Referee Report for the Doctoral Thesis:
Shape optimization in contact problems with friction
by Róbert Pathó

In virtually all areas of mechanical and civil engineering one is confronted with problems involving the contact of static and/or moving components. In light of this, the ability to obtain optimal shapes in the design of these components that take into account the effects of contact and friction is of great importance. An "optimal" shape might be one that is robust to various loading and contact regimes or one that minimizes/spreads the stresses within the workpieces uniformly. A suitable model for such phenomena is the well-known Signorini problem with Tresca or, the mathematically more challenging and physically more relevant, Coulomb friction. Given a reference load and initial contact region of a workpiece, one may view the Signorini problem with Tresca or Coulomb friction as the forward problem in an optimization or control framework.

Mathematically speaking, the Signorini problem with Tresca friction is a type of elliptic variational inequality of the second kind. The Signorini problem with Coulomb friction may be understood as the embedding of the Signorini problem with Tresca friction in a fixed-point iteration. As such this problem may be viewed as a quasivariational inequality with a generally non-unique solution set.

The main goal of the thesis is to generalize the shape optimization of 2D contact problems with Tresca and Coulomb laws of friction to cases in which the coefficient of friction may depend on the magnitude of tangential displacements. This is done in an effort to model, e.g., dynamic contact problems or the stick-slip motion of tectonic plates during earthquakes.

The candidate proposes and develops the following approach:

1. Analyze the existence in finite, and whenever possible infinite, dimensional formulations of the contact problems.

2. Derive sensitivity results of the possibly set-valued solution operators of the contact problems with respect to the shape of the contact region and loading term.

3. Formulate general shape optimization problems and utilize the sensitivity results to prove the existence of an optimal shape, i.e., contact region amongst a select class of suitable shapes.

4. Derive generalized differentiability results for the set-valued solutions via the limiting variational calculus of B.Mordukhovich

5. Use the generalized differentiability results to approximate or obtain elements of the Clarke subdifferential of the reduced objective functions.

6. Extend the Bundle-Trust algorithm of Schramm and Zowe, as described in the book by Outrata, Kočvara, and Zowe, to the current setting.

7. Demonstrate the viability of this approach via numerical examples.

Despite the obvious (expected) influence of the supervisors, the candidate's approach is innovative and effective, at least according to the examples in Chapter 4. In particular, he displays the mastery of an impressively broad array of techniques. These range from the theory of contact problems and their discretization with the finite element method to nonlinear analysis, e.g., fixed-point theory and sensitivity analysis of solution operators, to set-valued analysis, e.g., the Mordukhovich coderivative and Clarke generalized Jacobians, and finally, to nonsmooth optimization, by the extension of the BT-algorithm. In doing so, the candidate has demonstrated the abilities that are commensurate with the conferral of a doctoral degree in mathematics.

The thesis is clearly written with only a small handful of typographical errors and the overall layout of the topics follows a clear logic. This is not to say that the thesis or the approach for that matter is free of any critical issues. I have listed these point below in detail; many of which are of a minor nature. In spite of my points of critique, I highly recommend the committee to accept Mr. Pathó's thesis.

Prof. Dr. Thomas M. Surowiec
Berlin, October 2014.

Specific Comments and Questions

1. Pg.7,Pgr.3 "standard nonlinear optimization problem": MPECs are not really standard.
2. Pg.8,Pgr.3 "Fast minimization algorithms" : In what sense "fast?" Are there other approaches?
3. Pg.8,Pgr.1 "In the present thesis, we aim at ... -see [49]." Why is the Tresca case analyzed? You mention later that it does not sufficiently capture the true physics.
4. Pg.10,Pgr.2-. The notation is eventually understood in context (by someone with experience in such problems). However, it is not fully introduced, e.g., we are led to deduce for ourselves that $\mathbf{u}_t = \mathbf{u} - u_n\boldsymbol{\nu}$.
5. Pg.11,Pgr.1 "It says that no slip occurs until the shear stress does not attain a certain threshold value." It is difficult to understand what you mean in this sentence. Do you mean: "If sheer stress attains a certain value, then slip occurs?"
6. Pg.11,Eq.(1.7). Do you really need to use $C^{0,1}$ and $C^{1,1}$ to define U_{ad} and $\widetilde{U}_{ad}$, respectively, in the continuous setting? It seems that the gradient and second-order constraints would allow you to take, e.g., H^1, and later H^2, instead. I could imagine that this would make the proof of existence of an optimal shape somewhat easier.
7. Pg.13,Eqs.(1.10)-(1.12). $\mathbf{H}^1(\Omega(\alpha))$ should be $\mathbf{V}(\alpha)$. Where do you define $\mathbf{L}^2(\widehat{\Omega})$, $\mathbf{H}^1(\widehat{\Omega})$?
8. Pg.14,Sec.(1.1.2). What is the real advantage to using this saddle-point formulation? Once the problem is discretized, (1.13)/(1.18) can be solved with a semismooth Newton method that would (most likely) enjoy global linear and local superlinear convergence on every mesh. This would surely be faster than an Uzawa-type iteration. As your final numerical scheme requires you to obtain a feasible point for every function evaluation, solving the saddle-point problem could be potentially very expensive computationally as the mesh is refined.

9. Pg.17,Thm.4. This form of citation is unacceptable. The book by Ekeland and Temam is several hundred pages long and contains many results.

10. Pg.19,Pgr.2. '...'the mappings $\mathbb{A} : U_{ad} \to \mathbb{R}^{n\times n}$..." It would have been helpful at some point in the thesis to have seen several real examples of these mappings.

11. Pg.21,Lma.1. Considering that the friction and slip constants are fixed here, I am wondering if this proof of the discrete inf-sup condition new?

12. Pg.21,Thm.6. Statements like "...following existence and uniqueness results are not difficult to prove." are, in my opinion, not befitting of a doctoral thesis. Moreover, the citation in the proof should point to the actual result in [19].

13. Pg.26,Pgr. "...can be carried out by means of suitable fixed-point theorems applied to Φ_h." Which fixed-point theorems? Schauder?

14. Pg.26,Thm.9. How small is $\overline{C}_d$? Does this rule out any physically relevant examples?

15. Pg.29,Pgr.4. "...should be smooth enough..." Where in [11] does one find this result? Do there exist any such functions that fulfill these conditions?

16. Pg.30,Rmk.5. "As we shall see, such issues are not..." What issues do you mean? I assume that there are compactness issues.

17. Pg.32,Thm.11. "Continuity of $\tilde{\Psi}_\alpha^C$ is very easy to verify:..." The argumentation is too terse in my opinion. Is it so obvious that you obtain convergence of the *entire* sequence $\{(u^{(i)}, \boldsymbol{\lambda}^{(i)}\}$?

18. Pg.36,Lma.3. It would have been better to have provided the proof of this lemma rather than the brief sketch following it.

19. Pg.45,Sec.2.4.1. "$\widetilde{S}$ is not convex,..." Do you mean that graph$\widetilde{S}$ is not convex or that the images of $\widetilde{S}$ are not convex?

20. Pg.45,Pgr.2. "In general, bundle methods have turned out to be the method of choice..." Since your problem arises as the discretization of an infinite-dimensional optimization problem, it can only be considered small to medium scale for coarse mesh refinements. Ultimately, you have a large-scale problem.

21. Pg.49,Pgr.2. "Therefore, the computation of the desired subgradient..." This fact is rather critical. Indeed, you state that the algorithm might "collapse" in nonregular cases since true subgradients are not provided. Furthermore, you state that $\boldsymbol{\xi}$ should be replaced by a correct subgradient. However, where is it stated that you can obtain a correct subgradient and for that matter, if you can in fact obtain a correct subgradient, then why don't you always do this?

22. Pg.52.Prp.4. In the proof, you claim that the calculation of the regular normal cone is straightforward. Since this is not one of the usual objects that one encounters in nonlinear analysis, it would have been helpful if you had provided the proof.

23. Pg.54,Eq.(2.48) A reference to Theorm 23 in the appendix would be helpful.

24. Pg.61,Sec. 3.2.2. One usually would not define an asserted property in the proof that this property holds.

25. Pg.63,Pgr.1 "Otherwise a recovery step has to be made..." This connects back to one of my previous comments. How do you actually do this recovery step?

26. Pg.65,Eq.(3.21) "A straightforward calculation yields:..." Seeing as this is a thesis, I would expect such a calculation to be included.

27. Pg.75,Pgr.1 "...when updating the bundle.." A more detailed discussion would have been desirable.

28. Pg.75,Pgr.2 "We conclude this section..." [52] is almost surely the wrong citation.

29. Pg.75,Pgr.3 "At the moment, however, there seem..." I think that a more thorough discussion would have been more appropriate.

30. Pg.78,Pgr.3. "...and C_{lip}, which can be made arbitrarily small." How can you make the Lipschitz modulus of given data arbitrarily small?

31. Pg.79,Pgr.4. "...total number of nodes...1800" This is a rather coarse discretization.

32. Pg.80-. I have several remarks and questions:

    (a) You only report the number of outer iterations. From this it is difficult to see how well the algorithm truly performed.

    (b) Since you are solving a discretization of an infinite dimensional problem, one would expect to see how the algorithm performed over mesh refinements, e.g., does the number of outer iterations increase or remain (relatively) unchanged?

    (c) How many iterations were needed (on average) to obtain a feasible point, i.e., solve the saddle-point problem? Assuming you used an alternating fixed-point iteration to solve the saddle-point problem, you still need to solve a variational inequality of the second kind at every loop. How was this done and how many iterations were needed?

    (d) What tolerances were used in these sub-subproblems? What were the stopping criteria and which norms were employed?

    (e) Do we know if the "smallness" assumptions on C_{lip} were satisfied for any of the examples?

    (f) In Example 1, you use an objective function that contains a norm. What norm was used here?

    (g) Did you ever observe cases in which the mapping was not coderivatively regular? Or some form of "biactivity?"

(h) You claim that the "BT algorithm converges to a solution." What is meant here by solution? Do we know if $0 \in \partial_C f(x)$ at this point? On page 82, in the third paragraph, you seemed to have stopped upon observing a certain decrease in the objective function value. Do we have any indication that a C-stationarity system is at least fulfilled up to a given tolerance?

33. Finally, it is clear that the calculation of limiting coderivatives of $\widehat{Q}_\tau$ and $N_{\mathbb{R}^p_+}$ are difficult to obtain exactly. If one is to go from the 2D to the 3D setting, then I can only imagine that this will become more difficult. On the other hand, it appears that one can lift (3.6) into a larger space and rewrite the resulting problem as a system of nonsmooth equations, without multifunctions, using the $\max(0, x)$ and $\min(0, x)$ functions. I wonder if this would have made the calculation of the generalized derivatives easier to handle.