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BACHELOR THESIS

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Časově závislé procesy v teorii strun

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Časově závislé procesy v teorii strun

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Abstrakt: Teorie strunových polí je přístup k teorii strun využívající druhého kvantování. Tento přístup umožňuje popsat off-shell procesy, jako je například přechod mezi dvěma vakui, tzv. tachyonová kondenzace, včetně jeho dynamiky. V této práci jsou shrnuty některé základní nástroje a koncepty, které jsou potřeba ke zformulování takové teorie pro otevřené bosonové struny. Konkrétně popisujeme elementární metody konformní teorie pole a jednoduché koncepty perturbační teorie strun. Následně identifikujeme kinetický člen akce strunných polí z on-shell podmínky perturbační teorie strun a axiomaticky zavedeme algebraickou strukturu na jejím Hilbertově prostoru za účelem zavedení interakčních členů. Krátce zmiňujeme korespondenci mezi klasickými řešeními strunové teorie pole a různými pozadími teorie strun a uvádíme nástroje pro rekonstrukci pozádí z daného řešení ve formě hraničního stavu. Dále ukážeme, že takové pozadí se chová jako zdroj uzavřených strun. Práce je uzavřena výpočtem hraničního stavu pro řešení reprezentující světlupodobnou kondenzaci tachyonu.

Klíčová slova: Teorie strun, konformní teorie pole, hraniční stavy, teorie strunových polí, kondenzace tachyonů

Title: Time dependent processes in String Theory

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Abstract: String field theory is a second quantized approach to string theory able to describe off-shell processes like transition between vacua, so-called tachyon condensation, and its dynamics. In this work, we review the necessary tools and ingredients needed to formulate such theory for open bosonic string theory. Namely, we introduce elementary conformal field theory methods and some basics of perturbative string theory. We then recognize the kinetic term of string field action from on-shell condition of the perturbative string and axiomatically introduce an algebraic structure on its Hilbert space to construct interaction terms. We briefly discuss the connection of classical string field theory solutions with various string field backgrounds and present means of reconstructing the background from a given solution by calculating the boundary state. We show that such backgrounds act like a source of open string states. We conclude by calculation of such boundary state for a solution representing light-like tachyon rolling.

Keywords: String theory, conformal field theory, boundary states, string field theory, tachyon condensation

Content

Introduction

Open bosonic string theory plays a similar role in string theory as free scalar field in quantum field theory. It is not a model that we are ultimately interested in since it does not have some physical properties that we observe, most importantly such model does not have any fermions. But it is relatively simple and it captures a lot of structure of the framework we are working in. By studying it, we can develop useful tools and learn valuable lessons that will be put to use when we will study the theory of main interest, which is superstring theory.

In this thesis, we will focus on open strings whose ends are connected to a space filling D25-brane. The most important aspect of this theory for our purposes is the existence of tachyonic ground state of open string. As we know from quantum field theory, negative mass squared is not pathological to a field theory, it just means that the vacuum we are expanding the theory around is unstable, and the field will make a transition into a stable vacuum. The instability due to existence of tachyon was reconciled by Ashoke Sen, who identified it with the instability of the space filling D-brane. The transition to a stable vacuum (tachyon vacuum) is known as tachyon condensation.

A great bit of progress in this subject was due to Ashoke Sen, who postulated three conjectures about this process. These conjectures state the following [13], [14]:

- The energy difference between the original vacuum and the tachyon vacuum exactly matches the D25-brane tension, and therefore the space-filling brane ceases to exist.
- There are no physical perturbative open strings around tachyon vacuum.
- Lower dimensional D-branes can be obtained as solitonic solutions.

Since the transition between the vacua is a non-perturbative process, we need to formulate a second quantized theory. This is the string field theory that was first proposed by Edward Witten in [19]. In this framework, Sen's conjectures have been tested by many calculations and two of them were even proven analytically by Schnabl and Ellwood [11],[7].

Aside from the qualitative aspects of tachyon condensation, the dynamics of such process can also be studied. Solutions describing this transition have never been acquired in full generality only with some additional ansatz. Most of the work done assumed space homogeneity. However, this does not seem to be so physically relevant case as was argued by Schnabl and Hellerman in [8]. Instead, they proposed that the field depends only on one lightcone coordinate. Such solution represents a planar wave propagating through spacetime with speed of light leaving the tachyon vacuum behind it. For technical reasons such solution requires a nontrivial dilaton background. In this thesis, we will calculate the boundary state of such solution from gauge invariant overlaps using the Kudrna-Macaffari-Schnabl construction [6].

Our thesis is mostly an introductory material for the study of string field theory. The first chapter is dedicated to conformal field theory, which is the basic tool of string theory. We utilize a special case, the free boson, to illustrate these methods. This particular example was chosen because the string theory action principle, which is introduced in the second chapter, reduces to the one of free boson. We then continue to develop few other key ideas that are essential for the introduction of string field theory (SFT). In the third chapter, we finally introduce SFT for open strings connected to a space filling D-brane. We comment on the definitions of star product, boundary state construction as well as simple tools of solving the equations of motion. We continue with description of how boundary states act like sources of closed string states. At the end of this work, we construct the boundary state for above mentioned light-like solution, using the discussed methods.

1. Free boson conformal field theory

In order to get to basic string theory and then to string field theory, we need to lay down some tools that we will be using. The main tool, to which this chapter is dedicated, is the conformal field theory, which is a rather rich subject on its own. For this reason we will not present a general introduction; instead, we will go through the particular parts that are going to be useful to us. These are mainly fundamental definitions and results for the free boson CFT (conformal field theory) in two flat dimensions. A broader introduction to the subject with relevance to string theory can be found in [10],[17], which are the main sources for this chapter.

1.1 The complex plane

Conformal field theory in two dimensions is rather special, as we shall see later in this chapter, and much of this uniqueness is due to the possibility of describing the plane by complex numbers. Before we get to any field theory, let us explain how such description works.

Classical description of a flat plane is given by a metric and two Cartesian coordinates σ^1 and σ^2. We already know that we want to work with flat metric, but we can still choose between the Minkowski and the Euclidean signature. Since we study physical theories, we should consider the Minkowski case; however, the calculations are very similar in both cases and final results can be always related through Wick rotation (substitution $\sigma^2 = i\sigma^0$). Therefore, we are free to choose the signature, and since the Euclidean case is simpler and more elegant, we will use it to describe the plane.

The complex coordinates are defined in terms of the Cartesian ones as follows:

$$z = \sigma^1 + i\sigma^2, \quad \bar{z} = \sigma^1 - i\sigma^2. \tag{1.1}$$

The holomorphic derivatives are

$$\partial_z = \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{1.2}$$

Note that these derivatives obey following relations

$$\partial z = 1, \quad \partial \bar{z} = 0, \quad \bar{\partial} z = 0, \quad \bar{\partial}\bar{z} = 1, \tag{1.3}$$

and therefore $\bar{\partial}$ annihilates holomorphic and ∂ antiholomorphic functions.

If we would wick rotate to Minkowski space, the dependence on z would become the dependence on lightcone coordinate $\sigma^1 - \sigma^0$, and hence holomorphic functions which depend only on z are sometimes called left-moving and by analogous argument antiholomorphic functions are called right-moving.

The Euclidean metric in complex coordinates is

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} \tag{1.4}$$

and the measure element is

$$\mathrm{d}\sigma^1\mathrm{d}\sigma^2 = \frac{1}{2}\mathrm{d}z\mathrm{d}\bar{z}. \tag{1.5}$$

Since there is a factor of half in the expression for measure element, we use a slightly different definition for delta function

$$\int \mathrm{d}^2 z\ \delta(z, \bar{z}) = 1, \tag{1.6}$$

which differs from a standard definition by a factor of half coming from the measure. It satisfies $\delta(z, \bar{z}) = \frac{1}{2}\delta^2(\sigma^1, \sigma^2)$.

1.1.1 Conformal transformation

One of the amazing aspects of CFT in two dimensions is that the set of conformal transformations is in one to one correspondence with the set of holomorphic functions. To see this, we need to go a step back from our complex notation to a general manifold, where conformal transformations are defined as the class of transformations whose generators $\epsilon_a(x)$ satisfy the conformal Killing equation

$$\nabla_a \epsilon_b(x) - \nabla_b \epsilon_a(x) = \Omega^2(x) g_{ab}(x). \tag{1.7}$$

By taking the trace and substituting back, we can eliminate Ω:

$$\nabla_a \epsilon_b(x) - \nabla_b \epsilon_a(x) = \frac{2}{d} \nabla_c \epsilon^c(x) g_{ab}(x), \tag{1.8}$$

where d is the dimension of the manifold.

Now let us restrict ourselves to the case we are interested in, the flat 2 dimensional plane. The covariant derivatives become partial derivatives and the metric becomes Kronecker's delta:

$$\partial_a \epsilon_b(x) - \partial_b \epsilon_a(x) = \partial_c \epsilon^c(x) \delta_{ab}. \tag{1.9}$$

By evaluating this equation for specific indices, we get two independent constrains

$$\partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0, \tag{1.10}$$

$$\partial_1 \epsilon_1 - \partial_2 \epsilon_2 = 0. \tag{1.11}$$

These are exactly the Riemann-Cauchy equations. To satisfy them, ϵ^a must be a function of $\sigma^1 + i\sigma^2$ only or, in language of complex analysis, $\epsilon(z, \bar{z})$ is a holomorphic function. From the generators we can build up a finite transformation, and since the generators are holomorphic, the finite transformation will be as well. Therefore, for every holomorphic function $f(z)$ we have a transformation given as follows:

$$z \to w = f(z), \qquad \bar{z} \to \bar{w} = \bar{f}(\bar{z}). \tag{1.12}$$

The measure element transforms as $\mathrm{d}^2 z \to \left|\mathrm{d}f/\mathrm{d}z\right|^2 \mathrm{d}^2 z$.

The fact that holomorphic change of coordinates is a conformal transformation is actually the reason why is conformal field theory so powerful in two dimensions. There is a conformal transformation for every holomorphic function, and therefore the conformal group is infinite dimensional. In such case, the conformal invariance becomes so restrictive to enable solving exactly even nontrivial interacting models. Theories on higher dimensional flat spaces always have finite dimensional conformal group.

1.2 Free boson action

Dynamics of collection of free bosons in two flat dimensions with Euclidean metric signature is given by the following Lagrangian

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2 z \, (\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu). \tag{1.13}$$

The factor of $1/4\pi\alpha'$ may seem a little arbitrary at the moment, but its physical meaning will become clear when we come to string theory. In preparation for the following chapter, we are considering a collection of D fields indexed by μ. These will eventually represent coordinates in so-called target spacetime. If we write this action in terms of complex coordinates, it becomes

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \partial X^\mu \bar{\partial} X_\mu \tag{1.14}$$

yielding classical equations of motion

$$\partial\bar{\partial} X^\mu(z,\bar{z}) = 0. \tag{1.15}$$

Since the derivatives are interchangeable, it follows that $\partial X^\mu(z,\bar{z})$ is holomorphic and $\bar{\partial} X^\mu(z,\bar{z})$ is antiholomorphic and thus justifying shorthand notation $\partial X^\mu(z)$ and $\bar{\partial} X^\mu(\bar{z})$. We can also decompose $X^\mu(z,\bar{z})$ into holomorphic and antiholomorphic part

$$X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}). \tag{1.16}$$

We will often simplify formulae by writing them only for the holomorphic part since the antiholomorphic part is completely analogous, differing from the holomorphic part only by bars over partial derivatives and by the dependence on $\bar{z}$ instead of z.

1.3 Conformal properties of free boson

Now we would like to show that free boson is conformally invariant theory. That means the action has to be conformally invariant. Consider a general conformal transformation $z \to w = f(z)$. The respective terms in action transform as follows:

$$\begin{aligned} \frac{\partial X^\mu(z)}{\partial z} &\to \frac{\partial X^\mu(f(z))}{\partial z} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial X^\mu(w)}{\partial w}, \\ \mathrm{d}^2 z &\to \mathrm{d}^2 z = \left|\frac{\mathrm{d}z}{\mathrm{d}f}\right|^2 \mathrm{d}^2 w. \end{aligned} \tag{1.17}$$

From the antiholomorphic part we would get the complex conjugate of $\mathrm{d}f/\mathrm{d}z$, and thus together with the transformation of holomorphic part we get the reciprocal value of the Jacobian term from the measure, which as a result cancel each other.

1.3.1 Stress energy tensor

Stress energy tensor is defined as a conserved current associated with translation symmetry. We can obtain it, using the Noether's procedure. Such tensor is not uniquely given since we can make changes to it without breaking the conservation law (see Belinfante construction [?]). However, there is a useful trick for deriving a symmetric stress energy tensor for a theory. Tensor given by this construction is sometimes called covariant, and it is the one that we know from general relativity.

Suppose that the theory is minimally coupled to the worldsheet metric g_{ab} and that the theory no longer necessarily sits on flat space. Thus, we get following Langrangian

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \sqrt{g} \partial_a X^\mu \partial_b X_\mu g^{ab}, \tag{1.18}$$

where the indeces a and b run through z and $\bar{z}$. For such theory, the stress energy tensor given by the Noether's procedure can be obtained as:

$$T_{ab} = \frac{-2\pi}{\sqrt{g}} \frac{\partial \mathcal{L}}{\partial g^{ab}}, \tag{1.19}$$

where $\mathcal{L}$ is the Lagrangian density. For more comments on how this works see [16]. After calculating this quantity in this generalized theory, we get:

$$T_{ab} = -\frac{1}{\alpha'} \Big(\partial_a X_\mu \partial_b X_\mu - \frac{1}{2} g_{ab} \partial_c X^\mu \partial^c X_\mu \Big). \tag{1.20}$$

Now to get the stress-energy tensor of our theory, we substitute for g_{ab} the flat metric 1.4. By doing so, we find out that $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and

$$T_{zz} = -\frac{1}{\alpha'} \partial X^\mu \partial X_\mu, \tag{1.21}$$

$$T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} \bar{\partial} X^\mu \bar{\partial} X_\mu. \tag{1.22}$$

One of basic consequences of conformal symmetry is that the trace of the energy momentum tensor vanishes (this is actually result of scale invariance). The tracelessness of T_{ab} translates in complex coordinates as $T_{z\bar{z}} = 0$, which, as we have seen, holds for the tensor we just derived. The energy-momentum tensor is a conserved current, therefore we have $\partial_a T^{ab} = 0$. Together with tracelessness this gives us $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$. Being so we see that $T_{zz}(z, \bar{z})$ is a holomorphic function, so we adapt a shorthand notation $T_{zz}(z, \bar{z}) = T(z)$. Similarly, $T_{\bar{z}\bar{z}}$ is antiholomorphic and will be from now on referred to as $\bar{T}(\bar{z})$.

1.4 Quantum aspects of free boson

So far we have only dealt with classical aspects of our field theory. Now we would like to get into the quantum theory. Before we do so, let us stress out a difference in terminology from QFT (quantum field theory). In QFT, when we are talking about fields, we mean various physical fields that are present in action and are integrated over in path integrals. In CFT any local expression is referred to as a field. For example, the various fields that have role in free boson CFT would be $X^\mu(z, \bar{z})$, $\partial^n X^\mu(z)$ or even composite expressions like $\mathrm{e}^{iX^0}(z, \bar{z})$. We could as well consider any sums or products of these simple expressions. When we turn to quantum theory, fields are promoted to local operators, and the quantities of interest are the expectation values of such operators. The expectation values are defined as path integrals with time ordered insertions of corresponding local functionals:

$$\langle \mathcal{F}[X] \rangle = \int \mathrm{d}[X] \mathcal{F}[X] \mathrm{e}^{-S[X]}. \tag{1.23}$$

Expressions like this are often called, by analogy with statistical mechanics, correlation functions. Correlation function usually has a different normalization. For example, it could be divided by a path integral with insertion of identity.

As we mentioned earlier, we can also multiply local operators. We have already seen an important operator of this type: the stress-energy tensor. However, as we have seen many times in quantum field theory, if we naively promote the Hamiltonian to an operator and evaluate the expectation value, we find out that it is infinite. Thus, we introduce a notion of normal ordering defined in terms of raising and lowering operators to subtract the infinities. Here we face a similar problem: If we calculate the expectation value of $\partial X^\mu(z)\partial X_\mu(w)$, we find out that it diverges as $w \to z$. As a consequence, we would like to introduce normal ordering, but we have not made any contact with raising and lowering operators, thus we need a different notion of normal ordering. In order to do so, let us first calculate a typically divergent quantity: the propagator.

1.4.1 Normal ordering

To derive propagator of the theory $\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle$, we use the property of path integral that it annihilates total derivatives. Using this, we get:

$$\begin{aligned} 0 &= \int \mathrm{d}[X] \frac{\delta}{\delta X_\mu(z,\bar{z})} \left[\mathrm{e}^{-S[X]} X^\nu(w,\bar{w}) \right] \\ &= \int \mathrm{d}[X] \mathrm{e}^{-S[X]} \left[\eta_{\mu\nu}\delta(z-w,\bar{z}-\bar{w}) + \frac{1}{\pi\alpha'}\partial\bar{\partial}X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \right] \\ &= \left\langle \eta_{\mu\nu}\delta(z-w,\bar{z}-\bar{w}) + \frac{1}{\pi\alpha'}\partial\bar{\partial}X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \right\rangle, \end{aligned} \tag{1.24}$$

and thus we obtain

$$\left\langle \partial\bar{\partial}X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \right\rangle = -\left\langle \pi\alpha'\eta_{\mu\nu}\delta(z-w,\bar{z}-\bar{w}) \right\rangle. \tag{1.25}$$

This equation can be viewed as a differential equation for the propagator. By using following standard result

$$\partial\bar{\partial}\ln|z-w|^2 = 2\pi\delta(z-w,\bar{z}-\bar{w}), \tag{1.26}$$

we get the propagator

$$\left\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \right\rangle = -\frac{\alpha'}{2}\eta_{\mu\nu}\left\langle \ln|z-w|^2 \right\rangle. \tag{1.27}$$

As we would expect, the propagator diverges as $w \to z$. This expression is exactly of the type we had problem with, and we even know how it diverges. As a result, a straight forward ordering is at hand, we just subtract the divergent piece. Accordingly, we use the following expression as the definition of normal ordering, which will be denoted by enclosure in colons:

$$: X^\mu(z,\bar{z})X^\nu(w,\bar{w}) := X^\mu(z,\bar{z})X^\nu(w,\bar{w}) + \frac{\alpha'}{2}\eta_{\mu\nu}\ln|z-w|^2. \tag{1.28}$$

Such normal ordering is distributive over addition and commutes with differentiation. We can also understand this expression as a definition of a contraction, and thus generalize it, using Wick's theorem onto composite operators.

Normal ordered products have an important property, they satisfy the quantum analogue of equations of motion. In order to derive them, we use the same trick as in the derivation of propagator. We thus get the following:

$$\begin{aligned}
0 &= \int \mathrm{d}[X]\frac{\delta}{\delta X_\mu(z,\bar{z})}\mathrm{e}^{-S[X]} \\
&= -\int \mathrm{d}[X]\mathrm{e}^{-S[X]}\frac{\delta S}{\delta X_\mu(z,\bar{z})} \\
&= -\Big\langle \frac{\delta S}{\delta X_\mu(z,\bar{z})}\Big\rangle \\
&= \frac{1}{\pi\alpha'}\Big\langle \partial\bar{\partial}X^\mu(z,\bar{z})\Big\rangle.
\end{aligned} \tag{1.29}$$

This is the content of Ehrenfest theorem. In other words: Expectation values obey the classical equations of motions. We could do the same calculation with other insertions in the path integral. As long as these insertions are sufficiently far from $(z,\bar{z})$, the calculation would proceed exactly the same. However, this is no longer true if the insertions are at $(z,\bar{z})$. We have already seen this in equation 1.25. The additional operator gave rise to a delta function, and therefore it no longer satisfied the equations of motion. But if we act with $\partial\bar{\partial}$ on 1.27, we get:

$$\Big\langle \partial\bar{\partial} : X^\mu(z,\bar{z})X^\nu(w,\bar{w}) : \Big\rangle = 0, \tag{1.30}$$

and thus we see that such normal ordered product obeys the equations of motion. This has a very important consequence: Functions $\Big\langle : X^\mu(z,\bar{z})X^\nu(w,\bar{w}) : \Big\rangle$ are harmonic, and therefore they have no singularities. This will become important later. The fact that normal ordered products satisfy the equations of motion is a general property, which holds for any string of local operators. Some authors even consider it the defining property. Applying our results to the energy-momentum tensor, we get:

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu(z)\partial X_\mu(z) : \tag{1.31}$$

and similarly for the antiholomorphic $\bar{T}(\bar{z})$. These fields have zero expectation value.

1.4.2 Operator product expansion

The operator product expansion (OPE) is a statement describing how local operators act when they approach each other inside of correlation function. The main idea is to approximate a pair of operators by a possibly infinite sum of local operators with factors that depend only on the former pair of operators and their separation. Suppose that we have a given operator basis $\{\mathcal{O}_i(z,\bar{z})\}_i$, then equations we would like to obtain are of the following form:

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k c_{ij}^k(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}). \tag{1.32}$$

It is important to stress out how should we understand this: The equation at hand is an operator equation, and therefore it is meant to hold inside of time ordered correlation functions even though the brackets signifying it are not present.

In order to derive such expressions, we will make use of the property that normal ordered strings of operators satisfy the equations of motion as we showed earlier. Let us demonstrate how this works on an example where we already know how normal ordering looks. Given two operators $X_\mu(z,\bar{z})$ and $X_\nu(w,\bar{w})$, we get

$$X_\mu(z,\bar{z})X_\mu(w,\bar{w}) = -\frac{2}{\alpha'}\eta_{\mu\nu}\ln|z-w|^2 + :X_\mu(z,\bar{z})X_\mu(w,\bar{w}): . \tag{1.33}$$

Now we can treat the normal ordered pair of operators as a function of two variables z and w, and Laurent expand the dependence on z around the point w. When we do this, the quantum equation of motion 1.28 guarantees that this expansion will have no terms with mixed derivatives and no singularities, therefore all negative powers of $z-w$ drop out. Applying this, we get

$$\begin{aligned} X_\mu(z,\bar{z})X_\nu(w,\bar{w}) = &-\frac{2}{\alpha'}\eta_{\mu\nu}\ln|z-w|^2 + :X_\mu(w,\bar{w})X_\mu(w,\bar{w}): \\ &+\sum_{n=1}^{\infty}\frac{(z-w)^n}{n!} :\partial^n X_\mu(w,\bar{w})X_\nu(w,\bar{w}): \\ &+\sum_{n=1}^{\infty}\frac{(\bar{z}-\bar{w})^n}{n!} :\bar{\partial}^n X_\mu(w,\bar{w})X_\nu(w,\bar{w}):, \end{aligned} \tag{1.34}$$

which is equation of type 1.32. Note that the important part of these expressions is the one that has singular behavior.

1.4.3 Examples of OPE

Using Wick's theorem and the procedure above, we can derive the OPE for other local operators that will be of interest to us. These calculations are fairly simple, so we will not carry them out explicitly. The following is a list of OPEs that will be important later on.

The TT OPE:

$$T(z)T(w) = \frac{D/4}{(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \tag{1.35}$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{D/4}{(\bar{z}-\bar{w})^4} + \frac{\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \dots \tag{1.36}$$

Dots at the end signify presence of non-singular terms. The factor in front of $1/4(z-w)^4$ is called central charge (usually denoted c), and the fact that in this case it is equal to the number of fields is due to the type of theory we are working with. For the antiholomorphic part the factor may be different. It is analogically called antiholomorphic central charge and is denoted $\bar{c}$. It should be stressed out that the bar here is only a notation device and does not signify complex conjugation. The $T(z)\bar{T}(\bar{w})$ OPE is non-singular.

The $\partial X^\mu T$ OPE:

$$T(z)\partial X^\mu(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} + \dots \tag{1.37}$$

The $\bar{\partial} X^\mu \bar{T}$ OPE:

$$\bar{T}(\bar{z})\bar{\partial} X^\mu(\bar{w}) = \frac{\bar{\partial} X^\mu(w)}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}^2 X^\mu(w)}{\bar{z}-\bar{w}} + \dots \tag{1.38}$$

The $:\mathrm{e}^{ik_\mu X^\mu}:$ OPE with T and $\bar{T}$:

$$T(z) : \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w}) = \frac{\alpha' k_\mu k^\mu}{4} \frac{: \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w})}{(z-w)^2} + \frac{\partial : \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w})}{z-w} + \dots \tag{1.39}$$

$$\bar{T}(\bar{z}) : \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w}) = \frac{\alpha' k_\mu k^\mu}{4} \frac{: \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} : \mathrm{e}^{ik_\mu X^\mu} : (w,\bar{w})}{\bar{z}-\bar{w}} + \dots \tag{1.40}$$

1.4.4 Classification of operators

Now we would like to classify local operators of our theory by their transformation properties. These properties are, in two dimensions, very tightly connected to a particular form of OPEs with conserved current associated with a symmetry at hand, in our case the conformal symmetry. This connection is due to Ward identities and their special form in two dimensions. We will not go into details of this, but it can be found in [10].

1.4.5 Quasi-primary operators

Let us consider following map that is proportional to parameter ϵ:

$$z \to (1+\epsilon)z, \qquad \bar{z} \to (1+\bar{\epsilon})\bar{z}. \tag{1.41}$$

We say that an operator is quasi-primary of weight $(h,\tilde{h})$ if it transforms (in leading order of epsilon) as

$$\mathcal{O} \to (1 - \epsilon(h + z\partial) - \bar{\epsilon}(\tilde{h} + \bar{z}\bar{\partial}))\mathcal{O}. \tag{1.42}$$

As we mentioned earlier, transformation properties of operators translate into their OPE with stress-energy tensor. Transformation 1.42 implies following form of OPEs

$$T(z)\mathcal{O}(w,\bar{w}) = \dots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots \tag{1.43}$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \dots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots \tag{1.44}$$

Note that all of the OPEs that we have shown are of this type. Therefore, all operators mentioned above were quasi-primary.

Under finite transformation given by function $f(z)$, quasi-primary operators transform as follows:

$$\mathcal{O}(z,\bar{z}) \to \mathcal{O}(f(z),\bar{f}(\bar{z})) = \Big(\frac{\partial f}{\partial z}\Big)^{-h}\Big(\frac{\partial \bar{f}}{\partial \bar{z}}\Big)^{-\tilde{h}}\mathcal{O}(z,\bar{z}). \tag{1.45}$$

Conformal weights are closely connected to scaling dimension Δ and spin s, namely they are the difference and sum:

$$h = \frac{1}{2}(\Delta + s), \tag{1.46}$$
$$\tilde{h} = \frac{1}{2}(\Delta - s). \tag{1.47}$$

1.4.6 Primary operators

Primary operators are defined by the following form of OPEs

$$T(z)\mathcal{O}(w,\bar{w}) = h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots \tag{1.48}$$
$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots \tag{1.49}$$

Comparing these with 1.44, we find out that all primary operators are also quasi-primary. Note that operators $: \mathrm{e}^{ik_\mu X^\mu} :$, ∂X^μ and $\bar{\partial} X^\mu$ are all primary operators as we can check from the OPEs above. Conformal weights can be read from them as well.

1.4.7 Correlation functions

As we mentioned earlier, correlation functions are proportional to path integrals with insertions. We can evaluate them, using path integral methods. However, the conformal symmetry of the theory presents us with another option. The requirement of conformal invariance constrains the form of simple correlation functions. Thus, we can calculate them without any reference to path integrals or even action principles. Let us now look how this works for the simplest nontrivial correlator. Consider following correlation function of quasi-primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$:

$$\Big\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\Big\rangle. \tag{1.50}$$

Since the theory is invariant under translations and rotations, we obtain

$$\Big\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\Big\rangle = f(|z_1 - z_2|), \tag{1.51}$$

where f is an undetermined function. Now we can consider another symmetry of the theory, the scale invariance $z \to \lambda z$. Using 1.42, we get

$$\begin{aligned}\Big\langle \mathcal{O}_1(\lambda z_1,\lambda\bar{z}_1)\mathcal{O}_2(\lambda z_2,\lambda\bar{z}_2)\Big\rangle &= \Big(\frac{\partial\lambda z}{\partial z}\Big)^{-h_1-h_2}\Big(\frac{\partial\lambda\bar{z}}{\partial\bar{z}}\Big)^{-\tilde{h}_1-\tilde{h}_2}\Big\langle \mathcal{O}(z,_1\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\Big\rangle \\ &= \lambda^{-h_1-\tilde{h}_1-h_2-\tilde{h}_2}\Big\langle \mathcal{O}(z,_1\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\Big\rangle. \end{aligned}\tag{1.52}$$

We learn that

$$f(|z_1 - z_2|) = \lambda^{h_1+\tilde{h}_1+h_2+\tilde{h}_2} f(\lambda|z_1 - z_2|). \tag{1.53}$$

The only function that scales like this is $C_{12}|z_1 - z_2|^{-h_1-\tilde{h}_1-h_2-\tilde{h}_2}$, where C_{12} is multiplicative constant (structure constant), and thus we get

$$f(|z_1 - z_2|) = \frac{C_{12}}{|z_1 - z_2|^{h_1+\tilde{h}_1+h_2+\tilde{h}_2}}. \tag{1.54}$$

Since the correlator has to be rotational invariant, the total spin within it has to add up to zero. Therefore, the conformal weights have to satisfy $h_1 = h_2 = h$ and $\tilde{h}_1 = \tilde{h}_2 = \tilde{h}$ [9]. Otherwise the correlator vanishes.

Analogically, we could do this for a correlator with three insertions and arrive at a similar result. The form of correlations is not entirely fixed for four or higher point functions. Concrete results can be found in [9].

As we mentioned earlier, correlation functions can be calculated, using path integral methods, one of which is the generating functional method. We define such functional as:

$$Z[J] = \Big\langle \exp\big(2i \int \mathrm{d}^2 z J_\mu(z,\bar{z}) X^\mu(z,\bar{z})\big)\Big\rangle. \tag{1.55}$$

Calculation of such functional is quite technical, and we will not go through it. Instead, we will leave you with chosen correlator calculated by this method in [10] in order to illustrate the nature of their form. Note that the complex plane has the topology of a two-dimensional sphere. Other topologies will lead to different correlations. Therefore, we denote the topology as a subscript of the correlator. A correlator of product of exponentials $: \mathrm{e}^{ik_\mu X^\mu} :$ obtained by this methods gives us:

$$\Big\langle \prod_i : \mathrm{e}^{ik_i^\mu X_\mu} : (z_i, \bar{z}_i) \Big\rangle_{S_2} = iC_{S_2}(2\pi)^D \delta^D(\sum_i k_i^\mu) \prod_{i<j} |z_i - z_j|^{\alpha' k_\mu k^\mu}, \tag{1.56}$$

where C_{S_2} is a constant, which will not be of immediate interest to us. The delta function constrains the sum of ks to be zero. This can be interpreted as momentum conservation when we turn to string theory.

1.5 Hilbert space formalism

So far we have only considered local operators and their expectation values defined through path integrals. Now we would like to talk about states of the theory. To do so, we have to quantize it. However, we will not do it by means of canonical quantization; instead, we will introduce so called radial quantization.

1.5.1 Radial quantization

Let us consider a little change in our theory accompanied by a change of description. Consider a theory defined on a cylinder instead of a plane. We will introduce a complex coordinate describing such cylinder

$$\omega = \sigma + i\tau. \tag{1.57}$$

The $\tau \in (-\infty, \infty)$ coordinate is associated with time-like direction while $\sigma \in [0, 2\pi]$ is space-like, with endpoints identified. The choice of time-like coordinate on a cylinder is not unique, and this ambiguity has some consequences; however, we will not discuss them in this thesis.

We can consider a conformal mapping $z = \mathrm{e}^{-i\omega}$, which takes us from cylinder to the whole complex plane. Under this transformation the past infinity maps to the origin, lines of constant σ are now rays originating at zero shooting radially away, and lines of constant τ become concentric circles around the origin. Let us

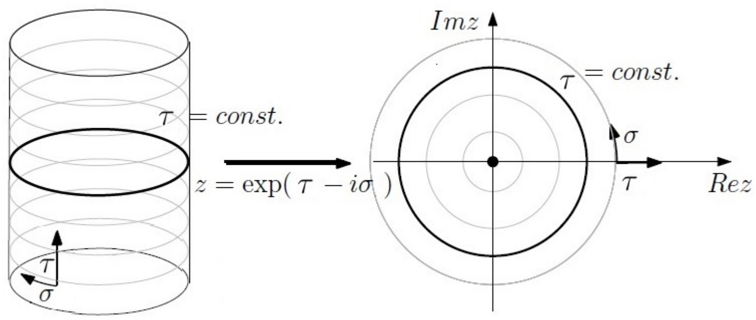

Figure 1.1: Mapping from a cylinder to the complex plane

now clarify why we have moved our theory from a flat plane to a cylinder and then mapped it back.

When quantizing a theory, we expand the field of interest into its Fourier modes and promote them to operators. Consider a holomorphic quasi-primary field ϕ of weight h:

$$\phi(\omega) = \sum_k a_k \mathrm{e}^{ik\omega}. \tag{1.58}$$

Now we apply the map from cylinder to a plane, the above expression then becomes:

$$\phi(z) = \sum_k \frac{a_k}{z^{k+h}}, \tag{1.59}$$

which is nothing but a Laurent expansion of the field ϕ defined on a plane. The upshot of this is that instead of Fourier coefficients we promote to operators the Laurent coefficients. Note that the labeling is shifted by h, so the zeroth mode ϕ_0 does not scale. This is not a conventional relabeling of the coefficients, the shift comes from the transformation properties of quasi primary operator under a conformal mapping 1.42.

Before we continue, there is one more thing that needs to be stressed out. When we define states on a plane, we have to remember that these states really live on a cylinder, where their evolution is governed by a Hamiltonian. After we map to a plane, the Hamiltonian becomes the dilation operator. This should not be surprising since scaling on a plane corresponds to shifts in τ back on cylinder.

1.5.2 Virasoro algebra

Let us now apply radial quantization on the holomorphic component of stress energy tensor $T(z)$.

$$T(z) = \sum_m \frac{L_m}{z^{m+2}} \tag{1.60}$$

The above expression can be inverted by a contour integral:

$$L_m = \frac{1}{2\pi i} \oint \mathrm{d}z z^{m+1} T(z). \tag{1.61}$$

The coefficients L_m are called Virasoro generators, and they satisfy the Virasoro algebra:

$$[L_m, L_n] = L_{m+n}(m-n) + \delta_{m+n,0}(m^2-1)m\frac{c}{12}, \tag{1.62}$$

where c is central charge. There is a corresponding set of operators $\bar{L}_m$ coming from the quantization of $\bar{T}(\bar{z})$. These commute with unbarred Ls.

The dilation operator D can be expressed in terms of these generators as

$$D = L_0 + \bar{L}_0. \tag{1.63}$$

1.5.3 State operator map

A very important aspect of conformal field theory in two dimension is the existence of a map between states and local operators. To see how this works, let us first comment on the states themselves.

In non-relativistic quantum theory of a point particle, a state of a system can be described using a wave function. Such wave function has a simple interpretation, it is the amplitude of locating the particle at point x. Given that we know the state of the system at time t_i, we can calculate the configuration at any time as

$$\psi(x_f, t_f) = \int \mathrm{d}x_i G(x_f, x_i, t_f, t_i)\psi(x_i, t_i), \tag{1.64}$$

where $G(x_f, x_i, t_f, t_i)$ is the propagator. We see that the wave function describing the initial state acts as a weighting factor. In the case of field theory, we do not have a wave function but a wave functional $\Psi[X]$ that describes amplitudes for the field configurations. Similarly, we can calculate the wave functional at later times, given that we know it at another.

$$\begin{aligned}\Psi[X_f](t_f) &= \int \mathrm{d}[X_i] G[X_f, X_i](t_i)\Psi[X_i, t_i] \\ &= \int \mathrm{d}[X_i] \int_{X(t_i)=X_i}^{X(t_f)=X_f} \mathrm{d}[X] \mathrm{e}^{-S[X]} \Psi[X_i, t_i]\end{aligned} \tag{1.65}$$

In radial quantization, time corresponds to a radial distance from the origin. In order to describe a configuration, we have to specify a weight along a circle of the radius corresponding with the initial time. This involves a non-local operator since it acts everywhere on a time slice. However, as we move to past infinity, the circles get smaller and smaller till they are mapped into the origin. But that is just a single point. By operating at this point, we can change the weighting, and thus specify the state at past infinity. We get

$$\Psi[X_f](t_f) = \int^{X(t_f)=X_f} \mathrm{d}[X]\mathrm{e}^{-S[X]}\mathcal{O}(0,0). \tag{1.66}$$

We will denote states corresponding to an operator as $|\mathcal{O}\rangle$. The simplest operator we have is the identity, and the resulting state plays a privileged role as well, it is the vacuum state of the theory denoted in the same spirit as $|1\rangle$. Acting with a local operator on states constructed this way means including them in path integral. Thus, it follows that $|\mathcal{O}\rangle = \mathcal{O}(0,0)\,|1\rangle$. Note that multiple insertions away from origin also represent states since we can use OPEs to move them into the origin.

Hand in hand with vacuum state, there comes a linear form $\langle 1|$ defined by its action on an arbitrary state given in terms of local operators:

$$\langle 1| \prod_i \mathcal{O}_i(z_i, \bar{z}_i)\,|1\rangle = \Big\langle \prod_i \mathcal{O}_i(z_i, \bar{z}_i) \Big\rangle. \tag{1.67}$$

1.5.4 Inner product

To every quasi primary state in our theory, we can construct a Hermitian conjugate. When we do this in Minkowski space, the space-time coordinates remain unchanged. In Euclidean space the situation is different. Time-like coordinate is now defined as Wick rotation of the time-like coordinate of Minkowski space-time, and thus carries an extra factor of i. This factor picks up a minus sign when Hermitian conjugated, and thus the time has to reverse its direction so that the original coordinate remains unchanged. In radial quantization this corresponds to inversion $r \to 1/r$, where r is the distance from origin. In complex coordinates this becomes $z \to 1/\bar{z}$. Hence, we define a Hermitian conjugate as

$$[\mathcal{O}(z, \bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\tilde{h}} \mathcal{O}(1/\bar{z}, 1/z). \tag{1.68}$$

Now we are ready to define Hermitian conjugate of a state

$$\begin{aligned} [\mathcal{O}(z, \bar{z}\,|1\rangle]^\dagger &= \langle 1|\,[\mathcal{O}(z, \bar{z}]^\dagger \\ &= \bar{z}^{2h} z^{2\tilde{h}}\,\langle 1|\,\mathcal{O}(1/\bar{z}, 1/z). \end{aligned} \tag{1.69}$$

When we conjugate operator that is inserted at the origin, the inversion takes it to infinity. For such case we define the insertion in terms of a limit.

Let us now check that inner product defined by such conjugation is a reasonable quantity. Consider two states $|\phi\rangle_1$, $|\phi\rangle_2$, then the inner product with itself is

$$\begin{aligned} \langle \phi_1 | \phi_2 \rangle &= \lim_{z,\bar{z}\to 0} \bar{z}^{-2h} z^{-2\tilde{h}}\,\langle 1|\,\phi_1(1/\bar{z}, 1/z)\phi_2(0,0)\,|1\rangle \\ &= \lim_{w,\bar{w}\to\infty} \bar{w}^{2h} w^{2\tilde{h}}\,\langle 1|\,\phi_1(\bar{w}, w)\phi_2(0,0)\,|1\rangle \\ &= \lim_{w,\bar{w}\to\infty} \bar{w}^{2h} w^{2\tilde{h}} \Big\langle \phi_1(\bar{w}, w)\phi_2(0,0) \Big\rangle. \end{aligned} \tag{1.70}$$

If we plugged in the two point function 1.54, all the w dependence would vanish, and thus the limit would be trivial. Therefore we see, that the inner product is well defined.

There is another inner product that is often used, the so called BPZ (Belavin-Polyakov-Zamolodchikov) product. It is defined by the following (for quasi primary operators):

$$\langle \phi_1 | \phi_2 \rangle = \lim_{z \to 0} \left(\frac{1}{z} \right)^{2h_1} \Big\langle \phi_1(-1/z) \phi_2(0) \Big\rangle. \tag{1.71}$$

Again, by using 1.54, we find out that the limit is well defined.

Both products are sometimes normalized to remove the structure constant coming from the correlator.

1.5.5 More on states and operators

Let us now examine the nature of states corresponding to particular operators. To interpret them, we need to expand them into modes where the interpretation is clear. In radial quantization we need to Laurent expand. The holomorphic fields that we have come across were the components of the stress energy tensor and the holomorphic and antiholomorphic derivatives of field X^μ. We have already seen the expansion of the stress energy tensor, the Virasoro generators, therefore we will only examine $\partial X^\mu(z)$. We will not go through the antiholomorphic part since it is analogous.

$$\partial X^\mu(z) = \sum_k \frac{\alpha_k^\mu}{z^{k+1}} \tag{1.72}$$

Let us now look how α^μs act on the vacuum state given by the state operator map as the insertion of identity. Note that the state $|1\rangle$ really stands for 1.66 with $\mathcal{O} = 1$.

$$\begin{aligned} \alpha_k^\mu |1\rangle &= \frac{1}{2\pi i} \oint \mathrm{d}z \partial X^\mu(z) z^k |1\rangle \\ &= \int^{X_f} \mathrm{d}[X] \mathrm{e}^{-S[X]} \frac{1}{2\pi i} \oint \mathrm{d}z \partial X^\mu(z) z^k \end{aligned} \tag{1.73}$$

In the case of $k \geq 0$, smooth functions ∂X^μ are annihilated by the contour integral. On the other hand any singularities cause the action to be infinite and thus vanish because of the weighting factor $\mathrm{e}^{-S[X]}$ [17]. Therefore we obtain

$$\alpha_k^\mu |1\rangle = 0 \qquad \forall m \geq 0. \tag{1.74}$$

The zeroth mode α_0^μ corresponds to a uniform increase of X^μ with time. When we get to string theory, this will have the interpretation of uniform movement through space, and thus it is identified (except for a constant) with the momentum operator. Since the vacuum state $|1\rangle$ is annihilated by α_0, we know that it carries zero momentum. In order to encapsulate this into our description, we invent a new label for our states $|\phi, p\rangle$ that keeps track of momenta. We will stray from our operator based notation of states in case of the vacuum. Instead of 1 denoting the insertion we will use standard 0. The original vacuum $|1\rangle$ will now be written as $|0, 0\rangle$.

One can easily check that the operator that changes the momentum is $:e^{ik_\mu X^\mu}:$. It satisfies

$$:e^{ik_\mu X^\mu}:(0,0)\,|0,0\rangle = |0,k\rangle\,. \tag{1.75}$$

1.6 Boundary CFT

Instead of a cylinder, we will now examine a theory defined on an infinite strip with Neumann boundary conditions. Conformal theories with boundaries are called boundary conformal theories (BCFT). The description of a strip is completely analogous with cylinder except that we do not make the σ coordinate periodic. It is conventional to consider $\sigma \in [0,\pi]$. With this choice the exact same map that took us from a cylinder to the complex plane now takes us to the upper half plane (UHP) with the real axis being the boundary. The analysis then continues very similarly. There are of course differences, which will be pointed out in this section.

1.6.1 Translation symmetry

One of the most obvious differences is the breakdown of translation invariance in direction normal to the boundary. We are still equipped with the stress energy tensor; however, it does not satisfy the full conservation law. Only the transverse part $T_{ab}t^b$ satisfies it, where t^b is a vector parallel to the boundary. Neumann boundary conditions are implemented by requiring that no energy flows through the edge. This translates to the following condition

$$T_{ab}t^b n^a(z) = 0 \qquad \forall \mathrm{Im} z = 0, \tag{1.76}$$

where the vector n^a is normal to the boundary. In components this becomes

$$T(z) = \bar{T}(\bar{z}) \qquad \forall \mathrm{Im} z = 0. \tag{1.77}$$

Since these components are holomorphic and antiholomorphic, they can be analytically continued to the other half of the plane. However, they have to satisfy the above condition on the real axis, and thus the continuation is

$$T(z) = \bar{T}(\bar{z}). \tag{1.78}$$

We found out that we have only independent component of the stress-energy tensor and only one set of Virasoro generators. What we have done here is known as the doubling trick and similar procedure can be applied in other cases as well. For example when calculating the propagator the following condition arises

$$\partial_\sigma G(z,\bar{z},w,\bar{w}) = 0 \qquad \text{for} \quad \sigma = 0. \tag{1.79}$$

Such problem can be solved by inclusion of an image charge. Doing so leaves us with

$$G(z,\bar{z},w,\bar{w}) = -\frac{\alpha'}{2}\ln|z-w|^2 - \frac{\alpha'}{2}\ln|z-\bar{w}|^2. \tag{1.80}$$

Note that it was the particular form of the divergence of the propagator that have lead us to the definition of normal ordering. With a different propagator there is also a different normal ordering - so called boundary normal ordering. Operators inserted on the boundary are always ordered this way. We denote this ordering by enclosure in $\substack{\star \\ \star}\ \substack{\star \\ \star}$.

$$\substack{\star \\ \star} X^\mu(y_1) X^\nu(y_2) \substack{\star \\ \star} = X^\mu(y_1) X^\nu(y_2) + 2\alpha' \ln|y_1 - y_2| \tag{1.81}$$

Note that ys are real since they are on the boundary. It is conventional to denote positions on the boundary by y and insertions away from it by z. It should be stressed out that the weight of boundary ordered operator is different from the normal ordered.

States can be constructed by state operator map in the same way as on the cylinder; however, origin lies on the boundary, so these operators are now boundary ordered.

There is yet another conformal mapping that is often used when working with theories on UHP. Using

$$f(z) = i\frac{z - i}{z + i}, \tag{1.82}$$

we can map UHP onto a unit disc. Under this transformation the past infinity maps into $-i$, the future infinity maps into i.

1.6.2 Boundary state

Theory on the strip is in many ways similar to the one on the cylinder. They are both governed by the same action, and thus their dynamics are the same away from boundary. At the boundary the theory on the strip gets constrained and so does the space of possible field configurations. It is not hard to see that the unbounded theory contains all configurations of the bounded case plus more. This is reflected by the fact that on the strip we only have one set of Virasoro generators in comparison to the cylinder where there are two of them. This leads us to the notion of describing the boundary field theory as unbounded while projecting out the configurations that do not satisfy imposed boundary conditions. In calculation of expectation values this would mean

$$\langle \mathcal{O} \rangle \to \langle \mathcal{O} \rangle_B \tag{1.83}$$

for any operator $\mathcal{O}$ of the theory. The index B signifies that we are requiring corresponding boundary conditions on the real axis. The boundary state is defined to do just this

$$\langle\langle B || \mathcal{O} \rangle = \langle \mathcal{O} \rangle_B. \tag{1.84}$$

1.7 Linear dilaton background

When we discussed the stress energy tensor, we have mentioned that there are changes that can be made to it without breaking the conservation law. However, this changes the conformal weights of quasi-primary operators, and thus we are dealing with a different theory. In this section we will review the results for this modified theory.

The modified stress-energy tensor is given as

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : + V^\mu \partial^2 X_\mu, \tag{1.85}$$

$$\bar{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : + V^\mu \bar{\partial}^2 X_\mu, \tag{1.86}$$

where V^μ is a fixed D-vector. Since there is a preferred direction, the theory is no longer covariant.

Changing the stress-energy tensor also changes the transformation properties of the fields. Under conformal mapping $z \to f(z)$ the field X^μ now transforms as

$$X^\mu(z, \bar{z}) \to X^\mu(f(z), f(z)^*) + \frac{\alpha'}{2} V^\mu \ln \left| \frac{\mathrm{d}f}{\mathrm{d}z} \right|^2. \tag{1.87}$$

If we impose Neumann boundary conditions, as we did on the strip, our symmetry group gets restricted only to transformations that satisfy additional condition $f(z)^* = f(\bar{z})$. This way the boundary remains intact. The fields on the boundary transform as

$$X^\mu(y) \to X^\mu(f(y)) + \alpha' V^\mu \ln \left| \frac{\mathrm{d}f}{\mathrm{d}y} \right|^2. \tag{1.88}$$

Since the central charge and the weights of quasi-primary operators were determined by calculating the OPE with stress-energy tensor, it should not be surprising that they change as well. For example, the weight of $: \mathrm{e}^{ik_\mu X^\mu} :$ is now

$$h = \alpha' \Big(\frac{k^2}{4} + i \frac{V^\mu k_\mu}{2} \Big). \tag{1.89}$$

Boundary ordered exponential ${}^\star_\star \mathrm{e}^{ik_\mu X^\mu} {}^\star_\star$ has weight

$$h = \alpha' \Big(k^2 + i V^\mu k_\mu \Big). \tag{1.90}$$

Correlators of the theory remain the same except that the momentum conservation gets modified. Let us illustrate this on a correlator of boundary insertions of exponentials on a disc.

$$\Big\langle \prod_i {}^\star_\star \mathrm{e}^{ik_i^\mu X_\mu} {}^\star_\star (y_i) \Big\rangle_{D_2} = i C_{D_2} (2\pi)^{26} \delta\big(\sum_i k_i^\mu + i V^\mu \big) \prod_{i<j} |y_i - y_j|^{2\alpha' k_i^\mu k_j^\nu \eta_{\mu\nu}}. \tag{1.91}$$

Note that the fact that we are dealing with disc topology does not mean we are indeed in the disc mapping. UHP has the same topology as a disc and is usually used to parametrize it. Therefore the insertions are meant to be on real axis. The delta function of complex argument is not well defined on its own and should be understood in its integral form. Because of the modification of the momentum conservation, the standard definition of BPZ product leads to

$$\langle 0, p | 0, q \rangle = C \delta^{26} (p^\mu + q^\mu + i V^\mu). \tag{1.92}$$

2. Basic string theory

In this chapter we will make first contact with string theory. We will motivate action for a string propagating through spacetime, and we will show how such action connects to the free boson CFT. We will also review BRST quantization (BRST refers to Becchi, Rouet, Stora and Tyutin). We will not however go into the spectrum of the string. This chapter mostly follows works of Polchinski [10]

2.1 Polyakov action

String theory studies physics of one dimensional object, a string. Such object, as it propagates, sweeps out a two dimensional world-sheet embedded in spacetime that can be described in terms of two parameters τ and σ, where σ is bounded. The embedding is then a collection of fields $X^\mu(\tau,\sigma)$, one for each spacetime direction. Since the choice of parametrization is rather arbitrary, we insist that the action describing the dynamics of such world-sheet does not depend on this choice, but only on the embedding itself. The simplest Poincare invariant action satisfying this condition is the area of the world-sheet [10]:

$$S = -\frac{1}{2\pi\alpha'}\int_M \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-h}, \tag{2.1}$$

where M denotes the world-sheet and $h = \det(h_{ab})$. The parameter α' is the Regge slope, which determines the tension of the string, but for the purposes of this thesis it will mostly tag along the calculations without having serious impact. h_{ab} is the induced metric on the world-sheet, given by

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu. \tag{2.2}$$

This is called the Nambu-Goto action and, as one can see, it is very analogous to the action describing the motion of free point particle. However, it is rather hard to work with because of the square root present. A more convenient alternative is the Polyakov action, which introduces an independent metric g_{ab} on the world-sheet. It is given by

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\sqrt{-g}\partial^a X^\mu \partial^b X_\mu g_{ab}, \tag{2.3}$$

where $g = \det(g_{ab})$.

As opposed to the Nambu-Goto action, this is rather nice looking expression and it even has an additional symmetry, the Weyl symmetry

$$\begin{aligned} X^\mu(\tau,\sigma) &\to X^\mu(\tau,\sigma), \\ g_{ab}(\tau,\sigma) &\to \mathrm{e}^{2\omega}(\tau,\sigma) g_{ab}(\tau,\sigma), \end{aligned} \tag{2.4}$$

along with the diffeomorphism (reparametrization) invariance and Poincare invariance. However, there is a downside as well, unlike before we do not yet know what such action represents. In order to find out, let us analyze the equations of motion that this action yields. We have two independent fields, thus we get two equations of motion:

$$0 = \partial_a(\sqrt{-g}\partial_b X^\mu g^{ab}) \tag{2.5}$$

$$g_{ab} = \frac{2}{g^{cd}\partial_c X^\nu \partial_d X_\nu}\partial_a X^\mu \partial_b X_\mu. \tag{2.6}$$

Substituting the equation for g_{ab} back into 2.3, we get back the original Nambu-Goto action. Therefore, we found that as long as the equation of motion for g_{ab} is satisfied the Polyakov action is equivalent to the Nambu-Goto action.

2.1.1 Open strings

As we mentioned, the σ coordinate of the string world-sheet is bounded, so in order to solve the equations of motion we have to impose boundary conditions. One option is to impose periodicity which would give us the closed string. Open string is achieved by imposing Dirichlet or Neumann conditions. These can differ for various directions of space. Dirichlet boundary conditions mean that the string is confined on so called D-brane of proper dimension, corresponding to the number of direction with Dirichlet boundary conditions. It is conventional to impose that $\sigma \in [0, 2\pi]$ for closed string and $\sigma \in [0, \pi]$ for open strings. The lesson to take from here is that the theory is not defined only by the action itself. But also by the choice of boundary conditions or in other words by specifying the background or a system of D-branes.

2.1.2 Conformal gauge

Symmetries of Polyakov action, except for Poincare invariance, are gauge symmetries. They reflect the fact that we have a redundant description of the system, and thus we can pose additional conditions on the fields. This process is called gauge fixing, and we can do this in a manner that will simplify our system considerably.

The complication in Polyakov action is the coupling to nontrivial world-sheet metric. However, as we can see, it does not depend on the metric so heavily since we can always make a Weyl transformation without affecting the action. We would like to use the Weyl invariance to somehow simplify this metric, ideally flatten it. Let us see what we can do by examining how curvature transforms under Weyl transformation. It is quite easy to check that for two metrics related by a Weyl transformation $g' = \mathrm{e}^{2\omega} g$ associated scalar curvatures satisfy [17]:

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega). \tag{2.7}$$

If we now set R' to zero, we get a differential equation for ω that can be solved. In general case, having zero scalar curvature does not mean that we have flattened the spacetime; however, in two dimensions the symmetries of Riemann tensor dictate that it is proportional to scalar curvature $R_{abcd} = R/2(g_{ac}g_{bd} - g_{ad}g_{bc})$. Therefore, we can always find a Weyl transformation that will make the metric flat. After that we can use reparametrization invariance to obtain the Minkowski metric. This choice is called the conformal gauge. By doing so, the action simplifies considerably:

$$S = -\frac{1}{4\pi\alpha'} \int_M \mathrm{d}\tau \mathrm{d}\sigma \partial^a X^\mu \partial^b X_\mu \eta_{ab}. \tag{2.8}$$

Thus, we effectively eliminated the dependence on the worldsheet metric, and therefore instead of two equations of motion we get only one. But there is a subtlety when performing such gauge fix. We still require that this action is equivalent to Nambu-Goto action, and therefore we require that equation 2.6 holds. Together with equation of motion for X fields we get:

$$\partial_a \partial^a X^\mu = 0, \tag{2.9}$$

$$\partial_\tau X_\mu \partial_\sigma X^\mu = 0, \tag{2.10}$$

$$(\partial_\tau X^\mu)^2 + (\partial_\sigma X^\mu)^2 = 0. \tag{2.11}$$

Now we are ready to make contact with previous chapter because, apart from the additional constraints, the action is the one of free boson on flat space and, as a result, everything we showed earlier for free boson can now be applied to strings.

Even though we have exhausted some gauge symmetry to fix the world-sheet metric to a flat one, there is still a lot of remnant symmetry. This should not be surprising now that we have identified the Polyakov action with the one of free boson which exerted conformal invariance. Hence, we see that we still have full conformal gauge symmetry at our disposal.

2.2 Polyakov path integral

Similarly as in chapter 1, we would like to define expectation values of local operators as path integrals with insertions. But in this case naive treatment like we used before would be wrong. We are dealing with a theory with a gauge symmetry, which means not every two field configurations are physically distinguishable. Configurations related by a gauge transformation are physically equivalent and should be counted only once. This is implemented by dividing the path integral by a volume of local gauge symmetry group. Thus we get

$$Z = \frac{1}{\text{Vol}} \int \mathrm{d}[X]\mathrm{d}[g]\mathrm{e}^{-S[X,g]}. \tag{2.12}$$

Note that in order to define the path integral we had to Wick rotate to Euclidean space, otherwise it would be ill-defined.

Our goal now is to separate the integration over physically equivalent configurations and over physically inequivalent ones. As in ordinary integrals, the change of coordinates is always accompanied by an insertion of Jacobian. Something similar happens in path integrals as well. The factor analogous to the Jacobian for this special case is called the Faddeev-Popov determinant. In order to calculate it, we adapt a simplifying notation that will combine Weyl and reparametrization transformations together.

We will denote a general action of a gauge transformation on a metric by a superscript ζ, which stands for the following:

$$g_{ab} \to g^\zeta_{ab} = g_{cd} \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} \mathrm{e}^{2\omega}. \tag{2.13}$$

Integration along the gauge orbit can now be written as integration over ζ. The measure for such integration is taken to be gauge invariant. Proof of its existence can be found in [10].

Inside integration along ζ all gauge related configurations are supposed to be counted as one. In order to implement this, we insert a delta functional inside the integral. We have freedom to choose which configuration out of gauge orbit will be counted. This choice is called the fiducial metric and is denoted by $\hat{g}$. The integral is then

$$\Delta_{FP}(g)\int \mathrm{d}[\zeta]\delta(g-\hat{g}^{\zeta})=1, \tag{2.14}$$

where $\Delta_{FP}(\hat{g})$ is called Faddeev-Popov determinant.

Now we would like to show that $\Delta_{FP}(g)$ is really the factor that we need, but before we do that, we have to prove that $\Delta_{FP}(g)$ is gauge invariant:

$$\begin{aligned}\Delta_{FP}^{-1}(g^{\zeta}) &= \int \mathrm{d}[\zeta']\delta(g^{\zeta}-\hat{g}^{\zeta\zeta'})\\ &= \int \mathrm{d}[\zeta']\delta(g-\hat{g}^{\zeta^{-1}\zeta'})\\ &= \int \mathrm{d}[\zeta'']\delta(g-\hat{g}^{\zeta''})\\ &= \Delta_{FP}^{-1}(g)\end{aligned} \tag{2.15}$$

which concludes the proof. In second line we used that the measure is invariant.

The expression 2.14 is equal to one, and therefore we can insert it inside 2.12 to get

$$Z=\frac{1}{\mathrm{Vol}}\int \mathrm{d}[X]\mathrm{d}[g]\mathrm{d}[\zeta]\Delta_{FP}(g)\delta(g-\hat{g}^{\zeta})\mathrm{e}^{-S[X,g]}. \tag{2.16}$$

The delta functional constrains the metric inside the action which allows us to carry out the integration along g:

$$\begin{aligned}Z &= \frac{1}{\mathrm{Vol}}\int \mathrm{d}[X]\mathrm{d}[g]\mathrm{d}[\zeta]\Delta_{FP}(\hat{g}^{\zeta})\delta(g-\hat{g}^{\zeta})\mathrm{e}^{-S[X,\hat{g}^{\zeta}]}\\ &= \frac{1}{\mathrm{Vol}}\int \mathrm{d}[X]\mathrm{d}[\zeta]\Delta_{FP}(\hat{g}^{\zeta})\mathrm{e}^{-S[X,\hat{g}^{\zeta}]}.\end{aligned} \tag{2.17}$$

The dependence on ζ is now only inside the action and in $\Delta_{FP}(g)$. But as we proved $\Delta_{FP}(g)$ is gauge invariant, and the action is also invariant. Therefore, nothing depends on ζ and we can integrate over gauge orbits. Such integration will give us the volume of the local gauge symmetry group which is exactly what we need to cancel the 1/Vol factor in front of the integral. Lastly we get

$$Z(\hat{g})=\int \mathrm{d}[X]\Delta_{FP}(\hat{g})\mathrm{e}^{-S[X,\hat{g}]}. \tag{2.18}$$

This is an integral over inequivalent configurations of the field, which is what we strove for. Note that Z does not really depend on the choice of $\hat{g}$, it just signifies what choice of fiducial metric we made.

We still have to explicitly calculate the Faddev-Poppov determinant $\Delta_{FP}(\hat{g})$, which is defined in 2.14. In order to do so, we have to analyze the delta functional. Since it is localized around $\hat{g}$, it is enough to work infinitesimally:

$$\begin{aligned}\Delta_{FP}^{-1}(\hat{g}) &= \int \mathrm{d}[\zeta]\delta(\hat{g} - \hat{g^\zeta}) \\ &= \int \mathrm{d}[\zeta]\delta(\delta\hat{g}^\zeta).\end{aligned} \tag{2.19}$$

The infinitesimal change $\delta\hat{g}^\zeta$ in terms of Weyl transformation and reparametrization respectively (as in 2.13) is:

$$\delta\hat{g}^\zeta_{ab} = 2\omega g_{ab} + \nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a. \tag{2.20}$$

Since the integration along ζ in 2.19 contributes only around $\hat{g}$, we can integrate along ω and $\delta\sigma_a$ instead of ζ. Thus we get

$$\Delta_{FP}^{-1}(\hat{g}) = \int \mathrm{d}[\omega]\mathrm{d}[\delta\sigma]\delta(2\omega g_{ab} + \nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a). \tag{2.21}$$

Now we substitute the integral representation of the delta functional:

$$\Delta_{FP}^{-1}(\hat{g}) = \int \mathrm{d}[\omega]\mathrm{d}[\delta\sigma]\mathrm{d}[\beta]\exp\Big(2\pi i\int \mathrm{d}^2\sigma\beta^{ab}\sqrt{\hat{g}}\big[2\omega g_{ab} + \nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a\big]\Big), \tag{2.22}$$

where β is symmetric tensor on the world-sheet. Integrating out ω leaves us with:

$$\Delta_{FP}^{-1}(\hat{g}) = \int \mathrm{d}[\delta\sigma]\mathrm{d}[\beta]\exp\Big(4\pi i\int \mathrm{d}^2\sigma\beta^{ab}\sqrt{\hat{g}}\nabla_a\delta\sigma_b\Big)\delta(2\beta^{ab}g_{ab}). \tag{2.23}$$

Outcoming delta functional forces the trace of β to be zero. We now redefine β to be traceless and symmetric, and the remaining delta functional just gets integrated to give a factor of one:

$$\Delta_{FP}^{-1}(\hat{g}) = \int \mathrm{d}[\delta\sigma]\mathrm{d}[\beta]\exp\Big(4\pi i\int \mathrm{d}^2\sigma\beta^{ab}\sqrt{\hat{g}}\nabla_a\delta\sigma_b\Big). \tag{2.24}$$

As discussed in [10], this path integral can be inverted by replacing the fields β_{ab} and $\delta\sigma_a$ by anti-commuting ghost fields:

$$\begin{aligned}\beta_{ab} &\to b_{ab}, \\ \delta\sigma_a &\to c_a.\end{aligned} \tag{2.25}$$

This leaves us with the final expression for the Faddeev-Popov determinant

$$\Delta_{FP}(\hat{g}) = \int \mathrm{d}[b]\mathrm{d}[c]\exp\Big(-\frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}b_{ab}\nabla^a c^b\Big), \tag{2.26}$$

where we have absorbed some multiplicative factors into the definition of b_{ab} and c_a. By substituting this back to 2.18, we get:

$$\Delta_{FP}(\hat{g}) = \int \mathrm{d}[X]\mathrm{d}[b]\mathrm{d}[c]\exp\Big(-\frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}b_{ab}\nabla^a c^b\Big)\mathrm{e}^{-S[X,\hat{g}]}. \tag{2.27}$$

Note that the ghosts are now on the same footing as the X_μ fields, and therefore should be treated as such. Accordingly, we define an action governing the dynamics of these fields as

$$S_g = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}b_{ab}\nabla^a c^b. \tag{2.28}$$

In conformal gauge this becomes

$$S_g = \frac{1}{2\pi}\int \mathrm{d}^2 z(b\partial c + \bar{b}\bar{\partial}\bar{c}). \tag{2.29}$$

2.3 bc ghost system

As a result of gauge fixing, we discovered new fields, which have to be included in path integration. Before we move on, let us briefly examine their dynamics. The action 2.29 yields the following equations of motion:

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0. \tag{2.30}$$

System exerts a new symmetry, the so called ghost symmetry, given by transformation $\delta b = i\epsilon b,\ \delta c = i\epsilon c$. Similarly for the antiholomorphic parts. This symmetry gives rise to a conserved current $j^g =: cb + \bar{c}\bar{b}:$. Note that the normal ordering here is different from the one in free boson theory. It is defined as follows:

$$: b(z)c(w) := b(z)c(w) - \frac{1}{z-w}. \tag{2.31}$$

We can assign ghost number to the fields in our theory $gh(c) = 1$, $gh(b) = -1$ and $gh(X^\mu) = 0$ which is conserved by the virtue of conservation law above.

The energy momentum tensor associated with this action is

$$T^g(z) = 2(\partial c)b + c\partial b, \tag{2.32}$$
$$\bar{T}^g(\bar{z}) = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}\bar{\partial}\bar{b}. \tag{2.33}$$

The central charge of the system is $c = \bar{c} = -26$. If we consider this system combined with free boson theory, we will find out that the central charges sum up giving us $c = \bar{c} = D - 26$. For consistency reasons that we have not mentioned, the central charge must be equal to zero. This condition gives us the critical dimension of bosonic string.

2.4 BRST quantization

In previous section we strove to define path integral with a fixed gauge to eliminate overcounting of physical states. By doing so, we acquired ghost fields governed by their own action, namely the b and c ghosts. Final path integral involves all these new fields, and as a result it exerts an extra symmetry, the BRST symmetry, which mixes the fields among themselves. With this new symmetry a new conserved current j_B appears. The current is given by [10]:

$$j_B = cT^X + : bc\partial c : + \frac{3}{2}\partial^2 c, \tag{2.34}$$

where T^X is stress energy tensor associated with X^μ fields. The $\bar{j}_B$ is analogical.

The BRST charge Q is then

$$Q = \frac{1}{2\pi i} \oint (\mathrm{d}z j_B - \mathrm{d}\bar{z}\bar{j}_B). \tag{2.35}$$

In order to calculate how BRST charge acts when operating on states of the theory, we will need to know some OPEs of j_B. These are:

$$j_B(z)c(w) = \frac{3}{(z-w)^3} + \frac{j^g}{(z-w)^2} + \frac{T^{X+g}}{z-w} \dots \tag{2.36}$$

$$j_B(z)b(w) = \frac{c(w)}{z-w} + \dots \tag{2.37}$$

where T^{X+g} is sum of stress energy tensors of ghost system and of the fields X^μ. The fields described by Polyakov action are sometimes referred to as matter fields since they were introduced to describe particles.

BRST charge has some very striking properties, first of them is its nilpotency:

$$Q^2 = 0. \tag{2.38}$$

This holds only when $D = 26$, but since that is the case we are considering, it does not pose a problem.

Another property is that $Q = Q^\dagger$. If it was not so, the Hermitian conjugate would have to correspond to a different symmetry. But there is non such symmetry.

Note that we have not yet identified the Hilbert space of our theory. Not every state that can be created from the vacuum by the action of various operators is necessarily physical. However, physical states $|\psi\rangle$ must satisfy $Q|\psi\rangle = 0$. By imposing this, we find out that there are still two copies of the spectrum. We can eliminate this redundancy by requiring $b_0|\psi\rangle = 0$. Note that this does not mean that every state satisfying these conditions is distinct physical state.

To see which states should be identified, consider a physical state $|\psi\rangle$ and an arbitrary state $|\chi\rangle$. Using these, we can construct a new state:

$$|\psi'\rangle = |\psi\rangle + Q|\chi\rangle. \tag{2.39}$$

Since $|\psi\rangle$ is physical and Q nilpotent, we find out that $|\psi'\rangle$ is also a physical state. Let us now examine the inner product with yet another physical state $|\phi\rangle$.

$$\begin{aligned}\langle\phi|\psi'\rangle &= \langle\phi|\psi\rangle + \langle\phi|Q|\chi\rangle \\ &= \langle\phi|\psi\rangle + \langle\phi|Q^\dagger|\chi\rangle \\ &= \langle\phi|\psi\rangle\end{aligned} \tag{2.40}$$

We see that new state $|\psi'\rangle$ has the same inner product with all other physical states of the theory. Therefore, states differing by a Q-exact term should be identified. As a result, we learn that the set of physical states is the cohomology of Q.

$$\mathcal{H}_{\text{physical}} = \mathcal{H}_{\text{closed}} \Big/ \mathcal{H}_{\text{exact}} \tag{2.41}$$

Note that we still consider only states satisfying $b_0|\psi\rangle = 0$.

3. Open string field theory

The development of our understanding of particles and their interaction comes hand in hand with means of describing them. One of the most fruitful steps in this chain was the invention of quantum field theory which described particles as disturbances of their respective fields. This became known as second quantization, while the classical quantum mechanical description is called first quantization. In previous chapters, we have employed many methods of QFT, but ultimately the results were of completely different nature. Various disturbances of the fields created particle types, not multitudes of particles in space. By comparing the perturbative string theory with point particle case, we find out that, despite all the work done, we are still at the level of first quantized theory. String field theory (SFT) is the second quantized string theory.

3.1 Analogy with QFT

Before we get to details of this framework, let us review some details about the derivation of second quantized theory on an example of point particle. Point particle, before second quantization, is described in terms of wave function that gives us the probability amplitudes of locating the particle at different points. Its dynamics are governed by an equation of motion (in non-relativistic quantum mechanics it is the Schrödinger equation) that constrains its possible evolution. This means that there are configurations that can never evolve into each other.

When second quantizing, we promote the wave function to an independent object and propose an action that would (at linearized level) reproduce the original equation of motion. By doing so, we make evolutions violating the equation of motion possible. These are known as off-shell processes while the ones that respect the equations of motion are called on-shell. Off-shell amplitudes, as we know from QFT, do contribute to scattering amplitudes; however, the end states must always be on-shell. In other words, the off-shell configurations are unphysical, and the original equation of motion defines what are the physical states. Note that there might be many possible actions that reproduce the correct equations of motion, the simplest of which is usually a free theory. Since such theory is of little physical interest, we include polynomial interaction terms in assigned action.

3.2 String field

In analogy with above procedure, we define string field to be an element of Hilbert space of the underlying theory which is given by the Polyakov action combined with bc ghost system. This however does not completely fix the theory, we also have to impose boundary conditions. By doing so, we define a reference theory, which we will denote BCFT_0. This is usually taken to be the theory of a string connected to a space-filling D-brane. That means imposing Neumann condition in all directions.

In QFT each particle was associated with a distinct field. The same can be done in string theory; however, since the string has an infinite tower of particle states, the number of fields is infinite. In this manner the string field can be decomposed as

$$|\Psi\rangle = \int \frac{\mathrm{d}^{26}k}{(2\pi)^{26})}\Big[t(k) + A_\mu(k)\alpha^\mu_{-1} + B(k)b_{-1}c_0 + \dots\Big]c_1\,|0,k\rangle\,. \tag{3.1}$$

The fields $t(k)$, $A_{\mu\nu}(k)$, etc. can be interpreted as Fourier transforms of fields corresponding to the operators they are multiplied with. Concretely $t(k)$ is the Fourier transform of tachyon field.

Now we would like to study the dynamics of string fields. We know that in perturbative theory the on-shell condition stated $Q\,|\Psi\rangle = 0$. Accordingly, we propose an action that would reproduce such equation of motion

$$S = -\langle\Psi|\,Q\,|\Psi\rangle\,, \tag{3.2}$$

where $\langle\Psi|$ is the BPZ conjugate.

One might ask what happened to the other imposed condition $b_0\,|\Psi\rangle = 0$. That becomes a gauge fixing condition in SFT called Feynman-Siegel gauge.

Now, to make things nontrivial, we want to introduce interaction terms. To do so, we need to promote the Hilbert space of string fields to an algebra by defining a product * among them.

3.2.1 Star product

Star product of string fields was first introduced axiomatically by Witten in [19] and was motivated by the picture of pair of string world-sheets joining into one. The axioms describing it are sometimes known as axioms of open string field theory. They state the following [18]:

- Associativity: $(|\Psi_1\rangle * |\Psi_2\rangle) * |\Psi_3\rangle = |\Psi_1\rangle * (|\Psi_2\rangle * |\Psi_3\rangle)$.
- Distributivity: $|\Psi_1\rangle * (|\Psi_2\rangle + |\Psi_3\rangle) = |\Psi_1\rangle * |\Psi_2\rangle + |\Psi_1\rangle * |\Psi_3\rangle$.
- Graded Leibniz rule: $Q(|\Psi_1\rangle * |\Psi_2\rangle) = (Q\,|\Psi_1\rangle) * |\Psi_2\rangle + (-1)^{gh_1}\,|\Psi_1\rangle * (Q\,|\Psi_2\rangle)$, where gh_1 is ghost number of $|\Psi_1\rangle$.
- Cyclicity: $\langle\Psi_1|\Psi_2 * \Psi_3\rangle = \langle\Psi_3|\Psi_1 * \Psi_2\rangle$.

There exists an identity element $|I\rangle$ of the star algebra that satisfies

$$|I\rangle * |\Psi\rangle = |\Psi\rangle * |I\rangle = |\Psi\rangle\,. \tag{3.3}$$

We will not show its explicit form. Identity string field is of ghost number zero.

In order to simplify our equations, we will replace $|\Psi\rangle$ by plain Ψ. However, meaning is the same. In spirit of this notation, we will introduce so called Witten integral, defined as:

$$\int \Psi_1 * \Psi_2 = \langle\Psi_1|\Psi_2\rangle\,. \tag{3.4}$$

Thus we will not need a notation for $\langle\Psi|$.

Witten integral annihilates Q exact terms as well as integrands whose total ghost number is different from three. This has an important consequence. The action 3.2 written in terms of Witten integral is

$$S = -\int \Psi * Q\Psi. \tag{3.5}$$

Consider the following expansion of string field Ψ:

$$\Psi = \sum_i \Psi_i, \tag{3.6}$$

where Ψ_i is a string fields of ghost number i. If we now plug this expansion into the action, all terms with ghost number different from three will vanish (note that Q has ghost number one). Thus we get:

$$S = \int \sum_i \Psi_i * Q\Psi_{2-i}. \tag{3.7}$$

Every field except Ψ_1 is linear in the action, and thus we find out that their dynamics are governed only by fields of different ghost number. The only field that has dynamics on its own is ghost field Ψ_1. Therefore, we can consistently restrict ourselves to string field of ghost number one.

Note that the algebra is in a way similar to the exterior algebra of forms with Q playing the role of exterior derivative and the ghost number being the degree of forms.

3.2.2 Interacting action

Now we are ready to introduce interacting terms into the string field action. In the same line of similarity with forms, Witten proposed Chern-Simons like action by adding a cubic vertex to action 3.2.

$$S = -\int (\frac{1}{2}\Psi * \Psi + \frac{g}{3}\Psi * \Psi * \Psi) \tag{3.8}$$

This action yields the following equations of motion:

$$Q\Psi + \Psi * \Psi = 0. \tag{3.9}$$

The action 3.9 exerts a gauge symmetry under the following transformation

$$\Psi \to \Psi + Q\Lambda + \Lambda * \Psi - \Psi * \Lambda, \tag{3.10}$$

where Λ is any string field of ghost number zero.

3.2.3 Representation of the star product

There are several representations of the star product that are being used; however, we will mention only one of them, the easiest to comprehend without the need to introduce new frames or concepts. This representation is sometimes known as split string formalism since it relies on a decomposition of the string to its left and right part. Let us know explain how this splitting works. After that we will write explicit formula for the star product.

A string field can be represented as a functional $\Psi[X]$ of the coordinates of string. Now imagine that we split the open string into three parts: the left part, the right part and the midpoint as follows [1]:

$$\begin{aligned} l^\mu(\sigma) &= X^\mu(\sigma) & 0 \le \sigma < \frac{\pi}{2}, \\ r^\mu(\sigma) &= X^\mu(\pi - \sigma) & \frac{\pi}{2} < \sigma \le \pi, \\ X^\mu_m &= X^\mu\Big(\frac{\pi}{2}\Big). \end{aligned} \tag{3.11}$$

String field can be rewritten as

$$\Psi[X^\mu] \to \Psi[x^\mu, l^\mu, r^\mu]. \tag{3.12}$$

Note that the above expression is of matrix like form with l^μ and r^μ playing the role of matrix indeces. Star product is represented as

$$(\Psi * \Phi)[x^\mu, l^\mu, r^\mu] = \int \mathrm{d}[k]\Psi[x^\mu, l^\mu, k^\mu]\Phi[x^\mu, k^\mu, r^\mu]. \tag{3.13}$$

This has a very similar form as matrix multiplication. The right part of the first string is identified with the left part of the second string and then we sum over all such configurations. The Witten integral in this representation takes the following form

$$\int \Psi * \Phi = \int \mathrm{d}[k]\mathrm{d}[j]\mathrm{d}x_m \Psi[x^\mu, j^\mu, k^\mu]\Phi[x^\mu, k^\mu, j^\mu]. \tag{3.14}$$

This is again analogous to an operation on matrices - the trace.

3.3 Background independence

As we mentioned in the previous section, the underlying theory is defined by the action principles, stress energy tensor and also the background present. One of the aims of string theory is to find formulation that would be background independent. String field theory almost does this. It is widely believed that possible backgrounds of perturbative string theory are in one to one correspondence with the solutions of the OSFT (open string field theory) equation of motion 3.9. One way of this correspondence is already known, for a given solution we know how to construct a boundary state which describes the background. This is the content of following section.

3.3.1 Boundary state construction

As we just mentioned, solution of the equations of motion corresponds to the backgrounds of the underlying string theory. Such background can be captured in terms of boundary state, and thus we should be able to construct this state from a given solution. There are several methods of doing this, we will however use the method proposed by Kudrna, Macafferri and Schnabl in [6], which utilizes so called Ellwood's conjecture. Let us make few comments about this conjecture before we get to the construction itself.

Ellwood's conjecture is a hypothetical relation between the on-shell part of the boundary state and gauge invariant quantities that are sometimes known as Ellwood's invariants. This relation was first proposed by Ellwood in [2] and was verified for particular solutions. Rigorous proof is yet to be found. To state the conjecture, we have to define Ellwood's invariants

Consider a string field Ψ and a primary operator $\mathcal{V} = c\bar{c}V^m$ of ghost number two with conformal weights $(0,0)$ satisfying $\{Q, \mathcal{V}\} = 0$. Note that c and $\bar{c}$ are ghost field from bc ghost system. V^m is an arbitrary matter primary field of weight $(1,1)$. The Ellwood's invariant associated with Ψ and $\mathcal{V}$ is defined as

$$\mathrm{Tr}_{\mathcal{V}}\Psi = \langle I|\, \mathcal{V}(i)\, |\Psi\rangle\,. \tag{3.15}$$

Since the above definition is linear in Ψ, we can show its gauge invariance by examining the respective terms of the transformation 3.10 one by one. The first term gives us the original overlap. The second term is

$$\mathrm{Tr}_{\mathcal{V}}Q\Lambda = \langle I|\, \mathcal{V}(i)\, |Q\Lambda\rangle\,. \tag{3.16}$$

From the definition of the identity it is clear that it is annihilated by Q. Then by using the constraint on the choice of $\mathcal{V}$, we can commute Q through $\mathcal{V}$, and then since Q is Hermitian, let it act on identity. Thus we get zero. The last term

$$\mathrm{Tr}_{\mathcal{V}}(\Lambda * \Psi - \Psi * \Lambda) \tag{3.17}$$

vanishes due to the cyclicity of star product.

The Ellwood's conjecture states that

$$\mathrm{Tr}_{\mathcal{V}}\Psi - \mathrm{Tr}_{\mathcal{V}}\Psi_{TV} = -\frac{1}{4\pi i}\,\langle\mathcal{V}|\, c_0^-\, ||B\rangle\rangle\,, \tag{3.18}$$

where Ψ_{TV} is the solution for tachyon vacuum and $c_0^- = c_0 - \bar{c}_0$.

By inserting all on-shell primaries (spinless of weight 0), we can determine the physical part of the boundary state; however, the off-shell part remains hidden. This problem was bypassed by the mentioned construction of Kudrna, Macafferri and Schnabl. In their work, they considered an auxiliary extension of the underlying theory $\mathrm{BCFT}' = \mathrm{BCFT}^0 \otimes \mathrm{BCFT}^{aux}$ where BCFT^{aux} is of central charge zero. By doing so we can consider primary operators V^m with weights $(1-h, 1-h)$ and multiply them by operators ω from the auxiliary sector of weight (h,h). Thus, the final operator is of weight $(1,1)$ and can be used in the Ellwood's conjecture. We can do this in such way that the one point function $\left\langle\omega(0)\right\rangle^{aux} = 1$. The Ellwood's conjecture 3.18 is not yet well defined. We have expanded our theory, but we did not expand the string field Ψ. Such extended solution is called lifted and is denoted as $\tilde{\Psi}$. For the class of solutions that will be of interest to us a particularly simple lift is possible:

$$\begin{aligned} |0\rangle &\to |0\rangle \otimes |0\rangle^{aux}\,, \\ L_n &\to L_n + L_n^{aux}. \end{aligned} \tag{3.19}$$

Generalized Ellwood's conjecture states that

$$\mathrm{Tr}_{\tilde{\mathcal{V}}}(\tilde{\Psi} - \tilde{\Psi}_{TV}) = -\frac{1}{4\pi i} \langle \tilde{\mathcal{V}} | \, c_0^- \, ||B\rangle\rangle \, , \tag{3.20}$$

where $\tilde{\mathcal{V}} = \mathcal{V} \otimes \omega$. Since this is true for any spinless matter primary field V, we can reconstruct the whole boundary state [6].

3.4 K, B, c subalgebra

One of very effective tools for solving 3.9 is the K, B, c algebra. It is generated by a set of three string fields K,B and c of ghost numbers $0, -1, 1$ respectively. B and c are grassman odd and K is grassman even. These fields satisfy the following [4]

$$\begin{aligned} \{B, c\} &= 1, \qquad [K, B] = 0, \qquad c^2 = B^2 = 0, \\ QK &= 0, \qquad Qc = cKc, \qquad QB = K, \end{aligned} \tag{3.21}$$

where all multiplication is in star product sense. Their explicit form is

$$\begin{aligned} K &= K_L^v \, |I\rangle \, , \\ B &= B_L^v \, |I\rangle \, , \\ c &= c_L^v \, |I\rangle \, , \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} K_L^v &= \int_L \frac{\mathrm{d}z}{2\pi i} v(z) T(z), \\ B_L^v &= \int_L \frac{\mathrm{d}z}{2\pi i} v(z) b(z), \\ c_L^v &= -\frac{1}{v(1)} c(1). \end{aligned} \tag{3.23}$$

The integrals along L are taken to surround a positive semicircle connecting $-i$ a i as in figure 3.1.

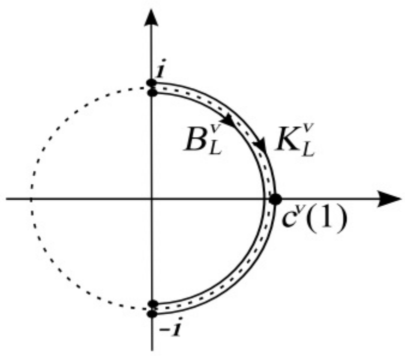

Figure 3.1: Definitions of K, B, c [4]

$v(z)$ is an arbitrary holomorphic field subjected to the following constraints:

$$\overline{v(z)} = \bar{z}^2 v\Big(\frac{1}{\bar{z}}\Big), \tag{3.24}$$

$$v(\pm i) = 0. \tag{3.25}$$

Such algebra can be supplemented by yet another field corresponding to an exactly marginal operator $J = cJ(1)\,|I\rangle$ [3]. This field satisfies

$$J^2 = 0, \qquad QJ = 0. \tag{3.26}$$

3.5 Marginal deformations

In this section, we will present a simple class of solutions generated by marginal operator (with nonsingular OPE with itself) deforming the underlying BCFT_0. These findings were presented in [12].

The equations of motion 3.9 can be solved perturbatively in parameter λ. The form of the solution is assumed to be

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \phi_n. \tag{3.27}$$

Substituting such ansatz into 3.9, we get an equation for each order of λ. At first level the equations yields $Q\phi_1 = 0$. Thus, for ϕ_1 we can choose any on-shell string field. For the n-th level we get

$$Q\phi_n = -(\phi_1\phi_{n-1} + \phi_2\phi_{n-2} + \cdots + \phi_{n-1}\phi_1). \tag{3.28}$$

In order to solve such equations, the right hand side has to be Q exact. This might not generally be the case; however, for the choice $\phi_1 = cJ(0)\,|0\rangle$, where $J(z)$ is exactly marginal, it is always so. Layer by layer these equations can be solved to give us the full solution. In terms of K, B, J string fields, this solution can be written as [3]

$$\Psi = \lambda F J \frac{1}{1 - \lambda B \frac{F^2-1}{K} J} F, \tag{3.29}$$

where F is an arbitrary function of K.

3.6 The closed string source

Similarly as we second quantized theory of open strings, we can do the same with closed strings. The procedure would be quite analogous until the point where we add interaction terms to the action. The interaction terms that can be added to the action are not arbitrary since particular choices will lead to inconsistencies. The fact that the action 3.8 is so simple is rather remarkable. For closed strings there is no such action, but we are still equipped with the non-interacting action which is given as follows [15]

$$S = -\frac{1}{Kg_s^2} \langle \Psi_c |\, c_0^- (Q + \bar{Q})\, |\Psi_c\rangle \,, \tag{3.30}$$

where K is a numerical constant, g_s is closed string coupling constant and $\bar{Q}$ is the antiholomorphic component of BRST charge. In open strings, there is just one independent component similarly as it is with stress-energy tensor. In presence of a D-brane, the action 3.30 can be coupled to the boundary state by including a source term

$$\langle \Psi_c | \, c_0^- \, || B \rangle\rangle \,. \tag{3.31}$$

The closed string field can be decomposed as follows

$$|\Psi_c\rangle = \int \frac{\mathrm{d}^{26}k}{(2\pi)^{26}} \Big[T(k) c_1 \bar{c}_1 + \Big(h_{\mu\nu} + b_{\mu\nu} \Big) \alpha^{\mu}_{-1} \bar{\alpha}^{\nu}_{-1} c_1 \bar{c}_1 + \ldots \Big] c_1 \, |0, k\rangle \,. \tag{3.32}$$

The fields $T(k)$, $h_{\mu\nu} = h_{\nu\mu}$ and $b_{\mu\nu} = -b_{\nu\mu}$ can be interpreted as Fourier transform of the closed tachyon field, graviton field and anti-symmetric tensor field respectively. Similarly, there is a general decomposition of the boundary state

$$||B\rangle\rangle = \int \frac{\mathrm{d}^{26}k}{(2\pi)^{26}} \Big[F(k) + \Big(A_{\mu\nu}(k) + C_{\mu\nu}(k) \Big) \alpha^{\mu}_{-1} \bar{\alpha}^{\nu}_{-1} + B(k)(b_{-1}\bar{c}_{-1} + \bar{b}_{-1} c_{-1}) + \ldots \Big]$$

$$(c_0 + \bar{c}_0) c_1 \bar{c}_1 \, |0, k\rangle \, .boundexpan \tag{3.33}$$

By substituting the above decompositions into 3.31, we obtain terms like this

$$\int \frac{\mathrm{d}^{26}k}{(2\pi(^{26}} \Big[h_{\mu\nu}(-k) \Big(B(k) \eta^{\mu\nu} + A^{\mu\nu}(k) \Big) + \ldots \Big]. \tag{3.34}$$

We see that the graviton field couples to $T^{\mu\nu} = B(k)\eta^{\mu\nu} + A^{\mu\nu}(k)$. The dynamics of $T^{\mu\nu}$ are determined by the corresponding solution of OSFT, and thus it acts like a source term for the graviton field. $T^{\mu\nu}$ can be identified as the stress-energy tensor. Other fields in the boundary state act like sources for other closed string fields.

3.7 Light-like tachyon condensation

The discussed solution for marginal deformation provides a one parameter family of string fields each of which satisfies the equations of motion 3.9. Using the operator $\mathrm{e}^{\pm X^0/\alpha'}$ of dimension $(1, 1)$, we recover a class of solutions referred to as rolling solutions. They represent the space-homogeneous decay of the underlying space filling D-brane. Another class is generated by the operator $\mathrm{e}^{\beta X^+}$, where $X^+ = (X^0 + X^1)/\sqrt{2}$. Note that such operator has dimension zero for all values of β. To remedy this, a linear dilaton background with nonzero V^+ can be introduced. Recalling 1.90, we can chose $\beta = 1/\alpha' V^+$ so that the weight is equal to $(1, 1)$. This solution was proposed by Hellerman and Schnabl in [8], where they also proved that such solution relaxes to the tachyon vacuum in late time asymptotic. They went on to calculate the source term for graviton. In this section, we will use the aforementioned construction to to uncover the full boundary state in terms of the original one (corresponding to Neumann boundary conditions).

3.7.1 Boundary state construction

The solution at hand can be written in form 3.29 as follows

$$\Psi = \lambda F c \mathrm{e}^{\beta X^+} \frac{1}{1 - \lambda B \frac{F^2 - 1}{K} c \mathrm{e}^{\beta X^+}} F. \tag{3.35}$$

The Ellwood's invariant for this class of solutions was calculated by Kishimoto in [5]. Given an on-shell weight zero primary field $\mathcal{V} = c\bar{c}V$ with V being purely matter primary, the gauge invariant overlap is given as

$$\mathrm{Tr}_{\mathcal{V}}(\Psi) = -\frac{1}{2\pi i}\Big\langle \mathcal{V}(0)c(1)\Big(\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big) - 1\big)\Big)\Big\rangle_{disk}. \tag{3.36}$$

Note that the insertion points are in a unit disc frame 1.82. After subtracting the tachyon vacuum, we obtain

$$\mathrm{Tr}_{\mathcal{V}}(\Psi - \Psi_{TV}) = -\frac{1}{2\pi i}\Big\langle \mathcal{V}(0)c(1)\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\Big\rangle_{disk}. \tag{3.37}$$

Following the procedure of [6], we expand our theory by a boundary field theory of central charge zero $\mathrm{BCFT}' = \mathrm{BCFT}^0 \otimes \mathrm{BCFT}^{aux}$. For solutions of the above form we can always perform a very simple lifting generated by

$$\tilde{\Psi} \to \Psi\Big|_{K\to K+K^{aux}}. \tag{3.38}$$

This corresponds to the lifting 3.19. Now we will lift the operator $\mathcal{V}$. We consider a purely matter primary field V^h of arbitrary weight h, and we multiply it by a field ω from the auxiliary sector, of weight $1-h$ to get $\tilde{\mathcal{V}} = c\bar{c}V^h\omega$. The generalized overlap now reads

$$\mathrm{Tr}_{\tilde{\mathcal{V}}}(\tilde{\Psi} - \tilde{\Psi}_{TV}) = -\frac{1}{2\pi i}\Big\langle \mathcal{V}(0)c(1)\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\Big\rangle_{disk}^{\mathrm{BCFT}'}, \tag{3.39}$$

where the superscript indicates that we are integrating over both the original and the auxiliary sector. The auxiliary, matter and ghost sectors factorize

$$\mathrm{Tr}_{\tilde{\mathcal{V}}}(\tilde{\Psi} - \tilde{\Psi}_{TV}) = -\frac{1}{2\pi i}\Big\langle V^h(0)\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\Big\rangle_{disk}^{matter} \big\langle c\bar{c}(0)c(1)\big\rangle_{disc}^{ghost}\big\langle\omega(0)\big\rangle_{disc}^{\mathrm{BCFT}^{aux}}. \tag{3.40}$$

We can choose $\langle\omega(0)\rangle = 1$ as we mentioned when we introduced this construction. The fact that the ghost sector factorizes is proven in [6]. We will use the following identities

$$\big\langle c\bar{c}(0)c(1)\big\rangle_{disc}^{ghost} = -1, \tag{3.41}$$

$$\big\langle c_0^- c\bar{c}(0)\big\rangle_{disc}^{ghost} = -2. \tag{3.42}$$

Substituting the above into 3.20, we obtain

$$\mathrm{Tr}_{\tilde{\mathcal{V}}}(\tilde{\Psi}-\tilde{\Psi}_{TV}) = -\frac{1}{4\pi i}\Big\langle c_0^- c\bar{c}(0)V^h(0)\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\big)\Big\rangle_{disk}^{BCFT^0}. \tag{3.43}$$

Substituting this into the generalized Ellwood conjecture 3.20, we get

$$\langle\tilde{\mathcal{V}}|\, c_0^-\, ||B\rangle\rangle = \Big\langle c_0^- c\bar{c}(0)V^h(0)\exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\Big\rangle_{disk}^{BCFT^0}. \tag{3.44}$$

Once this is true for any level matched primary, it follows that

$$||B\rangle\rangle = \exp\big(-\lambda\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\beta x^+}(e^{i\theta})\big)\,||B_0\rangle\rangle\,, \tag{3.45}$$

where $||B_0\rangle\rangle$ is the boundary state corresponding to Neumann boundary conditions in all directions or in other words to the space filling D-brane.

The boundary state is kind of an abstract object, and just by looking at its above form untrained eye probably cannot interpret it at all. To make sense of it, we could extract the coefficients in the expansion **??** and match particular linear combination with respective closed string fields. By Fourier transforming these, we obtain corresponding fields. Note that these are now fields in the 26-dimensional spacetime, not on the 2-dimensional world-sheet. We can already picture such objects (apart from them living in 26 dimensions) and interpret them.

Conclusion

In order to get to the heart of this thesis, which is the study of time dependent dynamics of an unstable space-filling D-brane, we had to lay down some basic tools. These are mostly the methods of conformal field theory which we have illustrated on the example of free boson theory. From there, we continued to introduce some basic perturbative string theory. By using the gauge symmetries of the action that governs string's dynamics, we uncovered that we can find a particular gauge fixing (conformal gauge) that simplifies our work significantly. It turned out that action for string is the free boson action. The gauge symmetries of the string had another consequence, the presence of reparametrization ghosts b and c and the BRST invariance accompanied by a conserved charge.

String field theory was presented as second quantization of string theory. We have tried to show it in analogy with QFT to make clear that this is not a strikingly new concept and to justify some basic terms. From the on-shell conditions of perturbative theory we uncovered the kinetic term of the string field action. Then we proposed a new action that includes interaction terms and yet yields the on-shell condition on linearized level. After this we presented a new outlook on string fields, particularly on classical solutions of the equation of motion given by the proposed action. The solutions represent D-brane configurations that form the background of perturbative string theory. This is quite spectacular finding since one of the aims of string theory is to find a background independent formulation. We elaborated on this correspondence with a general construction of a boundary state for a given solution that would describe the string's background.

Boundary states couple to closed string fields, and thus we saw that the decaying D-brane acts like a source of closed strings that would emerge in the process of tachyon condensation. We went on to calculate such state explicitly for a particular solution describing this decay and proposed a way for further examination - the extraction and interpretation of respective field from the boundary state decomposition. The next step would be the actual calculation of the spectrum of closed strings produced. This could yield some interesting cosmological consequences.

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List of Abbreviations

CFT	conformal field theory
BCFT	boundary conformal field theory
SFT	string field theory
OSFT	open string field theory
QFT	quantum field theory
BPZ	Belavin-Polyakov-Zamolodchikov
BRST	Becchi-Rouet-Stora-Tyutin
UHP	upper half plane