Charles University in Prague
Faculty of Mathematics and Physics

**DOCTORAL THESIS**

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# Exceptional Sets in Mathematical Analysis

Department of Mathematical Analysis

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Výjimečné množiny v matematické analýze

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Abstrakt: Tato práce sestává ze čtyř odborných článků. V prvním článku studujeme pojem $\sigma$-zdola pórovitých množin; hlavním výsledkem je konstrukce uzavřených množin $A, B \subset \mathbb{R}$, které nejsou $\sigma$-zdola pórovité a jejichž součin v $\mathbb{R}^2$ je zdola pórovitý. Ve druhém a třetím článku používáme množinově-teoretickou metodu založenou na Löwenheim-Skolemově větě (tzv. metodu elementárních submodelů) k důkazu separabilní determinovanosti jistých $\sigma$-ideálů množin v Banachových prostorech. Činíme tak nejprve pro pojmy $\sigma$-pórovitosti a $\sigma$-zdola pórovitosti (v článku druhém) a zjemněním použitých metod pak ve třetím článku dostaneme separabilní determinovanost dalších vlastností. V obou případech dostáváme zajímavé důsledky v podobě rozšíření vět známých pro separabilní prostory do kontextu neseparabilního; například: Libovolná spojitá konvexní funkce na Asplundově prostoru je fréchetovsky diferencovatelná ve všech bodech mimo kuželově malou (cone small) množinu. Čtvrtý článek zavádí následující pojem. Řekneme, že uzavřená množina $A \subset \mathbb{R}$ je $c$-odstranitelná, jestliže platí: Reálná funkce $f$ je konvexní na $\mathbb{R}^d$, kdykoliv je spojitá na $\mathbb{R}^d$ a lokálně konvexní na $\mathbb{R}^d \setminus A$. Podáváme nové postačující podmínky pro $c$-odstranitelnost společně s důkazem, že tyto jsou silnější než postačující podmínky známé dříve.

Klíčová slova: $\sigma$-pórovitá množina, elementární submodel, Banachův prostor, $c$-odstranitelná množina, konvexní funkce.

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Abstract: The present thesis consists of four research articles. In the first paper we study the notion of $\sigma$-lower porous set; our main result is the existence of two closed sets $A, B \subset \mathbb{R}$ which are not $\sigma$-lower porous, but their product in $\mathbb{R}^2$ is lower porous. In the second and third article we use a set-theoretical method of elementary submodels involving the Löwenheim-Skolem theorem to prove that certain $\sigma$-ideals of sets in Banach spaces are separably determined. In the second article we do so for $\sigma$-porous sets and $\sigma$-lower porous sets. In the next article we refine these methods obtaining separable determination of a wide class of $\sigma$-ideals. In both cases we derive interesting corollaries which extend known theorems in separable spaces to the nonseparable setting; for example, we obtain the following theorem. Any continuous convex function on an Asplund space is Fréchet differentiable outside a cone small set. In the fourth article we introduce the following notion. A closed set $A \subset \mathbb{R}^d$ is said to be $c$-removable if the following is true: Every real function on $\mathbb{R}^d$ is convex whenever it is continuous on $\mathbb{R}^d$ and locally convex on $\mathbb{R}^d \setminus A$. We then give new sufficient conditions for a set to be $c$-removable and we construct an example proving that these conditions are more general than those previously known.

Keywords: $\sigma$-porous set, elementary submodel, Banach space, $c$-removable set, convex function.

# Contents

# Introduction

This introduction consists of three sections. The aim of the first two sections is to provide a brief overview of historical development of the part of mathematical analysis to which this thesis contributes. No attempt was made to make this overview complete in any sense. The third section introduces the mathematical content of the present thesis.

## 0.1 Historical motivation

In mathematics, one often encounters the question of existence of certain objects, a question which can generally by approached from two directions: Either one can prove the existence of the object at question by providing a method for creating it from other objects whose existence has already been established (or assumed), or one can avoid explicit construction, often by proving that not only an object with desired properties exists, but that there is in fact an abundance of such objects.

It is probably clear to the reader that the former refers to what we call the *constructive proof* while the latter refers to the so-called *non-constructive* (or *existence*) *proof*. The distinction between the two is related to a number of philosophical questions, which we are not going to discuss here; note that *constructivism* is a mathematical philosophy that rejects all but constructive proofs by questioning basic principles such as the axiom of choice, the axiom of infinity or even the law of the excluded middle. In our branch of mathematics, however, it is sometimes impossible to avoid a non-constructive proof.

By non-constructive proof we usually mean one that makes use of the axiom of choice. Another example of non-constructive proof could be *proof by contradiction*. In this introduction we are more interested in the non-constructive scheme of proof indicated in the first paragraph: Proving that there are many objects with desired properties without providing an example. The following three classical examples illustrate our point:

**(a) Cardinality:** In 1874 Georg Cantor published the first proof of uncountability of real numbers. As a consequence he obtained that there are uncountably many transcendental numbers as it was easy to show that there are only countably many algebraic ones. This was a new proof of Liouville's result that there exist transcendental numbers. As a matter of fact, Cantor's work essentially contains two proofs of this fact which are closely related, but one is considered constructive (as it gives an explicit algorithm to find transcendental numbers), whereas the one based on the aforementioned *cardinality argument* is non-constructive. For us the important case is the latter.

**(b) Category:** In the beginning of the $19^{\text{th}}$ century a lot of attention was given to the notions of limit, continuity and derivative, and their precise definitions were gradually worked out and studied—most notably by Bernard Bolzano, Augustine-Louis Cauchy and later by Karl Weierstrass who gave the first "$\varepsilon$-$\delta$ definition" of limit (in the 1840s). In fact, the very notion of function still lacked a precise and universally accepted definition, and so did the field of real numbers. Basic facts, such as the Bolzano theorem asserting that a continuous real function of one real variable enjoys the intermediate value property, were provided with their first (almost) rigorous proofs.

In his publication from 1837, J. P. G. Lejeune Dirichlet was probably the first to give the now standard general definition of a function: $y$ is a function of $x$ when to each value of $x$ in a given interval there corresponds a unique value of $y$. He had already given a good example in 1829: The famous Dirichlet function is defined as the characteristic function of rationals and it is clearly not continuous at any point. It served mainly as an example of function which is not defined by a "nice" formula; at the time, such functions were by many considered abominations not worthy of study, or were simply disregarded as singular cases not interesting for applications. Nonetheless, it slowly became apparent that the classical definitions of functions were too restrictive.

A natural question which arose from the use of more general and more precise definitions was whether any continuous real function has (many) points of differentiability. Despite the increasing level of rigor in mathematics, Cauchy and nearly all mathematicians of his era believed (and "proved") that a continuous function must be differentiable except at isolated points. An early exception was Bolzano who did understand the distinction between continuity and differentiability, and already in 1834 published an example of a continuous nowhere differentiable function (which he had constructed in an unpublished manuscript from around 1820). His work, however, went unnoticed until after Weierstrass' famous example from 1874 (the Weierstrass Monster). There were a number of other examples of similar pathological functions given in the second half of the $19^{\text{th}}$ century.

All of the existence results for various pathological functions had one thing in common: They all gave explicit formulas, and were therefore constructive. Of course, in this case it was impossible to use a cardinality argument as in our example (a) to provide analogous non-constructive proof: The set of differentiable functions has the same cardinality as that of continuous ones. A more refined notion of "size" of sets was required for that—one that took into account the topological structure of the space of continuous functions.

In 1931 Stefan Banach and Stefan Mazurkiewicz independently proved the following:

**Theorem B** (Banach, Mazurkiewicz)**.** *Let $C[0,1]$ be the space of continuous real functions on $[0,1]$ endowed with the supremum norm. Then the set of all functions which are differentiable at some point in $[0,1]$ is of the first category in $C[0,1]$.*

Consequently, the set of nowhere differentiable continuous functions is nonempty by the Baire category theorem (proved by René-Louis Baire in his thesis in 1899). In other words, we obtain the existence result using a *Baire category argument*.

Similarly as in case (a) this does not only show the mere existence of such functions; in fact, the result says that "most" continuous functions are differentiable at no point. In the modern terminology we say that a *typical* continuous function is nowhere differentiable.

**(c) Measure:** In final decades of the $19^{\text{th}}$ century the theory of functions of real variable started to grow, motivated by the study of differentiation and integration properties of functions. However, no efforts in the direction of creating a more general theory of integration were made: The general opinion was that the concept of Riemann integral could not be generalized any further.

Nonetheless, the study of sets of discontinuities of functions led to the idea of quantifying how large these (or any other) sets are. The first attempts in this direction led to different concepts of *content*, most notably to the *Jordan content* (also *Peano-Jordan measure*). This is a set function which is an extension of the notion of size (length, area, volume) from line segments, squares or cubes to more general "well-behaved" shapes. Unfortunately, the family of "measurable sets" was unsatisfactorily small.

The next important step in the theory of content was made by Émile Borel who was the first to explicitly formulate the idea of $\sigma$-additivity as opposed to mere finite additivity. In his work from 1898 he essentially constructed by transfinite induction a $\sigma$-additive measure (in the modern sense) defined on Borel subsets of the real line such that the measure of each interval was its length.

For Borel, the purpose of this construction was to study sets of convergence of certain series; he did not think of using his idea in the theory of integral. The final step was done by Borel's student Henri Lebesgue in his thesis, which he published in 1902. Lebesgue constructed a general $\sigma$-additive measure on the real line (the Lebesgue measure), defined corresponding measurable sets and functions, and proved that the resulting notion of Lebesgue integral is more general than that of Riemann. The advantages of this new concept of integral followed largely from the $\sigma$-additivity of the measure, and the consequent large class of measurable functions.

Lebesgue devoted most of his efforts to the study of the connection between integral and primitive function. In particular, he was interested in the question for which functions $f : [a, b] \to \mathbb{R}$ holds the well-known formula

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Thus he came to the following theorem which is important for us:

**Theorem C** (Lebesgue)**.** *A monotone function $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ except for a Lebesgue null set.*

Consequently, all functions of bounded variation (in particular all Lipschitz functions) are differentiable almost everywhere. Indeed, for an $L$-Lipschitz function $f : [a, b] \to \mathbb{R}$ it is enough to consider the monotone function $g : x \mapsto f(x) + Lx$. To obtain the result for functions of bounded variation, one needs to know that every such function can be expressed as the difference of two monotone functions. The statement then follows from the fact that the union of two Lebesgue null sets is Lebesgue null. Though trivial in this case, this sort of argument will be useful to keep in mind later in this introduction.

Theorem C was the first general result on existence of derivatives of functions. It is interesting to note that unlike almost all the analogous results in higher dimensions which we shall mention in the sequel, this particular theorem for Lipschitz functions is optimal in the sense of the following result of Z. Zahorski [28]:

**Theorem Z.** *A subset of $\mathbb{R}$ is the set of points of non-differentiability of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ if and only if it is $G_{\delta\sigma}$-set of Lebesgue measure zero.*

Since there exists a Lebesgue null $G_\delta$-set which is residual (i.e. its complement is of the first category), it is clear that the Baire category method is not suitable to study sets of non-differentiability of Lipschitz function.

In the sequel we shall mention a number of similar "almost everywhere differentiability results for Lipschitz functions" and we shall usually refer to them as *Rademacher type theorems* (see below).

All three main results formulated in (a), (b) and (c) are existence results in the sense that they do not explicitly construct the objects at question, they simply state that objects with the desired property are, in fact, prevalent.

In case (a), "most" numbers are transcendental. Here "most" means "up to countably many".

In case (b), "most" continuous functions are nowhere differentiable. Here "most" means "up to a set of the first category".

Finally, in case (c), "most" points from the domain of a given Lipschitz function are points of differentiability. Here "most" means "up to a Lebesgue null set".

In all three cases we use some notion of "smallness of sets" and we use this notion to describe exactly how small the sets of "exceptional objects" are (in particular we obtain that they do not contain everything, thus proving the existence of objects with the opposite property). The main topic of this thesis is the study of certain notions of smallness of sets and the use of these notions to describe how small certain exceptional sets (i.e. sets of exceptional points) are. Probably the most important case of exceptional sets is that of points of non-differentiability of Lipschitz functions, and we will discuss this case in various settings. We shall often use the notion of $\sigma$-ideal:

**Definition.** Let $X$ be a set and $\mathcal{S}$ be a system of subsets of $X$. We say $\mathcal{S}$ is a *$\sigma$-ideal of subsets of $X$* if the following are true:

(i) If $A \in \mathcal{S}$ and $B \subset A$, then $B \in \mathcal{S}$;

(ii) If $A_n \in \mathcal{S}$ for each $n$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

A $\sigma$-ideal $\mathcal{S}$ of subsets of $X$ is called *nontrivial* if $X \notin \mathcal{S}$.

In all interesting situations discussed in this thesis (including (a), (b) and (c)), "smallness" of sets is understood with respect to some nontrivial $\sigma$-ideal $\mathcal{S}$ of sets, that is, a set $A$ will be considered "small" if $A \in \mathcal{S}$.

Let us recall also the definitions of two most important notions of derivative used in Banach spaces, the Gâteaux derivative and the Fréchet derivative.

**Definition.** Let $X$ and $Y$ be Banach spaces and $f : X \to Y$ be a mapping. The *Gâteaux derivative* of $f$ at a point $x_0 \in X$ is a bounded linear operator $T : X \to Y$ such that for every $u \in X$,

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu. \tag{1}$$

The operator $T$ is called the *Fréchet derivative* of $f$ at $x_0$ if it is the Gâteaux derivative of $f$ at $x_0$ and the limit in (1) is uniform with respect to $u$ from the unit ball (or unit sphere) in $X$.

The next important example of theorem stating the existence of (many) points of differentiability is the following theorem of Hans Rademacher from 1919:

**Theorem R** (Rademacher)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then $f$ is differentiable almost everywhere.*

The notions of Fréchet differentiability and Gâteaux differentiability coincide for Lipschitz functions on finite-dimensional spaces, and so we do not have to specify in which sense we understand the word "differentiable". Later on, we will always specify which of the two notions we mean.

Theorem R is not optimal as can be seen from the following (nontrivial) fact (see [18]):

*There exists a measure-zero set $A \subset \mathbb{R}^2$ such that each Lipschitz function on $\mathbb{R}^2$ has a point of differentiability in $A$.*

Questions related to sharpness of Rademacher theorem (in $\mathbb{R}^n$) have recently received considerable attention, but we are ultimately interested in the infinite-dimensional situation. Before we turn our attention to the intricacies of infinite dimension, it is, perhaps, worth to mention the following two theorems in the finite-dimensional setting:

**Theorem** (V. Stepanov, 1923)**.** *Any function $f : \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at almost all points at which it is Lipschitz. That is, the set*

$$\Big\{x \in \mathbb{R}^n;\ \limsup_{y\to x} \frac{|f(y) - f(x)|}{\|y - x\|} < \infty \quad \& \quad f'(x) \text{ does not exist}\Big\}$$

*is Lebesgue null.*

It is often possible to obtain a Stepanov type theorem from the corresponding Rademacher type theorem (in various infinite-dimensional settings; see e.g. the unpublished paper of J. Malý, L. Zajíček: *On Stepanov type differentiability theorems*).

The last example of differentiability theorem in finite dimension which we want to mention is due to F. Mignot and can be found in [20]. It states that a monotone multifunction on $\mathbb{R}^n$ is differentiable almost everywhere. It is interesting to note that unlike the Rademacher theorem, this theorem of Mignot has no infinite-dimensional analogue.

## 0.2 Modern development in infinite dimension

In order to even formulate a statement similar to the Rademacher theorem for functions defined on an infinite-dimensional Banach space $X$, we first need to define what do we mean by "almost everywhere". This is always done using some nontrivial $\sigma$-ideal $\mathcal{S}$ of subsets of $X$, and so the Rademacher type theorems are of the following form:

*Let $f : X \to Y$ be a Lipschitz mapping. Then $f$ is differentiable $\mathcal{S}$-almost everywhere. That is, there exists $A \in \mathcal{S}$ such that $f$ is differentiable at each $x \in X \setminus A$.*

One can formulate various statements of this type by imposing additional assumptions on the spaces $X$ and $Y$, by specifying in which sense do we mean the word "differentiable", by specifying the $\sigma$-ideal $\mathcal{S}$, or by assuming that $f$ is convex instead of Lipschitz. We shall now discuss some of the true statements of this form. The first theorems proved in the infinite-dimensional setting concerned continuous convex functions on (some) separable spaces:

**Theorem** (S. Mazur, 1933)**.** *Let $X$ be a separable Banach space. Then each continuous convex function on $X$ is Gâteaux differentiable on a dense $G_\delta$-subset of $X$.*

**Theorem** (E. Asplund, 1968, [2])**.** *Let $X$ be a Banach space with separable dual. Then each continuous convex function on $X$ is Fréchet differentiable on a dense $G_\delta$-subset of $X$.*

The latter theorem was proved for separable reflexive spaces already in 1963 by Joram Lindenstrauss, [14]. Note that both results are of the "Baire category type"; in the modern terminology of Luděk Zajíček, they are also called *generic results*. Note also that the assumption of $X$ having a separable dual cannot be relaxed to $X$ separable. Indeed (see e.g. [9] or [11]):

*On any separable Banach space with nonseparable dual there exists a nowhere Fréchet differentiable equivalent norm.*

It follows from Theorem Z and the remark below it that no generic result can hold for Fréchet derivatives of Lipschitz functions, and so Asplund's theorem cannot be generalized in this direction either. However, the statement can be strengthened by considering a smaller $\sigma$-ideal or nonseparable Asplund spaces as we shall see later.

Since the 1960s, a large number of similar results were proved by many mathematicians who used (or invented for that purpose) various methods in functional analysis, descriptive set theory, geometric measure theory, and recently even set theory and mathematical logic. Besides the fact that the question of generalizing the Rademacher theorem to infinite dimension is quite natural and interesting, there are two main reasons why there are so many results of this type: First, there are many settings in which one can consider the problem. It is possible to alter assumptions on the Banach spaces (with much more variety than in the finite-dimensional case) or the functions, or one can study different kinds of differentiability. We shall briefly discuss some of the branches in the sequel. Second, the infinite-dimensional problem is simply much more difficult; consequently, there are many partial results or results which strengthen or generalize previous ones. To give a taste of how complicated the situation is, let us state probably the most important of related open problems, which seems strikingly basic:

**Problem.** Let $H$ be an infinite-dimensional Hilbert space. Is it true that any three real-valued Lipschitz functions on $H$ have a common point of Fréchet differentiability?

Of course, the fact that this problem is still open means that it is not known whether in every Hilbert space there exists a $\sigma$-ideal of sets such that the corresponding Rademacher theorem for Fréchet derivative holds. What is known, however (and it is a highly non-trivial result of Joram Lindenstrauss, David Preiss and Jaroslav Tišer, [18, Chapters 13 and 16]), is that the answer is positive if we only consider two functions instead of three.

On the other hand, in 2003 J. Lindenstrauss and D. Preiss [16] proved a Rademacher type theorem for Fréchet derivative of Lipschitz functions on a class of Banach spaces including $c_0$ (see Case (7) below). The theorem is very deep and it works with the $\sigma$-ideal of $\Gamma$-null sets, which is tailor-made for the purpose by combining the notions of smallness in the sense of measure and category. One can readily see from the theorem that countably many Lipschitz functions on $c_0$ have a common point of Fréchet differentiability.

Let us now briefly discuss the most important aspects of the infinite-dimensional situation and the differences between diverse settings:

**Target space:** One of the possible ways to generalize Theorem C is to consider a target space more general than $\mathbb{R}$. But it was recognized already in 1930 that for some spaces $Y$ the differentiation theorem can fail even for Lipschitz functions from the unit interval to $Y$. For example, consider the function $f : [0,1] \to L_1[0,1]$ defined by $f : t \mapsto \mathbf{1}_{[0,1]}$ where $\mathbf{1}_{[0,t]}$ is the characteristic function of $[0,t]$. One can readily check that $f$ is an isometry (hence Lipschitz) and it is not differentiable at any point in $[0,1]$.

It follows that no Banach space containing a copy of $L_1[0,1]$ can serve as the target space in any Rademacher type theorem, and the same goes for $c_0$. The class of Banach spaces where a similar pathology does not appear was described and characterized already in the 1930s:

A Banach space $X$ is said to have the *Radon-Nikodým property* (we say also *RNP-space* for short) if every absolutely continuous $f : [0,1] \to X$ is differentiable almost everywhere.

There are many equivalent definitions of very different kinds which we are not going to discuss, but it is certainly useful to remember that all separable dual spaces enjoy the RNP (for more details see e.g. [11], [3] or [5]). It is obvious from the definition that $Y$ having the RNP is a necessary condition for any Rademacher type theorem with target space $Y$ to hold.

**Domain space:** In dealing with Fréchet differentiability it turns out that we also have to restrict the class of Banach spaces which can serve as domain spaces. For example, the norm on $\ell_1$ (which is a continuous convex function) is not Fréchet differentiable at any point. This shows that even when dealing with continuous convex functions (which are a subclass of locally Lipschitz functions), not all separable Banach spaces can be domain spaces in Rademacher type theorems for Fréchet derivative. The class of Asplund spaces is, in fact, defined by the validity of a Rademacher type theorem:

A Banach space $X$ is called *Asplund space* if each continuous convex function on $X$ is Fréchet differentiable on a residual set.

It is known that $X$ is Asplund if and only if each separable subspace has separable dual; in particular, if $X$ is separable, then $X$ is Asplund if and only if $X^*$ is separable. Similarly as the RNP, the property of being Asplund has many characterizations. As a matter of fact, $X$ is Asplund if and only if $X^*$ has the RNP.

No Rademacher type theorems for Fréchet differentiability can hold for non-Asplund spaces because on any non-Asplund Banach space there exists an equivalent norm which is nowhere Fréchet differentiable (see [11]).

For more details on Asplund spaces see e.g. [11] or [9].

**Discussion of possible settings:** The property of being Asplund and the

RNP aside, we have to answer four main questions in order to specify in which branch of the theory are we interested:

(i) Are we interested in *continuous convex* or *Lipschitz* functions?

(ii) Do we study *Gâteaux* or *Fréchet* differentiability?

(iii) Is the domain space *separable* or *nonseparable*?

(iv) Is the target space *the real line* or is it *infinite-dimensional*?

Obviously, it is easier to deal with continuous convex functions than general Lipschitz functions, it is easier to obtain Gâteaux than Fréchet differentiability, it is easier to work in separable spaces than in nonseparable ones, and it is easier to consider real-valued functions. Here by the word "easier" we mean that one is more likely to obtain strong results in that setting.

There are sixteen possible combinations of these assumptions, which makes the situation quite complicated already, and we are only considering the most basic cases (for instance, one can study other notions of derivative, e.g. the Hadamard derivative etc.). Fortunately, some cases are not extremely difficult (and have already been solved to a satisfactory degree), and strong results in some of the easier settings follow from more difficult ones. In the following discussion we are also going to avoid question (iv), touching it only very briefly at some points. This leaves us with eight different settings which we list using abbreviations; note that in cases with Fréchet differentiability we only consider Asplund spaces:

(1) Continuous convex & Gâteaux & separable;

(2) Continuous convex & Gâteaux & nonseparable;

(3) Continuous convex & Fréchet & separable Asplund;

(4) Continuous convex & Fréchet & nonseparable Asplund;

(5) Lipschitz & Gâteaux & separable;

(6) Lipschitz & Gâteaux & nonseparable;

(7) Lipschitz & Fréchet & separable Asplund;

(8) Lipschitz & Fréchet & nonseparable Asplund.

**Case (1):** Probably the easiest case; we have already mentioned Mazur's theorem from 1933 which in itself could be somewhat satisfactory. However, the current understanding of the matter is fairly complete thanks to the following theorem:

**Theorem** (L. Zajíček, [30])**.** *: Let $X$ be a separable Banach space and $A \subset X$. There is a convex continuous real-valued function on $X$ which is nowhere Gâteaux differentiable on $A$ if and only if $A$ is contained in a countable union of graphs of $\delta$-convex functions.*

**Case (2):** The situation is much more complicated than in the separable case. A Banach space is called *weak Asplund* if every continuous convex function on $X$ is Gâteaux differentiable on a residual set. There is no interesting characterization of weak Asplund spaces currently known; various sufficient conditions and related topics are studied in detail in the book [12]. It is obvious from the definition that every Asplund space is weak Asplund, and it follows from Mazur's 1933 theorem that every separable Banach space is weak Asplund. A less trivial example of a subclass of weak Asplund spaces is the class of weakly compactly generated spaces (WCG; see [12] for more details). The study of weak Asplund spaces is more topology oriented, and not too much related to the study of $\sigma$-ideals of small sets in Banach spaces.

**Case (3):** We already mentioned the result of Asplund from 1968 which states the generic differentiability of any continuous convex function on a Banach space with separable dual. This result was later improved on by L. Zajíček and D. Preiss who used more restrictive notions of smallness of sets. In particular, their joint work [24] introduces the notion of *angle small sets* in Banach spaces and establishes the corresponding *super-generic result* (i.e. a result where the notion of null sets is understood with respect to some $\sigma$-ideal of sets smaller than that of sets of the first category) which is close to being optimal (however the optimal result is not known even for Hilbert spaces). Angle small sets are in separable Banach spaces exactly the same as *cone small sets* which are studied and used in this thesis; see Definition 3.5.1.

In 2008 L. Zajíček considered the wider class of *approximately convex functions* (see Definition 3.6.1) and in [34] proved (among many other related results) the following super-generic result:

**Theorem** (L. Zajíček, [34])**.** *Let $X$ be a Banach space with separable dual, $G \subset X$ be an open set and $f : G \to \mathbb{R}$ be a continuous approximately convex function. Then the set $N_F(f)$ of all points $x \in G$ at which $f$ is not Fréchet differentiable is angle small.*

**Case (4):** The situation in nonseparable spaces is always trickier to handle, but in this case the above theorem of L. Zajíček can be extended to general Asplund spaces. This is proved in Chapter 3 of this thesis, Theorem 3.6.3. Note that this theorem uses the notion of cone smallness which in separable Banach spaces is equivalent to angle smallness (not so in nonseparable spaces). We shall discuss this result in more detail later in this introduction.

We remark that even for higher-dimensional target spaces $Y$ (i.e. $\dim(Y) > 1$) it is possible to consider convex mappings if $Y$ is an ordered Banach space (this is necessary in order to formulate a definition of convex operator to $Y$). This setting was studied in the 80s by Jonathan M. Borwein (see e.g. [4]) who obtained generic differentiability results (i.e. differentiability up to a set of the first category) and more recently by L. Zajíček and Libor Veselý (see [27]) with super-generic results.

**Case (5):** The simplest case for Lipschitz functions had its versions of Rademacher type theorems in the 70s:

*Let $X$ be a separable Banach space and $Y$ be a RNP-space. Then every Lipschitz function $f : X \to Y$ is Gâteaux differentiable outside a null set.*

Jens P. R. Christensen in 1972 proved this theorem for *Haar null sets* (which he introduced for the first time in non-locally compact abelian groups) in [6]. He was followed by Piotr Mankiewicz [19], Nachman Aronszajn [1] and Robert R. Phelps [21] who obtained the theorem for the $\sigma$-ideals of *cube null sets*, *Aronszajn null sets* and *Gauss null sets* respectively. These latter three $\sigma$-ideals are defined in quite different ways and it is a deep (and surprising) result of Marianna Csörnyei [7] that the three $\sigma$-ideals coincide in separable Banach spaces.

Although for some time it was thought that these results could not be strengthened, there were stronger results eventually obtained e.g. by D. Preiss and L. Zajíček in [25] and recently by D. Preiss in his paper *Gâteaux differentiability of cone-monotone and pointwise Lipschitz functions* where he proves results which are shown to be (close to) optimal.

**Case (6):** This setting seems to be rather open as some of the best results trivially follow from much stronger results on Fréchet differentiability. One of the reasons lies in the fact that results on Gâteaux differentiability cannot be separably reduced (a method which in some cases allows to deduce a nonseparable theorem from its separable version), a fact which lies in the non-uniform nature of Gâteaux derivative.

**Case (7):** Proving the existence of Fréchet derivatives is much more difficult and, in fact, for some time it was thought impossible: In the literature there were several published examples showing that Lipschitz functions need not be differentiable at any point even on separable Hilbert spaces. However, in 1979 R. R. Phelps and S. Fitzpatrick went through all such examples and found out they were all incorrect.

In 1990 D. Preiss published the main known result in this setting, the following celebrated theorem:

**Theorem** (D. Preiss, [22])**.** *Let $X$ be a Banach space with separable dual and $f : X \to \mathbb{R}$ be Lipschitz. Then $f$ is Fréchet differentiable on a dense set.*

However, this is not an “almost everywhere” type result (indeed, recall the problem stated in the beginning of this section).

So far the only Rademacher type theorems in this setting were given in [16]:

**Theorem** (J. Lindenstrauss and D. Preiss, [16])**.** *The following spaces have the property that every Lipschitz mapping of them into a RNP-space is Fréchet differentiable everywhere except on a $\Gamma$-null set: $C(K)$ for countable compact $K$, subspaces of $c_0$, the Tsirelson space.*

**Case (8):** The nonseparable situation does not attract a lot of attention in this case either. Probably the most interesting result is due to Marek Cúth [8, Theorem 6.18]: Using the separable reduction method of elementary submodels (which we explain and use in chapters 2 and 3 of this thesis) he generalized the above theorem of J. Lindenstrauss and D. Preiss to $C(K)$ for scattered compact $K$ and subspaces of $c_0(\Gamma)$ for arbitrary set $\Gamma$.

Of course, differentiation theorems are not the only kind of results which appear in this theory. There are a number of other results which contribute to our understanding of the matter. For instance, many results arose from the study of relations between and the properties of various $\sigma$-ideals of small sets. A good overview of properties of $\sigma$-ideals offers the book [3]; more recent results can be found in [18]. A good starting point on differentiation theorems of Lipschitz functions on Banach spaces offers the slightly outdated survey article [15].

Before we move on to discuss $\sigma$-porous sets, let us note the following aspect which most of the $\sigma$-ideals mentioned above have in common: They approximate the sets of non-differentiability "from above". For example, assume that $\mathcal{N}$ is the family of all sets of points of non-Gâteaux differentiability of Lipschitz functions on a given separable Banach space $X$. Assume also that a Rademacher type theorem holds in this setting with respect to a $\sigma$-ideal $\mathcal{M}$. Then $\mathcal{N} \subset \mathcal{M}$. In a sense, $\sigma$-porous sets can play the opposite role.

**$\sigma$-porous sets:** It is not a coincidence that different notions of $\sigma$-porosity have enjoyed decades of perpetual attention from many mathematicians who work in fields related to differentiation theory. A systematic study of $\sigma$-porous sets started in 1967 in [10], although the first to use some form of them (under a different terminology) was Arnaud Denjoy already in 1920. Today there is a vast theory developed around various notions of $\sigma$-porosity and related $\sigma$-ideals (see [31] and [33] for surveys of the area; note also the article [17]). They were studied from many angles, and found many interesting applications in real analysis and functional analysis.

An easy observation which could shed some light on the reasons for the usefulness of $\sigma$-porous functions is the following observation, which also explains in what sense do $\sigma$-porous sets play the opposite role to the previously discussed $\sigma$-ideals:

*Let $X$ be a Banach space. A set $E \subset X$ is porous if and only if the function $x \mapsto \operatorname{dist}(x, E)$ is not Fréchet differentiable at any point of $E$.*

In fact, the following is also true (cf. [23] or [18]):

*Let $E$ be a $\sigma$-porous subset of a separable Banach space $X$. Then there is a Lipschitz function $f : X \to \mathbb{R}$ which is not Fréchet differentiable at any point of $E$.*

Assume that $\mathcal{M}$ is a $\sigma$-ideal for which holds a Rademacher type theorem for Fréchet derivative and $X$ separable. Then it follows from the above facts that $\mathcal{M}$ contains all $\sigma$-porous sets.

We remark that the notion of $\sigma$-directional porosity (cf. [33]) corresponds to Gâteaux differentiability in the same way as $\sigma$-porosity corresponds to Fréchet differentiability.

## 0.3 Contributions of this thesis

The mathematical content of the present thesis is divided into four separate chapters. Each chapter corresponds to a research paper containing original results. We shall now briefly introduce each chapter.

### Chapter 1.

In the first chapter we study Cartesian products of $\sigma$-lower porous sets (a notion of porosity defined via limes inferior; see Definition 1.2.1). The work is motivated by a paper of L. Zajíček [32] where the following theorem is proved:

**Theorem** ([32, Theorem 1])**.** *Let $(X, \rho)$ and $(Y, \sigma)$ be topologically complete met-*

*ric spaces and let $A \subseteq X$ and $B \subseteq Y$ be non-$\sigma$-porous $G_\delta$-sets. Then the Cartesian product $A \times B$ is non-$\sigma$-porous in the space $(X \times Y, \rho_m)$ where $\rho_m$ is the maximum metric.*

It is a natural question to ask whether an analogous statement holds for lower porosity. We present a counterexample, showing that the answer is negative.

**Theorem.** *There exist closed non-$\sigma$-lower porous sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the Cartesian product $A \times B$ is lower porous in $\mathbb{R}^2$.*

However, if we strengthen the assumptions of the original conjecture, we obtain the following theorem. These two theorems together give us a fairly complete answer to our question.

**Theorem.** *Let $(X, \rho)$ and $(Y, \sigma)$ be topologically complete metric spaces. Assume that $A \subseteq X$ and $B \subseteq Y$ are Souslin sets in their respective spaces. If $A$ is non-$\sigma$-lower porous in $X$ and $B$ is non-$\sigma$-porous in $Y$ then the Cartesian product $A \times B$ is non-$\sigma$-lower porous in $X \times Y$ (with the maximum metric).*

**Chapter 2.**

In this chapter we establish separable determination theorems for $\sigma$-upper porous sets (the word "upper" can be omitted) and for $\sigma$-lower porous sets. We do so by employing the so-called *method of elementary submodels*. This is a set-theoretical method which involves the use of countable elementary structures and can be used in various branches of mathematics. This method has several advantages; most importantly it allows us to conveniently combine results obtained by this method. A disadvantage is the fact that in some areas of mathematics (including the study of $\sigma$-porous sets) it is not standard and requires a lot of getting used to. Further, theorems obtained by this method are formulated in a specific language, and thus are sometimes difficult to use or even correctly interpret.

Nevertheless, the method allowed us to obtain interesting new results whose statements do not involve elementary submodels. A very basic (and probably not very useful) example of separable determination result which we are able to obtain is the following:

**Theorem.** *Let $X$ be a Banach space and let $A \subset X$ be a Souslin set. Then for every separable subspace $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and*

*(i) $A$ is $\sigma$-upper porous if and only if $A \cap V$ is $\sigma$-upper porous in $V$;*

*(ii) $A$ is $\sigma$-lower porous if and only if $A \cap V$ is $\sigma$-lower porous in $V$.*

Note that this theorem essentially tells us that for any given Souslin set $A \subset X$ (whose porosity properties are not a priori known to us) we can find an "arbitrarily large" separable subspace which determines both the $\sigma$-upper porosity and $\sigma$-lower porosity of $A$. (Of course, this subspace depends on $A$.)

What is interesting is precisely the fact that the separable subspace $V$ can be found in such a way that it reflects both the aforementioned properties at the same time, and we achieve this desirable situation for free: The method of elementary submodels allows us to combine the (originally separate) results on $\sigma$-upper porosity and $\sigma$-lower porosity together.

As an application we combine the main result of this chapter on separable determination of $\sigma$-upper porosity with several auxiliary results (also obtained by the method of elementary submodels) from [8] to extend the following theorem of L. Zajíček to general Asplund spaces:

**Theorem** ([29, Theorem 2]). *Let $X$ be a Banach space with separable dual space and let $G \subset X$ be an open set. Let $f$ be a Lipschitz function on $G$ and let $A$ be the set of all the points $x \in G$ such that $f$ is Fréchet superdifferentiable at $x$ and $f$ is not Fréchet differentiable at $x$. Then $A$ is $\sigma$-upper porous.*

Let us briefly describe the steps in which we prove the result on separable determination of $\sigma$-upper porosity. Simplifying the situation, we can say that our aim is to prove that for a given set $A$ we can find a subspace $V$ such that the following is true:

$A$ is $\sigma$-upper porous in $X \iff A \cap V$ is $\sigma$-upper porous in $V$.

In applications of similar theorems, however, one only needs to use the implication ($\Longleftarrow$) (also, variants of the converse were known before). Let us rewrite what we want to prove:

(+) $A$ is non-$\sigma$-porous in $X \Longrightarrow A \cap V$ is non-$\sigma$-porous in $V$.

The obstacle in "separably reducing" non-$\sigma$-upper porosity is in the presence of the "$\sigma$" which makes the property somewhat global rather than described pointwise. In fact, it is quite non-trivial to prove non-$\sigma$-upper porosity of sets (not surprisingly, it is considerably harder than proving non-$\sigma$-lower porosity).

The standard tool, developed for that purpose by L. Zajíček, is the so-called Foran lemma which essentially states that if we have a system of sets intertwined in the "right" way (a *Foran system*), then each element of this system is non-$\sigma$-porous (cf. [32]). Most importantly, the "right" way in which the sets are intertwined is described by a condition which only involves pointwise porosity. This makes a Foran system (which by the Foran lemma consists only of non-$\sigma$-porous sets) actually easier to separably reduce than a single non-$\sigma$-porous set.

The proof of (+) now consists of the following steps:

(i) We have the non-$\sigma$-porous set $A$; find to it a Foran system $\mathcal{F}$ which contains (a part of) $A$. This means proving a "partial converse of the Foran lemma".

(ii) Separably reduce the Foran system $\mathcal{F}$. That is, find the subspace $V$ in such a way that the system $\mathcal{F}_V := \{F \cap V;\ F \in \mathcal{F}\}$ is a Foran system for upper porosity in $V$.

(iii) The Foran lemma applied on $\mathcal{F}_V$ now yields that all elements of $\mathcal{F}_V$ are non-$\sigma$-porous in $V$. In particular, $A \cap V$ is non-$\sigma$-porous in $V$ and the proof is finished.

**Chapter 3.**

The third chapter can be considered a loose continuation of Chapter 2. We use similar (but more refined) methods to obtain similar kinds of results. Our main aim is to further investigate separable determination of various properties of sets and functions in metric spaces (especially Banach spaces). This means, given a nonseparable metric space $X$ and a property of sets (or functions etc.) in $X$, we are interested whether certain statements about a property hold, provided that they hold in (some) separable subspaces of $X$.

The key method we use to obtain separable determination results uses countable elementary structures which we call elementary submodels. The reader ought to note, however, that there are other ways to tackle this topic. An example is the use of rich families of Banach spaces, which is described in detail e.g. in [18, Section 3.6]. Sometimes one can also opt to prove this sort of results in an "elementary way", in a sense imitating parts of the proof of the Löwenheim-Skolem theorem. This approach would be in many cases very complicated, but it can give a deeper insight.

We concentrated on the notion of *cone smallness* (see Definition 3.5.1) of subsets of Banach spaces, and succeeded in proving its separable determination. The proof goes in similar steps as the one used in Chapter 2 to prove the separable determination of $\sigma$-upper porosity, but this time we do not work with Foran systems, but rather with the new notion of Foran-Zajíček scheme, which is more refined. It essentially is a Souslin scheme on which we impose such additional assumptions that the proof of Foran lemma goes through. As a result we obtain an analogue for both the Foran lemma and its partial converse used in previous chapter, which works for a wider class of porosity-like relations and moreover works for general Souslin sets.

As an application of our result on separable determination of cone smallness we obtained the following:

**Theorem.** *Let $X$ be an Asplund space and $G \subset X$ be open. Let $f: G \to \mathbb{R}$ be a continuous and approximately convex function. Then the set of all points of $G$ at which $f$ is not Fréchet differentiable is cone small.*

This is an example of a *separable reduction theorem* which means that it was proved by reduction to the separable case, which was already known to be true.

**Chapter 4.**

The last chapter of this thesis is not related to the study of $\sigma$-ideals of sets. We study the existence and uniqueness of convex extensions of functions which are locally convex outside a given closed set in $\mathbb{R}^d$, an area of research which is surprisingly unexplored. We introduce the following notion:

**Definition.** We say that a closed set $A \subset \mathbb{R}^d$ is *c-removable* if the following is true: Every real function $f$ on $\mathbb{R}^d$ is convex whenever it is continuous on $\mathbb{R}^d$ and locally convex on $\mathbb{R}^d \setminus A$.

Intuitively, $c$-removable sets are "negligible" for convexity of continuous functions, and thus could also be called exceptional.

One of our aims in this chapter is to provide a sufficient condition for a closed subset of $\mathbb{R}^d$ to be $c$-removable which is more general than that of *interval thinness* of sets established (and introduced) in the article [26]. By doing so we solve an open problem posed in the same article. For more details on the motivation of this work and its connection to [26], see the rather detailed introduction to the chapter.

It could be interesting to note, that there are well-known examples of similar removability problems in mathematics. For example the notion of *removability* of compact subsets of $\mathbb{C}$ related to the Vitushkin's conjecture (see [13, Section 12.2]) is defined as follows:

A compact set $F \subset \mathbb{C}$ is said to be *removable* if, given any bounded open domain $V \subset \mathbb{C}$ containing $F$ and any bounded analytic (i.e. differentiable in the complex sense) function $f : V \setminus F \to \mathbb{C}$, then $f$ has an analytic extension to the whole of $V$.

The two definitions are rather similar, so it is, perhaps, not very surprising that the geometric measure theory is useful in the study of both these notions.

Finally, let us remark, that even though we were able to make some progress, our results are far from final and there are several interesting open problems. In particular, it is not clear to us whether there is a closed totally disconnected Lebesgue null set in $\mathbb{R}^2$ which is not $c$-removable (we prove that such sets exist if we do not require zero measure).

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# Chapter 1.
# Products of Non-$\sigma$-Lower Porous Sets

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**Abstract**

In the present article we provide an example of two closed non-$\sigma$-lower porous sets $A, B \subseteq \mathbb{R}$ such that the product $A \times B$ is lower porous. On the other hand, we prove the following: Let $X$ and $Y$ be topologically complete metric spaces, let $A \subseteq X$ be a non-$\sigma$-lower porous Souslin set and let $B \subseteq Y$ be a non-$\sigma$-porous Souslin set. Then the product $A \times B$ is non-$\sigma$-lower porous. We also provide a brief summary of some basic properties of lower porosity, including a simple characterization of Souslin non-$\sigma$-lower porous sets in topologically complete metric spaces.

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## 1.1 Introduction

In the present article we deal with Cartesian products of $\sigma$-lower porous sets. The work is motivated by a paper of L. Zajíček [5] where the following theorem is proved:

**Theorem Z** ([5, Theorem 1])**.** *Let $(X, \rho)$ and $(Y, \sigma)$ be topologically complete metric spaces and let $A \subseteq X$ and $B \subseteq Y$ be non-$\sigma$-porous $G_\delta$-sets. Then the Cartesian product $A \times B$ is non-$\sigma$-porous in the space $(X \times Y, \rho_m)$ where $\rho_m$ is the maximum metric.*

It is a natural question to ask whether an analogous statement holds for lower porosity (i.e. the notion of porosity defined by limes inferior rather than limes superior). In Section 4 we present a counterexample, showing that the answer is negative.

**Theorem 1.** *There exist closed non-$\sigma$-lower porous sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the Cartesian product $A \times B$ is lower porous in $\mathbb{R}^2$.*

However, if we strengthen the assumptions of the original conjecture, we obtain the following theorem. These two theorems together give us a fairly complete answer to our question.

**Theorem 2.** *Let $(X, \rho)$ and $(Y, \sigma)$ be topologically complete metric spaces. Assume that $A \subseteq X$ and $B \subseteq Y$ are Souslin sets in their respective spaces. If $A$ is non-$\sigma$-lower porous in $X$ and $B$ is non-$\sigma$-porous in $Y$ then the Cartesian product $A \times B$ is non-$\sigma$-lower porous in $X \times Y$ (with the maximum metric).*

It is easy to see that both aforementioned notions of $\sigma$-porosity are invariant with respect to bilipschitz homeomorphisms. Therefore, in all the previous theorems we can equip the product spaces with any metric which is "bilipschitz equivalent" to the maximum metric and the resulting statement will be true.

It is also fitting to give an explanation as to why in Theorem 2 we only require the sets $A$ and $B$ to be Souslin while in Theorem Z these are assumed to be of the type $G_\delta$. The reason is that we use two inscribing theorems (see 1.2.5 and 1.2.6) which, at the time Theorem Z was proved, had not yet been discovered. Of course, this means Theorem Z can be generalized to Souslin sets.

## 1.2 Some facts about $\sigma$-lower porosity and abstract porosity

The main aim of this section is to provide the reader with a self-contained collection of some basic facts about $\sigma$-lower porous sets (with some references to related articles). It might be of some independent interest, but we shall use these facts to prove our main results.

**Notation.** In the whole paper we shall denote by $B(x, r)$ the open ball with centre $x$ and radius $r$, by $\overline{A}$ the closure of the set $A$, and by $\partial A$ the boundary of $A$. As usual, for a set $X$ the symbol $2^X$ denotes the power set of $X$.

**Convention.** Unless stated otherwise, we shall consider all product spaces equipped with the maximum metric (i.e. for $x_1, x_2 \in (X, \rho)$ and $y_1, y_2 \in (Y, \sigma)$, $\rho_m(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max\{\rho(x_1, x_2), \sigma(y_1, y_2)\}$).

The following standard definitions of $\sigma$-porosity originate in a work of A. Denjoy from 1920; however, a systematic investigation of these sets (and the usage of the current nomenclature) has begun in 1967 with an article of E. P. Dolzhenko. For extensive information about $\sigma$-porous sets from various viewpoints we refer the reader to L. Zajíček's survey articles [3] and [6]. The notion of abstract porosity is defined for example in [5].

**Definition 1.2.1.** Let $(X, \rho)$ be a metric space, $M \subseteq X$, $x \in X$ and $R > 0$. We define

$$\gamma(x, R, M) = \sup\{r > 0 : \text{for some } z \in X,\ B(z, r) \subseteq B(x, R) \setminus M\},$$

$$\overline{p}(M, x) = \limsup_{R \to 0_+} \frac{2 \cdot \gamma(x, R, M)}{R}, \qquad \underline{p}(M, x) = \liminf_{R \to 0_+} \frac{2 \cdot \gamma(x, R, M)}{R}.$$

A set $M \subseteq X$ is *(upper) porous at $x$* if $\overline{p}(M, x) > 0$ and *lower porous at $x$* if $\underline{p}(M, x) > 0$.

Now assume $\mathbf{P}$ is a relation between points and subsets of $X$ (i.e. $\mathbf{P} \subseteq X \times 2^X$). The symbol $\mathbf{P}(x, A)$ where $x \in X$ and $A \subseteq X$ means that $\langle x, A\rangle \in \mathbf{P}$. We say that $\mathbf{P}$ is an *abstract porosity on $X$* if the following conditions are satisfied (for all $A \subseteq X$, $B \subseteq X$ and $x \in X$):

**(A1)** If $A \subseteq B \subseteq X$, $x \in X$ and $\mathbf{P}(x, B)$, then $\mathbf{P}(x, A)$.

**(A2)** $\mathbf{P}(x, A)$ if and only if there is an $r > 0$ such that $\mathbf{P}(x, A \cap B(x, r))$.

**(A3)** $\mathbf{P}(x, A)$ if and only if $\mathbf{P}(x, \overline{A})$.

Note that the relations which correspond (in the sense of the first point of the following list) to the notions of porosity and lower porosity are clearly abstract porosities. Let $\mathbf{P}$ be an abstract porosity on $X$. We say that $A \subseteq X$ is

- $\mathbf{P}$*-porous at $x \in X$* if $\mathbf{P}(x, A)$,
- $\mathbf{P}$*-porous (in $X$)* if $A$ is $\mathbf{P}$-porous at each of its points,
- $\sigma$-$\mathbf{P}$*-porous (in $X$)* if $A$ is a countable union of $\mathbf{P}$-porous sets,
- $\sigma$-$\mathbf{P}$*-porous at $x \in X$* if there is an $r > 0$ such that $A \cap B(x, r)$ is $\sigma$-$\mathbf{P}$-porous.

In case $\mathbf{P}$ corresponds to lower porosity we say *$A$ is lower porous*, *$\sigma$-lower porous* or *$\sigma$-lower porous at $x$*. If $\mathbf{P}$ corresponds to ordinary (upper) porosity, we simply omit the symbol $\mathbf{P}$ and write *$A$ is porous* etc. (however, in some cases we tend to add “upper” to avoid confusion).

**Remark 1.2.2.** If $(X, \rho)$ is a metric space and $\mathbf{P}$ is an abstract porosity on $X$, it is well-known that the family $\mathcal{I}$ of all $\sigma$-$\mathbf{P}$-porous sets in $X$ satisfies the following conditions:

(i) If $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$.

(ii) If $A_n \in \mathcal{I}$ for all $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{I}$.

The following two propositions are well-known (see the survey article [6]), but we shall provide the proofs for the sake of completeness. Proposition 1.2.4 gives us a method to recognize non-$\sigma$-lower porous sets

**Proposition 1.2.3.** *Let $(X, \rho)$ be a metric space and let $A \subseteq X$ be $\sigma$-lower porous. Then $A$ can be covered by a countable family of closed lower porous sets.*

*Proof.* Without loss of generality we can assume the set $A$ is lower porous. From the definition of lower porosity it is clear that for any $x \in A$ we can choose a positive number $h_0 = h_0(x)$ such that for all $h \in (0, h_0(x))$:

$$\frac{2 \cdot \gamma(x, h, A)}{h} > \frac{\underline{p}(x, A)}{2}.$$

Thus we have chosen a function $h_0 : A \longrightarrow (0, \infty)$. Set

$$A_n := \left\{ x \in A : h_0(x) > \frac{1}{n} \text{ and } \underline{p}(x, A) > \frac{1}{n} \right\};$$

then, clearly, $A = \bigcup_{n=1}^{\infty} A_n$. We shall now prove that for each $n \in \mathbb{N}$ the set $\overline{A_n}$ is lower porous. And since it is obvious that for any $x \in X$, $R > 0$ and $M \subseteq X$ the equality $\gamma(x, R, M) = \gamma(x, R, \overline{M})$ is true, we only need to show that the set $A_n$ is lower porous at each point $x \in \overline{A_n} \setminus A_n$.

To that end, choose a natural number $n$ and a point $x \in \overline{A_n} \setminus A_n$. Now, for an arbitrary $h \in \left(0, \frac{1}{n}\right)$ there is a point $y \in B\left(x, \frac{h}{2}\right) \cap A_n$ and from the definition of $A_n$ it follows that there is a point $z \in B\left(y, \frac{h}{2}\right)$ such that $B\left(z, \frac{h}{8n}\right) \subseteq B\left(y, \frac{h}{2}\right) \setminus A_n$. Thus $\gamma(x, h, A_n) \geq \frac{h}{8n}$ and

$$\liminf_{h \to 0_+} \frac{2 \cdot \gamma(x, h, A_n)}{h} \geq \frac{1}{4n} > 0.$$

□

**Proposition 1.2.4.** *Let $(X, \rho)$ be a metric space and let $F \subseteq X$ be a topologically complete subspace. Let there exist a set $D \subseteq F$ dense in $F$ such that $F$ is lower porous (in $X$) at no point of $D$. Then $F$ is not $\sigma$-lower porous in $X$.*

*Proof.* Assume to the contrary that $F$ is $\sigma$-lower porous. Proposition 1.2.3 gives us closed lower porous sets $F_n$ ($n \in \mathbb{N}$) such that $F \subseteq \bigcup_{n=1}^{\infty} F_n$. Hence $F = \bigcup_{n=1}^{\infty}(F_n \cap F)$ and the set $F_n \cap F$ is closed in $F$ for each natural $n$. Using the Baire theorem in the topologically complete space $F$ we obtain an open set $G \subseteq X$ such that $\emptyset \neq G \cap F \subseteq F_{n_0} \cap F$ for some natural number $n_0$. Thus $G \cap F$ (being a subset of $F_{n_0}$) is lower porous in $X$ and it follows that $F$ is lower porous at every point $x \in G \cap F$ (for $G$ is an open set). But the set $D$ is dense in $F$ so there exists a point $x \in D \cap G \cap F$ which is a contradiction with the assumption that $F$ is lower porous at no point of $D$. □

Now we formulate two rather deep inscribing theorems which will be used on various occasions throughout the paper. Their purpose is to obtain some of our statements about non-$\sigma$-porous and non-$\sigma$-lower porous sets for all Souslin sets instead of closed (or $G_\delta$) sets only.

**Theorem 1.2.5** ([8, Theorem 3.1])**.** *Let $(X, \rho)$ be a topologically complete metric space and let $S \subseteq X$ be a non-$\sigma$-porous Souslin set. Then there exists a closed non-$\sigma$-porous set $F \subseteq S$.*

**Theorem 1.2.6** ([7, Corollary 3.4])**.** *Let $(X, \rho)$ be a topologically complete metric space and let $S \subseteq X$ be a non-$\sigma$-lower porous Souslin set. Then there exists a closed non-$\sigma$-lower porous set $F \subseteq S$.*

We continue by recalling several basic definitions (cf. e.g. [4] and [1]) which we need in the following.

**Definition 1.2.7.** Let $(X, \rho)$ be a metric space and let $\mathbf{P}$ be an abstract porosity on $X$. If $A \subseteq X$ then by $K_{\mathbf{P}}(A)$ we denote the set of all $x \in A$ such that $A$ is not $\sigma$-$\mathbf{P}$-porous at $x$.

Recall that a family of sets $\mathcal{M} \subseteq 2^X$ is called

- *locally finite* if for each $x \in X$ there is an $r > 0$ such that the ball $B(x, r)$ intersects at most finitely many elements of $\mathcal{M}$,
- *discrete* if for each $x \in X$ there is an $r > 0$ such that the ball $B(x, r)$ intersects at most one element of $\mathcal{M}$,
- *$\sigma$-discrete* if it is a countable union of discrete families.

We say $\mathcal{M}$ is a *cover* of $X$ if $\bigcup \mathcal{M} = X$. Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of $X$. Then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for each $B \in \mathcal{V}$ there is a set $A \in \mathcal{U}$ such that $B \subseteq A$.

An elementary proof of the following Proposition 1.2.9 can be found as the proof of Lemma 3 in the article [4]; we give an alternative proof which is more transparent, but is not elementary since it uses the famous theorem of A. H. Stone about the paracompactness of metric spaces ([1, Theorem 4.4.1]). We will use the following easy lemma.

**Lemma 1.2.8.** *Let $(X, \rho)$ be a metric space and let $\mathbf{P}$ be an abstract porosity on $X$. Then:*

*(i) If $\mathcal{M}$ is a discrete family of $\mathbf{P}$-porous sets, then $\bigcup \mathcal{M}$ is $\mathbf{P}$-porous.*

*(ii) If $\mathcal{M}$ is a $\sigma$-discrete family of $\sigma$-$\mathbf{P}$-porous sets, then $\bigcup \mathcal{M}$ is $\sigma$-$\mathbf{P}$-porous.*

*Proof.* First, we shall prove assertion (i). Let $\mathcal{M}$ be a discrete family of $\mathbf{P}$-porous sets and let $x \in \bigcup \mathcal{M}$ be an arbitrary point; we shall prove that $\bigcup \mathcal{M}$ is $\mathbf{P}$-porous at $x$. Since the family $\mathcal{M}$ is discrete, there is an $r > 0$ and $M \in \mathcal{M}$ such that

$$\left(\bigcup \mathcal{M}\right) \cap B(x, r) = M \cap B(x, r). \tag{1.1}$$

The set $M$ is $\mathbf{P}$-porous and from **(A1)** (see 1.2.1) we have that so is $M \cap B(x, r)$. It follows from (1.1) and **(A2)** that also the sum $\bigcup \mathcal{M}$ is $\mathbf{P}$-porous at $x$.

To prove the second assertion, assume (clearly without loss of generality) $\mathcal{M}$ is a discrete family of $\sigma$-$\mathbf{P}$-porous sets. Each $M \in \mathcal{M}$ can be written in the form $M = \bigcup_{n=1}^{\infty} A_n^M$ where the set $A_n^M$ is $\mathbf{P}$-porous for any $n \in \mathbb{N}$. It is obvious that for each $n \in \mathbb{N}$ the family $\left\{A_n^M : M \in \mathcal{M}\right\}$ is discrete. Thus, using the first part of this lemma, we obtain the $\sigma$-**P**-porosity of

$$\bigcup \mathcal{M} = \bigcup_{n=1}^{\infty} \bigcup_{M \in \mathcal{M}} A_n^M. \qquad \square$$

**Proposition 1.2.9.** *Let $(X, \rho)$ be a metric space and let $\mathbf{P}$ be an abstract porosity on $X$. Assume the set $A \subseteq X$ is $\sigma$-$\mathbf{P}$-porous at each of its points. Then $A$ is $\sigma$-$\mathbf{P}$-porous.*

*Proof.* Set $A_n := \left\{x \in A : B\left(x, \frac{1}{n}\right) \cap A \text{ is } \sigma\text{-}\mathbf{P}\text{-porous}\right\}$; by the assumption, $A = \bigcup_{n=1}^{\infty} A_n$. Let us fix an arbitrary $k \in \mathbb{N}$ and prove that $A_k$ is $\sigma$-**P**-porous.

To that end, we define the open cover $\mathcal{U}$ of $X$ as $\mathcal{U} := \left\{B\left(x, \frac{1}{2k}\right) : x \in X\right\}$; it is easy to see that for each $B \in \mathcal{U}$ the set $B \cap A_k$ is $\sigma$-**P**-porous. Using the Stone Paracompactness Theorem we obtain a $\sigma$-discrete refinement $\mathcal{V}$ of $\mathcal{U}$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, we have that for each $G \in \mathcal{V}$ the set $G \cap A_k$ is $\sigma$-**P**-porous and it follows from Lemma 1.2.8 that $A_k = \bigcup \{G \cap A_k : G \in \mathcal{V}\}$ is $\sigma$-**P**-porous. This concludes the proof. $\square$

An immediate consequence of this result is the following.

**Corollary 1.2.10.** *Let $(X, \rho)$ be a metric space and let $\mathbf{P}$ be an abstract porosity on $X$. Assume the set $A \subseteq X$ is not $\sigma$-$\mathbf{P}$-porous. Then:*

*(i) $K_{\mathbf{P}}(A)$ is nonempty and closed in $A$.*

*(ii) The set $A \setminus K_{\mathbf{P}}(A)$ is $\sigma$-$\mathbf{P}$-porous.*

*(iii) $K_{\mathbf{P}}(K_{\mathbf{P}}(A)) = K_{\mathbf{P}}(A)$ (i.e. $K_{\mathbf{P}}(A)$ is $\sigma$-$\mathbf{P}$-porous at none of its points).*

*(iv) The set of all points at which $K_{\mathbf{P}}(A)$ is not $\mathbf{P}$-porous is dense in $K_{\mathbf{P}}(A)$.*

The proposition that follows now, provides a simple characterization of non-$\sigma$-lower porous Souslin sets. It can be regarded as an analogue for lower porosity to a partial converse of the Foran lemma which was proved by L. Zajíček (see [5, Corollary 1]); the mentioned result works for upper porosity and $G_\delta$ sets (but can, of course, be generalized to Souslin sets via Theorem 1.2.5).

**Proposition 1.2.11.** *Let $(X, \rho)$ be a topologically complete metric space and let $A \subseteq X$ be a Souslin set. Then the following statements are equivalent:*

*(i) $A$ is not $\sigma$-lower porous.*

*(ii) There exists a closed set $F \subseteq A$ and a set $D \subseteq F$ dense in $F$ such that $F$ is lower porous at no point of $D$.*

*Proof.* To prove the implication (i)⇒(ii) assume A is a non-$\sigma$-lower porous Souslin set; using Theorem 1.2.6 we can assume without loss of generality that $A$ is closed. Let $\mathbf{P}$ be the abstract porosity which corresponds to lower porosity in $X$. Now it suffices to take $F := K_{\mathbf{P}}(A)$, as all the desired properties of $F$ follow from Corollary 1.2.10.

To prove (ii)⇒(i) suppose that (ii) holds. Then Proposition 1.2.4 gives that $F$ is non-$\sigma$-lower porous, and thus so is $A$. $\square$

**Remark 1.2.12.**

(a) It could be interesting to note a connection of Proposition 1.2.11 to the article [2] (especially Section 5) where the notion of **P**-reducible sets is defined and studied. If **P** is an abstract porosity on a metric space $X$, we say that $A \subseteq X$ is **P**-reducible if each nonempty closed set $F \subseteq A$ contains a **P**-porous subset with nonempty relative interior in $F$. Now the statement of Proposition 1.2.11 can be reformulated as follows:

If $X$ is topologically complete and $\mathbf{L}$ is the relation corresponding to the notion of lower porosity on $X$, then a Souslin set $A \subseteq X$ is $\sigma$-lower porous if and only if it is **L**-reducible.

(b) Now let us briefly turn our attention to the general case. As Corollary 1.2.10 (iv) holds for any abstract porosity **P**, the following is true:

Let **P** be any abstract porosity on a metric space $X$ and let $A \subseteq X$ be closed. Then:

$$A \text{ is non-}\sigma\text{-}\mathbf{P}\text{-porous} \implies A \text{ is not } \mathbf{P}\text{-reducible.}$$

If $X$ is topologically complete and **P** corresponds to upper porosity on $X$, it suffices to assume the set $A$ to be Souslin (due to Theorem 1.2.5).

However, if **P** is such that an analogue of Proposition 1.2.4 for **P** does not hold (e.g., the upper porosity), then the other implication in the previous statement does not necessarily hold (see Example 3.1 or Corollary 5.3 with Proposition 5.1 of [2]). That is the reason why a more elaborate method of recognizing non-$\sigma$-upper porous sets had to be developed in order to prove Theorem Z from the introduction (the method of the Foran Lemma and its partial converse).

## 1.3 One positive result

**Theorem 1.3.1.** *Let $(X, \rho)$ and $(Y, \sigma)$ be topologically complete metric spaces. Assume the Souslin set $A \subseteq X$ is not $\sigma$-lower porous and the Souslin set $B \subseteq Y$ is not $\sigma$-porous. Then the Cartesian product $A \times B$ is not $\sigma$-lower porous in the space $X \times Y$ (with the maximum metric).*

*Proof.* Let $\mathbf{L} \subseteq X \times 2^X$ be the relation corresponding to the notion of lower porosity on $X$ (i.e. $\mathbf{L}(x, C)$ if and only if $C$ is lower porous at $x$) and let $\mathbf{U} \subseteq Y \times 2^Y$ be the relation corresponding to upper porosity on $Y$. Since both these relations are abstract porosities, from Corollary 1.2.10 we know that $K_{\mathbf{L}}(A) \neq \emptyset$ and $K_{\mathbf{U}}(B) \neq \emptyset$; without loss of generality we shall now assume that $A = K_{\mathbf{L}}(A)$ and $B = K_{\mathbf{U}}(B)$ and using Theorem 1.2.5 and Theorem 1.2.6 we may also assume that the sets $A$ and $B$ are closed in their spaces.

Denote by $A_1$ the set of all points of $A$ at which $A$ is not lower porous and by $B_1$ the set of all points of $B$ at which $B$ is not porous. From 1.2.10 we know that $A_1$ is dense in $A$ and $B_1$ is dense in $B$; thus $A_1 \times B_1$ is dense in $A \times B$. By Proposition 1.2.4, it suffices to prove that $A \times B$ is lower porous at no point of $A_1 \times B_1$. However, this is true due to Corollary 1.4.8, hence the proof is complete. $\square$

## 1.4 Counterexample

**Definition 1.4.1.** Denote $D_0 := \emptyset$ and for each $n \in \mathbb{N}$ we define the open set $D_n \subseteq (0,1)$ as

$$D_n := \bigcup_{i=0}^{3^{n-1}-1} \Big( \frac{1+3i}{3^n}, \frac{2+3i}{3^n} \Big).$$

Furthermore, for each $n \in \mathbb{N} \cup \{0\}$ we define

$$M_n := \partial D_n, \qquad A_n := [0,1] \setminus D_n.$$

Finally, if $I \subseteq \mathbb{N}$ is nonempty, we define

$$D_I := \bigcup_{n \in I} D_n, \qquad M_I := \bigcup_{n \in I} M_n, \qquad A_I := [0,1] \setminus D_I.$$

**Definition 1.4.2.** Let $(X, \rho)$ be a metric space and let $\varepsilon > 0$. Recall that $M \subseteq X$ is an *$\varepsilon$-net in $X$*, if for each point $x \in X$ there exists some $y \in M$ such that $\rho(x,y) \leq \varepsilon$.

The following facts are easy to see.

**Observation 1.4.3.**

*(i) For each $n \in \mathbb{N}$ the set $M_n$ is a $3^{-n}$-net in the interval $[0,1]$.*

*(ii) If $I \subseteq \mathbb{N}$ is infinite, then*

- $\overline{M_I} = [0,1]$,
- *$A_I$ is porous.*

*(iii) Whenever $m, n \in \mathbb{N}$, $m \neq n$, then we have $M_m \cap M_n = \emptyset$.*

*(iv) $M_n \cap D_m \neq \emptyset$ if and only if $m < n$.*

*(v) $A_{\mathbb{N}}$ is the ternary Cantor set.*

**Lemma 1.4.4.** *Let $I \subseteq \mathbb{N}$ be infinite and let $\emptyset \neq J \subseteq \mathbb{N}$. Then $M_I \cap A_J$ is dense in $A_J$.*

*Proof.* Choose an arbitrary $y \in A_J$ and $\varepsilon > 0$. Now find an $n_0 \in I$ such that $2 \cdot 3^{-n_0} < \varepsilon$ and denote $K := J \cap (0, n_0)$. On account of 1.4.3 (iv) it is true that $M_{n_0} \cap A_K = M_{n_0} \cap A_J$. Setting $n_1 := \max(K \cup \{0\})$ we have $n_1 < n_0$ and it is obvious that the components of $A_K$ are closed intervals whose length is at least $3^{-n_1}$. The set $M_{n_0}$ is a $3^{-n_0}$-net in $[0,1]$ (1.4.3 (i)) and $3^{-n_1} > 2 \cdot 3^{-n_0}$; from these two facts now easily follows that $M_{n_0}$ is a $(2 \cdot 3^{-n_0})$-net in $A_K$. This implies the existence of a point $z \in M_{n_0} \cap A_K = M_{n_0} \cap A_J \subseteq M_I \cap A_J$ such that $|z - y| \leq 2 \cdot 3^{-n_0} < \varepsilon$, which concludes the proof. □

**Definition 1.4.5.** Let $(X, \rho)$ be a metric space, let $A \subseteq X$ and let $x \in X$. We define the function $\delta_{A,x} : (0, \infty) \longrightarrow [0, \infty)$ as

$$\delta_{A,x}(h) := \frac{2 \cdot \gamma(x, h, A)}{h}.$$

**Lemma 1.4.6.** *Assume $I \subseteq \mathbb{N}$ is nonempty and let $x \in A_I$ and $n \in I$. Then for each $h \in \left[\frac{4}{3^{n+1}}, \frac{4}{3^n}\right]$ we have that $\delta_{A_I,x}(h) \geq \frac{1}{4}$.*

*Proof.* Let $I \subseteq \mathbb{N}$, $x \in A_I$ and $n \in I$ be given. Since $n \in I$, we have that $A_I \subseteq [0,1] \setminus D_n$ and thus $x \in [0,1] \setminus D_n$. From 1.4.3 (i) we know that the set $M_n = \partial D_n$ is a $3^{-n}$-net in the interval $[0,1]$ which implies that $\operatorname{dist}(x, D_n) \leq 3^{-n}$. From this and from the fact that $D_n$ consists of pairwise disjoint open intervals of length $3^{-n}$, it follows that for all $h \in \left[\frac{1}{3^n}, \frac{2}{3^n}\right]$ holds the inequality

$$2 \cdot \gamma(x, h, [0,1] \setminus D_n) \geq h - \frac{1}{3^n}.$$

What is more, for any $h > \frac{2}{3^n}$

$$2 \cdot \gamma(x, h, [0,1] \setminus D_n) \geq \frac{1}{3^n}.$$

Consequently

$$\delta_{A_I,x}(h) \geq \frac{2 \cdot \gamma(x, h, [0,1] \setminus D_n)}{h} \geq$$

$$\geq \begin{cases} \frac{1}{h}\left(h - \frac{1}{3^n}\right) \geq 1 - \frac{3^{n+1}}{4} \cdot \frac{1}{3^n} = \frac{1}{4} & \text{for } h \in \left[\frac{4}{3^{n+1}}, \frac{2}{3^n}\right], \\ \frac{1}{h} \cdot \frac{1}{3^n} \geq \frac{3^n}{4} \cdot \frac{1}{3^n} = \frac{1}{4} & \text{for } h \in \left[\frac{2}{3^n}, \frac{4}{3^n}\right]. \end{cases}$$

□

**Proposition 1.4.7.** *Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces and let us have sets $A \subseteq X$ and $B \subseteq Y$. Finally, let there be given points $x \in X$ and $y \in Y$. Then:*

*(i)* $\gamma(\langle x, y\rangle, h, A \times B) = \max\{\gamma(x, h, A), \gamma(y, h, B)\}$ *for any $h > 0$.*

*(ii)* $\delta_{A \times B, \langle x,y\rangle} = \max\{\delta_{A,x}, \delta_{B,y}\}$.

*Proof.* We shall prove assertion (i). Without loss of generality we may assume that $\alpha := \max\{\gamma(x, h, A), \gamma(y, h, B)\} = \gamma(x, h, A) > 0$. Choose arbitrary $h > 0$ and $\varepsilon \in (0, \alpha)$. By the definition of $\gamma(x, h, A)$, there exists a point $x_1 \in X$ such that $B(x_1, \alpha - \varepsilon) \subseteq B(x, h) \setminus A$. Thus,

$$B(\langle x_1, y\rangle, \alpha - \varepsilon) \subseteq B(\langle x, y\rangle, h) \setminus A \times B$$

and this means that

$$\gamma(\langle x, y\rangle, h, A \times B) \geq \alpha - \varepsilon = \max\{\gamma(x, h, A), \gamma(y, h, B)\} - \varepsilon.$$

To prove the opposite inequality we take arbitrary $h > 0$ and $\varepsilon > 0$ again. Setting $\beta := \gamma(\langle x, y\rangle, h, A \times B)$, we can assume that $\varepsilon < \beta$. Now find a point $\langle x_1, y_1\rangle \in X \times Y$ such that

$$G := B(\langle x_1, y_1\rangle, \beta - \varepsilon) \subseteq B(\langle x, y\rangle, h) \setminus A \times B.$$

Taking into account that $G = B(x_1, \beta - \varepsilon) \times B(y_1, \beta - \varepsilon)$ (for we consider the space $X \times Y$ with the maximum metric), this yields that

$$B(x_1, \beta - \varepsilon) \subseteq B(x, h) \setminus A \qquad \text{or} \qquad B(y_1, \beta - \varepsilon) \subseteq B(y, h) \setminus B.$$

This implies the following inequality which concludes the proof of (i):

$$\max\{\gamma(x, h, A), \gamma(y, h, B)\} \geq \beta - \varepsilon = \gamma(\langle x, y\rangle, h, A \times B) - \varepsilon.$$

The second assertion follows immediately from (i). □

**Corollary 1.4.8.** *Under the assumptions of Proposition 1.4.7 we have that if $A$ is not lower porous at $x$ and $B$ is not porous at $y$, then $A \times B$ is not lower porous at $\langle x, y\rangle$.*

*Proof.* Let $x$ and $y$ be as above. Then

$$\liminf_{h\to 0_+} \delta_{A,x}(h) = 0 \qquad \text{and} \qquad \limsup_{h\to 0_+} \delta_{B,y}(h) = 0.$$

From Proposition 1.4.7 we know that $\delta_{A\times B,\langle x,y\rangle} = \max\{\delta_{A,x}\,, \delta_{B,y}\}$, and so it is easy to see that $\liminf_{h\to 0_+} \delta_{A\times B,\langle x,y\rangle} = 0$, i.e., $A \times B$ is not lower porous at $\langle x, y\rangle$. □

**Remark 1.4.9.** Let $(X, \rho)$ be a metric space. If the set $A \subseteq X$ is not porous then neither is $A^2 = A \times A$ porous in $X^2$. The same statement is true for lower porosity or, in general, for any notion of porosity which is determined solely by the function $\delta_{A,x}(h)$.

Indeed, if we assume that the set $A$ is not porous at a certain point $x \in A$, then, since $\delta_{A,x} = \delta_{A^2,\langle x,x\rangle}$, it is clear that $A^2$ is not porous at $\langle x, x\rangle$. Clearly, the same argument works for many other notions of porosity – including, for example, lower porosity.

We shall now prove the main result of this section which implies Theorem 1 from the Introduction.

**Theorem 1.4.10.** *Let the set $I \subseteq \mathbb{N}$ be defined by the formula*

$$I := \bigcup_{i=1}^{\infty} \left[\, i^2, i^2 + i \,\right) \cap \mathbb{N}$$

*and let $J = \mathbb{N} \setminus I$. Then none of the closed sets $A_I$ and $A_J$ is $\sigma$-lower porous while the product $A_I \times A_J$ is lower porous.*

*Proof.* First, we shall prove that the set $A_J$ is not $\sigma$-lower porous; of course, the proof for $A_I$ would be analogous. Being a closed subspace of $\mathbb{R}$, $A_J$ is a topologically complete space. Hence, according to Proposition 1.2.4 it suffices to find a dense subset of $A_J$ at whose points the set $A_J$ is not lower porous. We claim that $M_I \cap A_J$ is such a set. Indeed, by Lemma 1.4.4, $M_I \cap A_J$ is dense in $A_J$; it only remains to be shown that $A_J$ is lower porous at no point of $M_I \cap A_J$.

To prove that, choose an arbitrary point $x \in M_I \cap A_J$ and let $n_0 \in I$ be the unique natural number such that $x \in M_{n_0}$ (the uniqueness of $n_0$ is clear from 1.4.3 (iii)). Now $x$ can be written in the form $\frac{k}{3^{n_0}}$, where $k \in \mathbb{N}$ is not divisible by 3. It follows that for each natural $j > n_0$

$$\operatorname{dist}(x, D_j) = \frac{1}{3^j}. \tag{1.2}$$

Moreover, since $x \in M_I \cap A_J$, for each natural $j < n_0$ we have

$$\operatorname{dist}(x, D_j) \geq \frac{1}{3^{n_0}}. \tag{1.3}$$

Now fix a natural number $i_0$ such that $i_0^2 > n_0$ and choose an arbitrary $i > i_0$. The inequalities (1.2) and (1.3) imply that

$$\operatorname{dist}\Big(x, \bigcup\{D_n : n \in J, n \le i^2 - 1\}\Big) = \frac{1}{3^{i^2-1}}. \tag{1.4}$$

From the definition of $J$ we see that $\{i^2, i^2+1, \ldots, i^2+i-1\} \cap J = \emptyset$. This fact, together with (1.4), implies that the longest interval contained in

$$\Big(x - \frac{1}{3^{i^2-1}}, x + \frac{1}{3^{i^2-1}}\Big)$$

and disjoint with $A_J$ is a component of $D_{i^2+i}$ (as $i^2 + i \in J$), and therefore its length is $3^{-(i^2+i)}$. That is,

$$\delta_{A_J,x}\Big(\frac{1}{3^{i^2-1}}\Big) = 3^{i^2-1} \cdot \frac{1}{3^{i^2+i}} = \frac{1}{3^{i+1}};$$

it follows that $\liminf_{h\to 0_+} \delta_{A_J,x}(h) = 0$ which means that $A_J$ is not lower porous at $x$.

To prove that the product $A_I \times A_J$ is lower porous, choose an arbitrary point $\langle x, y\rangle \in A_I \times A_J$. By Lemma 1.4.6 we have

$$\delta_{A_I,x}(h) \ge \frac{1}{4}, \quad \text{whenever} \quad h \in \bigcup_{n\in I}\Big[\frac{4}{3^{n+1}}, \frac{4}{3^n}\Big] =: F_I,$$

$$\text{and also} \quad \delta_{A_J,y}(h) \ge \frac{1}{4}, \quad \text{whenever} \quad h \in \bigcup_{n\in J}\Big[\frac{4}{3^{n+1}}, \frac{4}{3^n}\Big] =: F_J.$$

But $I \cup J = \mathbb{N}$, so $F_I \cup F_J = \big(0, \frac{4}{3}\big]$, and it immediately follows from Proposition 1.4.7 that $\liminf_{h\to 0_+} \delta_{A_I\times A_J,\langle x,y\rangle}(h) \ge \frac{1}{4}$, concluding the proof. □

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# Chapter 2.
# $\sigma$-Porosity is Separably Determined

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**Abstract**

We prove a separable reduction theorem for $\sigma$-porosity of Souslin sets. In particular, if $A$ is a Souslin subset in a Banach space $X$, then each separable subspace of $X$ can be enlarged to a separable subspace $V$ such that $A$ is $\sigma$-porous in $X$ if and only if $A \cap V$ is $\sigma$-porous in $V$. Such a result is proved for several types of $\sigma$-porosity. The proof is done using the method of elementary submodels, hence the results can be combined with other separable reduction theorems. As an application we extend a theorem of L. Zajíček on differentiability of Lipschitz functions on separable Asplund spaces to the nonseparable setting.

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## 2.1 Introduction

The aim of this article is to obtain separable reduction theorems for some classes of $\sigma$-porous sets by employing the method of elementary submodels. This is a set-theoretical method which can be used in various branches of mathematics. A. Dow in [2] illustrated the use of this method in topology, W. Kubiś in [4] used it in functional analysis, namely to construct projections on Banach spaces.

In this article we shall use the method of elementary submodels to prove Theorem 2.5.1 and Theorem 2.5.4 which have as a consequence for example the following:

**Theorem 2.1.1.** *Let $(X, \|\cdot\|)$ be a Banach space and let $A \subset X$ be a Souslin set. Then for every separable subspace $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and*

*(i) $A$ is $\sigma$-upper porous if and only if $A \cap V$ is $\sigma$-upper porous in the space $V$,*

*(ii) $A$ is $\sigma$-lower porous if and only if $A \cap V$ is $\sigma$-lower porous in the space $V$.*

As a consequence of Theorem 2.5.1 and [1, Theorem 5.7] we get the following:

**Theorem 2.1.2.** *Let $X, Y$ be Banach spaces, $G \subset X$ an open subset and $f : G \to Y$ be a function. Then for every separable subspace $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and that the following two conditions are equivalent:*

*(i) the set of the points where $f$ is not Fréchet differentiable is $\sigma$-upper porous,*

*(ii) the set of the points where $f \restriction V$ is not Fréchet differentiable is $\sigma$-upper porous in $V$.*

The first result is in a certain sense an improvement of the result of J. Lindenstrauss, D.Preiss and J.Tišer [6, Corrolary 3.6.7], from where only the implication $(i) \to (ii)$ follows. Moreover, we are able to easily extend results concerning points of non-differentiability from separable Banach spaces to the non-separable case. An example of such a result is Theorem 2.5.5 which has been proved in the article [9] – the generalization is in Theorem 2.5.6.

Let us recall the most relevant notions, definitions and notations:

**Notation.** We denote by $\omega$ the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$, by $\mathbb{R}_+$ the interval $(0, \infty)$ and $\mathbb{Q}_+$ stands for $\mathbb{R}_+ \cap \mathbb{Q}$. Whenever we say that a set is countable, we mean that the set is either finite or infinite and countable. If $f$ is a mapping then we denote by $\operatorname{Rng} f$ the range of $f$ and by $\operatorname{Dom} f$ the domain of $f$. By writing $f : X \to Y$ we mean that $f$ is a mapping with $\operatorname{Dom} f = X$ and $\operatorname{Rng} f \subset Y$. By the symbol $f\restriction_Z$ we denote the restriction of the mapping $f$ to the set $Z$.

If $(X, \rho)$ is a metric space, we denote by $U(x, r)$ the open ball (i.e. the set $\{y \in X : \rho(x, y) < r\}$) and by $d(x, A)$ the distance function from a set $A \subset X$ (i.e. $d(x, A) = \inf\{\rho(x, a); a \in A\}$). We shall consider normed linear spaces over the field of real numbers (but many results hold for complex spaces as well). If $X$ is a normed linear space, $X^*$ stands for the (continuous) dual space of $X$.

## 2.2 Elementary submodels

The method of elementary submodels enables us to find specific separable subspaces (of Banach spaces) which can be used for proofs of separable reduction theorems. In this section we briefly describe this method and recall some basic notions. More information can be found in [1] where this method is described in greater detail.

First, let us recall some definitions:

Let $N$ be a fixed set and $\phi$ a formula in the language of $ZFC$. Then the *relativization of* $\phi$ *to* $N$ is the formula $\phi^N$ which is obtained from $\phi$ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each quantifier of the form "$\exists x$" by "$\exists x \in N$".

For example, if

$$\phi = \forall x \, \forall y \, \exists z \, ((x \in z) \; \wedge \; (y \in z))$$

and $N = \{a, b\}$, then the relativization of $\phi$ to $N$ is

$$\phi^N = \forall x \in N \; \forall y \in N \; \exists z \in N \; ((x \in z) \; \wedge \; (y \in z)).$$

It is clear that $\phi$ is satisfied, but $\phi^N$ is not.

If $\phi(x_1, \ldots, x_n)$ is a formula with all free variables shown (i.e. a formula whose free variables are exactly $x_1, \ldots, x_n$) then $\phi$ *is absolute for* $N$ if and only if

$$\forall a_1, \ldots, a_n \in N \quad (\phi^N(a_1, \ldots, a_n) \leftrightarrow \phi(a_1, \ldots, a_n)).$$

The method is based mainly on the following set-theoretical theorem (a proof can be found in [5, Chapter IV, Theorem 7.8]).

**Theorem 2.2.1.** *Let $\phi_1, \ldots, \phi_n$ be any formulas and $X$ any set. Then there exists a set $M \supset X$ such, that*

$$(\phi_1, \ldots, \phi_n \textit{ are absolute for } M) \quad \wedge \quad (|M| \leq \max(\omega, |X|)).$$

Since the previous theorem will often be used throughout the paper, the following notation is useful.

**Definition.** Let $\phi_1, \ldots, \phi_n$ be any formulas and let $X$ be any countable set. Let $M \supset X$ be a countable set satisfying that $\phi_1, \ldots, \phi_n$ are absolute for $M$. Then we say that $M$ *is an elementary submodel for* $\phi_1, \ldots, \phi_n$ *containing* $X$. This is denoted by $M \prec (\phi_1, ..., \phi_n; \; X)$.

Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown and let $M$ be some elementary submodel for $\phi$. To use the absoluteness of $\phi$ for $M$ efficiently, we need to know that many sets are elements of $M$. The reason is that for $a_1, \ldots, a_n \in M$ we have $\phi(a_1, \ldots, a_n)$ if and only if $\phi^M(a_1, \ldots, a_n)$. Using the following lemma we can force the elementary submodel $M$ to contain all the required objects created (uniquely) from elements of $M$ (for a proof see [1, Lemma 2.5]).

**Lemma 2.2.2.** *Let $\phi(y, x_1, \ldots, x_n)$ be a formula with all free variables shown and let $X$ be a countable set. Let $M$ be a fixed set, $M \prec (\phi, \exists y \; \phi(y, x_1, \ldots, x_n); \; X)$ and let $a_1, \ldots, a_n \in M$ be such that there exists only one set $u$ satisfying $\phi(u, a_1, \ldots, a_n)$. Then $u \in M$.*

It would be very laborious and pointless to use only the basic language of the set theory. For example, we often write $x < y$ and we know that this is in fact a shortcut for the formula $\varphi(x, y, <)$ with all free variables shown. Therefore, in the following text we use this extended language of the set theory as we are used to. We shall also use the following convention.

**Convention 2.2.3.** *Whenever we say*

for any suitable elementary submodel $M$ (the following holds...),

*we mean that*

there exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set $Y$ such that for every $M \prec (\phi_1, \dots, \phi_n;\ Y)$ (the following holds...).

By using this new terminology we lose the information about the formulas $\phi_1, \dots, \phi_n$ and the set $Y$. This is, however, not important in applications.

**Remark 2.2.4.** We are able to combine any finite number of results we have proved using the technique of elementary submodels. This includes all the theorems starting with "For any suitable elementary submodel $M$ the following holds:" More precisely:

Let us have sentences $T_1(a), \dots, T_n(a)$. Assume that whenever an $i \in \{1, \dots, n\}$ is given, then for any suitable elementary submodel $M_i$ the sentence $T_i(M_i)$ is satisfied. Then it is easy to verify that for any suitable model $M$ the sentence

$$T_1(M)\ \wedge\ \dots\ \wedge\ T_n(M)$$

is satisfied (it suffices to combine all the lists of formulas and all the sets from the definition above).

Let us recall several more results about suitable elementary submodels (proofs can be found in [1, Chapters 2 and 3]):

**Proposition 2.2.5.** *For any suitable elementary submodel $M$ the following holds:*

*(i) If $A, B \in M$, then $A \cap B \in M$, $B \setminus A \in M$ and $A \cup B \in M$.*

*(ii) Let $f$ be a function such that $f \in M$. Then $\operatorname{Dom} f \in M$, $\operatorname{Rng} f \in M$ and for every $x \in \operatorname{Dom} f \cap M$, $f(x) \in M$.*

*(iii) Let $S$ be a finite set. Then $S \in M$ if and only if $S \subset M$.*

*(iv) Let $S \in M$ be a countable set. Then $S \subset M$.*

*(v) For every natural number $n > 0$ and for arbitrary $(n+1)$ sets $a_0, \dots, a_n$ it is true,that*

$$a_0, \dots, a_n \in M \leftrightarrow \langle a_0, \dots, a_n \rangle \in M.$$

**Notation 2.2.6.**

- If $A$ is a set, then by saying that an elementary model $M$ contains $A$ we mean that $A \in M$.

- If $(X, \rho)$ is a metric space (resp. $(X, +, \cdot, \|\cdot\|)$ is a normed linear space) and $M$ an elementary submodel, then by saying $M$ *contains* $X$ (or by writing $X \in M$) we mean that $(X, \rho) \in M$ (resp. $(X, +, \cdot, \|\cdot\|) \in M$).
- If $X$ is a topological space and $M$ an elementary submodel, then we denote by $X_M$ the set $\overline{X \cap M}$.

**Proposition 2.2.7.** *For any suitable elementary submodel $M$ the following holds:*

*(i) If $X$ is a metric space then whenever $M$ contains $X$, it is true that*

$$\forall r \in \mathbb{R}_+ \cap M \ \forall x \in X \cap M \quad U(x, r) \in M.$$

*(ii) If $X$ is a normed linear space then whenever $M$ contains $X$, it is true that*

$$X_M \text{ is closed separable subspace of } X.$$

**Convention 2.2.8.** *The proofs in the following text often begin in the same way. To avoid unnecessary repetitions, by saying* "Let us fix a $(*)$-elementary submodel $M$ [containing $A_1, \ldots, A_n$]" *we will understand the following:*

Let us have formulas $\varphi_1, \ldots, \varphi_m$ and a countable set $Y$ such that the elementary submodel $M \prec (\varphi_1, \ldots, \varphi_m;\ Y)$ is suitable for all the propositions from [1]. Add to them formulas marked with $(*)$ in all the preceding proofs from this paper and formulas marked with $(*)$ in the proof below (and all their subformulas). Denote such a list of formulas by $\phi_1, \ldots, \phi_k$. Let us fix a countable set $X$ containing the sets $Y$, $\omega$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$ and all the common operations and relations on real numbers ($+$, $-$, $\cdot$, $:$, $<$). Fix an elementary submodel $M$ for formulas $\phi_1, \ldots, \phi_k$ containing $X$ [such that $A_1, \ldots, A_n \in M$].

Thus, any $(*)$-elementary submodel $M$ is suitable for the results from [1] and all the preceding theorems and propositions from this paper, making it possible to use all of these results for $M$.

In order to demonstrate how this technique works, we prove the following two easy lemmas which we use later (the proof of the second lemma is also contained in the proof of Proposition 4.1 in [1]).

**Lemma 2.2.9.** *For any suitable elementary submodel $M$ the following holds: Whenever $A \in M$ is a nonempty set, then $A \cap M$ is nonempty.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$and fix some nonempty set $A \in M$. Then

$$\exists x \quad (x \in A). \tag{$*$}$$

This formula has only one free variable $A$ and the set $A$ is contained in $M$. Thus, due to the absoluteness of the formula above, there exists an $x \in M$ such that $x \in A$. $\square$

**Lemma 2.2.10.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a metric space, $B \subset X$. Then whenever $M$ contains $X$, $B$ and a set $D \subset B$, it is true that*

$$D \text{ is dense in } B \to D \cap M \text{ is dense in } B \cap X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ such that $B, D \in M$. If the set $B$ is empty then the proposition is obvious. Otherwise fix $b \in B \cap X_M$ and $r > 0$. Choose some $b_0 \in U(b, \frac{r}{2}) \cap M$ and a rational number $q \in (\rho(b, b_0), \frac{r}{2})$. Then $U(b_0, q) \subset U(b, r)$ and

$$\exists d \in D \quad (d \in U(b_0, q)). \tag{$*$}$$

In the preceding formula we use the shortcut $d \in U(b_0, q)$ which stands for $d \in X \ \wedge \ \rho(d, b_0) < q$. Free variables in this formula are $X, \rho, <, D, b_0, q$. Those are contained in $M$ and thus we can use the absoluteness to find a $d \in D \cap M$ such that $(d \in U(b_0, q))^M$. Using the absoluteness again we obtain that $d$ is an element of $U(b_0, q)$. Consequently,

$$U(b, r) \cap D \cap M \supset U(b_0, q) \cap D \cap M \neq \emptyset$$

and so the set $D \cap M$ is dense in $B \cap X_M$. $\square$

## 2.3 $\sigma$-porous sets

In this section we compile several known results concerning different notions of $\sigma$-porous sets. The usefulness of these facts for our needs will be apparent later; for more information about properties and applications of different types of porosity we refer the reader to survey articles [10] and [13]. On some occasions we shall also refer to the paper [8].

Let us begin by stating several basic definitions.

**Definition.** Let $(X, \rho)$ be a metric space, $A \subset X$, $x \in X$ and $R > 0$. Then we denote by $\gamma(x, R, A)$ the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $U(z, r) \subset U(x, R) \setminus A$. The set $A$ is called *upper porous at $x$ in the space $X$* if

$$\limsup_{R \to 0^+} \frac{\gamma(x, R, A)}{R} > 0.$$

In most cases it is clear which space $X$ we have in mind. Therefore we often omit the words "*in the space $X$*". (We shall apply this convention to other notions as well.)

Let $g$ be a strictly increasing and continuous real-valued function defined on $[0, h)$ (where $h > 0$) with $g(0) = 0$. We call such a function *porosity function.* We say that $A$ is *$\langle g \rangle$-porous at $x$ (in the space $X$)* if there exists a sequence of open balls $\{U(c_n, r_n)\}$ such that $c_n \to x$, $U(c_k, r_k) \cap A = \emptyset$ and $x \in U(c_k, g(r_k))$ for each $k$.

We say the set $A$ is $\langle g \rangle$*-porous* if it is $\langle g \rangle$*-porous* at each of its points and $\sigma$-$\langle g \rangle$*-porous* if it is a countable union of $\langle g \rangle$*-porous* sets. The set $A$ is *upper-porous* if it is *upper-porous* at each of its points and *$\sigma$-upper porous* if it is a countable union of *upper-porous* sets.

**Definition.** Let $(X, \rho)$ be a topologically complete metric space and let $g$ be a porosity function. We say that $\mathcal{F}$ is a *Foran system for $\langle g \rangle$-porosity in $X$* if the following conditions hold:

(i) $\mathcal{F}$ is a nonempty family of nonempty $G_\delta$ subsets of $X$.

(ii) For each $S \in \mathcal{F}$ and each open set $G \subset X$ with $S \cap G \neq \emptyset$ there exists $S^* \in \mathcal{F}$ such that $S^* \subset S \cap G$ and $S$ is $\langle g \rangle$-porous at no point of $S^*$.

**Proposition 2.3.1** (Foran Lemma)**.** *Let $(X, \rho)$ be a topologically complete metric space and let $\mathcal{F}$ be a Foran system for $\langle g \rangle$-porosity in $X$. Then no member of $\mathcal{F}$ is $\sigma$-$\langle g \rangle$-porous.*

This is a special case of the general Foran Lemma (see [12, Proposition 1]) which works for any porosity-like relation. Our definition of Foran system is, therefore, accordingly simplified as well. We also need the following.

**Notation 2.3.2.** By 3-porosity we mean $\langle g \rangle$-porosity where $g(x) = 3x$ for $x \in \mathbb{R}$.

**Lemma 2.3.3** ([12, Lemma E])**.** *Let $(X, \rho)$ be a metric space and let $A \subset X$. Then $A$ is $\sigma$-upper porous if and only if it is $\sigma$-3-porous.*

Another result from [12] which we shall use is the following partial converse of the Foran Lemma. For ordinary $\sigma$-upper porosity we can extend its validity from $G_\delta$ sets to Souslin sets using the inscribing Theorem 2.3.5 of J. Pelant and M. Zelený from the work [14].

It could be interesting to note that in case our metric space $X$ is locally compact, we can use a different inscribing theorem due to L. Zajíček and M. Zelený [15, Theorem 5.2] and obtain an extension of 2.3.4 to analytic sets for general $\sigma$-$\langle g \rangle$-porosity.

**Lemma 2.3.4** ([12, Corollary 1])**.** *Let $(X, \rho)$ be a topologically complete metric space, let $\emptyset \neq A \subset X$ be $G_\delta$ and let $g$ be a porosity function. Then $A$ is not $\sigma$-$\langle g \rangle$-porous if and only if it contains a member of a Foran system for $\langle g \rangle$-porosity.*

**Theorem 2.3.5** ([14, Theorem 3.1])**.** *Let $(X, \rho)$ be a topologically complete metric space and let $S \subseteq X$ be a non-$\sigma$-upper porous Souslin set. Then there exists a closed non-$\sigma$-upper porous set $F \subseteq S$.*

**Definition.** Let $(X, \rho)$ be a metric space, $A \subset X$ and $x \in X$. We say that $A$ is *lower porous at $x$* if

$$\liminf_{R \to 0^+} \frac{\gamma(x, R, A)}{R} > 0.$$

The set $A$ is *lower porous* if it is lower porous at each of its points and *$\sigma$-lower porous* if it is a countable union of lower porous sets.

Even though the Foran Lemma can be used for any notion of porosity, we have to use a different approach in the case of lower porosity. The reason is that unlike in the case of upper porosity, we were unable to separably reduce the property of not being *lower* porous at a point. Therefore, we use the following proposition.

**Proposition 2.3.6** ([7, Proposition 2.11])**.** *Let $(X, \rho)$ be a topologically complete metric space and let $A \subseteq X$ be a Souslin set. Then the following propositions are equivalent:*

*(i) $A$ is not $\sigma$-lower porous.*

*(ii) There exists a closed set $F \subseteq A$ and a set $D \subseteq F$ dense in $F$ such that $F$ is lower porous at no point of $D$.*

## 2.4 Auxiliary results

In this section we prove some preliminary statements which will be of use later. In general, for a space $X$ and a set $A \subset X$, we are trying to find a separable subspace $X_M \subset X$ with certain special properties. The first desired property is: Whenever $A$ is a member of a Foran system in $X$ then $A \cap X_M$ is a member of a Foran system in $X_M$. Together with Lemma 2.3.4 this will be essential to the proof of Theorem 2.5.1 about $\sigma$-upper porosity.

Also, in order to prove a result similar to 2.5.1 for $\sigma$-lower porosity, two auxiliary propositions (based on the ideas from [1]) are collected.

**Proposition 2.4.1.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a metric space and $g$ a porosity function. Then whenever $M$ contains $X$ and a set $A \subset X$, it is true that for every $x \in X_M$*

$$A \text{ is not } \langle g \rangle\text{-porous at } x \to A \cap X_M \text{ is not } \langle g \rangle\text{-porous at } x \text{ in the space } X_M.$$

*If $M$ contains also $g$, then*

$$A \text{ is not } \langle g \rangle\text{-porous} \to A \cap X_M \text{ is not } \langle g \rangle\text{-porous in the space } X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ and $A$ and fix some $x \in X_M$ such that $A$ is not $\langle g \rangle$-porous at $x$. Take sequences $\{c_n\}_{n \in \mathbb{N}} \subset X_M$ and $\{r_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ such that $c_n \to x$ and $x \in U(c_n, g(r_n))$ for all $n \in \mathbb{N}$. It is sufficient to show that there exists an $n \in \mathbb{N}$ satisfying $U(c_n, r_n) \cap A \cap X_M \neq \emptyset$. Since $A$ is not $\langle g \rangle$-porous, we can fix some $n \in \mathbb{N}$ such that $U(c_n, r_n) \cap A \neq \emptyset$. Take some $a \in A \cap U(c_n, r_n)$ and choose an $\varepsilon > 0$ such that $\rho(a, c_n) + 2\varepsilon < r_n$. Then take a point $c \in X \cap M \cap U(c_n, \varepsilon)$ and $q_n \in \mathbb{Q} \cap (\rho(a, c_n) + \varepsilon, r_n - \varepsilon)$. Hence,

$$\exists a \in A \quad (\rho(a, c) < q_n). \tag{$*$}$$

Thus, by the absoluteness, there exists an $a \in A \cap M$ such that

$$\rho(a, c_n) \leq \rho(a, c) + \varepsilon < q_n + \varepsilon < r_n.$$

Consequently, $a \in A \cap U(c_n, r_n) \cap M$ and thus the set $A \cap X_M$ is not $\langle g \rangle$-porous at $x$ in the space $X_M$.

If $A$ is not $\langle g \rangle$-porous then

$$\exists x \in A \quad (A \text{ is not } \langle g \rangle\text{-porous at } x). \tag{$*$}$$

Using the absoluteness and the already proved part we obtain an $x \in A \cap M$ such that $A \cap X_M$ is not $\langle g \rangle$-porous at $x$ in the space $X_M$. □

**Proposition 2.4.2.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a topologically complete metric space and $g$ a porosity function. Then whenever $M$ contains $X$, $g$ and a set $A \subset X$, it is true that if $A$ is a member of a Foran system for $\langle g \rangle$-porosity in $X$, then $A \cap X_M$ is a member of a Foran system for $\langle g \rangle$-porosity in $X_M$.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ such that $A \in M$ and let the following formula be true

$$\exists \mathcal{F}\ (\mathcal{F} \text{ is a Foran system for } \langle g\rangle\text{-porosity in } X \text{ such that } A \in \mathcal{F}). \qquad (*)$$

Notice that the preceding is a formula with all parameters in $M$. Thus, by the absoluteness, there exists an $\mathcal{F} \in M$ which is a Foran system for $\langle g\rangle$-porosity in $X$ with $A \in \mathcal{F}$. Set

$$\mathcal{F}' := \{S \cap X_M :\ S \in \mathcal{F} \cap M,\ S \cap X_M \neq \emptyset\}.$$

First we notice that, by Lemma 2.2.9, the set $A \cap M$ is nonempty; it follows that $A \cap X_M \in \mathcal{F}'$. Thus it suffices to establish that $\mathcal{F}'$ is a Foran system for $\langle g\rangle$-porosity in $X_M$. Clearly, $\mathcal{F}'$ is a nonempty family of nonempty $G_\delta$ subsets of $X_M$ so there only remains to be verified the second condition from the definition of Foran system.

To that end, take some $S \in \mathcal{F} \cap M$ such that $S \cap X_M \neq \emptyset$ (denote by $S_M$ the set $S \cap X_M$). Then take an arbitrary open set $G \subset X$ with $S_M \cap G \neq \emptyset$ and fix some $x \in S_M \cap G$ and $r \in \mathbb{Q}_+$ such that $U(x, r) \subset G$. Choose $x_0 \in U(x, \frac{r}{2}) \cap M$. Then $x \in U(x_0, \frac{r}{2}) \subset U(x, r)$; thus, $S \cap U(x_0, \frac{r}{2}) \neq \emptyset$. Using Propositions 2.2.5 and 2.2.7 we obtain that $S \cap U(x_0, \frac{r}{2}) \in M$.

Now, as $\mathcal{F}$ is a Foran system (in $X$), the following formula is true:

$$\exists S^* \in \mathcal{F} : (S^* \subset S \cap U(x_0, \tfrac{r}{2}), S \text{ is } \langle g\rangle\text{-porous at no point of } S^*) \qquad (*)$$

By the absoluteness, there exists an $S^* \in M$ satisfying the formula above. Using Lemma 2.2.9 we can see that $S^* \cap M \neq \emptyset$. Thus, $S^*$ is a member of $\mathcal{F}'$, $S^* \cap X_M \subset S_M \cap U(x_0, \frac{r}{2}) \subset S_M \cap G$ and by Proposition 2.4.1 above, $S_M$ is $\langle g\rangle$-porous at no point of $S^* \cap X_M$. Consequently, $A \cap X_M$ is, indeed, a member of a Foran system for $\langle g\rangle$-porosity in $X_M$ – the system $\mathcal{F}'$. □

**Remark 2.4.3.** Note that the last proof depends solely on our ability to separably reduce $\langle g\rangle$-porosity of a set at a point. It would work for any other type of porosity which fulfils this condition, e.g., the $(g)$-porosity (for the definition see [8] or [10]).

Before proceeding to the last section where we use the propositions above, let us briefly turn our attention to the matter of lower porosity and formulate two related facts:

**Lemma 2.4.4.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a metric space, $A \subset X$ and $d(\cdot, A) : X \to \mathbb{R}$ the function defined by the formula $d(\cdot, A)(x) := d(x, A)$. Then whenever $M$ contains $X$ and $A$ then $d(\cdot, A)$ is an element of $M$.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ such that $A \in M$. Then the lemma follows immediately from Lemma 2.2.2 and from the absoluteness of the following formula and its subformulas

$$\begin{aligned} &\exists d(\cdot, A) \quad (d(\cdot, A) \text{ is a function which maps every} \\ &\quad x \in X \text{ to the real number } \inf\{\rho(x, a);\ a \in A\}). \end{aligned} \qquad (*)$$

□

Finally, we present the following proposition (its proof is contained in the proof of Proposition 4.10 in [1]).

**Proposition 2.4.5.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a metric space and $A \subset X$. Then whenever $M$ contains $X$ and $A$, it is true that for every $x \in A \cap M$*

*$A$ is not lower porous at $x \rightarrow A \cap X_M$ is not lower porous at $x$ in the space $X_M$.*

Note, that this is exactly the moment, where we were unable to reduce the property of not being lower porous at a point. However, thanks to Proposition 2.3.6, this proposition will be sufficient.

## 2.5 Main results

In the main part of this article we show that the set properties "to be $\sigma$-upper porous" and "to be $\sigma$-lower porous" are separably determined. We formulate the related theorems in the language of elementary submodels (which is useful when we want to combine several results concerning elementary submodels together). However, we also formulate a corollary of these results in such a setting that no knowledge of elementary submodels is required (see Theorem 2.1.1).

Next, we show that these results may be useful for proving that some results concerning separable spaces hold in a nonseparable setting as well. This is demonstrated in Theorem 2.5.6.

First, let us show that $\sigma$-upper porosity is a separably determined notion.

**Theorem 2.5.1.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a topologically complete metric space, $g$ a porosity function and $A \subset X$ a Souslin set. Then whenever $M$ contains $X$ and $A$, it is true that*

$$A \text{ is } \sigma\text{-upper porous in } X \leftrightarrow A \cap X_M \text{ is } \sigma\text{-upper porous in } X_M.$$

*Moreover, if $A$ is $G_\delta$ and $M$ contains also $g$, then*

$$A \text{ is not } \sigma\text{-}\langle g\rangle\text{-porous in } X \rightarrow A \cap X_M \text{ is not } \sigma\text{-}\langle g\rangle\text{-porous in } X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ such that $g, A \in M$. Assume the set $A$ is of the type $G_\delta$; we shall prove the second part of the proposition first. Due to Lemma 2.3.4 and the absoluteness of the formula (and its subformulas)

$$\exists B \quad (B \subset A \text{ and } B \text{ is a member of a Foran system for } \langle g\rangle\text{-porosity}), \qquad (*)$$

we can assume that the set $A$ is a member of a Foran system $\mathcal{F}$ for $\langle g\rangle$-porosity. Hence the set $A \cap X_M$ is a member of a Foran system $\mathcal{F}'$ for $\langle g\rangle$-porosity in $X_M$ (Proposition 2.4.2) and thus is not $\sigma$-$\langle g\rangle$-porous in $X_M$ (Proposition 2.3.1).

The implication from the left to the right for $\sigma$-upper porosity follows immediately from Lemma 2.4.4 and [1, Corollary 4.13].

We shall prove the other implication indirectly; owing to Theorem 2.3.5 we can assume that $A$ is $G_\delta$ again (even closed). The result now follows from the already proved part and Lemma 2.3.3, using the absoluteness of the formula (and its subformulas)

$$\exists g \quad (g : \mathbb{R} \to \mathbb{R} \text{ is a function such that for all } x \in \mathbb{R} \text{ is } g(x) = 3x). \qquad (*)$$

□

**Remark 2.5.2.** It is not known to the authors whether the other implication for $\sigma$-$\langle g\rangle$-porosity holds. However, under the assumptions of the preceding theorem, it is true that whenever $A$ is $\sigma$-$\langle g\rangle$-porous then $A \cap X_M$ is $\sigma$-$\langle dg\rangle$-porous in $X_M$ for any $d > 2$. This may be established in the following way:

First, using the ideas presented in [1] (mainly Proposition 4.12 and Corrolary 4.13), we are able to see that if $A$ is $\sigma$-$(g, c)$-porous in $X$ (where $c > 0$; the definition is natural – see [8]), then $A \cap X_M$ is $\sigma$-$(g, c/2)$-porous in the space $X_M$.

Now let us assume the set $A$ is $\sigma$-$\langle g\rangle$-porous in $X$. Then [8, Lemma 3.1(ii)] implies it is $\sigma$-$(g, 1/2)$-porous in $X$ and thus $A \cap X_M$ is $\sigma$-$(g, 1/4)$-porous in the space $X_M$. In the nontrivial case when there exists a $\delta > 0$ such that $g(x) > x$ for all $x \in (0, \delta)$ (if that is not the case, then the notion of $\langle g\rangle$-porosity is usually not very interesting) it is not difficult to prove that $g$ satisfies the assumption from [8, Proposition 4.4]. Thus $A \cap X_M$ is $\sigma$-$(g, c)$-porous for any $c \in (0, 1/2)$. To pass back to $\langle\cdot\rangle$-porosity, we use a slightly refined version of [8, Lemma 3.1(i)] which for any $d > 1$ states that $(f, d)$-porosity of a given set $N$ at a given point $x$ implies $\langle f\rangle$-porosity of $N$ at $x$. We easily obtain that the set $A \cap X_M$ is $\sigma$-$\langle dg\rangle$-porous for any $d > 2$.

Moreover, under the additional assumption that there exists a $d > 2$ and a $\delta > 0$ such that $g(x) > dx$ for any $x \in (0, \delta)$, we are able to prove (similarly as above) that whenever $A$ is $\sigma$-$\langle g\rangle$-porous then $A \cap X_M$ is $\sigma$-$\langle g\rangle$-porous in $X_M$.

**Remark 2.5.3.** Under the assumptions of Theorem 2.5.1 the following holds: If $g$ is a porosity function such that for some $c > 0$ there is a $\delta > 0$ such that $cg(x) > x$ for all $x \in (0, \delta)$, then

$$A \text{ is } \sigma\text{-}(g)\text{-porous in } X \leftrightarrow A \cap X_M \text{ is } \sigma\text{-}(g)\text{-porous in } X_M.$$

This can be established as follows: Let $d = 12c$ and let $A$ be non-$\sigma$-$(g)$-porous in $X$. Then it is non-$\sigma$-$(dg, 1)$-porous and thus $A$ is non-$\sigma$-$\langle \frac{d}{2}g\rangle$-porous in $X$. Theorem 2.5.1 asserts that the same holds also for $A \cap X_M$ in $X_M$. Hence, $A \cap X_M$ is non-$\sigma$-$(\frac{d}{2}g, 2)$-porous ([8, Lemma 3.1(i)]), i.e., it is non $\sigma$-$(\frac{d}{12}g, \frac{1}{3})$-porous (in $X_M$). Now, since the function $\frac{d}{12}g = cg$ satisfies the assumption of [8, Proposition 4.4], we obtain that $A \cap X_M$ is not $\sigma$-$(g)$-porous in $X_M$.

For the proof of the other implication assume that the set $A$ is $\sigma$-$(g)$-porous. It is easy to see that there exist $(g, c_n)$-porous sets $A_n$ (with $c_n > 0$ for each $n \in \mathbb{N}$) such that $A = \bigcup_{n=1}^{\infty} A_n$. In the same way as in the previous remark we obtain that $A_n \cap X_M$ is $\left(g, \frac{c_n}{2}\right)$-porous in $X_M$ for each $n$. Hence, $A \cap X_M$ is $\sigma$-$(g)$-porous in $X_M$.

We shall now turn our attention to $\sigma$-lower porosity and show it is separably determined.

**Theorem 2.5.4.** *For any suitable elementary submodel $M$ the following holds: Let $(X, \rho)$ be a topologically complete metric space and let $A \subset X$ be a Souslin set. Then whenever $M$ contains $X$ and $A$, it is true that*

$$A \text{ is } \sigma\text{-lower porous in } X \leftrightarrow A \cap X_M \text{ is } \sigma\text{-lower porous in } X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ containing $X$ such that $A \in M$. Then the implication from the left to the right follows from [1, Corollary 4.13] and from Lemma 2.4.4.

To prove the opposite implication we use Proposition 2.3.6. Let us assume that the set $A$ is not $\sigma$-lower porous in $X$. Then

$$\exists F\ \exists D\quad (F \subset A \text{ is a nonempty closed set such that } D \subset F \text{ is dense in } F \text{ and } F \text{ is not lower porous at any point of } D). \qquad (*)$$

By the absoluteness of this formula (and its subformulas) above, we are able to find sets $F, D \in M$ satisfying the conditions above. Using Lemma 2.2.9 and Lemma 2.2.10 we can see that $F \cap M \neq \emptyset$ and $D \cap M$ is dense in $F \cap X_M$. Moreover, by Proposition 2.4.5, $F \cap X_M$ is not lower porous at any point of $D \cap M$. Thus, from Proposition 2.3.6 it follows that the set $A \cap X_M$ is not $\sigma$-lower porous in the space $X_M$. ☐

Theorem 2.1.1 from the introduction is just an easy consequence of Theorem 2.5.1, Theorem 2.5.4 and Proposition 2.2.7 since Convention 2.2.3 allows us to combine these three results; by doing that we obtain a theorem in the setting of Banach spaces which concerns both types of porosity. In a similar way, Theorem 2.1.2 follows from the Theorem 2.5.1, Theorem 5.7 in [1] and Proposition 2.2.7 (because the set of the points where a function is Fréchet differentiable is a $F_{\sigma\delta}$ set - see for example [1, Theorem 5.8]) ).

Finally, we give the following application of our results. In [9] the following theorem is proved (we use the more common terminology from [3]).

**Definition.** Let $(X, \|\cdot\|)$ be a Banach space and let $f$ be a real function defined on $X$. We say that $f$ is Fréchet superdifferentiable at $x \in X$ if and only if there exists $x^* \in X^*$ such that

$$\limsup_{h \to 0} \frac{(f(x+h) - f(x) - x^*(h))}{\|h\|} \leq 0.$$

**Theorem 2.5.5** ([9, Theorem 2])**.** *Let $(X, \|\cdot\|)$ be a Banach space with separable dual space and let $G \subset X$ be an open set. Let $f$ be a Lipschitz function on $G$ and let $A$ be the set of all the points $x \in G$ such that $f$ is Fréchet superdifferentiable at $x$ and $f$ is not Fréchet differentiable at $x$. Then $A$ is $\sigma$-upper porous.*

Using the method of elementary submodels, it is now easy to extend the validity of this result to general Asplund spaces.

**Theorem 2.5.6.** *Let $(X, \|\cdot\|)$ be an Asplund space and let $G \subset X$ be an open set. Let $f$ be a Lipschitz function on $G$ and let $A$ be the set of all the points $x \in G$ such that $f$ is Fréchet superdifferentiable at $x$ and $f$ is not Fréchet differentiable at $x$. Then $A$ is $\sigma$-upper porous.*

*Proof.* Let us denote by $D(f)$ the set of points where $f$ is Fréchet differentiable and by $S(f)$ the set of points where $f$ is Fréchet superdifferentiable. It easily follows from the article [11] (Section 4, Lemma 3 and Lemma 4) that the set $S(f)$ is Souslin. Now, using Theorem 2.5.1, Proposition 2.2.7 and [1, Theorem 5.7], take an elementary submodel $M$ satisfying:

- $X_M$ is a separable subspace of $X$,
- $D(f) \cap X_M = D(f \restriction_{X_M})$,
- $A$ is $\sigma$-upper porous if and only if $A \cap X_M$ is $\sigma$-upper porous in the space $X_M$.

Note that $A \cap X_M \subset \{x \in X_M; x \in S(f \restriction_{X_M}) \setminus D(f \restriction_{X_M})\}$ and that the set on the right side is $\sigma$-upper porous (because $X_M$ is a separable space with separable dual); thus the set $A$ is $\sigma$-upper porous. $\square$

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# Chapter 3.
# On Separable Determination of $\sigma$-**P**-Porous Sets in Banach Spaces

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**Abstract**

We use a method involving elementary submodels and a partial converse of Foran lemma to prove separable reduction theorems concerning Souslin $\sigma$-**P**-porous sets where **P** can be from a rather wide class of porosity-like relations in complete metric spaces. In particular, we separably reduce the notion of Souslin cone small set in Asplund spaces. As an application we prove that a continuous approximately convex function on an Asplund space is Fréchet differentiable up to a cone small set.

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## 3.1 Introduction

The present paper could be considered as a sequel to the articles [1] and [3]. Our main aim is to further investigate separable determination of various properties of sets and functions in metric spaces (especially Banach spaces). This means, given a nonseparable metric space $X$ and a property of sets (or functions etc.) in $X$, we are interested whether certain statements about a property hold, provided that they hold in (some) separable subspaces of $X$. More concretely, if $X$ is a metric space, we are interested in $\sigma$-$\mathbf{P}$-porous sets in $X$, where $\mathbf{P}$ is a porosity-like relation on $X$ (see the definition below).

The key method we use to obtain separable determination results uses countable elementary structures which we call elementary submodels. This method is described in Section 3.2. Further details and examples of the use of this method can be found in [1] and [3]. However, the reader ought to note that there are other ways to tackle this topic. An example is the use of rich families of Banach spaces, which is described in detail e.g. in [7, Section 3.6]. Sometimes one can also opt to prove this sort of results in an "elementary way", in a sense imitating parts of the proof of the Löwenheim-Skolem theorem. This approach would be in many cases very complicated, but it can give a deeper insight.

In the rather technical Section 3.3 we prove auxiliary results which we use in Section 3.4 to prove our general separable determination result, Proposition 3.4.6. The general scheme of our proof is rather similar to that of separable determination of $\sigma$-upper porosity of sets in [3] and involves the Foran lemma and its partial converse. The difference is that here we need no inscribing theorems as in [3] to prove the statement for all Souslin sets.

Section 3.5 contains the main result of this article, the separable determination of the notion of cone small sets in Asplund spaces (Theorem 3.5.10). In the last section we provide several applications of our results, most notably Theorem 3.6.3.

We recall the most relevant notions, definitions, and notations. We denote by $\omega$ the set of all natural numbers (including 0), by $\mathbb{N}$ the set $\omega \setminus \{0\}$, by $\mathbb{Q}$ the set of all rational numbers, by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}_+$ the interval $(0, \infty)$, and $\mathbb{Q}_+$ stands for $\mathbb{R}_+ \cap \mathbb{Q}$. We denote by $\omega^{<\omega}$ the set of all finite sequences of elements of $\omega$ including the empty one, and by $\omega^\omega$ the set of all infinite sequences in $\omega$. We use the convention that countable sets can be also finite. If $f$ is a mapping then we denote by $\operatorname{Rng} f$ the range of $f$ and by $\operatorname{Dom} f$ the domain of $f$. By writing $f\colon X \to Y$ we mean that $f$ is a mapping with $\operatorname{Dom} f = X$ and $\operatorname{Rng} f \subset Y$. By the symbol $f\restriction_Z$ we denote the restriction of the mapping $f$ to a set $Z \subset X$.

Let $(X, \rho)$ be for a while a metric space. We denote by $U_X(x, r)$ the open ball with centre $x$ and radius $r > 0$, i.e., the set $\{y \in X;\ \rho(x, y) < r\}$. We often write $U(x, r)$ instead of $U_X(x, r)$. If $A, B \subset X$ are nonempty sets, we denote by $\mathrm{d}(A, B)$ the distance between $A$ and $B$, i.e., $\mathrm{d}(A, B) := \inf\{\rho(a, b);\ a \in A, b \in B\}$. We shall consider normed linear spaces over the field of real numbers. If $Y$ is a normed linear space, $Y^*$ stands for the dual space of $Y$.

By a *point-set relation* on $X$ we understand any subset of the product $X \times 2^X$. If $\mathbf{R} \subset X \times 2^X$, then, instead of $(x, A) \in \mathbf{R}$, we shall write $\mathbf{R}(x, A)$. A relation $\mathbf{P} \subset X \times 2^X$ is called *porosity-like* if

(i) if $A \subset B$ and $\mathbf{P}(x, B)$, then $\mathbf{P}(x, A)$,

(ii) $\mathbf{P}(x, A)$ if and only if there is $r > 0$ such that $\mathbf{P}(x, A \cap U(x, r))$,

(iii) $\mathbf{P}(x, A)$ if and only if $\mathbf{P}(x, \overline{A})$.

We then say that a set $A$ is $\mathbf{P}$-*porous*, if $\mathbf{P}(x, A)$ for every $x \in A$. A set $A$ is called $\sigma$-$\mathbf{P}$-*porous*, if it is the union of countably many $\mathbf{P}$-porous sets.

**Example 3.1.1.** *Let $(X, \rho)$ be a metric space. We define a point-set relation $\mathbf{P}$ by*

$$\mathbf{P}(x, A) \text{ if and only if } A \text{ is upper porous at } x.$$

*It is easy to verify that $\mathbf{P}$ is a porosity-like relation on $X$.*

## 3.2 Elementary submodels

In this section we recall some basic notions and statements concerning the method of elementary submodels. A brief description of this method can be found in [3]; for a more detailed description see [1]. Let $N$ be a fixed set and $\phi$ a formula in the language of $ZFC$. By the *relativization of $\phi$ to $N$* we understand the formula $\phi^N$ which is obtained from $\phi$ by replacing each symbol of the form "$\forall x$" by "$\forall x \in N$" and each symbol of the form "$\exists x$" by "$\exists x \in N$". Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown, i.e., a formula whose free variables are exactly $x_1, \ldots, x_n$. We say that $\phi$ *is absolute for* $N$ if

$$\forall a_1, \ldots, a_n \in N\colon \big(\phi^N(a_1, \ldots, a_n) \leftrightarrow \phi(a_1, \ldots, a_n)\big).$$

The method is based mainly on the following theorem (a proof can be found in [6, Chapter IV, Theorem 7.8]). The cardinality of a set $A$ is denoted by $|A|$.

**Theorem 3.2.1.** *Let $\phi_1, \ldots, \phi_n$ be any formulas, and $X$ be any set. Then there exists a set $M \supset X$ such that $\phi_1, \ldots, \phi_n$ are absolute for $M$ and $|M| \leq \max(\aleph_0, |X|)$.*

Since the set from Theorem 3.2.1 will often be used, the following notation is useful.

**Definition 3.2.2.** Let $\phi_1, \ldots, \phi_n$ be any formulas, and let $X$ be any countable set. Let $M \supset X$ be a countable set such that $\phi_1, \ldots, \phi_n$ are absolute for $M$. Then we say that $M$ is an *elementary submodel for $\phi_1, \ldots, \phi_n$ containing $X$*. This is denoted by $M \prec (\phi_1, \ldots, \phi_n; X)$.

Let us emphasize that an elementary submodel in our terminology is always countable.

The fact that a certain formula is absolute for $M$ will always be used in order to satisfy the assumption of the following lemma. It is a statement similar to [1, Lemma 2.6]. Using this lemma we can force the elementary submodel $M$ to contain all needed objects constructed (uniquely) from elements of $M$.

**Lemma 3.2.3.** *Let $n \in \mathbb{N}$, let $\phi(y, x_1, \ldots, x_n)$ be a formula whoose all free variables are shown, and let $M$ be a fixed set such that both formulae $\phi$ and $\exists y\colon \phi(y, x_1, \ldots, x_n)$ are absolute for $M$. Assume there exist $a_1, \ldots, a_n \in M$ and $u$ satisfying $\phi(u, a_1, \ldots, a_n)$. Then there exists $a \in M$ such that $\phi(a, a_1, \ldots, a_n)$.*

*Proof.* Using the absoluteness of the formula $\exists y\colon \phi(y, x_1, \ldots, x_n)$ there exists $a \in M$ satisfying $\phi^M(a, a_1, \ldots, a_n)$. Using the absoluteness of $\phi$ we get, that for this $a \in M$ the formula $\phi(a, a_1, \ldots, a_n)$ holds. □

It would be very laborious and pointless to use only the basic language of the set theory. For example, having a function $f$, we often write $y = f(x)$ and we know that this is a shortcut for a formula with free variables $x$, $y$, and $f$.

Indeed, consider the formula

$$\varphi(x, y, z) = \forall a(a \in z \leftrightarrow (a = x \vee a = y)).$$

Then $\varphi(x, y, z)$ is true if and only if $z = \{x, y\}$. Recall that $y = f(x)$ means $\{\{x\}, \{x, y\}\} \in f$. Hence, $y = f(x)$ if and only if the following formula is true

$$\forall z(\forall a(a \in z \leftrightarrow \varphi(x, x, a) \vee \varphi(x, y, a)) \Rightarrow z \in f).$$

Therefore, in the following text we use this extended language of the set theory as we are used to. We shall also use the following convention.

**Convention 3.2.4.** *Whenever we say* "for any suitable elementary submodel $M$ the following holds ... " *we mean that "there exists a list of formulas $\phi_1, \ldots, \phi_n$ and a countable set $Y$ such that for every $M \prec (\phi_1, \ldots, \phi_n; Y)$ the following holds ... "*

By using this new terminology we lose the information about the formulas $\phi_1, \ldots, \phi_n$ and the set $Y$. However, this is not important in applications.

We recall several further results about suitable elementary submodels (all the proofs are based on Lemma 3.2.3 and they can be found in [1, Chapters 2 and 3]).

**Proposition 3.2.5.** *For any suitable elementary submodel $M$ the following holds.*

*(i) If $A, B \in M$, then $A \cap B \in M$, $B \setminus A \in M$ and $A \cup B \in M$.*

*(ii) Let $f$ be a function such that $f \in M$. Then $\operatorname{Dom} f \in M$, $\operatorname{Rng} f \in M$ and for every $x \in \operatorname{Dom} f \cap M$ we have $f(x) \in M$.*

*(iii) Let $S$ be a finite set. Then $S \in M$ if and only if $S \subset M$.*

*(iv) Let $S \in M$ be a countable set. Then $S \subset M$.*

*(v) For every $n \in \mathbb{N}$ and for arbitrary sets $a_0, \ldots, a_n$ it is true that $a_0, \ldots, a_n \in M$ if and only if $n$-tuple $(a_0, \ldots, a_n)$ is in $M$.*

**Convention 3.2.6.**

- *If $(X, \rho)$ is a metric space (resp. $(X, +, \cdot, \|\cdot\|)$ is a normed linear space) and $M$ is an elementary submodel, then by writing $X \in M$ we mean that $(X, \rho) \in M$ (resp. $(X, +, \cdot, \|\cdot\|) \in M$).*
- *If $X$ is a topological space and $M$ is an elementary submodel, then we denote by $X_M$ the set $\overline{X \cap M}$.*

**Proposition 3.2.7.** *For any suitable elementary submodel $M$ the following holds.*

*(i) If $X$ is a metric space then whenever $X \in M$, it is true that*

$$\forall r \in \mathbb{R}_+ \cap M \; \forall x \in X \cap M \colon U(x,r) \in M.$$

*(ii) If $X$ is a normed linear space and $X \in M$, then $X_M := \overline{X \cap M}$ is linear (closed and separable).*

**Convention 3.2.8.** *The proofs in the following text often begin in the same way. To avoid unnecessary repetitions, by saying* "Let us fix a $(*)$-elementary submodel $M$" *we will understand the following.*

*"Let us have formulas $\varphi_1, \dots, \varphi_m$ and a countable set $Y$ such that the elementary submodel $M \prec (\varphi_1, \dots, \varphi_m; Y)$ is suitable for all the propositions from [1] and [3]. Add to them the formulas marked with $(*)$ in all the preceding proofs from this paper and the formulas marked with $(*)$ in the proof below and all their subformulas. Denote such a list of formulas by $\psi_1, \dots, \psi_k$. Let us fix a countable set $X$ containing (as its elements) the sets $Y$, $\omega$, $\omega^\omega$, $\omega^{<\omega}$, $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$, and all the usual operations and relations on real numbers ($+$, $-$, $\cdot$, $:$, $<$). Fix an elementary submodel $M$ for formulas $\psi_1, \dots, \psi_k$ such that $X \in M$."*

Note that for the countable set $X \in M$ as above we get, by Proposition 3.2.5, $X \subset M$. Therefore, $Y \in M$ and, again by Proposition 3.2.5, $Y \subset M$. Thus, any $(*)$-elementary submodel $M$ is suitable for the results from [1], [3] and all the preceding theorems and propositions from this paper, making it possible to use all of these results for $M$. In order to demonstrate how this technique works, we prove the following lemma which we use later.

**Lemma 3.2.9.** *For any suitable elementary submodel $M$ the following holds. Let $(X, \rho)$ be a metric space and $\mathscr{F}$ be a countable collection of subsets of $X$. Then whenever $X \in M$ and $\mathscr{F} \subset M$, it is true that*

$$\bigcup \mathscr{F} \text{ is dense in } X \quad \Rightarrow \quad \bigcup \mathscr{F} \cap M \text{ is dense in } X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ with $X \in M$ such that $\mathscr{F} \subset \mathcal{P}(X) \cap M$ and $\bigcup \mathscr{F}$ is dense in $X$. In order to see that $\bigcup \mathscr{F} \cap X_M$ is dense in $X_M$, it is sufficient to prove that, for every $x \in X \cap M$ and $r \in \mathbb{Q}_+$, there exists $F \in \mathscr{F}$ such that $U(x,r) \cap X_M \cap F \neq \emptyset$. Fix some $x \in X \cap M$ and $r \in \mathbb{Q}_+$. Then there exists $F \in \mathscr{F}$ such that the following formula is satisfied

$$\exists y \colon (y \in F \;\wedge\; \rho(x,y) < r). \qquad (*)$$

The preceding formula has free variables $F$, $\rho$, $<$, $x$, and $r$. Those are in $M$; hence, by Lemma 3.2.3, there exists $y \in M$ such that $y \in F$ and $\rho(x,y) < r$. Consequently, $U(x,r) \cap X_M \cap F \neq \emptyset$. □

## 3.3 Foran-Zajíček scheme

We employ the following notation. Given $s, t \in \omega^{<\omega}$, we write $s \prec t$ if $t$ is an extension of $s$ (not necessarily proper). The concatenation of $s = (s_i)_{i<n} \in \omega^{<\omega}$ and $t = (t_i)_{i<n} \in \omega^{<\omega}$ is the sequence $s\hat{\ }t := (s_0, \dots, s_{n-1}, t_0, \dots, t_{n-1})$. If $s \in \omega^{<\omega}$ and $i \in \omega$, we write $s\hat{\ }i$ instead of $s\hat{\ }(i)$. If $\nu = (\nu_0, \nu_1, \nu_2, \dots) \in \omega^\omega$ and $n \in \omega$, then the symbol $\nu|n$ means the finite sequence $(\nu_0, \nu_1, \dots, \nu_{n-1})$. By $\nu|0$ we mean the empty sequence. If $t \in \omega^{<\omega}$, then the symbol $|t|$ denotes the length of $t$. By a *tree* we mean any subset $T$ of $\omega^{<\omega}$ such that for every $s \in \omega^{<\omega}$ and $t \in T$ with $s \prec t$, we have $s \in T$. We say that a tree $T$ is *pruned* if for every $t \in T$ there exists $n \in \omega$ such that $t\hat{\ }n \in T$.

Any family $\mathcal{A} = \{A(s);\ s \in \omega^{<\omega}\}$ of sets is called a *Souslin scheme*. Given such an $\mathcal{A}$ a *Souslin operation* $\mathcal{S}$ is defined by

$$\mathcal{S}(\mathcal{A}) := \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} A(\nu|n).$$

Sometimes we write $\mathcal{S}_s(A(s))$ instead of $\mathcal{S}(\mathcal{A})$. A Souslin scheme $\{A(s);\ s \in \omega^{<\omega}\}$ is called *monotone* if $A(s) \supset A(t)$ whenever $s, t \in \omega^{<\omega}$, and $s \prec t$. Finally, a subset $Y$ of a topological space $X$ is called a *Souslin* set (in $X$) if there exists a Souslin scheme $\mathcal{A}$ consisting of closed subsets of $X$ with $\mathcal{S}(\mathcal{A}) = Y$.

**Setting 3.3.1.** Throughout this section we will assume that $(X, \rho)$ is a complete metric space, $\mathbf{P}$ is a porosity-like relation on $X$, and $\mathcal{B}$ is a basis of open sets in $X$.

**Definition 3.3.2.** For any $A \subset X$ we define the following set operators:

$$\begin{aligned} \ker_{\mathbf{P}}(A) &:= A \setminus \bigcup\{U;\ U \subset X \text{ is open and } U \cap A \text{ is } \sigma\text{-}\mathbf{P}\text{-porous}\}, \\ N_{\mathbf{P}}(A) &:= \{x \in A;\ \neg\mathbf{P}(x, A)\}. \end{aligned}$$

The following lemma is easy to prove. Its assertions (i) and (ii) can be found, e.g., in [9].

**Lemma 3.3.3.** *Let $A \subset X$. Then we have*

*(i)* $A \setminus \ker_{\mathbf{P}}(A)$ *is $\sigma$-$\mathbf{P}$-porous,*

*(ii)* $\ker_{\mathbf{P}}(\ker_{\mathbf{P}}(A)) = \ker_{\mathbf{P}}(A)$*,*

*(iii) if $A \subset X$ is a Souslin set then* $\ker_{\mathbf{P}}(A)$ *is a Souslin set,*

*(iv) if $A \subset B \subset X$, $\ker_{\mathbf{P}}(B) = B$, and $B \setminus A$ is $\sigma$-$\mathbf{P}$-porous, then* $\ker_{\mathbf{P}}(A) = A$*,*

*(v)* $A \setminus N_{\mathbf{P}}(A)$ *is $\mathbf{P}$-porous.*

**Definition 3.3.4.** A Souslin scheme $\mathscr{F} = \{S(t);\ t \in \omega^{<\omega}\}$ consisting of nonempty subsets of $X$ is called $(\mathcal{B}, \mathbf{P})$*-Foran-Zajíček scheme in $X$* if for every $t \in \omega^{<\omega}$ and $k \in \omega$ we have

(i) $\bigcup_{j \in \omega} S(t\hat{\ }j)$ is a dense subset of $S(t)$,

(ii) $S(t)$ is $\mathbf{P}$-porous at no point of $S(t\hat{\ }k)$,

(iii) for any $\nu \in \omega^\omega$ and any sequence $(G_n)_{n \in \omega}$ of sets from $\mathcal{B}$ satisfying:

   (a) $\lim_{n \to \infty} \operatorname{diam} G_n = 0$,
   (b) $\overline{G_{n+1}} \subset G_n$ for every $n \in \omega$,
   (c) $S(\nu|n) \cap G_n \neq \emptyset$ for every $n \in \omega$,

we have

$$\bigcap_{n\in\omega}\big(S(\nu|n)\cap G_n\big)\neq\emptyset.$$

If there is no danger of confusion we will say just *Foran-Zajíček scheme.*

**Remark 3.3.5.** The definition of Foran-Zajíček scheme is inspired by the notion of Foran system in the form introduced by Zajíček in [13]. Foran systems provides a basic tool for constructions of small but non-$\sigma$-porous sets.

**Lemma 3.3.6.** *Let $\mathscr{F}=\{S(t);\ t\in\omega^{<\omega}\}$ be a Foran-Zajíček scheme in $X$. Then no element of $\mathscr{F}$ is $\sigma$-$\mathbf{P}$-porous.*

*Proof.* We mimic the standard proof which works for Foran systems, see [13, Lemma 4.3]. Clearly, it is sufficient to prove that $S(\emptyset)$ is not $\sigma$-$\mathbf{P}$-porous. Suppose on the contrary that $S(\emptyset)=\bigcup_{n=1}^{\infty}A_n$, where each $A_n$ is $\mathbf{P}$-porous. We set $A_0:=\emptyset$. We will construct $\nu=(\nu_0,\nu_1,\dots)\in\omega^\omega$ and a sequence of open sets $(G_n)_{n\in\omega}$ such that for every $n\in\omega$ we have

(a) $\operatorname{diam} G_n<2^{-n}$,

(b) $\overline{G_n}\subset G_{n-1}$ if $n>0$,

(c) $G_n\cap S(\nu|n)\neq\emptyset$,

(d) $S(\nu|n)\cap G_n\cap A_n=\emptyset$,

(e) $G_n\in\mathcal{B}$.

We will construct inductively $\nu_n$'s and $G_n$'s. If $n=0$, then we pick an open set $G_0\in\mathcal{B}$ intersecting $S(\emptyset)$ with $\operatorname{diam} G_0<1$. Then conditions (a)–(e) are clearly satisfied. Now suppose that we have already constructed $G_n$ and $s=(\nu_0,\dots,\nu_{n-1})$ for some fixed $n\in\omega$. Note that if $n=0$, then $s=\emptyset$. We distinguish two cases.

First suppose that $A_{n+1}$ is not dense in $S(s)\cap G_n$. Then we find a nonempty open set $G_{n+1}\in\mathcal{B}$ such that $G_{n+1}\cap S(s)\neq\emptyset$, $\overline{G_{n+1}}\subset G_n\setminus A_{n+1}$, and $\operatorname{diam} G_{n+1}<2^{-(n+1)}$. Further, using condition (i) from Definition 3.3.4 we find $\nu_n\in\omega$ such that $S(s\hat{\ }\nu_n)\cap G_{n+1}\neq\emptyset$.

Now suppose that $A_{n+1}$ is dense in $S(s)\cap G_n$. Find $\nu_n\in\omega$ so that $S(s\hat{\ }\nu_n)\cap G_n\neq\emptyset$ by condition (i) from Definition 3.3.4. We shall show that the intersection $S(s\hat{\ }\nu_n)\cap G_n\cap A_{n+1}$ is empty. Indeed, if there is $x$ in this set, then using the properties of $\mathbf{P}$, we subsequently have $\mathbf{P}(x,A_{n+1})$, $\mathbf{P}(x,\overline{A_{n+1}})$, $\mathbf{P}(x,S(s)\cap G_n)$, and finally $\mathbf{P}(x,S(s))$. Now (ii) in Definition 3.3.4 says that $x\notin S(s\hat{\ }\nu_n)$, a contradiction. It remains then to pick any open set $G_{n+1}\in\mathcal{B}$ such that $\overline{G_{n+1}}\subset G_n$, $G_{n+1}\cap S(s\hat{\ }\nu_n)\neq\emptyset$, and $\operatorname{diam} G_{n+1}<2^{-(n+1)}$. This finishes the construction of $\nu$ and $G_n$'s.

Since $\mathscr{F}$ is a Foran-Zajíček scheme there exists $x\in\bigcap_{n\in\omega}\big(S(\nu|n)\cap G_n\big)$. By (b) and (d) we have $x\in S(\emptyset)\setminus\bigcup_{n=1}^{\infty}A_n=\emptyset$, a contradiction. □

**Definition 3.3.7.** We say that a Souslin scheme $\mathcal{C}=\{C(s);\ s\in\omega^{<\omega}\}$ is *subordinate* to a Souslin scheme $\mathcal{A}=\{A(s);\ s\in\omega^{<\omega}\}$ (notation $\mathcal{C}\sqsubseteq\mathcal{A}$) if there exists a mapping $\varphi\colon\omega^{<\omega}\to\omega^{<\omega}$ such that for each $s\in\omega^{<\omega}$ we have

- $|\varphi(s)| = |s|$,
- if $t \in \omega^{<\omega}, s \prec t$, then $\varphi(s) \prec \varphi(t)$,
- $C(s) \subset A(\varphi(s))$.

**Definition 3.3.8.** Let $\mathcal{A} = \{A(s);\ s \in \omega^{<\omega}\}$ be a Souslin scheme. Denote $C(s) = \mathcal{S}_t(A(s\hat{\ }t))$, $s \in \omega^{<\omega}$, i.e.,

$$C(s) := \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} A(s\hat{\ }\nu|n).$$

We say that $\mathcal{A}$ is $\mathbf{P}$-*regular* if $\mathcal{A}$ is monotone and for every $s \in \omega^{<\omega}$ we have $\ker_{\mathbf{P}}(C(s)) = C(s) \neq \emptyset$.

**Lemma 3.3.9.** *Let $\mathcal{A}$ be a Souslin scheme consisting of closed sets and $C \subset \mathcal{S}(\mathcal{A})$ be a Souslin set. Then there exists a Souslin scheme $\mathcal{C}$ consisting of closed sets which is subordinate to $\mathcal{A}$ and $C = \mathcal{S}(\mathcal{C})$.*

*Proof.* Let $\mathcal{L} = \{L(s);\ s \in \omega^{<\omega}\}$ be a Souslin scheme consisting of closed sets with $\mathcal{S}(\mathcal{L}) = C$. Let $\mathcal{A} = \{A(s);\ s \in \omega^{<\omega}\}$. Fix a surjection $\psi = (\psi_1, \psi_2)\colon \omega \to \omega^2$. We define mappings $\varphi_1\colon \omega^{<\omega} \to \omega^{<\omega}$, $\varphi_2\colon \omega^{<\omega} \to \omega^{<\omega}$ by $\varphi_1(\emptyset) = \varphi_2(\emptyset) = \emptyset$ and

$$\varphi_1(s) := \big(\psi_1(s_0), \psi_1(s_1), \dots, \psi_1(s_{|s|-1})\big),$$
$$\varphi_2(s) := \big(\psi_2(s_0), \psi_2(s_1), \dots, \psi_2(s_{|s|-1})\big),$$

where $s = (s_0, \dots, s_{|s|-1}) \in \omega^{<\omega} \setminus \{\emptyset\}$. We define the desired scheme $\mathcal{C}$ by $C(s) := A(\varphi_1(s)) \cap L(\varphi_2(s))$. The scheme $\mathcal{C} := \{C(s);\ s \in \omega^{<\omega}\}$ consists of closed sets and is clearly subordinate to $\mathcal{A}$ via the mapping $\varphi_1$.

We shall verify the equality $C = \mathcal{S}(\mathcal{C})$. Let $x \in C$. Then there exist $\nu, \mu \in \omega^\omega$ such that $x \in L(\nu|k)$ and $x \in A(\mu|k)$ for every $k \in \omega$. Since $\psi$ is a surjection of $\omega$ onto $\omega^2$ we can find $\tau \in \omega^\omega$ such that $\varphi_1(\tau|k) = \nu|k$ and $\varphi_2(\tau|k) = \mu|k$ for every $k \in \omega$. Then we have $x \in C(\tau|k)$ for every $k \in \omega$. Consequently, $x \in \mathcal{S}(\mathcal{C})$.

Let $x \in \mathcal{S}(\mathcal{C})$. Find $\tau = (t_0, t_1, \dots) \in \omega^\omega$ so that $x \in C(\tau|k)$ for every $k \in \omega$. For $i = 1, 2$ we define $\mu_i := (\psi_i(t_0), \psi_i(t_1), \dots) \in \omega^\omega$. Then $x \in A(\mu_1|k) \cap L(\mu_2|k)$. Consequently, $x \in C$. Thus we have proved $C = \mathcal{S}(C)$. □

**Lemma 3.3.10.** *Let $\mathcal{A}$ be a Souslin scheme consisting of closed subsets of $X$ and $C \subset \mathcal{S}(\mathcal{A})$ be a nonempty Souslin set with $\ker_{\mathbf{P}}(C) = C$. Then there exists a $\mathbf{P}$-regular Souslin scheme $\mathcal{L} = \{L(s);\ s \in \omega^{<\omega}\}$ consisting of closed subsets of $X$ such that $\mathcal{L}$ is subordinate to $\mathcal{A}$ and $\mathcal{S}(\mathcal{L})$ is a dense subset of $C$.*

*Proof.* Let $\mathcal{A} = \{A(s);\ s \in \omega^{<\omega}\}$. Using Lemma 3.3.9 we find a Souslin scheme $\mathcal{D} = \{D(s);\ s \in \omega^{<\omega}\}$ consisting of closed subsets of $X$ which is subordinate to $\mathcal{A}$ and $\mathcal{S}(\mathcal{D}) = C$. Without any loss of generality we may assume that $\mathcal{D}$ is monotone. Indeed, one can define a scheme $\tilde{\mathcal{D}} = \{\tilde{D}(s);\ s \in \omega^{<\omega}\}$ by

$$\tilde{D}(s) := \bigcap_{k \leq |s|} D(s|k).$$

The new scheme consists of closed sets and satisfies $\tilde{D}(s) \subset D(s)$, thus it is subordinated to the scheme $\mathcal{A}$. Further we have $\mathcal{S}(\mathcal{D}) = \mathcal{S}(\tilde{\mathcal{D}})$, since for every $\alpha \in \omega^\omega$ we have

$$\bigcap_{j\in\omega} \tilde{D}(\alpha|j) = \bigcap_{j\in\omega} \bigcap_{k\leq j} D(\alpha|k) = \bigcap_{j\in\omega} D(\alpha|j),$$

thus

$$\mathcal{S}(\tilde{\mathcal{D}}) = \bigcup_{\alpha\in\omega^\omega} \bigcap_{j\in\omega} \tilde{D}(\alpha|j) = \bigcup_{\alpha\in\omega^\omega} \bigcap_{j\in\omega} D(\alpha|j) = \mathcal{S}(\mathcal{D}).$$

For $s \in \omega^{<\omega}$ we set

$$E(s) := \mathcal{S}_t(D(s\hat{\ }t)), \qquad H(s) := \ker_{\mathbf{P}}(E(s)),$$
$$P(s) := \overline{H(s)}, \qquad \text{and} \qquad Q(s) := \mathcal{S}_t(P(s\hat{\ }t)).$$

Our aim is to show that $\ker_{\mathbf{P}}(Q(s)) = Q(s)$ for every $s \in \omega^{<\omega}$.

For every $u \in \omega^{<\omega}$ we have

$$E(u) = \bigcup_{j\in\omega} E(u\hat{\ }j). \tag{3.1}$$

To verify (3.1) suppose that $x \in E(u)$. Then there exists $\alpha \in \omega^\omega$ such that for every $k \in \omega$ we have $x \in D(u\hat{\ }(\alpha|k))$. This implies $x \in E(u\hat{\ }\alpha(0))$. Now suppose that $x \in E(u\hat{\ }j)$ for some $j \in \omega$. Then there exists $\alpha \in \omega^\omega$ such that $x \in D(u\hat{\ }j\hat{\ }(\alpha|k))$ for every $k \in \omega$. We set $\beta = j\hat{\ }\alpha$. Using monotonicity of the scheme $\mathcal{D}$ we obtain $x \in D(u\hat{\ }(\beta|l))$ for every $l \in \omega$, therefore $x \in E(u)$.

Further, we have

$$H(u) \setminus \bigcup_{j\in\omega} H(u\hat{\ }j) \subset E(u) \setminus \bigcup_{j\in\omega} H(u\hat{\ }j)$$

$$= \Big(\bigcup_{j\in\omega} E(u\hat{\ }j)\Big) \setminus \Big(\bigcup_{j\in\omega} H(u\hat{\ }j)\Big) \subset \bigcup_{j\in\omega} \big(E(u\hat{\ }j) \setminus H(u\hat{\ }j)\big).$$

Since $E(u\hat{\ }j)\setminus H(u\hat{\ }j)$ is $\sigma$-$\mathbf{P}$-porous for every $j \in \omega$ (Lemma 3.3.3(i)), we conclude that the set $H(u) \setminus \bigcup_{j\in\omega} H(u\hat{\ }j)$ is $\sigma$-$\mathbf{P}$-porous. Fix for a while any $s \in \omega^{<\omega}$. We have

$$H(s) \setminus \mathcal{S}_t(H(s\hat{\ }t)) \subset \bigcup_{t\in\omega^{<\omega}} \Big(H(s\hat{\ }t) \setminus \bigcup_{j\in\omega} H(s\hat{\ }t\hat{\ }j)\Big). \tag{3.2}$$

Indeed, suppose that $x \in H(s)$ a $x \notin \bigcup_{t\in\omega^{<\omega}} \big(H(s\hat{\ }t) \setminus \bigcup_{j\in\omega} H(s\hat{\ }t\hat{\ }j)\big)$. Then $x \notin H(s)\setminus\bigcup_j H(s\hat{\ }j)$. Thus there exists $j_0 \in \omega$ such that $x \in H(s\hat{\ }j_0)$. Further we have $x \notin H(s\hat{\ }j_0)\setminus\bigcup_j H(s\hat{\ }j_0\hat{\ }j)$, therefore one can find $j_1 \in \omega$ with $x \in H(s\hat{\ }j_0\hat{\ }j_1)$. By induction we construct $\alpha = (j_0, j_1, j_2, \dots)$ such that $x \in H(s\hat{\ }\alpha|l)$ for every $l \in \omega$. This means that $x \in \mathcal{S}_t(H(s\hat{\ }t))$ and (3.2) holds.

Using (3.2) we get that $H(s) \setminus \mathcal{S}_t(H(s\hat{\ }t))$ is $\sigma$-$\mathbf{P}$-porous. Therefore

$$\ker_{\mathbf{P}}(\mathcal{S}_t(H(s\hat{\ }t))) = \mathcal{S}_t(H(s\hat{\ }t)) \tag{3.3}$$

by Lemma 3.3.3(iv). For every open set $V$ intersecting $H(s)$ the set $V \cap H(s)$ is not $\sigma$-$\mathbf{P}$-porous by definition. The set $H(s)\setminus\mathcal{S}_t(H(s\hat{\ }t))$ is $\sigma$-$\mathbf{P}$-porous. It implies that $\mathcal{S}_t(H(s\hat{\ }t))$ is a dense subset of $H(s)$. Observing

$$\mathcal{S}_t(H(s\hat{\ }t)) \subset \mathcal{S}_t(P(s\hat{\ }t)) = Q(s) \subset P(s) = \overline{H(s)} = \overline{\mathcal{S}_t(H(s\hat{\ }t))} \tag{3.4}$$

and using (3.3) we get $\ker_{\mathbf{P}}(Q(s)) = Q(s)$. Indeed, fix an open set $U$ with $U \cap Q(s) \neq \emptyset$. Using (3.4), $U \cap S_t(H(s\hat{\ }t)) \neq \emptyset$ and, by (3.3), $U \cap S_t(H(s\hat{\ }t))$ is not $\sigma$-$\mathbf{P}$-porous set. Hence, $U \cap Q(s)$ is not $\sigma$-$\mathbf{P}$-porous set and, as $U$ was an arbitrary open set intersecting $Q(s)$, $\ker_{\mathbf{P}}(Q(s)) = Q(s)$.

Further, we set $T := \{s \in \omega^{<\omega};\ P(s) \neq \emptyset\}$. The set $T$ is obviously a nonempty tree. Moreover, $T$ is pruned. Indeed, let $s \in T$, then $H(s) \neq \emptyset$ and thus $E(s)$ is non-$\sigma$-$\mathbf{P}$-porous. We have $E(s) = \bigcup_{n\in\omega} E(s\hat{\ }n)$ and therefore there exists $m \in \omega$ such that $E(s\hat{\ }m)$ is non-$\sigma$-$\mathbf{P}$-porous. Thus $P(s\hat{\ }m) \neq \emptyset$ and so $s\hat{\ }m \in T$.

We find a mapping $\varphi\colon \omega^{<\omega} \to T$ such that for every $s \in \omega^{<\omega}$ we have

- $|\varphi(s)| = |s|$,
- if $t \in \omega^{<\omega}$, $s \prec t$, then $\varphi(s) \prec \varphi(t)$,
- $\{\varphi(s\hat{\ }n);\ n \in \omega\} = \{\varphi(s)\hat{\ }k;\ k \in \omega\} \cap T$.

We have $\emptyset \in T$ since $T$ is nonempty. We set $\varphi(\emptyset) := \emptyset$. Suppose that $\varphi(s) \in T$ has been already defined for some $s \in \omega^{<\omega}$. The set $W := \{k \in \omega;\ \varphi(s)\hat{\ }k \in T\}$ is nonempty since $T$ is pruned. Thus we can find a mapping $\psi\colon \omega \to \omega$ such that $\psi(\omega) = W$. We define $\varphi(s\hat{\ }n) := \varphi(s)\hat{\ }\psi(n)$. This finishes the construction of $\varphi$. It is easy to check that the mapping $\varphi$ has all the required properties.

We set $L(s) := P(\varphi(s))$ and $\mathcal{L} := \{L(s);\ s \in \omega^{<\omega}\}$. The scheme $\{E(s);\ s \in \omega^{<\omega}\}$ is monotone since $\mathcal{D}$ is monotone. This easily gives that the scheme $\mathcal{L}$ is also monotone.

By the properties of $\varphi$ and the definition of $T$ we have $\mathcal{S}_t(L(s\hat{\ }t)) = Q(\varphi(s)) \neq \emptyset$ for every $s \in \omega^{<\omega}$. Indeed, let $x \in Q(\varphi(s))$ for some $s \in \omega^{<\omega}$. Then there exists $\nu \in \omega^\omega$ such that $x \in P(\varphi(s)\hat{\ }\nu|n)$ for every $n \in \omega$. Thus $P(\varphi(s)\hat{\ }\nu|n) \neq \emptyset$ for every $n \in \omega$. This means that $\varphi(s)\hat{\ }\nu|n \in T$ for every $n \in \omega$. Using the properties of $\varphi$ we find $\mu \in \omega^\omega$ such that $\varphi(s\hat{\ }\mu|n) = \varphi(s)\hat{\ }\nu|n$ for every $n \in \omega$. Thus we have $x \in \bigcap_{n\in\omega} P(\varphi(s)\hat{\ }\nu|n) = \bigcap_{n\in\omega} L(s\hat{\ }\mu|n) \subset \mathcal{S}_t(L(s\hat{\ }t))$. Let $x \in \mathcal{S}_t(L(s\hat{\ }t))$. Then there exists $\nu \in \omega^\omega$ such that $x \in L(s\hat{\ }\nu|n) = P(\varphi(s\hat{\ }\nu|n))$ for every $n \in \omega$. Using the properties of $\varphi$ again, we get $x \in \bigcap_{n\in\omega} P(\varphi(s\hat{\ }\nu|n)) \subset Q(\varphi(s))$. Finally, for $s \in \omega^{<\omega}$ we have $H(\varphi(s)) \neq \emptyset$ and $H(\varphi(s)) \setminus \mathcal{S}_t(H(\varphi(s)\hat{\ }t))$ is $\sigma$-$\mathbf{P}$-porous. Thus $\mathcal{S}_t(H(\varphi(s)\hat{\ }t)) \neq \emptyset$ and by (3.4) we get $Q(\varphi(s)) \neq \emptyset$. Thus $\mathcal{L}$ is $\mathbf{P}$-regular.

Clearly $\mathcal{L} \sqsubseteq \mathcal{D}$. Using the fact that $\mathcal{D} \sqsubseteq \mathcal{A}$, we get $\mathcal{L} \sqsubseteq \mathcal{A}$. It remains to verify that $\mathcal{S}(\mathcal{L})$ is dense in $C$. Since by definition we have $P(s) \subset D(s)$ for every $s \in \omega^{<\omega}$, we get $Q(\emptyset) \subset E(\emptyset) = C$. The set $\mathcal{S}_t(H(t))$ is a dense subset of $H(\emptyset)$. We get by (3.4) that $Q(\emptyset)$ is a dense subset of $H(\emptyset) = C$. This concludes the proof since $\mathcal{S}(\mathcal{L}) = Q(\emptyset)$. □

**Proposition 3.3.11.** *Suppose that* $\mathbf{P}$ *is such that* $N_{\mathbf{P}}(A)$ *is Souslin whenever* $A \subset X$ *is a Souslin set, and let* $S \subset X$ *be a Souslin non-*$\sigma$*-*$\mathbf{P}$*-porous set. Then there exists a* $(\mathcal{B}, \mathbf{P})$*-Foran-Zajíček scheme* $\mathscr{F}$ *in* $X$ *such that each element of* $\mathscr{F}$ *is a subset of* $S$*.*

*Proof.* For every $n \in \omega$ we will construct a $\mathbf{P}$-regular Souslin scheme $\mathcal{A}_n = \{A^n(s);\ s \in \omega^{<\omega}\}$ consisting of closed sets. For $s \in \omega^{<\omega}$ we denote $C^n(s) := \mathcal{S}_t(A^n(s\hat{\ }t))$. We require $\mathcal{S}(\mathcal{A}_0) \subset S$ and, for every $n \in \omega, n > 0$,

- $\mathcal{A}_n \sqsubseteq \mathcal{A}_{n-1}$ is witnessed by a mapping $\varphi_n\colon \omega^{<\omega} \to \omega^{<\omega}$,

- $\varphi_n(s) = s$ for every $s \in \omega^{<\omega}, |s| < n$.

Applying Lemma 3.3.10 to the Souslin scheme $\{X;\ s \in \omega^{<\omega}\}$ and the set $\ker_{\mathbf{P}}(S)$ we find a $\mathbf{P}$-regular Souslin scheme $\mathcal{A}_0$ consisting of closed sets with $\mathcal{S}(\mathcal{A}_0) \subset \ker_{\mathbf{P}}(S) \subset S$. Suppose that $n \in \omega, n > 0$, and we have already constructed the desired schemes $\mathcal{A}_j$ and the mappings $\varphi_j$, $j < n$. Fix $s \in \omega^{n-1}$ for a while. The set $C^{n-1}(s)$ is not $\sigma$-$\mathbf{P}$-porous by $\mathbf{P}$-regularity of $\mathcal{A}_{n-1}$ and therefore $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^{n-1}(s)))$ is not $\sigma$-$\mathbf{P}$-porous by Lemma 3.3.3(i),(v). Thus the set $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^{n-1}(s)))$ is nonempty. The set $C^{n-1}(s)$ is Souslin by the construction. Using the assumption we see that $N_{\mathbf{P}}(C^{n-1}(s))$ is Souslin and by Lemma 3.3.3(iii) we get that $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^{n-1}(s)))$ is Souslin. By Lemma 3.3.10 we find a $\mathbf{P}$-regular Souslin scheme $\mathcal{L}_s = \{L^s(t);\ t \in \omega^{<\omega}\}$ such that $\mathcal{S}(\mathcal{L}_s)$ is a dense subset of $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^{n-1}(s)))$ and $\mathcal{L}_s \sqsubseteq \{A^{n-1}(s\hat{}t);\ t \in \omega^{<\omega}\}$ is witnessed by a mapping $\varphi_n^s \colon \omega^{<\omega} \to \omega^{<\omega}$. Do so for every $s \in \omega^{n-1}$.

For $t = (t_0, \dots, t_{|t|-1}) \in \omega^{<\omega}$, we set

$$A^n(t) := \begin{cases} A^{n-1}(t), & |t| < n, \\ L^{t|(n-1)}(t_{n-1}, \dots, t_{|t|-1}), & |t| \geq n; \end{cases}$$

$$\varphi_n(t) := \begin{cases} t, & |t| < n, \\ t|(n-1)\hat{}\varphi_n^{t|(n-1)}(t_{n-1}, \dots, t_{|t|-1}), & |t| \geq n. \end{cases}$$

Further, we set $\mathcal{A}_n = \{A^n(t);\ t \in \omega^{<\omega}\}$. For every $t \in \omega^{<\omega}, |t| < n$, we have $A^n(t) = A^{n-1}(t) = A^{n-1}(\varphi(t))$. For every $t \in \omega^{<\omega}, |t| \geq n$, we have

$$A^n(t) = L^{t|(n-1)}(t_{n-1}, \dots, t_{|t|-1})$$
$$\subset A^{n-1}(t|(n-1)\hat{}\varphi_n^{t|(n-1)}(t_{n-1}, \dots, t_{|t|-1})) = A^{n-1}(\varphi_n(t)).$$

Thus $\mathcal{A}_n \sqsubseteq \mathcal{A}_{n-1}$ is witnessed by $\varphi_n$. This finishes the induction step.

We define $S(s) := C^{|s|}(s)$, $s \in \omega^{<\omega}$, and $\mathscr{F} := \{S(s);\ s \in \omega^{<\omega}\}$. We verify the conditions (i)–(iii) from Definition 3.3.4.

(i) Let $n \in \omega$ and $s \in \omega^n$. By definition we have $A^{n+1}(s\hat{}t) = L^s(t)$ and $C^{n+1}(s) = \mathcal{S}_t(A^{n+1}(s\hat{}t))$. Thus we get $C^{n+1}(s) = \mathcal{S}(\mathcal{L}_s)$. Since $\ker_{\mathbf{P}}(C^n(s)) = C^n(s)$, the set $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^n(s)))$ is dense in $C^n(s)$. Since $\mathcal{S}(\mathcal{L}_s)$ is dense in $\ker_{\mathbf{P}}(N_{\mathbf{P}}(C^n(s)))$, the set $\mathcal{S}(\mathcal{L}_s)$ is dense in $C^n(s)$. But

$$\bigcup_{j \in \omega} S(s\hat{}j) = \bigcup_{j \in \omega} C^{n+1}(s\hat{}j) = C^{n+1}(s) = \mathcal{S}(\mathcal{L}_s)$$

is a dense subset of $S(s) = C^n(s)$.

(ii) We have

$$C^{|t|+1}(t\hat{}k) = \mathcal{S}_u(A^{|t|+1}(t\hat{}k\hat{}u)) = \mathcal{S}_u(L^t(k\hat{}u)) \subset \mathcal{S}(\mathcal{L}_t) \subset \ker_{\mathbf{P}}(N_{\mathbf{P}}(C^{|t|}(t))).$$

Thus we can conclude $S(t\hat{}k) = C^{|t|+1}(t\hat{}k) \subset N_{\mathbf{P}}(C^{|t|}(t)) = N_{\mathbf{P}}(S(t))$ for every $t \in \omega^{<\omega}$ and $k \in \omega$.

(iii) Suppose that we have $\nu \in \omega^\omega$ and a sequence $(G_n)_{n\in\omega}$ of open sets in $\mathcal{B}$ such that

(a) $\lim_{n\to\infty} \operatorname{diam}(G_n) = 0$,

(b) $\overline{G_{n+1}} \subset G_n$ for every $n \in \omega$,

(c) $S(\nu|n) \cap G_n \neq \emptyset$ for every $n \in \omega$.

We have that $\bigcap_{n\in\omega} G_n = \{x\}$ for some $x \in X$, since $X$ is complete. Our task is to show that $x \in S(\nu|m)$ for every $m \in \omega$. Fix $m \in \omega$. For each $k \in \omega$ we pick $y_k \in S(\nu|k) \cap G_k$. We have $\lim y_k = x$. Fix $k \in \omega$, $k \geq m$. Observe that

$$C^n(\nu|n) = \mathcal{S}_t(A^n(\nu|n\hat{\ }t)) \subset A^n(\nu|n).$$

Then, for every $n \in \omega, n > k$, we have

$$\begin{aligned} y_n \in S(\nu|n) = C^n(\nu|n) \subset A^n(\nu|n) \\ \subset A^{n-1}(\varphi_n(\nu|n)) \subset \cdots \subset A^m(\varphi_{m+1} \circ \cdots \circ \varphi_n(\nu|n)). \end{aligned} \tag{3.5}$$

Since $\mathcal{A}_m$ is a $\mathbf{P}$-regular scheme, it is monotone. Using this fact and (3.5) we get

$$y_n \in A^m(\varphi_{m+1} \circ \cdots \circ \varphi_n(\nu|n)) \subset A^m(\varphi_{m+1} \circ \cdots \circ \varphi_n(\nu|k)). \tag{3.6}$$

Since $\varphi_j(\nu|k) = \nu|k$ for every $j \in \omega, j > k$, we get

$$A^m(\varphi_{m+1} \circ \cdots \circ \varphi_n(\nu|k)) = A^m(\varphi_{m+1} \circ \cdots \circ \varphi_{k+1}(\nu|k)).$$

Using this and (3.6) we get $x \in A^m(\varphi_{m+1} \circ \cdots \circ \varphi_{k+1}(\nu|k))$ since the latter set is closed. Since $\nu|m \prec \varphi_{m+1} \circ \cdots \circ \varphi_{k+1}(\nu|k)$ we can conclude that $x \in S(\nu|m)$. We verified that $\mathscr{F}$ is a $(\mathcal{B}, \mathbf{P})$-Foran-Zajíček scheme. □

## 3.4 Porosity-like relations

**Definition 3.4.1.** Let $X$ be a metric space and $\mathbf{R}$ be a point-set relation on $X$ (i.e., $\mathbf{R} \subseteq X \times \mathcal{P}(X)$). Let $M$ be a set and $\mathbf{R}'$ be a point-set relation on $X_M := \overline{X \cap M}$. We say that the set $M$ is a *pointwise* $(\mathbf{R} \to \mathbf{R}')$*-model* if

$$\forall A \in \mathcal{P}(X) \cap M \ \forall x \in X_M \colon \big(\mathbf{R}(x, A) \ \to \ \mathbf{R}'(x, A \cap X_M)\big).$$

Similarly, we define the notion of a *pointwise* $(\mathbf{R} \leftarrow \mathbf{R}')$*-model* and *pointwise* $(\mathbf{R} \leftrightarrow \mathbf{R}')$*-model*.

**Definition 3.4.2.** Let $X$ be a metric space and $\mathbf{P}$ be a porosity-like relation on $X$. Let $M$ be a set and $\mathbf{P}'$ be a porosity-like relation on $X_M := \overline{X \cap M}$. We say that the set $M$ is a $(\mathbf{P} \to \mathbf{P}')$*-model* if for every set $A \in \mathcal{P}(X) \cap M$

$$A \text{ is } \mathbf{P}\text{-porous in the space } X \ \to \ A \cap X_M \text{ is } \mathbf{P}'\text{-porous in the space } X_M.$$

We say that the set $M$ is a $(\sigma\text{-}\mathbf{P} \to \sigma\text{-}\mathbf{P}')$*-model* if for every set $A \in \mathcal{P}(X) \cap M$

$$A \text{ is } \sigma\text{-}\mathbf{P}\text{-porous in the space } X \ \to \ A \cap X_M \text{ is } \sigma\text{-}\mathbf{P}'\text{-porous in the space } X_M.$$

Similarly, we define the notion of $(\mathbf{P} \leftarrow \mathbf{P}')$*-model*, $(\mathbf{P} \leftrightarrow \mathbf{P}')$*-model*, and $(\sigma\text{-}\mathbf{P} \leftrightarrow \sigma\text{-}\mathbf{P}')$*-model*.

**Proposition 3.4.3.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be a metric space, $\mathbf{P}$ be a porosity-like relation on $X$ and $\mathbf{P}'$ be a porosity-like relation on $X_M$. Assume $X \in M$ and $\mathbf{P} \in M$.*

*(i) If $M$ is a pointwise $(\mathbf{P} \to \mathbf{P}')$-model, then $M$ is a $(\mathbf{P} \to \mathbf{P}')$-model.*

*(ii) If $M$ is a pointwise $(\mathbf{P} \leftarrow \mathbf{P}')$-model, then $M$ is a $(\mathbf{P} \leftarrow \mathbf{P}')$-model.*

*(iii) If $M$ is a $(\mathbf{P} \to \mathbf{P}')$-model, then $M$ is a $(\sigma\text{-}\mathbf{P} \to \sigma\text{-}\mathbf{P}')$-model.*

*In particular, if $M$ is a pointwise $(\mathbf{P} \leftrightarrow \mathbf{P}')$-model, then $M$ is a $(\mathbf{P} \leftrightarrow \mathbf{P}')$-model and a $(\sigma\text{-}\mathbf{P} \to \sigma\text{-}\mathbf{P}')$-model.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$ with $X$, $\mathbf{P} \in M$.

(i) The statement follows immediately from definitions (with $M$ an arbitrary set, not necessarily an elementary submodel).

(ii) Let us suppose $M$ is a pointwise $(\mathbf{P} \leftarrow \mathbf{P}')$-model and let us fix a non-$\mathbf{P}$-porous set $A \in \mathcal{P}(X) \cap M$. Consider the formula

$$\exists x \in A\colon (x, A) \notin \mathbf{P}, \tag{$*$}$$

with free variables $A$ and $\mathbf{P}$. Since $A \in M$, $\mathbf{P} \in M$, and the above formula is absolute for $M$, there exists by Lemma 3.2.3 a point $x \in A \cap M$ such that $(x, A) \notin \mathbf{P}$, i.e., $A$ is not $\mathbf{P}$-porous at $x$. Hence, by premise $A \cap X_M$ is not $\mathbf{P}'$-porous at $x$. Thus, $A \cap X_M$ is not $\mathbf{P}'$-porous in the space $X_M$ and (ii) holds.

(iii) Suppose that $A \in M \cap \mathcal{P}(X)$ is $\sigma$-$\mathbf{P}$-porous. Then the next formula is satisfied

$$\begin{aligned}\exists D\colon \big(&D \text{ is a function with } \operatorname{Dom} D = \mathbb{N},\ \forall n \in \mathbb{N}: \\ &D(n) \subset X \text{ is } \mathbf{P}\text{-porous set, and} A \subset \bigcup_{n \in \mathbb{N}} D(n)\big).\end{aligned} \tag{$*$}$$

Now by Lemma 3.2.3 we find $D \in M$ such that

$$\begin{aligned}&D \text{ is a function with } \operatorname{Dom} D = \mathbb{N},\ \forall n \in \mathbb{N}: \\ &D(n) \subset X \text{ is } \mathbf{P}\text{-porous set, and } A \subset \bigcup_{n \in \mathbb{N}} D(n).\end{aligned}$$

By Proposition 3.2.5 (ii), we have $D(n) \in M$ for every $n \in \mathbb{N}$. Since $M$ is a $(\mathbf{P} \to \mathbf{P}')$-model, we obtain that $D(n) \cap X_M$ is $\mathbf{P}'$-porous in $X_M$ for every $n \in \mathbb{N}$, hence, $A \cap X_M$ is $\sigma$-$\mathbf{P}'$-porous in $X_M$. □

**Lemma 3.4.4.** *Let $(X, \varrho)$ be a complete metric space and $(Y, \varrho)$ be a closed subset of it. Consider sequences $(y_n)_{n \in \mathbb{N}}$ in $Y$ and $r_n \to 0$ such that $\overline{U(y_{n+1}, r_{n+1}) \cap Y} \cap Y \subset U(y_n, r_n)$ for every $n \in \mathbb{N}$. Then there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that $\overline{U(y_{n_{k+1}}, r_{n_{k+1}})} \subset U(y_{n_k}, r_{n_k})$ for every $k \in \mathbb{N}$.*

*Proof.* We shall prove the following statement which implies the conclusion of the lemma: For each $k \in \mathbb{N}$ there is $l \in \mathbb{N}$, $l > k$ such that $\overline{U(y_l, r_l)} \subset U(y_k, r_k)$.

Assume this is not the case, i.e., there is a natural number $n_0$ such that $\overline{U(y_n, r_n)} \setminus U(y_{n_0}, r_{n_0}) \neq \emptyset$ for each natural number $n > n_0$. Choose a sequence $\{z_n\}_{n=n_0+1}^{\infty}$ such that $z_n \in \overline{U(y_n, r_n)} \setminus U(y_{n_0}, r_{n_0})$ for each $n > n_0$. From the assumptions it is obvious that the sequence $\{y_n\}_{n=1}^{\infty}$ is Cauchy and hence it has a limit $y \in Y$ (as $Y$ is complete). Since $\varrho(y_n, z_n) \le r_n$, it also follows from $r_n \to 0$ that $\lim_{n\to\infty} z_n = y$. Consequently, $y \notin U(y_{n_0}, r_{n_0})$ as $U(y_{n_0}, r_{n_0})$ is open and $z_n \notin U(y_{n_0}, r_{n_0})$ for any $n > n_0$.

On the other hand, the assumptions give that $\{y_n;\ n > n_0\} \subset U(y_{n_0+1}, r_{n_0+1}) \cap Y$ and so $y = \lim_{n\to\infty} y_n \in \overline{U(y_{n_0+1}, r_{n_0+1}) \cap Y} \cap Y \subset U(y_{n_0}, r_{n_0})$. This is a contradiction. □

**Proposition 3.4.5.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be a complete metric space, $A \subset X$ and $\mathbf{P}$ be a porosity-like relation on $X$. Let there exist a $(\mathcal{B}, \mathbf{P})$-Foran-Zajíček scheme $\mathscr{F}$ in $X$, where $\mathcal{B} := \{U(x,r);\ x \in X,\ r \in \mathbb{R}_+\}$, such that each element of $\mathscr{F}$ is a subset of $A$. Assume that $\{X, A, \mathbf{P}\} \subset M$ and $M$ is a pointwise $(\mathbf{P} \leftarrow \mathbf{P}')$-model for some porosity-like relation $\mathbf{P}'$ on $X_M$.*

*Then there exists a $(\mathcal{B}', \mathbf{P}')$-Foran-Zajíček scheme $\mathscr{F}'$ in $X_M$, where $\mathcal{B}' := \{U(x,r) \cap X_M;\ x \in X_M,\ r \in \mathbb{R}_+\}$, such that each element of $\mathscr{F}'$ is a subset of $A \cap X_M$.*

*Proof.* By the assumption, the following formula is true:

$$\begin{gathered}\exists S\ (S : \omega^{<\omega} \to \mathcal{P}(X) \text{ is such that } \{S(t);\ t \in \omega^{<\omega}\} \\ \text{is a } (\mathcal{B}, \mathbf{P})\text{-Foran-Zajíček scheme in } X \text{ and, for every } t \in \omega^{<\omega},\ S(t) \subset A).\end{gathered} \quad (*)$$

Using Lemma 3.2.3 and absoluteness of the preceding formula and its subformula for $M$, we find the corresponding $S \in M$. Consequently, for every $t \in \omega^{<\omega}$, by Proposition 3.2.5 (ii) we have $S(t) \in M$. Now it is sufficient to prove, that

$$\mathscr{F}' := \{S(t) \cap X_M;\ t \in \omega^{<\omega}\}$$

is a $(\mathcal{B}', \mathbf{P}')$-Foran-Zajíček scheme in $X_M$. Fix any $t \in \omega^{<\omega}$. By (i) in Definition 3.3.4 we have that $\bigcup_{j\in\omega} S(t\hat{}j)$ is a dense subset of $S(t)$. Hence, by Lemma 3.2.9 applied to the metric space $S(t)$, $\bigcup_{j\in\omega} S(t\hat{}j) \cap X_M$ is a dense subset of $\overline{S(t) \cap M}$. Using Lemma 3.2.9 again, $S(t) \cap M$ is a dense subset of $S(t) \cap X_M$. Thus, $\bigcup_{j\in\omega} S(t\hat{}j) \cap X_M$ is a dense subset of $S(t) \cap X_M$ and the condition (i) from Definition 3.3.4 is satisfied.

In order to prove (ii) for $\mathscr{F}'$, fix any $t \in \omega^{<\omega}$, any $k \in \omega$, and any $x \in S(t\hat{}k) \cap X_M$. From (ii) valid for $\mathscr{F}$, we know that $(x, S(t)) \notin \mathbf{P}$. Realizing that $S(t)$ is in $M$, the assumption implies that $(x, S(t) \cap X_M) \notin \mathbf{P}'$, that is, $S(t) \cap X_M$ is not $\mathbf{P}'$-porous at $x$.

In order to prove that (iii) holds, let us take some $\nu \in \omega^\omega$, a sequence $(x_n)_{n\in\omega}$ of elements of $X_M$ and a sequence $(r_n)_{n\in\omega}$ of numbers from $\mathbb{R}_+$ such that the open balls $G_n = U(x_n, r_n) \cap X_M$ satisfy conditions (a), (b), and (c) in the space $X_M$. It is easy to see that the radii $r_n$ can be chosen in such a way that $r_n \to 0$. Indeed, if we put $r'_n := \operatorname{diam} G_n + 1/n$, then $G_n = U(x_n, r'_n) \cap X_M$ and $r'_n \to 0$. Then Lemma 3.4.4 gives the existence of an increasing sequence of integers $(n_k)_{k=1}^{\infty}$ such that $\overline{U(x_{n_{k+1}}, r_{n_{k+1}})} \subset U(x_{n_k}, r_{n_k})$ for each $k$. Hence we have that the sequence $(U(x_{n_k}, r_{n_k}))_{k=1}^{\infty}$ satisfies condition (b) from Definition 3.3.4 with $n := k$ and the condition (a) follows from our assumption that $r_n \to 0$. Now we verify the condition (c). From the assumptions on $G_n$ we know that $U(x_{n(k)}, r_{n(k)}) \cap S(\nu\restriction k) \supset U(x_{n(k)}, r_{n(k)}) \cap S(\nu\restriction n(k)) \cap X_M \neq \emptyset$ for every $k \in \omega$

and so (as $\mathscr{F}$ is a $(\mathcal{B}, \mathbf{P})$-Foran-Zajíček scheme in $X$) we have that there exists $x \in \bigcap_{k=1}^{\infty} U(x_{n(k)}, r_{n(k)}) \cap S(\nu \restriction k)$. Since $\lim x_n = x$ by (a) and (b), we have $x \in X_M$. Consequently, $x \in \bigcap_{n=1}^{\infty}(G_n \cap S(\nu \restriction n) \cap X_M)$. This verifies (iii) from Definition 3.3.4. □

**Proposition 3.4.6.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be a complete metric space, $\mathbf{P}$ be a porosity-like relation on $X$, and $\mathbf{P}'$ be a porosity-like relation on $X_M$. Suppose that $\mathbf{P}$ has the property that $N_{\mathbf{P}}(S)$ is a Souslin set whenever $S \subset X$ is Souslin. Assume $A \subset X$ is a Souslin set and $\{X, \mathbf{P}, A\} \subset M$. Then whenever $M$ is a pointwise $(\mathbf{P} \leftrightarrow \mathbf{P}')$-model, then the following holds:*

$$A \text{ is } \mathbf{P}\text{-porous in the space } X \leftrightarrow A \cap X_M \text{ is } \mathbf{P}'\text{-porous in the space } X_M,$$
$$A \text{ is } \sigma\text{-}\mathbf{P}\text{-porous in the space } X \leftrightarrow A \cap X_M \text{ is } \sigma\text{-}\mathbf{P}'\text{-porous in the space } X_M.$$

*Proof.* By Proposition 3.4.3, it is sufficient to prove the implication from the right to the left in the second equivalence. Let us fix a Souslin set $A \subset X$ which is not $\sigma$-$\mathbf{P}$-porous and a $(*)$-elementary submodel $M$ with $\{X, \mathbf{P}, A\} \subset M$. We would like to verify that $A \cap X_M$ is not $\sigma$-$\mathbf{P}'$-porous in the space $X_M$. Let $\mathcal{B}$, $\mathcal{B}'$ be as in Proposition 3.4.5. By Proposition 3.3.11, there exists a $(\mathcal{B}, \mathbf{P})$-Foran-Zajíček scheme $\mathscr{F}$ in $X$ such that each element of $\mathscr{F}$ is a subset of $A$. Using Proposition 3.4.5, there exists a $(\mathcal{B}', \mathbf{P}')$-Foran-Zajíček scheme $\mathscr{F}'$ in $X_M$ such that each element of $\mathscr{F}'$ is a subset of $A \cap X_M$. Hence, by Lemma 3.3.6, $A \cap X_M$ is not $\sigma$-$\mathbf{P}'$-porous in the space $X_M$. □

**Remark 3.4.7.** Let $X$ be a complete metric space and $\mathbf{P}_{up}$ be the porosity-like relation defined by

$$\mathbf{P}_{up} := \{(x, A) \in X \times \mathcal{P}(X);\ A \text{ is upper porous at } x \text{ in } X\}$$

(for the definition of upper porosity, see for example [1]). Let us fix a $(*)$-elementary submodel $M$ with $\{X, \mathbf{P}_{up}\} \subset M$. Denote by $\mathbf{P}'_{up}$ the porosity-like relation defined by

$$\mathbf{P}'_{up} := \{(x, A) \in X_M \times \mathcal{P}(X_M);\ A \text{ is upper porous at } x \text{ in } X_M\}.$$

Then, by results from [1] and [3], $M$ is a pointwise $(\mathbf{P}_{up} \leftrightarrow \mathbf{P}'_{up})$-model. It is easy to see that $N_{\mathbf{P}_{up}}(S)$ is a Souslin set whenever $S \subset X$ is Souslin. Thus, by Proposition 3.4.6, $\sigma$-upper porosity is a separably determined property. This result has already been proved in [3]. However, a nontrivial inscribing theorem ([16, Theorem 3.1]) was needed in the proof there. The method of using Foran-Zajíček scheme in the general setting (see Section 3.3) enables us to avoid the usage of this result.

**Remark 3.4.8.** It is known to the authors that the notions of lower porosity, $\langle g \rangle$-porosity, and $(g)$-porosity satisfy also the assumptions of Proposition 3.4.6 (for definitions see [11]). Consequently, those porosities (and corresponding $\sigma$-porosities) are separably determined when considering Souslin sets in complete metric spaces. We do not present proofs of those results here since at this moment we see no interesting applications of them.

In next section we prove that the notion of $\alpha$-cone porosity in Asplund spaces satisfies the assumptions of Proposition 3.4.6 and, therefore, cone smallness is also separably determined.

**Question.** *Is the notion of $\sigma$-directional porosity (see [14] for the definition) is separably determined in the sense of Corollary 3.5.11?*

Note that the notion of $\sigma$-directional porosity is defined also in [7], but in a slightly different way which is equivalent to the definition from [14] for separable Banach spaces.

## 3.5 Cone porosity

In this section we prove that the notion of $\alpha$-cone porosity in Asplund spaces satisfies the assumptions of Proposition 3.4.6 and, therefore, cone smallness is separably determined.

**Definition 3.5.1.** Let $X$ be a Banach space. For $x^* \in X^* \setminus \{0\}$ and $\alpha \in [0,1)$ we define the *$\alpha$-cone*

$$C(x^*, \alpha) := \{x \in X;\ \alpha\|x\| \cdot \|x^*\| < x^*(x)\}.$$

Given $\alpha \in [0,1)$, a set $A \subset X$ is said to be *$\alpha$-cone porous* at $x \in X$ (in the space $X$) if there exist $R > 0$ such that for each $\varepsilon > 0$ there exists $z \in U(x,\varepsilon)$ and $x^* \in X^* \setminus \{0\}$ such that

$$U(x,R) \cap \big(z + C(x^*,\alpha)\big) \cap A = \emptyset.$$

It is easy to observe that $\alpha$-cone porosity is an example of the porosity-like relation. The corresponding relation is denoted by $\mathbf{P}_X^{\alpha\text{-}cone}$. A set in $X$ is said to be *cone small* if it is $\sigma$-$\mathbf{P}_X^{\alpha\text{-}cone}$-porous for each $\alpha \in (0,1)$. Given $\alpha \in [0,1)$, a set in $X$ is said to be *($\sigma$-)$\alpha$-cone porous* if it is ($\sigma$-)$\mathbf{P}_X^{\alpha\text{-}cone}$-porous.

The following lemma comes from [1, Lemma 4.14].

**Lemma 3.5.2.** *For any suitable elementary submodel $M$ the following holds. Let $(X,\rho)$ be a metric space and $f\colon X \to \mathbb{R}$ be a function. Then whenever $\{X,f\} \subset M$, it is true that for every $R > 0$ and $x \in X_M$ we have*

$$\sup_{u \in U(x,R)} f(u) = \sup_{u \in U(x,R)\cap X_M} f(u).$$

**Proposition 3.5.3.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be a Banach space and $\alpha \in [0,1)$. Then whenever $\{X,\alpha\} \subset M$, $M$ is a pointwise $(\mathbf{P}_X^{\alpha\text{-}cone} \to \mathbf{P}_{X_M}^{\alpha\text{-}cone})$-model.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$ with $\{X,\alpha\} \subset M$ and a set $A \in \mathcal{P}(X) \cap M$. Fix some $x \in X_M$ such that $A$ is $\alpha$-cone porous at $x$. This means that there exists a rational number $R > 0$ such that

$$\forall \varepsilon > 0\ \exists z \in U(x,\varepsilon)\ \exists x^* \in X^* \setminus \{0\}\colon U(x,R) \cap \big(z + C(x^*,\alpha)\big) \cap A = \emptyset.$$

We will show that this formula is true in the space $X_M$ with the constant $\frac{1}{4}R$ instead of $R$. Fix $\varepsilon \in \mathbb{Q}_+$. Put $\delta := \min\{\frac{1}{3}\varepsilon, \frac{1}{4}R\}$ and pick a point $x' \in U(x,\delta) \cap M$. Then it is easy to observe that the following formula is true

$$\exists z' \in U(x', \tfrac{2}{3}\varepsilon)\ \exists x^* \in X^* \setminus \{0\}\colon U(x', \tfrac{1}{2}R) \cap \big(z' + C(x^*,\alpha)\big) \cap A = \emptyset. \qquad (*)$$

(Indeed, it is enough to take a point $z' \in U(x, \frac{1}{3}\varepsilon) \subset U(x', \frac{2}{3}\varepsilon)$ and $x^* \in X^* \setminus \{0\}$ satisfying $U(x,R) \cap \big(z' + C(x^*, \alpha)\big) \cap A = \emptyset$ and to observe that $U(x', \frac{1}{2}R) \subset U(x,R)$.) Using the absoluteness of this formula (and its subformulas) we find $z'' \in U(x', \frac{2}{3}\varepsilon) \cap M \subset U(x, \varepsilon) \cap M$ and $f \in X^* \cap M \setminus \{0\}$ such that

$$U(x', \tfrac{R}{2}) \cap \big(z'' + C(f, \alpha)\big) \cap A = \emptyset. \tag{3.7}$$

By Lemma 3.5.2 we have $\|f\| = \|f\restriction_{X_M}\|$. Hence, the cone $C(f\restriction_{X_M}, \alpha)$ in the space $X_M$ equals to $C(f, \alpha) \cap X_M$. We need to verify that

$$U(x, \tfrac{R}{4}) \cap \big(z'' + C(f, \alpha)\big) \cap A \cap X_M = \emptyset.$$

Fix some $a \in A \cap X_M$ such that $\|x - a\| < \frac{1}{4}R$. Then $a$ is an element of $U(x', \frac{1}{2}R)$. By (3.7) we conclude $a \notin z'' + C(f\restriction_{X_M}, \alpha)$ and the proof is finished. □

In order to show the existence of a pointwise $(\mathbf{P}_X^{\alpha\text{-}cone} \leftarrow \mathbf{P}_{X_M}^{\alpha\text{-}cone})$-models we restrict our attention to Asplund spaces. We recall that a Banach space is *Asplund* if and only if every separable subspace of it has separable dual, see [4]. First, we need to prove that "functionals from a suitable elementary submodel $M$ are dense in $(X_M)^*$ when $X$ is an Asplund space". This seems to be a nontrivial result which might have other uses in separable reduction theorems. The proof of it can be done using the existence of a "projectional generator with domain $X$" in the dual space of an Asplund space $X$. In fact, it is sufficient to use only the first part of the proof of this statement from [5].

**Theorem 3.5.4.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be an Asplund space. Then whenever $X \in M$, it is true that*

$$\overline{\{x^*\restriction_{X_M};\ x^* \in X^* \cap M\}} = (X_M)^*.$$

*Proof.* The inclusion "$\subset$" is obvious. We show the opposite inclusion. It is proved in the second step of the proof of [5, Theorem 1] that there exist continuous mappings $D(n)\colon X \to X^*$, $n \in \mathbb{N}$, such that that, for every closed separable subspace $V \subset X$, we have

$$\overline{\operatorname{span}}\{D(n)(x)\restriction_V;\ n \in \mathbb{N}, x \in V\} = V^*. \tag{3.8}$$

Using the absoluteness of the following formula (and its subformula)

$$\begin{aligned}\exists D\colon \big(&D \text{ is a function, } \operatorname{Dom} D = \mathbb{N},\ D(n) \text{ are norm to norm continuous} \\ &\text{mappings from } X \text{ into } X^* \text{ and for every closed separable subspace} \qquad (*)\\ &V \text{ of } X \text{ we have } \overline{\operatorname{span}}\{D(n)(x)\restriction_V;\ n \in \mathbb{N}, x \in V\} = V^*\big),\end{aligned}$$

find $D \in M$ such that $(*)$ holds; then, by Proposition 3.2.5 (ii), $D(n) \in M$ for every $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$ and $x \in X \cap M$, we have, by Proposition 3.2.5 (ii), $D(n)(x) \in M$. Using the continuity of $D(n)$ for every $n \in \mathbb{N}$, we get

$$\{D(n)(x);\ n \in \mathbb{N}, x \in X_M\} \subset \overline{\{D(n)(x);\ n \in \mathbb{N}, x \in X \cap M\}} \subset \overline{X^* \cap M}.$$

Hence, using (3.8) with $V := X_M$, and the latter inclusion, we have

$$\begin{aligned}(X_M)^* = \overline{\operatorname{span}}\{D(n)(x)\restriction_{X_M};\ n \in \mathbb{N}, x \in X_M\} &\subset \overline{\{x^*\restriction_{X_M};\ x^* \in \overline{X^* \cap M}\}}\\ &\subset \overline{\{x^*\restriction_{X_M};\ x^* \in X^* \cap M\}}.\end{aligned}$$

□

As one can notice, the key part of the proof was the validity of (3.8). Note that there is an alternative way of proving (3.8) based on the argument of Ch. Stegall, see [2]. Now, we need to observe that it is enough to consider functionals from a dense subset of $X^*$ in the definition of $\alpha$-cone porosity.

**Lemma 3.5.5.** *Let $X$ be a Banach space and let $E \subset X$ and $D \subset X^*$ be norm-dense subsets. Let $A \subset X$ and $x \in X$ and $\alpha \in [0,1)$. Then $A$ is $\alpha$-cone porous at $x$, (if and) only if the following is true:*

$$\exists R \in \mathbb{Q}_+ \ \forall \varepsilon \in \mathbb{Q}_+ \ \exists y^* \in D \ \exists w \in U(x,\varepsilon) \cap E \colon U(x,R) \cap (w + C(y^*,\alpha)) \cap A = \emptyset.$$

*Proof.* The sufficiency of our condition is very easy to see. Let us, therefore, assume that a given set $A$ is $\alpha$-cone porous at a given point $x \in X$, and deduce from it the desired condition.

Since $A$ is $\alpha$-cone porous at $x$, it is easy to see that there exists $R \in \mathbb{Q}_+$ such that

$$\forall \varepsilon > 0 \ \exists x^* \in X \ \exists z \in U(x,\varepsilon) \colon U(x,R) \cap \big(z + C(x^*,\alpha)\big) \cap A = \emptyset. \tag{3.9}$$

Let $\varepsilon \in \mathbb{Q}_+$. Using (3.9) we find $x^* \in X^*$ and $z \in B(x, \min\{\varepsilon, R\})$ such that

$$U(x,R) \cap (z + C(x^*,\alpha)) \cap A = \emptyset.$$

Choose $w \in U(x, \min\{\varepsilon, R\}) \cap (z + C(x^*,\alpha)) \cap E$. Since $w - z \in C(x^*,\alpha)$, we have

$$x^*(w-z) - \alpha \|x^*\| \|w - z\| > 0.$$

Using the last inequality and the density of $D$ we find $y^* \in D$ such that

(a) $\|y^*\| \geq \|x^*\|$,

(b) $\|x^* - y^*\| < \frac{1}{2R}\big(x^*(w-z) - \alpha\|x^*\|\|w-z\|\big)$.

Now it is sufficient to prove that

$$U(x,R) \cap (w + C(y^*,\alpha)) \subset z + C(x^*,\alpha). \tag{3.10}$$

Indeed, since then we have

$$U(x,R) \cap (w + C(y^*,\alpha)) \cap A \subset U(x,R) \cap (z + C(x^*,\alpha)) \cap A = \emptyset.$$

To verify (3.10) take $u \in C(y^*,\alpha)$ with $w + u \in U(x,R)$. Then we have

$$\|u\| \leq \|u + w - x\| + \|x - w\| \leq 2R. \tag{3.11}$$

We compute

$$\begin{aligned} x^*(w+u-z) &= x^*(w-z) + y^*(u) + (x^* - y^*)(u) \\ &\geq x^*(w-z) + \alpha\|x^*\| \cdot \|u\| - \|x^* - y^*\| \cdot 2R \qquad \text{(by(3.11))} \\ &\geq \alpha\|x^*\| \cdot \|u\| + \alpha\|x^*\| \cdot \|w - z\| \qquad \text{(by (b))} \\ &\geq \alpha\|x^*\| \cdot \|u + w - z\|. \end{aligned}$$

This shows that $w + u - z \in C(x^*,\alpha)$. Consequently, we get $w + u \in z + C(x^*,\alpha)$. □

Now, we are ready to see the existence of a pointwise $(\mathbf{P}_X^{\alpha\text{-}cone} \leftrightarrow \mathbf{P}_{X_M}^{\alpha\text{-}cone})$-models in Asplund spaces.

**Proposition 3.5.6.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be an Asplund space and $\alpha \in [0,1) \cap \mathbb{Q}$. Then whenever $X \in M$, $M$ is a pointwise $(\mathbf{P}_X^{\alpha\text{-}cone} \leftrightarrow \mathbf{P}_{X_M}^{\alpha\text{-}cone})$-model.*

*Proof.* Let us fix a $(*)$-elementary submodel $M$ with $X \in M$ and a set $A \in \mathcal{P}(X) \cap M$. By Proposition 3.5.3, $M$ is a pointwise $(\mathbf{P}_X^{\alpha\text{-}cone} \to \mathbf{P}_{X_M}^{\alpha\text{-}cone})$-model. Fix some $x \in X_M$ such that $A$ is not $\alpha$-cone porous at $x$. We will show that $A \cap X_M$ is not $\alpha$-cone porous at $x$ in the space $X_M$. Notice that, by Lemma 3.5.2, $\|x^* \restriction_{X_M}\| = \|x^*\|$ for every $x^* \in X^* \cap M$. Hence, the cone $C(x^* \restriction_{X_M}, \alpha)$ in the space $X_M$ equals $C(x^*, \alpha) \cap X_M$. Thus, by Lemma 3.5.5 and Theorem 3.5.4, it is sufficient to prove that the following formula is true

$$\forall R \in \mathbb{Q}_+ \ \exists \varepsilon \in \mathbb{Q}_+ \ \forall z \in U(x,\varepsilon) \cap M \ \forall x^* \in (X^* \cap M) \setminus \{0\}\colon \\ U(x,R) \cap X_M \cap \big(z + C(x^*,\alpha)\big) \cap A \neq \emptyset.$$

Fix $R \in \mathbb{Q}_+$. As $A$ is not $\alpha$-cone porous at $x$, there exists $\varepsilon \in \mathbb{Q}_+$ such that

$$\forall z \in U(x,\varepsilon) \ \forall x^* \in X^* \setminus \{0\}\colon U(x, \tfrac{1}{3}R) \cap \big(z + C(x^*,\alpha)\big) \cap A \neq \emptyset. \tag{3.12}$$

Let us fix $z \in U(x,\varepsilon) \cap M$ and $x^* \in (X^* \cap M) \setminus \{0\}$. Find some $x' \in U(x, \frac{1}{3}R) \cap M$. Then $U(x, \frac{1}{3}R) \subset U(x', \frac{2}{3}R)$. By (3.12), the following formula is true

$$\exists a \in A\colon a \in \big(z + C(x^*,\alpha)\big) \cap U(x', \tfrac{2}{3}R). \tag{$*$}$$

Using the absoluteness of the formula (and its subformula) above, there exists $a \in A \cap M$ satisfying the formula above. Then $a \in U(x,R) \cap \big(z + C(x^*,\alpha)\big)$. Hence,

$$U(x,R) \cap X_M \cap \big(z + C(x^*,\alpha)\big) \cap A \neq \emptyset.$$

Thus, $A \cap X_M$ is not $\alpha$-cone porous at $x$ in the space $X_M$. This finishes the proof. □

In the remainder of the section we prove that the assumption on descriptive quality of $N_{\mathbf{P}}(S)$ from Proposition 3.4.6 is satisfied for the cone porosity. We begin with the following lemma.

**Lemma 3.5.7.** *Let $X$ be a Banach space, $x^* \in X^*$, $\alpha \in [0,1)$ and take $x \in C(x^*,\alpha)$. Then $\mathrm{d}(X \setminus C(x^*,\alpha), x + C(x^*,\alpha)) > 0$.*

*Proof.* It is easy to verify that $C(x^*,\alpha)$ is an open set and that it is a convex cone in the sense that for any two points $y, z$ from $C(x^*,\alpha)$ and any $c > 0$ the points $cy$ and $y + z$ also belong to $C(x^*,\alpha)$. Set $\delta := \mathrm{d}(x, X \setminus C(x^*,\alpha))$. The number $\delta$ is positive, since $C(x^*,\alpha)$ is open. Take any point $y \in x + C(x^*,\alpha)$ (then $y - x \in C(x^*,\alpha)$). Hence, $U(x,\delta) \subset C(x^*,\alpha)$, and so $U(y,\delta) = (y-x) + U(x,\delta) \subset C(x^*,\alpha)$. Since $y \in x + C(x^*,\alpha)$ was chosen arbitrarily, we conclude that $\mathrm{d}(X \setminus C(x^*,\alpha), x + C(x^*,\alpha)) \geq \delta > 0$. □

**Proposition 3.5.8.** *Let $X$ be a Banach space, $\alpha \in [0,1)$, and $A \subset X$ be any set. Then the set $S$ of all points $x \in X$ at which $A$ is $\alpha$-cone porous is of the type $G_{\delta\sigma}$.*

*Proof.* For $x, z \in X$, $R > 0$, $x^* \in X^* \setminus \{0\}$, and $\alpha \in [0,1)$ we set

$$T(x, R, z, x^*, \alpha) := U(x, R) \cap \big(z + C(x^*, \alpha)\big).$$

First we show that

$$S = \bigcup_{R \in \mathbb{Q}_+} \bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcup_{x^* \in X^* \setminus \{0\}} G(R, \varepsilon, x^*), \tag{3.13}$$

where

$$G(R, \varepsilon, x^*) := \{x \in X;\ \exists z \in U(x, \varepsilon)\colon\ \mathrm{d}(T(x, R, z, x^*, \alpha), A) > 0\}.$$

It is easy to see that the inclusion "$\supset$" holds. To prove "$\subset$" consider $x \in S$. Then we can find $R' > 0$ such that for every $\varepsilon > 0$ there are $z' \in U(x, \varepsilon)$ and $x^* \in X^* \setminus \{0\}$ such that $T(x, R', z', x^*, \alpha) \cap A = \emptyset$. Fix $R \in (0, R') \cap \mathbb{Q}$. Take any $\varepsilon \in \mathbb{Q}_+$. Then we find $z' \in U(x, \varepsilon)$ and $x^* \in X^* \setminus \{0\}$ such that $T(x, R, z', x^*, \alpha) \cap A = \emptyset$. Pick some $z \in \big(z' + C(x^*, \alpha)\big) \cap U(x, \varepsilon)$. Then we have $\mathrm{d}(T(x, R, z, x^*, \alpha), X \setminus U(x, R')) \geq R' - R > 0$ and by Lemma 3.5.7 we have that

$$\mathrm{d}\big(T(x, R, z, x^*, \alpha), X \setminus (z' + C(x^*, \alpha))\big) > 0.$$

Since

$$A \subset \big(X \setminus U(x, R')\big) \cup \big(X \setminus (z' + C(x^*, \alpha)\big),$$

we get $x \in G(R, \varepsilon, x^*)$ and the equality (3.13) is proved.

Now it is sufficient to show that the set $G(R, \varepsilon, x^*)$ is open. To this end fix $R > 0$, $\varepsilon > 0$, $x^* \in X^* \setminus \{0\}$ and consider $x \in G(R, \varepsilon, x^*)$. There exists $z \in U(x, \varepsilon)$ with $\mathrm{d}(T(x, R, z, x^*, \alpha), A) =: 2\eta > 0$. Fix any $x' \in U(x, \eta)$. We have

$$T(x', R, z + x' - x, x^*, \alpha) = (x' - x) + T(x, R, z, x^*, \alpha).$$

This gives $\mathrm{d}(T(x', R, z + x' - x, x^*, \alpha), A) \geq \eta > 0$. Since $z + x' - x \in U(x', \varepsilon)$ we have $x' \in G(R, \varepsilon, x^*)$. This implies $U(x, \eta) \subset G(R, \varepsilon, x^*)$ and we are done. ☐

**Corollary 3.5.9.** *Let $X$ be a Banach space, $\alpha \in [0,1)$, and $A \subset X$ be a Souslin set. Then the set $N_{\mathbf{P}_X^{\alpha\text{-}cone}}(A)$ is Souslin.*

**Theorem 3.5.10.** *For any suitable elementary submodel $M$ the following holds. Let $X$ be an Asplund space, $A \subset X$ be Souslin, and $\alpha \in [0,1) \cap \mathbb{Q}$. Then whenever $\{X, A\} \subset M$, the following are true:*

$$A \text{ is } \alpha\text{-cone porous in } X \leftrightarrow A \cap X_M \text{ is } \alpha\text{-cone porous in } X_M,$$
$$A \text{ is } \sigma\text{-}\alpha\text{-cone porous in } X \leftrightarrow A \cap X_M \text{ is } \sigma\text{-}\alpha\text{-cone porous in } X_M,$$
$$A \text{ is cone small in } X \leftrightarrow A \cap X_M \text{ is cone small in } X_M.$$

*Proof.* Let us fix a $(*)$-elementary submodel $M$ with $\{X, A\} \subset M$. Then the following formula is clearly true

$$\begin{gathered}\exists \mathbf{R} \text{ point-set relation on } X\ \forall x \in X\ \forall B \subset X\colon \\ (B \text{ is } \alpha\text{-cone porous at } x \leftrightarrow (x, B) \in \mathbf{R}).\end{gathered} \tag{$*$}$$

The absoluteness of this formula and its subformula implies that $\mathbf{P}_X^{\alpha\text{-}cone} \in M$. The first two parts of the theorem now follow using Propositions 3.4.6 and 3.5.6 and Corollary 3.5.9. The third equivalence follows from the second one via the definition of the cone smallness. ☐

Note that no one implication in the Theorem above is trivial. However, one can say that it is much harder to prove the implications "$\Leftarrow$". The reason is that in the proof we use the complicated notion of a Foran-Zajíček scheme from Section 3.3.

**Corollary 3.5.11.** *Let $X$ be an Asplund space and $A \subset X$ be a Souslin set. Then for every separable space $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and*

$$A \text{ is cone small in } X \leftrightarrow A \cap V \text{ is cone small in } V.$$

## 3.6 Applications

**Definition 3.6.1** ([8])**.** Let $X$ be a real Banach space, $G \subset X$ be open. A function $f\colon G \to \mathbb{R}$ is called *approximately convex at* $x_0 \in G$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon\lambda(1-\lambda)\|x-y\|$$

whenever $\lambda \in [0,1]$ and $x, y \in U(x_0, \delta)$. We say that $f$ is *approximately convex on* $G$ if it is approximately convex at each $x_0 \in G$.

**Remark 3.6.2.** The class of approximately convex functions includes semiconvex functions and strongly paraconvex functions (for definitions see, e.g., [15]).

We will apply our result about cone small sets to prove the following generalization of [15, Theorem 5.5] to nonseparable Asplund spaces. Note that the following theorem is also a strengthening of [15, Theorem 5.9] which states that a continuous approximately convex function on an Asplund space is Fréchet differentiable except for the points from the union of a cone small set and a $\sigma$-cone supported set. Note also that, unlike [15, Theorem 5.5], our Theorem 3.6.3 states that the exceptional set is cone small and not angle small. However, it is easy to prove that these two notions are equivalent if $X$ is separable.

**Theorem 3.6.3.** *Let $X$ be an Asplund space and $G \subset X$ be open. Let $f\colon G \to \mathbb{R}$ be a continuous and approximately convex function. Then the set of all points of $G$ at which $f$ is not Fréchet differentiable is cone small.*

To prove the theorem we will need several notions and a lemma. The notion of LAN mapping is defined and studied in [12].

**Definition 3.6.4.** Let $X$ be a Banach space and $G \subset X$ be open. We say a (singlevalued) mapping $g\colon G \to X^*$ is *LAN* (*locally almost nonincreasing*) if for any $a \in G$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in U(a, \delta)$ we have

$$(g(x_1) - g(x_2))(x_1 - x_2) \leq \varepsilon\|x_1 - x_2\|.$$

We say a multivalued mapping $T\colon G \to X^*$ is *submonotone on* $G$ if for any $a \in G$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in U(a, \delta)$, $x_1^* \in T(x_1)$, and $x_2^* \in T(x_2)$ we have

$$(x_1^* - x_2^*)(x_1 - x_2) \geq -\varepsilon\|x_1 - x_2\|.$$

Observe that $T$ is LAN if and only if $-T$ is singlevalued and submonotone.

The following lemma generalizes [12, Lemma 3] to general Asplund spaces.

**Lemma 3.6.5.** *Let $X$ be Asplund, $G \subset X$ be open and $g\colon G \to X^*$ be LAN. Then $g$ is continuous at all points of $G$ except those which belong to a cone small set.*

*Proof.* Denote by $A$ the set of all points of $G$ at which $g$ is not continuous (then $A$ is $F_\sigma$) and let us fix a $(*)$-elementary submodel $M$ with $\{X, g\} \subset M$. Then $X_M$ is a Banach space with separable dual and $g\restriction_{X_M}$ is clearly LAN. Denote by $B$ the set of all points of $G \cap X_M$ (the intersection is nonempty) at which $g\restriction_{X_M}$ is not continuous. By [12, Lemma 3], $B$ is angle small in $X_M$. But [1, Theorem 5.2] gives that $B = A \cap X_M$ and that $A \in M$. Hence, by Theorem 3.5.10, $A$ is cone small. □

*Proof of Theorem 3.6.3.* By [15, Lemma 2.5 (ii) and (iii)] the multivalued mapping

$$x \mapsto \partial^F f(x) := \left\{x^* \in X^*;\ \liminf_{h\to 0} \frac{f(x+h) - f(x) - x^*(h)}{\|h\|} \geq 0\right\}$$

is submonotone on $G$. Choose any selection $g$ of $\partial^F f$ on $G$; then $g$ is also submonotone. Lemma 3.6.5 implies that $g$ is continuous on $G$ up to a cone small set. Now, [15, Lemma 5.4] says that $f$ is Fréchet differentiable at points of continuity of $g$, concluding the proof. □

Another possible application of Theorem 3.5.10 is the following strengthening of [10, Proposition 4.2] (for definitions see [10]).

**Proposition 3.6.6.** *Let $Y$ be a countably Daniell ordered Banach space with the Radon-Nikodým property. Assume that*

*(a) either $X$ is a closed subspace of $c_0(\Delta)$, where $\Delta$ is an uncountable set,*

*(b) or $X = C(K)$, where $K$ is scattered compact topological space.*

*Let $A \subset X$ be an open convex set and $f\colon A \to Y$ be a continuous convex operator. Then $f$ is Fréchet differentiable on $A$ except for a cone small $\Gamma$-null set.*

The only difference from the original assertion is that, instead of $\sigma$-lower porous, we have the exceptional set cone small which is a stronger assertion. We will, however, omit the proof, as there is no difference from the proof in [10]; one just needs to use our Theorem 3.5.10 instead of [3, Theorem 5.4] which is an analogue of 3.5.10 for $\sigma$-lower porosity.

Note that we also obtain an analogue of [10, Proposition CR] for cone smallness. Since this could be of some independent interest, we formulate it below (see also [3, Theorem 1.2]).

**Proposition 3.6.7.** *Let $X, Y$ be Banach spaces, $G \subset X$ be an open set, and $f\colon G \to Y$ an arbitrary mapping. Then for every separable space $V_0 \subset X$ there exists a closed separable space $V \subset X$ such that $V_0 \subset V$ and that the following are equivalent:*

*(i) the set of all points where $f$ is not Fréchet differentiable is cone small in $X$,*

*(ii) the set of all points where $f\restriction_{V\cap G}$ is not Fréchet differentiable is cone small in $V$.*

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# Chapter 4.
# On Removable Sets For Convex Functions

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**Abstract**

In the present article we provide a sufficient condition for a closed set $F \in \mathbb{R}^d$ to have the following property which we call $c$-removability: Whenever a function $f : \mathbb{R}^d \to \mathbb{R}$ is locally convex on the complement of $F$, it is convex on the whole $\mathbb{R}^d$. We also prove that no generalized rectangle of positive Lebesgue measure in $\mathbb{R}^2$ is $c$-removable. Our results also answer the following question asked in an article by Jacek Tabor and Józef Tabor [*J. Math. Anal. Appl.* **365** (2010)]: Assume the closed set $F \subset \mathbb{R}^d$ is such that any locally convex function defined on $\mathbb{R}^d \setminus F$ has a unique convex extension on $\mathbb{R}^d$. Is $F$ necessarily intervally thin (a notion of smallness of sets defined by their "essential transparency" in every direction)? We prove the answer is negative by finding a counterexample in $\mathbb{R}^2$.

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## 4.1 Introduction

The present article is mostly motivated by the work [5] about negligible sets for convexity of functions in $\mathbb{R}^d$, where an interesting open problem was raised. We shall need the following notion introduced in [5].

A set $A \subset \mathbb{R}^d$ is called *intervally thin* if for any $x, y \in \mathbb{R}^d$ and any $\varepsilon > 0$ there exist $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$ such that $[x', y'] \cap A = \emptyset$.

**Problem TT.** Let $A \subset \mathbb{R}^n$ be closed. Suppose that for an arbitrary open set $U$ containing $A$ every locally convex function $f : U \setminus A \to \mathbb{R}$ has a unique locally convex extension on $U$. Is it then necessarily true that $A$ is intervally thin?

Arguably our main result is that the answer to this question is negative. Example 4.4.3 and Remark 4.4.4 provide a closed set $K$ which is not intervally thin, but which enjoys the "unique extension property for convex functions" (UEP) from Problem TT. We took the liberty of calling this set $K$ "the Holey Devil's Staircase" since it is the graph of the classical Cantor function (the Devil's Staircase) minus all the horizontal open line segments contained in the graph (in other words, it is the graph of the restriction of the Cantor function to the Cantor set).

One can readily verify that the Holey Devil's Staircase is not intervally thin. It is enough to consider the last intersection of the graph of the Cantor function with any line segment with endpoints in $(-\infty, 0) \times \left(0, \frac{1}{2}\right)$ and $(1, \infty) \times \left(\frac{1}{2}, 1\right)$; clearly, this intersection is an element of $K$.

To prove that $K$ has the UEP, is considerably more difficult and our effort in this direction has inspired a large part of this article.

The main result of [5] is essentially the following theorem. Note that since we restrict our attention to convex functions (as opposed to $\omega$-semiconvex functions studied in [5]), we change the formulation of the theorem accordingly:

**Theorem TT.** *Let $U$ be an open subset of $\mathbb{R}^d$ and let $A$ be a closed intervally thin subset of $U$. Let $f : U \setminus A \to \mathbb{R}$ be a locally convex function. Then $f$ has a unique locally convex extension on $U$.*

The proof of this theorem consists of two principal steps:

(1) First, one proves that there is a unique continuous extension; this is the more difficult part.

(2) Once one has the continuous extension, it is then easy to prove that it is locally convex.

Our aim is to apply this scheme to our set $K$. It turns out that in this case the easier step is (1); we only need a simple generalization of the corresponding theorem from [5]—which we have in Lemma 4.4.2.

Performing step (2) for $K$ is the crucial part and it motivates the introduction of $c$-removable sets with the consequent natural question: Which sets are $c$-removable?

**Definition 4.1.1.** We say that a closed set $A \subset \mathbb{R}^d$ is *$c$-removable* if the following is true: Every real function $f$ on $\mathbb{R}^d$ is convex whenever it is continuous on $\mathbb{R}^d$ and locally convex on $\mathbb{R}^d \setminus A$.

A consequence of Theorem TT is that all closed intervally thin sets are $c$-removable, but this fact does not help us. In $\mathbb{R}^2$ we were able to find a sufficient condition more general than interval thinness which covers also the case of our set $K$:

**Proposition 1.** *Let $K \subset \mathbb{R}^2$ be compact and intervally thin in two different directions. Assume that for a dense set of line segments $L \subset \mathbb{R}^2$ the cardinality of $K \cap L$ is at most countable. Then $K$ is c-removable.*

Here interval thinness of $K$ in a direction means that to any given line segment in that direction we can find arbitrarily close line segments contained in the complement of the set $K$. It is not difficult to see that for any closed set $K$ intervally thin in a direction $v$, any continuous function which is locally convex outside $K$ is necessarily convex on all lines parallel to $v$. Hence, the assumption of interval thinness of $K$ in two directions ensures that our function is convex in those two directions (i.e. is essentially separately convex) which we can use further in the proof—the key Lemma 4.3.1 tells us that a separately convex function cannot "have a concave angle" on any line.

The condition from the proposition may seem rather artificial, but it emerges quite naturally from our method of the proof. What is more, it is easily seen to be more general than interval thinness and is fulfilled by $K$. (Hence, the Holey Devil's Staircase is $c$-removable.) However, we were not able to generalize this condition to higher dimensions; instead, we used the geometric measure theory to obtain the following theorem which in $\mathbb{R}^2$ is strictly weaker than Proposition 1.

**Theorem 1.** *Let $M \subset \mathbb{R}^d$ be a compact set which is intervally thin in d linearly independent directions $n_1, \ldots, n_d$. Suppose that $M$ has $\sigma$-finite $(d-1)$-dimensional Hausdorff measure. Then $M$ is c-removable.*

This condition does not include interval thinness because there are intervally thin sets of positive $d$-dimensional measure in $\mathbb{R}^d$. For instance, in $\mathbb{R}^2$ one can construct such a set by taking the full unit square and digging in it countably many straight tunnels in such a way that the rest is intervally thin but still of positive measure.

Among other signs, also from this fact it seems rather obvious that this theorem is far from being a characterization of $c$-removable sets. In fact, it is not even clear whether all $c$-removable sets in $\mathbb{R}^2$ are totally disconnected; from the considerations contained in the second part of Section 5 it seems plausible that the Koch curve might be an example of a non-trivial $c$-removable continuum in $\mathbb{R}^2$. (Of course, such an example has to be rather complicated as it is not difficult to prove that no smooth curve in $\mathbb{R}^2$ is $c$-removable.)

On the other hand, we have the following.

**Theorem 2.** *Let $A, B \subseteq \mathbb{R}$ be closed sets of positive Lebesgue measure. Then $A \times B$ is not c-removable.*

This theorem is interesting only for $A$, $B$ totally disconnected (otherwise $A \times B$ contains a non-degenerated line segment and the statement is trivial, as explained in the proof). However, we do not know (and would like to know) whether e.g. the Cantor dust ($C \times C$ where $C$ is the Cantor set) is $c$-removable. As a matter of fact, possibly the most interesting of related open problems is:

**Problem.** Is there a closed totally disconnected Lebesgue null set in $\mathbb{R}^2$ which is not $c$-removable?

It is worth pointing out that Theorem 2 is related to the recent work [4] where a totally disconnected compact subset of $\mathbb{R}^2$ which is not $c$-removable is constructed. The construction is rather complicated, but the witnessing function has a compact support, making the example stronger. However, even Theorem 2 is enough to achieve the main goal of [4], which is to disprove a theorem by L. Pasqualini from 1938 [3, Theorem 51] stating that any totally disconnected compact set in $\mathbb{R}^2$ is $c$-removable. It was the connection to this old article what convinced us that, of the two steps involved in the proof of Theorem TT, the crucial one is actually the second.

## 4.2 Notation and basic facts

All spaces shall be equipped with the Euclidean metric. We denote by $B(x, \varepsilon)$ the open ball (with respect to the Euclidean metric) with the centre $x$ and radius $\varepsilon$. For a set $M \subset \mathbb{R}^d$, by $M^c$ we mean the complement of $M$ in $\mathbb{R}^d$. Since confusion is unlikely, the symbol $(x, y)$ denotes an open interval in $\mathbb{R}$ as well as the point in $\mathbb{R}^2$ with coordinates $x$ and $y$. The symbol $[x, y]$ denotes the line segment with endpoints $x$ and $y$ (when $x, y \in \mathbb{R}^d$, $d \geq 1$). By $\mathcal{H}^k$ we denote the $k$-dimensional Hausdorff measure. For $M \subset \mathbb{R}^d$ and $\alpha$ a countable ordinal we denote the $\alpha$-th Cantor-Bendixson derivative of $M$ by $M^{(\alpha)}$. The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1}$. For $v \in \mathbb{R}^d$ we denote the orthogonal complement of $v$ by $v^\perp$. The symbol $\operatorname{Lin} M$ denotes the linear span of $M \subset \mathbb{R}^d$. For a fixed $d \in \mathbb{N}$ denote the standard basis of $\mathbb{R}^d$ by $\{e_1, \ldots, e_d\}$.

Let $U \subset \mathbb{R}^d$ ($d \geq 1$) be open and $f : U \to \mathbb{R}$ be a function. We say that $f$ is *locally convex* on $U$ if for every $x \in U$ there is some $V \subset U$, an open convex neighbourhood of $x$, such that the function $f|_V$ is convex. It is easy to see that a locally convex function is convex on any convex set contained in its domain.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *separately convex*, if it is convex on all lines parallel to the coordinate axes.

The set $A \subset \mathbb{R}^d$ is called *$k$-rectifiable* if there exist countably many Lipschitz mappings $f_i : \mathbb{R}^k \to \mathbb{R}^d$ such that

$$\mathcal{H}^k\left(A \setminus \bigcup_{i=0}^{\infty} f_i\left(\mathbb{R}^k\right)\right) = 0.$$

Since we will work only with the case $k = d - 1$, we will call $(d-1)$-rectifiable sets just *rectifiable*.

Let $G(d, k)$ be the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the unique invariant probability measure $\nu_k^d$. Besides the Hausdorff measure we will also use the $k$-dimensional Favard measure (integralgeometric measure) $\mathcal{I}^k$ on $\mathbb{R}^d$ which is for a Borel set $M$ defined as

$$\mathcal{I}^k(M) = \frac{1}{\beta(d,k)} \int_{G(d,k)} \int_V \mathcal{H}^0(M \cap p_V^{-1}(y))\, d\mathcal{H}^k(y)\, d\nu_k^d(V),$$

where $p_V$ is the orthogonal projection to $V$ and the number $\beta(d, k)$ is a non-zero constant depending only on $d$ and $k$ whose precise value is not important for us.

We will also need the following properties of the Favard measures. Let $M \subset \mathbb{R}^d$ be a Borel set such that $\mathcal{H}^{d-1}(M) < \infty$. Then $M$ can be expressed as a union of a rectifiable set $R$ and a set $P$ satisfying $\mathcal{I}^{d-1}(P) = 0$ (c.f. [1, 3.3.13]). Moreover, each rectifiable set $R \subset \mathbb{R}^d$ satisfies $\mathcal{I}^{d-1}(R) = \mathcal{H}^{d-1}(R)$ (c.f. [1, 3.2.26]).

## 4.3 Separately convex functions

The following lemma is a variant of an unpublished observation by V. Šverák (see [6]). For the convenience of the reader we provide a proof as we were not able to find one in the literature.

**Lemma 4.3.1.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately convex function. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(t,t)$. Then*

$$\liminf_{t\to 0+} \frac{g(x+t) + g(x-t) - 2g(x)}{t} \geq 0 \tag{4.1}$$

*for every $x$.*

*Proof.* Aiming for a contradiction, assume that there is a separately convex function $f$ on $\mathbb{R}^2$ and $x \in \mathbb{R}$ such that inequality (4.1) with $g(t) = f(t,t)$ does not hold. There is no loss of generality in assuming that $x = 0$ and $g(0) = 0$, which in turn implies that there is a constant $c$ such that

$$\liminf_{t\to 0+} \frac{g(t) + g(-t)}{t} < c < 0.$$

We can suppose (possibly by multiplying $f$ by $\frac{1}{|c|}$) that $c = -1$. Hence, there is a sequence $t_n \searrow 0$ such that for each $n \in \mathbb{N}$,

$$\frac{g(t_n) + g(-t_n)}{t_n} \leq -1. \tag{4.2}$$

For $t > 0$ put

$$\sigma(t) := f(t,-t) + f(-t,t) \quad \text{and} \quad \rho(t) := f(t,t) + f(-t,-t).$$

Note that, since $f$ is separately convex,

$$\sigma(t) + \rho(t) \geq 0 \tag{4.3}$$

for every $t$. Now we shall prove the following:
**Claim:** *If for some $p$, $\sigma(t_n) \geq pt_n$, for all $n$, then $\sigma(t_n) \geq (p+2)t_n$ for all $n$.*

This is enough to prove the lemma since (4.3) and (4.2), together with the above claim, imply $\sigma(t_n) \geq L$ for every $n$ and every $L \in \mathbb{R}$, which is not possible.

To prove the claim, first observe that due to (4.2) we know that for each $n$,

$$\rho(t_n) \leq -t_n.$$

This implies for each $n$,

$$\frac{\sigma(t_n) - \rho(t_n)}{2t_n} \geq \frac{p+1}{2}.$$

By separate convexity of $f$, we get that

$$f(t_k, t_n) + f(-t_k, -t_n) \geq \rho(t_k) + (t_n - t_k) \cdot \frac{\sigma(t_k) - \rho(t_k)}{2t_k}$$

provided $k > n$. If we consider $k \to \infty$, we obtain

$$f(0, t_n) + f(0, -t_n) \geq t_n \frac{p+1}{2}.$$

Using the separate convexity of $f$ one more time together with (4.2) we get

$$\sigma(t_n) \geq f(0, t_n) + f(0, -t_n) + (f(0, t_n) + f(0, -t_n) - \rho(t_n)) \geq (p+2)t_n. \quad \square$$

**Definition 4.3.2.** The set $A \subset \mathbb{R}^d$ is called *intervally thin in direction* $v \in S^{d-1}$ if for any $x, y \in \mathbb{R}^d$ with $x - y$ parallel to $v$, and any $\varepsilon > 0$, there exist $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$ such that $[x', y'] \cap A = \emptyset$.

*Proof of Theorem 1.* First we need to prove the following claim:
**Claim:** *Let $M \subset \mathbb{R}^d$ be as in the statement of Theorem 1. For any non-convex continuous function $f : \mathbb{R}^d \to \mathbb{R}$, there is a line $L \subset \mathbb{R}^d$ such that $L \cap M$ is (at most) countable and $f|_L$ is non-convex. Moreover, $L$ can be found such that its direction $v \in S^{d-1}$ is not a linear combination of any $d-1$ vectors from $n_1, \ldots, n_d$.*

To prove the Claim, express $M$ as a countable union of sets $M_n$ satisfying $\mathcal{H}^{d-1}(M_n) < \infty$. By [1, 3.3.13] we can express each $M_n$ in the form $P_n \cup R_n$ with $\mathcal{I}^{d-1}(P_n) = 0$ and $R_n$ rectifiable.

Fix $n \in \mathbb{N}$. Using [1, 3.2.26] we see that

$$\mathcal{I}^{d-1}(R_n) = \mathcal{H}^{d-1}(R_n) < \infty.$$

This means, by the definition of the Favard measure, that for almost every $H \in G(d, d-1)$, almost every line perpendicular to $H$ intersects $R_n$ in at most finitely many points. In particular, almost every line intersects $R_n$ in finitely many points.

So, putting $P = \bigcup P_n$ and $R = \bigcup R_n$, we have that $\mathcal{I}^{d-1}(P) = 0$, and also that almost every line intersects $R$ in at most countably many points. Hence, almost every line intersects $M = P \cup R$ in at most countably many points.

Since $f$ is non-convex, the set $\mathcal{A}$ consisting of all lines $L$ such that $f|_L$ is non-convex has a positive measure. To finish the proof of the Claim we simply pick a line from $\mathcal{A}$ such that $L \cap M$ is at most countable, and it is obvious we can do so in such a way that the "moreover" part of the Claim holds as well.

Having proved the Claim, we shall now prove the theorem by induction in $d$. Assume first that $d = 2$; we shall proceed by contradiction. To that end, let there be given a set $M \subset \mathbb{R}^2$ as in the statement of the theorem and a non-convex continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is locally convex on $M^c$. Let us fix a line $L$ which the Claim gives us for $M$ and $f$, and let us fix a point $z \in L$.

Due to the last part of the Claim, we can suppose (possibly by composing $f$ with a suitable affine mapping) that $n_i = e_i$ $(i = 1, 2)$, $v = \frac{1}{\sqrt{2}}(1, 1)$ ($v$ is the direction of $L$ from the Claim) and that $z = (0, 0)$. In particular, these assumptions imply that $f$ is separately convex.

Put $K := M \cap L$; then $K$ is a countable compact. By Lemma 4.3.1 we know that if $f|_L$ is locally convex on $L \setminus N$ for some closed set $N \subset L$, then it is convex on a neighbourhood of any isolated point of $N$. Using this observation, one can readily prove by induction that $f|_L$ is locally convex on $L \setminus K^{(\alpha)}$ for every countable ordinal $\alpha$. But, since $K$ is a countable compact, there is a countable ordinal $\beta$ such that $K^{(\beta)} = \emptyset$ which is a contradiction with the assumption of $f|_L$ being non-convex. This finishes the proof for $d = 2$.

Suppose now that the theorem is true for every $d$ up to $k - 1 \geq 2$; we will prove that it is true for $d = k$ as well, again by contradiction. Again, let us have $M \subset \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ with $M$ as in the statement of the theorem and $f$ a non-convex continuous function on $\mathbb{R}^d$ which is locally convex on $M^c$. As before, we fix a corresponding line $L$ in direction $v \in S^{d-1}$ from the Claim, we fix a point $z \in L$, and assume without loss of generality that $n_i = e_i$ $(i = 1, \ldots, d)$, $v = \frac{1}{\sqrt{d}}(1, \ldots, 1) \in \mathbb{R}^d$ and that $z = (0, \ldots, 0) \in \mathbb{R}^d$.

Put $\nu := \frac{1}{\sqrt{d}}(0, 1, 1, \ldots, 1) \in \mathbb{R}^d$, $A(p) := pe_1 + \operatorname{Lin}\{e_2, \ldots, e_d\}$ and $L(p) := pe_1 + \operatorname{Lin}\{\nu\}$ for $p \in \mathbb{R}$. Then it is easy to verify that one of the following two statements is true:

(a) $f|_{L(p)}$ is non-convex for every $p$ from some interval $(a, b)$,

(b) $f|_{L(p)}$ is convex for every $p$.

Indeed, take any convergent sequence of real numbers $p_n \to p_\infty$ such that $f|_{L(p_n)}$ is convex for every $n$. Pick $t_1 < t_2 < t_3$ with $t_2 = \lambda t_1 + (1 - \lambda)t_3$ and define $x^i_\alpha = p_\alpha e_1 + t_i \nu$ for $i = 1, 2, 3$ and $\alpha \in \mathbb{N} \cup \{\infty\}$. For each $n$ we have

$$f(x^2_n) \leq \lambda f(x^1_n) + (1 - \lambda) f(x^3_n).$$

Since $f$ is continuous and $x^i_n \to x^i_\infty$ for $i = 1, 2, 3$, we obtain that

$$f(x^2_\infty) \leq \lambda f(x^1_\infty) + (1 - \lambda) f(x^3_\infty)$$

as well. Therefore the set $\{p \in \mathbb{R} : f|_{L(p)} \text{ is convex}\}$ is closed.

If $(a)$ holds, then by [2, Theorem 7.7] we know that $M \cap A(q)$ is of $\sigma$-finite $(d-2)$-dimensional Hausdorff measure for some $q \in (a, b)$. Applying the $(d-1)$-dimensional version of the theorem (which we assume to be true) to the function $f|_{A(q)}$, we obtain a contradiction.

On the other hand, $(b)$ is not possible either. Indeed, we can apply Lemma 4.3.1 to $f|_{\operatorname{Lin}\{v, e_1\}}$ the same way as in the proof of the case $d = 2$ and obtain a contradiction with the fact that $f$ is non-convex on the line $L$ which is contained in $\operatorname{Lin}\{v, e_1\}$. □

*Proof of Proposition 1.* Let us have a set $K$ as in Proposition 1, and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is locally convex on $K^c$. It is then easy to prove that $f$ is convex on all lines in the directions in which $K$ is intervally thin (use a limit argument similar to the one which was used in the above proof to show the dichotomy of (a) and (b)). As in the above proof for $d = 2$, we can now use Lemma 4.3.1 to show that $f$ is convex on each line segment from the (dense) set of line segments which intersect $K$ in at most countably many points. It follows, again by a limit argument, that $f$ is convex on all line segments, i.e. $f$ is convex on $\mathbb{R}^2$. □

## 4.4 Extensions of locally convex functions

**Definition 4.4.1.** We say that a set $A \subset \mathbb{R}^d$ is *totally disconnected in a direction* $v \in S^{d-1}$ if the set $A \cap l$ is totally disconnected for every line $l$ parallel to $v$.

The following lemma is a refinement of [5, Theorem 3.1] (note that the non-trivial part of the theorem is the existence of a unique continuous extension).

**Lemma 4.4.2.** *Suppose that $A \subset \mathbb{R}^d$ is closed and both totally disconnected and intervally thin in some direction $v \in S^{d-1}$. Let $U \subset \mathbb{R}^d$ be open. Then every function locally convex on $A^c \cap U$ admits a continuous extension to $U$.*

*Proof.* Let $f : U \to \mathbb{R}$ be locally convex on $A^c \cap U$. Choosing $x \in U$ (we can clearly assume $U \neq \emptyset$) and $\varepsilon > 0$, we need to prove that there is a $\delta > 0$ such that if $|x - a|, |x - b| \leq \delta$ and $a, b \in A^c \cap U$, then $|f(a) - f(b)| < \varepsilon$. Without any loss of generality we can suppose that $x = 0$ and that $v$ is parallel to one of the coordinate axes.

For $u \in \mathbb{R}^d$ and $r > 0$ put $l_u := u + \mathrm{Lin}\{v\}$ and $C(u, r) := u + [-r, r]^d$. Since $A$ is closed and totally disconnected in the direction $v$, we can find $\alpha > 0$ and $\frac{\alpha}{2} > \gamma > 0$ such that for $y := \alpha v$ we have $C(y, \gamma) \subset A^c \cap U$, $C(-y, \gamma) \subset A^c \cap U$, and such that the convex hull of $C(y, \gamma) \cup C(-y, \gamma)$ is contained in $U$. Since $f$ is locally convex on $A^c \cap U$ and therefore locally Lipschitz on $A^c \cap U$, there is $K > 0$ such that $f$ is K-Lipschitz on both $C(y, \gamma)$ and $C(-y, \gamma)$. Using the fact that $A$ is totally disconnected in the direction $v$ again, we can find $\min\left(\frac{\varepsilon}{25K}, \alpha - 2\gamma\right) > \lambda > 0$ such that for $z := \lambda v$ we have $z \in A^c \cap U$. Since $f$ is continuous on $A^c \cap U$, there is $\lambda > \delta > 0$ such that for every $u \in C(z, \delta)$ we have $|f(z) - f(u)| \leq \frac{\varepsilon}{4}$.

To obtain a contradiction, suppose that there are $a, b \in C(x, \delta) \cap A^c$ such that $|f(a) - f(b)| \geq \varepsilon$. Let $x_a$ and $x_b$ be the unique points in $(z + v^\perp) \cap l_a$ and $(z + v^\perp) \cap l_b$, respectively. Then $x_a, x_b \in C(z, \delta)$ and so $|f(z) - f(x_a)| \leq \frac{\varepsilon}{4}$ and $|f(z) - f(x_b)| \leq \frac{\varepsilon}{4}$. Moreover, one of the inequalities

$$f(a) - f(z) \geq \frac{\varepsilon}{2}, \ \ f(a) - f(z) \leq -\frac{\varepsilon}{2}, \ \ f(b) - f(z) \geq \frac{\varepsilon}{2}, \ \ f(b) - f(z) \leq -\frac{\varepsilon}{2}$$

must hold. Therefore, one of the inequalities

$$f(a) - f(x_a) \geq \frac{\varepsilon}{4}, \ \ f(a) - f(x_a) \leq -\frac{\varepsilon}{4}, \ \ f(b) - f(x_b) \geq \frac{\varepsilon}{4}, \ \ f(b) - f(x_b) \leq -\frac{\varepsilon}{4} \tag{4.4}$$

must hold as well.

Now, consider for instance the inequality $f(a) - f(x_a) \geq \frac{\varepsilon}{4}$. Since $A$ is intervally thin in the direction $v$, there are three collinear points $s_y \in C(-y, \gamma)$, $s_a \in C(x, \delta)$ and $s_{x_a} \in C(z, \delta)$ such that $[s_y, s_{x_a}] \subset U \setminus A$ and such that

$$|f(s_a) - f(a)| \leq \frac{\varepsilon}{16} \quad \text{and} \quad |f(s_{x_a}) - f(x_a)| \leq \frac{\varepsilon}{16}.$$

Then we have

$$|s_a - s_{x_a}| \leq |x - z| + 2\delta = \lambda + 2\delta \leq 3\lambda \leq \frac{3\varepsilon}{25K}. \tag{4.5}$$

Moreover,

$$\begin{aligned} f(s_a) - f(s_{x_a}) &\geq f(a) - f(x_a) - |f(s_{x_a}) - f(x_a)| - |f(s_a) - f(a)| \\ &\geq \frac{\varepsilon}{4} - \frac{\varepsilon}{16} - \frac{\varepsilon}{16} = \frac{\varepsilon}{8}. \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6) we obtain

$$\frac{f(s_a) - f(s_{x_a})}{|s_a - s_{x_a}|} \geq \frac{\varepsilon}{8} \cdot \frac{25K}{3\varepsilon} > K. \tag{4.7}$$

Choose an arbitrary $w \in ([s_y, s_{x_a}] \setminus \{s_y\}) \cap C(-y, \gamma)$. From the convexity of $f$ on $[s_y, s_{x_a}]$, the fact that $[s_y, w] \subset [s_y, s_a] \subset [s_y, s_{x_a}]$, (4.7) and the fact that $f$ is $K$-Lipschitz on $C(-y, \gamma)$ we obtain

$$K < \frac{f(s_a) - f(s_{x_a})}{|s_a - s_{x_a}|} \leq \frac{f(s_y) - f(w)}{|s_y - w|} \leq K,$$

which is not possible. The remaining cases in (4.4) can be proved following the same lines. □

**Example 4.4.3.** *There is a compact set $K \subset \mathbb{R}^2$ which is not intervally thin and such that for every $f : K^c \to \mathbb{R}$ locally convex on $K^c$ there is a convex extension $F : \mathbb{R}^2 \to \mathbb{R}$.*

*Proof.* Let $h : [0, 1] \to [0, 1]$ be the classical Cantor function (the Devil's Staircase) and let $C \subset [0, 1]$ be the Cantor ternary set. Now define the set $K$ as the graph of $h$ restricted to $C$ (so $K$ is the set named in the introduction as the Holey Devil's Staircase).

First note that $\mathcal{H}^1(K) < \mathcal{H}^1(\operatorname{graph} h) < \infty$ and that $K$ is intervally thin in directions $(1, 0)$ and $(0, 1)$. Therefore using Theorem 4.1 and Lemma 4.4.2 we obtain that $K$ has the desired extension property and so it remains to prove that $K$ is not intervally thin. Define $H : \mathbb{R} \to \mathbb{R}$ by $H = h$ on $[0, 1]$, $H = 0$ on $(-\infty, 0)$ and $H = 1$ on $(1, \infty)$. Then $\mathbb{R}^2 \setminus \operatorname{graph} H$ has two components, say $C^+$ and $C^-$, and therefore for any $x^\pm \in C^\pm$ the line segment $[x^+, x^-]$ intersects $\operatorname{graph} H$. Now take $x^+ \in B((-\frac{1}{3}, \frac{1}{3}), \frac{1}{4})$ and $x^- \in B((\frac{4}{3}, \frac{2}{3}), \frac{1}{4})$ and set

$$x := \sup\{a \in \mathbb{R} : \text{there exists } b \in \mathbb{R} \text{ such that } (a, b) \in [x^+, x^-] \cap \operatorname{graph} H\}.$$

Then $(x, H(x)) \in K$. □

**Remark 4.4.4.** Note that the set $K$ from Example 4.4.3 also provides an answer to Problem TT (see the introduction) in the negative. This can be seen from the fact that the argument used in the proof of Example 4.4.3 can be easily localized using the self affinity of $K$.

## 4.5 Two examples

In the first part of this section we prove Theorem 2 which provides us with a very natural class of examples of sets which are not $c$-removable. We shall need the following definitions.

Let $\beta, \varepsilon > 0$. Then we define the function $g_{\beta,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}$ by

$$g_{\beta,\varepsilon}(x, y) := \begin{cases} \beta y^2 - 2\varepsilon x - \varepsilon^2, & (x, y) \in (-\infty, -\varepsilon] \times \mathbb{R}, \\ \beta y^2 + x^2, & (x, y) \in (-\varepsilon, \varepsilon) \times \mathbb{R}, \\ \beta y^2 + 2\varepsilon x - \varepsilon^2, & (x, y) \in [\varepsilon, \infty) \times \mathbb{R}. \end{cases}$$

For $w \in \mathbb{R}$, set

$$g^w_{\beta,\varepsilon}(x,y) := g_{\beta,\varepsilon}(x-w,y) \quad \text{and} \quad h^w_{\beta,\varepsilon}(x,y) := g^w_{\beta,\varepsilon}(y,x).$$

One can readily verify that all the functions just defined are convex and $C^1$.

The Hessian matrix of $g^w_{\beta,\varepsilon}$ in $(-\infty, -\varepsilon + w) \times \mathbb{R}$, $(-\varepsilon + w, \varepsilon + w) \times \mathbb{R}$ and $(\varepsilon + w, \infty) \times \mathbb{R}$, respectively, is

$$\begin{pmatrix} 0 & 0 \\ 0 & 2\beta \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 2\beta \end{pmatrix}, \tag{4.8}$$

and similarly the Hessian matrix of $h^w_{\beta,\varepsilon}$ in $\mathbb{R} \times (-\infty, -\varepsilon + w)$, $\mathbb{R} \times (-\varepsilon + w, \varepsilon + w)$ and $\mathbb{R} \times (\varepsilon + w, \infty)$, respectively, is

$$\begin{pmatrix} 2\beta & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2\beta & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2\beta & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

Further, define the functions $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \{1,2,3,4\}$, as follows:

$$f_1(x,y) := \begin{cases} \frac{1}{12}x^2 + 4(y-1)^2, & (x,y) \in \mathbb{R} \times (1,\infty), \\ \frac{1}{12}x^2, & (x,y) \in \mathbb{R} \times (-\infty, 1], \end{cases}$$

and $f_2(x,y) := f_1(x,-y)$, $f_3(x,y) := f_1(y,x)$ and $f_4(x,y) := f_3(-x,y)$.

Again, it is easy to check that the functions $f_i$, $i = 1,2,3,4$ are $C^1$ and that the Hessian matrix of (e.g.) $f_1$ in $\mathbb{R} \times (1,\infty)$ and $\mathbb{R} \times (-\infty, 1)$, respectively, is

$$\begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.10}$$

It is also useful to note that the Hessian Matrix of $\varphi : (x,y) \mapsto -xy$ is

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{4.11}$$

**Lemma 4.5.1.** *Let there, for each $i \in \mathbb{N}$, be given $\varepsilon_i > 0$, $\beta_i \in \left(0, \frac{1}{80}\right)$ and $w_i \in (-1,1)$ such that $\sum \beta_i = \frac{1}{4}$, $\sum \varepsilon_i < \frac{1}{24}$ and $(w_i - \varepsilon_i, w_i + \varepsilon_i)$ are pairwise disjoint intervals contained in $[-1,1]$. Denote $g_i := g^{w_i}_{\beta_i,\varepsilon_i}$ and $h_i := h^{w_i}_{\beta_i,\varepsilon_i}$, define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as*

$$f(x,y) := \varphi(x,y) + \sum_{i=1}^{4} f_i(x,y) + \sum_{i=1}^{\infty} \left(g_i(x,y) + h_i(x,y)\right),$$

*and define the set $K \subset \mathbb{R}$ as*

$$K := [-1,1] \setminus \bigcup_{i=1}^{\infty} (w_i - \varepsilon_i, w_i + \varepsilon_i).$$

*Then $f$ has the following properties:*

1. *It is non-convex, since*

$$\frac{f(-1,-1) + f(1,1)}{2} < f(0,0);$$

*2. it is locally convex on $\mathbb{R}^2 \setminus K^2$ and continuous on $\mathbb{R}^2$.*

*Proof.* First, we need to check that $f$ is a well-defined function. To that end, it is sufficient to note that for any $R > 1$ and any $i \in \mathbb{N}$ the maximum of $g_i$ on $[-R, R]^2$ is attained at the point $(R, R)$ and is less than $\beta_i R^2 + 2\varepsilon_i 2R$. Obviously, the same is true for $h_i$, and so the infinite series in the definition of $f$ converges locally uniformly on $\mathbb{R}^2$.

Now, the Hessian matrix of $f$ at a point $(x, y) \in \mathbb{R}^2 \setminus K^2$ need not exist; however, all the summands in the definition of $f$, excluding $\varphi$, are convex functions. Since the sum of convex functions is convex, it is enough to prove that at any point $(x, y)$ outside $K^2$ we can find finitely many $f_i$'s, $g_i$'s and $h_i$'s such that their sum together with $-xy$ has a positively definite Hessian matrix at $(x, y)$.

This is easy to do if $(x, y) \notin [-1, 1]^2$; in this case we only need to use one of the $f_i$'s. For example, if $(x, y) \in \mathbb{R} \times (1, \infty)$, then we see from (4.10) and (4.11) that $\varphi + f_1$ is convex at $(x, y)$.

The "worst case" is that $(x, y) \in [-1, 1]^2$ lies in a single vertical stripe of the form $(w_k - \varepsilon_k, w_k + \varepsilon_k) \times \mathbb{R}$ (fix the $k$) and not in any of the horizontal stripes of the form $\mathbb{R} \times (w_i - \varepsilon_i, w_i + \varepsilon_i)$. In this case we find a finite set $F \subset \mathbb{N}$ such that $(x, y)$ does not lie on the boundary of any stripe of the form $\mathbb{R} \times (w_i - \varepsilon_i, w_i + \varepsilon_i)$ with $i \in F$ (which can happen at most twice) and such that

$$\alpha_F := \sum_{i \in F} \beta_i > \frac{9}{40}.$$

Let us now consider the function

$$f_F := \varphi + \sum_{i \in F} (g_i + h_i) + \begin{cases} g_k, & \text{if } k \notin F; \\ 0, & \text{if } k \in F. \end{cases}$$

Since we know that $(x, y)$ does not lie on the boundary of any of the stripes (horizontal or vertical) involved in the definition of $f_F$, the Hessian matrix of $f_F$ exists on a convex open neighbourhood $U$ of $(x, y)$ and it is easy to see from (4.8) and (4.9) that its determinant satisfies

$$\det\left(H_{f_F}(a, b)\right) \geq 4\left(\alpha_F + \alpha_F^2\right) - 1 > 4\left(\frac{9}{40} + \frac{81}{1600}\right) - 1 > 0, \quad (a, b) \in U.$$

As $\frac{\partial^2 f_F}{\partial x^2} > 0$, we obtain that $f_F$ is convex on $U$. On the other hand, $f - f_F$ is convex and therefore $f$ is convex on $U$; one can check the other cases in a similar way concluding the proof of property $(b)$.

It remains to verify property $(a)$. Denote

$$\delta_i := \frac{g_i(-1, -1) + g_i(1, 1)}{2} - g_i(0, 0).$$

An easy computation shows that

$$\delta_i = \beta_i + 2\varepsilon_i(1 - |w_i|)$$

and clearly the same is also true if we substitute all the occurrences of $g$ in the definition of $\delta_i$ by $h$. Hence,

$$\sum_{i=1}^{\infty} \delta_i = \sum_{i=1}^{\infty} \beta_i + 2\sum_{i=1}^{\infty} (\varepsilon_i(1 - |w_i|)) < \frac{1}{4} + 2\sum_{i=1}^{\infty} \varepsilon_i < \frac{1}{3}.$$

We also have

$$\sum_{i=1}^{4}\left(\frac{f_i(-1,-1)+f_i(1,1)}{2}-f_i(0,0)\right)=4\cdot\frac{1}{12}=\frac{1}{3}\,.$$

The last two facts clearly imply $(a)$. □

*Proof of Theorem 2.* If $A\times B$ contains a line segment then it is not $c$-removable as follows from [5, Example 2.1]. From now on, assume that the sets $A$ and $B$ are totally disconnected.

By the Lebesgue density theorem we can find points $a\in\mathbb{R}$ and $b\in\mathbb{R}$ which are density points of $A$ and $B$ respectively. Without loss of generality assume that $a=b=0$; since $A$ and $B$ are both closed, $0\in A\cap B$. Then 0 is clearly a point of density of $A\cap B\cap(-A)\cap(-B)$; consequently we can further assume that the sets $A$ and $B$ are symmetrical. We shall prove that $(A\cap B)^2$ is not $c$-removable.

Take an $r>0$ such that $\frac{1}{2r}\lambda\left(A\cap B\cap(-r,r)\right)>\frac{23}{24}$ and such that $r$ (and therefore also $-r$) is in $A\cap B$. Without loss of generality we can assume that $r=1$. Now, for $i\in\mathbb{N}$ take $w_i\in(-1,1)$ and $\varepsilon_i>0$ such that the intervals $(w_i-\varepsilon_i,w_i+\varepsilon_i)$ are pairwise disjoint and such that

$$[0,1]\setminus A\cap B=\bigcup_{i=1}^{\infty}(w_i-\varepsilon_i,w_i+\varepsilon_i).$$

The assumptions of Lemma 4.5.1 are now satisfied for any choice of positive numbers $\beta_i$, $i\in\mathbb{N}$, such that $\sum\beta_i=\frac{1}{4}$. □

The following two lemmas are concerned with the Koch curve and constitute a partial result regarding its $c$-removability. See also Problem 4.6.4.

**Lemma 4.5.2.**

$$\lim_{k\to\infty}\left(3^k\prod_{j=0}^{k-1}\frac{3^{j+1}+3}{3^{j+1}+1}-2\sum_{m=0}^{k-1}3^m\prod_{j=0}^{m-1}\frac{3^{j+1}+3}{3^{j+1}+1}\right)=\infty\,.$$

*Proof.* First, consider the following formula which follows easily by induction

$$\prod_{j=0}^{k-1}\frac{3^{j+1}+3}{3^{j+1}+1}=\frac{2\cdot 3^k}{3^k+1}\,.$$

Now

$$\begin{aligned}2\left(\frac{3^{2k}}{3^k+1}-2\sum_{m=0}^{k-1}\frac{3^{2m}}{3^m+1}\right)=&2\left(3^k-\frac{3^k}{3^k+1}-2\sum_{m=0}^{k-1}\left(3^m-\frac{3^m}{3^m+1}\right)\right)\\ =&2\left(3^k-2\sum_{m=0}^{k-1}3^m-\frac{3^k}{3^k+1}+2\sum_{m=0}^{k-1}\frac{3^m}{3^m+1}\right)\\ \geq&2\left(3^k-2\frac{3^k-1}{3-1}-1+2\sum_{m=0}^{k-1}\frac{3^m}{3^m+1}\right)\\ =&2\left(1-1+2\sum_{m=0}^{k-1}\frac{3^m}{3^m+1}\right)\geq 4\sum_{m=0}^{k-1}\frac{1}{2}=2k.\end{aligned}$$

□

**Lemma 4.5.3.** *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function locally convex on the complement of the Koch curve. Then $f$ is convex on every line parallel to the $y$-axis.*

*Proof.* First note that it is sufficient to prove the statement of the lemma for a dense set of lines parallel to the $y$-axis. Due to the self similarity of the Koch curve it is then sufficient to prove that there is no continuous function on $[0,3] \times [0, \frac{6}{\sqrt{3}}]$ such that $f(\frac{3}{2}, 0) + f(\frac{3}{2}, \frac{6}{\sqrt{3}}) < 2f(\frac{3}{2}, \frac{3}{\sqrt{3}})$. Here we consider the realization of the Koch curve with endpoints $(0,0)$ and $(3,0)$. For simplicity we will work in coordinates where the point $(\frac{3}{2}, \frac{3}{\sqrt{3}})$ is translated to the origin. For $i \in \mathbb{N}_0$ denote

$$a_i = \left(0, -\frac{\sqrt{3}}{3^{i+1}}\right), \quad b_i = \left(\frac{1}{2 \cdot 3^i}, -\frac{\sqrt{3}}{3^{i+1}}\right), \quad u_i = \left(\frac{1}{2 \cdot 3^i}, -\frac{\sqrt{3}}{3^{i+1}}\right),$$

$$z = \left(0, \frac{\sqrt{3}}{3}\right), \quad s_i = \left(\frac{1}{2 \cdot 3^i}, \frac{\sqrt{3}}{3}\right) \quad \text{and} \quad p_i = \left(\frac{1}{2 \cdot 3^i}, 0\right).$$

Modifying $f$ by adding an appropriate affine function and multiplying it by an appropriate constant we can suppose that $f(z) \leq 0$, $f(s_i) \leq 0$, $f(a_0) = 1$ and $f(p_i) \geq f(0,0) = 1$. Since $f$ is convex on $[s_i, b_i]$ for every $i$ we can write

$$\begin{aligned} f(b_i) \geq & f(u_{i+1}) + \frac{|b_i - u_{i+1}|}{|s_i - u_{i+1}|}(f(u_{i+1}) - f(s_i)) \geq \left(1 + \frac{\frac{\sqrt{3}}{3^{i+1}} - \frac{\sqrt{3}}{3^{i+2}}}{\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3^{i+2}}}\right) f(u_{i+1}) \\ = & \left(1 + \frac{2}{3^{i+1}+1}\right) f(u_{i+1}) = \frac{3^{i+1}+3}{3^{i+1}+1} f(u_{i+1}). \end{aligned} \tag{4.12}$$

Moreover, since $f$ is convex on $[a_i, u_i]$ for every $i$ we can write

$$\begin{aligned} f(u_i) \geq & f(b_i) + \frac{|b_i - u_i|}{|b_i - a_i|}(f(b_i) - f(a_i)) \\ \geq & f(b_i) + \left(\frac{\frac{1}{2 \cdot 3^i} - \frac{1}{2 \cdot 3^{i+1}}}{\frac{1}{2 \cdot 3^i}}\right)(f(b_i) - 1) = 3f(b_i) - 2. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) we then obtain

$$f(b_i) \geq \frac{3^{i+1}+3}{3^{i+1}+1}(3f(b_i) - 2),$$

and iterating for every $i < k$,

$$f(b_i) \geq 3^k \prod_{j=0}^{k-1} \frac{3^{j+1}+3}{3^{j+1}+1} f(b_{i+k}) - 2\sum_{m=0}^{k-1} 3^m \prod_{j=0}^{m-1} \frac{3^{j+1}+3}{3^{j+1}+1}.$$

Since $f$ is convex on $[s_i, b_i]$ we have for every $i$,

$$f(b_i) \geq f(p_{i+1}) + \frac{|s_i - p_{i+1}|}{|b_i - p_{i+1}|}(f(p_{i+1}) - f(s_i)) \geq f(p_{i+1}) \geq 1.$$

Finally, for $i = 0$ and any $k > 0$ we obtain

$$f(b_0) \geq 3^k \prod_{j=0}^{k-1} \frac{3^{j+1}+3}{3^{j+1}+1} - 2\sum_{m=0}^{k-1} 3^m \prod_{j=0}^{m-1} \frac{3^{j+1}+3}{3^{j+1}+1}$$

which is not possible due to Lemma 4.5.2 □

## 4.6 Open problems

The following general question is likely to be very difficult to answer, but naturally arises from the introduction of the notion of $c$-removability.

**Problem 4.6.1.** Is there any interesting characterization of $c$-removable sets?

However, there are several other interesting problems whose solutions might contribute to our understanding of the matter.

**Problem 4.6.2.** Is there a closed totally disconnected Lebesgue null set in $\mathbb{R}^2$ which is not $c$-removable?

**Problem 4.6.3.** Is the Cantor dust $c$-removable?

**Problem 4.6.4.** Is there a non-trivial $c$-removable continuum in $\mathbb{R}^2$?

Note that if one could prove that there is a dense set of lines intersecting the Koch curve in countably many points, the answer would be positive; this would follow from the proof of Theorem 4.1.

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