton such that $\|P_s\| \leq r$ for every $s \in \Gamma$.

We say that $\{P_s\}_{s\in\Gamma}$ is a *commutative projectional skeleton* if $P_s \circ P_t = P_t \circ P_s$ for every $s, t \in \Gamma$.

Definition 2.1.2. Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X and let $D(\mathfrak{s}) = \bigcup_{s\in\Gamma} P_s^*[X^*]$. Then we say that $D(\mathfrak{s})$ is induced by a projectional skeleton.

Recall that due to [33], we may always assume that every projectional skeleton is an r-projectional skeleton for some $r \geq 1$ (just by passing to a suitable cofinal subset of Γ).

Definition 2.1.3. A *retractional skeleton* in a compact space K is a family of retractions $\{r_s\}_{s\in\Gamma}$, indexed by an up-directed partially ordered set Γ, such that

(i) for every $x \in K$, $x = \lim_{s\in\Gamma} r_s(x)$ and $r_s[K]$ is metrizable for each $s \in \Gamma$,

(ii) $s \leq t \Rightarrow r_s = r_s \circ r_t = r_t \circ r_s$,

(iii) given $s_0 < s_1 < \cdots$ in Γ, $t = \sup_{n\in\omega} s_n$ exists and $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in K$.

We shall say that $\{r_s\}_{s\in\Gamma}$ is a *commutative retractional skeleton* if $r_s \circ r_t = r_t \circ r_s$ for every $s, t \in \Gamma$.

By $\mathcal{R}_0$ we denote the class of all compacta which have a retractional skeleton.

Definition 2.1.4. Let $\mathfrak{s} = \{r_s\}_{s\in\Gamma}$ be a retractional skeleton in a compact space K and let $D(\mathfrak{s}) = \bigcup_{s\in\Gamma} r_s[K]$. Then we say that $D(\mathfrak{s})$ is induced by a retractional skeleton in K.

The class of Banach spaces with a projectional skeleton (resp. class of compact spaces with a retractional skeleton) is closely related to the concept of Plichko spaces (resp. Valdivia compacta). By [33, Theorem 27], Plichko spaces are exactly spaces with a commutative projectional skeleton. By [34, Theorem 6.1], Valdivia compact spaces are exactly compact spaces with a commutative retractional skeleton. Moreover, it immediately follows from the proof of [34, Theorem 6.1] that whenever K is a Valdivia compact with a dense Σ-subset A, then A is induced by a commutative retractional skeleton in K.

When X is a Banach space with $\operatorname{dens} X = \aleph_1$, then X is a Plichko space if and only if it has a projectional skeleton. Similarly, if K is a compact space with weight $\leq \aleph_1$, then K is Valdivia if and only if K has a retractional skeleton. Indeed, in this case the projectional (resp. retractional) skeleton can be indexed by a well-ordered set $[0, \aleph_1)$, so it may be commutative.

An example of a compact space with a retractional skeleton which is not Valdivia is $[0, \omega_2]$ (see [34, Example 6.4]). An example of a space with a 1-projectional skeleton which is not Plichko is $\mathcal{C}([0, \omega_2])$ (see [30, Theorem 1]).

2.2 Main results

The following is a generalization of [26, Theorem 1].

Theorem 2.2.1. *The following conditions are equivalent for a Banach space $\langle X, \|\cdot\|\rangle$:*

(i) $(B_{\langle X^*, \|\cdot\|\rangle}, w^*)$ *is Corson;*

(ii) $\langle X, |||\cdot|||\rangle$ *has a 1-projectional skeleton for every equivalent norm* $|||\cdot|||$*;*

(iii) $(B_{\langle X^*, |||\cdot|||\rangle}, w^*)$ *has a retractional skeleton for every equivalent norm* $|||\cdot|||$*.*

The following is a generalization of [23, Theorem 3.1].

Theorem 2.2.2. *The following conditions are equivalent for a compact space K:*

(i) *K is a Corson compact;*

(ii) *every continuous image of K has a retractional skeleton.*

We will get the last theorem as a special case of Theorem 2.2.6 bellow. To formulate it in a simple general way, we use the class of compact spaces introduced in [27].

Definition 2.2.3. A compact Hausdorff space is said to belong to the class $\mathcal{G}\Omega$ if for every nonempty open subset $U \subset K$ the following holds.

> If U does not contain at least one G_δ point of K, then the one-point compactification of U contains a homeomorphic copy of $[0, \omega_1]$.

We will need also the following notion of property (M).

Definition 2.2.4. A compact space K is said to have the property (M) if every Radon probability measure on K has separable support.

The following two theorems are generalizations of [27, Theorem 1] and [27, Theorem 2].

Theorem 2.2.5. *The following conditions are equivalent for a compact space K from the class $\mathcal{G}\Omega$.*

(i) *K is a Corson compact with the property (M).*

(ii) *Every subspace of $\mathcal{C}(K)$ has a 1-projectional skeleton.*

(iii) *(B_{Y^*}, w^*) has a retractional skeleton for every subspace $Y \subset \mathcal{C}(K)$.*

In particular, the assumptions of this theorem are satisfied if K is a continuous image of a space with a retractional skeleton.

Theorem 2.2.6. *The following conditions are equivalent for a compact space K from the class $\mathcal{G}\Omega$.*

(i) *K is a Corson compact.*

(ii) *$\mathcal{C}(L)$ has a 1-projectional skeleton for every continuous image L of K.*

(iii) *$(B_{\mathcal{C}(L)^*}, w^*)$ has a retractional skeleton for every continuous image L of K.*

(iv) *$P(L)$ has a retractional skeleton for every continuous image L of K.*

In particular, the assumptions of this theorem are satisfied if K is a continuous image of a space with a retractional skeleton.

2.3 Properties of compact spaces with a retractional skeleton

In this section we first collect several important properties of sets induced by a retractional skeleton in a compact space. These properties are similar to properties of dense Σ-subsets of Valdivia compact spaces and the proofs are often done in a similar way. Having those results in hand, we deduce from them some properties of compact spaces with a retractional skeleton. These are similar to the ones of Valdivia compact spaces and the proofs are often done in the same way.

We start with the following theorem which sums up basic properties of sets induced by a retractional skeleton.

Theorem 2.3.1 ([33, Theorem 30])**.** *Assume D is induced by a retractional skeleton in K. Then:*

(i) *D is dense in K and for every countable set $A \subset D$, $\overline{A}$ is metrizable and contained in D;*

(ii) *D is a Fréchet-Urysohn space;*

(iii) *D is a normal space and $K = \beta D$.*

We continue with some consequences of Theorem 2.3.1. The following lemma is just an easy generalization of [28, Lemma 1.7]. The proof is identical, we only use Theorem 2.3.1 instead of [28, Lemma 1.6].

Lemma 2.3.2. *Let K be a compact space and A, B be two subsets induced by a retractional skeleton in K. If $M \subset K$ is a set such that $A \cap B \cap M$ is dense in M, then $A \cap M = B \cap M$. In particular, $A = B$ whenever $A \cap B$ is dense in K.*

As any set induced by a retractional skeleton in a compact space is countably compact, we get as a consequence of [28, Lemma 1.11] the following.

Lemma 2.3.3. *Assume D is induced by a retractional skeleton in a compact space K. Then $G \cap D$ is dense in G whenever $G \subset K$ is G_δ. In particular, if $x \in K$ is a G_δ point, then $x \in D$.*

Corollary 2.3.4. *Let K be a compact space with a dense set of G_δ points. Then there is at most one set D which is induced by a retractional skeleton in K.*

Proof. This follows immediately from Lemma 2.3.2 and Lemma 2.3.3. □

We continue with the following lemma.

Lemma 2.3.5. *Let K be a compact space, $F \subset K$ closed subset and let $D \subset K$ be such that D is induced by a retractional skeleton in K. If $D \cap F$ is dense in F, then $D \cap F$ is induced by a retractional skeleton in F.*

Proof. Let $\mathfrak{s} = \{r_s\}_{s\in\Gamma}$ be a retractional skeleton in K and put

$$\Gamma' = \{s \in \Gamma : \; r_s[F] \subset F\}.$$

In order to see $\mathfrak{s}' = \{r_s \restriction_F\}_{s\in\Gamma'}$ is a retractional skeleton in F, it is enough to prove that Γ' is a cofinal subset of Γ such that for every sequence $s_0 < s_1 < \cdots$ in Γ', $\sup_{n\in\omega} s_n \in \Gamma'$. Once this is proved, it is easy to notice that $D(\mathfrak{s}') = D \cap F$.

In order to verify that Γ' is a cofinal subset of Γ, fix $\gamma_0 \in \Gamma$ and put $C_{-1} = \emptyset$. We inductively find sequences $\{\gamma_n\}_{n\in\omega} \subset \Gamma$ and $\{C_n\}_{n\in\omega}$ in the following way. Having γ_n and C_{n-1}, we find a countable set $C_n \subset D \cap F$ such that $r_{\gamma_n}[C_n]$ is dense in $r_{\gamma_n}[D \cap F]$. Then, using (ii) and (iii) from the definition of a retractional skeleton and $C_n \subset D$, we find $\gamma_{n+1} > \gamma_n$ such that $C_n \subset r_{\gamma_{n+1}}[K]$. Put $t = \sup_{n\in\omega} \gamma_n$. Now we will prove that $r_t[D \cap F] \subset D \cap F$.

Fix a metric ρ in the space $r_t[K]$ and a point $x \in D \cap F$. Then, for every $n \in \omega$, we find a point $c_n \in C_n$ satisfying

$$\rho(r_{\gamma_k}(x), r_{\gamma_k}(c_n)) < \frac{1}{n}\,, \quad k \le n.$$

Such a point exists, because $\{z \in r_{\gamma_n}[D \cap F] : \; \forall k \le n : \rho(r_{\gamma_k}(x), r_{\gamma_k}(z)) < \frac{1}{n}\}$ is an open set in $r_{\gamma_n}[D \cap F]$ containing $r_{\gamma_n}(x)$; thus, it contains $r_n(c_n)$ for some $c_n \in C_n$. Passing to a subsequence if necessary, we may without loss of generality assume there is a point $c \in r_t[K]$ such that $c_n \to c$. Consequently, $\rho(r_{\gamma_k}(x), r_{\gamma_k}(c)) = 0$ for every $k \in \omega$. Hence,

$$r_t(x) = \lim_{k\to\infty} r_{\gamma_k}(x) = \lim_{k\to\infty} r_{\gamma_k}(c) = r_t(c) = \lim_{n\to\infty} r_t(c_n) = \lim_{n\to\infty} c_n \in D \cap F$$

and $r_t[D \cap F] \subset D \cap F$.

Using the density of $D \cap F$ in F, $t \in \Gamma'$ and Γ' is cofinal in Γ.

Having $s_0 < s_1 < \cdots$ in Γ' and $t = \sup_{n\in\omega} s_n$, it is obvious that for every $x \in F$, $r_t(x) = \lim_{n\to\infty} r_{s_n}(x) \in F$. Thus, $t \in \Gamma'$. □

Notice, that the preceding lemma is trivial in the case when D is a dense Σ-subset of K. However, for spaces with a retractional skeleton this required some work. The proof can also be done using the method of elementary submodels, namely Theorem 2.4.8. This alternative proof is much shorter, but its difficulty is hidden in Theorem 2.4.8 and in the method of elementary submodels.

The following lemma is a strengthening of Lemma 2.3.3. It is just an easy generalization of [27, Lemma 5]. Every set induced by a retractional skeleton in a compact space K satisfies (by Theorem 2.3.1 and Lemma 2.3.5) all the properties of dense Σ-subsets in K which are required in the proof from [27]. Hence, the proof of the lemma can be done in the same way as the proof of [27, Lemma 5].

Lemma 2.3.6. *Let K be a compact space and $G = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ where each U_n is an open subset of K. If D is induced by a retractional skeleton in K, then $G \cap D$ is dense in G. Consequently, $G \cap D$ is induced by a retractional skeleton in G.*

Now we collect several properties of compact spaces with a retractional skeleton which follow from the above results concerning sets induced by a retractional skeleton. As an easy corollary to Theorem 2.3.1 we get the following.

Corollary 2.3.7. *Let K be a compact space, $x \in K$ and Γ an uncountable set. Let $\{g_k\}_{k=1}^\infty$ and $\{f_\gamma\}_{\gamma\in\Gamma}$ be sets of G_δ points in K such that $x \in \overline{\{g_k\}_{k=1}^\infty} \cap \overline{\{f_\gamma\}_{\gamma\in\Gamma}}$ and no countable sequence from $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to x. Then K does not have a retractional skeleton.*

Proof. In order to get a contradiction, let us assume that a set D is induced by a retractional skeleton in K. Then $\{g_k\}_{k=1}^\infty \cup \{f_\gamma\}_{\gamma\in\Gamma} \subset D$, and since $x \in \overline{\{g_k\}_{k=1}^\infty}$, it follows that $x \in D$. Put $C = \{f_\gamma\}_{\gamma\in\Gamma} \subset D$. Then $x \in \overline{C}$, but no countable sequence from C converges to x. This is a contradiction with the fact that D is a Fréchet space. □

In [23, Example 3.4] there are some basic examples of compact spaces which are not Valdivia. Since they have both weight $\aleph_1$, they do not have a retractional skeleton either. We sum up these in the example bellow.

Example 2.3.8. (i) Let K_1 be the compact space obtained from $([0,\omega_1] \times \{0\}) \oplus ([0,\omega] \times \{1\})$ by identifying the points $(\omega_1, 0)$ and $(\omega, 1)$. Then $K_1 \notin \mathcal{R}_0$.

(ii) Let K_2 be the compact space obtained from $[0,\omega_1] \times \{0,1\}$ by identifying the points $(\omega_1, 0)$ and $(\omega_1, 1)$. Then $K_2 \notin \mathcal{R}_0$.

The following stability result follows immediately from Lemma 2.3.3, Lemma 2.3.5 and Lemma 2.3.6.

Theorem 2.3.9. *Let K be a compact space with a retractional skeleton. Then:*

(i) *every subset of K, which is the closure of an arbitrary union of G_δ sets, has a retractional skeleton as well;*

(ii) *if $G = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ with U_n open, then G has a retractional skeleton as well.*

We continue with a theorem from [22]. First, we recall some definitions (see [22]).

We denote by $\mathcal{R}$ the smallest class of compact spaces that contains all metrizable ones and that is closed under limits of continuous retractive inverse sequences. It is claimed in [33] that $\mathcal{R}_0 \subset \mathcal{R}$. A more detailed proof of this fact is contained in the proof of [34, Lemma 6.3] (the assumption on the commutativity of the skeleton is not needed to obtain $\mathcal{R}_0 \subset \mathcal{R}$).

We denote by $\mathcal{RC}$ the smallest class of compact spaces that contains all metrizable ones and that is closed under continuous images and inverse limits of transfinite sequences of retractions. Obviously, $\mathcal{R} \subset \mathcal{RC}$.

Theorem 2.3.10 ([22, Theorem 19.22])**.** *Let $K \in \mathcal{RC}$. Then either $[0, \omega_1]$ embeds into K or else K is Corson compact.*

Now we show what is the correspondence between compact spaces with a retractional skeleton and Corson compact spaces.

Theorem 2.3.11. *Let K be a compact space. Then it is a Corson compact if and only if K is induced by a retractional skeleton in K. Moreover, whenever D is a set induced by a retractional skeleton in a Corson compact K, then $D = K$.*

Proof. Let K be a Corson compact. Then, as mentioned above, it immediately follows from the proof of [34, Theorem 6.1] that K is induced by a commutative retractional skeleton in K. Moreover, whenever D is induced by a retractional skeleton in K, $D = K$ by Lemma 2.3.2.

If K is induced by a retractional skeleton in K, then K is Fréchet-Urysohn space; thus, $[0, \omega_1]$ does not embed into K. It follows from Theorem 2.3.10 that K is Corson. □

Corollary 2.3.12. *Assume D is induced by a retractional skeleton in a compact space K. Then every subset of D closed in K is Corson.*

Proof. This follows from Lemma 2.3.5 and from Theorem 2.3.11 above. □

The following lemma is an analogue to [27, Lemma 2].

Lemma 2.3.13. *Let K be a compact space such that $P(K)$ has a retractional skeleton. If we denote by G the set of all G_δ points of K, then $\overline{G}$ has a retractional skeleton as well.*

Proof. We use the same idea as in [27, Lemma 2]. Let us fix a set D induced by a retractional skeleton in $P(K)$. If $g \in G$ is a G_δ point of K, then it is easy to verify (see [28, Lemma 5.5]), that the Dirac measure δ_g supported by the point g is a G_δ point in $P(K)$; hence, by Lemma 2.3.3, $\delta_g \in D$. Thus, if we identify k with δ_k for every $k \in K$, $G \subset D$. Consequently, by Lemma 2.3.5, $\overline{G} \cap D$ is induced by a retractional skeleton in $\overline{G}$. □

Proposition 2.3.14. *Let $D \subset X^*$ be a set induced by a 1-projectional skeleton. Then there exists a convex symmetric set R, induced by a retractional skeleton in (B_{X^*}, w^*).*

Proof. Let $\{P_s\}_{s\in\Gamma}$ be a 1-projectional skeleton such that $D = \bigcup_{s\in\Gamma} P_s^*(X^*)$. Using only the definitions and [33, Lemma 10] it is easy to see that $\{P_s^* \restriction_{B_{X^*}}\}_{s\in\Gamma}$ is retractional skeleton in B_{X^*}. Now it remains to show that $R = \bigcup_{s\in\Gamma} P_s^*(B_{X^*})$ is convex and symmetric. It is easily checked that $R = D \cap B_{X^*}$. Now we observe that D is an up-directed union of linear sets; thus, it is linear. Consequently, R is convex and symmetric. □

Now we are ready to see that once we know (i)⇒(ii) in Theorem 2.2.5 (resp. Theorem 2.2.6), (iii) (resp. (iv)) is the strongest condition.

Proposition 2.3.15. *Let K be a compact space. Consider the following conditions.*

(i) *K has a retractional skeleton.*

(ii) *$\mathcal{C}(K)$ has a 1-projectional skeleton.*

(iii) *There is a convex symmetric set induced by a retractional skeleton in $(B_{\mathcal{C}(K)^*}, w^*)$.*

(iv) *$(B_{\mathcal{C}(K)^*}, w^*)$ has a retractional skeleton.*

(v) *There is a convex symmetric set induced by a retractional skeleton in $P(K)$.*

(vi) *$P(K)$ has a retractional skeleton.*

Then the following implications hold:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (vi), \quad (iii) \Rightarrow (v) \Rightarrow (vi).$$

Moreover, if K has a dense set of G_δ points, then all the conditions are equivalent.

Proof. The implication (i)⇒(ii) comes from [33, Proposition 28], (ii)⇒(iii) follows from Lemma 2.3.14, (iii)⇒(iv) and (v)⇒(vi) are obvious. For (iv)⇒(vi) and (iii)⇒(v) it is enough to observe that $P(K)$ is a closed G_δ set in $(B_{\mathcal{C}(K)^*}, w^*)$ and use Lemma 2.3.3 and Lemma 2.3.5. If K has a dense set of G_δ points, then (vi)⇒(i) follows from Lemma 2.3.13. □

Remark 2.3.16. It is known that the implication (ii)⇒(i) in Proposition 2.3.15 does not hold. There even exists a compact space K such that $\mathcal{C}(K)$ is 1-Plichko, but $K \notin \mathcal{R}$ (see [2, Theorem 3.2]). However, in the case of commutative skeletons (i.e. Plichko spaces and Valdivia compact spaces), it is true that (ii)⇔(iii)⇔(v) (see [28, Theorem 5.2]). The proof of this fact uses a characterization of dense subsets induced by a retractional skeleton in K by a topological property of the space $(\mathcal{C}(K), \tau_p(D))$ ($\tau_p(D)$ is the topology of the pointwise convergence on D). Thus, two natural questions arise. They are formulated at the end of this article (see Problem 1 and Question 1).

2.4 The method of elementary submodels

In this section we prove Lemma 2.4.9 and Lemma 2.4.10 using the method of elementary submodels. If one does not feel comfortable with this method, he can skip this section and use only its results. The knowledge of the method elementary submodels is not needed any further.

The method of elementary submodels is a set-theoretical method which can be used in various branches of mathematics. W.Kubiś in [33] used this method to create a projectional (resp. retractional) skeleton in certain Banach (resp. compact) spaces. In [4] the method has been slightly simplified and precised. We briefly recall some basic facts about the method and give a more detailed proof of Theorem 2.4.8 which is also proved in a slightly different form in [22, Theorem 19.16]. Finally, we prove Lemma 2.4.9 and Lemma 2.4.10.

First, let us recall some definitions. Let N be a fixed set and ϕ a formula in the language of ZFC. Then the *relativization of* ϕ *to* N is the formula ϕ^N which is obtained from ϕ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each quantifier of the form "$\exists x$" by "$\exists x \in N$".

If $\phi(x_1, \dots, x_n)$ is a formula with all free variables shown (i.e. a formula whose free variables are exactly $x_1, \dots, x_n$) then ϕ *is absolute for* N if and only if

$$\forall a_1, \dots, a_n \in N \quad (\phi^N(a_1, \dots, a_n) \leftrightarrow \phi(a_1, \dots, a_n)).$$

The method is based mainly on the following theorem (a proof can be found in [36, Theorem IV.7.8]).

Theorem 2.4.1. *Let $\phi_1, \dots, \phi_n$ be any formulas and X any set. Then there exists a set $M \supset X$ such, that*

$$(\phi_1, \dots, \phi_n \textit{ are absolute for } M) \quad \wedge \quad (|M| \leq \max(\omega, |X|)).$$

Since the set from Theorem 2.4.1 will often be used, the following notation is useful.

Definition 2.4.2. Let $\phi_1, \dots, \phi_n$ be any formulas and let X be any countable set. Let $M \supset X$ be a countable set satisfying that $\phi_1, \dots, \phi_n$ are absolute for M. Then we say that M *is an elementary submodel for* $\phi_1, \dots, \phi_n$ *containing* X. This is denoted by $M \prec (\phi_1, ..., \phi_n;\ X)$.

We shall also use the following convention.

Convention. Whenever we say

for any suitable elementary submodel M (the following holds...),

we mean that

there exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y such that for every $M \prec (\phi_1, \dots, \phi_n;\ Y)$ (the following holds...).

By using this new terminology we lose the information about the formulas $\phi_1, \dots, \phi_n$ and the set Y. This is, however, not important in applications.

Let us emphasize that a suitable elementary submodel is always countable. This is really needed in our applications; see, e.g., the proof of Theorem 2.4.8.

Let us recall several more results about suitable elementary submodels (proofs can be found in [4, Chapters 2 and 3]):

Lemma 2.4.3. *Let $\varphi_1, \dots, \varphi_n$ be a subformula closed list of formulas and let X be any countable set. Let $\{M_k\}_{k \in \omega}$ be a sequence of sets satisfying*

(i) $M_i \subset M_j, \quad i \leq j,$

(ii) $\forall k \in \omega :\ M_k \prec (\varphi_1, ..., \varphi_n;\ X).$

Then for $M := \bigcup_{k \in \omega} M_k$ it is true, that also $M \prec (\varphi_1, ..., \varphi_n;\ X)$.

Lemma 2.4.4. *For any suitable elementary submodel M the following holds.*

(i) *Let f be a function such that $f \in M$. Then* $\operatorname{Dom} f \in M$, $\operatorname{Rng} f \in M$ *and* $f(M) \subset M$.

(ii) *Let $S \in M$ be a countable set. Then $S \subset M$.*

Lemma 2.4.5. *Let $\phi(y, x_1, \ldots, x_n)$ be a formula with all free variables shown and let X be a countable set. Let M be a fixed set, $M \prec (\phi, \exists y \phi(y, x_1, \ldots, x_n);\ X)$ and let $a_1, \ldots, a_n \in M$ be such that there exists only one set u satisfying $\phi(u, a_1, \ldots, a_n)$. Then $u \in M$.*

Using the last lemma we can force the elementary submodel M to contain all the needed objects created (uniquely) from elements of M.

Given a compact space K and an arbitrary elementary submodel M we define the following equivalence relation $\sim_M$ on K:

$$x \sim_M y \quad \iff \quad (\forall f \in \mathcal{C}(K) \cap M) : \ f(x) = f(y).$$

We shall write $K/_M$ instead of $K/_{\sim_M}$ and we shall denote by q_K^M the canonical quotient map. It is not hard to check that $K/_M$ is a compact Hausdorff space.

In [4] it is proved that the following lemma holds (slightly different version may be also found in [33]).

Lemma 2.4.6. *Let K be a compact space. Then for any suitable elementary submodel M it is true that*

$$\overline{\mathcal{C}(K) \cap M} = \{\varphi \circ q_K^M;\ \varphi \in \mathcal{C}(K/_M)\}.$$

Consequently, we can identify $\overline{\mathcal{C}(K) \cap M}$ with the space $\mathcal{C}(K/_M)$.

We will need the following simple, but useful lemma.

Lemma 2.4.7. *For every suitable elementary submodel M the following holds: Let K be a compact metric space. Then whenever $K \in M$, $\mathcal{C}(K) \cap M$ separates points of K.*

Proof. Fix a suitable elementary submodel M such that $K \in M$. Then, using the absoluteness of the formula (and its subformula)

$$\exists D(D \text{ is a countable subset of } \mathcal{C}(K) \text{ separating points of } K),$$

there exists a countable set $D \in M$ separating points of K. By Lemma 2.4.4, $D \subset M$. Consequently, $\mathcal{C}(K) \cap M \supset D$ separates points of K. □

Finally, let us show how the method of elementary submodels is connected with the compact spaces with a retractional skeleton.

Theorem 2.4.8 ([22, Theorem 19.16])**.** *Let K be a compact space, and let D be its dense subset. The following properties are equivalent.*

(i) *There exists a set $D(\mathfrak{s})$ induced by a retractional skeleton in K such that $D \subset D(\mathfrak{s})$.*

(ii) *For every suitable elementary submodel M, the quotient map $q_K^M : K \to K/_M$ is one-to-one on $\overline{D \cap M}$.*

Under the assumption that D is countably closed, the conditions above are also equivalent to the following:

(iii) *D is induced by a retractional skeleton in K.*

Proof. First, let us suppose that (i) holds. Without loss of generality we assume that $D = D(\mathfrak{s})$ is induced by a retractional skeleton $\{r_s\}_{s\in\Gamma}$ in K. Define a mapping $r : \Gamma \to \mathcal{C}(K)$ by $r(s) = r_s$. Fix formulas $\varphi_1, \ldots, \varphi_n$ containing all the formulas (and their subformulas) marked by $(*)$ in the proof below and a countable set $Y \supset \{D, K, \Gamma, r\}$ such that whenever $M \prec (\varphi_1, ..., \varphi_n;\ Y)$, all the results mentioned above hold for M. Fix some $x, y \in \overline{D \cap M}$, $x \neq y$. Using the the absoluteness of the formula (and its subformulas)

$$\forall u, v \in \Gamma\ \exists w \in \Gamma\ w \geq u, v, \qquad (*)$$

the set $(\Gamma \cap M)$ is up-directed. Thus, there exists $t = \sup(\Gamma \cap M)$. Using the absoluteness of the formula (and its subformulas)

$$\forall x \in D\ \exists s \in \Gamma\ x \in r_s[K], \qquad (*)$$

$D \cap M \subset r_t[K]$. Thus, $\overline{D \cap M} \subset r_t[K]$. Now we find a sequence $s_0 < s_1 \cdots$ in $\Gamma \cap M$ such that $\sup_{n\in\omega} s_n = t$. Then $x = \lim_{n\to\infty} r_{s_n}(x)$ and $y = \lim_{n\to\infty} r_{s_n}(y)$. There exists an $n \in \mathbb{N}$ such that $r_{s_n}(x) \neq r_{s_n}(y)$. By Lemma 2.4.7, there is a function $f \in \mathcal{C}(r_{s_n}[K]) \cap M$ such that $f(r_{s_n}(x)) \neq f(r_{s_n}(y))$. By Lemma 2.4.4, $r(s_n) = r_{s_n} \in M$. Now, using the absoluteness of the formula (and its subformula)

$$\forall f, g \in \mathcal{C}(K)\ \exists h \in \mathcal{C}(K) \quad (h = f \circ g), \qquad (*)$$

$g = f \circ r_{s_n} \in \mathcal{C}(K) \cap M$ and $g(x) \neq g(y)$.

In order to prove (ii)⇒(i), fix formulas $\varphi_1, \ldots, \varphi_n$ and a countable set Y such that whenever $M \prec (\varphi_1, ..., \varphi_n;\ Y)$, q_K^M is one-to-one on $\overline{D \cap M}$, all the statements mentioned above about suitable models hold and all the formulas (and their subformulas) marked by $(*)$ below are absolute for M. We can without loss of generality assume that $K, D \in Y$ (if not, we just put $Y' = Y \cup \{K, D\}$). Fix $M \prec (\varphi_1, ..., \varphi_n;\ Y)$. In the following we write q^M instead of q_K^M. Observe that $q^M[D \cap M]$ is a dense subset of $K/_M$.

Indeed, using Lemma 2.4.6 it is not difficult to show that

$$\{\psi^{-1}(U) :\ \psi \in \mathcal{C}(K/_M),\ \psi \circ q^M \in \mathcal{C}(K) \cap M,\ U \text{ is an open rational interval}\}$$

is a basis of $K/_M$. Now if we take an open rational interval U and a function $\psi \in \mathcal{C}(K/_M)$ such that $\psi \circ q^M \in \mathcal{C}(K) \cap M$, then by the denseness of $q[D]$ in $K/_M$ there is a $d \in D$ such that $q^M(d) \in \psi^{-1}(U)$. Thus,

$$\exists d \in D \quad \psi(q^M(d)) \in U. \qquad (*)$$

Using the elementarity of M, there is a $d \in D \cap M$ such that $\psi(q^M(d)) \in U$. Hence, $q^M[D \cap M] \cap \psi^{-1}(U) \neq \emptyset$.

It follows that $q^M[\overline{D \cap M}] = K/_M$. If we denote $j^M = (q^M \restriction_{\overline{D\cap M}})^{-1}$, then j^M is a homeomorphism of $\overline{D \cap M}$ and $K/_M$. It follows that $r_M = j^M \circ q^M : K \to \overline{D \cap M}$ is a retraction onto.

By Theorem 2.4.1, there exists a set $R \supset Y \cup K \cup \{U : U \text{ is an open set in } K\}$ such that $\varphi_1, ..., \varphi_n$ are absolute for R. It follows from the proof of Theorem 2.4.1 (see [36, Theorem IV.7.8]) that for every countable set $Z \subset R$ there exists an $M \subset R$ such that $M \prec (\varphi_1, ..., \varphi_n;\ Z)$. Hence, by Lemma 2.4.3,

$$\Gamma = \{M \subset R :\ M \prec (\varphi_1, ..., \varphi_n;\ Y)\}$$

is a nonempty and up-directed set where the supremum of every increasing countable chain exists. We will verify that $\{r_M\}_{M\in\Gamma}$ is the retractional skeleton we are looking for.

Observe that $f(r_M(x)) = f(x)$ for every $f \in \overline{\mathcal{C}(K) \cap M}$ and $x \in K$. Indeed, every $f \in \overline{\mathcal{C}(K) \cap M}$ equals $\psi \circ q^M$ for some $\psi \in \mathcal{C}(K/_M)$. It follows that for every $x \in K$

$$f(r_M(x)) = \psi(q^M r_M(x)) = \psi(q^M j^M q^M(x)) = \psi(q^M(x)) = f(x).$$

Moreover, as q^M is one-to-one on $\overline{D \cap M}$, $\mathcal{C}(K) \cap M$ separates points of $\overline{D \cap M}$.

Fix some $M \in \Gamma$. The set $r_M[K] = \overline{D \cap M}$ is homeomorphic to $K/_M$; hence, it is metrizable. In order to verify (i) from the definition of a retractional skeleton, fix $x \in K$ and an open set $U \ni x$. Find $M \in \Gamma$ such that $x, U \in M$. Using the absoluteness of the formula (and its subformula)

$$\exists f \in \mathcal{C}(K) \quad (f(x) = 0\ \wedge\ \forall y \in U\ f(y) = 1), \tag{$*$}$$

for every $M \subset N \in \Gamma$ there is $f \in \mathcal{C}(K) \cap N$ such that $f(x) = 0$ and $f(y) = 1$ for $y \notin U$. Find a point $d \in \overline{D \cap N}$ such that $q^N(d) = q^N(x)$. Then $r_N(x) = d \in U$ (otherwise $f(r_N(x)) = 1$ which would be a contradiction because $f(r_N(x)) = f(x)$). Consequently, $x = \lim_{M\in\Gamma} r_M(x)$.

To verify (ii) from the definition of a retractional skeleton, fix $M \subset N$ from Γ. Then it is obvious that $r_N(r_M(x)) = r_M(x)$. Let us take a function $g \in \overline{\mathcal{C}(K) \cap M} \subset \overline{\mathcal{C}(K) \cap N}$ and a point $x \in K$. Then, by the argument above,

$$g(r_M(x)) = g(x) = g(r_N(x)) = g(r_M(r_N(x))).$$

As $\mathcal{C}(K) \cap M$ separates points of $\overline{D \cap M}$, $r_M(x) = r_M(r_N(x)$ holds as well.

Finally, take $M_0 \subset M_1 \subset \ldots$ in Γ, $M = \bigcup_{n\in\omega} M_n$ and $x \in K$. Fix $f \in \mathcal{C}(K) \cap M$ and find $n \in \mathbb{N}$ such that $f \in \mathcal{C}(K) \cap M_n$. It follows that for every $k \geq n$, $f(r_M(x)) = f(x) = f(r_{M_k}(x))$. Consequently, $\lim_{n\to\infty} f(r_{M_n}(x)) = f(r_M(x))$ for every $f \in \mathcal{C}(K) \cap M$; hence, for every $f \in \overline{\mathcal{C}(K) \cap M}$. By Lemma 2.4.6 and the fact that $\overline{D \cap M}$ is homeomorphic with $K/_M$, we may identify $\overline{\mathcal{C}(K) \cap M}$ with $\mathcal{C}(\overline{D \cap M})$ and $\lim_{n\to\infty} f(r_{M_n}(x)) = f(r_M(x))$ for every $f \in \mathcal{C}(\overline{D \cap M})$. It follows that $\lim_{n\to\infty} r_{M_n}(x) = r_M(x)$.

We have verified that $\mathfrak{s} = \{r_M\}_{M\in\Gamma}$ is a retractional skeleton. Obviously, $D(\mathfrak{s}) = \bigcup_{M\in\Gamma} \overline{D \cap M} \supset D$ and $D = D(\mathfrak{s})$ if D is a countably closed set. □

We end this section with two lemmas. These statements are similar to [23, Lemma 2.8] and [27, Lemma 6]. In proofs we use the method of elementary submodels (namely Theorem 2.4.8).

Lemma 2.4.9. *Let K be a compact space and $F \subset K$ be a metrizable closed set. Put $L = K \setminus F \cup \{F\}$ endowed with the quotient topology induced by the mapping $Q : K \to L$ defined by*

$$Q(x) = \begin{cases} x & x \in K \setminus F \\ F & x \in F. \end{cases}$$

If D is induced by a retractional skeleton in L and $Q^{-1}(D)$ is dense in K, then $Q^{-1}(D)$ is induced by a retractional skeleton in K.

Proof. Let us fix a suitable elementary submodel M such that $Q, K, F \in M$. Notice that $Q^{-1}(D)$ is countably closed. Thus, by Theorem 2.4.8, it is enough to verify that q_K^M is one-to-one on $\overline{Q^{-1}(D) \cap M}$. Fix two distinct points $x, y \in \overline{Q^{-1}(D) \cap M}$. Then (in the last inclusion we use Lemma 2.4.4)

$$Q(x), Q(y) \in Q(\overline{Q^{-1}(D) \cap M}) \subset \overline{Q(Q^{-1}(D) \cap M)} \subset \overline{D \cap Q(M)} \subset \overline{D \cap M}.$$

We distinguish two cases. If $Q(x) \neq Q(y)$, then by the assumption and Theorem 2.4.8, there exists a function $f \in \mathcal{C}(L) \cap M$ such that $f(Q(x)) \neq f(Q(y))$. Using the elementarity of M, $f \circ Q \in \mathcal{C}(K) \cap M$. Thus, the mapping $f \circ Q$ is the witness of the fact that $q_K^M(x) \neq q_K^M(y)$.

If $Q(x) = Q(y)$, then $x, y \in F$. By the elementarity of M, there is a countable set $S \in M$, $S \subset \mathcal{C}(K)$ such that S separates the points of F. By Lemma 2.4.4, $S \subset M$. Consequently, there exists a function $f \in C(K) \cap M$ such that $f(x) \neq f(y)$; hence, $q_K^M(x) \neq q_K^M(y)$. □

Lemma 2.4.10. *Let X be a Banach space and Y its subspace such that X/Y is separable. Let i denote the injection of Y into X and i^* its adjoint mapping. Let K be a w^*-compact subset of X^* and let $D \subset i^*(K)$ be a set induced by a retractional skeleton in $i^*(K)$. If $(i^*)^{-1}(D) \cap K$ is dense in K, then the set $(i^*)^{-1}(D) \cap K$ is induced by a retractional skeleton in K.*

Proof. Let us denote by Q the canonical quotient mapping from X onto X/Y. Then there is a countable set $S \subset X$ such that $Q(S)$ is dense in X/Y. By Theorem 2.4.8, it is sufficient to prove that for every suitable elementary submodel M such that $S, Y, K, X, i^* \in M$, the mapping q_K^M is one-to-one on $\overline{(i^*)^{-1}(D) \cap M} \cap K$. Fix two distinct points $x^*, y^* \in \overline{(i^*)^{-1}(D) \cap M} \cap K$. Then (in the last inclusion we use Lemma 2.4.4)

$$i^*(x), i^*(y) \in i^*(\overline{(i^*)^{-1}(D) \cap M}) \subset \overline{i^*((i^*)^{-1}(D) \cap M)} \subset \overline{D \cap i^*(M)} \subset \overline{D \cap M}.$$

We distinguish two cases. If $i^*(x^*) \neq i^*(y^*)$, then by the assumption and Theorem 2.4.8, there exists a function $f \in \mathcal{C}(i^*(K)) \cap M$ such that $f(i^*(x^*)) \neq f(i^*(y^*))$. Using the elementarity of M, $f \circ i^* \in \mathcal{C}(K) \cap M$. Thus, the mapping $f \circ i^*$ is the witness of the fact that $q_K^M(x^*) \neq q_K^M(y^*)$.

If $i^*(x^*) = i^*(y^*)$, then $0 \neq x^* - y^* \in Y^\perp$. Using the fact that $Q(S)$ is dense in X/Y, there exists a point $z \in S \subset M$ such that $x^* - y^*(z) \neq 0$. Thus, the point $z \restriction_K \in \mathcal{C}(K) \cap M$ is the witness of the fact that $q_K^M(x^*) \neq q_K^M(y^*)$. □

2.5 Auxiliary results

First, we give statements required in the proof of Theorem 2.2.1. We begin with a lemma which is well known.

Lemma 2.5.1. *Let $C \subset (X^*, w^*)$ be a countable compact. Then $\overline{\text{conv}}^{w^*} C$ is metrizable.*

Proof. As C is countable compact, it is metrizable. Hence, $P(C)$ is metrizable. Now we observe (see [26, Lemma 4]) that $\overline{\text{conv}}^{w^*}C$ is a continuous image of a metrizable compact space $P(C)$; thus, it is metrizable as well (see [13, Theorem 4.4.15]). $\square$

The following lemma and theorem were proved in the context of Valdivia compact spaces in [26, Proposition 3 and Theorem 1]. In [20] there are given proofs which work even for the setting of spaces from the class $\mathcal{R}_0$. Proofs contain some arguments which are not necessary, so we give simplified ones.

Lemma 2.5.2 (cf. [20, Lemma 5.52])**.** *Let X be a Banach space such that $[0,\omega_1]$ embeds into (B_{X^*}, w^*). Let us have a point $e \in X$ and $\varepsilon > 0$. Then there exists a w^*-compact and convex set $L \subset \{x^* \in X^* : x^*(e) = 0\} \cap \varepsilon B_{X^*}$ that does not have a retractional skeleton.*

Proof. There exists $\{f_\alpha\}_{0\leq\alpha\leq\omega_1} \subset (B_{X^*}, w^*)$ which is homeomorphic to $[0,\omega_1]$. We may without loss of generality assume that $f_{\omega_1} = 0$ and $\{f_\alpha\}_{0\leq\alpha\leq\omega_1} \subset (\varepsilon B_{X^*}, w^*)$. Moreover, fix a linearly independent set of points $\{e_k\}_{k=1}^\infty \subset X$ and functionals $\{g_k\}_{k=1}^\infty \subset \varepsilon B_{X^*}$ such that $\|g_k\| \leq \frac{1}{k}$, $g_k(e) = 0$ and $g_k(e_l) \neq 0$ if and only if $k = l$ (such a set of points and functionals exists - it is enough to create a biorthogonal system using the "Gram-Schmidt orthogonalization process", see [20, Lemma 1.21]). Since $f_\alpha(e) \to 0$ and, for all $k \in \mathbb{N}$, $f_\alpha(e_k) \to 0$ as $\alpha \to \omega_1$, there exists an $\alpha_0 < \omega_1$ such that $f_\alpha(e) = 0$ and $f_\alpha(e_k) = 0$ for all $k \in \mathbb{N}$, $\alpha \in [\alpha_0, \omega_1]$. Fix $\beta \in [\alpha_0, \omega_1)$. By Lemma 2.5.1, the set $L_\beta = \overline{\text{conv}}^{w^*}(\{g_k\}_{k=1}^\infty \cup \{f_\alpha\}_{\alpha\in[\alpha_0,\beta)})$ is metrizable and thus there exists $\gamma_\beta \in (\beta, \omega_1)$ such that $f_\zeta \notin L_\beta$ whenever $\gamma_\beta \leq \zeta < \omega_1$ (otherwise some uncountable set $\{f_\zeta\} \subset L_\beta$ would contain a sequence converging to f_{ω_1}, which is a contradiction). By the separation theorem, choose $y_\beta \in X$ such that $\sup y_\beta(L_\beta) < f_{\gamma_\beta}(y_\beta)$. Since $\lim_{\alpha\to\omega_1} f_\alpha(y_\beta) = 0$, $f_\alpha(y_\beta) = 0$ for α large enough. Based on the above, we inductively find an increasing set $\{i(\beta)\}_{\beta<\omega_1} \subset [\alpha_0, \omega_1)$ such that $\sup_\beta i(\beta) = \omega_1$ and for every non limit ordinal $\beta < \omega_1$ there exists $y_\beta \in X$ satisfying $0 \leq \sup y_\beta(L_{i(\beta)}) < f_{i(\beta)}(y_\beta)$ and $f_{i(\gamma)}(y_\beta) = 0$ for all $\gamma > \beta$. Let $L = \overline{\text{conv}}^{w^*}(\{g_k\}_{k=1}^\infty \cup \{f_{i(\beta)}\}_{\beta<\omega_1})$. Then, for every non limit ordinal $\beta < \omega_1$, $f_{i(\beta)}$ is w^*-exposed by y_β in L; hence, it is a w^*-G_δ point of L. Similarly, for all $k \in \mathbb{N}$, g_k is w^*-exposed by e_k in L and all the functionals g_k are w^*-G_δ points of L. By Corollary 2.3.7, L does not have a retractional skeleton. $\square$

Now we are ready to prove the following theorem, which will be used in the proof of Theorem 2.2.1.

Theorem 2.5.3 (cf. [20, Theorem 5.51])**.** *Let $\langle X, \|\cdot\|\rangle$ be a Banach space such that $[0,\omega_1]$ embeds into (B_{X^*}, w^*). Then there is, for any $\varepsilon \in (0,1)$, an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that $(1-\varepsilon)|\!|\!|\cdot|\!|\!| \leq \|\cdot\| \leq |\!|\!|\cdot|\!|\!|$ and $(B_{\langle X^*, |\!|\!|\cdot|\!|\!|\rangle}, w^*) \notin \mathcal{R}_0$.*

Proof. Let us take an arbitrary $e \in S_X$ and $\varepsilon \in (0,1)$. Then, by Lemma 2.5.2, there exists a w^*-compact and convex set $L \subset \ker(e) \cap \varepsilon B_{X^*}$ such that $L \notin \mathcal{R}_0$. Let us take an arbitrary $h \in S_{X^*}$ such that $h(e) = 1$. Then

$$B = \text{conv}\{(L+h) \cup (-L-h) \cup (1-\varepsilon)B_{X^*}\}$$

is a convex symmetric w^*-compact set such that $(1-\varepsilon)B_{X^*} \subset B \subset (1+\varepsilon)B_{X^*}$, so there is an equivalent norm $|\cdot|$ on X such that B is its dual unit ball. It remains to show that

B does not have a retractional skeleton (then we put $\|\!|\cdot\|\!| = (1+\varepsilon)|\cdot|$ and this finishes the proof). Observe that

$$L + h = \{f \in B : \ f(e) = 1\}.$$

Thus, $L+h$ is a w^*-closed w^*-G_δ subset of B and it does not have a retractional skeleton (because $L \notin \mathcal{R}_0$). By Theorem 2.3.9, $B \notin \mathcal{R}_0$. □

Now we give some preliminary results which will be used in the proof of Theorem 2.2.6. The following proposition is an analogue to [27, Proposition 1].

Proposition 2.5.4. *Let K be a compact space, G the set of all G_δ points of K. If $\overline{G}$ is not Corson, then there are points $a, b \in K$ such that $P(L) \notin \mathcal{R}_0$ where L is the quotient space made from K by identifying a and b.*

Proof. We use the same idea as in [27, Proposition 1]. If $\overline{G} \notin \mathcal{R}_0$, then we can take $a = b$ due to Lemma 2.3.13. Now suppose that $\overline{G}$ has a retractional skeleton. Let D be the unique set induced by a retractional skeleton in $\overline{G}$ (the set is unique by Corollary 2.3.4). Choose $a \in D$ a non-isolated point and $b \in \overline{G} \setminus D$ (such a point exists due to Theorem 2.3.11). Let L be the quotient space made from K by identifying a and b and let Q be the quotient mapping. Then $Q(\overline{G})$ does not have a retractional skeleton.

Indeed, in order to get a contradiction let $B \subset Q(\overline{G})$ be a set induced by a retractional skeleton in $Q(\overline{G})$. Choose in the space $\overline{G}$ open neighborhoods U and V of a and b respectively with $\overline{U} \cap \overline{V} = \emptyset$. Then $U' = Q(U \setminus \{a\})$ and $V' = Q(U \setminus \{a\})$ are disjoint open sets with $\overline{U'} \cap \overline{V'} = \{\{a, b\}\}$. By Lemma 2.3.6, $\{a, b\} \in B$. Lemma 2.4.9 shows that $(Q \restriction_{\overline{G}})^{-1}(B)$ is induced by a retractional skeleton in $\overline{G}$. By the uniqueness of D, $(Q \restriction_{\overline{G}})^{-1}(B) = D$. This is a contradiction, because $b \in (Q \restriction_{\overline{G}})^{-1}(B) \setminus D$.

Moreover, it is clear that $G \setminus \{a\}$ is dense in $\overline{G}$ and $Q(g)$ is a G_δ point in L for every $g \in G \setminus \{a\}$. Thus, $Q(\overline{G})$ is the closure of all the G_δ points in L. By Lemma 2.3.13, $P(L) \notin \mathcal{R}_0$. □

To deal with compact spaces without G_δ points we use again the same approach as in [27].

Proposition 2.5.5. *Let K be a compact space such that there are two disjoint homeomorphic closed nowhere dense sets $M, N \subset K$ such that $N \notin \mathcal{R}_0$. Then there is L, an at most two-to-one continuous image of K, such that $L \notin \mathcal{R}_0$. Moreover, if N has a dense set of (relatively) G_δ points, then $P(L) \notin \mathcal{R}_0$.*

Proof. We use the same idea as in [27, Proposition 2]. Let $h : M \to N$ be a homeomorphism and put $L = K \setminus M$ with the quotient topology defined by the mapping

$$\varphi(x) = \begin{cases} x & x \in K \setminus M \\ h(x) & x \in M. \end{cases}$$

There are disjoint open sets U', V' in K such that $U' \supset M$, $V' \supset N$ and $\overline{U'} \cap \overline{V'} = \emptyset$. Put $U = \varphi(U') \setminus N$ and $V = \varphi(V') \setminus N$. Then it follows from the definition of the quotient topology that U and V are disjoint open sets in L and it is easy to see that $\overline{U} \cap \overline{V} = N$. Let us assume that $L \in \mathcal{R}_0$. Then, by Theorem 2.3.9(ii), N has a retractional skeleton, which is a contradiction.

Finally, let us assume that N has a dense set of (relatively) G_δ points and $P(L) \in \mathcal{R}_0$. Copying word by word the arguments from [27, Proposition 2], we observe that $P(\overline{W})$ is of the form $\bigcap_{n\in\mathbb{N}} \overline{G_n}$ with G_n open in $P(L)$ whenever $W \subset L$ is open and that $P(N) = P(\overline{U}) \cap P(\overline{V})$. By Theorem 2.3.9(ii), $P(N)$ has a retractional skeleton. By Proposition 2.3.15, N has a retractional skeleton, which is a contradiction. □

The following corollary is a generalization of [27, Corollary 1]. The proof can be done just by copying word by word the arguments from [27], using Example 2.3.8 and Proposition 2.5.5 instead of [23, Example 3.4] and [27, Proposition 2].

Corollary 2.5.6. *Let K be a compact space which contains four pairwise disjoint nowhere dense homeomorphic copies of the ordinal segment $[0, \omega_1]$. Then there is L, at most four-to-one continuous image of K, such that $P(L) \notin \mathcal{R}_0$*

In the proof of Theorem 2.2.5, the following generalization of [27, Proposition 3] will be required.

Proposition 2.5.7. *Let K be a Corson compact space without property (M). Then there is a hyperplane $Y \subset \mathcal{C}(K)$ such that $B_{Y^*} \notin \mathcal{R}_0$.*

Proof. In [27, Proposition 3] it is observed that under the assumptions above, the following holds.

$P(K)$ has a dense set of G_δ points, and a dense Σ-subset A. This set A contains all the Dirac measures and there is a continuous measure $\mu \in P(K) \setminus A$. Take an arbitrary point k from the support of the measure μ and put

$$Y = \{f \in \mathcal{C}(K) : \ f(k) = \mu(f)\}.$$

Denote by i the inclusion of Y into $\mathcal{C}(K)$. Then $i^*(P(K))$ is a w^*-closed w^*-G_δ subset of B_{Y^*}.

Now, in [27, Proposition 3] it is proved that B_{Y^*} is not Valdivia. Let us see that it is not even in the class $\mathcal{R}_0$. For contradiction suppose $B_{Y^*} \in \mathcal{R}_0$.

Then, by Theorem 2.3.9, $i^*(P(K)) \in \mathcal{R}_0$. Let B be a set induced by a retractional skeleton in $i^*(P(K))$. It follows from [27, Lemma 7] that $C = (i^*)^{-1}(B) \cap P(K)$ is dense in $P(K)$. By Lemma 2.4.10, C is induced by a retractional skeleton in $P(K)$. As $P(K)$ has a dense set of G_δ points, $C = A$ by Corollary 2.3.4. But $\delta_k \in A = C$ and also $\mu \notin A = C$. This is a contradiction with $i^*(\delta_k) = i^*(\mu)$. □

Finally, we observe that continuous images of spaces from the class $\mathcal{R}_0$ belong to the class $\mathcal{G}\Omega$. The proof is again completely analogous to a similar result concerning Valdivia compacta [27, Proposition 4] (we only use Theorems 2.3.9 and 2.3.10 instead of [27, Lemma 5] and [24, Theorem 1]) and so it is omitted.

Proposition 2.5.8. *Lek K be a compact space which is a continuous image of a space from the class $\mathcal{R}_0$. Then K belongs to the class $\mathcal{G}\Omega$.*

2.6 Proofs of the main results and open problems

Without mentioning it any further, we will use two important results mentioned above. First, a Banach space is 1-Plichko if and only if it has a commutative 1-projectional skeleton. Next, a compact space is Valdivia if and only if it has a commutative retractional skeleton.

Proof of Theorem 2.2.1. The implication (i)⇒(ii) comes from [26, Theorem 1]. Obviously, (ii)⇒(iii) and (iii)⇒(iv). It follows from Proposition 2.3.15 that (ii)⇒(iii) holds. Finally, in order to prove (iii)⇒(i), let us assume that B_{X^*} is not a Corson compact. If $B_{X^*} \notin \mathcal{R}_0$, we are done. If $B_{X^*} \in \mathcal{R}_0$, we use Theorems 2.3.10 and 2.5.3 to find an equivalent norm $\|\!|\cdot|\!\|$ such that $(B_{\langle X^*,\|\!|\cdot|\!\|\rangle}, w^*) \notin \mathcal{R}_0$. □

Proof of Theorem 2.2.6. It follows from Proposition 2.3.15 that (i)⇒(ii)⇒(iii)⇒(iv) hold. Finally, let K be a non-Corson compact from the class $\mathcal{G}\Omega$. Let G be the set of G_δ points in K. If $\overline{G}$ is not Corson, we use Proposition 2.5.4 to get a two-to-one continuous image L of K such that $P(L) \notin \mathcal{R}_0$. If $\overline{G}$ is Corson, we copy word by word the arguments from the proof of (3)⇒(1) in [27, Theorem 2] to get a continuous image L_0 of K such that it contains four pairwise disjoint nowhere dense homeomorphic copies of $[0, \omega_1]$. Now it is enough to use Corollary 2.5.6. □

Proof of Theorem 2.2.5. The implication (i)⇒(ii) comes from [27, Theorem 1]. It follows from Proposition 2.3.15 that (ii)⇒(iii) holds. Suppose that (iii) holds. By Theorem 2.2.6, K is Corson. If it had not the property (M), we would get a contradiction with Proposition 2.5.7. □

Theorem 2.2.2 is just an immediate corollary of Theorem 2.2.6 and the well known fact that a continuous image of a Corson compact is again a Corson compact.

Finally, we state several open questions.

Given a compact space K and a dense subset $D \subset K$, let $\tau_p(D)$ denote the topology of the pointwise convergence on D (i.e. the weakest topology on $\mathcal{C}(K)$ such that $f \mapsto f(d)$ is continuous for every $d \in D$). Then D is a Σ-subset of K if and only if D is countably closed and $(\mathcal{C}(K), \tau_p(D))$ is primarily Lindelöf (see [25, Definition 1.2 and Theorem 2.1]).

Problem 1. Assume $D \subset K$ is a dense (resp. dense and countably closed) set in a compact space. Find a topological property (T) of $(\mathcal{C}(K), \tau_p(D))$ such that D is induced by a retractional skeleton in K if and only if $(\mathcal{C}(K), \tau_p(D))$ has the property (T).

For the motivation of the following question see Remark 2.3.16.

Question 1. Let K be a compact space. Consider the following conditions

(i) $\mathcal{C}(K)$ has a 1-projectional skeleton

(ii) There is a convex symmetric set induced by a retractional skeleton in $(B_{\mathcal{C}(K)^*}, w^*)$

(iii) There is a convex symmetric set induced by a retractional skeleton in $P(K)$

Is it true that (iii)⇒(ii) (resp. (ii)⇒(i), resp. (iii)⇒(i))?

Remark 2.6.1. During the review process of this paper, Question 1 has been answered in positive in [6, Theorem 4.1].

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3. Simultaneous projectional skeletons

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Abstract: We prove the existence of a simultaneous projectional skeleton for certain subspaces of $C(K)$ spaces. This generalizes a result on simultaneous projectional resolutions of identity proved by M. Valdivia. We collect some consequences of this result. In particular we give a new characterization of Asplund spaces using the notion of projectional skeleton.

3.1 Introduction

Systems of bounded linear projections on Banach spaces are an important tool for the study of the structure of nonseparable Banach spaces. They enable us to transfer properties from smaller (separable) spaces to larger ones.

One of the important concepts of such a system is a *projectional resolution of the identity* (PRI, for short); see, e.g. [20] and [16] for a definition and results on constructing a PRI in various classes of spaces.

However, an even better knowledge of the Banach space is provided by a *projectional skeleton*. The class of spaces with a projectional skeleton was introduced by W. Kubiś in [33]. Spaces with a 1-projectional skeleton not only have a PRI, but they form a $\mathcal{P}$-class; see, e.g., [20, Definition 3.45] and [22, Theorem 17.6]. Consequently, an inductive argument works well when "putting smaller pieces from PRI together" and we may prove that those spaces inherit certain structure from separable spaces. For example, every space with a projectional skeleton has a strong Markushevich basis and an LUR renorming; see, e.g., [20, Theorem 5.1] and [11, Theorem VII.1.8]. Moreover, it is possible to characterize some classes of other spaces (e.g. WLD, Plichko and Asplund spaces) in terms of a projectional skeleton; see [33] for more details.

One of the largest class of spaces admitting a PRI is related to Valdivia compact spaces.

Definition 3.1.1. Let Γ be a set. We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma;\ |\{\gamma \in \Gamma;\ x(\gamma) \neq 0\}| \leq \omega\}$. Given a compact K, $A \subset K$ is called a Σ-*subset* of K if there is a homeomorphic embedding $h : K \to [0,1]^\kappa$ such that $A = h^{-1}[\Sigma(\kappa)]$. A compact space K is said to be *Valdivia compact* if there exists a dense Σ-subset of K.

The following result is contained in [43]. Let us just note that there is proved even something more in [43], but we will be interested only in the following statement.

Theorem A. (See [43, Theorem 1].) *Let K be a Valdivia compact space with a dense Σ-subset A. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of $\tau_p(A)$-closed subspaces of $\mathcal{C}(K)$. If $\operatorname{dens}\mathcal{C}(K) = \mu$, then there is a PRI $\{P_\alpha;\ \omega \leq \alpha \leq \mu\}$ in $\mathcal{C}(K)$ such that $P_\alpha(Y_n) \subset Y_n$, $n \in \mathbb{N}, \omega \leq \alpha \leq \mu$.*

The system of projections as above is called “simultaneous projectional resolution of the identity” in [43]. In the present paper we generalize Theorem A using the notion of a skeleton.

Let us have a partially ordered set $(\Gamma, <)$. We say that it is *up-directed*, if for any $s, t \in \Gamma$, there is $u \in \Gamma$ such that $u \geq s$, $u \geq t$. We say that Γ is *σ-complete*, if for every increasing sequence $(s_n)_{n\in\mathbb{N}}$ in Γ, $\sup_{n\in\mathbb{N}} s_n$ exists.

Definition 3.1.2. A *projectional skeleton* in a Banach space X is a family of projections $\{P_s\}_{s\in\Gamma}$, indexed by an up-directed σ-complete partially ordered set Γ, such that:

(i) Each P_sX is separable.

(ii) $X = \bigcup_{s\in\Gamma} P_sX$.

(iii) $s \leq t \Rightarrow P_s = P_s \circ P_t = P_t \circ P_s$.

(iv) Given $s_1 < s_2 < \cdots$ in Γ and $t = \sup_{n\in\mathbb{N}} s_n$, $P_tX = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}X}$.

Given $r \geq 1$, we say that $\{P_s\}_{s\in\Gamma}$ is an *r-projectional skeleton* if it is a projectional skeleton such that $\|P_s\| \leq r$ for every $s \in \Gamma$.

We say that $\{P_s\}_{s\in\Gamma}$ is a *commutative projectional skeleton* if $P_s \circ P_t = P_t \circ P_s$ for any $s, t \in \Gamma$.

Remark 3.1.3. Having an r-projectional skeleton $\{P_s\}_{s\in\Gamma}$, an increasing sequence of indices $s_0 < s_1 < \cdots$ in Γ and $t = \sup_{n\in\mathbb{N}} s_n$, it is easy to verify that $P_t(x) = \lim_n P_{s_n}(x)$ for every $x \in X$; see [33, Lemma 10]. This statement holds even for an arbitrary projectional skeleton, not necessarily uniformly bounded, but this will not be needed any further. Recall that due to [33], we may always assume that every projectional skeleton is an r-projectional skeleton for some $r \geq 1$ (just by passing to a suitable cofinal subset of Γ).

In [34] there was introduced a class of compact spaces with a retractional skeleton and it was observed in [34], [33] and [5] that those spaces are more general than Valdivia compact spaces, but they share a lot of properties with them.

Definition 3.1.4. A *retractional skeleton* in a compact space K is a family of retractions $\mathfrak{s} = \{r_s\}_{s\in\Gamma}$, indexed by an up-directed σ-complete partially ordered set Γ, such that:

(i) $r_s[K]$ is metrizable for each $s \in \Gamma$.

(ii) For every $x \in K$, $x = \lim_{s\in\Gamma} r_s(x)$.

(iii) $s \leq t \Rightarrow r_s = r_s \circ r_t = r_t \circ r_s$.

(iv) Given $s_1 < s_2 < \cdots$ in Γ and $t = \sup_{n\in\mathbb{N}} s_n$, $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in K$.

We say that $\{r_s\}_{s\in\Gamma}$ is a *commutative retractional skeleton* if $r_s \circ r_t = r_t \circ r_s$ for any $s, t \in \Gamma$.
We say that $D(\mathfrak{s}) = \bigcup_{s\in\Gamma} r_s[K]$ is the *set induced by the retractional skeleton* $\{r_s\}_{s\in\Gamma}$.

By $\mathcal{R}_0$ we denote the class of all compacta which have a retractional skeleton.

The class of Banach spaces with a projectional skeleton (resp. class of compact spaces with a retractional skeleton) is closely related to the concept of Plichko spaces (resp. Valdivia compacta). By [33, Theorem 27], Plichko spaces are exactly spaces with a commutative projectional skeleton. By [34, Theorem 6.1], Valdivia compact spaces are exactly compact spaces with a commutative retractional skeleton. An example of a compact space with a retractional skeleton which is not Valdivia is $[0, \omega_2]$ (see [34, Example 6.4]). An example of a space with a 1-projectional skeleton which is not Plichko is $\mathcal{C}([0, \omega_2])$ (see [30, Theorem 1]).

The above mentioned generalization of the result from [43] is the following.

Theorem 3.1.5. *Let us have $K \in \mathcal{R}_0$ with $D \subset K$ induced by a retractional skeleton in K and countably many $\tau_p(D)$-closed subspaces $(Y_n)_{n\in\mathbb{N}}$ of $\mathcal{C}(K)$. Then there exists a 1-projectional skeleton $\{P_s\}_{s\in\Gamma}$ in $\mathcal{C}(K)$ such that $P_s(Y_n) \subset Y_n$, $n \in \mathbb{N}$, $s \in \Gamma$.*

In particular, for all $n \in \mathbb{N}$, $\{P_s \restriction_{Y_n}\}_{s\in\Gamma}$ is a 1-projectional skeleton in Y_n.

The statement with PRI, instead of a projectional skeleton, follows immediately from the proof of [22, Theorem 17.6]. Hence, this really is a generalization of Theorem A.

Moreover, we use the existence of a "simultaneous skeleton" to prove other statements concerning the structure of spaces with a projectional (resp. retractional) skeleton. We study a relationship between projectional and retractional skeletons. In particular, we give an answer to [5, Question 1]. We also study subspaces (resp. continuous images) of spaces with a projectional (resp. retractional) skeleton.

Using the above, we give the following characterization of Asplund spaces. A Banach space X is Asplund if and only if the dual space has a 1-projectional skeleton after every renorming of X if and only if the bidual unit ball has a retractional skeleton after every renorming of X. In particular, this gives an answer to [29, Question 1]. Let us just note that the answer has already been known to O. Kalenda before and it has been contained in one of his unpublished remarks.

The structure of the paper is as follows: first, we prove Theorem 3.1.5. Next, we use this result to study the relationship between projectional and retractional skeletons. Then we characterize those subspaces (resp. continuous images) of a space with a projectional (resp. retractional) skeleton, where a "natural projectional subskeleton" exists. Next, we give a new characterization of Asplund spaces. Finally, we show some more applications of the given results.

3.2 Preliminaries

We denote by ω the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$.

All topological spaces are assumed to be Hausdorff. Let T be a topological space. The closure of a set A we denote by $\overline{A}$. We say that $A \subset T$ is *countably closed* if $\overline{C} \subset A$ for every countable $C \subset A$. A topological space T is a *Fréchet-Urysohn space* if for every $A \subset T$ and every $x \in \overline{A}$ there is a sequence $x_n \in A$ with $x_n \to x$. We say that T is *countably compact* if every countable open cover of T has a finite subcover. If T is completely regular, we denote by βT the Stone-Čech compactification of T.

Let K be a compact space. By $\mathcal{C}(K)$ we denote the space of continuous functions on K. Given a dense set $D \subset K$, we denote by $\tau_p(D)$ the topology of the pointwise convergence on D; i.e., the weakest topology on $C(K)$ such that $\mathcal{C}(K) \ni f \mapsto f(d)$ is continuous for every $d \in D$. $P(K)$ stands for the space of probability measures with the w^*–topology; the w^*–topology is taken from the representation of $P(K)$ as a compact subset of $(\mathcal{C}(K)^*, w^*)$.

We shall consider Banach spaces over the field of real numbers (but many results hold for complex spaces as well). If X is a Banach space and $A \subset X$, we denote by $\operatorname{conv} A$ the convex hull of A. We write $A^\perp = \{x^* \in X^*;\ (\forall a \in A)\ x^*(a) = 0\}$. B_X is the unit ball in X; i.e., the set $\{x \in X;\ \|x\| \leq 1\}$. X^* stands for the (continuous) dual space of X. For a set $A \subset X^*$ we denote by $\overline{A}^{w^*}$ the *weak** closure of A. Given a set $D \subset X^*$ we denote by $\sigma(X, D)$ the weakest topology on X such that each functional from D is continuous.

A set $D \subset X^*$ is *r-norming* if

$$\|x\| \leq r. \sup\{|x^*(x)|;\ x^* \in D \cap B_{X^*}\},\ x \in X.$$

We say that a set $D \subset X^*$ is norming if it is r-norming for some $r \geq 1$.

Recall that a Banach space X is called *Plichko* (resp. 1-*Plichko*) if there are a linearly dense set $M \subset X$ and a norming (resp. 1-norming) set $D \subset X^*$ such that for every $x^* \in D$ the set $\{m \in M;\ x^*(m) \neq 0\}$ is countable.

Definition 3.2.1. Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X and let $D(\mathfrak{s}) = \bigcup_{s\in\Gamma} P_s^*[X^*]$. Then we say that $D(\mathfrak{s})$ *is induced by a projectional skeleton.*

Some properties of a set induced by a retractional skeleton in $K \in \mathcal{R}_0$ are similar to the properties of a "dense Σ-subset" in a Valdivia compact K. Bellow we collect some of the most important statements. They will be needed in what follows.

Lemma 3.2.2. *Assume D is induced by a retractional skeleton in K. Then:*

(i) *D is dense and countably closed in K.*

(ii) *$K = \beta D$ and D is a Fréchet-Urysohn space.*

(iii) *If $F \subset K$ is closed and $F \cap D$ is dense in F, then $F \cap D$ is induced by a retractional skeleton in F.*

(iv) *If $G \subset K$ is a G_δ set, then $G \cap D$ is dense in G. In particular, if $G \subset K$ is a closed G_δ set, then G has a retractional skeleton.*

(v) *If $E \subset D$ is a countably closed and dense set in K, then $E = D$.*

Proof. The statements (i) and (ii) are proved in [33], (iii) and (iv) are proved in [5]. In order to prove (v), we follow the lines of [28, Lemma 1.7]. Fix $x \in D$. Then $x \in \overline{E}$ and using the fact that D is Fréchet-Urysohn, there exists a sequence $x_n \in E$ with $x_n \to x$. As E is countably closed, $x \in E$. □

For other statements concerning similarities between Valdivia compacta and spaces with a retractional skeleton we refer to [5] where more details may be found.

The last statement of this section is the following lemma which we will need later.

Lemma 3.2.3. *Let $\varphi : D \to B$ be a continuous mapping, where D is a countably compact space and B is Fréchet-Urysohn. Then φ is closed.*

Proof. Let $A \subset D$ be a closed set. Then A is countably compact; hence, $\varphi(A)$ is countably compact. Fix $x \in \overline{\varphi(A)}$. As B is Fréchet-Urysohn, there exists a sequence $(x_n)_{n=1}^\infty \subset \varphi(A)$ with $x_n \to x$. Since $\varphi(A)$ is countably compact, there exists a subnet (x_ν) of the sequence $(x_n)_{n=1}^\infty$ such that $x_\nu \to y \in \varphi(A)$. It follows that $y = x \in \varphi(A)$. □

3.3 Simultaneous projectional skeletons

In this section we prove Theorem 3.1.5.

Assume D is induced by a retractional skeleton $\{r_s\}_{s\in\Gamma}$ in K. Let us define, for $s \in \Gamma$, the projection P_s by $P_s(f) = f \circ r_s$, $f \in \mathcal{C}(K)$. It is known that $\{P_s\}_{s\in\Gamma}$ is a 1-projectional skeleton in $\mathcal{C}(K)$. Now, let us fix a set $\Gamma' \subset \Gamma$. We would like to know if $\{P_s\}_{s\in\Gamma'}$ is still a 1-projectional skeleton. It is easily seen that a sufficient condition for this is that Γ' be unbounded and σ-closed in Γ in the sense of the following definition.

Definition 3.3.1. Let Γ be an up-directed σ-complete partially ordered set and $\Gamma' \subset \Gamma$. We say that Γ' is:

(i) *unbounded* (in Γ), if for every $s \in \Gamma$ there exists $t \in \Gamma'$ such that $s \leq t$;

(ii) *σ-closed* (in Γ), if for every increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ in Γ', $\sup s_n \in \Gamma'$.

Next, it is easy to check that whenever $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a sequence of unbounded and σ-closed sets in Γ, then $\bigcap_{n\in\mathbb{N}} \Gamma_n$ is again unbounded and σ-closed in Γ.

Let us fix a subspace Y of $\mathcal{C}(K)$. In order to see that there is a "simultaneous projectional skeleton for $\mathcal{C}(K)$ and Y", it is enough to find an unbounded and σ-closed set $\Gamma' \subset \Gamma$ such that $P_s(Y) \subset Y$ for every $s \in \Gamma'$. Then obviously $\{P_s\restriction_Y\}_{s\in\Gamma'}$ is 1-projectional skeleton in Y and $\{P_s\}_{s\in\Gamma'}$ is 1-projectional skeleton in $\mathcal{C}(K)$.

If we were able to find such an unbounded and σ-closed set $\Gamma' \subset \Gamma$ for every $\tau_p(D)$-closed subspace of $\mathcal{C}(K)$, then Theorem 3.1.5 would easily follow using the fact that we may intersect countably many unbounded and σ-closed sets as mentioned above.

This is done in the following proposition. The proof is quite technical and its idea comes from [43], where a similar statement concerning PRI is proved. In the proof we do not need Y to be a subspace, so we formulate it in a more general way.

Proposition 3.3.2. *Let K be a compact space with a retractional skeleton $\mathfrak{s} = \{r_s\}_{s\in\Gamma}$ and put $D = D(\mathfrak{s})$. Let Y be a $\tau_p(D)$-closed subset of $\mathcal{C}(K)$. Then there exists an unbounded and σ-closed set $\Gamma' \subset \Gamma$ such that, for every $t \in \Gamma'$ and $f \in Y$ we have $f \circ r_t \in Y$.*

Proof. In the proof we denote by $\mathcal{O}$ the set of all the open intervals in $\mathbb{R}$ with rational endpoints. If $K_1, \ldots, K_n$ are subsets of K and $o_1, \ldots, o_n \in \mathcal{O}$, we put

$$T(K_1, K_2, \ldots, K_n; o_1, o_2, \ldots, o_n) = \{f \in \mathcal{C}(K);\ f(K_i) \subset o_i \text{ for any } i = 1, 2, \ldots, n\}.$$

Let us define, for every $s \in \Gamma$, the projection $P_s : \mathcal{C}(K) \to \mathcal{C}(K)$ by $P_s(f) = f \circ r_s$, $f \in \mathcal{C}(K)$. By [33, Proposition 28], $\{P_s\}_{s\in\Gamma}$ is a 1-projectional skeleton in $\mathcal{C}(K)$. Put

$$\Gamma' = \{s \in \Gamma;\ P_s(Y) \subset Y\}.$$

Using Remark 3.1.3, it is easy to verify that Γ' is σ-closed set. In order to show that it is unbounded, let us fix some $s \in \Gamma$ and put $s_1 = s$. We inductively define increasing sequences $(s_n)_{n\in\mathbb{N}}$ in Γ and $(\mathcal{U}_n)_{n\in\mathbb{N}}$ in the following way.

Whenever $s_n \in \Gamma$ is given, let $\mathcal{U}_n$ be a countable basis of the topology on $r_{s_n}[K]$. For all $k \in \mathbb{N}$, $U_{j_1}, \ldots, U_{j_k} \in \mathcal{U}_n$, $o_{m_1}, \ldots, o_{m_k} \in \mathcal{O}$ we fix, if it exists, a set $\{x_1, \ldots, x_k\} \subset D$ such that

$$T(r_{s_n}^{-1}(U_{j_1}), r_{s_n}^{-1}(U_{j_2}), \ldots, r_{s_n}^{-1}(U_{j_k}); o_{m_1}, \ldots, o_{m_k}) \;\subset\; T(\{x_1\}, \{x_2\}, \ldots, \{x_k\}; o_{m_1}, \ldots, o_{m_k}),$$

and that the latter set is a subset of $\mathcal{C}(K) \setminus Y$.
Now, we find $s_{n+1} > s_n$ such that $r_{s_{n+1}}[K]$ contains all the points $\{x_1, \ldots, x_k\}$ corresponding to all

$$k \in \mathbb{N}, \quad U_{j_1}, \ldots, U_{j_k} \in \mathcal{U}_n, \quad o_{m_1}, \ldots, o_{m_k} \in \mathcal{O}.$$

We define $t = \sup s_n$. Now, it remains to show that $P_t(Y) \subset Y$. Arguing by contradiction, let us assume that there exists an $f \in Y$ such that $P_t(f) \notin Y$. Then there are $k \in \mathbb{N}$, $z_1, \ldots, z_k \in D$ and $o_1, \ldots, o_k \in \mathcal{O}$ with

$$P_t(f) \in T(\{z_1\}, \{z_2\}, \ldots, \{z_k\}; o_1, o_2, \ldots, o_k) \subset \mathcal{C}(K) \setminus Y.$$

Now, fix $\varepsilon > 0$ such that, for every $i \in \{1, \ldots, k\}$,

$$[P_t(f)(z_i) - 3\varepsilon, P_t(f)(z_i) + 3\varepsilon] \subset o_i.$$

Using the fact that $\{P_s\}_{s\in\Gamma}$ is a 1-projectional skeleton in $\mathcal{C}(K)$ and Remark 3.1.3, we find $n \in \mathbb{N}$ with $\|P_{s_n}(f) - P_t(f)\| < \varepsilon$. By the continuity of $P_{s_n}(f)\restriction_{r_{s_n}[K]} = f\restriction_{r_{s_n}[K]} \in \mathcal{C}(r_{s_n}[K])$, for every $i \in \{1, \ldots, k\}$, there is $U_i \in \mathcal{U}_n$ with $r_{s_n}(z_i) \in U_i$ and

$$f(U_i) \subset (P_{s_n}(f)(z_i) - \varepsilon, P_{s_n}(f)(z_i) + \varepsilon).$$

Thus, for every $x \in r_{s_n}^{-1}(U_i)$,

$$\begin{aligned}|P_t(f)(x) - P_t(f)(z_i)| \leq &|P_t(f)(x) - P_{s_n}(f)(x)| + |P_{s_n}(f)(x) - P_{s_n}(f)(z_i)| + \\ &|P_{s_n}(f)(z_i) - P_t(f)(z_i)| \leq 3\varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}P_t(f) \in T(r_{s_n}^{-1}(U_1), \ldots, r_{s_n}^{-1}(U_k); o_1, \ldots, o_k) &\subset T(\{z_1\}, \ldots, \{z_k\}; o_1, \ldots, o_k) \\ &\subset \mathcal{C}(K) \setminus Y.\end{aligned}$$

By the construction of the sequence s_n, there exists $\{x_1, \ldots, x_k\} \subset r_{s_{n+1}}[K]$ such that

$$\begin{aligned}T(r_{s_n}^{-1}(U_1), r_{s_n}^{-1}(U_2), \ldots, r_{s_n}^{-1}(U_k); o_1, \ldots, o_k) &\subset T(\{x_1\}, \{x_2\}, \ldots, \{x_k\}; o_1, \ldots, o_k) \\ &\subset \mathcal{C}(K) \setminus Y.\end{aligned}$$

Consequently,

$$f(x_i) = P_{s_{n+1}}(f)(x_i) = P_t(f)(x_i) \in o_i, \quad i = 1, \ldots, k,$$

and

$$f \in T(\{x_1\}, \{x_2\}, \ldots, \{x_k\}; o_1, \ldots, o_k) \subset \mathcal{C}(K) \setminus Y,$$

which is in contradiction with $f \in Y$. □

Let us recall that Theorem 3.1.5 easily follows from Proposition 3.3.2, as mentioned above. Moreover, we easily obtain the following more precise and more technical statement.

Corollary 3.3.3. *Assume D is induced by a retractional skeleton $\{r_s\}_{s\in\Gamma}$ in K. Let $\{P_s\}_{s\in\Gamma}$ be the 1-projectional skeleton in $\mathcal{C}(K)$ induced by $\{r_s\}_{s\in\Gamma}$; i.e., $P_s(f) = f \circ r_s$, $s \in \Gamma$, $f \in \mathcal{C}(K)$.*

Let $(F_n)_{n=1}^\infty$ be a sequence of closed subsets in K such that $F_n \cap D$ is dense in F_n for all $n \in \mathbb{N}$. Let $(Y_n)_{n=1}^\infty$ be a sequence of $\tau_p(D)$-closed subsets of $\mathcal{C}(K)$. Then there is an up-directed, unbounded and σ-closed set $\Gamma' \subset \Gamma$ such that, for all $n \in \mathbb{N}$, $r_s[F_n] \subset F_n$ and $P_s[Y_n] \subset Y_n$, for $s \in \Gamma'$.

In particular, for every $n \in \mathbb{N}$, $\{r_s\restriction_{F_n}\}_{s\in\Gamma'}$ is a retractional skeleton in F_n and $\{P_s\restriction_{Y_n}\}_{s\in\Gamma'}$ is a 1-projectional skeleton in Y_n if Y_n is a subspace.

Proof. Recall, that the intersection of countably many unbounded and σ-closed sets in Γ is again an unbounded and σ-closed set in Γ. Thus, it is enough to use Proposition 3.3.2 and the proof of [5, Lemma 3.5] to construct a sequence of unbounded and σ-closed sets $\{\Gamma_n\}_{n\in\mathbb{N}}$ such that, for every $s \in \Gamma_n$, $r_s[F_n] \subset F_n$ and $P_s[Y_n] \subset Y_n$, then setting $\Gamma' = \bigcap_{n\in\mathbb{N}} \Gamma_n$. □

3.4 Consequences of the existence of a simultaneous projectional skeleton

We use the existence of a "simultaneous projectional skeleton" to obtain certain new results concerning the structure of spaces with a projectional (resp. retractional) skeleton. Those are similar results to the ones from [28], concerning spaces with a commutative projectional (resp. retractional) skeleton; i.e., Plichko spaces and Valdivia compacta. Let us remark that Theorem 3.4.1 gives an answer to [5, Question 1].

The following two theorems give the relationship between 1-projectional and retractional skeletons.

Theorem 3.4.1. *Let K be a compact space. Then the following conditions are equivalent:*

(i) *$\mathcal{C}(K)$ has a 1-projectional skeleton.*

(ii) *There is a convex symmetric set induced by a retractional skeleton in $(B_{\mathcal{C}(K)^*}, w^*)$.*

(iii) *There is a convex set induced by a retractional skeleton in $(B_{\mathcal{C}(K)^*}, w^*)$.*

(iv) *There is a convex set induced by a retractional skeleton in $P(K)$.*

Theorem 3.4.2. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following conditions are equivalent:*

(i) X has a 1-projectional skeleton.

(ii) There is a convex symmetric set induced by a retractional skeleton in (B_{X^*}, w^*).

Moreover, if D is a 1-norming subspace of X^*, then:

(iii) *If D is a set induced by a 1-projectional skeleton in X, then $D \cap B_{X^*}$ is induced by a retractional skeleton in (B_{X^*}, w^*).*

(iv) *D is a subset of a set induced by a 1-projectional skeleton in X if and only if $D \cap B_{X^*}$ is a subset of a set induced by a retractional skeleton in (B_{X^*}, w^*).*

Let us note that by [31] there is a Banach space which has no PRI (and hence no 1-projectional skeleton) but whose dual unit ball is Valdivia (and hence it has a retractional skeleton). Thus, Theorem 3.4.2 does not hold without the assumption on convexity and symmetry in (ii). However, the answer to the following question seems to be unknown.

Question 2. Let $(X, \|\cdot\|)$ be a Banach space such that there is a convex set induced by a retractional skeleton in (B_{X^*}, w^*). Does X have a 1-projectional skeleton?

Proof of Theorem 3.4.1. Implication (i)⇒(ii) is proved in [5, Proposition 3.15], (ii)⇒(iii) is obvious and (iii)⇒(iv) follows from Lemma 3.2.2. Thus, it remains to prove (iv)⇒(i). Let us fix a convex set D induced by a retractional skeleton in $P(K)$. Let us consider the injection $I : \mathcal{C}(K) \to \mathcal{C}(P(K))$ defined by $I(f)(\mu) = \mu(f)$, $\mu \in P(K)$, $f \in \mathcal{C}(K)$. Notice, that $F \in \mathcal{C}(P(K))$ belongs to $I(\mathcal{C}(K))$ if and only if F is affine.

Indeed, obviously every $f \in I(\mathcal{C}(K))$ is affine. Moreover, if $F \in \mathcal{C}(P(K))$ is affine, we define $f \in \mathcal{C}(K)$ by $f(x) = F(\delta_x)$, where δ_x is the Dirac measure on K supported by $x \in K$. Then $I(f) = F$.

Moreover, $I(\mathcal{C}(K))$ is a $\tau_p(D)$-closed subset in $\mathcal{C}(P(K))$. Indeed, let $F_\nu \overset{\tau_p(D)}{\to} F$ where $F_\nu \in I(\mathcal{C}(K))$ and $F \in \mathcal{C}(P(K))$. Using the fact that D is convex and F_ν are affine, $F\restriction_D$ is affine. As D is dense in $P(K)$, F is affine and hence $F \in I(\mathcal{C}(K))$.

By Theorem 3.1.5, $I(\mathcal{C}(K))$ has a 1-projectional skeleton. As $I(\mathcal{C}(K))$ is isometric to $\mathcal{C}(K)$, $\mathcal{C}(K)$ has a 1-projectional skeleton as well. □

Let us recall the following well-known lemma. Its proof can be found for example in [25, Lemma 2.14].

Lemma 3.4.3. *Let X be a Banach space. Consider the isometry $I : X \to \mathcal{C}(B_{X^*}, w^*)$ defined by $I(x)(x^*) = x^*(x)$, $x \in X$, $x^* \in B_{X^*}$. Then $f \in \mathcal{C}(B_{X^*}, w^*)$ is an element of $I(X)$ if and only if f is affine and $f(0) = 0$.*

Moreover, if D is a dense convex symmetric set in B_{X^*}, then $I(X)$ is $\tau_p(D)$-closed subset in $\mathcal{C}(B_{X^*}, w^*)$.

Now we are ready to prove the second theorem.

Proof of Theorem 3.4.2. The implication (i)⇒(ii) and the assertion (iii) are proved in [5, Proposition 3.14]. The "only if" part in (iv) follows from (iii).

Let us continue by proving (ii)⇒(i). Fix a convex symmetric set D induced by a retractional skeleton in (B_{X^*}, w^*). Consider the isometry $I : X \to \mathcal{C}(B_{X^*}, w^*)$ defined by $I(x)(x^*) = x^*(x)$, $x \in X$, $x^* \in B_{X^*}$. By Lemma 3.4.3, $I(X)$ is a $\tau_p(D)$-closed subset in $\mathcal{C}(B_{X^*}, w^*)$. By Theorem 3.1.5, $I(X)$ has a 1-projectional skeleton. Thus, X has a 1-projectional skeleton and (ii)⇒(i) holds.

It remains to prove the "if" part of (iv). Let D be a subspace of X^* and $\mathfrak{s} = \{r_s\}_{s\in\Gamma}$ be a retractional skeleton in (B_{X^*}, w^*) with $D \cap B_{X^*} \subset D(\mathfrak{s})$. By Lemma 3.4.3, $I(X)$ is $\tau_p(D \cap B_{X^*})$-closed in $\mathcal{C}(B_{X^*}, w^*)$; hence, it is also $\tau_p(D(\mathfrak{s}))$-closed. By Proposition 3.3.2, we may without loss of generality assume that $\{P_s\restriction_{I(X)}\}_{s\in\Gamma}$ is a 1-projectional skeleton in $I(X)$, where $P_s(f)=f\circ r_s$, $s\in\Gamma$, $f\in\mathcal{C}(B_{X^*},w^*)$. Hence, $\mathfrak{s}_X=\{I^{-1}\circ P_s\circ I\}_{s\in\Gamma}$ is a 1-projectional skeleton in X. In order to verify that $D\subset D(\mathfrak{s}_X)$, fix $d\in D$ and $s\in\Gamma$ such that $r_s(d)=d$. Fix $x\in X$. Then

$$\begin{aligned}(I^{-1}\circ P_s\circ I)^*(d)(x)&=(d\circ I^{-1})\,(P_s(I(x)))=(d\circ I^{-1})\,(I(x)\circ r_s)\\&=(I(x)\circ r_s)(d)=I(x)(r_s(d))=I(x)(d)=d(x).\end{aligned}$$

Thus, $(I^{-1}\circ P_s\circ I)^*(d)=d$ and $d\in D(\mathfrak{s}_X)$. Hence, (iv) holds. □

The following two theorems give a finer idea on when a continuous image (resp. a subspace) of a space with a retractional (resp. projectional) skeleton has again a retractional (resp. projectional) skeleton.

Theorem 3.4.4. *Let $\varphi:K\to L$ be a continuous surjection between compact spaces. Assume D is induced by a retractional skeleton in K and put $B=\varphi(D)$. Then the following conditions are equivalent:*

(i) *B is induced by a retractional skeleton in L.*

(ii) *$\varphi^*C(L)=\{f\circ\varphi;\ f\in\mathcal{C}(L)\}$ is $\tau_p(D)$-closed in $\mathcal{C}(K)$.*

(iii) *$L=\beta B$ and B is a Fréchet-Urysohn space.*

(iv) *$L=\beta B$ and $\varphi\restriction_D$ is a quotient mapping of D onto B.*

Theorem 3.4.5. *Let $(X,\|\cdot\|)$ be a Banach space, $Y\subset X$ a subspace and $D\subset X^*$ a set induced by 1-projectional skeleton. Then the following conditions are equivalent:*

(i) *$D\restriction_Y$ is induced by a 1-projectional skeleton in Y.*

(ii) *$D\restriction_Y\cap B_{Y^*}$ is induced by a retractional skeleton in (B_{Y^*},w^*).*

(iii) *Y is $\sigma(X,D)$-closed in X.*

(iv) *$\beta((D\cap B_{X^*})\restriction_Y,w^*)=(B_{Y^*},w^*)$ and $((D\cap B_{X^*})\restriction_Y,w^*)$ is a Fréchet-Urysohn space.*

(v) *$\beta((D\cap B_{X^*})\restriction_Y,w^*)=(B_{Y^*},w^*)$ and $R:d\to d\restriction_Y$ is a quotient mapping of $(D\cap B_{X^*},w^*)$ onto its image in (B_{Y^*},w^*).*

Proof of Theorem 3.4.4. The assertion (i)⇒(iii) follows from Lemma 3.2.2. Assume (iii) is true. Then, using Lemma 3.2.3, $\varphi\restriction_D$ is closed, and therefore a quotient mapping. Hence, (iii)⇒(iv) is proved.

(iv)⇒(ii) Assume that (iv) holds and fix a net of functions f_ν from $\mathcal{C}(L)$ such that $f_\nu\circ\varphi\overset{\tau_p(D)}{\to}g\in\mathcal{C}(K)$. Now, define function f as $f(\varphi(d))=g(d)$, $d\in D$. As $\varphi\restriction_D$ is a quotient mapping, f is continuous and bounded (and defined on B). Hence, there is a continuous extension $\tilde{f}\in\mathcal{C}(L)$, $\tilde{f}\supset f$. As $\tilde{f}\circ\varphi=g$ on the dense set D, $\tilde{f}\circ\varphi=g$ on K and $g\in\varphi^*C(L)$. Thus, (iv)⇒(ii) is proved.

(ii)⇒(i) Assume that (ii) holds. Let $\mathfrak{s}=\{r_s\}_{s\in\Gamma}$ be a retractional skeleton in K such that $D(\mathfrak{s})=D$. By Proposition 3.3.2, we can without loss of generality assume that $\{P_s\restriction_{\varphi^*\mathcal{C}(L)}\}_{s\in\Gamma}$ is a 1-projectional skeleton in $\varphi^*\mathcal{C}(L)$, where $P_s(f)=f\circ r_s$, $f\in\mathcal{C}(K)$,

$s \in \Gamma$. In the rest of this proof we will denote by Y (resp. T_s) the space $\varphi^*\mathcal{C}(L)$ (resp. projections $P_s\restriction_{\varphi^*\mathcal{C}(L)}$). Recall that by [33], $\{T_s^*\restriction_{B_{Y^*}}\}_{s\in\Gamma}$ is a retractional skeleton in (B_{Y^*}, w^*); hence, $R = \bigcup_{s\in\Gamma} T_s^*(B_{Y^*})$ is induced by a retractional skeleton in (B_{Y^*}, w^*).

Observe, that L is homeomorphic to a subset of (B_{Y^*}, w^*). Indeed, let us define the mapping $h : L \to (B_{Y^*}, w^*)$ by $h(l) = \delta_{\varphi^{-1}(l)}\restriction_Y$, where $\delta_{\varphi^{-1}(l)}$ is the Dirac measure on K supported by a point from $\varphi^{-1}(l)$. It is easy to observe that h is a homeomorphism onto $h(L)$.

Now, we will verify that $h(\varphi(D)) \subset h(L) \cap R$. Fix $s \in \Gamma$ and $k \in K$. We would like to see that $\mu = h(\varphi(r_s(k))) \in R$. Hence, we need to see $T_s^*(\mu) = \mu$. Fix $f \in \mathcal{C}(L)$. Then $P_s(f \circ \varphi) \in Y$; hence, there exists $g \in \mathcal{C}(L)$ such that $f \circ \varphi \circ r_s = g \circ \varphi$. Moreover, $f \circ \varphi \circ r_s = f \circ \varphi$ on $r_s[K]$; thus, $g \circ \varphi = f \circ \varphi$ on $r_s[K]$. Now,

$$T_s^*(\mu)(f \circ \varphi) = \mu(f \circ \varphi \circ r_s) = \mu(g \circ \varphi) = (g \circ \varphi)(r_s(k)) = (f \circ \varphi)(r_s(k)) = \mu(f \circ \varphi),$$

and $T_s^*(\mu) = \mu$.

Using the above and the fact that $\varphi(D)$ is dense in L, $h(L) \cap R$ is dense in $h(L)$. By Lemma 3.2.2, $h(L) \cap R$ is induced by a retractional skeleton in $h(L)$. By Lemma 3.2.2 (v), $h(\varphi(D)) = h(L) \cap R$. Hence, $\varphi(D)$ is induced by a retractional skeleton in L. This finishes the proof. □

Proof of Theorem 3.4.5. By Theorem 3.4.2 (iii), (i)⇒(ii) is true.

(ii)⇒(iv) Let us assume that (ii) holds. Then $D\restriction_Y \cap B_{Y^*}$ (resp. $D \cap B_{X^*}$) is induced by a retractional skeleton in (B_{Y^*}, w^*) (resp. (B_{X^*}, w^*)). By Lemma 3.2.2, $D\restriction_Y \cap B_{Y^*}$ (resp. $D \cap B_{X^*}$) is dense and countably compact in (B_{Y^*}, w^*) (resp. (B_{X^*}, w^*)). Let us consider the injection $I : Y \hookrightarrow X$. Then I^* is $w^*{-}w^*$ continuous and $I^*(D \cap B_{X^*}) = (D \cap B_{X^*})\restriction_Y$ is dense and countably compact in (B_{Y^*}, w^*). By Lemma 3.2.2 (v), $(D \cap B_{X^*})\restriction_Y = D\restriction_Y \cap B_{Y^*}$ and (iv) holds.

Assume (iv) is true. Then, using Lemma 3.2.3, R is closed, and therefore a quotient mapping. Hence, (iv)⇒(v) is proved.

(v)⇒(iii) Assume that (v) holds and fix a net y_ν from Y such that $y_\nu \overset{\tau_p(D)}{\to} x \in X$. Now, define function y as $y(R(d)) = d(x)$, $d \in D$. Since R is a quotient mapping, y is continuous and bounded (and defined on $(D \cap B_{X^*})\restriction_Y$). Hence, there is a continuous extension $\tilde{y} \in \mathcal{C}(B_{Y^*}, w^*)$, $\tilde{y} \supset y$. As $\tilde{y} \circ R = x$ on the dense set $D \cap B_{X^*}$, $\tilde{y} \circ R = x$ on (B_{X^*}, w^*). Thus, $\tilde{y}$ is affine on B_{Y^*} and $\tilde{y}(0) = 0$. By Lemma 3.4.3, there exists $z \in Y$ such that $y^*(z) = \tilde{y}(y^*)$ for every $y^* \in B_{Y^*}$. Consequently, $x = z \in Y$ and Y is $\sigma(X, D)$-closed in X.

(iii)⇒(i) Let $\mathfrak{s} = \{P_s\}_s \in \Gamma$ be the 1-projectional skeleton in X such that $D = D(\mathfrak{s})$ and let Y be $\sigma(X, D)$-closed in X. Consider the isometry $I : X \to \mathcal{C}(B_{X^*}, w^*)$ defined by $I(x)(x^*) = x^*(x)$, $x \in X$, $x^* \in B_{X^*}$. By Theorem 3.4.2, $D \cap B_{X^*}$ is induced by a retractional skeleton in (B_{X^*}, w^*). By Lemma 3.4.3, $I(X)$ is a $\tau_p(D \cap B_{X^*})$-closed subset of $\mathcal{C}(B_{X^*}, w^*)$. We claim that $I(Y)$ is $\tau_p(D \cap B_{X^*})$-closed in $\mathcal{C}(B_{X^*}, w^*)$.

Indeed, let $I(y_\nu) \overset{\tau_p(D \cap B_{X^*})}{\longrightarrow} f$ where $y_\nu \in Y$ and $f \in \mathcal{C}(B_{X^*}, w^*)$. As $I(X)$ is $\tau_p(D \cap B_{X^*})$-closed, $f = I(x)$ for some $x \in X$. Now it is easy to observe that $y_\nu \overset{\sigma(X,D)}{\longrightarrow} x$; hence, $x \in Y$. Thus, $f = I(x) \in I(Y)$ and the claim is proved.

Recall that by [33], $\{P_s^*\restriction_{B_{X^*}}\}_{s\in\Gamma}$ is the retractional skeleton in (B_{X^*}, w^*) which induces the set $D \cap B_X^*$ and $\{T_s\}_{s\in\Gamma}$ is a projectional skeleton in $\mathcal{C}(B_{X^*}, w^*)$, where T_s is defined by

$T_s(f) = f \circ P_s^*\restriction_{B_{X^*}}$, $s \in \Gamma$, $f \in \mathcal{C}(B_{X^*}, w^*)$. By Proposition 3.3.2, we can without loss of generality assume that $T_s(I(Y)) \subset I(Y)$ for every $s \in \Gamma$. Thus, $\mathfrak{s}_Y = \{(I^{-1} \circ T_s \circ I)\restriction_Y\}_{s\in\Gamma}$ is a 1-projectional skeleton in Y. It is straightforward to check that, for every $s \in \Gamma$, $(I^{-1} \circ T_s \circ I)\restriction_X = P_s$. Thus, $\mathfrak{s}_Y = \{P_s\restriction_Y\}_{s\in\Gamma}$ is a 1-projectional skeleton in Y and $D(\mathfrak{s}_Y) = D\restriction_Y$. □

3.5 A new characterization of Asplund spaces

In [29] there has been introduced a new class of Banach spaces, (T). A Banach space X belongs to (T) if and only if B_X is contained in a "Σ-subset" of $(B_{X^{**}}, w^*)$; i.e., B_X is contained in a set induced by a commutative retractional skeleton. Recall that every space from (T) is Asplund. The class (T) has been used to prove some results concerning biduals of Asplund spaces. Namely, if the norm on a Banach space X is Kadec, then X is in (T) if and only if the bidual unit ball is a Valdivia compact space. There has been raised a question of whether X is Asplund whenever the bidual unit ball is Valdivia after every equivalent renorming of X. This problem has been solved by O. Kalenda in an unpublished remark, where it is proved that the answer to the problem is positive.

In the following we first observe that, by Theorem 3.4.2, the noncommutative version of the condition determining the class (T) gives a characterization of Asplund spaces. In this way, we may look at Asplund spaces as at the "noncommutative class (T)". Using this observation, we show that "commutative" results concerning the class (T) (including the unpublished remark) have their "noncommutative" versions concerning Asplund spaces. In particular, we show that a Banach space X is Asplund if and only if the bidual unit ball has a retractional skeleton after every equivalent renorming of X.

It remains open whether a Banach space X is in (T) whenever the bidual unit ball is Valdivia after every equivalent renorming of X. This question has been already raised in [29].

Let us start with the observation that Asplund spaces form exactly the "noncommutative class (T)".

Theorem 3.5.1. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following conditions are equivalent:*

(i) *X is Asplund.*

(ii) *X is a subset of a set induced by a 1-projectional skeleton in X^*.*

(iii) *B_X is a subset of a set induced by a retractional skeleton in $(B_{X^{**}}, w^*)$.*

Proof. The equivalence (i)⇔(ii) is proved in [33, Proposition 26] (for a simpler proof of (i)⇒(ii) see also [7]). The equivalence (ii)⇔(iii) follows from Theorem 3.4.2. □

Notice that, by [29, Example 4.10], $\mathcal{C}(K)^*$ has a commutative 1-projectional skeleton whenever K is a compact space. Thus, condition (ii) in Theorem 3.5.1 cannot be in general replaced by assuming that X^* has a 1-projectional skeleton.

However, if X has a Kadec norm, then the condition (ii) in Theorem 3.5.1 may be weakened in the above mentioned way. This follows from the following "noncommutative version" of [29, Theorem 4.9]. Recall that a norm is called *Kadec* if the norm and weak topologies coincide on the unit sphere, and that each locally uniformly rotund norm is Kadec, see e.g. [14, Exercise 8.45].

Proposition 3.5.2. *Assume that the norm on a Banach space $(X, \|\cdot\|)$ is Kadec. Then the following assertions are equivalent:*

(i) *X is Asplund.*

(ii) *X^* has a 1-projectional skeleton.*

(iii) *$(B_{X^{**}}, w^*)$ has a retractional skeleton.*

Proof. The implication (i)⇒(ii) follows from Theorem 3.5.1 and (ii)⇒(iii) is a consequence of Theorem 3.4.2. Let us assume D is a set induced by a retractional skeleton in $(B_{X^{**}}, w^*)$. Using Theorem 3.5.1, it is enough to show that $B_X \subset D$. In order to prove it, we follow the lines of the proof from [29, Theorem 4.9], using only Lemma 3.2.2 instead of [29, Lemma 2.4]. □

Now we give the new characterization of Asplund spaces we mentioned above.

Theorem 3.5.3. *Let X be a Banach space. Then the following assertions are equivalent:*

(i) *X is Asplund.*

(ii) *$(X, |\cdot|)^*$ has a 1-projectional skeleton for every equivalent norm $|\cdot|$ on X.*

(iii) *$(B_{(X,|\cdot|)^{**}}, w^*)$ has a retractional skeleton for every equivalent norm $|\cdot|$ on X.*

Let us recall that in [29] there is constructed an Asplund space X such that the bidual unit ball does not have a commutative retractional skeleton; i.e., is not Valdivia. Consequently, X^* does not have a commutative 1-projectional skeleton; i.e., X^* is not 1-Plichko. Thus, conditions (ii) and (iii) in Theorem 3.5.3 may not be replaced by its commutative versions. Therefore, the following question, raised already in [29], seems to be interesting. It would give a characterization of those spaces, which have a Valdivia bidual unit ball under every equivalent renorming of X.

Question 3. Suppose that X is a Banach space such that for every equivalent norm on X the bidual unit ball has a commutative retractional skeleton; i.e., it is Valdivia. Is X in the class (T)?

Now we are going to prove Theorem 3.5.3. First, we need the following statement. It is an analogy to the statement contained in the unpublished remark by O. Kalenda mentioned above, where the result is proved for the class of Valdivia compact spaces.

Lemma 3.5.4. *Let $(X, \|\cdot\|)$ be a Banach space such that $(B_{(X,|\cdot|)^{**},w^*}) \in \mathcal{R}_0$ whenever $|\cdot|$ is an equivalent norm on X. Then each subspace of X has the same property.*

Proof. In order to get a contradiction, let Y be a subspace of X with an equivalent norm $|\cdot|$ such that $B_{(Y,|\cdot|)^{**}}$ does not have a retractional skeleton. Then Y is a proper subspace of X and hence there are $f \in X^*$ and $x_0 \in X \setminus Y$ such that $f\restriction_Y = 0$ and $f(x_0) = 1$. The formula

$$\|x\|_1 = |f(x)| + \|x - f(x)x_0\|$$

clearly defines an equivalent norm on X.

In the following we will consider any Banach space canonically embedded in its second dual. Further, having a subspace Z of X, we may consider Z^{**} as a subspace of X^{**} (if $i : Z \to X$ is the identity, then i^{**} is a $w^* - w^*$ continuous linear isometry from Z^{**} onto $(Z^\perp)^\perp = \overline{Z}^{w^*}$; moreover, $X \cap Z^{**} = Z$).

Thus, $M = B_{(Y,|\cdot|)^{**}}$ can be viewed as a w^*-compact convex and symmetric subset of X^{**}. Put $N = \{F \in B_{(X,\|\cdot\|_1)^{**}};\ F(f) = 0\}$ and

$$B = \operatorname{conv}\{N \cup (M + x_0) \cup (M - x_0)\}.$$

Then B is a w^*-compact convex and symmetric subset of X^{**}. Let us fix a $c > 0$ such that $\|y\| \leq c|y|$ for every $y \in Y$. Then it is easy to verify that

$$\frac{1}{2}B_{(X,\|\cdot\|_1)^{**}} \subset B \subset (1 + c)B_{(X,\|\cdot\|_1)^{**}}.$$

Thus, there is an equivalent norm $\|\cdot\|_{**}$ on X^{**} such that B is the unit ball on $(X^{**}, \|\cdot\|_{**})$. Moreover, as B is w^*-closed, the norm $\|\cdot\|_{**}$ is a dual norm to some norm $\|\cdot\|_*$ on X^* and $B_\circ = \{x^* \in X^*;\ F(x^*) \leq 1 \text{ for } F \in B\}$ is the unit ball in $(X^*, \|\cdot\|_*)$ (see [11, Fact 5.4]). Notice that $B \cap X$ is w^*-dense in B.

Indeed, first we put $K = \{F \in X^{**};\ F(f) = 0\}$. Now we observe that $K = (\{x \in X;\ f(x) = 0\}^\perp)^\perp$; hence, we may identify K with $\{x \in X : f(x) = 0\}^{**}$. Then $N \cap X$ may be identified with $K \cap B_X$, which is dense in $N = K \cap B_{(X,\|\cdot\|_1)^{**}}$. Similarly, $M \cap X$ is dense in M. As $N \cap X$ (resp. $M \cap X$) is dense in N (resp. M) and $x_0 \in X$, $B \cap X$ is dense in B.

Consequently, $B_\circ$ is w^*-closed in X^* and, by [11, Fact 5.4], $\|\cdot\|_*$ is a dual norm to some norm on X. Hence, B is a bidual unit ball with respect to an equivalent norm on X. Now it suffices to observe that B does not have a retractional skeleton.

Let us suppose that B has a retractional skeleton. Then

$$x_0 + M = \{F \in B;\ F(f) = 1\}$$

is a w^*-closed w^*-G_δ subset of B; hence, by Lemma 3.2.2, it has a retractional skeleton. This is in contradiction with the choice of M. □

Proof of Theorem 3.5.3. The implication (i)⇒(ii) follows from Theorem 3.5.1 and (ii)⇒(iii) is a consequence of Theorem 3.4.2. Finally, suppose that X is not Asplund. Then there is a separable subspace $Y \subset X$ which is not Asplund. Let $|\cdot|$ be an equivalent Kadec norm on Y. By Proposition 3.5.2, $B_{(Y,|\cdot|)^{**}}$ does not have a retractional skeleton. Hence, by Lemma 3.5.4, there is an equivalent norm on X such that the bidual unit ball does not have a retractional skeleton. □

The following question has already been articulated in [29] and [1].

Question 4. Let X be an Asplund space. Is there an equivalent norm on X such that X^* has a commutative 1-projectional skeleton; i.e., is 1-Plichko, or equivalently has a countably 1-norming Markushevich basis?

3.6 Some more applications

In the last section we collect some more applications of the results contained in previous sections. They are straightforward analogies to results contained in [28], where similar statements are proved for Valdivia compact spaces and Plichko spaces.

First, we give some statements concerning open continuous surjections.

Lemma 3.6.1. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. If L has a dense set of G_δ points and D is a set induced by a retractional skeleton in K, then $\varphi(D)$ is a set induced by a retractional skeleton in L.*

Proof. In order to prove the lemma it is enough to follow the lines of the proof from [28, Lemma 3.23], using only the set induced by a retractional skeleton instead of the dense Σ-subset. Instead of [28, Theorem 3.22] and [28, Lemma 1.11] we use Theorem 3.4.4 and Lemma 3.2.2. □

As an immediate consequence we get the following theorem.

Theorem 3.6.2. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. If L has a dense set of G_δ points and $K \in \mathcal{R}_0$, then $L \in \mathcal{R}_0$.*

It is easy to check that any open continuous image of a compact space with a dense set of G_δ points has again this property (see [25, Lemma 4.3]). Thus, if $K \in \mathcal{R}_0$ has a dense set of G_δ points and φ is an open continuous surjection, then $\varphi(K) \in \mathcal{R}_0$.

However, some assumption on K is needed as there exists a Valdivia compact space K of weight $\aleph_1$ and an open continuous surjection φ such that $\varphi(K)$ is not Valdivia (and hence does not have a retractional skeleton); see [35] for more details.

Let us have a closer look at products.

Lemma 3.6.3. *Let K and L be nonempty compact spaces. If L has a dense set of G_δ points and $K \times L \in \mathcal{R}_0$, then both K and L have a retractional skeleton as well.*

Proof. In order to prove the lemma it is enough to follow the lines of the proof from [25, Proposition 4.7], using only Theorem 3.6.2 and Lemma 3.2.2 instead of [25, Theorem 4.5] and [25, Lemma 1.7]. □

Let us recall that the class $\mathcal{R}_0$ is closed under arbitrary products (see [33, Proposition 3.1]). Thus, the following theorem follows immediately.

Theorem 3.6.4. *Let $(K_\alpha)_{\alpha\in A}$ be a collection of nonempty compact spaces such that each K_α has a dense set of G_δ points. Then $\prod_{\alpha\in A} K_\alpha$ has a retractional skeleton if and only if each K_α has a retractional skeleton.*

However, the following question seems to be open.

Question 5. Suppose that K and L are compact spaces such that $K \times L$ has a retractional skeleton. Do both K and L have a retractional skeleton?

Concerning the stability of the class of spaces with a projectional skeleton, not much is known. Using the results of the previous sections we can obtain some information.

Theorem 3.6.5. *If X is a Banach space with a 1-projectional skeleton and $Y \subset X$ is a separable subspace, then X/Y has a 1-projectional skeleton.*

Proof. In the proof we follow the ideas from [28, Proposition 4.36]. Let D be a set induced by a 1-projectional skeleton in X. Then $D \cap B_{X^*}$ is induced by a retractional skeleton in B_{X^*}. As Y is separable, $Y^\perp$ is w^*-G_δ and w^*-closed subset of X^*. By Lemma 3.2.2, $D \cap B_{X^*} \cap Y^\perp$ is a convex symmetric set induced by a retractional skeleton in $B_{X^*} \cap Y^\perp$. Using the identification $(X/Y)^* = Y^\perp$ and Theorem 3.4.2, X/Y has a 1-projectional skeleton. □

Theorem 3.6.6. *Let X be a Banach space and $Y \subset X$ a closed subspace such that X/Y is separable. Then:*

(i) *If Y is complemented in X, then Y has a projectional skeleton if and only if X has a projectional skeleton.*

(ii) *If Y is 1-complemented and X has a 1-projectional skeleton, then Y has a 1-projectional skeleton.*

Proof. Assertion (ii) and the "if" part of (i) follow from Theorem 3.6.5. The converse in (i) follows from the fact that the class of spaces with a projectional skeleton is closed under ℓ_1-sums (see [33, Theorem 17]). □

However, the following question seems to be open.

Question 6. Does every 1-complemented subspace of a space with a 1-projectional skeleton have a 1-projectional skeleton as well?

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4. Rich families and elementary submodels

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Abstract: We compare two methods of proving separable reduction theorems in functional analysis – the method of rich families and the method of elementary submodels. We show that any result proved using rich families holds also when formulated with elementary submodels and the converse is true in spaces with fundamental minimal system an in spaces of density $\aleph_1$. We do not know whether the converse is true in general. We apply our results to show that a projectional skeleton may be without loss of generality indexed by ranges of its projections.

4.1 Introduction

For the study of nonseparable Banach spaces, construction of a separable subspace with certain property is sometimes important. It enables us to transfer properties from smaller (separable) spaces to larger ones.

One of the important approaches is the "separable reduction". By a separable reduction we usually mean the possibility to extend the validity of a statement from separable spaces to the nonseparable setting without knowing the proof of the statement in the separable case. This method has been used e.g. in [4, 9, 15, 21, 41, 38, 40, 46]. The proof of separable reduction theorems depends on a "separable determination": a statement ϕ concerning a nonseparable Banach space X is here considered to be *separably determined* if

$$\text{The statement } \phi \text{ holds in } X \iff \forall F \in \mathcal{F}: \text{ The statement } \phi \text{ holds in } F,$$

where $\mathcal{F}$ is a sufficiently large family of separable subspaces of X; typically, for any separable subspace of X there is a bigger subspace from $\mathcal{F}$. Note that in the literature sometimes a statement is considered to be "separably determined" if only the implication from the left to the right above holds. We refer to a nice introduction from [15], where more details about the history (and not only that) may be found.

Although in applications one makes the final deduction using just one separable subspace, it is convenient to know that the family $\mathcal{F}$ is large in order to join finitely many arguments together. Hence, given a statement ϕ, we are trying to construct a large family of separable subspaces $\mathcal{F}$ so that the above holds for ϕ. There are several approaches to this. One of them is the concept of *rich families* introduced in [3] by Borwein, Moors and further used e.g. in [41, 38, 40, 46].

Definition 4.1.1. Let X be a Banach space. A family $\mathcal{F}$ of separable subspaces of X is called *rich* if

(i) each separable subspace of X is contained in an element of $\mathcal{F}$ and

(ii) for every increasing sequence F_i in $\mathcal{F}$, $\overline{\bigcup_{i=1}^{\infty} F_i}$ belongs to $\mathcal{F}$.

Another concept is the "method of suitable models" (or method of elementary submodels) used in [4, 9, 10]. The class $\mathcal{F}$ then consists of spaces of the form $\overline{X \cap M}$, where M is a suitable model; i.e. a set for which certain finite list of formulas is absolute and which contains some countable set given in advance. We refer to the next section and [4], where more details may be found.

In the present paper we investigate the relationship between these two methods. This relationship is summed up in Theorem 4.3.3 which says that a statement which is separably determined by the method of rich families is also separably determined by the method of suitable models; under certain additional conditions the converse holds as well. However, we do not know whether these two methods are equivalent in general. It is not clear even in case $X = \mathcal{C}(K)$ where K is a Boolean space (i.e. zero-dimensional Hausdorff compact space); see Question 9.

A key tool to prove the mentioned theorem is a procedure of creating a rich family of separable subspaces, where every space from the family is of the form $\overline{X \cap M}$ for a suitable model; see Theorem 4.2.9. It seems to be a challenging problem whether the assumptions in Theorem 4.2.9 are really needed; see Question 8.

We apply the main theorem to clarify and simplify the definition of a projectional skeleton. This notion was introduced by W. Kubiś in [33] and further investigated in [5, 6, 17, 32]. Let us give the original definition.

Definition 4.1.2. A *projectional skeleton* in a Banach space X is a family of bounded projections $\{P_s\}_{s\in\Gamma}$ indexed by an up-directed partially ordered set Γ satisfying the following conditions:

(i) Each P_sX is separable.

(ii) $X = \bigcup_{s\in\Gamma} P_sX$.

(iii) $s \leq t \Rightarrow P_s = P_s \circ P_t = P_t \circ P_s$.

(iv) Given $s_1 < s_2 < \cdots$ in Γ, $t = \sup_{n\in\omega} s_n$ exists and $P_tX = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}X}$.

Given $r \geq 1$, we say that $\{P_s\}_{s\in\Gamma}$ is an *r-projectional skeleton* if it is a projectional skeleton such that $\|P_s\| \leq r$ for every $s \in \Gamma$.

We say that $\{P_s\}_{s\in\Gamma}$ is a *commutative projectional skeleton* if $P_s \circ P_t = P_t \circ P_s$ for any $s, t \in \Gamma$.

The notion of a projectional skeleton is an important tool to study nonseparable Banach spaces since it provides a decomposition of such a space to separable pieces. There is a hope that if we glue them together, their properties will be preserved by the nonseparable Banach space we started with. It is known that inductive arguments work well in a space with a projectional skeleton. Consequently, every space with a projectional skeleton has a strong Markushevich basis and an LUR renorming; see the introduction of [6] for more details.

One can ask, what is the mysterious index set Γ in the definition of a projectional skeleton above. It is claimed already in [32] that we may always assume that the projectional skeleton is indexed by ranges of its projections. This statement is used also in [17]. However, the proof is not complete and we fill in the gap in the last section of this paper; see Theorem 4.4.3. We give two proofs of Theorem 4.4.3 - one uses the results above and one is absolutely self-contained.

Let us recall the most relevant notions, definitions and notations: We denote by ω the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$. Whenever we say that a set is countable, we mean that the set is either finite or infinite and countable. If A is a set, we denote by $[A]^{\leq\omega}$ the set of all its countable subsets. If f is a mapping then we denote by $\operatorname{Rng} f$ the range of f and by $\operatorname{Dom} f$ the domain of f. By writing $f : X \to Y$ we mean that f is a mapping with $\operatorname{Dom} f = X$ and $\operatorname{Rng} f \subset Y$. By the symbol $f\restriction_Z$ we denote the restriction of the mapping f to the set Z.

We shall consider Banach spaces over the field of real numbers (but many results hold for complex spaces as well). B_X is the unit ball in X; i.e., the set $\{x \in X :;\ \|x\| \leq 1\}$. X^* stands for the (continuous) dual space of X. For a set $A \subset X^*$ we write $A_\perp = \{x \in X;\ (\forall a \in A)\ a(x) = 0\}$. If $\{x_i\}_{i\in I}$ is a family of vectors in the Banach space X, we denote by $[x_i;\ i \in I]$ the closed linear hull of $\{x_i\}_{i\in I}$. A set $D \subset X^*$ is *r-norming* if, for every $x \in X$, $\|x\| \leq r. \sup\{|x^*(x)| :\ x^* \in D \cap B_{X^*}\}$.

4.2 Rich families generated by suitable models

In this section we give a method of generating rich families of separable subspaces with certain additional properties. The method is based on the proof of the famous Löwenheim–Skolem Theorem; see e.g. [36, Chapter IV Theorem 7.8]. In Theorem 4.2.9 we show that if X is a Banach space with a fundamental minimal system or with $\operatorname{dens} X = \aleph_1$, then the method generates rich families of separable subspaces. However, it is not known to the authors whether the condition on a Banach space X is necessary.

Let us first recall some definitions:

Let N be a fixed set and φ a formula in the language of ZFC. Then the *relativization of φ to N* is the formula φ^N which is obtained from φ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each quantifier of the form "$\exists x$" by "$\exists x \in N$".

If $\varphi(x_1, \dots, x_n)$ is a formula with all free variables shown (i.e., a formula whose free variables are exactly $x_1, \dots, x_n$) then φ *is absolute for* N if and only if

$$\forall a_1, \dots, a_n \in N \quad (\varphi^N(a_1, \dots, a_n) \leftrightarrow \varphi(a_1, \dots, a_n)).$$

A list of formulas, $\varphi_1, \dots, \varphi_n$, is said to be *subformula closed* if and only if every subformula of a formula in the list is also contained in the list.

Any formula in the set theory can be written using symbols $\in, =, \wedge, \vee, \neg, \to, \leftrightarrow, \exists, (,), [,]$ and symbols for variables. Let us assume a subformula closed list of formulas $\varphi_1, \dots, \varphi_n$ is written in this way. Then it is not difficult to show, that the absoluteness of $\varphi_1, \dots, \varphi_n$ for N in other words says, that those formulas don't create any new sets in N. This result is contained in the following lemma (a proof can be found in [36, Chapter IV, Lemma 7.3]):

Lemma 4.2.1. *Let N be a set and $\varphi_1, \dots, \varphi_n$ subformula closed list of formulas (formulas containing only symbols $\in, =, \wedge, \vee, \neg, \to, \leftrightarrow, \exists, (,), [,]$ and symbols for variables). Then the following are equivalent:*

(i) $\varphi_1, \dots, \varphi_n$ are absolute for N

(ii) Whenever φ_i is of the form $\exists x \varphi_j(x, y_1, \dots y_l)$ (with all free variables shown), then

$$\forall y_1, \dots y_l \in N \left[\exists x \; (\varphi_j(x, y_1, \dots y_l)) \rightarrow (\exists x \in N)(\varphi_j(x, y_1, \dots y_l))\right]$$

The method of creating rich families is based on the proof of the Löwenheim–Skolem Theorem and on the following statement (for a proof see e.g. [36, Chapter IV, Theorem 7.8]).

Theorem 4.2.2. *Let $\varphi_1, \dots, \varphi_n$ be any formulas and Y any set. Then there exists a set $M \supset Y$ such, that*

$$(\varphi_1, \dots, \varphi_n \text{ are absolute for } M) \quad \wedge \quad (|M| \leq \max(\omega, |Y|)).$$

The set with the properties from previous theorem will be often used throughout the paper. Therefore we will use the following definition.

Definition 4.2.3. Let $\varphi_1, \dots, \varphi_n$ be any formulas and let Y be any countable set. Let $M \supset Y$ be a countable set satisfying that $\varphi_1, \dots, \varphi_n$ are absolute for M. Then we say that *M is a suitable model for $\varphi_1, \dots, \varphi_n$ containing Y*. We denote this by $M \prec (\varphi_1, \dots, \varphi_n;\; Y)$.

If X is a topological space and $M \prec (\varphi_1, \dots, \varphi_n;\; Y)$, we denote by X_M the set $\overline{X \cap M}$.

We will repeatedly use some results from [4] which are summed up in the following lemma; see [4, Proposition 2.9, 2.10 and 3.2].

Lemma 4.2.4. *There are formulas $\theta_1, \dots, \theta_m$ and a countable set Y_0 such that any $M \prec (\theta_1, \dots, \theta_n;\; Y_0)$ satisfies the following conditions:*

- *If $f \in M$ is a mapping, then $\mathrm{Dom}(f) \in M$, $\mathrm{Rng}(f) \in M$ and $f[M] \subset M$.*
- *If $A \in M$ is a countable set, then $A \subset M$.*
- *If $\langle X, +, \cdot \rangle \in M$ is a vector space, then $X \cap M$ is a $\mathbb{Q}$-linear subspace of X.*

In the sequel we will always assume that the formulas from the previous lemma are contained in the respective list of formulas and that the set Y_0 is contained in the respective countable set.

We will try to find out conditions under which suitable models generate nice rich families in the following sense.

Definition 4.2.5. Let X be a Banach space. We say that *suitable models generate nice rich families in X*, if the following holds:

> Whenever Y is a countable set and $\varphi_1, \dots, \varphi_n$ is a list of formulas, there exists a family $\mathcal{M}$ satisfying the following conditions:
>
> (i) For every $M \in \mathcal{M}$, $M \prec (\varphi_1, \dots, \varphi_n; Y)$.
>
> (ii) The set $\{X_M;\; M \in \mathcal{M}\}$ is a rich family of separable subspaces in X.
>
> (iii) $\forall M, N \in \mathcal{M}: \quad M \subset N \iff \overline{X \cap M} \subset \overline{X \cap N}$.

The main obstacle is to find a family $\mathcal{M}$ of suitable models such that $\{X_M;\ M \in \mathcal{M}\}$ covers the space X and the condition (iii) in the definition above is satisfied.

First, we recall the prescription on creating suitable models using fixed "Skolem function" (the prescription comes from the proof of the Löwenheim–Skolem Theorem). Sets created with a fixed Skolem function will be used later in order to find the family $\mathcal{M}$ of suitable models.

Let us have a subformula closed list of formulas $\varphi_1, \ldots, \varphi_n$ and a set R such that $\varphi_1, \ldots, \varphi_n$ are absolute for R. Fix some well-ordering $\lhd$ on the set R. Now, for the given list of formulas $\varphi_1, \ldots, \varphi_n$, set R and well-ordering $\lhd$, we define the *Skolem function* ψ *for* $\varphi_1, \ldots, \varphi_n$*,* R *and* $\lhd$ in the following way:

1. For every $i \in \{1, \ldots, n\}$, we denote by l_i the number of all the free variables in the formula φ_i and we define a mapping $H_i : R^{l_i} \to R$ in the following way:
   - if $l_i = 0$, then $R^{l_i} = \{\emptyset\}$ and $H_i(\emptyset)$ is the $\lhd$-least element of R.
   - if $l_i > 0$ and $(r_1, \ldots, r_{l_i}) \in R^{l_i}$ is fixed, then:
     - if $\varphi_i = \exists x \varphi_j(x, v_1, \ldots, v_{l_i})$ for some formula φ_j and if there is some $x \in R$ such that $\varphi_j(x, r_1, \ldots, r_{l_i})$ holds, then then $H_i(r_1, \ldots, r_{l_i})$ is the $\lhd$-least of such elements.
     - in all the other cases, $H_i(r_1, \ldots, r_{l_i})$ is the $\lhd$-least element of R.
2. Now, we define $\psi : [R]^{\leq\omega} \to [R]^{\leq\omega}$ in the following way:
   - Fix a countable set $A \subset R$ and put $A_0 = A$.
   - Having A_k, we define A_{k+1} by
   $$A_{k+1} = A_k \cup \bigcup \{H_i(a_1, \ldots, a_{l_i});\ i = 1, \ldots, n\ , (a_1, \ldots, a_{l_i}) \in (A_k)^{l_i}\}.$$
   - Now, we define $\psi(A) = \bigcup_{k\in\omega} A_k$.

Lemma 4.2.6. *Let* ψ *be a Skolem function for* $\varphi_1, \ldots, \varphi_n$*,* R *and* $\lhd$*. Then*

(i) *For every* $A \in [R]^{\leq\omega}$*,* $\psi(A)$ *is a countable set,* $\varphi_1, \ldots, \varphi_n$ *are absolute for* $\psi(A)$ *and* $\psi(A) \supset A$*.*

(ii) *The mapping* ψ *is idempotent; i.e.* $\psi \circ \psi = \psi$*.*

(iii) *The mapping* ψ *is monotone; i.e.* $\psi(A) \subset \psi(B)$ *whenever* $A, B \in [R]^{\leq\omega}$ *are such that* $A \subset B$*.*

(iv) *For every countable family* $\mathcal{F}$ *consisting of sets from* $[R]^{\leq\omega}$*,* $\psi(\bigcup \mathcal{F}) = \bigcup_{F\in\mathcal{F}} \psi(F)$*.*

(v) *Let* $J \subset R$ *be an arbitrary set. Then, for every* $A \in [J]^{\leq\omega}$ *and* $B \in [R]^{\leq\omega}$*,*
$$\psi(A) \cap J \subset \psi(B) \iff \psi(A) \subset \psi(B).$$

Proof. It follows immediately from the definition of the Skolem function ψ that (ii) and (iii) holds. By Lemma 4.2.1 and the definition of the Skolem function ψ, (i) holds.

Let $\mathcal{F}$ be as in (iv). Then it follows from (iii) that $\psi(\bigcup \mathcal{F}) \supset \bigcup_{F\in\mathcal{F}} \psi(F)$. In order to prove the other inclusion, notice that $\bigcup \mathcal{F} \subset \bigcup_{F\in\mathcal{F}} \psi(F)$. By (iii) and (ii) respectively, $\psi(\bigcup \mathcal{F}) \subset \psi(\bigcup_{F\in\mathcal{F}} \psi(F)) = \bigcup_{F\in\mathcal{F}} \psi(\psi(F)) = \bigcup_{F\in\mathcal{F}} \psi(F)$. Thus, (iv) holds.

Let J, A, B be as in (v). Then the implication from the right to the left is obvious. In order to prove the opposite one, let us suppose that $\psi(A) \cap J \subset \psi(B)$. By (i), $\psi(A) \cap J \supset A \cap J = A$. Hence, $\psi(B) \supset A$. Using (ii) and (iii), $\psi(B) \supset \psi(A)$. □

Under certain condition, the submodels created from the Skolem function above generate rich family of separable subspaces. The condition is contained in the following lemma. However, it is open to the authors whether this condition is necessary; see Question 7 bellow Lemma 4.2.8. We start with a definition.

Definition 4.2.7. Let $\{x_i\}_{i\in I}$ be a family of vectors in the Banach space X. The family of vectors $\{x_i\}_{i\in I}$ is said to be a *fundamental minimal* system if $[x_i;\ i \in I] = X$ and, for every $i \in I$, $x_i \notin [x_j;\ j \in I, j \neq i]$.

Lemma 4.2.8. *Let X be a Banach space. Let $\{x_i\}_{i\in I}$ be a system of vectors such that $[x_i;\ i \in I] = X$ and let there exist a countable set Y and formulas $\varphi_1, \ldots, \varphi_n$ such that for every $M \prec (\varphi_1, \ldots, \varphi_n; Y)$,*

$$x_i \in [x_j;\ j \in M \cap I] \Rightarrow i \in M.$$

Then suitable models generate nice rich families in X.

Proof. Let us define a mapping $f : I \to X$ by $f(i) = x_i$, $i \in I$. Without loss of generality we may suppose that $f \in Y$ and $[I]^{\leq\omega} \in Y$. Suppose further that Y contains the set Y_0 from Lemma 4.2.4 and the list of formulas $\varphi_1, \ldots, \varphi_n$ contains the formulas from Lemma 4.2.4 and the formulas marked below by $(*)$ and their subformulas.

We will show that if $M \prec (\varphi_1, \ldots, \varphi_n; Y)$, then X_M is a separable subspace of X and $X_M = [x_i;\ i \in M \cap I]$.

Indeed, let $M \prec (\varphi_1, \ldots, \varphi_n; Y)$ be arbitrary. By Lemma 4.2.4 we have $f[I \cap M] \subset X \cap M$ and $X \cap M$ is a $\mathbb{Q}$-linear set. Thus, X_M is a separable subspace of X and $[x_i;\ i \in M \cap I] \subset X_M$. Further, for any $x \in X \cap M$ we have

$$\exists A \in [I]^{\leq\omega} : x \in [x_i;\ i \in A] \tag{$*$}$$

By absoluteness of this formula there is such an A in M. By Lemma 4.2.4 we have $A \subset M$. Hence, $X \cap M \subset [x_i;\ i \in I \cap M]$ and $X_M \subset [x_i;\ i \in I \cap M]$.

In order to verify that suitable models generate nice rich families in X, fix a countable set Z and formulas $\phi_1, \ldots, \phi_k$. By adding countably many sets to Z and by adding finitely many formulas to $\phi_1, \ldots, \phi_k$, we may assume that $Z \supset Y$ and all the formulas $\varphi_1, \ldots, \varphi_n$ are contained in $\phi_1, \ldots, \phi_k$.

By Theorem 4.2.2, there exists a set $R \supset Z \cup \mathcal{P}(X)$ such that $\phi_1, ..., \phi_k$ are absolute for R. Fix a well-ordering $\lhd$ on R and let ψ be the Skolem function for $\phi_1, \ldots, \phi_k$, R and $\lhd$. Put

$$\mathcal{M} := \{\psi(A \cup Z);\ A \in [I]^{\leq\omega}\}.$$

By Lemma 4.2.6 (i), for every $M \in \mathcal{M}$, $M \prec (\phi_1, \ldots, \phi_k; Z)$. We claim that

$$\mathcal{F} = \{X_M;\ M \in \mathcal{M}\}$$

is the rich family of subspaces we are looking for.

It is easy to see that for every separable subspace $T \subset X$ there exists $M \in \mathcal{M}$ such that $T \subset X_M$ (it is enough to take any countable set D such that $[x_i;\ i \in D] \supset T$ and put $M = \psi(D \cup Z)$).

Fix $M, N \in \mathcal{M}$. Obviously, $X_M \subset X_N$ whenever $M \subset N$. On the other hand, let us assume that $X_M \subset X_N$; hence, $[x_i;\ i \in I \cap M] \subset [x_i;\ i \in I \cap N]$. By the assumption, $M \cap I \subset I \cap N$. By Lemma 4.2.6 (v), $M \subset N$. Hence, $X_M \subset X_N$ if and only if $M \subset N$.

Let us have a chain of spaces from $\mathcal{F}$, $X_{M_1} \subset X_{M_2} \subset \ldots$ where $M_n \in \mathcal{M}$ for every $n \in \mathbb{N}$. By the above, $M_1 \subset M_2 \subset \ldots$. By Lemma 4.2.6 (iv), $\bigcup_{n\in\mathbb{N}} M_n \in \mathcal{M}$. Now, it is easy to verify (see e.g. [4, Lemma 3.4]) that $\overline{\bigcup_{i=1}^{\infty} X_{M_i}} = X_{\bigcup_{n\in\mathbb{N}} M_n} \in \mathcal{F}$. Thus, $\mathcal{F}$ is a rich family of separable subspaces. This finishes the proof. □

Question 7. Let X be a Banach space. In order to verify that "suitable models generate nice rich families in X", it is sufficient to verify that the system of vectors $\{x_i\}_{i\in I}$ from Lemma 4.2.8 exists. Is this condition also necessary?

In the following we summarize two important cases when condition of Lemma 4.2.8 is satisfied. We do not know an example of a Banach space X where suitable models do not generate rich families in X.

Theorem 4.2.9. *Let X be a Banach space. Let X have a fundamental minimal system $\{x_i\}_{i\in I}$ or $\operatorname{dens} X = \aleph_1$. Then suitable models generate nice rich families in X.*

Proof. It is enough to verify that in both cases the assumptions of Lemma 4.2.8 are satisfied.

First, let us assume that $\{x_i\}_{i\in I}$ is a fundamental minimal system. Let us fix an arbitrary set $J \subset I$. Then

$$i \notin J \Rightarrow x_i \notin [x_j;\ j \in J].$$

Hence, the assumption of Lemma 4.2.8 is obviously satisfied (M need not be a suitable model but an arbitrary set).

Now, let us assume that $\operatorname{dens} X = \aleph_1$ and fix a family of vectors $\{x_\alpha\}_{\alpha<\omega_1}$ such that $[x_\alpha;\ \alpha < \omega_1] = X$ and $x_\alpha \notin [x_\beta;\ \beta < \alpha]$ for every $\alpha < \aleph_1$. Such a system can be easily constructed by transfinite induction.

Let us fix a list of formulas $\varphi_1, \ldots, \varphi_n$ which contains the formulas from Lemma 4.2.4 and a countable set Y containing the set Y_0 from Lemma 4.2.4 and the mapping $\alpha \mapsto x_\alpha$. Let $M \prec (\varphi_1, \ldots, \varphi_n; Y)$ be arbitrary.

Then $\omega_1 \in M$ (as the domain of the above mentioned mapping due to Lemma 4.2.4) and, moreover, for each $\alpha \in \omega_1 \cap M$ we have $\alpha \subset M$ (again by Lemma 4.2.4). Set $\gamma = \sup \omega_1 \cap M$. Then in fact $\gamma = \omega_1 \cap M$. Therefore for any $\alpha \in \omega_1$ we have

$$x_\alpha \in [x_\beta : \beta \in \omega_1 \cap M] \Leftrightarrow x_\alpha \in [x_\beta : \beta < \gamma] \Leftrightarrow \alpha < \gamma \Leftrightarrow \alpha \in M.$$

Indeed, the first and the third equivalences follow from the above proven fact $\gamma = \omega_1 \cap M$ and the second one follows from the properties of the family $\{x_\alpha\}_{\alpha<\omega_1}$. This verifies the assumption of Lemma 4.2.8, so the proof is completed. □

Let us recall that it is undecidable in ZFC whether every Banach space with $\operatorname{dens} X = \aleph_1$ has a fundamental minimal system; see e.g. [20, Section 4.4] and [42, Corollary 6]. We know only one example of a Banach space with $\operatorname{dens} X > \aleph_1$ and without a fundamental minimal system. It is the space $\ell_\infty^c(\Gamma)$ for an index set Γ of cardinality greater than continuum (we denote by $\ell_\infty^c(\Gamma)$ the closed subspace of $\ell_\infty(\Gamma)$ consisting of vectors of countable support); see e.g. [20, Theorem 4.26]. However, even in this case it is not clear to the authors whether suitable models generate nice rich families. Hence, the following question seems to be open.

Question 8. Do suitable models generate nice rich families in every Banach space? Do suitable models generate nice rich families in $\ell_\infty^c(\Gamma)$?

4.3 Separable reduction theorems

In this section we investigate the relationship between the "method of suitable models" and the "method of rich families" in separable determination theorems. In order to formulate our results as theorems we need the following precise definitions. They are necessary to substitute metamathematical notions "property" or "statement". They do not cover all the conceivable situations but they cover the important cases where separable reductions were investigated.

Definition 4.3.1. Let $\phi(X, A, f, y_1, \ldots, y_k)$ be a statement concerning the Banach space X, the set A and the function f. More precisely, we consider $\phi(X, +, \cdot, \|\cdot\|, A, f, y_1, \ldots, y_k)$, where

$$\phi(X, +, \cdot, \|\cdot\|, A, f, y_1, \ldots, y_k) = \langle X, +, \cdot, \|\cdot\|\rangle \text{ is a Banach space}, A \subset X, f \text{ is a function}, \operatorname{Dom}(f) \subset X \text{ and } \tilde{\phi}(X, +, \cdot, \|\cdot\|, A, f, y_1, \ldots, y_k)$$

for a formula $\tilde{\phi}(u_1, u_2, u_3, u_4, v, w, y_1, \ldots, y_k)$ with all free variables shown.

Let us fix some constants $C_1, \ldots, C_k$, a Banach space X, $A \subset X$ and a function f with $\operatorname{Dom}(f) \subset X$. We say the statement $\phi(X, A, f, C_1, \ldots, C_k)$ *is separably determined by the method of rich families*, if there exists a rich family $\mathcal{F}$ of separable subspaces of X such that, for every $F \in \mathcal{F}$,

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(F, A \cap F, f\restriction_F, C_1, \ldots, C_k).$$

We say the statement $\phi(X, A, f, C_1, \ldots, C_k)$ *is separably determined by the method of suitable models*, if there exists a countable set Y and a finite list of formulas $\varphi_1, \ldots, \varphi_n$ such that whenever $M \prec (\varphi_1, \ldots, \varphi_n; Y)$ and $\{X, +, \cdot, \|\cdot\|, A, f, C_1, \ldots, C_k\} \subset M$, then

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(X_M, A \cap X_M, f\restriction_{X_M}, C_1, \ldots, C_k).$$

In order to see that statements separably determined by the method of rich families are separably determined by the method of suitable models, we need the following.

Proposition 4.3.2. *There exists a list of formulas $\varphi_1, \ldots, \varphi_n$ and a countable set Y such that for every $M \prec (\varphi_1, \ldots, \varphi_n;\ Y)$ the following holds: Let $\langle X, +, \cdot, \|\cdot\|\rangle$ be a Banach space and $\mathcal{F}$ a rich family of separable subspaces of X. Then whenever $\{X, +, \cdot, \|\cdot\|, \mathcal{F}\} \subset M$, $X_M \in \mathcal{F}$.*

Proof. Fix a subformula-closed list of formulas $\varphi_1, \dots, \varphi_n$ containing formulas from Lemma 4.2.4, the axiom of power set and the formulas marked below by $(*)$. Fix a countable set Y containing the set Y_0 from Lemma 4.2.4. Fix $M \prec (\varphi_1, \dots, \varphi_n;\ Y)$, X and $\mathcal{F}$ as above. Assume that $\{X, +, \cdot, \|\cdot\|, \mathcal{F}\} \subset M$.

First, we will show that

$$X_M = \overline{\bigcup\{F;\ F \in \mathcal{F} \cap M\}}. \tag{4.1}$$

Indeed, let us fix $x \in X \cap M$. Then, using the absoluteness of the formula (and its subformula)

$$\exists F \quad (F \in \mathcal{F}\ \wedge\ x \in F), \tag{$*$}$$

there exists $F \in \mathcal{F} \cap M$ such that $x \in F$. Hence,

$$X_M = \overline{X \cap M} \subset \overline{\bigcup\{F;\ F \in \mathcal{F} \cap M\}}.$$

In order to see the opposite inclusion holds, let us fix $F \in \mathcal{F} \cap M$. Using the absoluteness of the formula (and its subformula)

$$\exists D \quad (D \text{ is a countable dense set in } F), \tag{$*$}$$

there exists a countable set $D \in M$ such that $\overline{D} = F$. Hence, $D \subset M$ and $F = \overline{D} \subset X_M$. Consequently, (4.1) holds.

Now, $\mathcal{F} \cap M$ is a nonempty, up-directed set. Indeed, $\mathcal{F} \cap M \neq \emptyset$ follows from the absoluteness of the formula (and its subformula)

$$\exists F \quad (F \in \mathcal{F}). \tag{$*$}$$

Let us fix $F, G \in \mathcal{F} \cap M$. By the absoluteness of the formula (and its subformula)

$$\exists H \quad (H \in \mathcal{F}\ \wedge\ F \cup G \subset H), \tag{$*$}$$

there exists $H \in \mathcal{F} \cap M$ such that $F \cup G \subset H$. Thus, $\mathcal{F} \cap M$ is an up-directed set.

Hence, there is an increasing sequence of sets $\{F_n\}_{n\in\mathbb{N}}$ from $\mathcal{F} \cap M$ such that $\bigcup \mathcal{F} \cap M = \bigcup_{n\in\mathbb{N}} F_n$. Consequently,

$$X_M = \overline{\bigcup\{F;\ F \in \mathcal{F} \cap M\}} = \overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{F}.$$

□

Now we are ready to formulate and prove the main result.

Theorem 4.3.3. *Let X be a Banach space, $A \subset X$ and f a function with* $\mathrm{Dom}(f) \subset X$. *Let $\phi(X, A, f, C_1, \dots, C_k)$ be a statement concerning the Banach space X, the set A, the function f and constants $C_1, \dots, C_k$. Consider the following conditions*

(i) $\phi(X, A, f, C_1, \dots, C_k)$ *is separably determined by the method of suitable models.*

(ii) $\phi(X, A, f, C_1, \dots, C_k)$ *is separably determined by the method of rich families.*

Then (ii) implies (i). Moreover, if X have a fundamental minimal system $\{x_i\}_{i\in I}$ or $\operatorname{dens} X = \aleph_1$*, then both conditions are equivalent.*

Proof. Let us assume that (ii) holds. Let us fix the countable set Y and the list of formulas $\varphi_1, \ldots, \varphi_n$ from Proposition 4.3.2 and add to them the following formula and its subformulas

$$\exists \mathcal{F} \quad (\mathcal{F} \text{ is a rich family of separable subspaces of } X \text{ such that, for every } F \in \mathcal{F}, \\ \phi(X, A, f, C_1, \ldots, C_k) \iff \phi(F, A \cap F, f \restriction_F, C_1, \ldots, C_k)). \tag{4.2}$$

Denote such an extended list of formulas by $\phi_1, \ldots, \phi_k$. Fix $M \prec (\phi_1, \ldots, \phi_k; Y)$ with $\{X, +, \cdot, \|\cdot\|, A, f, C_1, \ldots, C_k\} \subset M$. By (ii) and the absoluteness of the formula (4.2) (and its subformula), there exists a rich family $\mathcal{F} \in M$ of separable subspaces of X such that, for every $F \in \mathcal{F}$,

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(F, A \cap F, f \restriction_F, C_1, \ldots, C_k).$$

By Proposition 4.3.2, $X_M \in \mathcal{F}$. Hence,

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(X_M, A \cap X_M, f \restriction_{X_M}, C_1, \ldots, C_k).$$

As M was an arbitrary set with $M \prec (\phi_1, \ldots, \phi_k; Y)$ and $\{X, +, \cdot, \|\cdot\|, A, f, C_1, \ldots, C_k\} \subset M$, (i) holds.

Now, let us prove (i)$\Rightarrow$(ii) in the "moreover" part. By (i), there is a countable set $Z = Y \cup \{X, +, \cdot, \|\cdot\|, A, f, C_1, \ldots, C_k\}$ and a finite list of formulas $\varphi_1, \ldots, \varphi_n$ such that, for every $M \prec (\varphi_1, \ldots, \varphi_n; Z)$,

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(X_M, A \cap X_M, f \restriction_{X_M}, C_1, \ldots, C_k).$$

By Theorem 4.2.9, there exists a family $\mathcal{M}$ such that, for every $M \in \mathcal{M}$, $M \prec (\varphi_1, \ldots, \varphi_n; Z)$ and $\mathcal{F} = \{X_M;\ M \in \mathcal{M}\}$ is a rich family of separable subspaces in X. Hence, for every $F \in \mathcal{F}$,

$$\phi(X, A, f, C_1, \ldots, C_k) \iff \phi(F, A \cap F, f \restriction_F, C_1, \ldots, C_k).$$

Consequently, (ii) holds. □

Let us show some consequences of the above. First, applying Theorem 4.3.3 on a result from [9], we get the following; see e.g. [9] for the definition of a σ-upper porous set.

Corollary 4.3.4. *Let X, Y be Banach spaces and $f : X \to Y$ be a function. Let X have a fundamental minimal system or* $\operatorname{dens} X = \aleph_1$*. Then there exists a rich family $\mathcal{F}$ of separable spaces of X such that for every $F \in \mathcal{F}$ the following two conditions are equivalent:*

(i) the set of the points where f is not Fréchet differentiable is σ-upper porous,

(ii) the set of the points where $f \restriction_F$ is not Fréchet differentiable is σ-upper porous in F.

Proof. Let us denote by $\phi(X, f, Y)$ the formula "the set of the points where f is not Fréchet differentiable is σ-upper porous in X". By [9] (see the proof of Theorem 1.2 which is given just bellow the proof of Theorem 5.4), $\phi(X, f, Y)$ is separably determined by the method of suitable models. By Theorem 4.3.3, $\phi(X, f, Y)$ is separably determined by the method of rich families. □

Applying Theorem 4.3.3 on a result from [18], we get the following result concerning nonseparable Gurariĭ spaces. A Banach space X is said to be a Gurariĭ space if, for every pair of finite-dimensional spaces $S \subset T$, for every isometric embedding $f : S \to X$ and for every $\varepsilon > 0$, there exists an ε-isometric embedding $g : T \to X$ such that $g \restriction_S = f$. Let us recall there exists exactly one (up to isometry) separable Gurariĭ space. For a survey on Gurariĭ spaces, see e.g. [18].

Corollary 4.3.5. *There exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y such that for every $M \prec (\phi_1, \dots, \phi_n;\ Y)$ the following holds: Let X be a Banach space with $\{X, +, \cdot, \|\cdot\|\} \subset M$. Then X is a Gurariĭ space if and only if X_M is a Gurariĭ space.*

Proof. Let us denote by $\phi(X)$ the formula "X is a Gurariĭ space". By [18, Theorem 3.4], $\phi(X)$ is separably determined by the method of rich families. By Theorem 4.3.3, $\phi(X)$ is separably determined by the method of suitable models. □

Remark 4.3.6. Let X be a Banach space, $A \subset X$, f a function with domain in X and $\phi(X, A, f)$ a statement determined by the method of suitable models. It is known to the authors that if $\phi(X, A, f)$ holds, then there does not exist a rich family of separable subspaces $\mathcal{F}$ such that, for every $F \in \mathcal{F}$, $\neg\phi(F, A \cap F, f \restriction_F)$ holds. The argument is as follows. Arguing by contradiction, let us fix such a rich family of separable subspaces $\mathcal{F}$. By the assumption, there is a countable set Y and a finite list of formulas $\varphi_1 \dots, \varphi_n$ such that, whenever $M \prec (\varphi_1 \dots, \varphi_n; Y)$, $\phi(X_M, A \cap X_M, f \restriction_{X_M})$ holds. We inductively find sequences $(F_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ such that

- for every $n \in \mathbb{N}$, $M_n \prec (\varphi_1 \dots, \varphi_n; Y)$ and $F_n \in \mathcal{F}$;
- $M_1 \subset M_2 \subset \dots$ and $F_1 \subset X_{M_1} \subset F_2 \subset X_{M_2} \subset \dots$.

Put $M = \bigcup_{n=1}^{\infty} M_n$. It is easy to check that $M \prec (\varphi_1 \dots, \varphi_n; Y)$ and $X_M = \overline{\bigcup_{n=1}^{\infty} X_{M_n}}$; see, e.g., [4]. Thus, $X_M = \overline{\bigcup_{n=1}^{\infty} F_n} \in \mathcal{F}$ and $\phi(X_M, A \cap X_M, f \restriction_{X_M})$ holds. This is a contradiction.

However, the following question remains open.

Question 9. Is the method of suitable models equivalent to the method of rich families? More precisely, does (i)⇒(ii) hold in Theorem 4.3.3 if we do not assume that X has a fundamental minimal system or is of density $\aleph_1$? Does it hold at least for $\mathcal{C}(K)$ spaces? Does it hold for $\mathcal{C}(K)$ spaces, where K is a Boolean space?

4.4 Projectional skeletons

In the last section we apply the previous results to give a characterization of spaces with a projectional skeleton. We prove that a projectional skeleton may be without loss of generality considered to be simple in the sense of the following definition.

Definition 4.4.1. A *simple projectional skeleton* in a Banach space X is a family of bounded projections $\{P_F\}_{F\in\mathcal{F}}$ indexed by a rich family of separable subspaces $\mathcal{F}$ satisfying the following conditions:

(i) for every $F \in \mathcal{F}$, $P_F(X) = F$;

(ii) if $E \subset F$ in $\mathcal{F}$, then $P_E = P_E \circ P_F = P_F \circ P_E$.

Given $r \geq 1$, we say that $\{P_F\}_{F\in\mathcal{F}}$ is an *simple r-projectional skeleton* if it is a simple projectional skeleton such that $\|P_F\| \leq r$ for every $F \in \mathcal{F}$.

We say that $\{P_F\}_{F\in\mathcal{F}}$ is a *simple commutative projectional skeleton* if $P_E \circ P_F = P_F \circ P_E$ for any $E, F \in \mathcal{F}$.

Definition 4.4.2. Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a (simple) projectional skeleton in a Banach space X and let $D(\mathfrak{s}) = \bigcup_{s\in\Gamma} P_s^*[X^*]$. Then we say that $D(\mathfrak{s})$ *is induced by a (simple) projectional skeleton.*

Theorem 4.4.3. *Let X be a Banach space and let $D \subset X^*$ be an r-norming subspace ($r \geq 1$). The following properties are equivalent:*

(i) *X has a (commutative) r-projectional skeleton $\mathfrak{s}$ with $D = D(\mathfrak{s})$.*

(ii) *X has a simple (commutative) r-projectional skeleton $\mathfrak{s}$ with $D = D(\mathfrak{s})$.*

As we have remarked above, it is claimed already in [32] that such a statement holds. However, the proof contains a gap as it is not clear why $X_M \subset X_N$ should imply $M \subset N$ (for M, N suitable models). Here, using the preceding results, we fill in the gap. Moreover, as this result could interest broader audience (not familiar with elementary submodels), we give a proof which is absolutely self-contained. Of course, the idea of both proofs is the same. In the second proof we avoid the use of suitable models by giving in fact the proof of Theorem 4.2.2 in the concrete case.

Proof 1 - using the method of suitable models. It is obvious that every simple projectional skeleton is a projectional skeleton. Thus, (i) easily follows from (ii).

Let us assume $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ is an r-projectional skeleton in X and $D = D(\mathfrak{s})$. It follows immediately from the proof of [33, Lemma 14] that there exist a countable set Y and a finite list of formulas $\varphi_1, \dots, \varphi_n$ such that whenever $M \prec (\varphi_1, \dots, \varphi_n; Y)$, there exists a projection P_M with $\|P_M\| \leq r$, $P_M(X) = X_M$ and $\ker(P_M) = (D \cap M)_\perp$.

Recall, that every space with a projectional skeleton has a Markushevich basis. Every Markushevich basis is a fundamental minimal system (this is immediate from the definition, see [20, Definition 1.7]). Hence, by Theorem 4.2.9, there exists a family $\mathcal{M}$ such that

- for every $M \in \mathcal{M}$, $M \prec (\varphi_1, \dots, \varphi_n; Y)$,
- the set $\mathcal{F} = \{X_M;\ M \in \mathcal{M}\}$ is a rich family of separable subspaces in X and
- $\forall M, N \in \mathcal{M}: \quad M \subset N \iff \overline{X \cap M} \subset \overline{X \cap N}$.

$\{P_M\}_{M\in\mathcal{M}}$ can be equivalently indexed as $\{P_M\}_{X_M\in\mathcal{F}}$. Let us fix $M,N\in\mathcal{M}$ with $X_M\subset X_N$. Then obviously $P_M(X)\subset P_N(X)$ and $P_M=P_N\circ P_M$. Moreover, $M\subset N$. Thus, $\ker(P_M)=(D\cap M)_\perp\supset(D\cap N)_\perp=\ker(P_N)$; hence, $P_M=P_M\circ P_N$. Consequently, $\mathfrak{s}'=\{P_M\}_{X_M\in\mathcal{F}}$ is a simple r-projectional skeleton in X. It is clear that $D(\mathfrak{s}')=\bigcup_{M\in\mathcal{M}}P_M^*(X^*)=\bigcup_{M\in\mathcal{M}}\overline{D\cap M}^{w^*}$. As any set induced by a projectional skeleton is countably closed, we have that $D(\mathfrak{s}')\subset D$. By [33, Corollary 19], $D(\mathfrak{s}')=D$.

It follows immediately from the proof of [33, Lemma 14] that, for every $M\in\mathcal{M}$, projection P_M equals P_s for some $s\in\Gamma$ (more precisely, $s=\sup(\Gamma\cap M)$). Hence, $\mathfrak{s}'$ is commutative whenever $\mathfrak{s}$ is commutative. $\square$

Proof 2 - not using the method of suitable models. It is obvious that every simple projectional skeleton is a projectional skeleton. Thus, (i) easily follows from (ii).

Let us assume $\mathfrak{s}=\{P_s\}_{s\in\Gamma}$ is an r-projectional skeleton in X and $D=D(\mathfrak{s})$. Fix a Markushevich basis $\{x_i\}_{i\in I}$ on X.

For every $A\in[I]^{\leq\omega}$ we will find sets $I(A)\in[I]^{\leq\omega}$, $D(A)\in[D]^{\leq\omega}$ and an up-directed set $\Gamma(A)\in[\Gamma]^{\leq\omega}$ such that if we put $t_A=\sup(\Gamma(A))$ and $X_A=[x_i;\ i\in I(A)]$ then

(a) for every $A,B\in[I]^{\leq\omega}$, if $X_A=X_B$ then $\Gamma(A)=\Gamma(B)$;

(b) for every $A\in[I]^{\leq\omega}$, $A\subset I(A)$;

(c) for every sequence of sets $(A_n)_{n=1}^\infty$ from $[I]^{\leq\omega}$, $X_{\bigcup_{n\in\mathbb{N}}A_n}=\overline{\bigcup_{n\in\mathbb{N}}X_{A_n}}$;

(d) for every $A\in[I]^{\leq\omega}$, $\operatorname{Rng}(P_{t_A})=X_A$ and $\ker(P_{t_A})=D(A)_\perp$;

(e) for every $A,B\in[I]^{\leq\omega}$, if $X_A\subset X_B$ then $D(A)\subset D(B)$.

Let us first verify that $\mathcal{F}=\{X_A;\ A\in[I]^{\leq\omega}\}$ is a rich family of separable subspaces and $\mathfrak{s}'=\{P_F\}_{F\in\mathcal{F}}$ is a simple r-projectional skeleton, where $P_{X_A}=P_{t_A}$ for every $X_A\in\mathcal{F}$ (by (a), P_{X_A} is well-defined), $D=D(\mathfrak{s}')$ and the skeleton $\mathfrak{s}'$ is commutative whenever $\mathfrak{s}$ is commutative.

Having a separable subspace Y find $A\in[I]^{\leq\omega}$ such that $Y\subset[x_i;\ i\in A]$. By (b), $Y\subset X_A$. It follows immediately from (c) that, for every increasing sequence F_i in $\mathcal{F}$, $\overline{\bigcup_{i=1}^\infty F_i}$ belongs to $\mathcal{F}$. Hence, $\mathcal{F}$ is a rich family of separable subspaces. By (d), $\operatorname{Rng}(P_F)=F$ for every $F\in\mathcal{F}$. Fix $E,F\in\mathcal{F}$ with $E\subset F$. Then $P_E=P_F\circ P_E$, because $\operatorname{Rng}(P_E)\subset\operatorname{Rng}(P_F)$. By (e) and (d), $\ker(P_F)\subset\ker(P_E)$; thus, $P_E=P_E\circ P_F$. Hence, $\mathfrak{s}'$ is a simple projectional skeleton. For every $F\in\mathcal{F}$, $\|P_F\|\leq r$ and $\mathfrak{s}'$ is commutative whenever $\mathfrak{s}$ is commutative. Moreover, $D(\mathfrak{s}')=\bigcup_{A\in[I]^{\leq\omega}}\overline{D(A)}^{w^*}$. As any set induced by a projectional skeleton is countably closed, we have that $D(\mathfrak{s}')\subset D$. By [33, Corollary 19], $D(\mathfrak{s}')=D$.

It remains to construct, for every $A\in[I]^{\leq\omega}$, sets $I(A)\in[I]^{\leq\omega}$, $D(A)\in[D]^{\leq\omega}$ and an up-directed set $\Gamma(A)\in[\Gamma]^{\leq\omega}$ satisfying the conditions above. In the following we say a set $E\subset X^*$ is r-norming for $Z\subset X$ if, for every $x\in Z$, $\|x\|\leq r\sup\{x^*(x);\ x^*\in B_{X^*}\cap E\}$. Fix a well-ordering $\lhd$ on the set $D\cup\Gamma\cup[I]^{\leq\omega}$

Fix $A\in[I]^{\leq\omega}$. Now, we inductively define sequences of countable sets $(I_n^A)_{n=1}^\infty$, $(\Gamma_n^A)_{n=1}^\infty$, $(D_n^A)_{n=1}^\infty$ in the following way. We put $I_1^A=A$, $\Gamma_1^A=\emptyset$ and $D_1^A=\emptyset$. Let us assume that $(I_n^A)_{n=1}^k$, $(\Gamma_n^A)_{n=1}^k$ and $(D_n^A)_{n=1}^k$ were already defined. We put $X_k^{\mathbb{Q}}=\mathbb{Q}$-span$\{x_i;\ i\in I_k^A\}$. This a countable set. In order to satisfy (d) and (b), it is enough to define I_{k+1}^A, Γ_{k+1}^A and D_{k+1}^A in such a way that those sequences are increasing and

- D^A_{k+1} is r-norming for $X^{\mathbb{Q}}_k$,
- Γ^A_{k+1} is an up-directed set satisfying $X^{\mathbb{Q}}_k \subset \bigcup_{s\in\Gamma^A_{k+1}} P_s(X)$ and $D^A_{k+1} \subset \bigcup_{s\in\Gamma^A_{k+1}} P^*_s(X^*)$,
- I^A_{k+1} is a set satisfying $\bigcup_{s\in\Gamma^A_{k+1}} P_s(X) \subset [x_i;\ i\in I^A_{k+1}]$.

However, in order to satisfy (a), (c) and (e), this construction must be done in a more precise way. Now, we define

$$D^A_{k+1} = D_k \cup \bigcup_{l\in\mathbb{N}} \bigcup_{x\in X^{\mathbb{Q}}_k} \{d\in D;\ d \text{ is the } \lhd\text{-least element of } D \text{ with } \|d\|\le 1 \text{ and } \|x\|\le r|d(x)|+1/l\}.$$

Having defined D^A_{k+1}, we put $\Gamma^A_{k+1} = \bigcup_{m\in\mathbb{N}} \Gamma^A_{k+1,m}$ where $\Gamma^A_{k+1,m}$ are inductively defined as follows.

$$\Gamma^A_{k+1,1} = \Gamma_k \cup \bigcup_{x\in X^{\mathbb{Q}}_k} \{s\in\Gamma;\ s \text{ is the } \lhd\text{-least element of } \Gamma \text{ satisfying } x\in P_s(X)\} \cup \bigcup_{d\in D_{k+1}} \{s\in\Gamma;\ s \text{ is the } \lhd\text{-least element of } \Gamma \text{ satisfying } d\in P^*_s(X^*)\},$$

$$\Gamma^A_{k+1,m+1} = \Gamma^A_{k+1,m} \cup \bigcup_{u,v\in\Gamma^A_{k+1,m}} \{s\in\Gamma;\ s \text{ is the } \lhd\text{-least element of } \Gamma \text{ satisfying } s\ge u,v\}.$$

Finally, we put

$$I^A_{k+1} = I^A_k \cup \bigcup_{s\in\Gamma^A_{k+1}} \{\cup S;\ S\in[I]^{\le\omega} \text{ is the } \lhd\text{-least element of } [I]^{\le\omega} \text{ satisfying } P_s(X)\subset[x_i;\ i\in S]\}.$$

Now $I(A) = \bigcup_{n\in\mathbb{N}} I^A_n$, $\Gamma(A) = \bigcup_{n\in\mathbb{N}} \Gamma^A_n$ and $D(A) = \bigcup_{n\in\mathbb{N}} D^A_n$. It follows immediately from the construction above that

(1) $D(A)$ is r-norming for X_A,

(2) $\Gamma(A)$ is an up-directed set satisfying $X_A \subset \overline{\bigcup_{s\in\Gamma(A)} P_s(X)}$ and $D(A) \subset \bigcup_{s\in\Gamma(A)} P^*_s(X^*)$,

(3) $\bigcup_{s\in\Gamma(A)} P_s(X) \subset [x_i;\ i\in I(A)] = X_A$,

(4) for every $A,B\in[I]^{\le\omega}$, if $X_A\subset X_B$ then $I(A)\subset I(B)$, $D(A)\subset D(B)$ and $\Gamma(A)\subset\Gamma(B)$,

(5) for every $A\in[I]^{\le\omega}$, $A\subset I(A)$,

(6) for every $A\in[I]^{\le\omega}$, $I(A) = I(I(A))$,

(7) for every $A,B\in[I]^{\le\omega}$ with $A\subset B$, $I(A)\subset I(B)$.

Now, (a) and (e) follow from (4); (b) from (5). From (2) and (3) it follows that

$$P_{t_A}(X) = \overline{\bigcup_{s\in\Gamma(A)} P_s(X)} = X_A.$$

In order to see that $\ker(P_{t_A}) = D(A)_\perp$, let us fix $x \in \ker(P_{t_A})$ and $d \in D(A)$. By (2), there exists $s \in \Gamma(A)$ with $P_s^*(d) = d$; hence, $d(x) = P_s^*(d)(x) = d(P_s x) = d(P_s \circ P_{t_A} x) = d(0) = 0$ and $x \in D(A)_\perp$. Thus, $\ker(P_{t_A}) \subset D(A)_\perp$. Moreover, since D_A is r-norming for $X_A = \mathrm{Rng}(P_{t_A})$, $\mathrm{Rng}(P_{t_A}) \cap D(A)_\perp = \{0\}$. Consequently, $\ker(P_{t_A}) = D(A)_\perp$. We have verified that (d) holds.

It remains to verify (c). Let us fix a sequence of sets $(A_n)_{n=1}^\infty$ from $[I]^{\leq\omega}$. By (6) and (7), similarly as in the proof of Lemma 4.2.6 (iv), $I(\bigcup_n A_n) = \bigcup_{n=1}^\infty I(A_n)$. Hence,

$$X_{\bigcup_{n\in\mathbb{N}} A_n} = [x_i;\ i \in I(\cup_{n\in\mathbb{N}} A_n)] = [x_i;\ i \in \cup_{n\in\mathbb{N}} I(A_n)] = \overline{\cup_{n\in\mathbb{N}}[x_i;\ i \in I(A_n)]} = \overline{\bigcup_{n\in\mathbb{N}} X_{A_n}}$$

and (c) holds. □

Remark 4.4.4. Inspecting the proof of Theorem 4.4.3, it is immediate that we can without loss of generality assume the projections in a projectional skeleton are all created from suitable models. More precisely, let $\varphi_1, \ldots, \varphi_n$ be a finite list of formulas and Y a countable set. Let X be a Banach space and let $D \subset X^*$ be an r-norming subspace ($r \geq 1$). Then the following assertions are equivalent:

(i) X has a (commutative) r-projectional skeleton $\mathfrak{s}$ with $D = D(\mathfrak{s})$.

(ii) X has a simple (commutative) r-projectional skeleton $\{P_F\}_{F\in\mathcal{F}}$ with $D = D(\mathfrak{s})$ such that, for every $F \in \mathcal{F}$, there exists $M \prec (\varphi_1, \ldots, \varphi_n; Y)$ satisfying $\mathrm{Rng}\, P_F = X_M$ and $\ker P_F = (D \cap M)_\perp$.

Remark 4.4.5. If X has a commutative projectional skeleton, we may construct a simple commutative skeleton in a more transparent way. However, some nontrivial results are used. We refer reader to [22, 34, 6] where necessary definitions used bellow may be found. First, X has a commutative 1-projectional skeleton under an equivalent renorming; hence, by [33, Proposition 29], we may without loss of generality assume (B_{X^*}, w^*) has a commutative retractional skeleton. By [34, Theorem 6.1], (B_{X^*}, w^*) is a Valdivia compact space. For an arbitrary Valdivia compact K it is easy to construct a simple commutative projectional skeleton in the space $\mathcal{C}(K)$.

Indeed, let us follow the ideas and notation used in the proof of [22, Theorem 19.14]. Let $K \subset [0,1]^\kappa$ be such that $\Sigma(\kappa) \cap K$ is dense in K. We call a set $T \subset \kappa$ *admissible* if $x \cdot \chi_T \in K$ for every $x \in K$ and we denote by Γ the set of all countable admissible subsets of κ. For every $S \in \Gamma$, we define $r_S : K \to K$ by $r_S(x) = x \cdot \chi_S$, $x \in K$. Then $\{r_S\}_{S\in\Gamma}$ is a commutative retractional skeleton. Now, for every $S \in \Gamma$, we define $P_S : \mathcal{C}(K) \to \mathcal{C}(K)$ by $P_S(f) = f \circ r_S$, $f \in \mathcal{C}(K)$. Then $\mathfrak{s} = \{P_S\}_{S\in\Gamma}$ is a commutative projectional skeleton. Moreover, assuming (without loss of generality) that $S = T$ whenever $r_S = r_T$, it is not difficult to verify that $\mathrm{Rng}(P_S) \subset \mathrm{Rng}(P_T)$ if and only if $S \subset T$; hence, $\mathfrak{s}$ is a simple commutative projectional skeleton.

It follows that $\mathcal{C}(B_{X^*}, w^*)$ has a simple projectional skeleton $\{P_S\}_{S\in\Gamma}$. By [6, Proposition 3.1 and Lemma 4.4], we may without loss of generality assume that $\{P_S \restriction_X\}_{S\in\Gamma}$ is a simple projectional skeleton in X.

We used nontrivial results [34, Theorem 6.1] and [6, Proposition 3.1]; however, the construction of the simple projectional skeleton itself comes from the Valdivia case which is easier to handle.

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