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BACHELOR THESIS

Michael Bílý

Transformations of ODEs into gradient systems in stationary points

Department of Mathematical Analysis

Supervisor of the bachelor thesis:	RNDr. Tomáš Bárta, Ph.D.
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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Transformations of ODEs into gradient systems in stationary points

Autor: Michael Bílý

Katedra: Katedra matematické analýzy

Vedoucí bakalářské práce: RNDr. Tomáš Bárta, Ph.D., Katedra matematické analýzy

Abstrakt: Tato bakalářská práce navazuje na článek Bárta, Chill a Fašangová [1]. V tomto článku bylo ukázáno, že každá obyčejná diferenciální rovnice s Lyapunovskou funkcí je i gradientovým systémem. Toto bylo ukázáno pro určitou Riemannovskou metriku na množině nestacionárních bodů. V této práci odvodíme nutné a postačující podmínky aby tato metrika měla spojité rozšíření do izolovaného stacionárního bodu a tedy aby ODR byla gradientovým systémem na celém definičním oboru.

Klíčová slova: Gradientové systémy, riemannovská metrika, obyčejné diferencilní rovnice

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Author: Michael Bílý

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Abstract: This bachelor thesis follows article by Bárta, Chill a Fašangová [1]. It is proven there that every ordinary differential equation with a strict Lyapunov function is in fact a gradient system for certain Riemannian metric on the set of all non-equilibrium points. We will try to determine necessary and sufficient conditions for this Riemannian metric to have continuous extension to isolated equilibrium point so that the ODE is gradient system on the whole domain.

Keywords: Gradient systems, Riemannian metric, ordinary differential equations

Contents

Chapter 1

Introduction

The dissipative systems and namely gradient systems form the fundamental part of theory of ordinary and partial differential equations. The properties of dissipativity and gradient structure provide us tools to prove not just existence and uniqueness of solution but also let us obtain qualitative results about their regularity and long-time behaviour, like the stabilization or non-stabilization of results. For more on the subject of dissipative and gradient systems read Chill and Fašangová [2] and Teschl [3].

This bachelor thesis follows article by Bárta, Chill and Fašangová [1]. It is proven there that every ordinary differential equation

$$\dot{x} + F(x) = 0$$

with a strict Lyapunov function is in fact a gradient system for certain Riemannian metric on the set of all non-equilibrium points.

The gradient $\nabla_g \mathcal{E}$ of a function $\mathcal{E}$ is by definition dependant on the Riemannian metric g of ambient space, given as a unique vector such that

$$\langle \mathcal{E}', X \rangle = \langle \nabla_g \mathcal{E}, X \rangle_g \quad \text{for all } X \in \mathbb{R}^n.$$

If $\mathcal{E}$ is a strict Lyapunov function of the ODE, i.e.

$$\langle \mathcal{E}', F \rangle > 0 \quad \text{whenever } F \neq 0,$$

one can choose the metric $\tilde{g}$ such as $F = \nabla_{\tilde{g}} \mathcal{E}$ on the set of all non-equilibrium points. This implies that $F^{\perp_{\tilde{g}}} = \ker \mathcal{E}'$ because

$$\langle \mathcal{E}', X \rangle = \langle \nabla_{\tilde{g}} \mathcal{E}, X \rangle_{\tilde{g}} = \langle F, X \rangle_{\tilde{g}}\,.$$

If we consider the decomposition of vector $X = X_0 + X_1$ where $X_0 \in \ker \mathcal{E}'$ and $X_1 \in \langle F \rangle$ so

$$X_0 = X - X_1 \qquad \textit{and} \qquad X_1 = \frac{\langle \mathcal{E}', X \rangle}{\langle \mathcal{E}', F \rangle} F,$$

we get that

$$\langle X, Y \rangle_{\tilde{g}} = \langle X_0, Y_0 \rangle_{\tilde{g}} + \langle X_1, Y_1 \rangle_{\tilde{g}} = \langle X_0, Y_0 \rangle_{\tilde{g}} + \frac{\langle \mathcal{E}', X \rangle}{\langle \mathcal{E}', F \rangle} \frac{\langle \mathcal{E}', Y \rangle}{\langle \mathcal{E}', F \rangle} \langle F, F \rangle_{\tilde{g}}$$

$$= \langle X_0, Y_0 \rangle_{\tilde{g}} + \frac{1}{\langle \mathcal{E}', F \rangle} \langle \mathcal{E}', X \rangle \langle \mathcal{E}', Y \rangle .$$

From this we can see that the product $\langle X, Y \rangle_{\tilde{g}}$ is uniquely determined by functions F and $\mathcal{E}$ if one of the two vectors X, Y is from subspace $\langle F \rangle$. On $\ker \mathcal{E}' \times \ker \mathcal{E}'$ one has free choice of $\tilde{g}$. If the Riemannian metric $\tilde{g}$ can be continuously extended to an equilibrium point how does $\tilde{g}$ have to be defined on $\ker \mathcal{E}' \times \ker \mathcal{E}'$? Is it always possible to define it so that $\tilde{g}$ extends continuously? These questions were left unanswered and this bachelor thesis tries to answer them.

In the first part we assume that $\mathcal{E}''$ and F' exist and F' is regular in an equilibrium point. With this we develop necessary conditions for $\tilde{g}$ to have continuous extension, further we show that these are also sufficient. In the second part we move to a general case and let F' be singular then we develop necessary conditions. We will illustrate the theory on examples to show the use of often ugly conditions and try to explain that they are justified.

1.1 Definitions

Throughout this thesis we denote $\|\cdot\|$ the Euclidean norm and use this definition of ODE.

Definition 1. *We consider Ordinary Differential Equation (ODE) in the form of*

$$\dot{x} + F(x) = 0 \qquad \text{for } x \in \Omega, \tag{1.1}$$

where Ω is a connected open subset of $\mathbb{R}^n$, $n \geq 2$, and F is a continuous vector field on Ω.

Definition 2 (Gradient)**.** *For a function $\mathcal{E} : \Omega \to \mathbb{R}$ and a Riemannian metric g we define gradient $\nabla_g \mathcal{E}(x)$ the vector representation of the linear functional $\mathcal{E}'(x)$ in the scalar product $g(x)$, i.e.*

$$\langle \mathcal{E}'(x), X \rangle = \langle \nabla_g \mathcal{E}(x), X \rangle_{g(x)} \quad \text{for all } X \in \mathbb{R}^n.$$

Existence and uniqueness of gradient is due to representation theorem - see Theorem 10 in Appendix.

We use this definition of *strict Lyapunov function.*

Definition 3 (Lyapunov function)**.** *We say that a scalar function $\mathcal{E} \in C^1(\Omega)$ is a strict Lyapunov function for the ODE* (1.1) *if*

$$\langle \mathcal{E}'(x), F(x) \rangle > 0 \quad \text{whenever } x \in \Omega \quad \text{and} \quad F(x) \neq 0.$$

Every strict Lyapunov function is non-increasing along solutions of (1.1), and if it is constant on some solution then that solution must be stationary. On the other hand the gradient systems on Ω equipped with a Riemannian metric g

$$\dot{x} + \nabla_g \mathcal{E}(x) = 0$$

are prototype examples of dissipative system with Lyapunov function $\mathcal{E}$ itself.

M denotes the set of all equilibrium points of F i.e. $M := \{u \in \Omega : F(u) = 0\}$ and $\bar{x}$ denotes an isolated equilibrium point. We will examine only isolated point $\bar{x}$ of M in Ω, that means there is δ such that $\bar{x}$ is the only equilibrium point in $U(\bar{x}, \delta)$. We can reduce Ω to this U and assume that $\bar{x}$ is the only equilibrium point.

Definition 4 (Scalar product norm)**.** *On the set of all bilinear forms we define a norm*

$$\|a\| = \sup_{\|u\|=1, \|v\|=1} |a(u,v)| .$$

The subset of all scalar products is naturally equipped with metric induced by this norm.

Definition 5 (Riemannian metric)**.** *A continuous function g on Ω that assigns every $x \in \Omega$ a scalar product $g(x)$ is called a Riemannian metric on Ω.*

$\mathcal{L}(X, Y)$ denotes a set of all linear functions from a vector space X to a vector space Y. If $Y = \mathbb{R}$ then we write $\mathcal{L}(X)$. Due to Theorem 11 in Appendix for every Riemannian metric g there is an *Euclidean representation* $L : \Omega \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such as

$$\langle X, Y \rangle_{g(x)} = \langle L(x)X, Y \rangle_2 \quad \text{for all } X, Y \in \mathbb{R}^n, x \in \Omega.$$

Continuity of g can be seen as continuity of L or continuity of

$$x \to \langle X, Y \rangle_{g(x)}$$

for all vectors X, Y (or continuous vector fields) due to Theorem 12 in Appendix.

Definition 6 (Derivative)**.** *For a function $F : \Omega \to \mathbb{R}^m$ and $x \in \Omega$ we define $F^0(x) = F(x)$ and for $k \geq 1$ we define $F^k(x)$ the unique k-multilinear mapping to $\mathbb{R}^m$ (i.e. $\mathcal{L}(\mathbb{R}^m, \mathcal{L}(\dots, \mathbb{R}^m))$) satisfying*

$$\lim_{h \to 0} \frac{F^{k-1}(x + hu)(v) - F^{k-1}(x)(v) - hF^k(x)(v, u)}{h} = 0,$$

for all $v \in \mathbb{R}^{n(k-1)}$, $u \in \mathbb{R}^n$, if it exists.

We will write F' and F'' instead of F^1 and F^2, $F^k(x)u_1u_2 \dots u_k$ instead of $F^k(x)(u_1, u_2 \dots, u_k)$ and if $u := u_1 = u_2 = \dots = u_k$ then we simply write $F^k(x)u^k$, no confusion can be made.
Of course for $u_j = (u_j^1, \dots, u_j^n)$ in the standard basis we know that

$$(F^k(x)u_1u_2 \dots u_k)_j = \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} \frac{\partial^k F_j}{\partial x_{i_1} \dots \partial x_{i_k}}(x) u_1^{i_1} \dots u_k^{i_k}.$$

Taylor expansion theorem is stated in Appendix 9.

And finally we stick to the notation of duality between tangent and cotangent vectors from the article [1]. That means for $u' \in (\mathbb{R}^n)'$ and $v \in \mathbb{R}^n$ we denote $\langle u', v \rangle$ as the cotangent vector u' applied on the vector v. However in Taylor expansion of a function into $(\mathbb{R}^n)'$ we will omit the duality notation, so we simply write

$$\mathcal{E}'(x) = \mathcal{E}'(0) + \mathcal{E}''(0)x + \frac{1}{2!}\mathcal{E}^3(0)x^2 + \dots + \frac{1}{k!}\mathcal{E}^{k+1}(0)x^k + o(\|x\|^k).$$

Here all $\mathcal{E}^{j+1}(0)x^j$ are elements of $(\mathbb{R}^n)'$ as well as the $o(\|x\|^k)$ so it makes sense to write

$$\langle \mathcal{E}''(0)x, X\rangle \quad \text{and} \quad \left\langle o(\|x\|^k), X\right\rangle.$$

1.2 Motivational example

From the original paper [1] we know that if the ODE (1.1) is a gradient system with $\nabla_{\tilde{g}}\mathcal{E} = F$ then the Riemannian metric $\tilde{g}$ is given on the corresponding subspace $\langle F(x)\rangle$, the definition follows:

Definition 7. *For any strict Lyapunov function $\mathcal{E}$ of* (1.1), *$x \in \Omega$ which is not a equilibrium point of F and $X, Y \in \mathbb{R}^n$. We take the decomposition $X = X_0 + X_1$ where $X_0 \in \ker \mathcal{E}'(x)$ and $X_1 \in \langle F(x)\rangle$ so*

$$X_0 = X - X_1 \qquad \text{and} \qquad X_1 = \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle} F(x)$$

and we define $\tilde{g}$

$$\langle X, Y\rangle_{\tilde{g}(x)} = \langle X_0, Y_0\rangle_{g(x)} + \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle} \langle \mathcal{E}'(x), X\rangle \langle \mathcal{E}'(x), Y\rangle,$$

where g is an arbitrary Riemannian metric.

Our goal is to determine the necessary and sufficient conditions on $\mathcal{E}$ and g so that the corresponding $\tilde{g}$ extends continuously to an equilibrium point $\bar{x} \in M$. We will start with an example where $\tilde{g}$ cannot be continuously extended to the origin.

Example 1. *Let $\Omega = \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix} \quad \text{and} \quad \mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(x^2 + y^2).$$

Then $\tilde{g}$ does not have continuous extension to the origin for any g.

Clearly $\mathcal{E}$ is a strict Lyapunov function of ODE (1.1) and the origin is the only equilibrium point of F.

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix}, F\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} x - y \\ x + y \end{pmatrix} = x^2 - xy + xy + y^2 = x^2 + y^2.$$

In points $\begin{pmatrix} h \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ h \end{pmatrix}$ we get

$$F\begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} h \\ h \end{pmatrix} \quad \text{and} \quad F\begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} -h \\ h \end{pmatrix}.$$

Now we take X and Y such that they are in the corresponding subspaces, e.g.

$$X := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

From the Definition 7 of $\tilde{g}$ we can see that $X_0 = 0$ in point $\begin{pmatrix} h \\ 0 \end{pmatrix}$ and $Y_0 = 0$ in point $\begin{pmatrix} 0 \\ h \end{pmatrix}$, since

$$X \in \left\langle F \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle \quad \text{and} \quad Y \in \left\langle F \begin{pmatrix} 0 \\ h \end{pmatrix} \right\rangle.$$

Now the metric $\tilde{g}$ in these points and directions does not depend on metric g. Using the definition of $\tilde{g}$ we get

$$\begin{aligned}
\langle X, Y \rangle_{\tilde{g}(h,0)} &= \frac{1}{\left\langle \mathcal{E}' \begin{pmatrix} h \\ 0 \end{pmatrix}, F \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle} \left\langle \mathcal{E}' \begin{pmatrix} h \\ 0 \end{pmatrix}, X \right\rangle \left\langle \mathcal{E}' \begin{pmatrix} h \\ 0 \end{pmatrix}, Y \right\rangle \\
&= \frac{1}{h^2} \begin{pmatrix} h \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{h^2} h(-h) = -1, \\
\langle X, Y \rangle_{\tilde{g}(0,h)} &= \frac{1}{\left\langle \mathcal{E}' \begin{pmatrix} 0 \\ h \end{pmatrix}, F \begin{pmatrix} 0 \\ h \end{pmatrix} \right\rangle} \left\langle \mathcal{E}' \begin{pmatrix} 0 \\ h \end{pmatrix}, X \right\rangle \left\langle \mathcal{E}' \begin{pmatrix} 0 \\ h \end{pmatrix}, Y \right\rangle \\
&= \frac{1}{h^2} \begin{pmatrix} 0 \\ h \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{h^2} h \cdot h = 1.
\end{aligned}$$

Considering the limit for $h \to 0^+$ we can see that the metric $\tilde{g}$ does not have continuous extension to the origin.

❑

We can see that for any metric g the extension cannot exist. On the other hand the Example from [1] shows that even if $\mathcal{E}$ admits extension it does not admit it for all g. We will present this later in Example 2.
We will now study why this happens.

Chapter 2

Main Result

In this chapter we focus on the case when $\mathcal{E}''(\bar{x})$ and $F'(\bar{x})$ exist and $F'(\bar{x})$ is regular. Because we can reduce our domain we can without loss of generality assume that $\bar{x} = 0 \in M$ is the only equilibrium point of F.

We begin with two lemmas that reveal how $\mathcal{E}'(0)$ and $\mathcal{E}''(0)F'(0)^{-1}$ behave.

Lemma 1. *If $F'(0)$ exists and is regular then $\mathcal{E}'(0) = 0$.*

Proof. We take a fixed u such that $\|u\| = 1$ then we set $x = hu$ where h is a real number such that $x \in \Omega$. Then for all such h we have

$$\langle \mathcal{E}'(x), F(x) \rangle > 0, \quad \text{i.e.} \quad \left\langle \mathcal{E}'(x), \frac{F(hu)}{|h|} \right\rangle > 0,$$

since $\mathcal{E}$ is a strict Lyapunov function. Now taking the limit for $h \to 0^+$ and using $F(0) = 0$ we get

$$\langle \mathcal{E}'(0), F'(0)u \rangle \geq 0.$$

On the other hand, if $h \to 0^-$ we get

$$\langle \mathcal{E}'(0), F'(0)(-u) \rangle \geq 0,$$

so

$$\langle \mathcal{E}'(0), F(0)u \rangle = 0.$$

Since u was arbitrary and $F'(0)$ is regular, we get $\mathcal{E}'(0) = 0$.

❑

Lemma 2. *If $\mathcal{E}''(0)$ exists then for any $X \in \mathbb{R}^n$ we get*

$$\left\langle \mathcal{E}''(0)F'(0)^{-1}X, X \right\rangle \geq 0.$$

Proof. We know that $F(x) = F'(0)x + o(\|x\|)$ and $\mathcal{E}'(x) = \mathcal{E}''(0)x + o(\|x\|)$ so

$$\langle \mathcal{E}'(x), F(x) \rangle = \langle \mathcal{E}''(0)x, F'(0)x \rangle + \langle \mathcal{E}''(0)x, o(\|x\|) \rangle$$
$$+ \langle o(\|x\|), F'(0)x \rangle + \langle o(\|x\|), o(\|x\|) \rangle .$$

We take $\|u\| = 1$ and put $x = hu$ so $h = \|x\|$. Since $\mathcal{E}$ is a strict Lyapunov function

$$\langle \mathcal{E}'(x), F(x) \rangle > 0 \quad \text{for all } x \in \Omega \setminus \{0\},$$

using Taylor series and dividing by h^2 we get

$$\left\langle \mathcal{E}''(0)\frac{x}{h}, F'(0)\frac{x}{h}\right\rangle + \left\langle \mathcal{E}''(0)\frac{x}{h}, \frac{o(\|x\|)}{h}\right\rangle + \left\langle \frac{o(\|x\|)}{h}, F'(0)\frac{x}{h}\right\rangle + \left\langle \frac{o(\|x\|)}{h}, \frac{o(\|x\|)}{h}\right\rangle > 0.$$

Taking the limit for $h \to 0^+$ we get

$$\langle \mathcal{E}''(0)u, F'(0)u\rangle \geq 0. \tag{2.1}$$

Finally for $X = 0$ the assertion of the Lemma is clear. For $X \neq 0$ we take $u = \frac{1}{\|F'(0)^{-1}X\|}F'(0)^{-1}X$. Inserting this u into (2.1) we obtain

$$\frac{1}{\|F'(0)^{-1}X\|^2}\left\langle \mathcal{E}''(0)F'(0)^{-1}X, X\right\rangle \geq 0,$$

multiplying inequality by $\|F'(0)^{-1}X\|^2$ we get wanted result.

❑

2.1 Necessary conditions

Now we assume that there is a Riemannian metric $\tilde{g}$ such that $\nabla_{\tilde{g}}\mathcal{E} = F$ in Ω. Then for any vector $X \in \mathbb{R}^n$ following must hold

$$\langle F(x), X\rangle_{\tilde{g}(x)} = \langle \nabla_{\tilde{g}}\mathcal{E}(x), X\rangle_{\tilde{g}(x)} = \langle \mathcal{E}'(x), X\rangle.$$

Now we consider x in the form of hu where $u \neq 0$. We divide this equation by h and use $F(0) = 0$ and $\mathcal{E}'(0) = 0$ to get

$$\left\langle \frac{F(hu) - F(0)}{h}, X\right\rangle_{\tilde{g}(hu)} = \left\langle \frac{\mathcal{E}'(hu) - \mathcal{E}'(0)}{h}, X\right\rangle,$$

taking the limit for $h \to 0$ we get

$$\langle F'(0)u, X\rangle_{\tilde{g}(0)} = \langle \mathcal{E}''(0)u, X\rangle. \tag{2.2}$$

(Here we used property of continuity of metric $\tilde{g}$ 1.1.)
For any Y we take $u = F'(0)^{-1}Y$ and inserting into (2.2)

$$\langle Y, X\rangle_{\tilde{g}(0)} = \left\langle \mathcal{E}''(0)F'(0)^{-1}Y, X\right\rangle$$

From properties of scalar product it follows that

$$\left\langle \mathcal{E}''(0)F'(0)^{-1}X, X\right\rangle > 0 \quad \text{for all } X \in \mathbb{R}^n \setminus \{0\},$$

and $\langle \mathcal{E}''(0)F'(0)^{-1}\cdot, \cdot\rangle$ must be a symmetric bilinear form.

This leads us to formulate following condition on functions $\mathcal{E}$ and F:

Definition 8. *Let $\bar{x} \in \Omega$ and let $\mathcal{E} : \Omega \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}^n$ be two functions such that $F'(\bar{x})$ and $\mathcal{E}''(\bar{x})$ exist and $F'(\bar{x})$ is regular. If*

$$\langle X, Y \rangle \to \left\langle \mathcal{E}''(\bar{x})F'(\bar{x})^{-1}X, Y \right\rangle$$

is a scalar product, i.e.

$$\left\langle \mathcal{E}''(\bar{x})F'(\bar{x})^{-1}X, Y \right\rangle = \left\langle \mathcal{E}''(\bar{x})F'(\bar{x})^{-1}Y, X \right\rangle \quad \text{for all } X, Y \in \mathbb{R}^n,$$

$$\left\langle \mathcal{E}''(\bar{x})F'(\bar{x})^{-1}X, X \right\rangle > 0 \quad \text{for all } X \in \mathbb{R}^n \setminus \{0\}\,.$$

We say that functions $\mathcal{E}$ and F satisfy condition (♠) in $\bar{x}$.

Now back to our Example 1 we can see why it did not work:

$$\mathcal{E}''\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F'\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathcal{E}''\begin{pmatrix} 0 \\ 0 \end{pmatrix}\left(F'\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

which is clearly not a symmetric matrix so $\tilde{g}$ does not have continuous extension to the origin.
Now we can look at the example from the original article [1]:

Example 2. *Let $\Omega = \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ 2y \end{pmatrix} \quad \textit{and} \quad \mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} := \frac{1}{2}(x^2 + y^2),$$

then there is a Riemannian metric g such that $\tilde{g}$ allows continuous extension to the origin.

Again the origin is the only equilibrium point of F and for $(x, y) \neq (0, 0)$ we get

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix}, F\begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = x^2 + 2y^2 > 0.$$

So $\mathcal{E}$ is a strict Lyapunov function.

$$\mathcal{E}''(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\mathcal{E}''\begin{pmatrix} 0 \\ 0 \end{pmatrix}\left(F'\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

so functions $\mathcal{E}$ and F satisfy (♠) in $\bar{x}$. It was shown that if we take g to be the Euclidean metric then the corresponding metric $\tilde{g}$ does not have a continuous extension to the origin.
However if we take g to be given by

$$\langle X, Y \rangle_{g(x)} = X_1 Y_1 + \frac{1}{2} X_2 Y_2,$$

then corresponding $\tilde{g}$ does have a continuous extension to the origin.

❑

Now we will try to answer the question: If $\mathcal{E}$ and F satisfy condition (♠), for which metric g does corresponding metric $\tilde{g}$ have a continuous extension?

The following Theorem shows that only metric g that has chance to define continuous metric $\tilde{g}$ has to be in form of necessary condition for $\tilde{g}$.

Theorem 3. *Let $\bar{x} \in \Omega$ be isolated point of M in Ω and let $\mathcal{E} : \Omega \to \mathbb{R}$ be a strict Lyapunov function for (1.1). If functions $\mathcal{E}$ and F satisfy condition (♠) in $\bar{x}$ and g is a Riemannian metric on Ω such that the corresponding Riemannian metric $\tilde{g}$ from the Definition 7 has continuous extension to $\bar{x}$. Then*

$$\langle X, Y \rangle_{g(\bar{x})} = \left\langle \mathcal{E}''(\bar{x}) F'(\bar{x})^{-1} X, Y \right\rangle.$$

Proof. WLOG $\bar{x} = 0$ is the only equilibrium point of F.
We have already shown that

$$\langle X, Y \rangle_{\tilde{g}(0)} = \left\langle \mathcal{E}''(0) F'(0)^{-1} X, Y \right\rangle.$$

So we want to show that

$$\langle X, Y \rangle_{\tilde{g}(0)} = \langle X, Y \rangle_{g(0)} \quad \text{for all } X, Y \in \mathbb{R}^n$$

Since for any Riemannian metric h it holds that

$$\langle X, Y \rangle_h = \frac{\|X + Y\|_h^2 - \|X - Y\|_h^2}{2},$$

we can see that it is sufficient to show

$$\|X\|_{\tilde{g}(0)}^2 = \|X\|_{g(0)}^2 \quad \text{for all } X \in \mathbb{R}^n.$$

We fix $\|u\| = 1$ and put $x = hu$ where h is positive number such that $x \in \Omega$. We start with definition of $\tilde{g}$ in x

$$\langle X, Y \rangle_{\tilde{g}(x)} = \left\langle X - \frac{\langle \mathcal{E}'(x), X \rangle}{\langle \mathcal{E}'(x), F(x) \rangle} F(x), Y - \frac{\langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle} F(x) \right\rangle_{g(x)}$$

$$+ \frac{1}{\langle \mathcal{E}'(x), F(x) \rangle} \langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle \quad \text{for all } X, Y \in \mathbb{R}^n \text{ and } 0 \neq x \in \Omega.$$

Then

$$\langle X, Y \rangle_{\tilde{g}(x)} = \langle X, Y \rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), X \rangle}{\langle \mathcal{E}'(x), F(x) \rangle} \langle Y, F(x) \rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle} \langle F(x), X \rangle_{g(x)}$$

$$+ \frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle^2} \|F(x)\|_{g(x)}^2 + \frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle}$$

$$= \langle X, Y \rangle_{g(x)} - \frac{\left\langle \frac{\mathcal{E}'(hu)}{h}, X \right\rangle}{\left\langle \frac{\mathcal{E}'(hu)}{h}, \frac{F(hu)}{h} \right\rangle} \left\langle Y, \frac{F(hu)}{h} \right\rangle_{g(x)} - \frac{\left\langle \frac{\mathcal{E}'(hu)}{h}, Y \right\rangle}{\left\langle \frac{\mathcal{E}'(hu)}{h}, \frac{F(hu)}{h} \right\rangle} \left\langle \frac{F(hu)}{h}, X \right\rangle_{g(x)}$$

$$+\frac{\left\langle\frac{\mathcal{E}'(hu)}{h},X\right\rangle\left\langle\frac{\mathcal{E}'(hu)}{h},Y\right\rangle}{\left\langle\frac{\mathcal{E}'(hu)}{h},\frac{F(hu)}{h}\right\rangle^2}\left\|\frac{F(hu)}{h}\right\|_{g(x)}^2+\frac{\left\langle\frac{\mathcal{E}'(hu)}{h},X\right\rangle\left\langle\frac{\mathcal{E}'(hu)}{h},Y\right\rangle}{\left\langle\frac{\mathcal{E}'(hu)}{h},\frac{F(hu)}{h}\right\rangle}.$$

Now we take the limit for $h\to 0$ using the facts that $\mathcal{E}'(0)=0$ from Lemma 1 and $F(0)=0$. We get

$$\langle X,Y\rangle_{\tilde g(0)}=\langle X,Y\rangle_{g(0)}-\frac{\langle\mathcal{E}''(0)u,X\rangle}{\langle\mathcal{E}''(0)u,F'(0)u\rangle}\langle Y,F'(0)u\rangle_{g(0)}$$

$$-\frac{\langle\mathcal{E}''(0)u,Y\rangle}{\langle\mathcal{E}''(0)u,F'(0)u\rangle}\langle F'(0)u,X\rangle_{g(0)}$$

$$+\frac{\langle\mathcal{E}''(0)u,X\rangle\langle\mathcal{E}''(0)u,Y\rangle}{\langle\mathcal{E}''(0)u,F'(0)u\rangle^2}\|F'(0)u\|_{g(0)}^2+\frac{\langle\mathcal{E}''(0)u,X\rangle\langle\mathcal{E}''(0)u,Y\rangle}{\langle\mathcal{E}''(0)u,F'(0)u\rangle}.$$

Let us denote $Z:=F'(0)u$, since u was arbitrary and $F'(0)$ is a regular (from (♠)), we have that Z can have arbitrary direction.

By form of $\tilde g$ in 0 we can see that

$$\langle\mathcal{E}''(0)u,X\rangle=\langle Z,X\rangle_{\tilde g(0)}\quad\text{and}\quad\langle\mathcal{E}''(0)u,F'(0)u\rangle=\|Z\|_{\tilde g(0)}^2.$$

We continue

$$\langle X,Y\rangle_{\tilde g(0)}=\langle X,Y\rangle_{g(0)}-\frac{\langle Z,X\rangle_{\tilde g(0)}}{\|Z\|_{\tilde g(0)}^2}\langle Y,Z\rangle_{g(0)}-\frac{\langle Z,Y\rangle_{\tilde g(0)}}{\|Z\|_{\tilde g(0)}^2}\langle Z,X\rangle_{g(0)}$$

$$+\frac{\langle Z,X\rangle_{\tilde g(0)}\langle Z,Y\rangle_{\tilde g(0)}}{\|Z\|_{\tilde g(0)}^4}\|Z\|_{g(0)}^2+\frac{\langle Z,X\rangle_{\tilde g(0)}\langle Z,Y\rangle_{\tilde g(0)}}{\|Z\|_{\tilde g(0)}^2},$$

now we plug $X=Y$ and choose Z such that $X\perp_{\tilde g(0)}Z$ which is always possible for $n\geq 2$. This finally yields that

$$\|X\|_{\tilde g(0)}^2=\|X\|_{g(0)}^2$$

and the claim is proven.

❑

Back to our Example 2 we can see that

$$\langle X,Y\rangle_{g(x)}=X_1Y_1+\frac{1}{2}X_2Y_2=\left\langle\mathcal{E}''(0)F'(0)^{-1}X,Y\right\rangle$$

so this example can work.

2.2 Sufficient conditions

Now if we want to ask whether these are also sufficient conditions. So if $\mathcal{E}$ and F satisfy (♠) and g has corresponding form in equilibrium point, does $\tilde g$ have a continuous extension to this point? The answer is yes! But before we prove it we need two lemmas:

Lemma 4. *Let 0 be an isolated point of M in Ω and let $X : \Omega \to \mathbb{R}^n$ be a continuous vector field. If functions $\mathcal{E}$ and F satisfy condition (♠) in 0. Then there exist constants $C_X, d > 0$ such that $d \leq \left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, F'(0)\frac{x}{\|x\|}\right\rangle$ and $\left|\left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, X(x)\right\rangle\right| \leq C_X$ for all $x \neq 0$ from some neighbourhood of 0.*

Proof. Consider function $u \to \langle \mathcal{E}''(0)u, F'(0)u\rangle$ on the unit sphere in $\mathbb{R}^n$. The unit sphere is compact so the function attains its minimum $d := \min\limits_{\|u\|=1} \langle \mathcal{E}''(0)u, F'(0)u\rangle$. We can see that d is non-negative because of the positive definiteness from (♠) and is not zero because $F'(0)$ is a regular from the same property.
We can estimate second expression by the Cauchy-Schwarz inequality

$$\left|\left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, X(x)\right\rangle\right| \leq \|\mathcal{E}''(0)\| \, \|X(x)\| \,.$$

Since $\|X\|$ is continuous it attains its maximum on some δ-ball around 0 so we take $C_X := \max\limits_{\|x\|\leq\delta} \|X(x)\| \, \|\mathcal{E}''(0)\|$.

❑

Lemma 5. *Let $A \in \mathcal{L}(\mathbb{R}^n)$, $X : \Omega \to \mathbb{R}^n$ be continuous vector field, h be a Riemannian metric on Ω and $0 \in \Omega$. Then*

$$\lim_{x\to 0}\left(\left\langle A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_{h(x)} - \left\langle A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_{h(0)}\right) = 0$$

and

$$\lim_{x\to 0}\left(\left\langle A\frac{x}{\|x\|}, X(x)\right\rangle_{h(x)} - \left\langle A\frac{x}{\|x\|}, X(x)\right\rangle_{h(0)}\right) = 0.$$

Proof. We take $L : \Omega \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ the representation of h in the Euclidean scalar product from 11, that is

$$\langle X, Y\rangle_{h(x)} = \langle L(x)X, Y\rangle_2 \quad \text{for all } X, Y \in \mathbb{R}^{n.}$$

The first term rewrites as

$$\left|\left\langle A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_{h(x)} - \left\langle A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_{h(0)}\right| = \left|\left\langle (L(x) - L(0))\, A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_2\right|$$

$$= \left|\left\langle (L(x) - L(0))\, A\frac{x}{\|x\|}, A\frac{x}{\|x\|}\right\rangle_2\right| \leq \|L(x) - L(0)\| \, \|A\|^2 \,,$$

where we used the CS inequality. The limit now follows from continuity of L.

Similarly we consider second term

$$\left|\left\langle A\frac{x}{\|x\|}, X(x)\right\rangle_{h(x)} - \left\langle A\frac{x}{\|x\|}, X(x)\right\rangle_{h(0)}\right| = \left|\left\langle (L(x) - L(0))\, A\frac{x}{\|x\|}, X(x)\right\rangle_2\right|$$

$$= \left| \left\langle (L(x) - L(0))\, A \frac{x}{\|x\|}, X(x) \right\rangle_2 \right| \le \|L(x) - L(0)\| \, \|A\| \, \|X(x)\| .$$

The limit now follows from continuity of L and X.

❑

And here comes the main result.

Theorem 6. *Let $\mathcal{E} : \Omega \to \mathbb{R}$ be a strict Lyapunov function for* (1.1)*, $\bar{x} \in \Omega$ be isolated point of M in Ω. If functions $\mathcal{E}$ and F satisfy condition (♠) in $\bar{x}$ and g is a Riemannian metric on Ω such that $\langle X, Y \rangle_{g(\bar{x})} = \langle \mathcal{E}''(\bar{x})(F'(\bar{x}))^{-1} X, Y \rangle$ for all $X, Y \in \mathbb{R}^n$. Then the corresponding Riemannian metric $\tilde{g}$ defined by (7) has continuous extension to $\bar{x}$ and this extension is $g(\bar{x})$.*

Proof. WLOG $\bar{x} = 0$ is the only equilibrium point of F.

Note that from definition of g in 0 we have

$$\langle \mathcal{E}''(0)x, F'(0)x \rangle = \langle F'(0)x, F'(0)x \rangle_{g(0)} = \|F'(0)x\|^2_{g(0)} \tag{2.3}$$

and

$$\langle \mathcal{E}''(0)x, X \rangle = \langle F'(0)x, X \rangle_{g(0)} . \tag{2.4}$$

We need to show that if we plug $\tilde{g}(0) := g(0)$ then $\tilde{g}$ is continuous.

By Theorem 12 a metric r on Ω is continuous if the function

$$x \to \langle X, Y \rangle_{r(x)}$$

is continuous for all vectors.

So we consider any two vectors $X, Y \in \mathbb{R}^n$. First we are going to derive three limits and then put them together to get our result. First we will show that

$$\lim_{x \to 0} \frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right) = 0 \tag{2.5}$$

We have

$$\lim_{x \to 0} \frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right)$$

$$= \lim_{x \to 0} \frac{\langle \mathcal{E}''(0)x + o(\|x\|), X \rangle \langle \mathcal{E}''(0)x + o(\|x\|), Y \rangle}{\langle \mathcal{E}''(0)x + o(\|x\|), F'(0)x + o(\|x\|) \rangle^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right),$$

using arithmetic of limits and boundedness of $\frac{1}{\left\langle \mathcal{E}''(0) \frac{x}{\|x\|}, F'(0) \frac{x}{\|x\|} \right\rangle}$ we get

$$= \lim_{x \to 0} \frac{\|x\|^2 \langle \mathcal{E}''(0)x, X \rangle \langle \mathcal{E}''(0)x, Y \rangle}{\langle \mathcal{E}''(0)x, F'(0)x \rangle^2} \frac{1}{\|x\|^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right).$$

Using Lemma 4 we can see that $\left| \frac{\|x\|^2 \langle \mathcal{E}''(0)x, X \rangle \langle \mathcal{E}''(0)x, Y \rangle}{\langle \mathcal{E}''(0)x, F'(0)x \rangle^2} \right|$ is bounded by $\frac{C_X C_Y}{d^2}$. So it is sufficient to prove that $\frac{1}{\|x\|^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right)$ goes to 0.

$$\lim_{x \to 0} \frac{1}{\|x\|^2} \left(\|F(x)\|^2_{g(x)} - \langle \mathcal{E}'(x), F(x) \rangle \right)$$

$$= \lim_{x\to 0}\Big(\left\| F'(0)\frac{x}{\|x\|} + \frac{o(\|x\|)}{\|x\|}\right\|^2_{g(x)} - \left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, F'(0)\frac{x}{\|x\|}\right\rangle - \left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, \frac{o(\|x\|)}{\|x\|}\right\rangle - \left\langle \frac{o(\|x\|)}{\|x\|}, F'(0)\frac{x}{\|x\|}\right\rangle - \left\langle \frac{o(\|x\|)}{\|x\|}, \frac{o(\|x\|)}{\|x\|}\right\rangle \Big),$$

using (2.3) and arithmetic of limits we get

$$= \lim_{x\to 0}\left(\left\| F'(0)\frac{x}{\|x\|}\right\|^2_{g(x)} - \left\| F'(0)\frac{x}{\|x\|}\right\|^2_{g(0)} \right) = 0,$$

which is true because of Lemma 5.

Now in similar fashion we prove that

$$\lim_{x\to 0}\frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), X\rangle - \langle F(x), X\rangle_{g(x)}\right) = 0 \tag{2.6}$$

We have

$$\lim_{x\to 0}\frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), X\rangle - \langle F(x), X\rangle_{g(x)}\right)$$
$$= \lim_{x\to 0}\frac{\|x\|\langle \mathcal{E}''(0)x, Y\rangle}{\langle \mathcal{E}''(0)x, F'(0)x\rangle}\frac{1}{\|x\|}\left(\langle \mathcal{E}''(0)x, X\rangle - \langle F'(0)x, X\rangle_{g(x)}\right).$$

Again using Lemma 4 we can see that $\left|\frac{\|x\|\langle \mathcal{E}''(0)x, Y\rangle}{\langle \mathcal{E}''(0)x, F'(0)x\rangle}\right|$ is bounded by $\frac{C_Y}{d}$. So it is sufficient to prove $\frac{1}{\|x\|}\left(\langle \mathcal{E}''(0)x, X\rangle - \langle F'(0)x, X\rangle_{g(x)}\right)$ goes to 0. By (2.4) and Lemma 5 we have

$$\lim_{x\to 0}\left(\left\langle \mathcal{E}''(0)\frac{x}{\|x\|}, X\right\rangle - \left\langle F'(0)\frac{x}{\|x\|}, X\right\rangle_{g(x)}\right)$$
$$= \lim_{x\to 0}\left(\left\langle F'(0)\frac{x}{\|x\|}, X\right\rangle_{g(0)} - \left\langle F'(0)\frac{x}{\|x\|}, X\right\rangle_{g(x)}\right) = 0.$$

Analogically

$$\lim_{x\to o}\frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), Y\rangle - \langle F(x), Y\rangle_{g(x)}\right) = 0. \tag{2.7}$$

Now we will show that $\langle X, Y\rangle_{\tilde{g}(x)}$ goes to $\langle X, Y\rangle_{g(0)}$ as x goes to 0.
We fix $\varepsilon > 0$. From continuity of g and (2.5),(2.6),(2.7) there are $\delta_1, \delta_2, \delta_3$ and δ_4 such that

$$\left| \langle X, Y\rangle_{g(x)} - \langle X, Y\rangle_{g(0)}\right| < \frac{\varepsilon}{4}$$

for all $x \in P(0, \delta_1)$.

$$\left| \frac{\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle^2}\|F(x)\|^2_{g(x)} - \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle\right| < \frac{\varepsilon}{4}$$

for all $x \in P(0, \delta_2)$.

$$\left| \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), Y\rangle\text{-}\langle F(x), Y\rangle_{g(x)}\right)\right| < \frac{\varepsilon}{4}$$

for all $x \in P(0,\delta_3)$.

$$\left|\frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), X\rangle - \langle F(x), X\rangle_{g(x)}\right)\right| < \frac{\varepsilon}{4}$$

for all $x \in P(0,\delta_4)$.

Now the following holds for all $x \in P(0, \min\{\delta_1,\delta_2,\delta_3,\delta_4\})$.

We take the Definition 7 of $\tilde{g}$:

$$\begin{aligned}
\langle X, Y\rangle_{\tilde{g}(x)} &= \langle X_0, Y_0\rangle_{g(x)} + \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle \\
&= \left\langle X - \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}F(x), Y - \frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}F(x)\right\rangle_{g(x)} \\
&\quad + \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle \\
&= \langle X, Y\rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle F(x), Y\rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle X, F(x)\rangle_{g(x)} \\
&\quad + \frac{\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle^2}\|F(x)\|^2_{g(x)} + \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle.
\end{aligned}$$

Finally

$$\begin{aligned}
\left|\langle X, Y\rangle_{\tilde{g}(x)} - \langle X, Y\rangle_{g(0)}\right| &\le \left|\langle X, Y\rangle_{g(x)} - \langle X, Y\rangle_{g(0)}\right| \\
&+ \left| - \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle F(x), Y\rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle X, F(x)\rangle_{g(x)}\right. \\
&\left. + \frac{\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle^2}\|F(x)\|^2_{g(x)} + \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle\right| \\
&< \frac{\varepsilon}{4} + \left|\frac{\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle^2}\|F(x)\|^2_{g(x)} - \frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle\right| \\
&+ \left|\frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle - \frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle F(x), Y\rangle_{g(x)}\right| \\
&+ \left|\frac{1}{\langle \mathcal{E}'(x), F(x)\rangle}\langle \mathcal{E}'(x), X\rangle\langle \mathcal{E}'(x), Y\rangle - \frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\langle F(x), X\rangle_{g(x)}\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left|\frac{\langle \mathcal{E}'(x), X\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), Y\rangle - \langle F(x), Y\rangle_{g(x)}\right)\right| \\
&+ \left|\frac{\langle \mathcal{E}'(x), Y\rangle}{\langle \mathcal{E}'(x), F(x)\rangle}\left(\langle \mathcal{E}'(x), X\rangle - \langle F(x), X\rangle_{g(x)}\right)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Since this was for arbitrary X, Y we get that $\tilde{g}$ is continuous.

❑

Back to Example 1, if there is a Lyapunov function $\mathcal{E}$ such that

$$\mathcal{E}''(0) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

then

$$\mathcal{E}''(0)F'(0)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$$

which is symmetric and positive definite matrix so F can be gradient of such function.

Chapter 3

General case

Now we will try to ease the restraining assumption that $F'(0)$ is regular. But then our advance in determination of the necessary condition would fail on multiple occasions. First of all we cannot prove that $\langle \mathcal{E}'(0), X\rangle = 0$ for all $X \notin \operatorname{Im} F'(0)$. We can still prove that $\langle \mathcal{E}''(0)u, F'(0)u\rangle \geq 0$ for all $u \in \mathbb{R}^n$ but it is not now as interesting as before.

3.1 Necessary conditions

We can proceed on how our Riemannian metric $\tilde{g}$ has to look like if $\nabla_{\tilde{g}}\mathcal{E} = F$. By setting $x = hu$ where $u \neq 0$ is fixed and $h \in \mathbb{R}$ such that $x \in \Omega$ we get

$$\langle F(x), X\rangle_{\tilde{g}(x)} = \langle \mathcal{E}'(x), X\rangle$$

must hold for all $X \in \mathbb{R}^n$. Considering limit $h \to 0^+$ we get

$$\langle \mathcal{E}'(0), X\rangle = \langle 0, X\rangle_{\tilde{g}(0)} = 0.$$

So if $\tilde{g}$ has continuous extension to 0 we get that $\mathcal{E}'(0)$ is 0 after all.
Now we know what happens for u such that $F'(0)u \neq 0$. Now we assume that $F''(0)$ exists and $F'(0)u = 0$, then using Taylor series

$$\left\langle \frac{h^2}{2} F''(0)u^2 + o(\|x\|^2), X \right\rangle_{\tilde{g}(x)} = \langle h\mathcal{E}''(0)u + o(\|x\|), X\rangle\,,$$

dividing by h and using limit $h \to 0$ we get

$$\langle 0, X\rangle_{\tilde{g}(0)} = 0 = \langle \mathcal{E}''(0)u, X\rangle\,.$$

So we get $\mathcal{E}''(0)u = 0$. This leads us to formulate following condition on F and $\mathcal{E}$: Consider F and $\mathcal{E}$ such that for fixed $u \neq 0$ there are $k \in \mathbb{N}$ such that $F^k(0)u^k \neq 0$ and $m \in \mathbb{N}$ such that $\mathcal{E}^{m+1}(0)u^m \neq 0$. We take minimal such k and m then $k = m$. Proof follows in next Lemma.

Lemma 7. *If there exists continuous extension of $\tilde{g}$ to 0 then $k = m$*

Proof. Suppose that $k < m$ then

$$\langle F(x), X\rangle_{\tilde{g}(x)} = \left\langle \frac{h^k}{k!} F^k(0)u^k + o(\|x\|^k), X \right\rangle_{\tilde{g}(x)}$$

$$= \langle \mathcal{E}'(x), X \rangle = \left\langle o(\|x\|^k), X \right\rangle,$$

so dividing by h^k and using $h \to 0$

$$\left\langle \frac{1}{k!} F^k(0) u^k, X \right\rangle_{\tilde{g}(0)} = \langle 0, X \rangle = 0,$$

so $F^k(0)u^k = 0$ which is contradiction with definition of k.
For $k > m$ we get contradiction the other way around so $k = m$.

❑

Because we assume 0 is isolated point of M we require that for every $x \neq 0$ there exists such k that $F^k(0)x^k \neq 0$, otherwise we would have $F(hx) = 0$ for all sufficiently small h (this is only when F is analytical function but we require this anyway). So from now on we will assume that for every $x \neq 0$ there exists such k.

Again fix u we have

$$\left\langle \frac{h^k}{k!} F^k(0) u^k + o(\|x\|^k), X \right\rangle_{\tilde{g}(x)} = \left\langle \frac{h^k}{k!} \mathcal{E}^{k+1}(0) u^k + o(\|x\|^k), X \right\rangle.$$

Dividing by $\frac{k!}{h^k}$ sending $h \to 0^+$ we get

$$\left\langle F^k(0) u^k, X \right\rangle_{\tilde{g}(0)} = \left\langle \mathcal{E}^{k+1}(0) u^k, X \right\rangle \quad \text{for all } X \in \mathbb{R}^n.$$

We can now finally define analogy of condition (♠). But first we define k as function of x:

Definition 9. *Let $\bar{x} \in \Omega$ and let $\mathcal{E} : \Omega \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}^n$ be functions such that for all $x \in \mathbb{R}^n \setminus \{0\}$ there exists $k, m \in \mathbb{N}$ for which $F^k(\bar{x})x^k \neq 0$ and $\mathcal{E}^{m+1}(\bar{x})x^m \neq 0$. Then we define $c_F(x)$ and $c_{\mathcal{E}'}(x)$ as the smallest such k and m. ($\bar{x}$ is to be understood from context)*

Definition 10. *Let $F : \Omega \to \mathbb{R}^n$ be a vector field and $\mathcal{E} : \Omega \to \mathbb{R}$ be a strict Lyapunov function of ODR (1.1), $\bar{x} \in \Omega$ be an isolated point of M in Ω and $\mathcal{E}'(\bar{x}) = 0$. If functions c_F and $c_{\mathcal{E}'}$ are well-defined, bounded, $c := c_F \equiv c_{\mathcal{E}'}$ and there exists a scalar product r such that for all $u \neq 0$*

$$\left\langle F^{c(u)}(\bar{x}) u^{c(u)}, X \right\rangle_r = \left\langle \mathcal{E}^{c(u)+1}(\bar{x}) u^{c(u)}, X \right\rangle \quad \textit{holds for all } X \in \mathbb{R}^n$$

then we say functions F and $\mathcal{E}$ satisfy condition (♣) in $\bar{x}$.

Remark. From definition of r and Theorem 13 we can see that it is uniquely determined only if the dimension of $\left\langle \{F^{c(u)}(\bar{x}) u^{c(u)}; \|u\| = 1\} \right\rangle$ is n.

Remark. If $c(u) \equiv 1$ or equivalently $F'(0)$ is regular, then condition (♣) is same as (♠).

Remark. This condition also says that function F and $\mathcal{E}$ are similar in the sense that their n and $n+1$ derivatives share zero directions.

As we showed this condition is necessary for $\tilde{g}$ to have continuous extension to $\bar{x}$.

Now we will try to show why boundedness of c is reasonable assumption: Assume that F is analytic function and c_F is not bounded. Note that $c_F(x) = c_F(\frac{x}{\|x\|})$ for all $x \neq 0$ so we can think of c_F as function from unit sphere. If it is not bounded then there is sequence $\{u_k\}_{k=1}^{\infty}$ such that $c_F(u_k)$ is increasing. Unit sphere is a compact so there is convergent subsequence (we will assume that $\{u_k\}_{k=1}^{\infty}$ itself is convergent to some u). Now let δ be such number that F is sum of its Taylor series on δ-ball around 0. Then for $h < \delta$

$$F(hu_k) = \sum_{m=c_F(u_k)}^{\infty} \frac{h^m}{m!} F^m(0) u_k^m.$$

Taking the limit for $k \to \infty$ and using the uniform convergence to exchange limit and sum we get $F(hu) = 0$. This means that 0 cannot be isolated point of M.

We show example where the condition (♣) is trivially not met.

Example 3. *Let $\Omega = \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y^2 \end{pmatrix} \quad \textit{and} \quad \mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y.$$

Then $\tilde{g}$ does not have continuous extension to the origin.

The origin is clearly the only stationary point of F and for all $(x, y) \neq (0, 0)$ we have

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix}, F\begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \begin{pmatrix} 2x \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ y^2 \end{pmatrix} = 2x^2 + y^2 > 0.$$

So $\mathcal{E}$ is a strict Lyapunov function. But $\mathcal{E}'(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0$ so $\tilde{g}$ cannot have continuous extension to the origin.

❑

Also the dimension of $\left\langle \{F^{c(u)}(\bar{x})u^{c(u)}; \|u\| = 1\} \right\rangle$ can be 1 so r does not have to be uniquely determined as the following example illustrates.

Example 4. *Let $\Omega = \mathbb{R}^n$, $A \in \mathbb{R}^n$ be a fixed vector and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by*

$$F(x) = \frac{1}{2} \|x\|^2 A.$$

We can see that

$$F(x) = \frac{1}{2} F''(0) x^2,$$

so $\left\langle \{F^{c(u)}(0)u^{c(u)}; \|u\| = 1\} \right\rangle = \langle \{F''(0)u^2; \|u\| = 1\} \rangle = \langle A \rangle$.

❑

Finally we examine an example where $c_F = c_{\mathcal{E}}$ but the metric r from (♣) does not exist.

Example 5. *Let F be the function from previous Example and $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\mathcal{E}(x) = \frac{1}{6}\sum_{j=1}^{n} a_j x_j^3,$$

where a_j are coordinates of the vector A from previous example and assume that $a_j \neq 0$ for all $j \in \{1, \cdots, n\}$. Then the metric $\tilde{g}$ does not have continuous extension to 0 for any metric g.

We have

$$\langle \mathcal{E}'(x), F(x) \rangle = \frac{1}{2}\|x\|^2 \sum_{j=1}^{n} a_j^2 x_j^2 > 0,$$

so $\mathcal{E}$ is a strict Lyapunov function and $\mathcal{E}'(0) = 0$, further

$$\mathcal{E}'(x) = \frac{1}{2}\mathcal{E}'''(0)x^2 = \frac{1}{2}\begin{pmatrix} a_1 x_1^2 \\ \vdots \\ a_n x_n^2 \end{pmatrix} \neq 0$$

for all $x \neq 0$. From this it is clear that $c_F = c_{\mathcal{E}'} = 2$. But

$$\left\langle F''(0)x^2, X \right\rangle_r = \left\langle \|x\|^2 A, X \right\rangle_r = \|x\|^2 \langle A, X \rangle_r\,,$$

so it is constant for $\|x\| = 1$. On the other hand

$$\left\langle \mathcal{E}'''(0)x^2, X \right\rangle = \left\langle \begin{pmatrix} a_1 x_1^2 \\ \vdots \\ a_n x_n^2 \end{pmatrix}, X \right\rangle$$

so it attains different values for $\|x\| = 1$ thus the metric r from (♣) cannot exist. ❑

Now we can state analogy to Theorem 3 but we have to make extra assumption that images of derivatives of F are rich. We will make the proof shorter because it is very analogical to proof of Theorem 3.

Theorem 8. *Let $\mathcal{E} : \Omega \to \mathbb{R}$ be a strict Lyapunov function for (1.1), $\bar{x} \in \Omega$ be isolated point of M in Ω. If functions $\mathcal{E}$ and F satisfy condition (♣) in $\bar{x}$, the set $\{F^{c(u)}(\bar{x})u^{c(u)}; u \neq 0\}$ contains at least three pairwise independent vectors and g is a Riemannian metric on Ω such that the corresponding Riemannian metric $\tilde{g}$ from Definition (7) has continuous extension to $\bar{x}$ then $g(\bar{x}) = \tilde{g}(\bar{x})$.*

Proof. WLOG $\bar{x} = 0$ is the only equilibrium point of F.
We know that $\tilde{g}(0)$ has to be in the form of r from (♣).
We take $x = hu$ for fixed $u \neq 0$ and again rewrite definition of $\tilde{g}$:

$$\langle X, Y \rangle_{\tilde{g}(x)} = \langle X, Y \rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), X \rangle}{\langle \mathcal{E}'(x), F(x) \rangle}\langle Y, F(x) \rangle_{g(x)} - \frac{\langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle}\langle F(x), X \rangle_{g(x)}$$

$$+\frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle^2}\|F(x)\|_{g(x)}^2 + \frac{\langle \mathcal{E}'(x), X \rangle \langle \mathcal{E}'(x), Y \rangle}{\langle \mathcal{E}'(x), F(x) \rangle}.$$

Denote $Z := F^{c(u)}(0)u^{c(u)}$ then taking the limit for $h \to 0$ and using property of $\tilde{g}(0)$ from (♣) we get

$$\langle X, Y\rangle_{\tilde{g}(0)} = \langle X, Y\rangle_{g(0)} - \frac{\langle Z, X\rangle_{\tilde{g}(0)}}{\|Z\|^2_{\tilde{g}(0)}} \langle Y, Z\rangle_{g(0)} - \frac{\langle Z, Y\rangle_{\tilde{g}(0)}}{\|Z\|^2_{\tilde{g}(0)}} \langle Z, X\rangle_{g(0)}$$

$$+ \frac{\langle Z, X\rangle_{\tilde{g}(0)} \langle Z, Y\rangle_{\tilde{g}(0)}}{\|Z\|^4_{\tilde{g}(0)}} \|Z\|^2_{g(0)} + \frac{\langle Z, X\rangle_{\tilde{g}(0)} \langle Z, Y\rangle_{\tilde{g}(0)}}{\|Z\|^2_{\tilde{g}(0)}},$$

now we plug X, Y such that $X, Y \in Z^{\perp_{\tilde{g}(0)}}$, this finally yields

$$\langle X, Y\rangle_{\tilde{g}(0)} = \langle X, Y\rangle_{g(0)}$$

for all X, Y from subspaces $Z^{\perp_{\tilde{g}(0)}}$ for three pairwise independent vectors Z. Equality of $g(0)$ and $\tilde{g}(0)$ now follows from Theorem 14 in Appendix.

❑

The case when $\{Z \in \mathbb{R}^n; Z = F^{c(u)}(\bar{x})u^{c(u)}, \|u\| = 1\}$ does contain less then three independent vectors is beyond this thesis. We can only see that if r is uniquely determined for $n \geq 3$ then our assumption holds.

3.2 Examples of possible extension

In this section we shall examine few examples to suggest that these might be also sufficient conditions for $\tilde{g}$ to have continuous extension.

Unfortunately we are not able to prove analogy to Theorem 6. The reason why we cannot follow proof of Theorem 6 is that we do not have positive boundedness from below of function $F^{c(x)}\left(\frac{x}{\|x\|}\right)^{c(x)}$ as it can go to zero as $c(x)$ changes value.

The following two examples show how to determine the metric r from (♣) from derivatives of F and $\mathcal{E}$ and also show trivial extension to the origin.

Example 6. *If we replace $\mathcal{E}$ in Example 3 with*

$$\mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^3,$$

then there is g such that $\tilde{g}$ has continuous extension to the origin.

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix}, F\begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \begin{pmatrix} 2x \\ 3y^2 \end{pmatrix}^T \begin{pmatrix} x \\ y^2 \end{pmatrix} = 2x^2 + 3y^4 > 0.$$

We can see that $\mathcal{E}'(0) = 0$ and

$$F'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{E}''(0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

so

$$F'(0)x \neq 0 \quad \text{and} \quad \mathcal{E}''(0)x \neq 0 \quad \text{for all } x \notin \langle(0, 1)\rangle$$

further for $u = (0,1)$ we get

$$\mathcal{E}'''(0)u^2 = \begin{pmatrix} u_1 u_1 \frac{\partial^3 \mathcal{E}}{\partial x^3} + 2u_1 u_2 \frac{\partial^3 \mathcal{E}}{\partial x^2 \partial y} + u_2 u_2 \frac{\partial^3 \mathcal{E}}{\partial x \partial y^2} \\ u_1 u_1 \frac{\partial^3 \mathcal{E}}{\partial x^2 \partial y} + 2u_1 u_2 \frac{\partial^3 \mathcal{E}}{\partial x \partial y^2} + u_2 u_2 \frac{\partial^3 \mathcal{E}}{\partial y^3} \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \neq 0$$

and

$$F''(0)u^2 = \begin{pmatrix} u_1 u_1 \frac{\partial^2 F}{\partial x^2} + 2u_1 u_2 \frac{\partial^2 F}{\partial x \partial y} + u_2 u_2 \frac{\partial^2 F}{\partial y^2} \\ u_1 u_1 \frac{\partial^2 F}{\partial x^2} + 2u_1 u_2 \frac{\partial^2 F}{\partial x \partial y} + u_2 u_2 \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \neq 0.$$

So the metric $\tilde{g}$ has to fulfil for all $u = (u_1, u_2)$

$$\langle F'(0)u, X\rangle_{\tilde{g}(0)} = \left\langle \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, X \right\rangle_{\tilde{g}(0)} = \langle \mathcal{E}''(0)u, X\rangle = \left\langle \begin{pmatrix} 2u_1 \\ 0 \end{pmatrix}, X \right\rangle = 2u_1 X_1$$

and for $u = (0, u_2)$

$$\langle F''(0)u^2, X\rangle_{\tilde{g}(0)} = \left\langle \begin{pmatrix} 0 \\ 2u_2^2 \end{pmatrix}, X \right\rangle_{\tilde{g}(0)} = \langle \mathcal{E}'''(0)u^2, X\rangle = \left\langle \begin{pmatrix} 0 \\ 6u_2^2 \end{pmatrix}, X \right\rangle = 6u_2^2 X_2.$$

From this we can see that metric r given by

$$\langle X, Y\rangle_r = 2X_1Y_1 + 3X_2Y_2$$

satisfies that. Indeed if we take $g(x) \equiv r$ we get that $\tilde{g} \equiv g$ and so $\tilde{g}$ has continuous extention to the origin, namly $g(0)$.

❑

Similarly:

Example 7. *Let $\Omega = \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y^3 \end{pmatrix} \quad \text{and} \quad \mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^4.$$

Then there is g such that $\tilde{g}$ has continuous extension to the origin.

The origin is clearly the only stationary point of F and for all $(x,y) \neq (0,0)$ we have

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix}, \left(F\begin{pmatrix} x \\ y \end{pmatrix}\right)\right\rangle = \begin{pmatrix} 2x \\ 4y^3 \end{pmatrix}^T \begin{pmatrix} x \\ y^3 \end{pmatrix} = 2x^2 + 4y^6 > 0.$$

So $\mathcal{E}$ is a strict Lyapunov function. It is easy to see that $\mathcal{E}'(0) = 0$ and $c_F(x) = c_{\mathcal{E}'}(x)$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

$$F'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{E}''(0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $u = (0,1)$ we have

$$F''(0)u^2 = \mathcal{E}'''(0)u^2 = 0,$$

but

$$\mathcal{E}^4(0)u^3 = \begin{pmatrix} 0 \\ u_2^3 \frac{\partial^4 \mathcal{E}}{\partial y^4} \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \end{pmatrix} \neq 0$$

and

$$F'''(0)u^3 = \begin{pmatrix} 0 \\ u_2^3 \frac{\partial^3 F}{\partial y^3} \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \neq 0.$$

So this time we get for g defined by

$$\langle X, Y \rangle_{g(x)} = 2X_1Y_1 + 4X_2Y_2,$$

that $\tilde{g} \equiv g$.

❑

These examples however do not meet assumptions of Theorem 8 because $\{Z \in \mathbb{R}^n; Z = F^{c(u)}(\bar{x})u^{c(u)}, \|u\| = 1\}$ contains only two independent directions. This might suggest that this condition is not necessary for Theorem 8. We can also see that if $n = 2$ and $rank(F'(0)) = 1$ then this assumption is never met.

We will end with example when $\tilde{g}$ can be extended even if $g \neq \tilde{g}$.

Example 8. *Let $\Omega = \mathbb{R}^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y^5 \\ y^2 + x^6 \end{pmatrix} \quad \text{and} \quad \mathcal{E}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}x^2 + \frac{1}{3}y^3.$$

Then for g given as standard Euclidean product $\tilde{g}$ has continuous extension.

The origin is isolated point of M in $\mathbb{R}^2$ and

$$\left\langle \mathcal{E}'\begin{pmatrix} x \\ y \end{pmatrix} F\begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \begin{pmatrix} x \\ y^2 \end{pmatrix}^T \begin{pmatrix} x + y^5 \\ y^2 + x^6 \end{pmatrix} = x^2 + xy^5 + y^4 + y^2x^6 > 0$$

on some $P(0)$ so $\mathcal{E}$ is Lyapunov function on P. We can see that Euclidean scalar product satisfies condition (♣) since

$$\mathcal{E}'(0) = 0,\ \mathcal{E}''(0)x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix},\ F'(0)x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix},$$

$$\mathcal{E}'''(0)x^2 = \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix} \text{ and } F''(0)x^2 = \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix}.$$

From definition of $\tilde{g}$ we get

$$\langle X, Y \rangle_{\tilde{g}(x)} - \langle X, Y \rangle_2 = \frac{\langle \mathcal{E}', X \rangle \langle \mathcal{E}', Y \rangle}{\langle \mathcal{E}', F \rangle^2} \|F\|^2$$

$$+ \frac{\langle \mathcal{E}', X \rangle \langle \mathcal{E}', Y \rangle}{\langle \mathcal{E}', F \rangle} - \frac{\langle \mathcal{E}', X \rangle \langle F, Y \rangle_2}{\langle \mathcal{E}', F \rangle} - \frac{\langle \mathcal{E}', Y \rangle \langle F, X \rangle_2}{\langle \mathcal{E}', F \rangle}$$

$$= \frac{(xX_1 + y^2X_2)(xY_1 + y^2Y_2)}{(x^2 + xy^5 + y^4 + y^2x^6)^2}(x^2 + y^4 + 2xy^5 + 2y^2x^6 + x^{12} + y^{10})$$

$$+ \frac{(xX_1 + y^2X_2)(xY_1 + y^2Y_2)}{x^2 + xy^5 + y^4 + y^2x^6}$$

$$- \frac{(xX_1 + y^2X_2)((x + y^5)Y_1 + (y^2 + x^6)Y_2)}{x^2 + xy^5 + y^4 + y^2x^6}$$

$$-\frac{((x+y^5)X_1+(y^2+x^6)X_2)(xY_1+y^2Y_2)}{x^2+xy^5+y^4+y^2x^6}$$

$$=\frac{(xX_1+y^2X_2)(xY_1+y^2Y_2)}{(x^2+xy^5+y^4+y^2x^6)^2}(xy^5+y^2x^6+x^{12}+y^{10})$$

$$-\frac{(xX_1+y^2X_2)(y^5Y_1+x^6Y_2)}{x^2+xy^5+y^4+y^2x^6}-\frac{(y^5X_1+x^6X_2)(xY_1+y^2Y_2)}{x^2+xy^5+y^4+y^2x^6}.$$

Taking the limit for $(x,y)\to 0$ last three expressions go to 0 so $\tilde{g}$ can be extended to 0 by $g(0)$ which is the Euclidean product.

❏

Chapter 4

Conclusion

We found that if F' is regular in stationary point $\bar{x}$ then necessary and sufficient condition for $\tilde{g}$ to have continuous extension into $\bar{x}$ is that $\mathcal{E}'' \cdot F'^{-1}$ defines a scalar product in $\bar{x}$ and the arbitrary metric g has to be chosen to be equal to $\mathcal{E}''(\bar{x})F'(\bar{x})^{-1}$.

If $F'(\bar{x})$ is not regular then the necessary condition is that derivatives of F and $\mathcal{E}'$ must share zero directions and there must be a scalar product r such that

$$\left\langle F^{c(u)}(\bar{x})u^{c(u)}, X\right\rangle_{r(\bar{x})} = \left\langle \mathcal{E}^{c(u)+1}(\bar{x})u^{c(u)}, X\right\rangle \quad \text{for all } X \in \mathbb{R}^n.$$

With additional assumption we proved that g has to take form of this r in $\bar{x}$. Unfortunately we were not able to show that these are also sufficient conditions.

Appendix

We will state some theorems that we used.

Theorem 9 (Taylor series)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be function and $F^k(x)$ exists then*

$$F(x+h) = F(x)+F'(x)h+\ldots+\frac{1}{k!}F^k(x)h^k+o(\|h\|^k) = \sum_{j=1}^{k}\frac{1}{j!}F^j(x)h^j+o(\|h\|^k).$$

Proof. Follows from definition of derivative.

❑

Theorem 10. *Let g be a scalar product on $\mathbb{R}^n$. For every linear functional $u' \in (\mathbb{R}^n)'$ there exists a unique vector $u \in \mathbb{R}^n$ such that*

$$u'(v) = \langle u, v \rangle_g \quad \text{for all } v \in \mathbb{R}^n.$$

We say that $u' \in (\mathbb{R}^n)'$ is represented by the element $u \in \mathbb{R}^n$ with respect to the scalar product g.

Proof. For proof see [2, Lemma 2.1].

❑

Theorem 11 (Euclidean representation)**.** *Let r be a Riemannian metric on an open set $\Omega \subset \mathbb{R}^n$. Then there exists continuous function $L : \Omega \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that*

$$\langle X, Y \rangle_{r(x)} = \langle L(x)X, Y \rangle_2 \quad \text{for all } X, Y \in \mathbb{R}^n, x \in \Omega,$$

the linear map $L(x)$ is symmetric and positive definite. We will call L representation of r in Euclidean product.

Proof. For proof see [2, Lemma 2.3].

❑

Theorem 12. *A metric g is continuous on Ω if and only if for all vectors X, Y the function*

$$x \to \langle X, Y \rangle_{g(x)}$$

is a continuous function.

Proof. If g is continuous we will show that $x \to \langle X, Y \rangle_{g(x)}$ is continuous in any point $\bar{x} \in \Omega$. For L the Euclidean representation of g from previous Theorem we have

$$\left| \langle X, Y \rangle_{g(x)} - \langle X, Y \rangle_{g(\bar{x})} \right| = |\langle (L(x) - L(\bar{x})X, Y \rangle_2|$$

$$\leq \|L(x) - L(\bar{x})\| \, \|X\| \, \|Y\| \, .$$

So if g is continuous then L is continuous by previous Theorem and so is $x \to \langle X, Y \rangle_{g(x)}$.

On the other hand if $x \to \langle X, Y \rangle_{g(x)}$ is continuous then plugging e_i, e_j elements of standard basis for X and Y we get that $L(x)_{ij}$ (elements of matrix representation of $L(x)$) are continuous. This means that L is continuous too and so is g.

❑

Theorem 13 (Scalar product uniqueness)**.** *Let $\{U_j \in \mathbb{R}^n; 1 \leq j \leq k\}$ be k independent vectors, if $k = n$ and two scalar products r and h satisfy*

$$\langle U_j, X \rangle_r = \langle U_j, X \rangle_h \quad \textit{for all } X \in \mathbb{R}^n, 1 \leq j \leq k$$

then $r = h$. If $k < n$ then this is not necessarily true.

Proof. Consider basis that contains U_j as first k elements. We take Gramm matrices R and H of r and h with respect to this basis. By plugging U_j for X we get that for $k = n$ these metrics are equal, thus products are equal. For $k < n$ the elements on positions (i, j) where $k + 1 \leq i, j \leq n$ can be different.

❑

Theorem 14 (Scalar product equality)**.** *Let $Z_1, Z_2, Z_3 \in \mathbb{R}^n$ be three pairwise independent vectors and let r, h be two scalar products. If they satisfy*

$$\langle X_1, Y_1 \rangle_r = \langle X_1, Y_1 \rangle_h \quad \textit{for all } X_1, Y_1 \in Z_1^{\perp},$$

$$\langle X_2, Y_2 \rangle_r = \langle X_2, Y_2 \rangle_h \quad \textit{for all } X_2, Y_2 \in Z_2^{\perp},$$

and

$$\langle X_3, Y_3 \rangle_r = \langle X_3, Y_3 \rangle_h \quad \textit{for all } X_3, Y_3 \in Z_3^{\perp},$$

then $h = r$. Theorem does not hold for only two vectors Z_1, Z_2.

Proof. Consider Gramm matrices $H = \{h_{ij}\}_{i,j=1}^n$ and $R = \{r_{ij}\}_{i,j=1}^n$ of h and r with basis that has Z_1 as first and Z_2 as last element. If we consider vectors in this base then the subspace $Z_1^{\perp}$ consists of elements that have 0 as first coordinate, similarly $Z_2^{\perp}$ consists of elements with 0 as last coordinate. By plugging vectors $(0, \ldots, 1, \ldots, 0)$ we get that $h_{ij} = r_{ij}$ except for $h_{1n} = h_{n1}, r_{1n} = r_{n1}$. Because Z_3 is independent of Z_1 and Z_2 there exists an element of $Z_3^{\perp}$ such that its coordinates are $(a, \ldots, b)$ where $a, b \neq 0$. Plugging it in we get $h_{n1} = r_{n1}$ so matrices are equal and so are the scalar products. We can see that without third vector the elements on positions $(1, n)$ and $(n, 1)$ can be different.

❑

Bibliography

[1] Tomáš Bárta, Ralph Chill, and Eva Fašangová. Every ordinary differential equation with a strict lyapunov function is a gradient system. *Monatsh Math*, 166(1):57–64, April 2012.

[2] Ralph Chill and Eva Fašangová. *Gradient Systems.* Matfyzpress, Prague 2010.

[3] G. Teschl. *Ordinary differential equations and dynamical systems.* AMS Graduate Studies in Mathematics 140, 2012.